

McKean-Vlasov diffusion and the well-posedness of the Hookean bead-spring-chain model for dilute polymeric fluids



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Declaration of Authorship

This is to certify that to the best of my knowledge, the content of this thesis is my own work except where otherwise indicated. This thesis has not been submitted for any degree or other purposes. I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Abstract

We reformulate a general class of classical bead-spring-chain models for dilute polymeric fluids, with Hookean spring potentials, as McKean–Vlasov diffusion. This results in a coupled system of partial differential equations involving the unsteady incompressible linearized Navier–Stokes equations, referred to as the Oseen system, for the velocity and the pressure of the fluid, with a source term which is a nonlinear function of the probability density function, and a second-order degenerate parabolic Fokker–Planck equation, whose transport terms depend on the velocity field, for the probability density function. We show that this coupled Oseen–Fokker–Planck system has a large-data global weak solution. We then perform a rigorous passage to the limit as the masses of the beads in the bead-spring-chain converge to zero, which is shown in particular to result in equilibration in momentum space. The limiting problem is then used to perform a rigorous derivation of the Hookean bead-spring-chain model for dilute polymeric fluids, which has the interesting feature that, if the flow domain is bounded, then so is the associated configuration space domain and the associated Kramers stress tensor is defined by integration over this bounded configuration domain. We close by establishing the short-time existence of strong solutions to the corresponding Navier–Stokes–Fokker–Planck system.

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Nomenclature

(\cdot, \cdot)	the inner product of $L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$
$(\psi_k)_{k \geq 1}$	a complete orthogonal basis in $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$
0	d -component zero (column-)vector
B	linear transformation from $(q, x) \in D^J \times \Omega$ to $r \in \Omega^{J+1}$, so that $r = B(q, x)$
D	bounded, balanced, convex domain $D := \Omega - \Omega$, neighbourhood of $0 \in \mathbb{R}^d$, and $D \subset [-L, L]^d$ for some $L > 0$
$E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$	separable predual, with Young's function $\Psi(r) = e^{ r } - r - 1$; the (Banach) space $E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is defined as the closure, in the norm of the Orlicz space $L_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, of the set of all real-valued bounded measurable functions defined on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d}$
E_j	vector $E_j(r, v, t) := \frac{1}{\epsilon}((\mathcal{L}r)_j + u(r_j, t)) - \frac{\beta^2}{\epsilon^2}v_j$
$F = F(q_j)$	spring force connecting the beads
H	spring constant, $H > 0$
J	number of linear chains of beads, $J \in \mathbb{N}^*$
$L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$	Maxwellian-weighted Orlicz space, with Young's function $\Phi(r) = \mathcal{F}(1 + r)$
$L^p(0, T; X)$	Bochner space of p -integrable X -valued functions
$L^p(\omega)$	Lebesgue space on $\omega \subset \mathbb{R}^n$, $p \in [1, \infty]$, $n \in \mathbb{N}$
$L_{\text{div}}^p(\omega)$	subspace of divergence-free functions in $L^p(\omega)$
$L_{\text{loc}}^p(\omega)$	space of locally $L^p(\omega)$ functions
$L_0^p(\omega)$	space of zero boundary trace functions in $L^p(\omega)$
$L_{0,\text{div}}^p(\omega)$	subspace of divergence-free functions in $L_0^p(\omega)$
$M(v)$	the Maxwellian
$M^{-1}(W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'$	dual of $W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, $s > (J + 1)d + 1$ with respect to the duality pairing $\langle M \cdot, \cdot \rangle_{W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}$, $s > (J + 1)d + 1$

P_N	the orthogonal projector in $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ onto \mathcal{X}_N
T	final time in $(0, \infty)$
U	given spring potential, $U \in \mathcal{C}^{0,1}([0, b]; \mathbb{R})$, with $b := \sup_{p \in D} p $
W	Wiener process
$W_M^{1,2}(\mathbb{R}^{(J+1)d})$	the Maxwellian-weighted Sobolev space on $\mathbb{R}^{(J+1)d}$
$W^{s,p}(\omega)$	Sobolev space on ω , $s \in \mathbb{N}$, $p \in [1, \infty]$
$W_0^{s,p}(\omega)$	space of zero boundary trace functions in $W^{s,p}(\omega)$
$W_{0,\text{div}}^{s,p}(\omega)$	subspace of divergence-free functions in $W_0^{s,p}(\omega)$
$[x]_{\pm}$	the nonnegative and nonpositive parts of x
\mathbb{E}^x	conditional expectation
\mathbb{K}	an elastic extra stress tensor representing the contribution of the polymeric stress to the Cauchy stress
Ω	the bounded open convex flow domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$
Ξ	Young's function
α	a regularization parameter, $\alpha \in (0, 1]$
\mathcal{A}	nonlinear operator
$\mathcal{D}(\overline{D})$	space of distributions on \overline{D} with $\mathcal{D}(\overline{D}) := \mathcal{C}_0^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T])$
$\mathcal{D}_0(\overline{D})$	the space of functions $\varphi \in \mathcal{D}(\overline{D})$ such that $\varphi = 0$ on $\Sigma_0 \times (0, T)$, where $\Sigma_0 := \bigcup_{j=1}^{J+1} \left\{ (r, v) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d}, v_j \cdot n(r_j) = 0 \right\}$
\mathcal{L}	$(J+1) \times (J+1)$ block-matrix
\mathcal{L}_0^*	adjoint operator of \mathcal{L}_0
ϵ	small mass of the $J+1$ beads
$\gamma\varrho$	trace of ϱ on the boundary $\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T)$, $j = 1, \dots, J+1$
$\gamma_t\varrho$	trace $\gamma_t\varrho = \varrho(\cdot, t)$ of ϱ on the section $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times \{t\}$ for all $t \in [0, T]$
λ	constant factor characteristic of the stiffness of the springs, $\lambda > 0$
$\langle M\cdot, \cdot \rangle_{W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}$	duality pairing, $s > (J+1)d + 1$
$\langle \cdot, \cdot \rangle$	duality pairing between $(W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'$ and $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ with respect to $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ as pivot space
\mathbb{I}	the $d \times d$ identity matrix, $\mathbb{I} \in \mathbb{R}^{d \times d}$

\mathbb{O}	the $d \times d$ zero matrix, $\mathbb{O} \in \mathbb{R}^{d \times d}$
$\mathbb{R}^{d \times d}$	space of real $d \times d$ matrices
$\mathbb{R}_{\text{symm}}^{d \times d}$	subset of symmetric matrices in $\mathbb{R}^{d \times d}$
\mathbb{R}^d	space of all d -dimensional vectors
$\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$	space of continuous functions on $\mathbb{R}_{\geq 0}$
$\mathcal{C}_*^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d}))$	the imposition of the specular boundary condition on all functions that belong to the function space $\mathcal{C}^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d}))$
$\mathcal{C}_0^\infty(\omega)$	space of smooth, compactly supported functions on ω
\mathcal{C}^k	space of functions which are k -times differentiable and the k -th derivative is continuous
$\mathcal{C}^{0,\gamma}(\overline{\Omega})^d$	Hölder space $\mathcal{C}^{0,\gamma}(\overline{\Omega})^d$ for $0 < \gamma < 1 - \frac{d}{\sigma}$, $\sigma > d$
$\mathcal{C}_w([0, T]; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$	the linear space of weakly continuous mappings from $[0, T]$ into $L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$
$\mathcal{C}_{w*}([0, T]; X)$	the linear space of weakly-* continuous mappings from $[0, T]$ into X
$\mathcal{D}'(0, T; L^2(\Omega)/\mathbb{R})$	space of distributions
\mathcal{R}	Rouse matrix
\mathcal{X}_N	the span of the set $\{\psi_1, \dots, \psi_N\}$
$\mathfrak{B}_1, \dots, \mathfrak{B}_{J+1}$	linear chain of $J + 1$ beads
DB	Jacobian matrix of the transformation B
$D\Phi$	the Jacobian of the (bijective) mapping Φ
T	absolute temperature
$\nabla \cdot$	divergence operator
∇_r	$(J + 1)d$ -component column vectors gradient $\nabla_r := (\partial_{r_1}^\top, \dots, \partial_{r_{J+1}}^\top)^\top$
∇_v	$(J + 1)d$ -component column vectors gradient $\nabla_v := (\partial_{v_1}^\top, \dots, \partial_{v_{J+1}}^\top)^\top$
$\nu(r_j)$	the unit outward normal (column-)vector to $\partial\Omega$ at $r_j \in \partial\Omega$, for $j = 1, \dots, J + 1$
\otimes	tensor product between vectors or product σ -algebra
$\overline{\Omega}$	closure of the domain Ω
$\partial\Omega$	boundary of the domain Ω
$\partial\Omega^{(j)}$	$(J + 1)$ -fold Cartesian product
∂_t	partial time derivative
∂_{v_j}	the (d -component) gradient operator with respect to $v_j \in \mathbb{R}^d$
$\partial_{v_j} \cdot$	the divergence operator with respect to v_j
$\partial_{v_j}^2$	the Laplace operator with respect to v_j , $\partial_{v_j}^2 = \partial_{v_j} \cdot \partial_{v_j}$
π	is the pressure, $\pi : \Omega \times (0, T] \rightarrow \mathbb{R}$
φ	test function

ϱ	the probability density function associated with the law of a diffusion process for $(r, v) \varrho : (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T] \mapsto \varrho(r, v, t) \in \mathbb{R}_{\geq 0}$
ζ	drag coefficient
b	bounded divergence-free velocity field
d	spatial dimension, $d \in \{2, 3\}$
d_Ω	distance function $d = d_\Omega \in W^{2,\infty}(\mathbb{R}^d)$
e_i	i -th unit vector in \mathbb{R}^d , $i \in 1, \dots, d$
g^*	the convex conjugate of the convex function $g : a \in \mathbb{R} \mapsto g(a) \in (-\infty, +\infty]$
$n(z)$	the gradient field $n(z) = \nabla_z d(z)$ defined in $\bar{\Omega}$ which coincides with the unit outward normal vector to Ω at every point of $\partial\Omega$
q	beads distances vector, $q = q(r) := (q_1^T, \dots, q_J^T)^T$, where $q_j = q_j(r) := r_{j+1} - r_j$ for $j = 1, \dots, J$
q_j	distance between two beads \mathfrak{B}_j and \mathfrak{B}_{j+1} , $j = 1, \dots, J + 1$
r	beads positions vector, $r := (r_1^T, \dots, r_{J+1}^T)^T$, where $r_j \in \Omega$ for $j = 1, \dots, J + 1$
r_1, \dots, r_{J+1}	positions of the $J + 1$ beads $\mathfrak{B}_1, \dots, \mathfrak{B}_{J+1}$ respectively which are considered to be points
t	time variable, $t \in (0, T]$
u	the fluid velocity field, $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^d$
u_0	given initial velocity field, $u_0 \in W_0^{1-2/z, z}(\Omega)^d$, with $z = d + \vartheta$ for some $\vartheta \in (0, 1)$
v	beads velocities vector, $v := (v_1^T, \dots, v_{J+1}^T)^T$, where $v_j \in \mathbb{R}^d$ for $j = 1, \dots, J + 1$
$v_*^{(j)}$	specular velocity, $j = 1, \dots, J + 1$
x	spatial variable, $x := \frac{1}{J+1} \sum_{j=1}^{J+1} r_j \in \Omega$
$C^{0,1}([0, b]; \mathbb{R})$	space of Lipschitz continuous functions on $[0, b]$ with values in \mathbb{R}
\det	determinant of a matrix
$ \cdot $	the absolute value of a real number, the Euclidean norm of a vector, or the Frobenius norm of a square matrix, depending on the context

Chapter 1

Introduction

1.1 Evolution of polymeric fluid modelling: a literature review

Polymer liquid motion has fascinated many generations of scientists. Many years of research have been devoted to the study of polymeric fluids, whose motion cannot be described simply by the Navier–Stokes equations. One of the most popular polymer models is the so called bead-spring dumbbell model: the high-molecular weight polymer is described as $J + 1$ beads connected by J massless harmonic springs. Each bead may represent a subchain that contains several repeated units. The potential energy of such a chain can be calculated from the positions of each bead in space, or from the relative distance between the beads. There are two classical types of bead-spring dumbbell model for dilute polymers: one models polymer molecules with beads connected by elastic springs and the other with beads connected by rigid rods. A model of the first type which has Hookean springs and neglects hydrodynamic interaction between the beads is the Rouse model. In this case, a simple harmonic potential is assumed, that is, the virtual springs follow Hooke’s law of elasticity, which implies that they are infinitely extendable with a linear elastic response. A further example of a model of the first type are models which have nonlinear springs that are finitely extensible nonlinear and elastic, called FENE models. Several other of kinetic models both dumbbell-type and rigid-rod-type in the presence of inertial forces have been derived in the literature, but we focus in this thesis on Hookean bead-spring-chain models.

1.1.1 A historical survey

Many years of research effort gave rapid advances in this field. The advances in physics and mathematics have resulted in a hierarchy of kinetic models. However, many challenging problems in both theory and applications remain. In physics, Kramers [38] developed in 1944 a kinetic theory for dilute polymeric fluids undergoing potential flow. He assumed zero-shear viscosity and considered a freely rotating bead-rod chain without preaveraged hydrodynamic interactions. Kramers' work constituted the first attempt at micromodelling polymers and paved the way for major contributions in the field of dilute polymeric fluids from many physicists and mathematicians throughout the 20th century. Other authors, such as Zimm [66] in 1956, have developed a model of Hookean springs with preaveraged hydrodynamic interaction. Rouse proposed a model in 1953 that was a key breakthrough in the field [56]. In the Rouse model, the polymer molecules suspended in the solvent are represented by linear chains of $J + 1$ identical, spherical massless "beads", subjected to Brownian noise, connected by J Hookean "springs". The solvent is modelled as an incompressible, Newtonian fluid, which is completely characterized by its viscosity. Since the Rouse model was developed for dilute polymer solutions, no interactions between different chains are considered. The "beads" do not represent individual monomers but rather chain segments of twenty or more monomers: the beads need to be large compared to the solvent molecules in order to justify the continuum description of the solvent in which the beads are immersed. The Rouse model describes the conformational dynamics of an ideal polymer chain. The model has been widely used by polymer chemists for interpreting linear viscoelastic measurements, and it has had considerable impact on the direction of experimental programs. The dumbbell version of the model ($J = 2$) was previously investigated by J. J. Hermans [32] in 1943. El-Kareh and Leal [1] proposed in 1989 a model that is an approximate macroscopic closure of a FENE-type microscopic-macroscopic model with centre-of-mass diffusion. The abovementioned polymeric fluid models that were developed in polymer physics, were followed by several attempts to pursue a mathematical analysis of these models.

1.1.2 Mathematical analysis of polymeric fluid models

The role of mathematics in polymer kinetic theory for dilute polymeric fluids has come a long way since the development of its models in physics. Several partial differential equation analysis research papers contributed to the evolution of this

research subject. An early contribution to the existence and uniqueness of local-in-time solutions to a family of dumbbell type polymeric flow models is due to Renardy [55]. Renardy proved in 1991 a local existence and uniqueness theorem in the case of both infinitely extensible and finitely extensible dumbbells in the absence of a solvent. Technical assumptions were made about the spring force, which however didn't include the FENE model. Li, Zhang and Zhang [65] in 2004 extended the work of Renardy [55] but excluded the case of a FENE spring. In their paper they proved the well-posedness of coupled kinetic-hydrodynamic models for polymeric fluids. These models differ from traditional hydrodynamic models by taking explicitly into account the micromechanical structure of the polymers. E, Li and Zhang [22] in 2004 also proved local existence results for solutions of the nonlinear dumbbell equations. The dumbbell model they considered is a coupled hydrodynamic-kinetic model for polymeric fluids where the configuration of the dumbbells is described by stochastic differential equations. They proved well-posedness for general nonlinear spring laws with smooth potential by directly deriving a priori estimates on the stochastic equation satisfied by a Brownian configuration field. Zhang and Zhang [65] in 2006 also proved local existence results for solutions of the nonlinear dumbbell equations. They coupled the momentum-continuity pair with a Fokker-Planck equation. The main results were local existence, uniqueness and regularity theorems for the FENE model in certain parameter range. All these works required high regularity of the initial data. Moreover, Zhang and Zhang [65] showed that, subject to the technical limitations highlighted in the paper, a preassigned boundary condition for the FENE-type Fokker-Planck equation was unnecessary as a result of the singularity on the boundary of the configuration domain. A similar conclusion was drawn by Liu and Liu [45] in 2008 in the case of FENE models under a steady flow field when the FENE exponent $b \geq 2$. It was shown that $b = 2$ is a critical value in the sense that for $b < 2$ a boundary condition is necessary and when $b \geq 2$ specification of the boundary distribution becomes redundant. Jourdain, Lelievre and Le Bris [35] in 2003 studied the FENE dumbbell model in the case of a simple shear flow and b sufficiently large. Existence of a unique solution to the FENE Langevin equation was proved, and a local-in-time existence and uniqueness result for the system coupling the stochastic differential equation and the linear momentum equation was deduced. The long-time behavior of some micro-macro models for polymeric fluids was investigated by Jourdain, Le Bris, Lelievre and Otto [34] in 2006, and entropy inequalities were used to prove exponential convergence to equilibrium for FENE dumbbells and a sufficiently smooth flow. Moreover, they have only obtained complete results in

the case of a homogeneous stationary flow. They were not able to obtain the same result for Hookean dumbbells, nor in the case of a nonhomogeneous stationary flow. Lions and Masmoudi [43] in 2007 had previously based their proof of global-in-time weak solutions to the corotational Oldroyd–B model on propagation of compactness and subsequently adopted a similar approach in order to prove global existence of weak solutions for the corotational FENE dumbbell model. Their proof was based on propagation of compactness, namely if one takes a sequence of weak solutions which converges weakly and such that the initial data converges strongly then the weak limit is also a solution. Use of the antisymmetric part of the velocity gradient enabled better estimates to be obtained for the probability density function ϱ . Global existence for smooth solutions for the coupled microscopic-macroscopic two-dimensional corotational FENE model has also been established by Lin, Zhang and Zhang [40] in 2008. Masmoudi [47] in 2008 proved local and global well-posedness for the FENE dumbbell model for a very general class of potentials. Indeed, in prior local or global well-posedness results, conditions on the strength of the singularity (or on the parameter b) were made. He also proved global existence for the FENE dumbbell model if the initial state was close to equilibrium and for the corotational FENE dumbbell model in two dimensions.

In 2007, Barrett and Süli [6] derived a version of the model with centre-of-mass diffusion in the case of $J = 1$. The article by Schieber [58] is concerned with generalized dumbbell models with centre-of-mass diffusion, and the paper of Degond and Liu [19] justified the presence of the centre-of-mass diffusion term through asymptotic analysis. In the literature, standard derivations of bead-spring models tend to omit the centre-of-mass diffusion term on the grounds that it is several orders of magnitude smaller than the other terms in the equation, hence negligible.

Several works are concerned with analogous questions to the ones considered here in the case of bead-spring chains with FENE-type potentials. The FENE model is used to model long-chained polymers. It models long polymer molecules by connecting a sequence of beads with nonlinear springs. The FENE model is typically used for shear thinning fluids. In the work of Zhang and Zhang [65], the local existence of regular solutions to FENE-type dumbbell models has been shown. In 2008, Masmoudi, Zhang and Zhang [48] considered the corotational model, only in dimension $d = 2$ and a degenerate parabolic Fokker–Planck equation. In 2010, Barrett and Süli [7] considered a general Hookean-type model, in both dimensions two and three involving a parabolic Fokker–Planck equation. In 2018, Barrett and Süli [10] considered

a general Hookean model, in dimension $d = 2$; and subsolutions with defect measure for the dimension $d = 3$ and a parabolic Fokker–Planck equation.

Although almost all mathematical studies of the well-posedness of the micro-macro equations for polymeric fluids are limited to homogeneous or locally homogeneous flows an exception is the study of Barrett and Süli [6] in 2007. Here, the authors worked with the coupled macroscopic, nonhomogeneous Fokker–Planck equation system for the bead-spring model and established the existence of global-in-time weak solutions for a general class of spring-force potentials including that for the FENE spring. The directional Friedrichs mollifiers in the Kramers expression for the stress were replaced, however, by their isotropic counterparts to simplify the analysis. Earlier, Barrett, Schwab and Süli [5] in 2005 had proved the existence of global-in-time weak solutions to the coupled locally homogeneous system, with an x -mollified velocity gradient in the Fokker–Planck equation and an x -mollified probability density function ϱ in the Kramers expression for the stress. In 2012, Barrett and Süli [8] proved the existence of global-in-time weak solutions to a general class of coupled bead–spring chain models that arise from the kinetic theory of dilute solutions of nonhomogeneous polymeric fluids with variable density and viscosity and FENE spring potentials. Barrett and Süli [9] in 2016 proved the existence of global-in-time weak solutions to the FENE type model involving the unsteady, compressible, isentropic, isothermal Navier-Stokes system in a bounded domain. More recently in 2018, Barrett and Süli [10] explored the existence of global weak solutions to the Hookean dumbbell model, a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymers, involving the unsteady incompressible Navier–Stokes equations in a bounded domain in two or three space dimensions, coupled to a Fokker–Planck-type parabolic equation. They proved the existence of large-data global weak solutions in the case of two space dimensions. In three space dimensions, they proved the existence of large-data global weak subsolutions to the model, which are weak solutions with a defect measure. The main obstacle in proving the existence of global weak solutions to the Hookean bead-spring-chain model is due to the failure of a compactness argument, used in passing to a limit on the extra-stress tensor in the existence proof.

1.1.3 What have we learned from previous works?

There have been major contributions and breakthroughs in research of polymeric fluids by physicists and mathematicians. The advances and certain apparent confusions

contained in previous works offered valuable insights that helped us to define the direction that we have taken in our work.

There are two sources of apparent confusion in the literature. The first one is whether the Fokker–Planck equation is parabolic or degenerate parabolic and the second one is whether the configuration space is unbounded even when the flow domain is bounded. In this work, the idea to overcome the failure of a compactness argument encountered in earlier works, as for example in [10], is to revisit the derivation of the Hookean bead-spring chain model from a mathematically rigorous perspective. An interesting outcome of our work, reported here, is that, in fact, Kramer’s expression for the polymeric extra stress tensor for the bead-spring-chain model involves integration over a bounded domain instead of the whole space whenever the flow-domain is bounded.

1.2 Aim and outline

In this thesis, we perform the mathematical analysis of a set of partial differential equations which arise in mathematical models of dilute polymeric fluids. More precisely, we study mathematically the Navier–Stokes–Fokker–Planck systems arising in a class of bead-spring-chain models. We establish the existence of global-in-time weak solutions to a large class of bead-spring chain models with Hookean-type spring potentials, a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymer solutions. The solvent is an incompressible Newtonian fluid confined to a bounded open Lipschitz domain Ω in \mathbb{R}^d , $d \in \{2, 3\}$, with a \mathcal{C}^2 boundary. The conservation of momentum and mass equations for the solvent then have the form of the incompressible Navier–Stokes equations in which the elastic extra-stress tensor $\mathbb{K} : \Omega \times (0, T] \rightarrow \mathbb{R}_{\text{symm}}^{d \times d}$ (the polymeric part of the Cauchy stress tensor) appears as a source term:

$$\partial_t u + (b \cdot \nabla)u - \mu \Delta u + \nabla \pi = \nabla \cdot \mathbb{K} \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.1a)$$

$$\nabla \cdot u = 0 \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.1b)$$

$$u(x, t) = 0 \quad \text{for } (x, t) \in \partial\Omega \times (0, T], \quad (1.1c)$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \Omega, \quad (1.1d)$$

In the equations (1.1), $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^d$ denotes the velocity field, and $\pi : \Omega \times (0, T] \rightarrow \mathbb{R}$ is the pressure; b is a divergence-free vector field. In a bead-spring chain model, consisting of $J + 1$ beads coupled with J elastic springs to represent a polymer chain, the extra-stress tensor \mathbb{K} is defined by the Kramers expression as a weighted average of ϱ , the probability density function of the conformation

vector $q = (q_1^T, \dots, q_J^T)^T \in \mathbb{R}^{Jd}$ of the chain, with q_i representing the d -component orientation vector of the i -th spring. The class of spring forces under consideration here are of the form

$$F(q) = \lambda U'(|q|) q, \quad \text{for } q \in \mathbb{R}^{Jd},$$

where $\lambda > 0$ is a *spring constant*, characteristic of the stiffness of the spring, and U is a given spring potential. The equation satisfied by ψ is the following second-order parabolic equation, the Fokker–Planck equation, whose transport coefficients depend on the velocity field u :

$$\begin{aligned} \partial_t \psi + u \cdot \nabla \psi + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u) q_j \psi) \\ - \beta \sum_{i,j=1}^J \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta \psi = 0, \end{aligned} \quad (1.2a)$$

$$\psi(x, q, 0) = \psi_0(x, q), \quad (1.2b)$$

$$\nabla \psi(x, q, t) \cdot n_x(x) = 0 \quad \text{for all } (x, q, t) \in \partial\Omega \times D^J \times (0, T], \quad (1.2c)$$

$$\sum_{i=1}^J \left[\beta \mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) - ((\nabla u_{(0)}) q_j \psi) \right] \cdot n_{q_j} = 0, \quad (1.2d)$$

for all $(x, q, t) \in \Omega \times (D \times \dots \times \partial D \times \dots \times D) \times (0, T]$, $j = 1, \dots, J$, where n_{q_j} is the unit outward normal vector to ∂D for the j -th copy of the domain D of admissible conformation vectors in the Cartesian product $D^J = D \times \dots \times D$. The symmetric positive definite block matrix $\mathcal{R} = (\mathcal{R}_{ij})_{i,j=1}^{Jd}$ of size $Jd \times Jd$ is referred to as the *Rouse matrix* and

$$\mathfrak{M}(q) := (2\pi\beta)^{-\frac{1}{2}Jd} \exp(-|q|^2/2\beta), \quad \text{where } q = (q_1^T, \dots, q_J^T)^T \in D^J,$$

is the the Maxwellian with $\beta > 0$. A notable feature of equation (1.2a) in our model compared to classical Fokker–Planck equations for bead-spring models in the literature is the presence of the x -dissipative centre-of-mass diffusion term $\frac{\beta}{J+1} \Delta \psi$ in the Fokker–Planck equation (1.2a). The omission of the centre-of-mass diffusion term in the case of a heterogeneous solvent velocity is a mathematically counter-productive model reduction. When the centre-of-mass diffusion term is absent, the Fokker–Planck equation becomes a degenerate parabolic equation with a hyperbolic behaviour with respect to (x, t) . Since the study of weak solutions to the coupled problem requires one to work with velocity fields u that have very limited Sobolev regularity $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$, one is then forced into the technically unpleasant framework of hyperbolically degenerate parabolic equations with rough

transport coefficients. Here we show that the positive centre-of-mass diffusion coefficient $\frac{\beta}{J+1}$ is not a mathematical artifact but is in fact the outcome of the physical derivation of the model.

Our goal is to show the existence of global-in-time large-data weak solutions to these equations in the regime when polymer molecules are modelled as Hookean bead-spring chains, which are suspended in a Newtonian fluid. We reformulate a general class of classical bead-spring-chain models for dilute polymeric fluids, with Hookean spring potentials, as McKean–Vlasov diffusion. This results in a coupled system of partial differential equations involving the unsteady incompressible linearized Navier–Stokes equations, referred to as the Oseen system, for the velocity and the pressure of the fluid, with a source term which is a nonlinear function of the probability density function, and a second-order degenerate parabolic Fokker–Planck equation, whose transport terms depend on the velocity field, for the probability density function. We prove that this coupled Oseen–Fokker–Planck system has a large-data global weak solution. We then perform a rigorous passage to the limit as the masses of the beads in the bead-spring-chain converge to zero, which we show in particular to result in what is referred to in the polymer physics literature as equilibration in momentum space. The limiting problem is then used to perform a rigorous derivation of the Hookean bead-spring-chain model for dilute polymeric fluids, which has the interesting feature that, if the flow domain is bounded, then so is the associated configuration space domain and the associated Kramers stress tensor is defined by integration over this bounded configuration domain. This work represents the first mathematically rigorous derivation of this polymer model, which was originally proposed in the early 1940’s by the prominent Dutch theoretical physicist Hans Kramers (1894–1952).

1.3 Modelling

We consider a set of partial differential equations arising in a class of bead-spring-chain models for dilute polymeric fluids, where long polymer molecules immersed in a viscous incompressible Newtonian fluid are idealized as linear chains of $J + 1$ beads $\mathfrak{B}_1, \dots, \mathfrak{B}_{J+1}$, each with small mass ϵ , which are considered to be points positioned at r_1, \dots, r_{J+1} , respectively, in the flow domain $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$; \mathfrak{B}_{j+1} and \mathfrak{B}_j are assumed to be connected with an elastic spring with spring force $F = F(q_j)$, where $q_j = r_{j+1} - r_j$, $j = 1, \dots, J$. Models of this type involve the coupling of the incompressible Navier–Stokes equations with a Fokker–Planck equation.

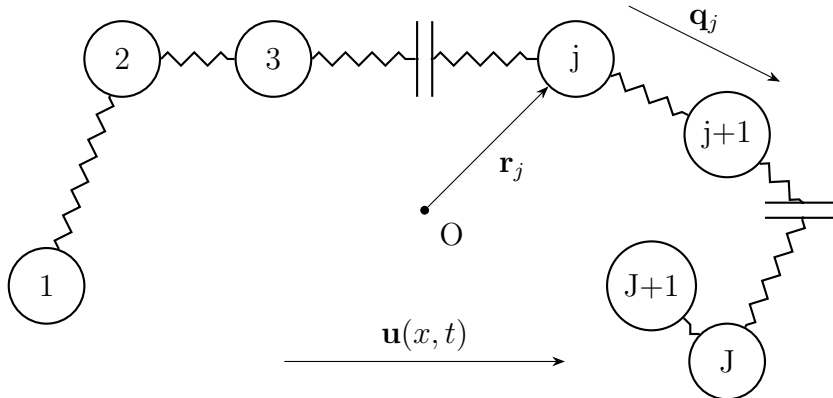


Figure 1.1: Noninteracting polymer chains, immersed into an incompressible Newtonian solvent with flow velocity u , are modelled by using dumbbells, each dumbbell representing a polymer chain. A dumbbell is a pair of beads, with centers of mass located, with respect to some fixed origin, connected with an elastic spring. The dumbbell is characterized by the position r of its center of mass and its elongation vector q .

Here we pursue an alternative line of investigation, which has to the best of our knowledge not, so far, been considered in connection with models of dilute polymeric fluids: we shall recast the model in terms of McKean–Vlasov diffusion, in the sense that the stochastic differential equation appearing in the model will have coefficients that depend on the distribution of the solution itself. As our objective here is to understand the impact of the McKean–Vlasov diffusion on the model rather than dealing with the usual technical difficulties associated with the presence of the non-linear convection term in the Navier–Stokes equation, we shall consider instead a linearization of the Navier–Stokes equation about a bounded divergence-free velocity field b , resulting in a linearized Navier–Stokes equation, usually referred to as the Oseen equation, whose right-hand side contains the divergence of an elastic extra stress tensor \mathbb{K} , representing the contribution of the polymeric stress to the Cauchy stress.

More precisely, we shall consider the following unsteady Oseen system on the space-time domain $\bar{\Omega} \times [0, T]$, where Ω is a bounded open convex domain in \mathbb{R}^d , $d \in \{2, 3\}$, with a \mathcal{C}^2 boundary, and $T > 0$:

$$\partial_t u + (b \cdot \nabla)u - \mu \Delta u + \nabla \pi = \nabla \cdot \mathbb{K} \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.3a)$$

$$\nabla \cdot u = 0 \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.3b)$$

$$u(x, t) = 0 \quad \text{for } (x, t) \in \partial\Omega \times (0, T], \quad (1.3c)$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \Omega, \quad (1.3d)$$

with

$$\mathbb{K}(x, t) := \mathbb{E}^x \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \quad \text{for } (x, t) \in \Omega \times (0, T], \quad J \geq 1, \quad (1.4)$$

where F is a spring force vector and \mathbb{E}^x denotes conditional expectation in a sense to be made precise below. We shall assume without loss of generality that $0 \in \mathbb{R}^d$ is the centroid, $\frac{1}{|\Omega|} \int_{\Omega} x \, dx$, of Ω .

In the equations (1.3), $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^d$ denotes the velocity field, and $\pi : \Omega \times (0, T] \rightarrow \mathbb{R}$ is the pressure; b is a divergence-free (in the sense of distributions on Ω) vector field, $b \in L^\infty(0, T; L^\infty(\Omega)^d)$ (see, however, Remark 2.6.1 concerning the weakening of this assumption); $u_0 \in W_0^{1-2/z, z}(\Omega)^d$, with $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, is a divergence-free (in the sense of distributions on Ω) initial velocity field; $\mu > 0$ is the viscosity coefficient; $\mathbb{K} : \Omega \times (0, T] \rightarrow \mathbb{R}_{\text{symm}}^{d \times d}$ is the elastic extra stress tensor involving the conditional expectation \mathbb{E}^x , which we now define. To this end we introduce the following notations:

$$\begin{aligned} r &:= (r_1^T, \dots, r_{J+1}^T)^T, & \text{where } r_j \in \Omega \text{ for } j = 1, \dots, J+1, \\ v &:= (v_1^T, \dots, v_{J+1}^T)^T, & \text{where } v_j \in \mathbb{R}^d \text{ for } j = 1, \dots, J+1, \\ q = q(r) &:= (q_1^T, \dots, q_J^T)^T, & \text{where } q_j = q_j(r) := r_{j+1} - r_j \text{ for } j = 1, \dots, J. \end{aligned}$$

We note here that

$$q_j \in D := \Omega - \Omega = \{\omega_1 - \omega_2 : \omega_1, \omega_2 \in \Omega\}, \quad \text{for } j = 1, \dots, J;$$

by definition, D is a bounded, balanced, convex neighbourhood of $0 \in \mathbb{R}^d$, and $D \subset [-L, L]^d$ for some $L > 0$; where here a balanced set is any set S such that $aS \subseteq S$ for all scalars a satisfying $|a| \leq 1$. Furthermore, we let

$$x := \frac{1}{J+1} \sum_{j=1}^{J+1} r_j.$$

Thanks to the assumed convexity of Ω , $x \in \Omega$ for any $r_1, \dots, r_{J+1} \in \Omega$.

Let

$$\varrho : (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T] \mapsto \varrho(r, v, t) \in \mathbb{R}_{\geq 0}$$

be the probability density function associated with the law of a diffusion process for (r, v) , which we shall define below; the law depends on ϱ itself through the function u and is therefore a McKean–Vlasov diffusion process.

Now, given $F \in L^\infty(D; \mathbb{R}^d)$, we define $\mathbb{E}(\sum_{j=1}^J F(q_j) \otimes q_j) : (0, T] \rightarrow \mathbb{R}_{\text{symm}}^{d \times d}$ by

$$\left(\mathbb{E} \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \right) (t) := \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^J F(q_j(r)) \otimes q_j(r) \right) \varrho(r, v, t) \, dr \, dv,$$

$$t \in (0, T],$$

and we perform a change of variables in this integral, replacing integration over $r \in \Omega^{J+1}$ by integration over $(q, x) \in D^J \times \Omega$. To this end, we note that the mapping $r \in \Omega^{J+1} \mapsto (q, x) \in D^J \times \Omega$ is one-to-one and onto. Denoting by B the linear transformation from $(q, x) \in D^J \times \Omega$ to $r \in \Omega^{J+1}$, so that $r = B(q, x)$, and letting DB denote the Jacobian matrix of the transformation, we have that

$$\left(\mathbb{E} \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \right) (t)$$

$$= \int_{D^J \times \Omega \times \mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \varrho(B(q, x), v, t) |\det DB| \, dq \, dx \, dv, \quad t \in (0, T].$$

Henceforth $|\cdot|$ will signify the absolute value of a real number, the Euclidean norm of a vector, or the Frobenius norm of a square matrix, depending on the context.

We note that the Cartesian product of $K \geq 1$ bounded open convex sets in \mathbb{R}^d is a bounded open convex set in \mathbb{R}^{Kd} (cf. [33], p.23), and that by Corollary 1.2.2.3 in [29], a bounded open convex set in a Euclidean space has Lipschitz boundary, so Ω^{J+1} and D^J are (convex) Lipschitz domains in $\mathbb{R}^{(J+1)d}$ and \mathbb{R}^{Jd} , respectively.

The class of spring forces under consideration here are of the form

$$F(q_j) = \lambda U'(|q_j|) q_j, \quad \text{for } q_j \in D, \quad j = 1, \dots, J,$$

where $\lambda > 0$ is a *spring constant*, characteristic of the stiffness of the spring, and U is a given spring potential, $U \in \mathcal{C}^{0,1}([0, b]; \mathbb{R})$, with $b := \sup_{p \in D} |p|$. For example, $U(s) = s$ corresponds to a model with Hookean springs, which we shall hereafter focus on in the rest of the thesis. Clearly, since Ω is bounded, the same is true of D and therefore $0 < b < \infty$.

The conditional expectation \mathbb{E}^x , which is the expectation under \mathbb{E} conditional on

$$\frac{1}{J+1} \sum_{j=1}^{J+1} r_j = x,$$

is then defined as follows: for $(x, t) \in \Omega \times (0, T]$,

$$\begin{aligned} & \left(\mathbb{E}^x \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \right) (x, t) \\ & := \frac{\int_{D^J \times \mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \varrho(B(q, x), v, t) |\det DB| \, dq \, dv}{\int_{D^J \times \mathbb{R}^{(J+1)d}} \varrho(B(q, x), v, t) |\det DB| \, dq \, dv}. \end{aligned}$$

Since DB is independent of q and v , the factor $|\det DB|$ cancels in the numerator, which is a $d \times d$ symmetric positive semidefinite matrix function, and in the denominator, and the expression for the above conditional expectation is thereby simplified to

$$\begin{aligned} & \left(\mathbb{E}^x \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \right) (x, t) = \frac{\int_{D^J \times \mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \varrho(B(q, x), v, t) \, dq \, dv}{\int_{D^J \times \mathbb{R}^{(J+1)d}} \varrho(B(q, x), v, t) \, dq \, dv}, \\ & (x, t) \in \Omega \times (0, T]. \end{aligned}$$

We note that if the denominator vanishes at a point $(x_0, t_0) \in \Omega \times (0, T]$, then, since ϱ is a nonnegative function, necessarily $\varrho(B(q, x_0), v, t_0) = 0$ for a.e. $(q, v) \in D^J \times \mathbb{R}^{(J+1)d}$, and therefore the numerator also vanishes at (x_0, t_0) . We shall adopt the convention that the ratio $0/0$ is, by definition, equal to 0.

Hence, now with $|\cdot|$ signifying the Frobenius matrix norm on $\mathbb{R}^{d \times d}$,

$$\left| \left(\mathbb{E}^x \left(\sum_{j=1}^J F(q_j) \otimes q_j \right) \right) (x, t) \right| \leq \text{ess. sup}_{q \in D^J} \sum_{j=1}^J |F(q_j) \otimes q_j| \quad \forall (x, t) \in \Omega \times (0, T],$$

whereby, recalling (1.4),

$$\|\mathbb{K}\|_{L^\infty(0, T; L^\infty(\Omega))} \leq \sum_{j=1}^J \|F(q_j) \otimes q_j\|_{L^\infty(D^J)} < \infty. \quad (1.5)$$

We note that, given ϱ , we may write $u(x, t) = (\mathcal{A}\varrho)(x, t)$, where the nonlinear operator \mathcal{A} involves composition of the ratio of two integral operators (as in the definition of the conditional expectation \mathbb{E}^x above), the divergence operator $\nabla \cdot$, and the solution operator for the time-dependent Oseen problem. As the velocity field $u = \mathcal{A}\varrho$ appears as a coefficient in the Fokker–Planck equation for the probability density function ϱ , it follows that it is, in fact, in the present context, a *nonlinear* partial differential equation for ϱ .

We will show that this coupled Oseen–Fokker–Planck system has a large-data global weak solution; having done so, we shall perform a rigorous analysis of the

small-mass limit, $\epsilon \rightarrow 0_+$, corresponding to passage to the limit as the masses of the beads $\mathfrak{B}_1, \dots, \mathfrak{B}_{J+1}$ in the bead-spring-chain converge to zero, leading to a rigorous derivation of the Hookean bead-spring-chain model.

We proceed to define the McKean–Vlasov diffusion. Let

$$\begin{aligned} \mathcal{U}(r, t; \varrho) &:= \left(u(r_1, t)^\top, \dots, u(r_{J+1}, t)^\top \right)^\top \\ &= \left((\mathcal{A}\varrho)(r_1, t)^\top, \dots, (\mathcal{A}\varrho)(r_{J+1}, t)^\top \right)^\top, \end{aligned}$$

with \mathcal{A} as indicated above, and consider the SDE

$$\epsilon^2 \ddot{r} = \mathcal{L}r + \zeta (\mathcal{U}(r, t; \varrho) - \dot{r}) + \sqrt{2\beta} \dot{W}.$$

Here $\epsilon^2 > 0$ signifies the mass of an individual bead in the chain, $\beta = k\mathbb{T}\zeta > 0$, where k is the Boltzmann constant, \mathbb{T} is the absolute temperature and ζ is the drag coefficient. Furthermore, \mathcal{L} is the following $(J+1) \times (J+1)$ block-matrix (analogous to a discrete Laplacian, corresponding to a homogeneous Neumann boundary condition) with $d \times d$ matrices as its entries, associated with a Hookean bead-spring-chain:

$$\lambda \begin{pmatrix} -\mathbb{I} & \mathbb{I} & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{I} & -2\mathbb{I} & \mathbb{I} & \ddots & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & -2\mathbb{I} & \mathbb{I} & \mathbb{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \mathbb{O} & \dots & \mathbb{I} & -2\mathbb{I} & \mathbb{I} \\ \mathbb{O} & \dots & \mathbb{O} & \mathbb{I} & -\mathbb{I} \end{pmatrix},$$

where $\lambda > 0$ is a constant factor characteristic of the stiffness of the springs, the block $\mathbb{I} \in \mathbb{R}^{d \times d}$ is the $d \times d$ identity matrix, and the block $\mathbb{O} \in \mathbb{R}^{d \times d}$ is the $d \times d$ zero matrix. Thus, \mathcal{L} is a $(J+1)d \times (J+1)d$ matrix, in fact.

As the parameter λ plays no role in the discussion that will follow, we set $\lambda = 1$; similarly, we set $\zeta = 1$. The SDE may then be rewritten as the first-order system

$$\begin{aligned} \epsilon \dot{r} &= v, \\ \epsilon \dot{v} &= \mathcal{L}r + \mathcal{U}(r, t; \varrho) - \epsilon^{-1}v + \sqrt{2\beta} \dot{W}. \end{aligned} \tag{1.6}$$

Then, $\varrho(r, v, t)$ solves the Fokker–Planck equation, stated in the next section, associated with this system. For (1.6) to be meaningful, it is clearly necessary that the function $(r, t) \in \Omega \times [0, T] \mapsto u(r, t) \in \mathbb{R}^d$ satisfies the Carathéodory condition: i.e., it is *continuous* with respect to r for a.e. $t \in [0, T]$ and *measurable* with respect to t for every $r \in \Omega$. This requirement is consistent with the underlying modelling assumption that the background fluid (i.e. the solvent), in which the polymer molecules are immersed, represents a ‘continuum’ relative to the scale of the polymer molecules.

To define the Fokker–Planck equation we mimic the procedure in [54] and introduce the following differential operators, noting that those with suffix 0 are independent of u (which is considered to be fixed for the moment), whilst those with suffix 1 are not:

$$\begin{aligned}
\mathcal{L}_{0,j}\varphi &:= -v_j \cdot \partial_{v_j}\varphi + \beta \partial_{v_j}^2\varphi, & j = 1, \dots, J+1, \\
\mathcal{L}_{1,j}(u)\varphi &:= v_j \cdot \partial_{r_j}\varphi + ((\mathcal{L}r)_j + u(r_j, t)) \cdot \partial_{v_j}\varphi, & j = 1, \dots, J+1, \\
\mathcal{L}_{0,j}^*\varphi &:= \partial_{v_j} \cdot (v_j\varphi) + \beta \partial_{v_j}^2\varphi, & j = 1, \dots, J+1, \\
\mathcal{L}_{1,j}^*(u)\varphi &:= -v_j \cdot \partial_{r_j}\varphi - ((\mathcal{L}r)_j + u(r_j, t)) \cdot \partial_{v_j}\varphi, & j = 1, \dots, J+1, \\
\mathcal{L}_0\varphi &:= \sum_{j=1}^{J+1} \mathcal{L}_{0,j}\varphi, \\
\mathcal{L}_1(u)\varphi &:= \sum_{j=1}^{J+1} \mathcal{L}_{1,j}(u)(\varphi).
\end{aligned}$$

In these expressions ∂_{v_j} denotes the (d -component) gradient operator with respect to $v_j \in \mathbb{R}^d$, $\partial_{v_j} \cdot$ denotes the divergence operator with respect to v_j , and $\partial_{v_j}^2 = \partial_{v_j} \cdot \partial_{v_j}$ is the Laplace operator with respect to v_j . We further note that $\mathcal{L}_{0,j}$ has a one-dimensional null-space spanned by the real-valued constant function that is identically equal to 1 with respect to v_j , denoted by $\mathbb{I}(v_j)$, and its adjoint $\mathcal{L}_{0,j}^*$ has null-space spanned by the function

$$g(v_j) := (2\pi\beta)^{-\frac{1}{2}} \exp(-|v_j|^2/2\beta).$$

Observe also that, for $g(s) = (2\pi\beta)^{-\frac{1}{2}} \exp(-s^2/2\beta)$, $s \in \mathbb{R}$, and with $'$ denoting differentiation with respect to the variable s , we have that

$$(sg(s))' + \beta g''(s) = 0,$$

implying that

$$(sg'(s))' + \beta(g'(s))'' = -g'(s). \tag{1.7}$$

Finally, we note that \mathcal{L}_0 has a one-dimensional null-space spanned by the constant function with respect to $v = (v_1^T, \dots, v_{J+1}^T)^T$, denoted by $\mathbb{I}(v) = \prod_{j=1}^{J+1} \mathbb{I}(v_j)$, and its adjoint \mathcal{L}_0^* has null-space spanned by the function

$$\varrho_\infty(v) = \prod_{j=1}^{J+1} g(v_j). \tag{1.8}$$

1.4 Summary of the main theorems, results and conclusions

In this section we will summarise the main theorems, results and conclusions of the thesis.

The main theorems of this thesis are the following:

We define $\mathcal{F} \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$, by

$$\mathcal{F}(s) := s(\log s - 1) + 1, \quad s \in \mathbb{R}_{> 0}, \quad \text{with } \mathcal{F}(0) := 1.$$

We shall assume that the initial datum ϱ_0 satisfies

$$\begin{aligned} \varrho_0 \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}), \quad \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \varrho_0(r, v) \, dr \, dv = 1, \\ M\mathcal{F}(\widehat{\varrho}_0) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}); \end{aligned} \quad (1.9)$$

Theorem 1.4.1. *The probability density function denoted by $\varrho = \varrho(r, v, t)$ solves the nonlinear partial differential equation*

$$\partial_t \varrho - \frac{\beta}{\epsilon^2} \left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (v_j \varrho) + \beta \partial_{v_j}^2 \varrho \right) + \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho + ((\mathcal{L}r)_j + u(r_j, t)) \cdot \partial_{v_j} \varrho \right) = 0, \quad (1.10)$$

$$\text{for all } (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T],$$

$$\varrho(r, v, 0) = \varrho_0(r, v) \quad \text{for all } (r, v) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d}. \quad (1.11)$$

Proof. The proof is a straightforward consequence of Itô's formula (see appendix A). \square

The equation (1.10) should be supplemented with a boundary condition; here, for the sake of simplicity of the exposition, we shall consider a specular boundary condition with respect to the independent variable r , which we shall state below.

Before formulating the specular boundary condition considered here, we require some additional notation. We let

$$\partial\Omega^{(j)} := \Omega \times \cdots \times \Omega \times \partial\Omega \times \Omega \times \cdots \times \Omega, \quad j = 1, \dots, J+1,$$

with $\partial\Omega$ appearing at the j -th position in this $(J+1)$ -fold Cartesian product. Clearly, $\bigcup_{j=1}^{J+1} \overline{\partial\Omega^{(j)}} = \partial\Omega^{J+1}$. Let, further,

$$\nu^{(j)}(r) := (0^{\text{T}}, \dots, 0^{\text{T}}, (\nu(r_j))^{\text{T}}, 0^{\text{T}}, \dots, 0^{\text{T}})^{\text{T}} \in \mathbb{R}^{(J+1)d},$$

where, for $r = (r_1, \dots, r_{J+1}) \in \partial\Omega^{(j)}$, the nonzero entry $\nu(r_j) \in \mathbb{R}^d$ appearing at the j -th position is the unit outward normal (column-)vector to $\partial\Omega$ at $r_j \in \partial\Omega$, for $j = 1, \dots, J+1$, and 0 is a d -component zero (column-)vector. With this notation, we then impose the following *specular boundary condition* for ϱ on $\partial\Omega^{(j)}$, $j = 1, \dots, J+1$:

$$\varrho(r, v, t) = \varrho(r, v_*^{(j)}, t) \quad \text{for all } (r, v, t) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T], \quad (1.12)$$

with $v \cdot \nu^{(j)}(r) < 0$ and where

$$v_*^{(j)} = v_*^{(j)}(r, v) := v - 2(v \cdot \nu^{(j)}(r)) \nu^{(j)}(r), \quad j = 1, \dots, J+1,$$

is the *specular velocity*; clearly, $v_*^{(j)} \cdot \nu^{(j)}(r) = -v \cdot \nu^{(j)}(r)$. This boundary condition on ϱ means that if the j -th bead in the chain (r_1, \dots, r_{J+1}) hits the boundary with velocity vector $v_j \in \mathbb{R}^d$ it is reflected with velocity vector $v_j - 2(v_j \cdot \nu(r_j)) \nu(r_j) \in \mathbb{R}^d$. With respect to the independent variable $v = (v_1^T, \dots, v_{J+1}^T)^T$ the domain of definition of ϱ is $\mathbb{R}^{(J+1)d}$. The behaviour of ϱ as a function of v in the limit of $|v| \rightarrow \infty$ is dictated by the requirement that $\varrho(\cdot, \cdot, t) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})$, for each fixed $t \in (0, T]$.

We also consider the Maxwellian $M(v) := \varrho_\infty(v)$ (cf. (1.8)) and define

$$\widehat{\varrho} := \frac{\varrho}{M} \quad \text{and} \quad \widehat{\varrho}_0 := \frac{\varrho_0}{M}.$$

Further, we define $\mathcal{F} \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$, by

$$\mathcal{F}(s) := s(\log s - 1) + 1, \quad s \in \mathbb{R}_{> 0}, \quad \text{with } \mathcal{F}(0) := 1.$$

In order to formulate Theorem 1.4.2, we require some additional notation. The imposition of the specular boundary condition on all functions that belong to a certain function space will be indicated by the subscript $*$ in our notation for the particular function space. For example,

$$\mathcal{C}_*^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d})) = \left\{ \varphi \in \mathcal{C}^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d})) : \varphi(r, v) = \varphi(r, v_*^{(j)}) \right. \\ \left. \text{for all } (r, v) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d}, \text{ with } v \cdot \nu^{(j)}(r) < 0, \right. \\ \left. j = 1, \dots, J+1 \right\}.$$

We also define $\mathcal{C}_w([0, T]; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ as the linear space of weakly continuous mappings from $[0, T]$ into $L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, and $W_M^{1,2}(\mathbb{R}^{(J+1)d})$ signifying the Maxwellian-weighted Sobolev space on $\mathbb{R}^{(J+1)d}$:

$$W_M^{1,2}(\mathbb{R}^{(J+1)d}) := \left\{ \varphi \in L_{\text{loc}}^1(\mathbb{R}^{(J+1)d}) : \|\varphi\|_{W_M^{1,2}(\mathbb{R}^{(J+1)d})}^2 \right. \\ \left. := \int_{\mathbb{R}^{(J+1)d}} M(v) \left(|\varphi(v)|^2 + \sum_{j=1}^{J+1} |\partial_{v_j} \varphi(v)|^2 \right) dv < \infty \right\},$$

(with analogous notation for all other Maxwellian-weighted Lebesgue and Sobolev spaces).

Theorem 1.4.2. *Assume that the initial datum ϱ_0 (cf. (A.2)) satisfies*

$$\begin{aligned} \varrho_0 \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}), \quad \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \varrho_0(r, v) \, dr \, dv = 1, \\ M\mathcal{F}(\widehat{\varrho}_0) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}); \end{aligned} \quad (1.13)$$

and that the initial datum u_0 (cf. (1.3)) satisfies $u_0 \in W_0^{1-2/z, z}(\Omega)^d$, with $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, and is divergence-free. Then, there exist functions $u = u_\epsilon$ and $\widehat{\varrho} = \widehat{\varrho}_\epsilon$, such that

$$u_\epsilon \in \mathcal{C}([0, T]; L^\sigma(\Omega)^d) \cap L^2(0, T; W_0^{1, \sigma}(\Omega)^d) \cap W^{1, 2}(0, T; W^{-1, \sigma}(\Omega)^d),$$

with $\sigma = \min(\widehat{\sigma}, z) > d$, $\widehat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, is a weak solution to the Oseen system (1.3), and $\widehat{\varrho}_\epsilon$ with

$$\mathcal{F}(\widehat{\varrho}_\epsilon) \in L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

$$\nabla_v \sqrt{\widehat{\varrho}_\epsilon} \in L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$

$$\begin{aligned} \nabla_v \widehat{\varrho}_\epsilon \in L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{and} \quad M \partial_t \widehat{\varrho}_\epsilon \in L^2(0, T; (W^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ s > (J + 1)d + 1, \end{aligned}$$

satisfies the following weak form of the Fokker–Planck equation: for all $t \in (0, T]$,

$$\begin{aligned} \int_0^t \langle M \partial_\tau \widehat{\varrho}_\epsilon(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle \, d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_\epsilon \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\ - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\epsilon \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \right) \\ - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u_\epsilon(r_j, \tau)) \widehat{\varrho}_\epsilon \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) = 0 \\ \forall \varphi \in L^2(0, T; W_{*, M}^{1, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ s > (J + 1)d + 1. \end{aligned} \quad (1.14)$$

Furthermore $\widehat{\varrho}_\epsilon(\cdot, \cdot, 0) = \widehat{\varrho}_0(\cdot, \cdot)$ in the sense of $\mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$, and

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}_\epsilon(r, v, t) \, dr \, dv = \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}_0(r, v) \, dr \, dv = 1 \quad \forall t \in (0, T]. \quad (1.15)$$

In addition, $\widehat{\varrho}_\epsilon$ satisfies the following energy inequality:

$$\begin{aligned} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_\epsilon(t)) \, dv \, dr + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\epsilon|^2}{\widehat{\varrho}_\epsilon} \, dv \, dr \, d\tau \\ \leq C \left[1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0) \, dv \, dr \right], \end{aligned} \quad (1.16)$$

where $C = C(\|u_0\|_{W^{1-\frac{2}{\sigma}, \sigma}(\Omega)}, \|b\|_{L^\infty(0, T; L^\infty(\Omega))})$, $\sigma = \min(\widehat{\sigma}, z) > d$, $\widehat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, in particular, C is independent of $\epsilon > 0$.

Proof. We refer the reader to Chapter 2. \square

Next, we rigorously identify the small-mass limit of the system, corresponding to passage to the limit $\epsilon \rightarrow 0_+$.

Theorem 1.4.3. *The small-mass limit of the coupled Oseen–Fokker–Planck system under consideration satisfies the following coupled problem: the velocity-pressure pair $(u_{(0)}, \pi_{(0)})$ solves the Oseen system*

$$\begin{aligned} \partial_t u_{(0)} + (b \cdot \nabla) u_{(0)} - \mu \Delta u_{(0)} + \nabla \pi_{(0)} &= \nabla \cdot \mathbb{K}_{(0)} && \text{for } (x, t) \in \Omega \times (0, T], \\ \nabla \cdot u_{(0)} &= 0 && \text{for } (x, t) \in \Omega \times (0, T], \\ u_{(0)}(x, t) &= 0 && \text{for } (x, t) \in \partial\Omega \times (0, T], \\ u_{(0)}(x, 0) &= u_0(x) && \text{for } x \in \Omega, \end{aligned} \quad (1.17)$$

with

$$\mathbb{K}_{(0)}(x, t) := \frac{\int_{D^J} \sum_{j=1}^J (F(q_j) \otimes q_j) \eta(B(q, x), t) \, dq}{\int_{D^J} \eta(B(q, x), t) \, dq} \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.18)$$

and the nonnegative function η , with $\int_{\Omega^{J+1}} \eta(r, t) \, dr = 1$ for all $t \in [0, T]$, solves the following parabolic initial-boundary-value problem:

$$\partial_t \eta = \sum_{j=1}^{J+1} \left(\beta \partial_{r_j}^2 \eta - \partial_{r_j} \cdot \left(\eta ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \right) \right) \quad \text{in } \Omega^{J+1} \times (0, T], \quad (1.19)$$

$$\eta(\cdot, 0) = \widehat{\varrho}_0 \in L^2(\Omega^{J+1}; \mathbb{R}_{\geq 0}), \quad (1.20)$$

subject to the weakly imposed boundary condition $\mathcal{J}_j \cdot \nu(r_j) = 0$ on $\partial\Omega^{(j)} \times (0, T]$ for $j = 1, \dots, J+1$, and the following zero-normal-flux boundary condition on η :

$$\left(\beta \partial_{r_j} \eta - \eta ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \right) \cdot \nu(r_j) = 0 \quad \text{on } \partial\Omega^{(j)} \times (0, T] \text{ for } j = 1, \dots, J+1. \quad (1.21)$$

Proof. We refer the reader to Chapter 4. \square

Then, we explore the connection between the small-mass-limit problem and the classical Hookean bead-spring-chain model for dilute polymeric fluids. To this end, we consider the symmetric positive definite block matrix \mathcal{R} of size $Jd \times Jd$ referred to as the *Rouse matrix*:

$$\mathcal{R} = \begin{pmatrix} 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \mathbb{O} \\ -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \ddots & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \mathbb{O} & \dots & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} \\ \mathbb{O} & \dots & \mathbb{O} & -\mathbb{I} & 2\mathbb{I} \end{pmatrix},$$

and let

$$\mathfrak{M}(q) := (2\pi\beta)^{-\frac{1}{2}Jd} \exp(-|q|^2/2\beta), \quad \text{where } q = (q_1^T, \dots, q_J^T)^T \in D^J.$$

Hence,

$$-\sum_{j=1}^{J+1} (\beta \partial_{r_j}^2 \eta - \partial_{r_j} \cdot (\eta(\mathcal{L}r)_j)) = -\beta \partial_q \cdot \left[\mathcal{R} \mathfrak{M}(q) \partial_q \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \partial_x^2 \psi. \quad (1.22)$$

Here, we do not need to restrict ourselves to the Oseen system and we consider the Navier–Stokes system.

Theorem 1.4.4. *The function $u_{(0)}$ satisfies the Navier–Stokes equation:*

$$\partial_t u_{(0)} + (u_{(0)} \cdot \nabla) u_{(0)} - \mu \Delta u_{(0)} + \nabla \pi = \nabla \cdot \mathbb{K}_{(0)} \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.23a)$$

$$\nabla \cdot u_{(0)} = 0 \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (1.23b)$$

$$u_{(0)}(x, t) = 0 \quad \text{for } (x, t) \in \partial\Omega \times (0, T], \quad (1.23c)$$

$$u_{(0)}(x, 0) = u_0(x) \quad \text{for } x \in \Omega, \quad (1.23d)$$

where

$$\mathbb{K}_{(0)} = n \int_{D^J} \sum_{j=1}^J (F(q_j) \otimes q_j) \psi(x, q, t) \, dq, \quad (1.24)$$

with $\mathbb{K}_0 : \Omega \times (0, T] \rightarrow \mathbb{R}_{\text{symm}}^{d \times d}$ is the elastic extra stress tensor, which satisfies:

$$\|\mathbb{K}_{(0)}\|_{L^\infty(0, T; L^\infty(\Omega))} < \infty. \quad (1.25)$$

The function ψ satisfies the Fokker–Planck equation for the classical Hookean bead-spring-chain model with centre-of-mass diffusion:

$$\begin{aligned} \partial_t \psi + u_{(0)} \cdot \nabla \psi + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u_{(0)})_{q_j} \psi) \\ - \beta \sum_{i,j=1}^J \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta \psi = 0, \end{aligned} \quad (1.26)$$

together with the initial condition

$$\psi(x, q, 0) = \psi_0(x, q), \quad (1.27)$$

where $\psi_0(x, q) := \hat{\varrho}_0(B(q, x)) \in L^2(\Omega^{J+1}; \mathbb{R}_{\geq 0})$, and the following zero-normal-flux boundary conditions on $\partial\Omega \times D^J \times (0, T]$ and $\Omega \times \partial D^J \times (0, T]$:

$$\nabla \psi(x, q, t) \cdot n_x(x) = 0 \quad \text{for all } (x, q, t) \in \partial\Omega \times D^J \times (0, T], \quad (1.28)$$

where n_x is the unit outward normal vector to $\partial\Omega$, and

$$\sum_{i=1}^J \left[\beta \mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) - ((\nabla u_{(0)})_{q_j} \psi) \right] \cdot n_{q_j} = 0 \quad (1.29)$$

for all $(x, q, t) \in \Omega \times (D \times \cdots \times \partial D \times \cdots \times D) \times (0, T]$, $j = 1, \dots, J$, where n_{q_j} is the unit outward normal vector to ∂D for the j th copy of the domain D in the Cartesian product $D^J = D \times \cdots \times D$.

Proof. We refer the reader to Chapter 5. □

We then show the existence of global weak solutions to the Hookean bead-spring-chain model. Hence, we state our assumptions on the initial conditions. For the initial velocity u_0 we assume that

$$u_0 \in L^2_{0,\text{div}}(\Omega). \quad (1.30)$$

For $\hat{\psi}_0 := \frac{\psi_0}{\mathfrak{M}(q)}$, where ψ_0 is the initial value of the probability density function ψ , we assume that

$$\hat{\psi}_0 \geq 0 \text{ a.e. in } \Omega \times D^J, \quad \hat{\psi}_0 \ln \hat{\psi}_0 \in L^1_M(\Omega \times D^J), \quad (1.31)$$

and in addition we require that

$$\varrho_0 \in L^\infty(\Omega), \quad \text{where } \varrho_0(x) := \int_{D^J} \mathfrak{M}(q) \hat{\psi}_0(x, q) \, dq. \quad (1.32)$$

Theorem 1.4.5. *Let $J \in \mathbb{N}$ be arbitrary. Assume that the initial data $u_0, \hat{\psi}_0$ satisfy (1.30)–(1.32). Then, there exist $(u_{(0)}, \mathbb{K}_{(0)}, \hat{\psi})$ satisfying the weak formulations of the system (1.23a) and (1.26) such that*

$$u_{(0)} \in L^\infty(0, T; L^2_{0, \text{div}}(\Omega)^d) \cap L^2(0, T; W^{1,2}_0(\Omega)^d) \cap W^{1,2}(0, T; W^{-1,2}_{0, \text{div}}(\Omega)^d), \quad (1.33)$$

$$\mathbb{K}_{(0)} \in L^2(0, T; L^2(\Omega)^{d \times d}), \quad (1.34)$$

and

$$\begin{aligned} \hat{\psi} &\in L^\infty(\Omega \times (0, T); L^1_{\mathfrak{M}(\mathbf{q})}(D^J)) \cap L^2(0, T; W^{1,1}_{\mathfrak{M}(\mathbf{q})}(\Omega \times D^J)), \\ \hat{\psi} &\geq 0 \text{ a.e. in } \Omega \times D^J \times (0, T), \\ \mathfrak{M}(\mathbf{q}) \hat{\psi} &\in W^{1,1}(0, T; W^{-1,1}(\Omega \times D^J)), \\ \hat{\psi} \log \hat{\psi} &\in L^\infty(0, T; L^1_{\mathfrak{M}(\mathbf{q})}(\Omega \times D^J)). \end{aligned} \quad (1.35)$$

Furthermore, the initial data are attained strongly in $L^2(\Omega) \times L^1_{\mathfrak{M}(\mathbf{q})}(\Omega \times D^J)$, i.e.

$$\lim_{t \rightarrow 0^+} \|u_{(0)}(\cdot, t) - u_0(\cdot)\|_{L^2(\Omega)}^2 + \|\hat{\psi}(\cdot, t) - \hat{\psi}_0(\cdot)\|_{L^1_{\mathfrak{M}(\mathbf{q})}(\Omega \times D^J)} = 0. \quad (1.36)$$

Moreover, for all $t \in (0, T)$ the following energy inequality holds in a weak sense:

$$\begin{aligned} &\frac{d}{dt} \left(\int_{\Omega \times D^J} n \mathfrak{M}(\mathbf{q}) \hat{\psi} \log \hat{\psi} \, dx \, dq + \frac{1}{2} \|u_{(0)}\|_{L^2(\Omega)}^2 \right) + \|\nabla u_{(0)}\|_{L^2(\Omega)}^2 \\ &\quad + 4n \left(\mathfrak{M}(\mathbf{q}) \nabla \sqrt{\hat{\psi}}, \nabla \sqrt{\hat{\psi}} \right)_{\Omega \times D^J} \\ &\quad + 4n \left(\mathcal{R} \mathfrak{M}(\mathbf{q}) \partial_q \sqrt{\hat{\psi}}, \partial_q \sqrt{\hat{\psi}} \right)_{\Omega \times D^J} \leq 0. \end{aligned} \quad (1.37)$$

The main results of this thesis are the following:

- Rigorous derivation of the Hookean bead-spring chain model;
- Existence of global weak solutions to the Oseen–Fokker–Planck system;
- Rigorous passage to small-mass limit and equilibration in momentum space.

The conclusions of this work are the following:

- If the flow domain Ω is bounded, then so is the high-dimensional configuration space domain $D^N := D \times \cdots \times D$, where $D := \Omega - \Omega$.

- Before passing to the small-mass limit, the Fokker–Planck equation is degenerate parabolic. After passing to the small mass limit, the Fokker–Planck equation, posed on $\Omega \times D^N \times [0, T]$, is parabolic.
- We rigorously prove an assertion, deduced by Schieber and Ottinger (1988) using formal asymptotics, that passage to the small-mass limit implies equilibration in momentum space.

1.5 Structure of the thesis

This work is the extended version of the paper [63], which is in preparation for submission for journal publication. The thesis is structured as follows: In Chapter 2 we show, for a fixed velocity field u , the existence of a global weak solution to the Fokker–Planck equation, subject to a specular boundary condition. The argument is based on a parabolic regularization of the (hypoelliptic) Fokker–Planck equation, and passage to the limit with the parabolic regularization parameter. In Section 2.3 we then return to the original coupled Oseen–Fokker–Planck system and use an iterative process between the Oseen equation and the Fokker–Planck equation to show the existence of large-data global weak solutions to the coupled problem for any non-negative L^1 initial datum with finite initial relative entropy for the Fokker–Planck equation, and any (distributionally) divergence-free initial datum $u_0 \in W_0^{1-2/z, z}(\Omega)^d$, with $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, for the Oseen equation. The latter regularity hypothesis on u_0 will then ensure the continuity of the velocity field u with respect to its spatial variable, alluded to in the last sentence of the previous paragraph, via maximal regularity theory for the unsteady Stokes system. Indeed, the fact that in the case of the Oseen system we are able to guarantee, through the above regularity hypothesis on u_0 , that the velocity field u belongs to the function space $u \in L^2(0, T; L^\infty(\Omega)^d)$ plays a crucial role in our proofs.

The proofs use a variety of compactness arguments for infinite sequences of approximate solutions. Passage to the limit in the extra stress tensor \mathbb{K} , whose divergence appears on the right-hand side of the Oseen equation, is nontrivial as \mathbb{K} depends nonlinearly on the probability density function; to this end, we shall show the strong convergence of the sequence of approximating probability density functions using techniques developed by DiPerna & Lions for the Fokker–Planck–Boltzmann system and related hypoelliptic PDEs (see, in particular, the Appendix in [20]). In Chapter 3 we show, by using the existence of a trace on the boundary of our domain, that the

solution to the Fokker–Planck equation attains the weakly imposed specular boundary condition in a strong sense. In Chapter 4 we then focus on the second objective of the thesis: we rigorously identify the small-mass limit of the system, as $\epsilon \rightarrow 0_+$. Once again, passage to the limit in the extra stress tensor \mathbb{K} , whose divergence appears on the right-hand side of the Oseen equation, is the main source of technical difficulties, as we require strong convergence of the approximating sequence of probability density functions, as $\epsilon \rightarrow 0_+$. Motivated by an argument in the work of Carrillo & Goudon [15], which first appeared in the context of diffusion asymptotics for hyperbolic problems in the work of Marcati & Milani [46], and was then applied in the framework of kinetic equations by Lions & Toscani [44] and Goudon & Poupaud [28], we shall use a compensated compactness argument based on the Div-Curl lemma to prove weak convergence, which we then strengthen to the desired strong convergence result, enabling us to identify the small-mass limit, as $\epsilon \rightarrow 0_+$. We prove in particular that passage to the small-mass limit results in equilibration in momentum space, in a sense to be made precise. This enables us to make mathematically rigorous various formal asymptotic calculations from the polymer physics literature asserting that passage to the small-mass limit implies equilibration in momentum space. In Chapter 5 we relate the resulting small-mass-limit model to the classical Hookean bead-spring-chain model for dilute solutions of polymeric fluids. In Chapter 6, we prove the existence of global weak solutions to the Hookean–bead–spring–chain model. In Appendix B, we consider the extension of the results to the Navier–Stokes–Fokker–Planck system.

Chapter 2

Existence of solutions to the Fokker–Planck equation

The probability density function associated with (1.6) is denoted by $\varrho = \varrho(r, v, t)$; formally it solves the *nonlinear* partial differential equation

$$\partial_t \varrho = \frac{\beta}{\epsilon^2} \mathcal{L}_0^* \varrho + \frac{1}{\epsilon} \mathcal{L}_1(u)^* \varrho. \quad (2.1)$$

In case it is not apparent, we emphasize that the nonlinearity enters into the equation through the dependence of the velocity field u on the probability density function ϱ , since u is the solution of the Oseen equation whose right-hand side depends on ϱ through the presence of the conditional expectation there. Substituting the defining expressions for \mathcal{L}_0^* and $\mathcal{L}_1(u)^*$ into (2.1) yields

$$\partial_t \varrho - \frac{\beta}{\epsilon^2} \left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (v_j \varrho) + \beta \partial_{v_j}^2 \varrho \right) + \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho + ((\mathcal{L}r)_j + u(r_j, t)) \cdot \partial_{v_j} \varrho \right) = 0, \quad (2.2)$$

$$\text{for all } (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T],$$

$$\varrho(r, v, 0) = \varrho_0(r, v) \quad \text{for all } (r, v) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d}. \quad (2.3)$$

The equation (2.2) should be supplemented with a boundary condition; here, for the sake of simplicity of the exposition, we shall consider a specular boundary condition with respect to the independent variable r , which we shall state below. More complicated boundary conditions can of course be used to model the interaction between the wall $\partial\Omega$ and the beads in the bead-spring-chain; for example, a Maxwell-type boundary condition (proposed by Maxwell [49] in 1879 as a phenomenological law by splitting the reflection operator into a local reflection operator and a diffuse

reflection operator) may be considered, as in [51]: it involves a boundary trace operator that is a convex linear combination of a specular boundary trace operator, describing local reflection by the wall, and a diffuse reflection operator.

Before formulating the specular boundary condition considered here, we require some additional notation. We let

$$\partial\Omega^{(j)} := \Omega \times \cdots \times \Omega \times \partial\Omega \times \Omega \times \cdots \times \Omega, \quad j = 1, \dots, J+1,$$

with $\partial\Omega$ appearing at the j -th position in this $(J+1)$ -fold Cartesian product. Clearly, $\bigcup_{j=1}^{J+1} \overline{\partial\Omega^{(j)}} = \partial\Omega^{J+1}$. Let, further,

$$\nu^{(j)}(r) := (0^T, \dots, 0^T, (\nu(r_j))^T, 0^T, \dots, 0^T)^T \in \mathbb{R}^{(J+1)d},$$

where, for $r = (r_1, \dots, r_{J+1}) \in \partial\Omega^{(j)}$, the nonzero entry $\nu(r_j) \in \mathbb{R}^d$ appearing at the j -th position is the unit outward normal (column-)vector to $\partial\Omega$ at $r_j \in \partial\Omega$, for $j = 1, \dots, J+1$, and 0 is a d -component zero (column-)vector. With this notation, we then impose the following *specular boundary condition* for ϱ on $\partial\Omega^{(j)}$, $j = 1, \dots, J+1$:

$$\varrho(r, v, t) = \varrho(r, v_*^{(j)}, t) \quad \text{for all } (r, v, t) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T], \text{ with } v \cdot \nu^{(j)}(r) < 0, \quad (2.4)$$

where

$$v_*^{(j)} = v_*^{(j)}(r, v) := v - 2(v \cdot \nu^{(j)}(r)) \nu^{(j)}(r), \quad j = 1, \dots, J+1,$$

is the *specular velocity*; clearly, $v_*^{(j)} \cdot \nu^{(j)}(r) = -v \cdot \nu^{(j)}(r)$. This boundary condition on ϱ means that if the j -th bead in the chain (r_1, \dots, r_{J+1}) hits the boundary with velocity vector $v_j \in \mathbb{R}^d$ it is reflected with velocity vector $v_j - 2(v_j \cdot \nu(r_j)) \nu(r_j) \in \mathbb{R}^d$. With respect to the independent variable $v = (v_1^T, \dots, v_{J+1}^T)^T$ the domain of definition of ϱ is $\mathbb{R}^{(J+1)d}$. The behaviour of ϱ as a function of v in the limit of $|v| \rightarrow \infty$ is dictated by the requirement that $\varrho(\cdot, \cdot, t) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})$, for each fixed $t \in (0, T]$.

In order to state the weak formulation of this problem we consider the Maxwellian $M(v) := \varrho_\infty(v)$ and define

$$\widehat{\varrho} := \frac{\varrho}{M} \quad \text{and} \quad \widehat{\varrho}_0 := \frac{\varrho_0}{M}.$$

Further, we define $\mathcal{F} \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$, by

$$\mathcal{F}(s) := s(\log s - 1) + 1, \quad s \in \mathbb{R}_{> 0}, \quad \text{with } \mathcal{F}(0) := 1.$$

The function \mathcal{F} is nonnegative, strictly convex, and has superlinear growth as $s \rightarrow +\infty$, i.e.

$$\lim_{s \rightarrow +\infty} \frac{\mathcal{F}(s)}{s} = +\infty.$$

We shall assume that the initial datum ϱ_0 (cf. (2.3)) satisfies

$$\begin{aligned} \varrho_0 \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}), \quad \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \varrho_0(r, v) \, dr \, dv = 1, \\ M\mathcal{F}(\widehat{\varrho}_0) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}); \end{aligned} \quad (2.5)$$

in other words, the initial probability density function ϱ_0 is assumed to have finite relative entropy with respect to the Maxwellian M .

We shall also *assume* throughout this section that $u \in L^2(0, T; W_0^{1, \sigma}(\Omega)^d)$ for some $\sigma > d$, and $\nabla \cdot u = 0$ a.e. in $\Omega \times (0, T)$, and that u is *given and held fixed*. We shall show later on that, under the assumptions on u_0 (cf. the paragraph following eq. (1.4)), the function u does indeed possess this regularity; in fact, we will see that $\sigma = \min(\hat{\sigma}, z)$, where $\hat{\sigma} := 2 + \frac{4}{d} > d$ for $d = 2, 3$, and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, whereby $\sigma > d$ for $d = 2, 3$, as is being assumed here. As a consequence of the assumed regularity of u , by Sobolev embedding, $u \in L^2(0, T; L^\infty(\Omega)^d)$.

We (formally) multiply the equation (2.2) by a function

$$\varphi \in W^{1,1}(0, T; \mathcal{C}^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d})));$$

and, assuming for the moment that ϱ is sufficiently smooth and satisfies the specular boundary condition (2.4), we integrate the resulting equality over $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T]$, and then integrate by parts with respect to each of the independent variables:

$$\begin{aligned} & \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}(r, v, T) \varphi(r, v, T) \, dv \, dr - \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho} \partial_\tau \varphi \, dv \, dr \, d\tau \\ & + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho} \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho} \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \right) \\ & + \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho} \varphi \, ds(r) \, d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho} \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\ & = \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \varphi(r, v, 0) \, dv \, dr, \end{aligned} \quad (2.6)$$

for all $\varphi \in W^{1,1}(0, T; \mathcal{C}^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d})))$. We focus our attention on the fifth integral on the left-hand side:

$$\begin{aligned} & \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho} \varphi \, dv \, ds(r) = \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)}} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho} \varphi \, ds(r) \, dv \\ &= \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_j \cdot \nu(r_j) > 0} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho} \varphi \, ds(r) \, dv \\ &+ \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_j \cdot \nu(r_j) < 0} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho} \varphi \, ds(r) \, dv. \end{aligned}$$

Now, since $|v_*^{(j)}|^2 = |v|^2$ and $v_*^{(j)} \cdot \nu^{(j)}(r) = -v \cdot \nu^{(j)}(r) = -v_j \cdot \nu(r_j)$, and using the specular boundary condition satisfied by $\widehat{\varrho}$, we have for the second integral on the right-hand side of the last equality that

$$\begin{aligned} & \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_j \cdot \nu(r_j) < 0} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho}(r, v, t) \varphi(r, v, t) \, ds(r) \, dv \\ &= \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_j \cdot \nu(r_j) < 0} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho}(r, v_*^{(j)}, t) \varphi(r, v, t) \, ds(r) \, dv \\ &= - \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : -v_j \cdot \nu(r_j) > 0} M(v) (-v_j \cdot \nu(r_j)) \widehat{\varrho}(r, v_*^{(j)}, t) \varphi(r, v, t) \, ds(r) \, dv \\ &= - \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_*^{(j)} \cdot \nu^{(j)}(r) > 0} M(v_*^{(j)}) (v_*^{(j)} \cdot \nu^{(j)}(r)) \widehat{\varrho}(r, v_*^{(j)}, t) \varphi(r, v, t) \, ds(r) \, dv. \end{aligned}$$

Assuming that the test function φ satisfies the specular boundary condition:

$$\varphi(r, v, t) = \varphi(r, v_*^{(j)}, t) \quad \forall (r, v, t) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T], \quad (2.7)$$

such that $v \cdot \nu^{(j)}(r) < 0, \quad j = 1, \dots, J+1,$

we then have, for all $j = 1, \dots, J+1$, that

$$\begin{aligned} & \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_j \cdot \nu(r_j) < 0} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho}(r, v, t) \varphi(r, v, t) \, ds(r) \, dv \\ &= - \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_*^{(j)} \cdot \nu^{(j)}(r) > 0} M(v_*^{(j)}) (v_*^{(j)} \cdot \nu^{(j)}(r)) \widehat{\varrho}(r, v_*^{(j)}, t) \varphi(r, v_*^{(j)}, t) \, ds(r) \, dv \\ &= - \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)}} M(v_*^{(j)}) (v_*^{(j)} \cdot \nu^{(j)}(r))_+ \widehat{\varrho}(r, v_*^{(j)}, t) \varphi(r, v_*^{(j)}, t) \, ds(r) \, dv \\ &= - \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} M(v_*^{(j)}) (v_*^{(j)} \cdot \nu^{(j)}(r))_+ \widehat{\varrho}(r, v_*^{(j)}, t) \varphi(r, v_*^{(j)}, t) \, dv \, ds(r). \end{aligned}$$

Since, for $r \in \partial\Omega^{(j)}$ fixed, the absolute value of the Jacobian $D\Phi$ of the (bijective) mapping

$$\Phi : v \in \mathbb{R}^{(J+1)d} \mapsto v_*^{(j)}(r, v) \in \mathbb{R}^{(J+1)d}$$

is equal to 1, whereby, for $r \in \partial\Omega^{(j)}$ fixed, $dv_*^{(j)} = |D\Phi| dv = dv$, by treating $v_*^{(j)}$ as a dummy variable in the last integral and renaming it into v , and noting again that $v \cdot \nu^{(j)}(r) = v_j \cdot \nu(r_j)$, it follows that, for all $j = 1, \dots, J+1$,

$$\begin{aligned} & \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_j \cdot \nu(r_j) < 0} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho}(r, v, t) \varphi(r, v, t) ds(r) dv \\ &= - \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} M(v) (v_j \cdot \nu(r_j))_+ \widehat{\varrho}(r, v, t) \varphi(r, v, t) dv ds(r) \\ &= - \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)}} M(v) (v_j \cdot \nu(r_j))_+ \widehat{\varrho}(r, v, t) \varphi(r, v, t) ds(r) dv \\ &= - \int_{\mathbb{R}^{(J+1)d}} \int_{\partial\Omega^{(j)} : v_j \cdot \nu(r_j) > 0} M(v) (v_j \cdot \nu(r_j)) \widehat{\varrho}(r, v, t) \varphi(r, v, t) ds(r) dv. \end{aligned}$$

Hence, provided that the test function $\varphi \in W^{1,1}(0, T; \mathcal{C}^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d})))$ appearing in (2.6) satisfies the specular boundary condition (2.7), the fifth integral in (2.6) will vanish. We shall therefore assume that this is indeed the case and will work with such test functions φ , whereby the absence of the fifth integral from (2.6) can be seen as a weak imposition of the specular boundary condition (2.4) for $\widehat{\varrho}$ (and, equivalently, for ϱ). The imposition of the specular boundary condition on all functions that belong to a certain function space will be indicated by the subscript $*$ in our notation for the particular function space. For example,

$$\begin{aligned} \mathcal{C}_*^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d})) &= \left\{ \varphi \in \mathcal{C}^\infty(\overline{\Omega}^{J+1}; \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d})) : \varphi(r, v) = \varphi(r, v_*^{(j)}) \right. \\ &\quad \text{for all } (r, v) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d}, \text{ with } v \cdot \nu^{(j)}(r) < 0, \\ &\quad \left. j = 1, \dots, J+1 \right\}. \end{aligned}$$

Thus, by eliminating the fifth integral from (2.6), we are led to the following problem: for a fixed divergence-free function $u \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$ with $\sigma > d$, we seek a function $\widehat{\varrho} \geq 0$ such that

$$\begin{aligned} M\widehat{\varrho} &\in \mathcal{C}_w([0, T]; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ M\mathcal{F}(\widehat{\varrho}) &\in L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad \sqrt{\widehat{\varrho}} \in L^2(0, T; L^2(\Omega^{J+1}; W_M^{1,2}(\mathbb{R}^{(J+1)d}))), \end{aligned}$$

with $\mathcal{C}_w([0, T]; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ being the linear space of weakly continuous mappings from $[0, T]$ into $L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, and $W_M^{1,2}(\mathbb{R}^{(J+1)d})$ signifying the Maxwellian-weighted Sobolev space on $\mathbb{R}^{(J+1)d}$:

$$\begin{aligned} W_M^{1,2}(\mathbb{R}^{(J+1)d}) &:= \left\{ \varphi \in L_{\text{loc}}^1(\mathbb{R}^{(J+1)d}) : \|\varphi\|_{W_M^{1,2}(\mathbb{R}^{(J+1)d})}^2 \right. \\ &\quad \left. := \int_{\mathbb{R}^{(J+1)d}} M(v) \left(|\varphi(v)|^2 + \sum_{j=1}^{J+1} |\partial_{v_j} \varphi(v)|^2 \right) dv < \infty \right\} \end{aligned}$$

(with analogous notation for all other Maxwellian-weighted Lebesgue and Sobolev spaces), such that

$$\begin{aligned}
& \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}(r, v, T) \varphi(r, v, T) \, dv \, dr \\
& - \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}(r, v, \tau) \partial_\tau \varphi(r, v, \tau) \, dv \, dr \, d\tau \\
& + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho} \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho} \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \right) \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho} \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\
& = \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \varphi(r, v, 0) \, dv \, dr \\
& \quad \forall \varphi \in W^{1,1}(0, T; W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \tag{2.8}
\end{aligned}$$

where $s > (J+1)d + 1$. We note that for $s > (J+1)d + 1$, by Sobolev embedding,

$$W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \hookrightarrow W_*^{1,\infty}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}).$$

We emphasize here again that the specular boundary condition is imposed weakly, through the *omission* of the fifth integral from (2.6) (and, thereby, through the *absence* of the corresponding term from (2.8)) and the choice of the test functions φ in $W^{1,1}(0, T; W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. This helps us to circumvent at this point the question whether $\widehat{\varrho}$ is regular enough to satisfy (2.4) in the (stronger) sense of a trace theorem on $\partial\Omega$. The existence of a trace in a stronger sense will be shown later, in Chapter 3.

2.1 Existence of solutions to a parabolic regularization of (2.8)

We begin by considering a parabolic regularization of the weak formulation (2.8): for a fixed divergence-free function $u \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$ with $\sigma > d$, and with $\alpha \in (0, 1]$ a regularization parameter that will be eventually sent to 0, we shall seek a function

$$\varrho_\alpha \in \mathcal{C}([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \cap L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$$

such that

$$\begin{aligned}
& \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, T) \varphi(r, v, T) \, dv \, dr \\
& - \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, \tau) \partial_\tau \varphi(r, v, \tau) \, dv \, dr \, d\tau \\
& + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \right) \\
& + \alpha \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\
& = \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \varphi(r, v, 0) \, dv \, dr \\
& \quad \forall \varphi \in W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \tag{2.9}
\end{aligned}$$

where, *in addition* to our earlier assumption (2.5) on the initial datum, we shall *temporarily assume* that

$$\widehat{\varrho}_0 \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}).$$

This additional assumption will be required in order to enable passage to the limit $\alpha \rightarrow 0_+$. In the final step of the existence proof, discussed in Section 2.3, this additional assumption on $\widehat{\varrho}_0$ will be removed, and the final global existence result for the coupled Oseen–Fokker–Planck system will be shown to hold assuming (2.5) only.

To show the existence of a solution to (2.9), note that $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, the normed linear space of all functions contained in the Maxwellian-weighted Sobolev space $W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ satisfying the specular boundary condition on $\partial\Omega$ in the sense of the trace theorem, is a separable Hilbert space, as it is a closed linear subspace of $W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, which is a separable Hilbert space (cf. Theorem 8.10.2 on p.418 in the monograph of Kufner, John & Fučík [39]). Furthermore, since $W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is compactly embedded into the space $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ (cf. Appendix D in [7]), $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is also compactly embedded into $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$. Thus, by a variant of the Hilbert–Schmidt theorem (cf. Lemma 5.1 in [25]), there exists a complete orthogonal basis $(\psi_k)_{k \geq 1}$ in $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, which is complete and orthonormal in $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$; the

function $\psi_k \in W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ solves the following eigenvalue problem:

$$\begin{aligned} (\psi_k, \eta)_{W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} &= \lambda_k (\psi_k, \eta)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} & \forall \eta \in W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}), \\ k &= 1, 2, \dots; & \|\psi_k\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} = 1. \end{aligned}$$

Let $\mathcal{X}_N := \text{span}\{\psi_1, \dots, \psi_N\}$ and denote by P_N the orthogonal projector in $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ onto \mathcal{X}_N . Suppose further that $w \in W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, with

$$w = \sum_{k=1}^{\infty} \alpha_k \psi_k.$$

As $(w - P_N w, \psi_j)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} = 0$ for all $j = 1, \dots, N$, thanks to the orthonormality of the functions ψ_k , $k \geq 1$, in $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, it follows that

$$P_N w = \sum_{k=1}^N \alpha_k \psi_k.$$

Thus, by the orthogonality of the ψ_k in $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, Parseval's identity implies that

$$\begin{aligned} \|P_N w\|_{W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 &= \sum_{k=1}^N \alpha_k^2 \|\psi_k\|_{W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \leq \sum_{k=1}^{\infty} \alpha_k^2 \|\psi_k\|_{W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \\ &= \|w\|_{W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \quad \forall w \in W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}). \end{aligned} \tag{2.10}$$

We shall seek a function

$$\widehat{\varrho}_{\alpha,N}(r, v, t) = \sum_{k=1}^N A_{k,N}(t) \psi_k(r, v) \tag{2.11}$$

such that

$$\begin{aligned} &\left(\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \int M(v) \widehat{\varrho}_{\alpha,N}(r, v, T) \psi_\ell(r, v) \, dv \, dr \right) \phi(T) \\ &\quad - \int_0^T \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \int M(v) \widehat{\varrho}_{\alpha,N}(r, v, \tau) \psi_\ell(r, v) \partial_\tau \phi(\tau) \, dv \, dr \, d\tau \\ &\quad + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \int M(v) \partial_{v_j} \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} \psi_\ell(r, v) \phi(\tau) \, dv \, dr \, d\tau \right) \\ &\quad - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \int M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \psi_\ell(r, v) \phi(\tau) \, dv \, dr \, d\tau \right) \end{aligned}$$

$$\begin{aligned}
& + \alpha \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_{\alpha, N} \cdot \partial_{r_j} \psi_\ell(r, v) \phi(\tau) \, dv \, dr \, d\tau \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_{\alpha, N} \cdot \partial_{v_j} \psi_\ell(r, v) \phi(\tau) \, dv \, dr \, d\tau \right) \\
& = \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \psi_\ell(r, v) \, dv \, dr \right) \phi(0) \\
& \quad \forall \ell \in \{1, \dots, N\} \text{ and } \forall \phi \in W^{1,2}(0, T). \tag{2.12}
\end{aligned}$$

Substituting (2.11) into (2.12) yields

$$\begin{aligned}
& A_{\ell, N}(T) \phi(T) - \int_0^T A_{\ell, N}(\tau) \partial_\tau \phi(\tau) \, d\tau \\
& + \int_0^T \sum_{k=1}^N A_{k, N}(\tau) \left(\frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \psi_k(r, v) \cdot \partial_{v_j} \psi_\ell(r, v) \, dv \, dr \right) \phi(\tau) \, d\tau \\
& + \int_0^T \sum_{k=1}^N A_{k, N}(\tau) \left(-\frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \psi_k(r, v) \cdot \partial_{r_j} \psi_\ell(r, v) \, dv \, dr \right) \phi(\tau) \, d\tau \\
& + \int_0^T \sum_{k=1}^N A_{k, N}(\tau) \left(\alpha \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \psi_k(r, v) \cdot \partial_{r_j} \psi_\ell(r, v) \, dv \, dr \right) \phi(\tau) \, d\tau \\
& + \int_0^T \sum_{k=1}^N A_{k, N}(\tau) \\
& \times \left(-\frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \psi_k(r, v) \cdot \partial_{v_j} \psi_\ell(r, v) \, dv \, dr \right) \phi(\tau) \, d\tau \\
& = \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \psi_\ell(r, v) \, dv \, dr \right) \phi(0),
\end{aligned}$$

for all $\ell \in \{1, \dots, N\}$ and for all $\phi \in W^{1,2}(0, T)$.

Denoting the sum of the terms in the brackets in the second, third and fourth line by $G_{\ell, k}$, and the term in the outer pair of brackets in the fifth line by $H_{\ell, k}(\tau)$, we have that

$$\begin{aligned}
A_{\ell,N}(T) \phi(T) - \int_0^T A_{\ell,N}(\tau) \partial_\tau \phi(\tau) d\tau + \int_0^T \sum_{k=1}^N (G_{\ell,k} + H_{\ell,k}(\tau)) A_{k,N}(\tau) \phi(\tau) d\tau \\
= \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \psi_\ell(r, v) dv dr \phi(0). \quad (2.13)
\end{aligned}$$

As it will transpire from the discussion that follows, $|G_{\ell,k}| < \infty$ and $|H_{\ell,k}(\tau)| < \infty$ for a.e. $\tau \in (0, T]$, and for all $\ell, k = 1, \dots, N$.

The above is the weak form of the following initial-value problem for a system of linear ODEs:

$$\begin{aligned}
\frac{d}{dt} A_{\ell,N}(t) + \sum_{k=1}^N (G_{\ell,k} + H_{\ell,k}(t)) A_{k,N}(t) &= 0, \quad t \in (0, T], \\
A_{\ell,N}(0) &= \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \psi_\ell(r, v) dv dr, \quad \ell = 1, \dots, N.
\end{aligned} \quad (2.14)$$

As $(G_{\ell,k})_{\ell,k=1}^N$ is a constant matrix, the existence of a solution to this system of linear ODEs will follow from Carathéodory's theorem once we have shown that $t \in (0, T) \mapsto H_{\ell,k}(t) \in \mathbb{R}$ is measurable and a (matrix) norm of the matrix $(H_{\ell,k}(t))_{\ell,k=1}^N$ is dominated by $h(t)$, where $h \in L^1(0, T)$. As a matter of fact, once this has been shown, the uniqueness of the solution to this system of ODEs will also follow, by Gronwall's lemma, thanks to the linearity of the system.

To this end, it suffices to note that, since by hypothesis $u \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$ for some $\sigma > d$, Fubini's theorem implies that all entries of the matrix $(H_{\ell,k}(t))_{\ell,k=1}^N$ are measurable functions of $t \in (0, T]$; furthermore, there exists a positive constant $C_0 = C_0(J, N)$ such that

$$\begin{aligned}
& \int_0^T \left| \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) u(r_j, \tau) \psi_k(r, v) \cdot \partial_{v_j} \psi_\ell(r, v) dv dr \right| d\tau \\
& \leq \sum_{j=1}^{J+1} \int_0^T \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |u(r_j, \tau) \psi_k(r, v)|^2 dv dr \right)^{\frac{1}{2}} \\
& \quad \times \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |\partial_{v_j} \psi_\ell(r, v)|^2 dv dr \right)^{\frac{1}{2}} d\tau \\
& \leq \|u\|_{L^1(0,T;L^\infty(\Omega))} \sum_{j=1}^{J+1} \max_{1 \leq \ell \leq N} \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |\partial_{v_j} \psi_\ell(r, v)|^2 dv dr \right)^{\frac{1}{2}} \\
& = C_0 \|u\|_{L^1(0,T;L^\infty(\Omega))}.
\end{aligned}$$

This then implies the existence of a measurable function $h \in L^1(0, T)$ such that the (matrix) norm of the matrix $(H_{\ell,k}(t))_{\ell,k=1}^N$ is dominated by $h(t)$, where $h \in L^1(0, T)$; take,

for example, $h(t) := \frac{C}{\epsilon}(1 + \|u(t)\|_{L^\infty(\Omega)})$, where C is a sufficiently large constant. Hence, Carathéodory's theorem implies the existence of a solution $A_{\ell,N} \in W^{1,1}(0,T)$ (which is, consequently, absolutely continuous on $[0,T]$), $\ell = 1, \dots, N$, to (2.13), and by Gronwall's lemma the solution to (2.13) is unique. In fact, since $H_{\ell,k} \in L^\infty(0,T)$, $\ell, k = 1, \dots, N$, it follows that $A_{\ell,N} \in W^{1,\infty}(0,T)$, $\ell = 1, \dots, N$; cf. (2.14). Thus, by noting (2.11), we deduce that the finite-dimensional problem (2.12) has a unique solution

$$\widehat{\varrho}_{\alpha,N} \in W^{1,\infty}(0,T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})).$$

Next, for any $t \in (0,T)$ fixed, and $h \in (0, T-t)$, consider the function

$$\chi_{t,h}(\tau) := \min \left\{ 1, \left(\frac{1}{h}(t - \tau) + 1 \right)_+ \right\}, \quad \tau \in [0, T].$$

Clearly, $\tau \mapsto \chi_{t,h}(\tau)$ is a continuous piecewise linear function defined on $[0, T]$, which is identically 1 on $[0, t]$, identically 0 on $[t+h, T]$, and has slope $-1/h$ on $[t, t+h]$. Taking $\phi = \chi_{t,h} A_{\ell,N} \in W^{1,\infty}(0, T)$ in (2.12) with $t \in (0, T)$ fixed and passing to the limit $h \rightarrow 0_+$, we have that

$$\begin{aligned} & \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha,N}(r, v, t) \psi_\ell(r, v) A_{\ell,N}(t) \, dv \, dr \\ & - \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha,N}(r, v, \tau) \psi_\ell(r, v) \partial_\tau A_{\ell,N}(\tau) \, dv \, dr \, d\tau \\ & + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} \psi_\ell(r, v) A_{\ell,N}(\tau) \, dv \, dr \, d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \psi_\ell(r, v) A_{\ell,N}(\tau) \, dv \, dr \, d\tau \right) \\ & + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \psi_\ell(r, v) A_{\ell,N}(\tau) \, dv \, dr \, d\tau \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} \psi_\ell(r, v) A_{\ell,N}(\tau) \, dv \, dr \, d\tau \right) \\ & = \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \psi_\ell(r, v) \, dv \, dr \right) A_{\ell,N}(0), \end{aligned} \tag{2.15}$$

for all $\ell \in \{1, \dots, N\}$ and $\forall \phi \in W^{1,2}(0, T)$.

Summing (2.15) through $\ell = 1, \dots, N$ and recalling (2.11) then yields

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha,N}^2(r, v, t) dv dr \\
& + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |\partial_{v_j} \widehat{\varrho}_{\alpha,N}|^2 dv dr d\tau \right) \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \widehat{\varrho}_{\alpha,N} dv dr d\tau \right) \\
& + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |\partial_{r_j} \widehat{\varrho}_{\alpha,N}|^2 dv dr d\tau \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} \widehat{\varrho}_{\alpha,N} dv dr d\tau \right) \\
& = \frac{1}{2} \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |\widehat{\varrho}_0(r, v)|^2 dv dr \right) \quad \forall t \in (0, T). \tag{2.16}
\end{aligned}$$

Let us denote by T_1 and T_2 the terms in the third and fifth line of (2.16), respectively; our objective is to bound these by quantities that can be absorbed into the remaining terms on the left-hand side. That will then result in uniform-in- N bounds on various norms of $\widehat{\varrho}_{\alpha,N}$, which will allow us to pass to the limit $N \rightarrow \infty$ in the Galerkin approximation.

We shall show below that $M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \widehat{\varrho}_{\alpha,N} \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$. Taking this for granted for the moment, we have that

$$\begin{aligned}
T_1 & := -\frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \widehat{\varrho}_{\alpha,N} dv dr d\tau \\
& = -\frac{1}{2\epsilon} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \cdot \partial_{r_j} (|\widehat{\varrho}_{\alpha,N}|^2) dv dr d\tau \\
& = -\frac{1}{2\epsilon} \sum_{j=1}^{J+1} \int_0^t \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} M(v) (v_j \cdot \nu(r_j)) |\widehat{\varrho}_{\alpha,N}|^2 dv ds(r) d\tau = 0,
\end{aligned}$$

because $\widehat{\varrho}_{\alpha,N} \in W^{1,\infty}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. It therefore remains to show that $M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \widehat{\varrho}_{\alpha,N}$ belongs to $L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$. Since the function $\sqrt{M(v)} \partial_{r_j} \widehat{\varrho}_{\alpha,N} \in L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, it suffices to show that $\sqrt{M(v)} v_j \widehat{\varrho}_{\alpha,N}$ belongs to $L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$.

To this end, we first recall the logarithmic Young's inequality

$$ab \leq e^a + b(\log b - 1) \quad \forall a, b \in \mathbb{R}_{\geq 0}.$$

This follows from the following Fenchel–Young inequality:

$$ab \leq g(a) + g^*(b) \quad \forall a, b \in \mathbb{R}_{\geq 0},$$

involving the convex function $g : a \in \mathbb{R} \mapsto g(a) \in (-\infty, +\infty]$ and its convex conjugate g^* , defined by $g^*(b) := \sup_{a \in \mathbb{R}} (ab - g(a))$, with $g(a) = e^a$ and

$$g^*(b) = \begin{cases} +\infty & \text{if } b < 0; \\ 0 & \text{if } b = 0; \\ b(\log b - 1) & \text{if } b > 0, \end{cases}$$

with the resulting inequality then restricted to $\mathbb{R}_{\geq 0}$. Consequently, recalling that $\mathcal{F}(s) = s(\log s - 1) + 1$ for $s > 0$ and $\mathcal{F}(0) := 0$, we have that

$$ab \leq e^a - 1 + \mathcal{F}(b) \quad \forall a, b \in \mathbb{R}_{\geq 0}. \quad (2.17)$$

Hence, with $a = \frac{1}{4\beta} |v_j|^2$ and $b = \|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2$, we have that

$$\frac{1}{4\beta} |v_j|^2 \|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 \leq \mathcal{F} \left(\|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 \right) + e^{\frac{1}{4\beta} |v_j|^2} - 1,$$

and therefore, upon multiplication by $M(v)$ and omitting the final, negative term from the right-hand side, we deduce that

$$\begin{aligned} & \frac{1}{4\beta} M(v) |v_j|^2 \|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 \\ & \leq M(v) \mathcal{F} \left(\|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 \right) + (2\pi\beta)^{-\frac{(J+1)d}{2}} e^{-\frac{1}{4\beta} |v_j|^2} \prod_{\substack{k=1 \\ k \neq j}}^{J+1} e^{-\frac{1}{2\beta} |v_k|^2} \\ & = M(v) \left[\|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 (\log \|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 - 1) + 1 \right] \\ & \quad + (2\pi\beta)^{-\frac{(J+1)d}{2}} e^{-\frac{1}{4\beta} |v_j|^2} \prod_{\substack{k=1 \\ k \neq j}}^{J+1} e^{-\frac{1}{2\beta} |v_k|^2} \\ & \leq M(v) \|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 \log \|\widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times (0, T))}^2 \\ & \quad + \left[M(v) + (2\pi\beta)^{-\frac{(J+1)d}{2}} e^{-\frac{1}{4\beta} |v_j|^2} \prod_{\substack{k=1 \\ k \neq j}}^{J+1} e^{-\frac{1}{2\beta} |v_k|^2} \right]. \end{aligned}$$

Integrating this over $\mathbb{R}^{(J+1)d}$ and applying Gross' logarithmic Sobolev inequality to the first term on the right-hand side yields (c.f. [30], particularly (1.2) there multiplied by 2, and (1.1) with $n = (J+1)d$) gives

$$\begin{aligned}
& \frac{1}{4\beta} \int_{\mathbb{R}^{(J+1)d}} M(v) |v_j|^2 \|\widehat{\varrho}_{\alpha,N}\|_{L^2(\Omega^{J+1} \times (0,T))}^2 dv \\
& \leq \int_{\mathbb{R}^{(J+1)d}} M(v) \|\widehat{\varrho}_{\alpha,N}\|_{L^2(\Omega^{J+1} \times (0,T))}^2 \log \|\widehat{\varrho}_{\alpha,N}\|_{L^2(\Omega^{J+1} \times (0,T))}^2 dv \\
& \quad + \int_{\mathbb{R}^{(J+1)d}} \left[M(v) + (2\pi\beta)^{-\frac{(J+1)d}{2}} e^{-\frac{1}{4\beta}|v_j|^2} \prod_{\substack{k=1 \\ k \neq j}}^{J+1} e^{-\frac{1}{2\beta}|v_k|^2} \right] dv \\
& \leq 2 \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} M(v) |\partial_{v_j} \widehat{\varrho}_{\alpha,N}|_{L^2(\Omega^{J+1} \times (0,T))}^2 dv \\
& \quad + \|\widehat{\varrho}_{\alpha,N}\|_{L^2_M(\mathbb{R}^{(J+1)d}; L^2(\Omega^{J+1} \times (0,T)))}^2 \log \|\widehat{\varrho}_{\alpha,N}\|_{L^2_M(\mathbb{R}^{(J+1)d}; L^2(\Omega^{J+1} \times (0,T)))}^2 \\
& \quad + \int_{\mathbb{R}^{(J+1)d}} \left[M(v) + (2\pi\beta)^{-\frac{(J+1)d}{2}} e^{-\frac{1}{4\beta}|v_j|^2} \prod_{\substack{k=1 \\ k \neq j}}^{J+1} e^{-\frac{1}{2\beta}|v_k|^2} \right] dv \\
& \leq 2 \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} M(v) \left(\int_{\Omega^{J+1} \times (0,T)} |\partial_{v_j} \widehat{\varrho}_{\alpha,N}|^2 dr d\tau \right) dv \\
& \quad + \|\widehat{\varrho}_{\alpha,N}\|_{L^2(0,T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))}^2 \log \|\widehat{\varrho}_{\alpha,N}\|_{L^2(0,T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))}^2 \\
& \quad + \int_{\mathbb{R}^{(J+1)d}} \left[M(v) + (2\pi\beta)^{-\frac{(J+1)d}{2}} e^{-\frac{1}{4\beta}|v_j|^2} \prod_{\substack{k=1 \\ k \neq j}}^{J+1} e^{-\frac{1}{2\beta}|v_k|^2} \right] dv. \tag{2.18}
\end{aligned}$$

The term in the square brackets on the right-hand side is trivially in $L^1(\mathbb{R}^{(J+1)d})$. Furthermore, both $\sqrt{M(v)} \widehat{\varrho}_{\alpha,N}$ and $\sqrt{M(v)} \partial_{v_j} \widehat{\varrho}_{\alpha,N}$ belong to $L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$ for all $j = 1, \dots, J+1$. Thus we have shown that $M(v) |v_j|^2 |\widehat{\varrho}_{\alpha,N}|^2 \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$; hence, $\sqrt{M(v)} v_j \widehat{\varrho}_{\alpha,N}$ belongs to $L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, as required. This completes the proof of the assertion that $\mathsf{T}_1 = 0$.

Let us now turn our attention to the term

$$\mathsf{T}_2 := -\frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} \widehat{\varrho}_{\alpha,N} dv dr d\tau \right).$$

We have, by the Cauchy–Schwarz inequality, the triangle inequality, and noting that $|(\mathcal{L}r)_j| \leq 4\sqrt{d}L$, that

$$\begin{aligned}
\mathsf{T}_2 & \leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} (|(\mathcal{L}r)_j| + |u(r_j, \tau)|) \widehat{\varrho}_{\alpha,N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
& \quad \times \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha,N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{T}_2 &\leq \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} (|(\mathcal{L}r)_j| + |u(r_j, \tau)|) \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\leq \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} |(\mathcal{L}r)_j| \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&+ \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} |u(r_j, \tau)| \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\leq \frac{1}{\epsilon} 4\sqrt{(J+1)d}L \left(\int_0^t \|\sqrt{M} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&+ \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} |u(r_j, \tau)| \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}}.
\end{aligned}$$

We shall focus our attention on the second term in the square brackets on the right-hand side:

$$\begin{aligned}
&\int_0^t \|\sqrt{M} |u(r_j, \tau)| \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \\
&= \int_0^t \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) |u(r_j, \tau)|^2 \widehat{\varrho}_{\alpha, N}^2(r, v, \tau) dr dv d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
& \int_0^t \|\sqrt{M} |u(r_j, \tau)| \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau \\
&= \int_0^t \int_{\Omega^{J+1}} |u(r_j, \tau)|^2 \left(\int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha, N}^2(r, v, \tau) \mathrm{d}v \right) \mathrm{d}r \mathrm{d}\tau \\
&\leq \int_0^t \|u(\cdot, \tau)\|_{L^\infty(\Omega)}^2 \left(\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha, N}^2(r, v, \tau) \mathrm{d}v \right) \mathrm{d}r \mathrm{d}\tau.
\end{aligned}$$

Thus, we have the following bound:

$$\begin{aligned}
& \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} |u(r_j, \tau)| \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\leq \sqrt{J+1} \left(\int_0^t \|u(\cdot, \tau)\|_{L^\infty(\Omega)}^2 \|\sqrt{M} \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, \tau)\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau \right)^{\frac{1}{2}}. \quad (2.19)
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{T}_2 &\leq \frac{1}{\epsilon} C(L, J) \left(\int_0^t (1 + \|u(\cdot, \tau)\|_{L^\infty(\Omega)}^2) \|\sqrt{M} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau \right)^{\frac{1}{2}}.
\end{aligned}$$

Returning with this bound to (2.16), we have that

$$\begin{aligned}
& \frac{1}{2} \|\sqrt{M} \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, t)\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \\
&+ \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau + \quad (2.20)
\end{aligned}$$

$$\begin{aligned}
& \alpha \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{r_j} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau \\
&\leq \frac{1}{2} \|\sqrt{M} \widehat{\varrho}_0\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \\
&\quad + \frac{1}{2\beta^2} C(L, J)^2 \left(\int_0^t (1 + \|u(\cdot, \tau)\|_{L^\infty(\Omega)}^2) \|\sqrt{M} \widehat{\varrho}_{\alpha, N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \mathrm{d}\tau \right) \\
&\forall t \in (0, T]. \quad (2.21)
\end{aligned}$$

Hence, by Gronwall's lemma,

$$\begin{aligned}
& \|\sqrt{M} \widehat{\varrho}_{\alpha,N}(\cdot, \cdot, t)\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \\
& + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha,N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \\
& + 2\alpha \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{r_j} \widehat{\varrho}_{\alpha,N}\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \\
& \leq \|\sqrt{M} \widehat{\varrho}_0\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \exp\left(\frac{1}{\beta^2} C(L, J)^2 (T + \|u\|_{L^2(0,T;L^\infty(\Omega))}^2)\right) \\
& \quad \forall t \in (0, T].
\end{aligned} \tag{2.22}$$

Thus, for $\alpha \in (0, 1]$ fixed, we deduce the following uniform bounds with respect to N :

$$\begin{aligned}
& \|\widehat{\varrho}_{\alpha,N}\|_{L^\infty(0,T;L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq C(L, J, T, \epsilon, \|u\|_{L^2(0,T;L^\infty(\Omega))}) \|\widehat{\varrho}_0\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}, \\
& \|\partial_{v_j} \widehat{\varrho}_{\alpha,N}\|_{L^2(0,T;L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq C(L, J, T, \epsilon, \|u\|_{L^2(0,T;L^\infty(\Omega))}) \|\widehat{\varrho}_0\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}, \\
& \sqrt{\alpha} \|\partial_{r_j} \widehat{\varrho}_{\alpha,N}\|_{L^2(0,T;L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq C(L, J, T, \epsilon, \|u\|_{L^2(0,T;L^\infty(\Omega))}) \|\widehat{\varrho}_0\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})},
\end{aligned} \tag{2.23}$$

for all $j \in \{1, \dots, J+1\}$. Furthermore, by (2.18),

$$\|v_j \widehat{\varrho}_{\alpha,N}\|_{L^2(0,T;L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq C(L, J, T, \epsilon, \|u\|_{L^2(0,T;L^\infty(\Omega))}) \|\widehat{\varrho}_0\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}, \tag{2.24}$$

for all $j \in \{1, \dots, J+1\}$; as $\beta > 0$ is considered to be fixed throughout, the dependence of the constants on β has not been (and will not be) indicated.

Next, we shall exploit the bounds stated in (2.23) and (2.24) to derive a uniform-in- N bound on $\partial_t \widehat{\varrho}_{\alpha,N}$ in the function space $L^2(0, T; (W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))')$. Let us first note that

$$\begin{aligned}
& \|\partial_t \widehat{\varrho}_{\alpha,N}(t)\|_{(W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'} \\
& = \sup_{w \in W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}), \|w\|_{W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \leq 1} (M \partial_t \widehat{\varrho}_{\alpha,N}(t), w) \\
& = \sup_{w \in W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}), \|w\|_{W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \leq 1} (M \partial_t \widehat{\varrho}_{\alpha,N}(t), P_N w),
\end{aligned}$$

where (\cdot, \cdot) denotes the inner product of $L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$. By reversing the partial integration with respect to τ in (2.13), we deduce, for all $t \in (0, T]$, that

$$\begin{aligned}
& \int_0^t (M \partial_\tau \widehat{\varrho}_{\alpha,N}(\cdot, \cdot, \tau), \psi_\ell(\cdot, \cdot)) \phi(\tau) d\tau + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t (M \partial_{v_j} \widehat{\varrho}_{\alpha,N}(\cdot, \cdot, \tau), \partial_{v_j} \psi_\ell(\cdot, \cdot)) \phi(\tau) d\tau \\
& - \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t (M v_j \widehat{\varrho}_{\alpha,N}(\cdot, \cdot, \tau), \partial_{r_j} \psi_\ell(\cdot, \cdot)) \phi(\tau) d\tau
\end{aligned}$$

$$\begin{aligned}
& + \alpha \sum_{j=1}^{J+1} \int_0^t (M \partial_{r_j} \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, \tau), \partial_{r_j} \psi_\ell(\cdot, \cdot)) \phi(\tau) \, d\tau \\
& - \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t (M ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, \tau), \partial_{v_j} \psi_\ell(\cdot, \cdot)) \phi(\tau) \, d\tau = 0 \\
& \forall \ell \in \{1, \dots, N\} \text{ and } \forall \phi \in W^{1,2}(0, T). \tag{2.25}
\end{aligned}$$

Hence, thanks to the density of $W^{1,2}(0, T)$ in $L^p(0, T)$ for all $p \in [1, \infty)$, and recalling the fundamental lemma of the calculus of variations (du Bois-Reymond's lemma), we have that

$$\begin{aligned}
& (M \partial_t \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, t), \psi_\ell(\cdot, \cdot)) + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} (M \partial_{v_j} \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, t), \partial_{v_j} \psi_\ell(\cdot, \cdot)) \\
& - \frac{1}{\epsilon} \sum_{j=1}^{J+1} (M v_j \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, t), \partial_{r_j} \psi_\ell(\cdot, \cdot)) + \alpha \sum_{j=1}^{J+1} (M \partial_{r_j} \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, t), \partial_{r_j} \psi_\ell(\cdot, \cdot)) \\
& - \frac{1}{\epsilon} \sum_{j=1}^{J+1} (M ((\mathcal{L}r)_j + u(r_j, t)) \widehat{\varrho}_{\alpha, N}(\cdot, \cdot, t), \partial_{v_j} \psi_\ell(\cdot, \cdot)) = 0 \\
& \forall \ell \in \{1, \dots, N\} \text{ and a.e. } t \in (0, T].
\end{aligned}$$

This then implies that

$$\begin{aligned}
(M \partial_t \widehat{\varrho}_{\alpha, N}(t), P_N w) & = -\frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} (M \partial_{v_j} \widehat{\varrho}_{\alpha, N}(t), \partial_{v_j} P_N w) \\
& + \frac{1}{\epsilon} \sum_{j=1}^{J+1} (M v_j \widehat{\varrho}_{\alpha, N}(t), \partial_{r_j} P_N w) - \alpha \sum_{j=1}^{J+1} (M \partial_{r_j} \widehat{\varrho}_{\alpha, N}(t), \partial_{r_j} P_N w) \\
& + \frac{1}{\epsilon} \sum_{j=1}^{J+1} (M ((\mathcal{L}r)_j + u(r_j, t)) \widehat{\varrho}_{\alpha, N}(t), \partial_{v_j} P_N w) \\
& =: S_1(t) + S_2(t) + S_3(t) + S_4(t) \quad \forall \ell \in \{1, \dots, N\} \text{ and a.e. } t \in (0, T].
\end{aligned}$$

The terms $S_1(t)$ and $S_3(t)$ are easy to bound: for a.e. $t \in (0, T]$,

$$\begin{aligned}
|S_1(t)| & \leq \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_{\alpha, N}(t)\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}} \\
& \quad \times \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{v_j} P_N w\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}}
\end{aligned}$$

and

$$|S_3(t)| \leq \alpha \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{r_j} \widehat{\varrho}_{\alpha, N}(t)\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}} \\ \times \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{r_j} P_N w\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}}.$$

Thus, by (2.23)₂ we have that,

$$\int_0^T |S_1(t)|^2 dt \leq C \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{v_j} P_N w\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}}$$

and, by (2.23)₃,

$$\int_0^T |S_3(t)|^2 dt \leq C\sqrt{\alpha} \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{r_j} P_N w\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}},$$

where C is a positive constant, independent of N and α .

For the term $S_2(t)$, we have, for a.e. $t \in (0, T]$, that

$$|S_2(t)| \leq \alpha \left(\sum_{j=1}^{J+1} \|\sqrt{M} |v_j| \widehat{\varrho}_{\alpha, N}(t)\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}} \\ \times \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{r_j} P_N w\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}}.$$

Now, (2.24) implies that

$$\frac{1}{4\beta} \int_0^T \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) |v_j|^2 |\widehat{\varrho}_{\alpha, N}|^2 dr dv d\tau \leq C, \quad (2.26)$$

where the constant C is independent of N and α , and therefore

$$\int_0^T |S_2(t)|^2 dt \leq C\alpha \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{r_j} P_N w\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}},$$

where C is independent of N and α .

Finally, thanks to (2.19) and (2.23), we have that

$$\int_0^T |S_4(t)|^2 dt \leq C \left(\sum_{j=1}^{J+1} \|\sqrt{M} \partial_{v_j} P_N w\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \right)^{\frac{1}{2}},$$

where, again, C is independent of N and α .

By collecting the bounds on S_1, \dots, S_4 , noting (2.10), and recalling that $\alpha \in (0, 1]$, we deduce that

$$\int_0^T \|\partial_t \widehat{\varrho}_{\alpha, N}(t)\|_{(W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'}^2 dt \leq C(L, J, T, \epsilon, \|u\|_{L^2(0,T;L^\infty(\Omega))}).$$

Hence,

$$\|\partial_t \widehat{\varrho}_{\alpha, N}\|_{L^2(0,T;(W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))')} \leq C(L, J, T, \epsilon, \|u\|_{L^2(0,T;L^\infty(\Omega))}), \quad (2.27)$$

as required.

The bounds (2.23), (2.24), (2.27) in conjunction with the compact embedding of $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ into the function space $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ and the Aubin–Lions lemma (cf. [59]) imply the existence of a subsequence (not indicated) of $(\widehat{\varrho}_{\alpha, N})_{N \geq 1}$ and of an element

$$\begin{aligned} \widehat{\varrho}_\alpha \in L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \cap L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \\ \cap W^{1,2}(0, T; (W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))') \end{aligned}$$

such that, as $N \rightarrow \infty$,

$$\begin{aligned} \widehat{\varrho}_{\alpha, N} &\rightharpoonup^* \widehat{\varrho}_\alpha && \text{weakly* in } L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \widehat{\varrho}_{\alpha, N} &\rightarrow \widehat{\varrho}_\alpha && \text{strongly in } L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \widehat{\varrho}_{\alpha, N} &\rightharpoonup \widehat{\varrho}_\alpha && \text{weakly in } L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ |v_j| \widehat{\varrho}_{\alpha, N} &\rightharpoonup |v_j| \widehat{\varrho}_\alpha && \text{weakly in } L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \partial_t \widehat{\varrho}_{\alpha, N} &\rightharpoonup \partial_t \widehat{\varrho}_\alpha && \text{weakly in } L^2(0, T; (W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'). \end{aligned} \quad (2.28)$$

Thanks to the density of $W_{0,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ in $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ (cf. Appendix A in [7]) and noting that $W_{0,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \subset W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, it follows that $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is dense in the space $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$. Thus, the Hilbert space $V := W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is continuously and densely embedded into the Hilbert space $H := L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$. Hence, according to the function space interpolation result (2.41) in Lions & Magenes [41], $[V, V']_{1/2} = H$, and therefore Theorem 3.1 in [41] yields the continuous embedding

$$\begin{aligned} L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \cap W^{1,2}(0, T; (W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))') \\ \hookrightarrow \mathcal{C}([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \end{aligned}$$

which then implies that

$$\begin{aligned} \widehat{\varrho}_\alpha \in \mathcal{C}([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \lim_{N \rightarrow \infty} (\widehat{\varrho}_{\alpha, N}(\cdot, \cdot, t) - \widehat{\varrho}_\alpha(\cdot, \cdot, t), \eta)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \rightarrow 0 \\ \forall \eta \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \quad \forall t \in [0, T]. \end{aligned} \quad (2.29)$$

By passing to the limit $N \rightarrow \infty$ in (2.22), using the weak convergence results (2.28) in conjunction with the weak lower-semicontinuity of the norm function, we deduce that $\widehat{\varrho}_\alpha$

satisfies the following energy inequality:

$$\begin{aligned}
& \|\sqrt{M} \widehat{\varrho}_\alpha(\cdot, \cdot, t)\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \\
& + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} \widehat{\varrho}_\alpha\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau + 2\alpha \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{r_j} \widehat{\varrho}_\alpha\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \\
& \leq \|\sqrt{M} \widehat{\varrho}_0\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \exp\left(\frac{1}{\beta^2} C(L, J)^2 (T + \|u\|_{L^2(0, T; L^\infty(\Omega))}^2)\right) \\
& \quad \forall t \in (0, T].
\end{aligned} \tag{2.30}$$

Furthermore, by replacing $\varphi(r, v, \tau)$ with $\varphi(r, v, \tau) \chi_{t, h}(\tau)$ in (2.9), for $t \in (0, T]$ fixed, and passing to the limit $h \rightarrow 0_+$, we deduce that

$$\begin{aligned}
& \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, t) \varphi(r, v, t) dv dr \\
& - \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, \tau) \partial_\tau \varphi(r, v, \tau) dv dr d\tau \\
& + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\
& + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi dv dr d\tau \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\
& = \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \varphi(r, v, 0) dv dr \\
& \quad \forall \varphi \in W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})).
\end{aligned} \tag{2.31}$$

By letting $t \rightarrow 0_+$ in the weak formulation (2.31), recalling that $\widehat{\varrho}_\alpha \in \mathcal{C}([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ and noting that $W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \hookrightarrow \mathcal{C}([0, T]; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, it follows that

$$\begin{aligned}
& \lim_{t \rightarrow 0_+} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, t) \varphi(r, v, t) dv dr \\
& = \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, 0) \varphi(r, v, 0) dv dr.
\end{aligned}$$

Then,

$$\begin{aligned}
& \lim_{t \rightarrow 0_+} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, t) \varphi(r, v, t) \, dv \, dr \\
&= \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \varphi(r, v, 0) \, dv \, dr \\
& \quad \forall \varphi \in W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})).
\end{aligned}$$

As was noted in the paragraph preceding (2.29), $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is continuously and densely embedded into $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, so we deduce from the above, with $\varphi(\cdot, \cdot, t) \equiv \eta(\cdot, \cdot) \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, that

$$\lim_{t \rightarrow 0_+} (\widehat{\varrho}_\alpha(t) - \widehat{\varrho}_0, \eta)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} = 0 \quad \forall \eta \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}). \quad (2.32)$$

This weak attainment of the initial datum $\widehat{\varrho}_0$ by the solution $\widehat{\varrho}_\alpha$ can be strengthened, in fact. By letting $t \rightarrow 0_+$ in (2.30), it follows that

$$\lim_{t \rightarrow 0_+} \|\widehat{\varrho}_\alpha(t)\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \leq \|\widehat{\varrho}_0\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2.$$

Hence, and noting (2.32),

$$\begin{aligned}
& \lim_{t \rightarrow 0_+} \|\widehat{\varrho}_\alpha(t) - \widehat{\varrho}_0\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 = \lim_{t \rightarrow 0_+} (\widehat{\varrho}_\alpha(t) - \widehat{\varrho}_0, \widehat{\varrho}_\alpha(t) - \widehat{\varrho}_0)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
&= \lim_{t \rightarrow 0_+} (\widehat{\varrho}_\alpha(t), \widehat{\varrho}_\alpha(t) - \widehat{\varrho}_0)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
&= \lim_{t \rightarrow 0_+} \|\widehat{\varrho}_\alpha(t)\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 - \lim_{t \rightarrow 0_+} (\widehat{\varrho}_\alpha(t), \widehat{\varrho}_0)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
&= \lim_{t \rightarrow 0_+} \|\widehat{\varrho}_\alpha(t)\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 - (\widehat{\varrho}_0, \widehat{\varrho}_0)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \leq 0,
\end{aligned}$$

which, by the nonnegativity of the norm, then implies the following strong attainment of the initial datum:

$$\lim_{t \rightarrow 0_+} \|\widehat{\varrho}_\alpha(t) - \widehat{\varrho}_0\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 = 0. \quad (2.33)$$

Having thus shown that $\widehat{\varrho}_\alpha$ satisfies the given initial condition, we shall now pass to the limit $N \rightarrow \infty$ in the Galerkin approximation, in order to show that $\widehat{\varrho}_\alpha$ is in fact a weak solution to the parabolic regularization (2.9) of (2.8).

Given any (fixed) $\varphi \in W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, we consider the function $\varphi_N \in W^{1,2}(0, T; \mathcal{X}_N)$, defined by

$$\varphi_N(r, v, t) := \sum_{k=1}^N \beta_{k,N}(t) \psi_k(r, v),$$

where $\beta_{k,N} \in W^{1,2}(0, T)$ is defined by

$$\beta_{k,N}(t) = (\varphi(\cdot, \cdot, t), \psi_k(\cdot, \cdot))_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}, \quad k = 1, \dots, N; \quad N \geq 1.$$

Hence,

$$\lim_{N \rightarrow \infty} \|\varphi - \varphi_N\|_{W^{1,2}(0,T;W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} = 0. \quad (2.34)$$

Next, for $\varphi \in W^{1,2}(0,T;W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ fixed and $\varphi_N \in W^{1,2}(0,T;\mathcal{X}_N)$ as defined above, we rewrite (2.12) in the following equivalent form:

$$\begin{aligned} & \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha,N}(r, v, T) \varphi(r, v, T) \, dv \, dr \\ & - \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha,N}(r, v, \tau) \partial_\tau \varphi(r, v, \tau) \, dv \, dr \, d\tau \\ & + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \right) \\ & + \alpha \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\ = & \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \varphi(r, v, 0) \, dv \, dr \\ & - \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) (\varphi - \varphi_N)(r, v, 0) \, dv \, dr \\ & + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha,N}(r, v, T) (\varphi - \varphi_N)(r, v, T) \, dv \, dr \\ & - \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_{\alpha,N}(r, v, \tau) \partial_\tau (\varphi - \varphi_N)(r, v, \tau) \, dv \, dr \, d\tau \\ & + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j} (\varphi - \varphi_N) \, dv \, dr \, d\tau \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j}(\varphi - \varphi_N) dv dr d\tau \right) \\
& + \alpha \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_{\alpha,N} \cdot \partial_{r_j}(\varphi - \varphi_N) dv dr d\tau \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_{\alpha,N} \cdot \partial_{v_j}(\varphi - \varphi_N) dv dr d\tau \right).
\end{aligned} \tag{2.35}$$

Now, using the convergence results (2.28), (2.29), the uniform bounds (2.19), (2.23), (2.26), together with the strong convergence (2.34), passage to the limit $N \rightarrow \infty$ in (2.35) yields that the function

$$\widehat{\varrho}_\alpha \in \mathcal{C}([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \cap L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$$

which satisfies (2.9). Indeed, as $N \rightarrow \infty$, all terms on the right-hand side of (2.35), except the first, converge to zero, while each of the terms on the left-hand side converges to its counterpart with $\widehat{\varrho}_{N,\alpha}$ replaced by $\widehat{\varrho}_\alpha$. Thus we have shown that, for any $\alpha \in (0, 1]$, $\widehat{\varrho}_\alpha$ is a solution of (2.9), and the energy inequality (2.30) holds. It is important to note for the purpose of the discussion in the next section that the right-hand side of the inequality (2.30) is independent of α ; therefore, (2.30) implies that

$$\widehat{\varrho}_\alpha \quad \text{is bounded in } L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \tag{2.36}$$

$$\partial_{v_j} \widehat{\varrho}_\alpha \quad \text{is bounded in } L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{for all } j = 1, \dots, J+1, \tag{2.37}$$

$$\sqrt{\alpha} \partial_{r_j} \widehat{\varrho}_\alpha \quad \text{is bounded in } L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{for all } j = 1, \dots, J+1, \tag{2.38}$$

provided that $\widehat{\varrho}_0 \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ and $u \in L^2(0, T; L^\infty(\Omega)^d)$. Furthermore, weak lower semicontinuity of the norm function, (2.24) and (2.28)₄ imply that

$$|v_j| \widehat{\varrho}_\alpha \quad \text{is bounded in } L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{for all } j = 1, \dots, J+1. \tag{2.39}$$

Similarly, (2.27) implies that

$$\partial_t \widehat{\varrho}_\alpha \quad \text{is bounded in } L^2(0, T; (W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'). \tag{2.40}$$

We are now ready to pass to the limit $\alpha \rightarrow 0_+$.

2.2 Passage to the limit with the parabolic regularization parameter

The next step in our argument is passage to the limit $\alpha \rightarrow 0_+$ in (2.9). We begin by noting that (2.36)–(2.39) imply that

$$\widehat{\varrho}_\alpha \rightharpoonup \widehat{\varrho} \quad \text{weak* in } L^\infty(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad (2.41a)$$

$$\partial_{v_j} \widehat{\varrho}_\alpha \rightharpoonup \partial_{v_j} \widehat{\varrho} \quad \text{weakly in } L^2(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{for all } j = 1, \dots, J+1, \quad (2.41b)$$

$$\alpha \partial_{r_j} \widehat{\varrho}_\alpha \rightarrow 0 \quad \text{strongly in } L^2(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{for all } j = 1, \dots, J+1, \quad (2.41c)$$

$$|v_j| \widehat{\varrho}_\alpha \rightharpoonup |v_j| \widehat{\varrho} \quad \text{weakly in } L^2(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{for all } j = 1, \dots, J+1, \quad (2.41d)$$

provided that $\widehat{\varrho}_0 \in L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ and $u \in L^2(0, T; L^\infty(\Omega)^d)$.

Next, we shall prove that $\widehat{\varrho}_\alpha \geq 0$ a.e. on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T]$ for all $\alpha \in (0, 1]$, and that

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, t) \, dr \, dv = 1 \quad \forall t \in [0, T]. \quad (2.42)$$

The proof of the latter assertion is straightforward: for $t = 0$ it follows from (2.5); for $t \in (0, T]$ (fixed), we take $\varphi(r, v, \tau) \equiv 1$ in (2.9) and note (2.5) to deduce that

$$\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha(r, v, t) \, dv \, dr = \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) \, dv \, dr = 1 \quad \forall t \in (0, T].$$

Before embarking on the proof of the nonnegativity of $\widehat{\varrho}_\alpha$ we shall first extend the set of test functions

$$W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$$

appearing in (2.31) to

$$L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$$

by using a density argument. We begin by rewriting (2.31) as follows:

$$\begin{aligned} & \int_0^t \langle M \partial_\tau \widehat{\varrho}_\alpha(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle \, d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \right) \\ & + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) = 0 \\
& \quad \forall \varphi \in W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \forall t \in (0, T],
\end{aligned} \tag{2.43}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $(W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'$ and $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ with respect to $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ as pivot space, into which $W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is continuously and densely embedded; hence $\langle \eta, \phi \rangle$ and $(\eta, \phi)_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}$ are identified when $\eta \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ and $\phi \in W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$. We note that for

$$\begin{aligned}
& \widehat{\varrho}_\alpha \in \mathcal{C}([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \cap L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \\
& \quad \cap W^{1,2}(0, T; (W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))')
\end{aligned}$$

fixed, each of the terms in (2.43) is a bounded linear functional of $\varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. As the Hilbert space $W^{1,2}(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ is continuously and densely embedded into the Hilbert space $L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, we deduce that

$$\begin{aligned}
& \widehat{\varrho}_\alpha \in \mathcal{C}([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \cap L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \\
& \quad \cap W^{1,2}(0, T; (W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))')
\end{aligned}$$

satisfies the following weak formulation:

$$\begin{aligned}
& \int_0^t \langle M \partial_\tau \widehat{\varrho}_\alpha(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle \, d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \right) \\
& + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_\alpha \cdot \partial_{r_j} \varphi \, dv \, dr \, d\tau \\
& - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_\alpha \cdot \partial_{v_j} \varphi \, dv \, dr \, d\tau \right) = 0 \\
& \quad \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \forall t \in (0, T].
\end{aligned} \tag{2.44}$$

We prove the nonnegativity of $\widehat{\varrho}_\alpha$ by using Stampacchia's truncation method. Let $[x]_\pm$ denote the nonnegative and nonpositive parts of x , i.e., $[x]_\pm := \frac{1}{2}(x \pm |x|)$. Thus, $x = [x]_+ + [x]_-$ and $x[x]_- = ([x]_-)^2$.

By taking $\varphi = [\widehat{\varrho}_\alpha]_-$ in (2.44) (which belongs to the function space $L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$), because $\widehat{\varrho}_\alpha$ belongs to this space), we have that

$$\begin{aligned} & \int_0^t \frac{1}{2} \frac{d}{dt} \|\sqrt{M} [\widehat{\varrho}_\alpha(\tau)]_-\|^2 d\tau + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} [\widehat{\varrho}_\alpha]_-\|^2 d\tau + \alpha \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{r_j} [\widehat{\varrho}_\alpha]_-\|^2 d\tau \\ &= \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t (M v_j [\widehat{\varrho}_\alpha]_-, \partial_{r_j} [\widehat{\varrho}_\alpha]_-) d\tau \\ & \quad + \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t (M ((\mathcal{L}r)_j + u(r_j, \tau)) [\widehat{\varrho}_\alpha]_-, \partial_{v_j} [\widehat{\varrho}_\alpha]_-) d\tau \quad \forall t \in (0, T), \end{aligned}$$

subject to the initial condition $[\widehat{\varrho}_\alpha(0)]_- = [\widehat{\varrho}_0]_- = 0$. Therefore, for all $t \in (0, T]$,

$$\begin{aligned} & \frac{1}{2} \|\sqrt{M} [\widehat{\varrho}_\alpha(t)]_-\|^2 + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{v_j} [\widehat{\varrho}_\alpha]_-\|^2 d\tau + \alpha \sum_{j=1}^{J+1} \int_0^t \|\sqrt{M} \partial_{r_j} [\widehat{\varrho}_\alpha]_-\|^2 d\tau \\ &= \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t (M v_j [\widehat{\varrho}_\alpha]_-, \partial_{r_j} [\widehat{\varrho}_\alpha]_-) d\tau \\ & \quad + \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t (M ((\mathcal{L}r)_j + u(r_j, \tau)) [\widehat{\varrho}_\alpha]_-, \partial_{v_j} [\widehat{\varrho}_\alpha]_-) d\tau. \end{aligned} \tag{2.45}$$

Next we apply the Cauchy–Schwarz inequality to each of the two terms on the right-hand side of (2.45). We then repeat the calculations that resulted in the bounds (2.18) and (2.19), but now with $\widehat{\varrho}_{\alpha,N}$ replaced by $[\widehat{\varrho}_\alpha]_-$ in those bounds, insert the resulting bounds into the right-hand side of (2.45), absorb the terms containing norms of derivatives of $[\widehat{\varrho}_\alpha]_-$ into the left-hand side, and apply Gronwall's lemma to deduce that $\|\sqrt{M} [\widehat{\varrho}_\alpha(t)]_-\|^2 = 0$ for all $t \in [0, T]$. Consequently $\widehat{\varrho}_\alpha \geq 0$ a.e. on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T]$, as has been asserted. Finally we note that an identical procedure can be used to deduce that $\widehat{\varrho}_\alpha$ is the unique weak solution of (2.44) satisfying the initial condition $\widehat{\varrho}_\alpha(0) = \widehat{\varrho}_0$.

The expression appearing on the right-hand side of (2.30) involves the $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\widehat{\varrho}_0$, whereas, ultimately, we would like to make use of the weaker hypotheses, stated in (2.5), only. As a matter of fact, in the next section we will require an analogous inequality whose right-hand side involves the $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\mathcal{F}(\widehat{\varrho}_0)$ rather than the $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\widehat{\varrho}_0$. Thus, before passing to the limit $\alpha \rightarrow 0_+$ by using the weak convergence results stated in (2.41a)–(2.41d), we shall derive additional bounds on $\widehat{\varrho}_\alpha$, which involve the $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\mathcal{F}(\widehat{\varrho}_0)$ rather than the $L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\widehat{\varrho}_0$. The resulting bounds will also play an important role in the next section, where we focus on the coupled Oseen–Fokker–Planck system. The argument is based on the relative entropy method. Briefly, the procedure involves choosing $\mathcal{F}'(\widehat{\varrho}_\alpha + \gamma)$ as test function in (2.44), with $\gamma > 0$, and passing to the limit $\gamma \rightarrow 0_+$; ideally,

we would like to choose $\mathcal{F}'(\widehat{\varrho}_\alpha)$ as test function in (2.44), however since $\widehat{\varrho}_\alpha \geq 0$ a.e. on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T]$, and $\widehat{\varrho}_\alpha$ is potentially equal to 0 on a subset of positive measure, there is no guarantee that $\mathcal{F}'(\widehat{\varrho}_\alpha) = \log \widehat{\varrho}_\alpha$ is a.e. finite. Thus we shall, instead, test with $\mathcal{F}'(\widehat{\varrho}_\alpha + \gamma) = \log(\widehat{\varrho}_\alpha + \gamma)$, and once we have obtained the necessary bounds we shall pass to the limit $\gamma \rightarrow 0_+$, which will then be followed by passage to the limit with $\alpha \rightarrow 0_+$.

We begin by noting that since $\widehat{\varrho}_\alpha \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ and $\widehat{\varrho}_\alpha \geq 0$, also

$$\mathcal{F}'(\widehat{\varrho}_\alpha + \gamma) = \log(\widehat{\varrho}_\alpha + \gamma) \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})).$$

Hence, for all $t \in (0, T]$, (2.44) yields

$$\begin{aligned} & \int_0^t \langle M \partial_\tau \widehat{\varrho}_\alpha(\cdot, \cdot, \tau), \mathcal{F}'(\widehat{\varrho}_\alpha(\cdot, \cdot, \tau) + \gamma) \rangle d\tau \\ & + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_\alpha \cdot \partial_{v_j} \mathcal{F}'(\widehat{\varrho}_\alpha(r, v, \tau) + \gamma) dv dr d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\alpha \cdot \partial_{r_j} \mathcal{F}'(\widehat{\varrho}_\alpha(r, v, \tau) + \gamma) dv dr d\tau \right) \\ & + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{r_j} \widehat{\varrho}_\alpha \cdot \partial_{r_j} \mathcal{F}'(\widehat{\varrho}_\alpha(r, v, \tau) + \gamma) dv dr d\tau \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_\alpha \cdot \partial_{v_j} \mathcal{F}'(\widehat{\varrho}_\alpha(r, v, \tau) + \gamma) dv dr d\tau \right) \\ & = 0. \end{aligned}$$

Thus, we have that

$$\begin{aligned} & \langle M \mathcal{F}(\widehat{\varrho}_\alpha(\cdot, \cdot, t) + \gamma), 1 \rangle - \langle M \mathcal{F}(\widehat{\varrho}_0(\cdot, \cdot) + \gamma), 1 \rangle \\ & + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha(r, v, \tau) + \gamma} dv dr d\tau \\ & + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{r_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha(r, v, \tau) + \gamma} dv dr d\tau \\ & = \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \frac{\widehat{\varrho}_\alpha}{\widehat{\varrho}_\alpha(r, v, \tau) + \gamma} \cdot \partial_{r_j} \widehat{\varrho}_\alpha dv dr d\tau \\ & + \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \frac{\widehat{\varrho}_\alpha}{\widehat{\varrho}_\alpha(r, v, \tau) + \gamma} \cdot \partial_{v_j} \widehat{\varrho}_\alpha(r, v, \tau) dv dr d\tau. \end{aligned}$$

Similarly to the term T_1 encountered earlier in the argument following (2.16), the first term on the right-hand side is equal to zero. This can be seen by interchanging the order of the integrals over Ω^{J+1} and $\mathbb{R}^{(J+1)d}$, observing that

$$\frac{\widehat{\varrho}_\alpha}{\widehat{\varrho}_\alpha(r, v, \tau) + \gamma} \partial_{r_j} \widehat{\varrho}_\alpha = \partial_{r_j} [\widehat{\varrho}_\alpha - \gamma \log(\widehat{\varrho}_\alpha + \gamma)]$$

and

$$\widehat{\varrho}_\alpha - \gamma \log(\widehat{\varrho}_\alpha + \gamma) \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$

and performing integration by parts with respect to r_j . The resulting equality can be rewritten as follows:

$$\begin{aligned} & \langle M \mathcal{F}(\widehat{\varrho}_\alpha(\cdot, \cdot, t) + \gamma), 1 \rangle + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha + \gamma} dv dr d\tau \\ & + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{r_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha + \gamma} dv dr d\tau \\ & = \langle M \mathcal{F}(\widehat{\varrho}_0(\cdot, \cdot) + \gamma), 1 \rangle \\ & + \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \frac{\widehat{\varrho}_\alpha}{\widehat{\varrho}_\alpha + \gamma} \cdot \partial_{v_j} \widehat{\varrho}_\alpha dv dr d\tau =: R_1 + R_2. \end{aligned}$$

We begin by considering R_2 . As

$$\left| \frac{\widehat{\varrho}_\alpha}{\widehat{\varrho}_\alpha + \gamma} \cdot \partial_{v_j} \widehat{\varrho}_\alpha \right| \leq \frac{\widehat{\varrho}_\alpha}{(\widehat{\varrho}_\alpha + \gamma)^{\frac{1}{2}}} \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|}{(\widehat{\varrho}_\alpha + \gamma)^{\frac{1}{2}}} \leq (\widehat{\varrho}_\alpha)^{\frac{1}{2}} \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|}{(\widehat{\varrho}_\alpha + \gamma)^{\frac{1}{2}}},$$

it follows by the Cauchy–Schwarz inequality that

$$\begin{aligned} R_2 & \leq \left(\frac{1}{\beta^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |(\mathcal{L}r)_j + u(r_j, \tau)|^2 \widehat{\varrho}_\alpha dv dr d\tau \right)^{\frac{1}{2}} \\ & \quad \times \left(\frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha + \gamma} dv dr d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \langle M \mathcal{F}(\widehat{\varrho}_\alpha(\cdot, \cdot, t) + \gamma), 1 \rangle + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha + \gamma} dv dr d\tau \\ & + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{r_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha + \gamma} dv dr d\tau \\ & \leq R_1 + \frac{1}{2\beta^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |(\mathcal{L}r)_j + u(r_j, \tau)|^2 \widehat{\varrho}_\alpha dv dr d\tau \right). \end{aligned} \quad (2.46)$$

The second term on the right-hand side of (2.46) is, thanks to (2.42), bounded as follows:

$$\begin{aligned}
& \frac{1}{2\beta^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |(\mathcal{L}r)_j + u(r_j, \tau)|^2 \widehat{\varrho}_\alpha \, dv \, dr \, d\tau \right) \\
& \leq \frac{1}{2\beta^2} \sum_{j=1}^{J+1} \int_0^T \left[\text{ess. sup}_{r \in \Omega^{J+1}} |(\mathcal{L}r)_j + u(r_j, \tau)|^2 \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\alpha \, dv \, dr \right] d\tau \\
& = \frac{1}{2\beta^2} \sum_{j=1}^{J+1} \int_0^T [\text{ess. sup}_{r \in \Omega^{J+1}} |(\mathcal{L}r)_j + u(r_j, \tau)|^2] d\tau \\
& \leq C(J, T)(1 + \|u\|_{L^2(0, T; L^\infty(\Omega))}^2), \tag{2.47}
\end{aligned}$$

where, again, the dependence of the constant $C(J, T)$ on β has been suppressed. Substituting (2.47) into (2.46) we deduce that

$$\begin{aligned}
& \langle M \mathcal{F}(\widehat{\varrho}_\alpha(\cdot, \cdot, t) + \gamma), 1 \rangle + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha + \gamma} \, dv \, dr \, d\tau \\
& \quad + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{r_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha + \gamma} \, dv \, dr \, d\tau \\
& \leq R_1 + C(J, T)(1 + \|u\|_{L^2(0, T; L^\infty(\Omega))}^2). \tag{2.48}
\end{aligned}$$

Let us now focus on the term R_1 . As

$$\begin{aligned}
R_1 &= \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0 + \gamma) \, dv \, dr \\
&= \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) [\widehat{\varrho}_0 (\log(\widehat{\varrho}_0 + \gamma) - 1) + 1] \, dv \, dr \\
& \quad + \gamma \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) (\log(\widehat{\varrho}_0 + \gamma) - 1) \, dv \, dr,
\end{aligned}$$

the dominated convergence theorem implies that the second summand on the right-hand side converges to 0 as $\gamma \rightarrow 0_+$, while the first summand, again by the dominated convergence theorem, converges to

$$\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) [\widehat{\varrho}_0 (\log \widehat{\varrho}_0 - 1) + 1] \, dv \, dr = \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0) \, dv \, dr.$$

Returning with this information to (2.46) we can now pass to the limit $\gamma \rightarrow 0_+$ there using, in the first term on the left-hand side, Fatou's lemma, and in the second and third term on

the left-hand side the monotone convergence theorem. Hence,

$$\begin{aligned}
& \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_\alpha(t)) \, dv \, dr + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha} \, dv \, dr \, d\tau \\
& \quad + \alpha \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{r_j} \widehat{\varrho}_\alpha|^2}{\widehat{\varrho}_\alpha} \, dv \, dr \, d\tau \\
& \leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0) \, dv \, dr + C(\beta, J, T)(1 + \|u\|_{L^2(0,T;L^\infty(\Omega))}^2). \tag{2.49}
\end{aligned}$$

and therefore (2.49) is the desired bound on $\widehat{\varrho}_\alpha$ that is uniform in α .

We shall also require a bound on the time derivative of $\widehat{\varrho}_\alpha$ that is uniform in α , which we shall now derive, using (2.49). Thanks to (2.43), we have that

$$\begin{aligned}
& \left| \int_0^T \langle M \partial_\tau \widehat{\varrho}_\alpha(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle \, d\tau \right| \leq \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |\partial_{v_j} \widehat{\varrho}_\alpha| |\partial_{v_j} \varphi| \, dv \, dr \, d\tau \\
& \quad + \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |v_j| \widehat{\varrho}_\alpha |\partial_{r_j} \varphi| \, dv \, dr \, d\tau \\
& \quad + \alpha \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) |\partial_{r_j} \widehat{\varrho}_\alpha| |\partial_{r_j} \varphi| \, dv \, dr \, d\tau \\
& \quad + \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) (|\mathcal{L}r|_j + |u(r_j, \tau)|) \widehat{\varrho}_\alpha |\partial_{v_j} \varphi| \, dv \, dr \, d\tau \\
& =: Q_1 + Q_2 + Q_3 + Q_4 \quad \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \forall t \in (0, T].
\end{aligned}$$

Next, we shall bound each of the terms Q_1, \dots, Q_4 . Thanks to (2.42), (2.49) and Sobolev embedding,

$$\begin{aligned}
Q_1 & \leq \frac{2\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \sqrt{\widehat{\varrho}_\alpha} |\partial_{v_j} \sqrt{\widehat{\varrho}_\alpha}| |\partial_{v_j} \varphi| \, dv \, dr \, d\tau \\
& \leq \frac{2\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^T \|\widehat{\varrho}_\alpha\|_{L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \|\partial_{v_j} \sqrt{\widehat{\varrho}_\alpha}\|_{L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
& \quad \times \|\partial_{v_j} \varphi\|_{L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \, d\tau.
\end{aligned}$$

Then,

$$\begin{aligned}
Q_1 &\leq \frac{2\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^T \|\partial_{v_j} \sqrt{\widehat{\varrho}_\alpha}\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\quad \times \left(\sum_{j=1}^{J+1} \int_0^T \|\partial_{v_j} \varphi\|_{L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}} \\
&\leq C \|\varphi\|_{L^2(0,T;W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \\
&\quad \forall \varphi \in L^2(0,T;W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d+1,
\end{aligned}$$

where C is a positive constant, independent of $\alpha \in (0, 1]$. By an identical argument,

$$\begin{aligned}
Q_3 &\leq C \|\varphi\|_{L^2(0,T;W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \\
&\quad \forall \varphi \in L^2(0,T;W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d+1,
\end{aligned}$$

where C is a positive constant, independent of $\alpha \in (0, 1]$. Next,

$$Q_2 \leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \| |v_j| \widehat{\varrho}_\alpha \|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \|\partial_{r_j} \varphi\|_{L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} d\tau.$$

Now, by Cauchy's inequality and the inequality (2.17), we have

$$\begin{aligned}
\| |v_j| \widehat{\varrho}_\alpha \|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} &= \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) |v_j| \widehat{\varrho}_\alpha dr dv \\
&\leq \frac{1}{2} \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) (1 + |v_j|^2) \widehat{\varrho}_\alpha dr dv \\
&\leq \frac{1}{2} L^{(J+1)d} + 2\beta \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \frac{|v_j|^2}{4\beta} \widehat{\varrho}_\alpha dr dv \\
&\leq \frac{1}{2} L^{(J+1)d} + 2\beta \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) (e^{\frac{1}{4\beta}|v_j|^2} - 1) dr dv \\
&\quad + 2\beta \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_\alpha) dr dv,
\end{aligned}$$

and hence, by (2.49),

$$\| |v_j| \widehat{\varrho}_\alpha \|_{L^\infty(0,T;L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq \frac{1}{2} \|(1 + |v_j|^2) \widehat{\varrho}_\alpha\|_{L^\infty(0,T;L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq C,$$

where C is a positive constant, independent of α , which then implies that

$$\begin{aligned}
Q_2 &\leq C \|\varphi\|_{L^2(0,T;W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \\
&\quad \forall \varphi \in L^2(0,T;W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d+1,
\end{aligned}$$

where C is a positive constant, independent of $\alpha \in (0, 1]$.

It remains to bound Q_4 ; proceeding in the same way as in (2.47), we deduce that

$$\begin{aligned} Q_4 &\leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \|(|(\mathcal{L}r)_j| + |u(r_j, \tau)|) \widehat{\varrho}_\alpha\|_{L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \|\partial_{v_j} \varphi\|_{L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} d\tau \\ &\leq C(\epsilon, J, T)(1 + \|u\|_{L^2(0, T; L^\infty(\Omega))}) \left(\sum_{j=1}^{J+1} \int_0^T \|\partial_{v_j} \varphi\|_{L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 d\tau \right)^{\frac{1}{2}}. \end{aligned}$$

Thus we have shown that

$$\begin{aligned} Q_4 &\leq C \|\varphi\|_{L^2(0, T; W^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \\ \forall \varphi &\in L^2(0, T; W_*^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d + 1, \end{aligned}$$

where C is a positive constant, independent of $\alpha \in (0, 1]$.

By collecting the bounds on Q_1, \dots, Q_4 , we have that

$$\begin{aligned} \left| \int_0^T \langle M \partial_\tau \widehat{\varrho}_\alpha(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle d\tau \right| &\leq C \|\varphi\|_{L^2(0, T; W^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \\ \forall \varphi &\in L^2(0, T; W_*^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d + 1, \end{aligned}$$

where C is a positive constant, independent of $\alpha \in (0, 1]$, which then implies the following uniform bound on the time derivative of $\widehat{\varrho}_\alpha$:

$$\|M \partial_t \widehat{\varrho}_\alpha\|_{L^2(0, T; (W_*^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))')} \leq C, \quad s > (J+1)d + 1, \quad (2.50)$$

where C is a positive constant, independent of $\alpha \in (0, 1]$.

For future reference, we collect here the various uniform bounds we have derived on $\widehat{\varrho}_\alpha$, $\alpha \in (0, 1]$:

$$\mathcal{F}(\widehat{\varrho}_\alpha) \quad \text{is bounded in } L^\infty(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad (2.51a)$$

$$\sum_{j=1}^{J+1} |\partial_{v_j} \sqrt{\widehat{\varrho}_\alpha}|^2 \quad \text{is bounded in } L^1(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad (2.51b)$$

$$\alpha \sum_{j=1}^{J+1} |\partial_{r_j} \sqrt{\widehat{\varrho}_\alpha}|^2 \quad \text{is bounded in } L^1(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad (2.51c)$$

$$\begin{aligned} M \partial_t \widehat{\varrho}_\alpha &\quad \text{is bounded in } L^2(0, T; (W_*^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ s > (J+1)d + 1, &\quad (2.51d) \end{aligned}$$

$$\widehat{\varrho}_\alpha \geq 0 \quad \text{and} \quad \|\widehat{\varrho}_\alpha(t)\|_{L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} = \|\widehat{\varrho}_0\|_{L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}, \quad t \in [0, T], \quad (2.51e)$$

$$(1 + |v_j|^2) \widehat{\varrho}_\alpha \quad \text{is bounded in } L^\infty(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad j = 1, \dots, J+1, \quad (2.51f)$$

$$|(\mathcal{L}r)_j + u(r_j, \tau)| \widehat{\varrho}_\alpha \quad \text{is bounded in } L^2(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad j = 1, \dots, J+1. \quad (2.51g)$$

By writing $\widehat{\varrho} = (\sqrt{\widehat{\varrho}})^2$, it then also follows from (2.51b), (2.51c) and (2.51e) that

$$\nabla_v \widehat{\varrho}_\alpha \quad \text{is bounded in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad (2.51h)$$

$$\alpha^{\frac{1}{2}} \nabla_r \widehat{\varrho}_\alpha \quad \text{is bounded in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad (2.51i)$$

where $\nabla_v := (\partial_{v_1}^T, \dots, \partial_{v_{J+1}}^T)^T$ and $\nabla_r := (\partial_{r_1}^T, \dots, \partial_{r_{J+1}}^T)^T$ are $(J+1)d$ -component column vectors.

We proceed by considering the Maxwellian-weighted Orlicz space $L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, with Young's function $\Phi(r) = \mathcal{F}(1 + |r|)$ (cf. Kufner, John & Fučík [39], Sec. 3.18.2). This has a separable predual $E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, with Young's function $\Psi(r) = e^{|r|} - |r| - 1$; the (Banach) space $E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is defined as the closure, in the norm of the Orlicz space $L_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, of the set of all real-valued bounded measurable functions defined on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d}$. As there exists a constant K such that $\mathcal{F}(1+r) \leq K(1+\mathcal{F}(r))$ for all $r \geq 0$, it follows from (2.51a) that the sequence $(\mathcal{F}(1 + \widehat{\varrho}_\alpha))_{\alpha > 0}$ is bounded in $L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. Hence, $\widehat{\varrho}_\alpha$ is bounded in $L^\infty(0, T; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) = L^\infty(0, T; (E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))')$. By the Banach–Alaoglu theorem, there exists a subsequence (not indicated) of the sequence $(\widehat{\varrho}_\alpha)_{\alpha > 0}$ and a

$$\begin{aligned} \widehat{\varrho} &\in L^\infty(0, T; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \\ (\text{whereby also } \mathcal{F}(\widehat{\varrho}) &\in L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))) \end{aligned} \quad (2.52)$$

such that

$$\widehat{\varrho}_\alpha \rightharpoonup \widehat{\varrho} \quad \text{weakly* in } L^\infty(0, T; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) = L^\infty(0, T; (E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'). \quad (2.53)$$

As, by definition, $L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \subset E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, it follows in particular that

$$\widehat{\varrho}_\alpha \rightharpoonup \widehat{\varrho} \quad \text{weakly in } L^p(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \forall p \in [1, \infty). \quad (2.54)$$

The convergence results (2.41a)–(2.41d) and (2.51a)–(2.51i) now imply the existence of

$$\widehat{\varrho} \in L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad \widehat{\varrho} \geq 0,$$

with

$$\begin{aligned} \nabla_v \widehat{\varrho} &\in L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{and} \quad M \partial_t \widehat{\varrho} \in L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ & \hspace{20em} s > (J+1)d + 1, \end{aligned}$$

such that

$$\begin{aligned} \widehat{\varrho}_\alpha &\rightharpoonup \widehat{\varrho} && \text{weakly* in } L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \nabla_v \widehat{\varrho}_\alpha &\rightharpoonup \nabla_v \widehat{\varrho} && \text{weakly in } L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \alpha \nabla_r \widehat{\varrho}_\alpha &\rightarrow 0 && \text{strongly in } L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ M \partial_t \widehat{\varrho}_\alpha &\rightharpoonup M \partial_t \widehat{\varrho} && \text{weakly in } L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ & && s > (J+1)d + 1, \\ v_j \widehat{\varrho}_\alpha &\rightharpoonup v_j \widehat{\varrho} && \text{weakly in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ & && j = 1, \dots, J+1, \\ ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho}_\alpha &\rightharpoonup ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho} && \text{weakly in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ & && j = 1, \dots, J+1. \end{aligned}$$

Using these convergence results, passage to the limit $\alpha \rightarrow 0_+$ in (2.44) implies the existence of

$$\widehat{\varrho} \in L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad \widehat{\varrho} \geq 0,$$

with

$$\begin{aligned} \nabla_v \widehat{\varrho} \in L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \text{ and } M \partial_t \widehat{\varrho} \in L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ s > (J+1)d + 1, \end{aligned}$$

f satisfying the following weak form of the Fokker–Planck equation: for all $t \in (0, T]$,

$$\begin{aligned} \int_0^t \langle M \partial_\tau \widehat{\varrho}(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho} \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\ - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho} \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\ - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho} \cdot \partial_{v_j} \varphi dv dr d\tau \right) = 0 \\ \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ s > (J+1)d + 1. \end{aligned} \tag{2.55}$$

It remains to discuss the attainment of the initial condition by $\widehat{\varrho}$. To this end, we require the following lemma.

Lemma 2.2.1. *Let X and Y be Banach spaces.*

(a) *If the space X is reflexive and is continuously embedded in the space Y , then*

$$L^\infty(0, T; X) \cap \mathcal{C}_w([0, T]; Y) = \mathcal{C}_w([0, T]; X).$$

(b) *If X has separable predual E and Y has predual F such that F is continuously embedded in E , then*

$$L^\infty(0, T; X) \cap \mathcal{C}_{w*}([0, T]; Y) = \mathcal{C}_{w*}([0, T]; X).$$

Part (a) is due to Strauss [62] (cf. Lions & Magenes [42], Lemma 8.1, Ch. 3, Sec. 8.4); part (b) is proved analogously, *via* the sequential Banach–Alaoglu theorem.

We shall prove that $\widehat{\varrho} \in \mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. Let us first recall that, thanks to (2.52),

$$\begin{aligned} \widehat{\varrho} \in L^\infty(0, T; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \\ \text{(whereby also } \mathcal{F}(\widehat{\varrho}) \in L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \text{)}, \end{aligned}$$

and, also,

$$\widehat{\varrho} \in W^{1,2}(0, T; M^{-1}(W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))), \quad s > (J+1)d + 1.$$

We then apply Lemma 2.2.1(b) by taking:

- $X := L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, the Maxwellian weighted Orlicz space with Young's function

$$\Phi(r) = \mathcal{F}(1 + |r|)$$

(cf. Kufner, John & Fučík [39], Sec. 3.18.2) whose separable predual

$$E := E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$$

has Young's function

$$\Psi(r) = e^{|r|} - |r| - 1;$$

- and $Y := M^{-1}(W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'$ whose predual with respect to the duality pairing

$$\langle M \cdot, \cdot \rangle_{W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}, \quad s > (J+1)d + 1,$$

is

$$F := W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}), \quad s > (J+1)d + 1,$$

and noting that $\mathcal{C}_{w*}([0; T]; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \subset \mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. This last inclusion and that $F \hookrightarrow E$ are proved by adapting Def. 3.6.1. and Thm. 3.2.3 in Kufner, John & Fučík [39] to the measure $M(v) dv dr$ to show that $L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \hookrightarrow L_M^\Xi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ for any Young's function Ξ , and then adapting Theorem 3.17.7 *ibid.* to deduce that

$$F \hookrightarrow L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \hookrightarrow E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) = E.$$

The abstract framework in Temam [64], Ch. 3, Sec. 4 then implies that $\widehat{\varrho}$ satisfies $\widehat{\varrho}(\cdot, \cdot, 0) = \widehat{\varrho}_0(\cdot, \cdot)$ in the sense of $\mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$.

By taking $\varphi \equiv 1$ in (2.55), we have that

$$\langle M \widehat{\varrho}(\cdot, \cdot, t), 1 \rangle - \langle M \widehat{\varrho}(\cdot, \cdot, 0), 1 \rangle = \int_0^t \langle M \partial_\tau \widehat{\varrho}(\cdot, \cdot, \tau), 1 \rangle d\tau = 0.$$

Hence,

$$\langle M \widehat{\varrho}(\cdot, \cdot, t), 1 \rangle - \langle M \widehat{\varrho}_0(\cdot, \cdot), 1 \rangle = 0,$$

and this then gives

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}(r, v, t) dr dv = \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0(r, v) dr dv = 1 \quad \forall t \in (0, T],$$

which, together with $\widehat{\varrho} \geq 0$, implies that

$$\varrho = M \widehat{\varrho} \in L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$$

is a probability density function, as required.

Noting that the function \mathcal{F} is nonnegative and convex, for each fixed $\gamma \in (0, 1]$ the first term on the left-hand side of (2.48) is weakly lower-semicontinuous in $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, as $\alpha \rightarrow 0_+$ (cf. Theorem 3.20 in [18]). Similarly, since $\xi \in \mathbb{R} \mapsto |\xi|^2$, with $y \geq 0$, is a nonnegative convex function, we have weak lower semicontinuity of the second term on

the left-hand side of (2.48) (cf. Corollary 3.24 in [18]) for each $\gamma \in (0, 1]$. By passing to the limit $\alpha \rightarrow 0_+$ in (2.48), and then passing to the limit $\gamma \rightarrow 0_+$ using the dominated convergence theorem in the first term on the left-hand side and the monotone convergence theorem in the second term on the left-hand side, we deduce that $\widehat{\varrho}$ satisfies the following energy inequality:

$$\begin{aligned} & \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}(t)) \, dv \, dr + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}|^2}{\widehat{\varrho}} \, dv \, dr \, d\tau \\ & \leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0) \, dv \, dr + \frac{16}{\beta} (J+1)d L^2 T + \frac{1}{\beta} (J+1) \|u\|_{L^2(0,T;L^\infty(\Omega))}^2. \end{aligned} \quad (2.56)$$

It is important to note here that, although we had supposed that $\widehat{\varrho}_0 \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})$, the upper bound in (2.56) only depends on the $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\mathcal{F}(\widehat{\varrho}_0)$, the $L^2(0, T; L^\infty(\Omega)^d)$ norm of u , and the constants d, β, J, L, T , all of which are independent of ϵ .

2.3 Existence of solutions to the coupled Oseen–Fokker–Planck system

We now return to the full system stated in the Introduction, our objective being to prove the existence of large-data global weak solutions to the coupled Oseen–Fokker–Planck system. To this end, we formulate an iterative process, by defining the sequence of functions $(u^{(k)}, \widehat{\varrho}^{(k)})$, for $k = 1, 2, \dots$, as follows. We set $u^{(1)} \equiv 0$. Given a divergence-free $u^{(k)} \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$, for some $k \geq 1$ and $\sigma > d$, we define $\widehat{\varrho}^{(k)}$ as the weak solution (in a sense to be made precise below) of the Fokker–Planck equation:

$$\begin{aligned} & M \partial_t \widehat{\varrho}^{(k)} - \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (M \partial_{v_j} \widehat{\varrho}^{(k)}) \right) \\ & + \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} M v_j \cdot \partial_{r_j} \widehat{\varrho}^{(k)} + ((\mathcal{L}r)_j + u^{(k)}(r_j, t)) \cdot \partial_{v_j} (M \widehat{\varrho}^{(k)}) \right) = 0, \end{aligned} \quad (2.57)$$

$$\text{for all } (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T],$$

$$\widehat{\varrho}^{(k)}(r, v, 0) = \widehat{\varrho}_0^{(k)}(r, v) \quad \text{for all } (r, v) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d}, \quad (2.58)$$

subject to a (weakly imposed) specular boundary condition with respect to the independent variable r . The precise specification of the initial datum $\widehat{\varrho}_0^{(k)}$ in terms of $\widehat{\varrho}_0$ will be detailed in the next subsection. Having determined $\widehat{\varrho}^{(k)}$ from this problem, we shall find the next velocity field iterate $u^{(k+1)}$ by solving, with $\widehat{\varrho}^{(k)}$ fixed, the Oseen system (cf. (2.65) below). We shall prove that one can extract a subsequence from the sequence of iterates $((u^{(k)}, \widehat{\varrho}^{(k)}))_{k \geq 1}$, which converges to a solution $(u, \widehat{\varrho})$ of the coupled Oseen–Fokker–Planck system in the limit of $k \rightarrow \infty$.

2.4 Definition of the initial data

First, we define the sequence of initial data $(\widehat{\varrho}_0^{(k)})_{k \geq 1}$ appearing in (2.58). Given $\widehat{\varrho}_0$ as in (2.5), and letting

$$G_k(s) := \frac{s}{1 + k^{-\frac{1}{4}} \sqrt{s}}, \quad s \in [0, \infty),$$

we define

$$\widehat{\varrho}_0^{(k)} := G_k(\widehat{\varrho}_0), \quad k = 1, 2, \dots$$

The purpose of this construction, which can be seen as a renormalization of the initial datum $\widehat{\varrho}_0$, is to ensure that, under the original hypotheses, (2.5), on $\widehat{\varrho}_0$, the functions $\widehat{\varrho}_0^{(k)}$ thus defined possess the following properties:

$$\widehat{\varrho}_0^{(k)} \in L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}), \quad \text{for each fixed } k \geq 1, \quad (2.59a)$$

$$M\mathcal{F}(\widehat{\varrho}_0^{(k)}) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}), \quad \text{for each fixed } k \geq 1, \quad (2.59b)$$

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0^{(k)} \, dr \, dv \leq 1, \quad \text{for each fixed } k \geq 1, \quad (2.59c)$$

$$\widehat{\varrho}_0^{(k)} \in L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}) \quad \text{for each fixed } k \geq 1, \quad (2.59d)$$

and, possibly for a subsequence only (not indicated),

$$\widehat{\varrho}_0^{(k)} \rightarrow \widehat{\varrho}_0 \quad \text{strongly in } L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \text{ as } k \rightarrow \infty, \quad (2.59e)$$

$$\mathcal{F}(\widehat{\varrho}_0^{(k)}) \rightarrow \mathcal{F}(\widehat{\varrho}_0) \quad \text{strongly in } L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \text{ as } k \rightarrow \infty. \quad (2.59f)$$

We shall now proceed to show that these properties do indeed hold; having done so, we shall explain their relevance in the proof of our main result.

That $\widehat{\varrho}_0^{(k)} \geq 0$ for all $k \geq 1$ is a direct consequence of its definition and the assumed nonnegativity of $\widehat{\varrho}_0$ (cf. (2.5)). By (2.5), and noting that $0 \leq G_k(s) \leq s$, (2.59a) and (2.59c) directly follow. The assertion (2.59b) is also immediate by noting that $\mathcal{F}(\widehat{\varrho}_0^{(k)}) = \mathcal{F}(G_k(\widehat{\varrho}_0)) \leq \max\{1, \mathcal{F}(\widehat{\varrho}_0)\}$. We therefore proceed to prove the inclusion (2.59d).

We have, for each $k \geq 1$, that

$$\|\widehat{\varrho}_0^{(k)}\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 = \|G_k(\widehat{\varrho}_0)\|_{L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}^2 \leq k^{\frac{1}{2}} \|\widehat{\varrho}_0\|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}. \quad (2.60)$$

Thus we have verified (2.59d).

Next,

$$\begin{aligned} \|\widehat{\varrho}_0^{(k)} - \widehat{\varrho}_0\|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} &= \|G_k(\widehat{\varrho}_0) - \widehat{\varrho}_0\|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\ &\leq \left\| \frac{\widehat{\varrho}_0 k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \right\|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}. \end{aligned} \quad (2.61)$$

Clearly,

$$\frac{\widehat{\varrho}_0 \sqrt{\widehat{\varrho}_0}}{k^{\frac{1}{4}} + \sqrt{\widehat{\varrho}_0}} \rightarrow 0 \quad \text{a.e. on } \Omega^{J+1} \times \mathbb{R}^{(J+1)d}.$$

Also, trivially,

$$0 \leq \frac{\widehat{\varrho}_0 k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \leq \widehat{\varrho}_0 \in L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}).$$

Hence, by the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \left\| \frac{\widehat{\varrho}_0 k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \right\|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} = 0.$$

By passing to the limit $k \rightarrow \infty$ in (2.61) we then deduce that

$$\lim_{k \rightarrow \infty} \|\widehat{\varrho}_0^{(k)} - \widehat{\varrho}_0\|_{L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} = 0.$$

Thus we have shown (2.59e).

To prove (2.59f), thanks to (2.59e), it suffices to show that, as $k \rightarrow \infty$,

$$\frac{\widehat{\varrho}_0}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \log \left(\frac{\widehat{\varrho}_0}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \right) \rightarrow \widehat{\varrho}_0 \log \widehat{\varrho}_0 \quad \text{strongly in } L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}).$$

To this end we write

$$\frac{\widehat{\varrho}_0}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \log \left(\frac{\widehat{\varrho}_0}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \right) = \frac{\widehat{\varrho}_0 \log \widehat{\varrho}_0}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} - \widehat{\varrho}_0 \frac{\log \left(1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0} \right)}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}}. \quad (2.62)$$

We shall show below that the first fraction on the right-hand side of the equality (2.62) converges to $\widehat{\varrho}_0 \log \widehat{\varrho}_0$ strongly in $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ while the second fraction converges to 0 strongly in $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, and that will complete the proof of (2.59f). Indeed, that the second fraction on the right-hand side of (2.62) converges to 0 strongly in $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ follows directly from the dominated convergence theorem by noting that

$$\left| \frac{\log \left(1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0} \right)}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} \right| \leq \frac{1}{e} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{\log \left(1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0} \right)}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} = 0$$

a.e. on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d}$.

Focusing now on the first fraction on the right-hand side of (2.62), we consider

$$\frac{\widehat{\varrho}_0 \log \widehat{\varrho}_0}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}} - \widehat{\varrho}_0 \log \widehat{\varrho}_0 = -\widehat{\varrho}_0 \log \widehat{\varrho}_0 \frac{k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}}{1 + k^{-\frac{1}{4}} \sqrt{\widehat{\varrho}_0}}.$$

The term on the right-hand side of this equality converges to 0 strongly in $L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ as $k \rightarrow \infty$, thanks to the dominated convergence theorem. That completes the proof of (2.59f).

The significance of (2.59a)–(2.59f) is that these are precisely the properties which we used in the previous section to prove, for a fixed divergence-free velocity field u , contained in $L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$, $\sigma > d$, the existence of a solution $\widehat{\varrho}$ to the Fokker–Planck equation, subject to such initial data for $\widehat{\varrho}$.

2.5 Existence of a solution to the initial-value problem (2.57), (2.58)

Having verified all of (2.59a)–(2.59f), the arguments developed in Section 2 imply the existence of a weak solution $\widehat{\varrho}^{(k)}$ to the problem (2.57) for a given divergence-free $u^{(k)} \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$ with $\sigma > d$. More precisely, there exists a

$$\widehat{\varrho}^{(k)} \in \mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

with

$$\begin{aligned} \nabla_v \widehat{\varrho}^{(k)} &\in L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad M \partial_t \widehat{\varrho}^{(k)} \in L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ s &> (J+1)d + 1, \end{aligned}$$

and satisfying

$$\begin{aligned} v_j \widehat{\varrho}^{(k)} &\in L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad j = 1, \dots, J+1, \\ ((\mathcal{L}r)_j + u^{(k)}(r_j, \tau)) \widehat{\varrho}^{(k)} &\in L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad j = 1, \dots, J+1, \end{aligned}$$

such that, for all $t \in (0, T]$:

$$\begin{aligned} &\int_0^t \langle M \partial_\tau \widehat{\varrho}^{(k)}(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}^{(k)} \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\ &- \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}^{(k)} \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\ &- \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u^{(k)}(r_j, \tau)) \widehat{\varrho}^{(k)} \cdot \partial_{v_j} \varphi dv dr d\tau \right) = 0 \\ &\forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d + 1. \end{aligned} \tag{2.63}$$

Furthermore $\widehat{\varrho}^{(k)}(\cdot, \cdot, 0) = \widehat{\varrho}_0^{(k)}(\cdot, \cdot)$ in the sense of $\mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$, and

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}^{(k)}(r, v, t) dr dv = \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_0^{(k)}(r, v) dr dv \leq 1, \quad t \in (0, T].$$

In addition, thanks to (2.56), $\widehat{\varrho}^{(k)}$ satisfies the following energy inequality:

$$\begin{aligned} &\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}^{(k)}(t)) dv dr + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}^{(k)}|^2}{\widehat{\varrho}^{(k)}} dv dr d\tau \\ &\leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0^{(k)}) dv dr + \frac{16}{\beta} (J+1)d L^2 T + \frac{1}{\beta} (J+1) \|u^{(k)}\|_{L^2(0,T;L^\infty(\Omega))}^2. \end{aligned} \tag{2.64}$$

It is important to note here that the upper bound in (2.64) only involves the $L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\mathcal{F}(\widehat{\varrho}_0^{(k)})$, which, thanks to (2.59f), converges to $\mathcal{F}(\widehat{\varrho}_0)$ strongly in $L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, as $k \rightarrow \infty$; and the $L^2(0, T; L^\infty(\Omega)^d)$ norm of $u^{(k)}$, which we shall now bound by a constant, independent of k and of ϵ . Once we have done so, (2.64) will yield a uniform-in- k (and ϵ -uniform) bound on the $L^\infty(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ norm of $\mathcal{F}(\widehat{\varrho}^{(k)})$ and the $L^2(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ norm of $\nabla_v \sqrt{\widehat{\varrho}^{(k)}}$, which will, together with the strong convergence of $\widehat{\varrho}^{(k)}$ to $\widehat{\varrho}$ in $L^1(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, which we shall also prove, yield the convergence results required to pass to the limit in the weak form of (2.57) as $k \rightarrow \infty$.

2.6 Existence of a solution to the Oseen system

Having shown the existence of a solution $\widehat{\varrho}^{(k)}$ to (2.57), (2.58) for a given divergence-free $u^{(k)} \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$ with $\sigma > d$, we define $(u^{(k+1)}, \pi^{(k+1)})$, with $u^{(k+1)} \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; W_0^{1,2}(\Omega)^d)$, and $\pi^{(k+1)} \in \mathcal{D}'(0, T; L^2(\Omega)/\mathbb{R})$ as the weak solution of the unsteady Oseen system:

$$\begin{aligned} \partial_t u^{(k+1)} + (b \cdot \nabla) u^{(k+1)} - \mu \Delta u^{(k+1)} + \nabla \pi^{(k+1)} &= \nabla \cdot \mathbb{K}^{(k)} & \text{for } (x, t) \in \Omega \times (0, T], \\ \nabla \cdot u^{(k+1)} &= 0 & \text{for } (x, t) \in \Omega \times (0, T], \\ u^{(k+1)}(x, 0) &= u_0(x) & \text{for } x \in \Omega, \end{aligned} \tag{2.65}$$

where $u_0 \in W_0^{1-2/z, z}(\Omega)^d$, with $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, is divergence-free, and

$$\begin{aligned} \mathbb{K}^{(k)}(x, t) &:= \frac{\int_{D^J \times \mathbb{R}^{(J+1)d}} \sum_{j=1}^J (F(q_j) \otimes q_j) M \widehat{\varrho}^{(k)}(B(q, x), v, t) dq dv}{\int_{D^J \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}^{(k)}(B(q, x), v, t) dq dv}, \\ (x, t) &\in \Omega \times (0, T]. \end{aligned}$$

Thanks to (1.5),

$$\|\mathbb{K}^{(k)}\|_{L^\infty(0, T; L^\infty(\Omega))} \leq C, \tag{2.66}$$

where C is a positive constant, independent of k . Thus, there exist a $\mathbb{K} \in L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^{d \times d}_{\text{symm}}))$ (to be identified), and a subsequence, not indicated, such that

$$\mathbb{K}^{(k)} \rightarrow \mathbb{K} \quad \text{weak* in } L^\infty(0, T; L^\infty(\Omega; \mathbb{R}^{d \times d}_{\text{symm}})) \text{ as } k \rightarrow \infty. \tag{2.67}$$

As $W_0^{1-2/z, z}(\Omega)^d \hookrightarrow L^2(\Omega)^d$ for $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, by standard arguments from the analysis of the incompressible Navier–Stokes equations (cf., for example, [64], Chpt. III) we deduce from (2.66) that there exists a unique weak solution $(u^{(k+1)}, \pi^{(k+1)})$ to the Oseen system with $u^{(k+1)} \in L^\infty(0, T; L^2(\Omega)^d) \cap L^2(0, T; W_0^{1,2}(\Omega)^d)$, and

$$\|u^{(k+1)}\|_{L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))} \leq C(1 + \|u_0\|_{L^2(\Omega)}),$$

where C is independent of k . Hence, by interpolation,¹

$$\|u^{(k+1)}\|_{L^{\hat{\sigma}}(Q_T)} \leq C \quad \text{where} \quad \begin{cases} \hat{\sigma} = 4 & \text{when } d = 2, \\ \hat{\sigma} = \frac{10}{3} & \text{when } d = 3, \end{cases}$$

where $Q_T := \Omega \times (0, T)$. Therefore, also,

$$\|b \otimes u^{(k+1)}\|_{L^{\hat{\sigma}}(Q_T)} \leq C \quad \text{where} \quad \begin{cases} \hat{\sigma} = 4 & \text{when } d = 2, \\ \hat{\sigma} = \frac{10}{3} & \text{when } d = 3. \end{cases}$$

Remark 2.6.1. *We note here in passing that the regularity hypothesis $b \in L^\infty(0, T; L^\infty(\Omega)^d)$ that was used here to deduce the last inequality can be weakened to assuming instead that $b \in L^s(0, T; L^s(\Omega)^d)$ for some $s > 2d(d+2)/(2(d+2) - d^2)$, $d = 2, 3$. The latter weaker assumption on b results in $\|b \otimes u^{(k+1)}\|_{L^{\hat{\sigma}}(Q_T)} \leq C$ for some $\hat{\sigma} > d$, which then still suffices to draw the same conclusions to the ones below. Perhaps divergence-free parabolic Lipschitz truncation could be used to relax this further.*

Continuing with our stronger but simpler assumption that $b \in L^\infty(0, T; L^\infty(\Omega)^d)$, we have that

$$\|\mathbb{K}^{(k)} - b \otimes u^{(k+1)}\|_{L^{\hat{\sigma}}(Q_T)} \leq C \quad \text{where} \quad \begin{cases} \hat{\sigma} = 4 & \text{when } d = 2, \\ \hat{\sigma} = \frac{10}{3} & \text{when } d = 3. \end{cases}$$

Clearly, $\hat{\sigma} = 2 + \frac{4}{d}$, $d = 2, 3$.

We shall now show that the divergence-free function $u^{(k+1)}$ possesses additional regularity, in the sense that $u^{(k+1)} \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$, with $\sigma := \min(\hat{\sigma}, z)$; we note that this fixes the value of σ , and it is clear that $\sigma > d$, as is required by the arguments contained in the statement of the Fokker–Planck equation and in Section 2. To do so, we shall move the convective term in the Oseen equation to the right-hand side of the equation, resulting in an unsteady Stokes system with source term $\nabla \cdot (\mathbb{K}^{(k)} - b \otimes u^{(k+1)})$. This then enables us to apply the regularity result for the unsteady Stokes system stated in [37] (cf. pp. 3067–3069 therein, in particular), which guarantees the existence of a positive constant $C = C_\sigma$, independent of k , such that

$$\|u^{(k+1)}\|_{W_\sigma^{1,\frac{1}{2}}(Q_T)} \leq C \left(\|\mathbb{K}^{(k)} - b \otimes u^{(k+1)}\|_{L^\sigma(Q_T)} + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)} \right),$$

where $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$, with $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, and

$$W_\sigma^{1,\frac{1}{2}}(Q_T) := L^\sigma(0, T; W_0^{1,\sigma}(\Omega)^d) \cap W^{1/2,\sigma}(0, T; L^\sigma(\Omega)^d).$$

As $W_\sigma^{1,\frac{1}{2}}(Q_T) \hookrightarrow L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$, it follows that

$$\|u^{(k+1)}\|_{L^2(0,T;W_0^{1,\sigma}(\Omega))} \leq C(1 + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}), \quad (2.68)$$

where $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$, $d = 2, 3$, and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$.

¹By the Gagliardo–Nirenberg inequality, $\|v\|_{L^4(\Omega)} \leq C\|v\|_{L^2(\Omega)}^{1/2}\|v\|_{W^{1,2}(\Omega)}^{1/2}$ for $d = 2$, and $\|v\|_{L^{10/3}(\Omega)} \leq C\|v\|_{L^2(\Omega)}^{2/5}\|v\|_{W^{1,2}(\Omega)}^{3/5}$ for $n = 3$. Hence, by the application of Hölder’s inequality, $\|v\|_{L^4(0,T;L^4(\Omega))} \leq C\|v\|_{L^\infty(0,T;L^2(\Omega))}^{1/2}\|v\|_{L^2(0,T;W^{1,2}(\Omega))}^{1/2}$ for $d = 2$ and $\|v\|_{L^{10/3}(0,T;L^{10/3}(\Omega))} \leq C\|v\|_{L^\infty(0,T;L^2(\Omega))}^{2/5}\|v\|_{L^2(0,T;W^{1,2}(\Omega))}^{3/5}$ for $d = 3$.

2.7 Passage to the limit $k \rightarrow \infty$

We deduce from (2.68) and (2.65) that

$$\begin{aligned} u^{(k)} &\rightarrow u && \text{weakly in } L^2(0, T; W_0^{1,\sigma}(\Omega)^d) \text{ as } k \rightarrow \infty, && \sigma > d, \\ u^{(k)} &\rightarrow u && \text{weakly in } W^{1,2}(0, T; W^{-1,\sigma}(\Omega)^d) \text{ as } k \rightarrow \infty, && \sigma > d, \\ u^{(k)} &\rightarrow u && \text{strongly in } L^2(0, T; \mathcal{C}^{0,\gamma}(\bar{\Omega})^d) \text{ as } k \rightarrow \infty, && 0 < \gamma < 1 - \frac{d}{\sigma}, \quad \sigma > d, \end{aligned} \quad (2.69)$$

where the last result follows, via the Aubin–Lions lemma, thanks to the compact embedding of the Sobolev space $W_0^{1,\sigma}(\Omega)^d$ into the Hölder space $\mathcal{C}^{0,\gamma}(\bar{\Omega})^d$ for $0 < \gamma < 1 - \frac{d}{\sigma}$, $\sigma > d$. Using (2.67) and (2.69) it is now straightforward to pass to the limit in (2.65).

All that remains to be done is to identify the weak* limit \mathbb{K} of the sequence $(\mathbb{K}^{(k)})_{k \geq 0}$ in terms of the limit $\widehat{\varrho}$ of the sequence $(\widehat{\varrho}^{(k)})_{k \geq 0}$. As $\mathbb{K}^{(k)}$ has the form

$$\frac{\mathfrak{A}^{(k)}}{\mathfrak{B}^{(k)}}, \quad k = 0, 1, \dots,$$

the limit \mathbb{K} is anticipated to be of the form

$$\frac{\mathfrak{A}}{\mathfrak{B}},$$

where

$$\begin{aligned} \mathfrak{A}^{(k)} &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} \sum_{j=1}^J (F(q_j) \otimes q_j) M \widehat{\varrho}^{(k)}(B(q, x), v, t) \, dq \, dv, \\ \mathfrak{B}^{(k)} &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}^{(k)}(B(q, x), v, t) \, dq \, dv, \\ \mathfrak{A} &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} \sum_{j=1}^J (F(q_j) \otimes q_j) M \widehat{\varrho}(B(q, x), v, t) \, dq \, dv, \\ \mathfrak{B} &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}(B(q, x), v, t) \, dq \, dv. \end{aligned}$$

The identification of the limit \mathbb{K} proceeds as follows. First we need to prove strong convergence of the sequence $(\widehat{\varrho}^{(k)})_{k \geq 0}$. As we are now required to work under the original hypotheses on the initial condition, stated in (2.5), rather than the stronger assumption used for the parabolic regularization of the Fokker–Planck equation, we can no longer use our earlier argument. In other words, the only piece of information we are allowed to use at this point is the energy inequality (2.64), in conjunction with the bound on $(u^{(k)})_{k \geq 0}$ supplied by (2.68).

We therefore argue as follows. Since we have by now already passed to the limit $\alpha \rightarrow 0_+$, and have thereby removed the r -diffusion term from the Fokker–Planck equation, we can

rewrite (2.57) as

$$\begin{aligned}
M\partial_t\widehat{\varrho}^{(k)} - \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (M\partial_{v_j}\widehat{\varrho}^{(k)}) \right) + \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} Mv_j \cdot \partial_{r_j}\widehat{\varrho}^{(k)} \right) \\
= -\frac{1}{\epsilon} \sum_{j=1}^{J+1} \left(((\mathcal{L}r)_j + u^{(k)}(r_j, t)) \cdot \partial_{v_j}(M\widehat{\varrho}^{(k)}) \right), \\
\text{in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))
\end{aligned} \tag{2.70}$$

(i.e., in the sense of distributions on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)$), and we can exploit the fact that the differential operator appearing on the left-hand side of (2.70) is hypoelliptic. Thus we can replicate the argument appearing in the Appendix of the work of DiPerna & Lions [20], concerning strong L^1 compactness of a sequence of solutions to a hypoelliptic equation driven by a sequence of source terms that is equibounded in L^1 and has uniform decay as $|v| \rightarrow \infty$ in a sense to be made precise below. Having done so, we will deduce the strong convergence of the sequence $(\widehat{\varrho}^{(k)})_{k \geq 0}$ in the function space $L^1(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$; i.e.

$$\widehat{\varrho}^{(k)} \rightarrow \widehat{\varrho} \quad \text{strongly in } L^1(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{as } k \rightarrow \infty. \tag{2.71}$$

To this end, we will first show that the expression appearing on the right-hand side of (2.70) is bounded in the norm of $L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, uniformly with respect to k . Clearly, for any $j \in \{1, \dots, J+1\}$, and $k \geq 1$,

$$\begin{aligned}
& \|((\mathcal{L}r)_j + u^{(k)}) \cdot \partial_{v_j}(M\widehat{\varrho}^{(k)})\|_{L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \\
& \leq C \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)} \right) \|\partial_{v_j}(M\widehat{\varrho}^{(k)}(\cdot, \cdot, t))\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\
& \leq C \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)} \right) \|M|v_j|\widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\
& \quad + C \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)} \right) \|M\partial_{v_j}\widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt.
\end{aligned} \tag{2.72}$$

The first term on the right-hand side of (2.72) is bounded, using (2.17), (2.64), (2.59f), and (2.68), as follows:

$$\begin{aligned}
\|M|v_j|\widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} & \leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} (M(v)(e^{|v_j|} - 1) + M(v)\mathcal{F}(\widehat{\varrho}^{(k)}(t))) dv dr \\
& \leq C \left(1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\mathcal{F}(\widehat{\varrho}^{(k)}(t)) dv dr \right) \\
& \leq C,
\end{aligned}$$

where C is a positive constant, independent of k ; hence, noting (2.68),

$$\begin{aligned} & \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|M |v_j| \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\ & \leq C \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) dt \leq C, \end{aligned} \quad (2.73)$$

where C is a positive constant, independent of k .

The second term on the right-hand side of (2.72) is bounded as follows. First, using the Cauchy–Schwarz inequality with respect to r and v we have that

$$\begin{aligned} \|M \partial_{v_j} \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} &= \left\| M \partial_{v_j} \left(\sqrt{\widehat{\varrho}^{(k)}(\cdot, \cdot, t)} \right)^2 \right\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\ &\leq 2 \|M \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\ &\quad \times \left\| M \partial_{v_j} \sqrt{\widehat{\varrho}^{(k)}(\cdot, \cdot, t)} \right\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}. \end{aligned}$$

Hence, now using the Cauchy–Schwarz inequality with respect to t , we deduce that

$$\begin{aligned} & \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|M \partial_{v_j} \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\ & \leq 2 \|M \widehat{\varrho}^{(k)}\|_{L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \\ & \quad \times \left\| M \partial_{v_j} \sqrt{\widehat{\varrho}^{(k)}(\cdot, \cdot, t)} \right\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\ & \leq 2 \|M \widehat{\varrho}^{(k)}\|_{L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \|1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\|_{L^2(0, T)} \\ & \quad \times \left\| M \partial_{v_j} \sqrt{\widehat{\varrho}^{(k)}} \right\|_{L^2(0, T; L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))}. \end{aligned}$$

Thus, by noting the uniform bounds (2.64), (2.59f), and (2.68), we have that

$$\int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|M \partial_{v_j} \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \leq C, \quad (2.74)$$

where C is a positive constant, independent of k .

Using (2.73) and (2.74) in (2.72), we then deduce that the expression on the right-hand side of (2.70) is bounded in $L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, uniformly with respect to k .

Next, we show that the sequence of functions appearing on the right-hand side of (2.70) has the following additional (‘equiboundedness’) property: for each $j \in \{1, \dots, J+1\}$,

$$\lim_{R \rightarrow \infty} \sup_{k \geq 1} \|\chi_{|v| \geq R}(\cdot) ((\mathcal{L}r)_j + u^{(k)}) \cdot \partial_{v_j} (M \widehat{\varrho}^{(k)})\|_{L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} = 0, \quad (2.75)$$

where $\chi_{|v| \geq R}$ is the characteristic function of the set of all $v \in \mathbb{R}^{(J+1)d}$ such that $|v| \geq R$, with $|\cdot|$ signifying the Euclidean norm on $\mathbb{R}^{(J+1)d}$. Similarly as in (2.72), we have that

$$\begin{aligned}
& \|\chi_{|v| \geq R}(\cdot) ((\mathcal{L}r)_j + u^{(k)}) \cdot \partial_{v_j} (M \widehat{\varrho}^{(k)})\|_{L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \\
& \leq C \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|\chi_{|v| \geq R}(\cdot) \partial_{v_j} (M \widehat{\varrho}^{(k)}(\cdot, \cdot, t))\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\
& \leq C \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|\chi_{|v| \geq R}(\cdot) M |v_j| \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\
& \quad + C \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|\chi_{|v| \geq R}(\cdot) M \partial_{v_j} \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt.
\end{aligned} \tag{2.76}$$

The first term on the right-hand side of (2.76) is bounded as follows. We first note that, for $|v| \geq R > 0$, by (2.17),

$$\begin{aligned}
0 \leq M(v) |v_j| \widehat{\varrho}^{(k)} & \leq M(v) |v| \widehat{\varrho}^{(k)} \leq \frac{4\beta}{R} M(v) \frac{|v|^2}{4\beta} \widehat{\varrho}^{(k)} \\
& \leq \frac{4\beta}{R} \left(M(v) (e^{\frac{|v|^2}{4\beta}} - 1) + M(v) \mathcal{F}(\widehat{\varrho}^{(k)}) \right).
\end{aligned}$$

Therefore, using (2.64), (2.59f), and (2.68), we have that

$$\begin{aligned}
& \|\chi_{|v| \geq R}(\cdot) M |v_j| \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
& \leq \frac{4\beta}{R} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} (M(v) (e^{\frac{|v|^2}{4\beta}} - 1) + M(v) \mathcal{F}(\widehat{\varrho}^{(k)}(t))) dv dr \\
& \leq \frac{C}{R} \left(1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}^{(k)}(t)) dv dr \right) \\
& \leq \frac{C}{R},
\end{aligned}$$

where C is a positive constant, independent of k ; hence, noting (2.68) again,

$$\int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|\chi_{|v| \geq R}(\cdot) M |v_j| \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \leq \frac{C}{R}, \tag{2.77}$$

where C is a positive constant, independent of k .

The second term on the right-hand side of (2.76) is bounded as follows. First, using the Cauchy–Schwarz inequality with respect to r and v we have that

$$\begin{aligned}
& \|\chi_{|v| \geq R}(\cdot) M \partial_{v_j} \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
& = \left\| \chi_{|v| \geq R}(\cdot) M \partial_{v_j} \left(\sqrt{\widehat{\varrho}^{(k)}(\cdot, \cdot, t)} \right)^2 \right\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \\
& \leq 2 \|\chi_{|v| \geq R}(\cdot) M \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} \left\| M \partial_{v_j} \sqrt{\widehat{\varrho}^{(k)}(\cdot, \cdot, t)} \right\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})}.
\end{aligned}$$

Hence, now using the Cauchy–Schwarz inequality with respect to t , we deduce that

$$\begin{aligned}
& \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|\chi_{|v|\geq R}(\cdot) M \partial_{v_j} \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\
& \leq 2 \|\chi_{|v|\geq R}(\cdot) M \widehat{\varrho}^{(k)}\|_{L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \\
& \quad \times \left\| M \partial_{v_j} \sqrt{\widehat{\varrho}^{(k)}(\cdot, \cdot, t)} \right\|_{L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \\
& \leq 2 \|\chi_{|v|\geq R}(\cdot) M \widehat{\varrho}^{(k)}\|_{L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \|1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\|_{L^2(0, T)} \\
& \quad \times \left\| M \partial_{v_j} \sqrt{\widehat{\varrho}^{(k)}} \right\|_{L^2(0, T; L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))}.
\end{aligned}$$

However, for $|v| \geq R > 0$, by (2.17),

$$0 \leq M(v) \widehat{\varrho}^{(k)} \leq \frac{4\beta}{R^2} M(v) \frac{|v|^2}{4\beta} \widehat{\varrho}^{(k)} \leq \frac{4\beta}{R^2} \left(M(v) \left(e^{\frac{|v|^2}{4\beta}} - 1 \right) + M(v) \mathcal{F}(\widehat{\varrho}^{(k)}) \right),$$

and therefore, by noting the uniform bounds (2.64), (2.59f), and (2.68), we have that

$$\|\chi_{|v|\geq R}(\cdot) M \widehat{\varrho}^{(k)}\|_{L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq \frac{C}{R^2},$$

where C is a positive constant, independent of k . Thus, by noting the uniform bound (2.68), we have that

$$\int_0^T \left(1 + \|u^{(k)}(\cdot, t)\|_{L^\infty(\Omega)}\right) \|\chi_{|v|\geq R}(\cdot) M \partial_{v_j} \widehat{\varrho}^{(k)}(\cdot, \cdot, t)\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} dt \leq \frac{C}{R^2}, \quad (2.78)$$

where C is a positive constant, independent of k . Hence, using (2.77) and (2.78) in (2.76), we obtain

$$\|\chi_{|v|\geq R}(\cdot) ((\mathcal{L}r)_j + u^{(k)}) \cdot \partial_{v_j} (M \widehat{\varrho}^{(k)})\|_{L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))} \leq \frac{C}{R},$$

where C is a positive constant, independent of k , and therefore (2.75) directly follows.

Furthermore, we note that, similarly to the argument preceding (2.77), for $|v| \geq R > 0$, by (2.17), we have that

$$0 \leq M(v) \widehat{\varrho}_0^{(k)} \leq \frac{4\beta}{R^2} M(v) \frac{|v|^2}{4\beta} \widehat{\varrho}_0^{(k)} \leq \frac{4\beta}{R^2} \left(M(v) \left(e^{\frac{|v|^2}{4\beta}} - 1 \right) + M(v) \mathcal{F}(\widehat{\varrho}_0^{(k)}) \right).$$

Therefore, using (2.64), (2.59f), and (2.68), we have that

$$\begin{aligned}
\|\chi_{|v|\geq R}(\cdot) M \widehat{\varrho}_0^{(k)}\|_{L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})} & \leq \frac{4\beta}{R^2} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left(M(v) \left(e^{\frac{|v|^2}{4\beta}} - 1 \right) + M(v) \mathcal{F}(\widehat{\varrho}_0^{(k)}) \right) dv dr \\
& \leq \frac{C}{R^2} \left(1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0^{(k)}) dv dr \right) \\
& \leq \frac{C}{R^2}, \tag{2.79}
\end{aligned}$$

where C is a positive constant, independent of k .

To summarize, we have shown that the sequence on the right-hand side of (2.70) is bounded in the norm of $L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, uniformly with respect to k . We have also shown that (2.75) and (2.79) hold. Having done so, we have verified the conditions stated under (A.4) and (A.5) in the Appendix of DiPerna & Lions [20]. The properties listed under (A.1)–(A.3) in [20] follow from properties of the fundamental solution of the hypoelliptic operator on the left-hand side of (2.70), and can be verified by recalling the explicit expression for the fundamental solution (see, for example, Section II.1 in [12]). Having checked each of (A.1)–(A.5) in [20], an identical argument to the one in the Appendix of [20] yields the strong convergence of $(M\widehat{\varrho}^{(k)})_{k \geq 0}$ to $M\widehat{\varrho}$ in the norm of $L^1(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, as stated in (2.71), and hence, thanks to the boundedness of this sequence in $L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ (which follows from (2.64), (2.59f) and (2.69)₃), strong convergence of $(M\widehat{\varrho}^{(k)})_{k \geq 0}$ to $M\widehat{\varrho}$ in $L^p(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ also follows, for all $p \in [1, \infty)$; equivalently, $(\widehat{\varrho}^{(k)})_{k \geq 0}$ converges to $\widehat{\varrho}$ in $L^p(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ for all $p \in [1, \infty)$.

We are now ready for the identification of the weak* limit \mathbb{K} of the sequence $(\mathbb{K}^{(k)})_{k \geq 0}$ in terms of $\widehat{\varrho}$. The argument consists of the following six steps.

- (i) The strong convergence (2.71) of the sequence $(\widehat{\varrho}^{(k)})_{k \geq 0}$ in $L^1(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ implies a.e. convergence of (a subsequence, not indicated, of) $\mathfrak{A}^{(k)}$ to \mathfrak{A} on $\Omega \times (0, T)$. Let us show that this is indeed the case: since the Jacobian $|\det B|$ is constant and $F \in L^\infty(D^J; \mathbb{R}^d)$, it follows from (2.71) by performing the change of variables $r = B(q, x)$ that, for any $j \in \{1, \dots, J\}$, also

$$\int_0^T \int_\Omega \int_{D^J \times \mathbb{R}^{(J+1)d}} |F(q_j) \otimes q_j| |\widehat{\varrho}^{(k)}(B(q, x), v, t) - \widehat{\varrho}(B(q, x), v, t)| M(v) dq dv dx dt \rightarrow 0.$$

This then implies that there exists a subsequence, not indicated, such that

$$\int_{D^J \times \mathbb{R}^{(J+1)d}} |F(q_j) \otimes q_j| |\widehat{\varrho}^{(k)}(B(q, x), v, t) - \widehat{\varrho}(B(q, x), v, t)| M(v) dq dv \rightarrow 0$$

for a.e. $(x, t) \in \Omega \times (0, T)$. Indeed, by defining, for each $j \in \{1, \dots, J\}$,

$$\delta_{kj}(x, t) := \int_{D^J \times \mathbb{R}^{(J+1)d}} |F(q_j) \otimes q_j| |\widehat{\varrho}^{(k)}(B(q, x), v, t) - \widehat{\varrho}(B(q, x), v, t)| M(v) dq dv,$$

Tonelli's theorem yields that $\delta_{kj} \in L^1(\Omega \times (0, T); \mathbb{R}_{\geq 0})$ for all $k \geq 1$. As,

$$\|\delta_{kj}\|_{L^1(\Omega \times (0, T))} = \int_0^T \int_\Omega \delta_{kj}(q, t) dq dt \rightarrow 0,$$

there exists a subsequence of $(\delta_{kj})_{k \geq 1}$, not indicated, such that $\delta_{kj}(x, t) \rightarrow 0_+$ for a.e. $(x, t) \in \Omega \times (0, T)$, for each $j \in \{1, \dots, J\}$,

- (ii) Analogously, $\mathfrak{B}^{(k)}$ converges to \mathfrak{B} a.e. on $\Omega \times (0, T)$.

- (iii) Now (i) and (ii) imply that $\mathfrak{A}^{(k)}/\mathfrak{B}^{(k)}$ converges to $\mathfrak{A}/\mathfrak{B}$ a.e. on $\Omega \times (0, T)$.
- (iv) Since $|\mathfrak{A}^{(k)}/\mathfrak{B}^{(k)}| \leq C$, where C is a positive constant, independent of k , the dominated convergence theorem yields that $\int_E \mathfrak{A}^{(k)}/\mathfrak{B}^{(k)} dx dt$ converges to $\int_E \mathfrak{A}/\mathfrak{B} dx dt$ for every measurable set $E \subset \Omega \times (0, T)$.
- (v) Next, (iv) together with the fact that $(\mathfrak{A}^{(k)}/\mathfrak{B}^{(k)})_{k \geq 0}$ is bounded in $L^\infty(\Omega \times (0, T))$, implies weak* convergence in $L^\infty(\Omega \times (0, T))$ of $\mathfrak{A}^{(k)}/\mathfrak{B}^{(k)}$ to $\mathfrak{A}/\mathfrak{B}$ thanks to Corollary 2.49 in [26].
- (vi) However, (2.67) states that $\mathfrak{A}^{(k)}/\mathfrak{B}^{(k)}$ converges weakly* to \mathbb{K} , in $L^\infty(\Omega \times (0, T))$. Therefore, by uniqueness of the weak* limit, $\mathbb{K} = \mathfrak{A}/\mathfrak{B}$.

Thus we have shown that

$$\mathbb{K} = \frac{\mathfrak{A}}{\mathfrak{B}} = \frac{\int_{D^J \times \mathbb{R}^{(J+1)d}} \sum_{j=1}^J (F(q_j) \otimes q_j) M \widehat{\varrho}(B(q, x), v, t) dq dv}{\int_{D^J \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}(B(q, x), v, t) dq dv}.$$

Finally, we can pass to the limit $k \rightarrow \infty$ in the sequence of Fokker–Planck equations (2.63). As this part of the proof is very similar to the passage to the limit $\alpha \rightarrow 0_+$ expounded in the previous section, we confine ourselves to summarizing the main points.

The strong convergence result (2.71) and the energy inequality (2.64) imply the existence of

$$\widehat{\varrho} \in L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

with

$$\begin{aligned} \nabla_v \sqrt{\widehat{\varrho}} &\in L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \nabla_v \widehat{\varrho} &\in L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{and} \quad M \partial_t \widehat{\varrho} \in L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ & \quad s > (J+1)d + 1, \end{aligned}$$

such that, as $k \rightarrow \infty$,

$$\begin{aligned} \widehat{\varrho}^{(k)} &\rightarrow \widehat{\varrho} && \begin{cases} \text{weakly* in } L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \text{strongly in } L^p(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \\ \text{for all } p \in [1, \infty), \end{cases} \\ \nabla_v \widehat{\varrho}^{(k)} &\rightharpoonup \nabla_v \widehat{\varrho} && \text{weakly in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ M \partial_t \widehat{\varrho}^{(k)} &\rightharpoonup M \partial_t \widehat{\varrho} && \text{weakly in } L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \\ & && s > (J+1)d + 1, \\ v_j \widehat{\varrho}^{(k)} &\rightharpoonup v_j \widehat{\varrho} && \text{weakly in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ & && j = 1, \dots, J+1, \\ ((\mathcal{L}r)_j + u^{(k)}(r_j, \tau)) \widehat{\varrho}^{(k)} &\rightharpoonup ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho} && \text{weakly in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ & && j = 1, \dots, J+1. \end{aligned}$$

Using these convergence results, passage to the limit $k \rightarrow \infty$ in (2.63) implies the existence of

$$\widehat{\varrho} \in L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

with

$$\begin{aligned} \nabla_v \sqrt{\widehat{\varrho}} &\in L^2(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ \nabla_v \widehat{\varrho} &\in L^2(0, T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{and} \quad M \partial_t \widehat{\varrho} \in L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))), \\ s &> (J+1)d + 1, \end{aligned}$$

satisfying the following weak form of the Fokker–Planck equation:

$$\begin{aligned} &\int_0^t \langle M \partial_\tau \widehat{\varrho}(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho} \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\ &- \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho} \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\ &- \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u(r_j, \tau)) \widehat{\varrho} \cdot \partial_{v_j} \varphi dv dr d\tau \right) = 0 \\ &\forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ &s > (J+1)d + 1, \quad \forall t \in (0, T]. \end{aligned} \tag{2.80}$$

Furthermore $\widehat{\varrho}(\cdot, \cdot, 0) = \widehat{\varrho}_0(\cdot, \cdot)$ in the sense of $\mathcal{C}_w([0, T]; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$, and

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}(r, v, t) dr dv = \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}_0(r, v) dr dv = 1.$$

In addition, $\widehat{\varrho}$ satisfies the following energy inequality:

$$\begin{aligned} &\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}(t)) dv dr + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}|^2}{\widehat{\varrho}} dv dr d\tau \\ &\leq C \left[1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0) dv dr \right], \end{aligned}$$

where $C = C(\|u_0\|_{W^{1-\frac{2}{\sigma}, \sigma}(\Omega)}, \|b\|_{L^\infty(0, T; L^\infty(\Omega))})$, $\sigma := \min(\widehat{\sigma}, z) > d$, with $\widehat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$. In particular, C is independent of $\epsilon > 0$.

This then completes the proof of the existence of large-data global weak solutions to the coupled Oseen–Fokker–Planck system under consideration, for all $\epsilon > 0$.

Chapter 3

Trace theorems for the solution of the Fokker–Planck equation

In this section, by using similar arguments as in [50], we prove that the solution to the Fokker–Planck equation has a unique trace on the boundary of our domain, which is defined thanks to a Green’s formula. We then use this result to prove that the specular boundary condition is attained in a strong sense by the solution. To this end, given the vector

$$E_j = E_j(r, v, t) := \frac{1}{\epsilon}((\mathcal{L}r)_j + u(r_j, t)) - \frac{\beta^2}{\epsilon^2}v_j$$

and a weak solution $\varrho = \varrho(r, v, t)$ of the Fokker–Planck equation

$$\begin{aligned} \Lambda_{E_j}(\varrho) := \partial_t \varrho + \sum_{j=1}^{J+1} E_j(r, v, t) \cdot \partial_{v_j} \varrho + \sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, t)) \varrho - \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \partial_{v_j}^2 \varrho \\ + \frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho = 0, \end{aligned} \quad (3.1)$$

$$\text{for all } (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T],$$

$$\varrho(r, v, 0) = \varrho_0(r, v) \quad \text{for all } (r, v) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d}, \quad (3.2)$$

satisfying the specular boundary condition in a weak sense, we show that ϱ has a trace $\gamma\varrho$ on the boundary $\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T)$, $j = 1, \dots, J+1$, and a trace $\gamma_t\varrho = \varrho(\cdot, t)$ on the section $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times \{t\}$ for all $t \in [0, T]$. These trace functions will be shown to be well-defined thanks to a Green’s formula, which we shall now discuss.

In the previous section we showed that $\varrho \in L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$ is a solution to (3.1) in the sense of distributions, i.e.,

$$\int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \varrho \Lambda_{E_j}^*(\varphi) \, dv \, dr \, d\tau = 0, \quad (3.3)$$

for all test functions $\varphi \in \mathcal{D}(\overline{D}) := \mathcal{C}_0^\infty(\overline{\Omega^{J+1}} \times \mathbb{R}^{(J+1)d} \times [0, T])$, where we have set:

$$\begin{aligned} \Lambda_{E_j}^*(\varrho) = & \partial_t \varphi + \sum_{j=1}^{J+1} E_j(r, v, t) \cdot \partial_{v_j} \varphi + \sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, t)) \varphi + \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \partial_{v_j}^2 \varphi \\ & + \frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varphi. \end{aligned} \quad (3.4)$$

From the previous section we know that $u \in L^2(0, T; W^{1, \sigma}(\Omega)^d)$, with $\sigma > d$. Since, by Morrey's inequality, $W^{1, \sigma}(\Omega) \hookrightarrow L^\infty(\Omega)$, we have in particular that $u \in L^1(0, T; L^\infty(\Omega)^d)$. We thereby deduce that $E_j \in L^1(0, T; L^\infty_{\text{loc}}(\Omega; W^{1, \infty}_{\text{loc}}(\mathbb{R}^d)))^d$ and $\partial_{v_j} \cdot E_j \in L^\infty(0, T; L^\infty_{\text{loc}}(\Omega \times \mathbb{R}^d))$ for all $j = 1, \dots, J+1$.

We shall suppose henceforth that the initial datum ϱ_0 for the Fokker–Planck equation has the following factorized form: $\varrho_0(r, v) = M(v) \widehat{\varrho}_0(r)$, where $\widehat{\varrho}_0$ is a nonnegative function of r only, such that $\int_{\Omega^{J+1}} \widehat{\varrho}_0(r) \, dr = 1$, and

$$\widehat{\varrho}_0 \in L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}).$$

Under this hypothesis it directly follows that

$$\widehat{\varrho} \in L^\infty(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

and

$$\varrho \in L^\infty(0, T; L^2_{M^{-1}}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

and consequently, since $M^{-1}(v) \geq (2\pi\beta)^{\frac{1}{2}(J+1)}$ for all $v \in \mathbb{R}^{(J+1)d}$, that

$$\varrho \in L^\infty(0, T; L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})).$$

Remark 3.0.1. *To show that $\widehat{\varrho}_0 \in L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})$ implies $\widehat{\varrho} \in L^\infty(0, T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$, one has to follow a similar line of argument as in Section 2, Subsection 2.2. Indeed, it suffices to test equation (2.44) with the function*

$$\widehat{\varrho}_\alpha \in L^2(0, T; W_{*, M}^{1, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$

rather than

$$\mathcal{F}'(\widehat{\varrho}_\alpha + \gamma) = \log(\widehat{\varrho}_\alpha + \gamma) \in L^2(0, T; W_{*, M}^{1, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$

where $\widehat{\varrho}_\alpha \geq 0$ and $\gamma > 0$, and pass to the limit $\alpha \rightarrow 0_+$ in the equation satisfied by $\widehat{\varrho}_\alpha$ using the bounds resulting from the corresponding energy estimate.

3.1 Statement of the trace theorem

Theorem 3.1.1. *Let $\varrho \in L^\infty(0, T; L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$ be a solution of equation (3.3). Then, for every $t \in [0, T]$, there exists a $\gamma_t \varrho \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ and a $\gamma \varrho$ defined on $\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T)$ for $j = 1, \dots, J+1$, such that:*

$$\gamma_t \varrho \in C([0, T]; L^1_{\text{loc}}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$$

and

$$\gamma \varrho \in L^1_{\text{loc}}(\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times [0, T], (v_j \cdot n(r_j))^2 \, dv \, ds(r) \, d\tau),$$

for $j = 1, \dots, J+1$, and which satisfy the Green's formula

$$\begin{aligned} & \int_{t_0}^{t_1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \varrho \Lambda_{E_j}^*(\varphi) \, dv \, dr \, d\tau \\ &= \left[\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \varrho(\cdot, \tau) \varphi \, dv \, dr \right]_{t_0}^{t_1} + \sum_{j=1}^{J+1} \int_{t_0}^{t_1} \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} (v_j \cdot n(r_j)) \gamma \varrho \varphi \, dv \, ds(r) \, d\tau \end{aligned} \quad (3.5)$$

for all $t_0, t_1 \in [0, T]$ and for all test functions $\varphi \in \mathcal{D}_0(\overline{D})$, the space of functions $\varphi \in \mathcal{D}(\overline{D})$ such that $\varphi = 0$ on $\Sigma_0 \times (0, T)$, where $\Sigma_0 := \bigcup_{j=1}^{J+1} \left\{ (r, v) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d}, v_j \cdot n(r_j) = 0 \right\}$ and we have used the notation $\mathcal{D}(\overline{D}) := \mathcal{C}_0^\infty(\overline{\Omega}^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T])$.

Let us first introduce some additional notation. Since $\partial\Omega$ is \mathcal{C}^2 , Ω is locally on one side of $\partial\Omega$ and there exists a function $d = d_\Omega \in W^{2,\infty}(\mathbb{R}^d)$ such that for all z in an interior neighbourhood of $\partial\Omega$ one has

$$d(z) = -\text{dist}(z, \partial\Omega).$$

We define in $\overline{\Omega}$ the gradient field

$$n(z) = \nabla_z d(z),$$

which coincides with the unit outward normal vector to Ω at every point of $\partial\Omega$. Hence, the unit outward normal (column-)vector to $\partial\Omega_j$ at $r_j \in \partial\Omega_j$, for $j = 1, \dots, J+1$, is

$$n(r_j) = \nabla_{r_j} d(r_j) = \partial_{r_j} d(r_j).$$

Here, the set Ω_j still denotes Ω ; by assigning it the index j , however, we wish to emphasize by our notation that in the consideration of the distance to the boundary of Ω , the distance of the coordinate $r_j \in \Omega_j$ is measured to the boundary $\partial\Omega_j$ of the set that contains it.

We consider

$$d\mu_i = |n(r_j) \cdot v_j|^i \, dv \, ds(r) \, dt, \quad i = 1, 2,$$

which are measures defined on $\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T)$. For a given real $R > 0$, we define the sets

$$B_R = \{y \in \mathbb{R}^d : |y| < R\}, \quad \mathcal{O} := \Omega^{J+1} \times \mathbb{R}^{(J+1)d}, \quad \mathcal{D} := \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T),$$

$$\mathcal{O}_R := (\Omega \cap B_R)^{J+1} \times B_R^{J+1}, \quad \mathcal{D}_R := \mathcal{O}_R \times (0, T).$$

We shall also use the abbreviation $L_R^{a,b}$ for the function space $L^a(0, T; L^b(\mathcal{O}_R))$, and $L_{\text{loc}}^{a,b}$ for the function space $L^a(0, T; L_{\text{loc}}^b(\mathcal{O}))$.

Proof of Theorem 3.1.1. The proof of the theorem will be performed in three steps. First, we obtain two a priori estimates assuming that the solution of equation (3.1) is smooth. Then, following the method proposed by DiPerna & Lions in [21], we approach the weak solution ϱ of equation (3.1) by a sequence of regular functions $(\varrho_k)_{k \geq 1}$, which are solutions

of equation (3.1) with an error term r_k that vanishes at infinity; these regular functions satisfy the two a priori estimates from the first step. Finally, we deduce the existence of a trace by passing to the limit.

STEP 1: A PRIORI ESTIMATES. In this step, we derive two a priori estimates. We first assume that

$$\varrho \in W_{\text{loc}}^{1,1}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$$

so that the following manipulations are admissible. We consider three functions that we shall specify later: $\psi \in C^1(\mathbb{R})$ nondecreasing with $\psi(0) = 0$, $\Phi = \Phi(r, v, t) \in \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d} \times \mathbb{R}^{(J+1)d} \times [0, T])$ and $\tilde{\beta} \in C^1(\mathbb{R})$, and we fix $t_0, t_1 \in [0, T]$. Below, we shall write ψ for $\psi(v_j \cdot n(r_j))$. We use Green's formula together with equation (3.1) to get

$$\begin{aligned} & \left[\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \tilde{\beta}(\varrho) \psi \Phi \, dv \, dr \right]_{t_0}^{t_1} + \sum_{j=1}^{J+1} \int_{t_0}^{t_1} \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} (v_j \cdot n(r_j)) \tilde{\beta}(\varrho) \psi \Phi \, dv \, ds(r) \, d\tau \\ &= \int_{t_0}^{t_1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \Lambda_{E_j}(\tilde{\beta}(\varrho(r, v, \tau))) \Phi(r, v, \tau) \psi(v_j \cdot n(r_j)) \, dv \, dr \, d\tau \\ &= \int_{t_0}^{t_1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left\{ \Phi \psi \Lambda_{E_j}(\tilde{\beta}(\varrho(r, v, \tau))) + \tilde{\beta}(\varrho) \Phi \Lambda_{E_j}(\psi(v_j \cdot n(r_j))) \right. \\ &\quad \left. + \tilde{\beta}(\varrho) \psi \Lambda_{E_j} \Phi(r, v, \tau) \right\} \, dv \, dr \, d\tau \\ &= \int_{t_0}^{t_1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left\{ \Phi \psi \tilde{\beta}'(\varrho) \Lambda_{E_j} \varrho + \tilde{\beta}(\varrho) \Phi \psi'(v_j \cdot n(r_j)) \Lambda_{E_j}(v_j \cdot n(r_j)) \right. \\ &\quad \left. + \tilde{\beta}(\varrho) \psi \Lambda_{E_j} \Phi(r, v, \tau) \right\} \, dv \, dr \, d\tau \tag{3.6} \\ &= \int_{t_0}^{t_1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left\{ \tilde{\beta}(\varrho) \Phi \left[\psi'(v_j \cdot n(r_j)) \right. \right. \\ &\quad \times \left(\frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j^T D^2 d_\Omega v_j + \sum_{j=1}^{J+1} E_j(r, v, \tau) \cdot n(r_j) \right) \\ &\quad \left. \left. + \sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, \tau)) \psi(v_j \cdot n(r_j)) \right] + \tilde{\beta}(\varrho) \psi \Lambda_{E_j} \Phi \right\} \, dv \, dr \, d\tau. \end{aligned}$$

We now fix $t_0 \in [0, T]$, a compact set K of $\mathcal{O} := \Omega^{J+1} \times \mathbb{R}^{(J+1)d}$, $\psi(z) = 1$ and $\tilde{\beta} = \tilde{\beta}_\epsilon$ where $\tilde{\beta}_\epsilon$ is a sequence of smooth even and nonnegative functions such that $\tilde{\beta}_\epsilon(0) = 0$ and $\tilde{\beta}_\epsilon(y) \rightarrow |y|$, for all $y \in \mathbb{R}$. We can then choose $\Phi \in \mathcal{C}_0^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ in such a way that $0 \leq \Phi \leq 1$ in $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)$, $\Phi = 1$ on K and we denote by $R > 0$ a real number satisfying $\text{supp } \Phi \subset \mathcal{O}_R$. The identity (3.6) then implies that, for all $t \in [0, T]$,

$$\begin{aligned}
\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \tilde{\beta}_{\tilde{\epsilon}}(\varrho(\cdot, t_1)) \Phi \, dv \, dr &= \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \tilde{\beta}_{\tilde{\epsilon}}(\varrho(\cdot, t_0)) \Phi \, dv \, dr \\
&+ \int_{t_0}^{t_1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, \tau)) \, dv \, dr \, d\tau \\
&+ \int_{t_0}^{t_1} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \Lambda_{E_j} \Phi \, dv \, dr \, d\tau \\
&\leq \|\tilde{\beta}_{\tilde{\epsilon}}(\varrho(\cdot, t_0))\|_{L^1_R} + C_R \int_0^T \sum_{j=1}^{J+1} \|\partial_{v_j} \cdot E_j(\cdot, \cdot, \tau)\|_{L^\infty_R} \|\tilde{\beta}_{\tilde{\epsilon}}(\varrho)(\cdot, \tau)\|_{L^1_R} \, d\tau \\
&+ C_{R, \epsilon, \beta, J} \|\nabla \Phi\|_{L^\infty} \int_0^T \left(1 + \sum_{j=1}^{J+1} \|E_j(\cdot, \cdot, \tau)\|_{L^\infty_R} \right) \|\tilde{\beta}_{\tilde{\epsilon}}(\varrho)(\cdot, \tau)\|_{L^1_R} \, d\tau \\
&+ C_{R, \epsilon, \beta, J} \|\Delta \Phi\|_{L^\infty} \int_0^T \|\tilde{\beta}_{\tilde{\epsilon}}(\varrho)(\cdot, \tau)\|_{L^1_R} \, d\tau.
\end{aligned}$$

Letting $\tilde{\epsilon}$ tend to 0, we deduce our first a priori estimate:

$$\begin{aligned}
\sup_{\tau \in [0, T]} \|\varrho(\cdot, \tau)\|_{L^1(K)} &\leq \|\varrho(\cdot, t_0)\|_{L^1_R} + C_R \int_0^T \sum_{j=1}^{J+1} \|\partial_{v_j} \cdot E_j(\cdot, \cdot, \tau)\|_{L^\infty_R} \|\varrho(\cdot, \tau)\|_{L^1_R} \, d\tau \\
&+ C_{R, \epsilon, \beta, J} \int_0^T \left(1 + \sum_{j=1}^{J+1} \|E_j(\cdot, \cdot, \tau)\|_{L^\infty_R} \right) \|\varrho(\cdot, \tau)\|_{L^1_R} \, d\tau.
\end{aligned} \tag{3.7}$$

Let us now fix a compact subset K of $\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d}$, $\psi(z) = z$, $t_0 = 0$, $t_1 = T$, with $\tilde{\beta}$ as before. We choose $\Phi \in C_0^\infty(\bar{\Omega}^{J+1} \times \mathbb{R}^{(J+1)d})$ in such a way that $0 \leq \Phi \leq 1$ in \mathcal{O} , $\Phi \equiv 1$ on K , and we denote by $R > 0$ a real number satisfying $\text{supp } \Phi \subset B_R \times B_R$.

We then deduce from the identity (3.6) a second a priori estimate:

$$\begin{aligned}
&\sum_{j=1}^{J+1} \int_0^T \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} (v_j \cdot n(r_j))^2 \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \Phi \, dv \, ds(r) \, d\tau \\
&= - \left[\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} (v_j \cdot n(r_j)) \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \Phi \, dv \, dr \right]_0^T \\
&+ \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left\{ \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \Phi \left[\frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j^T D^2 d\Omega v_j + \sum_{j=1}^{J+1} E_j(r, v, \tau) \cdot n(r_j) \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{J+1} (v_j \cdot n(r_j)) (\partial_{v_j} \cdot E_j(r, v, \tau)) \right] + (v_j \cdot n(r_j)) \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \Lambda_{E_j} \Phi \right\} \, dv \, dr \, d\tau
\end{aligned}$$

$$\begin{aligned}
&\leq R(\|\tilde{\beta}_{\tilde{\epsilon}}(\varrho(\cdot, T))\|_{L^1} + \|\tilde{\beta}_{\tilde{\epsilon}}(\varrho(\cdot, 0))\|_{L^1}) \\
&\quad + \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left\{ \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \Phi \left[\frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j^T D^2 d_{\Omega} v_j + \sum_{j=1}^{J+1} E_j(r, v, \tau) \cdot n(r_j) \right. \right. \\
&\quad \left. \left. + \sum_{j=1}^{J+1} (v_j \cdot n(r_j)) (\partial_{v_j} \cdot E_j(r, v, \tau)) \right] + (v_j \cdot n(r_j)) \tilde{\beta}_{\tilde{\epsilon}}(\varrho) \Lambda_{E_j} \Phi \right\} dv dr d\tau \\
&\quad + C_R \int_0^T \sum_{j=1}^{J+1} \|\partial_{v_j} \cdot E_j(\cdot, \cdot, \tau)\|_{L_R^\infty} \|\tilde{\beta}_{\tilde{\epsilon}}(\varrho)(\cdot, \tau)\|_{L_R^1} d\tau \\
&\quad + C_{R, \epsilon, \beta, J} \int_0^T \left(1 + \sum_{j=1}^{J+1} \|E_j(\cdot, \cdot, \tau)\|_{L_R^\infty} \right) \|\tilde{\beta}_{\tilde{\epsilon}}(\varrho)(\cdot, \tau)\|_{L_R^1} d\tau.
\end{aligned}$$

Letting $\tilde{\epsilon}$ tend to 0, we then have that

$$\begin{aligned}
&\|\varrho\|_{L^1([0, T] \times K, d\mu_2)} \\
&\leq R(\|\varrho(\cdot, T)\|_{L^1} + \|\varrho(\cdot, 0)\|_{L^1}) + C_R \int_0^T \sum_{j=1}^{J+1} \|\partial_{v_j} \cdot E_j(\cdot, \cdot, \tau)\|_{L_R^\infty} \|\varrho(\cdot, \tau)\|_{L_R^1} d\tau \quad (3.8) \\
&\quad + C_{R, \epsilon, \beta, J} \int_0^T \left(1 + \sum_{j=1}^{J+1} \|E_j(\cdot, \cdot, \tau)\|_{L_R^\infty} \right) \|\varrho(\cdot, \tau)\|_{L_R^1} d\tau.
\end{aligned}$$

STEP 2: REGULARIZATION. In this step, we prove the following lemma, which states that ϱ can be approximated by a sequence ϱ_k of regular functions, defined on $\overline{\mathcal{O}} \times [0, T]$, and we solve (3.1) with an error term r_k , which tends to 0 as $k \rightarrow \infty$. Given the sequence of mollifiers $(\omega_k)_{k \geq 1}$ defined by

$$\omega_k(z) = k^d \omega(kz), \quad k \in \mathbb{N}, \quad \omega \in C^\infty(\mathbb{R}^d; \mathbb{R}_{\geq 0}), \quad \text{supp } \omega \subset B_1, \quad \int_{\mathbb{R}^d} \omega(z) dz = 1,$$

where \mathbb{N} is the set of all positive integers, we introduce the sequence of regularized functions

$$\tilde{\varrho}_k = \varrho \star_{r, k} \omega_k \star_v \omega_k,$$

where \star_v denotes the usual convolution; thus,

$$\begin{aligned}
(u \star_v H_k)(v) &:= \int_{\mathbb{R}^{(J+1)d}} u(\eta) H_k(v - \eta) d\eta \\
&= \int_{\mathbb{R}^{(J+1)d}} u(\eta) \prod_{j=1}^{J+1} h_k(v_j - \eta_j) d\eta \\
&= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} u(\eta_1, \dots, \eta_{J+1}) \prod_{j=1}^{J+1} h_k(v_j - \eta_j) d\eta_1 \cdots d\eta_{J+1},
\end{aligned}$$

for any function $u \in L^1(\mathbb{R}^{(J+1)d})$ and a function $H_k(v) := \prod_{j=1}^{J+1} h_k(v_j)$, $v := (v_1^\top, \dots, v_{J+1}^\top)^\top \in \mathbb{R}^{(J+1)d}$, where $v_j \in \mathbb{R}^d$ for $j = 1, \dots, J+1$, $h_k \in L^1(\mathbb{R}^d)$, $\text{supp } h_k \subset B_{\frac{1}{k}}$. We have that the convolution above is well-defined since $H_k \in L^1(\mathbb{R}^{(J+1)d})$. Hence, by Young's inequality for convolutions, $u *_v H_k \in L^1(\mathbb{R}^{(J+1)d})$ and

$$\|u *_v H_k\|_{L^1(\mathbb{R}^{(J+1)d})} \leq \|u\|_{L^1(\mathbb{R}^{(J+1)d})} \|H_k\|_{L^1(\mathbb{R}^{(J+1)d})}.$$

Now, let $u \in L^1_{\text{loc}}(\overline{\Omega^{J+1}})$. We extend u by 0 to the complement of Ω^{J+1} and we denote by $\star_{r,k}$ the convolution–translation defined by:

$$\begin{aligned} (u \star_{r,k} H_k)(r) &:= \int_{\mathbb{R}^{(J+1)d}} u(y) H_k\left(r - \frac{2}{k}n(r) - y\right) dy \\ &= \int_{\mathbb{R}^{(J+1)d}} u(y) \prod_{j=1}^{J+1} h_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) dy \\ &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} u(y_1, \dots, y_{J+1}) \prod_{j=1}^{J+1} h_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) dy_1 \cdots dy_{J+1}, \end{aligned}$$

where $h_k \in L^1(\mathbb{R}^d)$, $\text{supp } h_k \subset B_{\frac{1}{k}}$, $r := (r_1^\top, \dots, r_{J+1}^\top)^\top \in \Omega^{J+1}$, $r_j \in \overline{\Omega_j} \subset \mathbb{R}^d$ for all $j = 1, \dots, J+1$. The point of using a convolution–translation is to ensure that the variable y stays in the interior of the domain Ω^{J+1} , so that we do not create bad discontinuities in the derivatives of u at the boundary of the domain. Indeed, since the mollifiers h_k are compactly supported in $B_{\frac{1}{k}}$, we have that $y_j \in B(r_j - \frac{2}{k}n(r_j), \frac{1}{k})$, for all $j = 1, \dots, J+1$. Set $\tilde{r}_j := r_j - \frac{2}{k}n(r_j)$ and $d(y_j, \partial\Omega_j) := \inf\{|y_j - z| : z \in \partial\Omega_j\}$ for all $j = 1, \dots, J+1$. Hence $|y_j - \tilde{r}_j| < \frac{1}{k}$ and for $z \in \partial\Omega_j$:

$$\begin{aligned} |y_j - z| &= |y_j - \tilde{r}_j + \tilde{r}_j - z| \\ &\geq \left| |y_j - \tilde{r}_j| - |z - \tilde{r}_j| \right|. \end{aligned}$$

Since $|z - \tilde{r}_j| = d(\tilde{r}_j, \partial\Omega_j) > \frac{2}{k}$, we obtain that $|y_j - \tilde{r}_j| - |z - \tilde{r}_j| < \frac{1}{k} - \frac{2}{k} = -\frac{1}{k}$. Thus we deduce that $|y_j - z| > \frac{1}{k} > 0$. This implies $d(y_j, \partial\Omega_j) > 0$, for all $j = 1, \dots, J+1$. Hence, y_j is in the interior of Ω_j for all $j = 1, \dots, J+1$, which implies that y is in the interior of Ω^{J+1} .

Lemma 3.1.2. *For each $k \in \mathbb{N}$ there exists a function*

$$\varrho_k \in C(\overline{\Omega^{J+1}} \times \mathbb{R}^{(J+1)d} \times [0, T]) \cap W^{1,1}(0, T; W^{1,\infty}_{\text{loc}}(\overline{\mathcal{O}})),$$

such that the sequence ϱ_k satisfies:

$$\begin{aligned} \varrho_k &\text{ is bounded in } L^\infty(0, T; L^2_{\text{loc}}(\overline{\mathcal{O}})), \\ \varrho_k &\rightarrow \varrho \text{ in } L^a(0, T; L^2_{\text{loc}}(\overline{\mathcal{O}})) \quad \forall a \in [1, \infty) \end{aligned} \tag{3.9}$$

and

$$\Lambda_{E_j} \varrho_k = r_k \text{ in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)), \tag{3.10}$$

where r_k converges to 0 in $L^1_{\text{loc}}(\mathcal{O} \times [0, T])$.

Proof. The proof of this lemma is inspired by the work [21] of DiPerna & Lions. By considering ϱ as a function of t , y and η , i.e., $\varrho = \varrho(y, \eta, t)$, we multiply equation (3.1) by the test function

$$\prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k (v_m - \eta_m) \in \mathcal{C}_0^\infty(\Omega_y^{J+1} \times \mathbb{R}_\eta^{(J+1)d})$$

for fixed $r \in \overline{\Omega^{J+1}}$ and $v \in \mathbb{R}^{(J+1)d}$, and integrate over y and η . We get

$$\begin{aligned} \partial_t \tilde{\varrho}_k &= - \left(\sum_{j=1}^{J+1} E_j(r, v, t) \cdot \partial_{v_j} \varrho \right) \star_{r,k} \omega_k \star_v \omega_k - \left(\sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, t)) \varrho \right) \star_{r,k} \omega_k \star_v \omega_k \\ &+ \left(\frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \partial_{v_j}^2 \varrho \right) \star_{r,k} \omega_k \star_v \omega_k - \left(\frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho \right) \star_{r,k} \omega_k \star_v \omega_k \\ &\in L^1(0, T; W_{\text{loc}}^{1,\infty}(\mathcal{O})), \end{aligned} \tag{3.11}$$

and, in particular, $\tilde{\varrho}_k \in W^{1,1}(0, T; W_{\text{loc}}^{1,\infty}(\mathcal{O}))$.

Let us define ϱ_k to be the continuous representative of $\tilde{\varrho}_k$ in the class of functions almost everywhere equal to $\tilde{\varrho}_k$. Then ϱ_k solves (3.10) with

$$r_k = r_k^1(\varrho) + r_k^2(\varrho) + r_k^3(\varrho) + r_k^4(\varrho),$$

where

$$\begin{aligned} r_k^1(\varrho) &= \sum_{j=1}^{J+1} E_j(r, v, t) \cdot \partial_{v_j} \varrho_k - \left(\sum_{j=1}^{J+1} E_j(r, v, t) \cdot \partial_{v_j} \varrho \right) \star_{r,k} \omega_k \star_v \omega_k, \\ r_k^2(\varrho) &= \sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, t)) \varrho_k - \left(\sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, t)) \varrho \right) \star_{r,k} \omega_k \star_v \omega_k, \\ r_k^3(\varrho) &= \left(\frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \partial_{v_j}^2 \varrho \right) \star_{r,k} \omega_k \star_v \omega_k - \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \partial_{v_j}^2 \varrho_k, \\ r_k^4(\varrho) &= \frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho_k - \left(\frac{1}{\epsilon} \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho \right) \star_{r,k} \omega_k \star_v \omega_k. \end{aligned}$$

We have to prove that $r_k^1(\varrho)$, $r_k^2(\varrho)$, $r_k^3(\varrho)$ and $r_k^4(\varrho)$ all converge to 0 in L_{loc}^1 . Let us remark that if ϱ is smooth then one has

$$\partial_{r_j} (\varrho \star_{r,k} \omega_k) = \left(I - \frac{2}{k} D^2 d_\Omega \right) (\partial_{r_j} \varrho) \star_{r,k} \omega_k,$$

and therefore

$$r_k^4(\varrho) \rightarrow_{k \rightarrow \infty} 0 \quad \text{in } L_{\text{loc}}^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)). \tag{3.12}$$

Indeed, we have that

$$\begin{aligned}
\partial_{r_j}(\varrho \star_{r,k} \omega_k) &= \partial_{r_j} \left(\int_{\Omega^{J+1}} \varrho(y, v, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dy \right) \\
&= \int_{\Omega^{J+1}} \varrho(y, v, t) \partial_{r_j} \left(\prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right) dy \\
&= \int_{\Omega^{J+1}} \varrho(y, v, t) \partial_{r_j} \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dy \\
&= \left(I - \frac{2}{k} D^2 d\Omega \right) \int_{\Omega^{J+1}} \varrho(y, v, t) \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dy \\
&= - \left(I - \frac{2}{k} D^2 d\Omega \right) \int_{\Omega^{J+1}} \varrho(y, v, t) \partial_{y_j} \left(\omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right) \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dy \\
&= \left(I - \frac{2}{k} D^2 d\Omega \right) \int_{\Omega^{J+1}} \partial_{y_j} \varrho(y, v, t) \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dy \\
&= \left(I - \frac{2}{k} D^2 d\Omega \right) \int_{\Omega^{J+1}} \partial_{y_j} \varrho(y, v, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dy \\
&= \left(I - \frac{2}{k} D^2 d\Omega \right) (\partial_{r_j} \varrho) \star_{r,k} \omega_k.
\end{aligned}$$

To deal with a general $\varrho \in L_{\text{loc}}^{\infty,1}$ we begin by proving an a priori estimate. One has

$$\begin{aligned}
r_k^4(\varrho) &= \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ v_j \cdot \partial_{r_j} \left[\varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right] \right. \\
&\quad \left. - \eta_j \cdot \partial_{y_j} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right\} dy d\eta.
\end{aligned}$$

By differentiation with respect to r_j in the first integrand and integration by parts in the

second integrand we obtain

$$\begin{aligned}
r_k^4(\varrho) &= \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ v_j \cdot \left[\varrho(y, \eta, t) \left(I - \frac{2}{k} D^2 d\Omega \right) \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \right. \\
&\quad \times \left. \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right] \\
&\quad - \eta_j \cdot \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \varrho(y, \eta, t) \\
&\quad \times \left. \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right\} dy d\eta \\
&= \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
&\quad \times \left\{ (v_j - \eta_j) \cdot \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \\
&\quad \left. - \frac{2}{k} v_j \cdot (D^2 d\Omega) \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right\} dy d\eta.
\end{aligned}$$

Lemma 3.1.3. *There exists a constant C , which only depends on R and $d\Omega$, such that the following bound holds:*

$$\|r_k^4(\varrho)\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}.$$

Proof. The proof proceeds as follows. We note that

$$\begin{aligned}
\|r_k^4(\varrho)\|_{L^1(\mathcal{D}_R)} &= \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \left| \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right. \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \left\{ (v_j - \eta_j) \cdot \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \\
&\quad \left. \left. - \frac{2}{k} v_j \cdot (D^2 d\Omega) \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right\} dy d\eta \right| dv dr dt \\
&\leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right. \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \times \left\{ (v_j - \eta_j) \cdot \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \\
&\quad \left. \left. - \frac{2}{k} v_j \cdot (D^2 d\Omega) \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right\} \right| dy d\eta dv dr dt
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right. \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k\left(r_l - \frac{2}{k}n(r_l) - y_l\right) \\
&\quad \times \left. \left\{ (v_j - \eta_j) \cdot \nabla \omega_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) \right\} \right| dy d\eta dv dr dt \\
&+ \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \frac{2}{k} \varrho(y, \eta, t) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right. \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k\left(r_l - \frac{2}{k}n(r_l) - y_l\right) \\
&\quad \times \left. \left\{ v_j \cdot (D^2 d_\Omega) \nabla \omega_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) \right\} \right| dy d\eta dv dr dt \\
&\leq I_1 + I_2,
\end{aligned} \tag{3.13}$$

where we set

$$\begin{aligned}
I_1 := &\frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right. \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k\left(r_l - \frac{2}{k}n(r_l) - y_l\right) \\
&\quad \times \left. \left\{ (v_j - \eta_j) \cdot \nabla \omega_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) \right\} \right| dy d\eta dv dr dt
\end{aligned}$$

and

$$\begin{aligned}
I_2 := &\frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \frac{2}{k} \varrho(y, \eta, t) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right. \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k\left(r_l - \frac{2}{k}n(r_l) - y_l\right) \\
&\quad \times \left. \left\{ v_j \cdot (D^2 d_\Omega) \nabla \omega_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) \right\} \right| dy d\eta dv dr dt.
\end{aligned}$$

We have the following upper bound on I_1 :

$$\begin{aligned}
I_1 &\leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\
&\quad \times \prod_{l=1, l \neq j}^{J+1} \omega_k\left(r_l - \frac{2}{k}n(r_l) - y_l\right) \\
&\quad \times \left| \omega_k(v_j - \eta_j)(v_j - \eta_j) \right| \left| \nabla \omega_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) \right| dy d\eta dv dr dt.
\end{aligned}$$

Now, using the Fubini–Tonelli theorem, we obtain:

$$\begin{aligned}
I_1 &\leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{\Omega \cap B_R} \int_{B_R} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \left(\prod_{m=1, m \neq j}^{J+1} \int_{B_R} \omega_k(v_m - \eta_m) dv_m \right) \\
&\quad \times \left(\prod_{l=1, l \neq j}^{J+1} \int_{\Omega \cap B_R} \omega_k\left(r_l - \frac{2}{k}n(r_l) - y_l\right) dr_l \right) \left| \omega_k(v_j - \eta_j)(v_j - \eta_j) \right| \\
&\quad \times \left| \nabla \omega_k\left(r_j - \frac{2}{k}n(r_j) - y_j\right) \right| dy d\eta dv_j dr_j dt.
\end{aligned} \tag{3.14}$$

First, using the change of variable $z = v_m - \eta_m$, which implies that $dz = dv_m$, we obtain the following upper bound:

$$\begin{aligned}
\prod_{m=1, m \neq j}^{J+1} \int_{B_R} \omega_k(v_m - \eta_m) dv_m &= \prod_{m=1, m \neq j}^{J+1} \int_{\mathbb{R}^d} \omega_k(z) dz \\
&= \prod_{m=1, m \neq j}^{J+1} \int_{\mathbb{R}^d} k^d \omega(kz) dz \\
&= \prod_{m=1, m \neq j}^{J+1} \int_{\mathbb{R}^d} \omega(z) dz \\
&= 1.
\end{aligned} \tag{3.15}$$

Remark 3.1.4. *Let us also remark that since $|v_m - \eta_m| \leq \frac{1}{k}$ for all $m = 1, \dots, J+1$ and $k \geq 1$, i.e., $\frac{1}{k} \leq 1$, which imply that $|\eta_m| \leq |v_m| + \frac{1}{k} \leq R+1$, for all $m = 1, \dots, J+1$, we have that $\eta \in B_{R+1}^{J+1}$.*

Next, we perform the change of variable $s_l = r_l - \frac{2}{k}n(r_l) - y_l$, which implies that $ds_l = (I - \frac{2}{k}\nabla_{r_l}n(r_l))dr_l$, i.e., $dr_l = (I - \frac{2}{k}\nabla_{r_l}n(r_l))^{-1}ds_l$ for all $l = 1, \dots, J+1$. For $l \in \{1, \dots, J+1\}$ fixed (and therefore not explicitly indicated), we set $A := \frac{2}{k}\nabla_{r_l}n(r_l)$. We have that

$$|A| \leq \frac{2 \|\nabla n\|_{L^\infty(\mathbb{R}^d)}}{k}.$$

Hence $|A| < 1$ for all $k > 2 \|\nabla n\|_{L^\infty(\mathbb{R}^d)} =: b$. In that case,

$$\begin{aligned}
|(I - A)^{-1}| &= \left| \sum_{n=0}^{\infty} A^n \right| \\
&\leq \sum_{n=0}^{\infty} |A|^n \\
&\leq \sum_{n=0}^{\infty} \left(\frac{2 \|\nabla n\|_{L^\infty(\mathbb{R}^d)}}{k} \right)^n \\
&= \frac{1}{1 - \frac{2 \|\nabla n\|_{L^\infty(\mathbb{R}^d)}}{k}} \\
&= \frac{k}{k - 2 \|\nabla n\|_{L^\infty(\mathbb{R}^d)}}.
\end{aligned}$$

We have that the function $g : x \mapsto \frac{x}{x-b}$, where $x > b$, is strictly monotonic decreasing. Indeed, $g'(x) = \frac{-b}{(x-b)^2} < 0$ for all $x > b$. Hence, taking $x \geq 2b$, the maximum of g is achieved at $x = 2b$, where $g(x) = 2$.

Therefore, by choosing $k \geq 2b$, we get

$$|(I - A)^{-1}| \leq 2;$$

thus, the change of variable gives

$$\begin{aligned}
\prod_{l=1, l \neq j}^{J+1} \int_{\Omega \cap B_R} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dr_l &\leq \prod_{l=1, l \neq j}^{J+1} 2 \int_{\mathbb{R}^d} \omega_k(s_l) ds_l \\
&= \prod_{l=1, l \neq j}^{J+1} 2 \int_{\mathbb{R}^d} k^d \omega(k s_l) ds_l \\
&= \prod_{l=1, l \neq j}^{J+1} 2 \int_{\mathbb{R}^d} \omega(z) dz \\
&\leq 2^J.
\end{aligned} \tag{3.16}$$

Remark 3.1.5. *Let us also remark that since $|r_l - \frac{2}{k} n(r_l) - y_l| \leq \frac{1}{k}$, for all $m = 1, \dots, J+1$, and for $k \geq 3$, i.e., $\frac{3}{k} \leq 1$, which imply that $|y_l| \leq |r_l| + \frac{3}{k} \leq R + 1$, we have that $y \in (\Omega \cap B_{R+1})^{J+1}$.*

Hence, using estimates (3.15) and (3.16) together with Remarks 3.1.4 and 3.1.5 in (3.14), we get for all $k \geq \max(2b, 3)$ that

$$\begin{aligned}
I_1 &\leq \frac{2^J}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{\Omega \cap B_R} \int_{B_R} \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \left| \omega_k(v_j - \eta_j)(v_j - \eta_j) \right| \\
&\quad \times \left| \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right| dy d\eta dv_j dr_j dt
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{2^J}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{B_R} \varrho(y, \eta, t) \left(\int_{B_R} \left| \omega_k(v_j - \eta_j)(v_j - \eta_j) \right| dv_j \right) \\
&\quad \times \left(\int_{\Omega \cap B_R} \left| \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right| dr_j \right) dy d\eta dt.
\end{aligned} \tag{3.17}$$

Again, by performing the same change of variable as in (3.16), we obtain, for a constant $C_1 > 0$, independent of k , the following bound:

$$\begin{aligned}
\int_{\Omega \cap B_R} \left| \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right| dr_j &\leq 2 \int_{\mathbb{R}^d} |\nabla \omega_k(\xi)| d\xi \\
&= 2 \int_{\mathbb{R}^d} k^{d+1} |\nabla \omega(k\xi)| d\xi \\
&= 2k \int_{\mathbb{R}^d} |\nabla \omega(z)| dz \\
&\leq 2k C_1.
\end{aligned} \tag{3.18}$$

Finally, noting that $w_k \in C_0^\infty(\mathbb{R}^d)$, we obtain, for a constant $C_2 > 0$, independent of k , that

$$\begin{aligned}
\int_{B_R} \left| \omega_k(v_j - \eta_j)(v_j - \eta_j) \right| dv_j &\leq \int_{\mathbb{R}^d} \omega_k(\xi) |\xi| d\xi \\
&= \int_{\mathbb{R}^d} k^d \omega(k\xi) |\xi| d\xi \\
&= \frac{1}{k} \int_{\mathbb{R}^d} \omega(z) |z| dz \\
&\leq \frac{C_2}{k}.
\end{aligned} \tag{3.19}$$

We use (3.18) and (3.19) in (3.17), for a constant $C > 0$, independent of k , we get the following bound on the term I_1 :

$$\begin{aligned}
I_1 &\leq \frac{2^J}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{B_R} \varrho(y, \eta, t) \left(\int_{B_R} \left| \omega_k(v_j - \eta_j)(v_j - \eta_j) \right| dv_j \right) \\
&\quad \times \left(\int_{\Omega \cap B_R} \left| \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right| dr_j \right) dy d\eta dt \\
&\leq \frac{2^{J+1} C_1 C_2}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) dy d\eta dt \\
&\leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}.
\end{aligned} \tag{3.20}$$

Now, using the same technique as for the bound on I_1 , we get the following bound on the term I_2 :

$$\begin{aligned}
I_2 &\leq \frac{4|B_R|}{\epsilon k} \sum_{j=1}^{J+1} \int_0^T \int_{\Omega \cap B_R} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} |D^2 d_\Omega| \varrho(y, \eta, t) \left(\prod_{m=1}^{J+1} \int_{B_R} \omega_k(v_m - \eta_m) dv_m \right) \\
&\quad \times \left(\prod_{l=1, l \neq j}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dr_l \right) \left| \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right| dy d\eta dr_j dt \\
&\leq \frac{2^{J+2} |B_R| \|D^2 d_\Omega\|_{L^\infty(\mathbb{R}^d)}}{\epsilon k} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \\
&\quad \times \left(\int_{\Omega \cap B_R} \left| \nabla \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right| dr_j \right) dy d\eta dt \\
&\leq \frac{2^{J+2} C_1 |B_R| \|D^2 d_\Omega\|_{L^\infty(\mathbb{R}^d)}}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) dy d\eta dt \\
&\leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}.
\end{aligned} \tag{3.21}$$

In conclusion, using (3.20) and (3.21) in (3.13), we get, with a constant C that only depends on R and $d(r)$, the following bound:

$$\begin{aligned}
\|r_k^4(\varrho)\|_{L^1(\mathcal{D}_R)} &\leq I_1 + I_2 \\
&\leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}.
\end{aligned} \tag{3.22}$$

That completes the proof of the lemma. \square

Next, for $\varrho \in L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$ we argue by density; in other words, we consider a sequence $(\varrho_\epsilon)_{\epsilon>0}$ of smooth functions such that $\varrho_\epsilon \rightarrow \varrho$ in $L^1_{\text{loc}}(\overline{\mathcal{D}})$, and we note the following obvious decomposition of the function $r_k^4(\varrho)$:

$$r_k^4(\varrho) = r_k^4(\varrho_\epsilon) + r_k^4(\varrho - \varrho_\epsilon),$$

which obviously converges to 0 in $L^1_{\text{loc}}(\overline{\mathcal{D}})$ as $k \rightarrow \infty$ and $\epsilon \rightarrow 0$, thanks to (3.12) and (3.22).

The convergence of the sequence $r_k^1(\varrho)$ to 0, as $k \rightarrow \infty$, has been proved, in a simpler case, in the article of DiPerna & Lions [21], Lemma II.1. By following a similar line of argument as in [21], we proceed as follows. If ϱ and E_j are sufficiently smooth, we have that

$$r_k^1(\varrho) = \sum_{j=1}^{J+1} E_j(r, v, t) \cdot \partial_{v_j} \varrho_k - \left(\sum_{j=1}^{J+1} E_j(r, v, t) \cdot \partial_{v_j} \varrho \right) \star_{r,k} \omega_k \star_v \omega_k$$

$$\begin{aligned}
&= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ E_j(r, v, t) \cdot \partial_{v_j} \left[\varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \right. \\
&\quad \times \left. \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right] - E_j(y, \eta, t) \cdot \partial_{\eta_j} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
&\quad \times \left. \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right\} dy d\eta.
\end{aligned}$$

By performing integration by parts in the second integrand, we get

$$\begin{aligned}
r_k^1(\varrho) &= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ E_j(r, v, t) \cdot \nabla \omega_k(v_j - \eta_j) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\
&\quad \times \varrho(y, \eta, t) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) + \partial_{\eta_j} \cdot \left[E_j(y, \eta, t) \omega_k(v_j - \eta_j) \right] \\
&\quad \times \varrho(y, \eta, t) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \left. \right\} dy d\eta \\
&= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ E_j(r, v, t) \cdot \nabla \omega_k(v_j - \eta_j) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\
&\quad \times \varrho(y, \eta, t) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) + \partial_{\eta_j} \cdot \left[E_j(y, \eta, t) \right] \varrho(y, \eta, t) \\
&\quad \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) - E_j(y, \eta, t) \cdot \nabla \omega_k(v_j - \eta_j) \\
&\quad \times \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \left. \right\} dy d\eta \\
&= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\
&\quad \times \left\{ \left[E_j(r, v, t) - E_j(y, \eta, t) \right] \cdot \nabla \omega_k(v_j - \eta_j) \right\} dy d\eta \\
&\quad + \left(\sum_{j=1}^{J+1} \varrho(r, v, t) (\partial_{v_j} \cdot E_j(r, v, t)) \right) \star_{r,k} \omega_k \star_v \omega_k.
\end{aligned}$$

The second term on the right-hand side converges to

$$\sum_{j=1}^{J+1} \varrho(r, v, t) (\partial_{v_j} \cdot E_j(r, v, t))$$

in L_{loc}^1 , as k tends to ∞ , by standard results for convolutions.

For the first integral, on the one hand, we have that

$$\begin{aligned}
& \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\
& \quad \times E_j(r, v, t) \cdot \nabla \omega_k(v_j - \eta_j) \, dy \, d\eta \\
&= - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
& \quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) E_j(r, v, t) \cdot \partial_{\eta_j} (\omega_k(v_j - \eta_j)) \, dy \, d\eta \\
&= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \varrho(y, \eta, t) \cdot E_j(r, v, t) \omega_k(v_j - \eta_j) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
& \quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \, dy \, d\eta \\
&= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \varrho(y, \eta, t) \cdot E_j(r, v, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
& \quad \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \, dy \, d\eta,
\end{aligned}$$

which converges to

$$\sum_{j=1}^{J+1} \partial_{v_j} \varrho(r, v, t) \cdot E_j(r, v, t)$$

in L^1_{loc} as k tends to ∞ . On the other hand,

$$\begin{aligned}
& - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\
& \quad \times E_j(y, \eta, t) \cdot \nabla \omega_k(v_j - \eta_j) \, dy \, d\eta \\
&= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\
& \quad \times E_j(y, \eta, t) \cdot \partial_{\eta_j} (\omega_k(v_j - \eta_j)) \, dy \, d\eta \\
&= - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \cdot [\varrho(y, \eta, t) E_j(y, \eta, t)] \omega_k(v_j - \eta_j) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
& \quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \, dy \, d\eta
\end{aligned}$$

$$\begin{aligned}
&= - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \cdot [\varrho(y, \eta, t) E_j(y, \eta, t)] \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
&\quad \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \, dy \, d\eta \\
&= - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \varrho(y, \eta, t) \cdot E_j(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
&\quad \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \, dy \, d\eta \\
&\quad - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \partial_{\eta_j} \cdot E_j(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
&\quad \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \, dy \, d\eta,
\end{aligned}$$

which converges to

$$- \sum_{j=1}^{J+1} \partial_{v_j} \varrho(r, v, t) \cdot E_j(r, v, t) - \sum_{j=1}^{J+1} \varrho(r, v, t) (\partial_{v_j} \cdot E_j(r, v, t))$$

in L^1_{loc} as k tends to ∞ . Hence r_k^1 converges in L^1_{loc} to 0 as k tends to ∞ . The general case follows by means of a density argument, using the inequality stated in the next lemma.

Lemma 3.1.6. *There exists a constant C , which only depends on R and d_Ω , such that the following bound holds:*

$$\|r_k^1(\varrho)\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^2(\mathcal{D}_{R+1})}.$$

Proof. We begin by noting that

$$\begin{aligned}
r_k^1(\varrho) &= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\
&\quad \times \left\{ [E_j(r, v, t) - E_j(y, \eta, t)] \cdot \nabla \omega_k(v_j - \eta_j) \right\} \, dy \, d\eta \\
&\quad + \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \cdot E_j(y, \eta, t) \\
&\quad \times \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \varrho(y, \eta, t) \, dy \, d\eta \\
&=: I_1 + I_2,
\end{aligned}$$

where we set

$$I_1 := \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \times \left\{ [E_j(r, v, t) - E_j(y, \eta, t)] \cdot \nabla \omega_k(v_j - \eta_j) \right\} dy d\eta, \quad (3.23)$$

and

$$I_2 := \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \cdot E_j(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \varrho(y, \eta, t) dy d\eta. \quad (3.24)$$

Let us recall that

$$E_j = E_j(r, v, t) := \frac{1}{\epsilon} ((\mathcal{L}r)_j + u(r_j, t)) - \frac{\beta^2}{\epsilon^2} v_j.$$

Hence,

$$E_j(r, v, t) - E_j(y, \eta, t) = \frac{1}{\epsilon} ((\mathcal{L}r)_j - (\mathcal{L}y)_j) + \frac{1}{\epsilon} (u(r_j, t) - u(y_j, t)) - \frac{\beta^2}{\epsilon^2} (v_j - \eta_j). \quad (3.25)$$

Using (3.25) in (3.23), we get

$$I_1 := \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \times \left\{ \left[\frac{1}{\epsilon} ((\mathcal{L}r)_j - (\mathcal{L}y)_j) + \frac{1}{\epsilon} (u(r_j, t) - u(y_j, t)) - \frac{\beta^2}{\epsilon^2} (v_j - \eta_j) \right] \cdot \nabla \omega_k(v_j - \eta_j) \right\} dy d\eta \quad (3.26)$$

$$=: I_1^1 + I_1^2 + I_1^3,$$

where we set

$$I_1^1 := \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \times \frac{1}{\epsilon} ((\mathcal{L}r)_j - (\mathcal{L}y)_j) \cdot \nabla \omega_k(v_j - \eta_j) dy d\eta,$$

$$I_1^2 := \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \times \frac{1}{\epsilon} (u(r_j, t) - u(y_j, t)) \cdot \nabla \omega_k(v_j - \eta_j) dy d\eta,$$

and

$$I_1^3 := - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{\beta^2}{\epsilon^2} (v_j - \eta_j) \cdot \nabla \omega_k(v_j - \eta_j) dy d\eta.$$

On the one hand, we have that

$$\|I_1^3\|_{L^1(\mathcal{D}_R)} = \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \left| \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ \left. \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \frac{\beta^2}{\epsilon^2} (v_j - \eta_j) \cdot \nabla \omega_k(v_j - \eta_j) dy d\eta \right| dv dr dt \\ \leq \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ \left. \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \frac{\beta^2}{\epsilon^2} (v_j - \eta_j) \cdot \nabla \omega_k(v_j - \eta_j) \right| dy d\eta dv dr dt.$$

Noting that $|v_j - \eta_j| \leq \frac{1}{k}$ for all $j = 1, \dots, J+1$, we then deduce that

$$\|I_1^3\|_{L^1(\mathcal{D}_R)} \leq \frac{\beta^2}{k \epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ \left. \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \nabla \omega_k(v_j - \eta_j) \right| dy d\eta dv dr dt \\ \leq \frac{\beta^2}{k \epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ \left. \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \nabla \omega_k(v_j - \eta_j) \right| dy d\eta dv dr dt \\ \leq \frac{\beta^2}{k \epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \left(\prod_{l=1}^{J+1} \int_{\Omega \cap B_R} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) dr_l \right) \\ \times \left(\prod_{m=1, m \neq j}^{J+1} \int_{B_R} \omega_k(v_m - \eta_m) dv_m \right) \\ \times \left(\int_{B_R} |\nabla \omega_k(v_j - \eta_j)| dv_j \right) dy d\eta dt$$

$$\begin{aligned}
&\leq \frac{2^{J+1} \beta^2}{k \epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{B_R} \varrho(y, \eta, t) \left(\int_{B_R} |\nabla \omega_k(v_j - \eta_j)| dv_j \right) dy d\eta dt \\
&\leq \frac{2^{J+1} \beta^2}{k \epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{\mathbb{R}^d} \varrho(y, \eta, t) \left(\int_{\mathbb{R}^d} |\nabla \omega_k(\xi)| d\xi \right) dy d\eta dt \\
&= \frac{2^{J+1} \beta^2}{k \epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{\mathbb{R}^d} \varrho(y, \eta, t) \left(\int_{\mathbb{R}^d} k^{d+1} |\nabla \omega(k\xi)| d\xi \right) dy d\eta dt \\
&= \frac{2^{J+1} \beta^2}{k \epsilon^2} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{\mathbb{R}^d} \varrho(y, \eta, t) \left(\int_{\mathbb{R}^d} k |\nabla \omega(z)| dz \right) dy d\eta dt \\
&\leq \frac{2^{J+1} (J+1) \beta^2 C}{\epsilon^2} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{\mathbb{R}^d} \varrho(y, \eta, t) dy d\eta dt \\
&\leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})},
\end{aligned}$$

where $C > 0$ is a constant independent of k . On the other hand, we have that

$$\begin{aligned}
\|I_1^1\|_{L^1(\mathcal{D}_R)} &= \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \left| \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\
&\quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) ((\mathcal{L}r)_j - (\mathcal{L}y)_j) \\
&\quad \left. \cdot \nabla \omega_k(v_j - \eta_j) dy d\eta \right| dv dr dt \\
&\leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\
&\quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) ((\mathcal{L}r)_j - (\mathcal{L}y)_j) \\
&\quad \left. \cdot \nabla \omega_k(v_j - \eta_j) \right| dy d\eta dv dr dt,
\end{aligned}$$

and noting that $|(\mathcal{L}r)_j - (\mathcal{L}y)_j| \leq |r_j - y_j| \leq \frac{3}{k}$ for all $j = 1, \dots, J+1$, we get

$$\begin{aligned} \|I_1^1\|_{L^1(\mathcal{D}_R)} &\leq \frac{3}{k\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k\left(r_l - \frac{2}{k}n(r_l) - y_l\right) \\ &\quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \left| \nabla \omega_k(v_j - \eta_j) \right| dy d\eta dv dr dt \\ &\leq \frac{2^{J+1}(J+1)C}{\epsilon} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) dy d\eta dt \\ &\leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}, \end{aligned}$$

where $C > 0$ is a constant independent of k .

In order to show that, for a constant $C > 0$ independent of k , we have $\|I_1\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^2(\mathcal{D}_{R+1})}$, it remains to prove that $\|I_1^2\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^2(\mathcal{D}_{R+1})}$. Using the fundamental theorem of calculus we have, for any sufficiently smooth real-valued function u , that

$$u(r_j, t) - u(y_j, t) = \int_0^1 \nabla u\left(y_j + h(r_j - y_j)\right)(r_j - y_j) dh.$$

Now, for $u \in L^2(0, T; W^{1,\sigma}(\Omega)^d)$, with $\sigma > d$, a solution to the Oseen system (1.3), we define

$$\tilde{u}(\cdot, t) = \begin{cases} u(\cdot, t) & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^d \setminus \Omega. \end{cases}$$

Let us first study the smoothed functions

$$u^\delta(\cdot, t) := \tilde{u} *_x \omega_\delta(\cdot, t) \quad \text{for } \delta > 0,$$

where $\omega_\delta(x) := \frac{1}{\delta^d} \omega\left(\frac{x}{\delta}\right)$ denotes the usual mollifier.

Lemma 3.1.7. *We have that*

$$u^\delta \rightarrow \tilde{u} \text{ in } L^2(\mathbb{R}^d \times (0, T)) \text{ as } \delta \rightarrow 0.$$

Proof. To prove this, we first note that if \tilde{u} is smooth, then, for $x \in \mathbb{R}^d$,

$$\begin{aligned} u^\delta(x, t) - \tilde{u}(x, t) &= \frac{1}{\delta^d} \int_{B(x, \delta)} \omega\left(\frac{x-z}{\delta}\right) (\tilde{u}(z, t) - \tilde{u}(x, t)) dz \\ &= \int_{B(0,1)} \omega(y) (\tilde{u}(x + \delta y, t) - \tilde{u}(x, t)) dy \\ &= \int_{B(0,1)} \omega(y) \int_0^1 \frac{d}{dh} (\tilde{u}(x + h \delta y, t)) dh dy \\ &= \delta \int_{B(0,1)} \omega(y) \int_0^1 \nabla \tilde{u}(x + h \delta y, t) \cdot y dh dy. \end{aligned}$$

Hence,

$$|u^\delta(x, t) - \tilde{u}(x, t)| \leq \delta \int_0^1 \int_{B(0,1)} |\nabla \tilde{u}(x + h \delta y, t)| dy dh,$$

and using the change of variable $z = x + h \delta y$,

$$\begin{aligned} |u^\delta(x, t) - \tilde{u}(x, t)| &\leq \delta \int_0^1 \frac{1}{(h \delta)^d} \int_{B(x, h\delta)} |\nabla \tilde{u}(z, t)| dz dh \\ &\leq C |B| \delta \mathcal{M}_{|\nabla \tilde{u}|}(x, t), \end{aligned}$$

where $\mathcal{M}_f(x) := \sup_{r>0} \int_{B(x,r)} |f(y)| dy$ is the Hardy–Littlewood maximal function.

Thus

$$\begin{aligned} \int_{\mathbb{R}^d} |u^\delta(x, t) - \tilde{u}(x, t)|^2 dx &\leq C \delta^2 \int_{\mathbb{R}^d} |\mathcal{M}_{|\nabla \tilde{u}|}(x, t)|^2 dx \\ &\leq C \delta^2 \int_{\mathbb{R}^d} |\nabla \tilde{u}(x, t)|^2 dx, \end{aligned} \tag{3.27}$$

where we have used a classical strong (p, p) -bound on the maximal function, which asserts that the maximal operator is bounded in $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$, i.e.,

$$\|\mathcal{M}_f\|_p \leq c \|f\|_p,$$

where $c = c(d, p) > 0$ is a constant. Now, for the smooth functions

$$\tilde{\tilde{u}}(\cdot, t) := \tilde{u} *_x \omega_\lambda(\cdot, t) \quad \text{with } \lambda > 0$$

and

$$u^\delta(\cdot, t) := \tilde{\tilde{u}} *_x \omega_\delta(\cdot, t) \quad \text{with } \delta > 0,$$

from estimate (3.27), we still have

$$\int_{\mathbb{R}^d} |u^\delta(x, t) - \tilde{\tilde{u}}(x, t)|^2 dx \leq C \delta^2 \int_{\mathbb{R}^d} |\nabla \tilde{\tilde{u}}(x, t)|^2 dx.$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^d} |((\tilde{u} *_x \omega_\lambda) *_x \omega_\delta)(x, t) - \tilde{u} *_x \omega_\lambda(x, t)|^2 dx &\leq C \delta^2 \int_{\mathbb{R}^d} |\nabla(\tilde{u} *_x \omega_\lambda)(x, t)|^2 dx \\ &\leq C \delta^2 \int_{\mathbb{R}^d} |\nabla \tilde{u}(x, t)|^2 dx. \end{aligned} \tag{3.28}$$

Also, since for a.e. $t \in (0, T)$

$$\tilde{\tilde{u}}(\cdot, t) \in L^\sigma(\mathbb{R}^d),$$

there exists a subsequence $(\lambda_{k_t})_{t \geq 0}$ such that

$$\tilde{u} *_x \omega_{\lambda_{k_t}}(\cdot, t) \rightarrow_{\lambda_{k_t} \rightarrow 0^+} \tilde{u}(\cdot, t) \quad \text{a.e. in } \mathbb{R}^d.$$

Hence, on the one hand,

$$(\tilde{u} *_x \omega_\delta) *_x \omega_{\lambda_{k_t}}(\cdot, t) \rightarrow_{\lambda_{k_t} \rightarrow 0^+} \tilde{u} *_x \omega_\delta(\cdot, t) \quad \text{a.e. in } \mathbb{R}^d,$$

on the other hand, since

$$(\tilde{u} *_x \omega_\lambda) *_x \omega_\delta = (\tilde{u} *_x \omega_\delta) *_x \omega_\lambda,$$

using Fatou's lemma in (3.28), we get

$$\begin{aligned} & \int_{\mathbb{R}^d} \liminf_{\lambda_{k_t} \rightarrow 0^+} |((\tilde{u} *_x \omega_\lambda) *_x \omega_\delta)(x, t) - \tilde{u} *_x \omega_\lambda(x, t)|^2 dx \\ & \leq \liminf_{\lambda_{k_t} \rightarrow 0^+} \int_{\mathbb{R}^d} |((\tilde{u} *_x \omega_\lambda) *_x \omega_\delta)(x, t) - \tilde{u} *_x \omega_\lambda(x, t)|^2 dx \\ & \leq C\delta^2 \int_{\mathbb{R}^d} |\nabla \tilde{u}(x, t)|^2 dx, \end{aligned}$$

that is,

$$\begin{aligned} \int_{\mathbb{R}^d} |(\tilde{u} *_x \omega_\delta)(x, t) - \tilde{u}(x, t)|^2 dx & \leq C\delta^2 \int_{\mathbb{R}^d} |\nabla \tilde{u}(x, t)|^2 dx \\ & \leq C\delta^2(1 + \|u_0\|_{L^2(\Omega)}^2). \end{aligned}$$

However, since

$$C\delta^2(1 + \|u_0\|_{L^2(\Omega)}^2) \rightarrow_{\delta \rightarrow 0^+} 0,$$

which implies that

$$\int_{\mathbb{R}^d} |(\tilde{u} *_x \omega_\delta)(x, t) - \tilde{u}(x, t)|^2 dx \rightarrow_{\delta \rightarrow 0^+} 0$$

uniformly in $t \in [0, T]$, by Lebesgue's dominated convergence theorem we then have that

$$\int_0^T \int_{\mathbb{R}^d} |(\tilde{u} *_x \omega_\delta)(x, t) - \tilde{u}(x, t)|^2 dx dt \rightarrow_{\delta \rightarrow 0^+} 0;$$

that is,

$$u^\delta \rightarrow \tilde{u} \text{ in } L^2(\mathbb{R}^d \times (0, T)) \text{ as } \delta \rightarrow 0,$$

as was asserted above. □

Hence, we deduce that there exists a subsequence $(\delta_p)_{p > 0}$ such that

$$(\tilde{u} *_x \omega_{\delta_p})(x, t) - \tilde{u} \rightarrow_{\delta_p \rightarrow 0^+} 0 \quad \text{for a.e. } (x, t) \in \mathbb{R}^d \times (0, T).$$

Set $u_p(x, t) := (\tilde{u} *_x \omega_{\delta_p})(x, t)$. Since

$$I_1^2 := \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{1}{\epsilon} (u(r_j, t) - u(y_j, t)) \cdot \nabla \omega_k(v_j - \eta_j) \, dy \, d\eta,$$

this implies, by Fatou's lemma, that

$$|I_1^2| \leq \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{1}{\epsilon} |u(r_j, t) - u(y_j, t)| |\nabla \omega_k(v_j - \eta_j)| \, dy \, d\eta \\ \leq \liminf_{\delta_p \rightarrow 0} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{1}{\epsilon} |u_p(r_j, t) - u_p(y_j, t)| \cdot \nabla \omega_k(v_j - \eta_j) \, dy \, d\eta \\ := \liminf_{\delta_p \rightarrow 0} I_1^{2,p}. \tag{3.29}$$

Moreover,

$$I_1^{2,p} \leq \frac{3}{k} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{1}{\epsilon} \left| \left(\int_0^1 \nabla u_p(y_j + h(r_j - y_j)) \, dh \right) \right| |\nabla \omega_k(v_j - \eta_j)| \, dy \, d\eta.$$

By noting that $(\nabla \omega)_k(z) = k^d \nabla \omega(kz)$, which implies that $k(\nabla \omega)_k(z) = \nabla \omega_k(z)$, we get

$$|I_1^{2,p}| \leq 3 \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{1}{\epsilon} \left| \left(\int_0^1 \nabla u_p(y_j + (r_j - y_j)) \, dh \right) \right| |(\nabla \omega)_k(v_j - \eta_j)| \, dy \, d\eta.$$

Hence we have that

$$\begin{aligned}
\|I_1^{2,p}\|_{L^1(\mathcal{D}_R)} &\leq \frac{3 \cdot 2^J}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \left(\int_{\Omega \cap B_R} \left[\omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \right. \\
&\quad \times \left. \left. \int_0^1 \left| \nabla u_p \left(y_j + h(r_j - y_j) \right) \right| dh \right] dr_j \right) \\
&\quad \times \left(\int_{B_R} |(\nabla \omega)_k(v_j - \eta_j)| dv_j \right) dy d\eta dt.
\end{aligned} \tag{3.30}$$

On the one hand,

$$\begin{aligned}
\int_{B_R} |(\nabla \omega)_k(v_j - \eta_j)| dv_j &\leq \int_{\mathbb{R}^d} |(\nabla \omega)_k(\xi)| d\xi \\
&= \int_{\mathbb{R}^d} k^d |\nabla \omega(k\xi)| d\xi \\
&= \int_{\mathbb{R}^d} |\nabla \omega(z)| dz \\
&\leq C,
\end{aligned} \tag{3.31}$$

where C is a positive constant, independent of k . Furthermore, we have that

$$\begin{aligned}
&\int_{\Omega \cap B_R} \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \left[\int_0^1 \left| \nabla u_p \left(y_j + h(r_j - y_j) \right) \right| dh \right] dr_j \\
&= \int_0^1 \int_{\Omega \cap B_R} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \left| \nabla u_p \left(y_j + h(r_j - y_j) \right) \right| dr_j dh.
\end{aligned}$$

Thus, by using the change of variable $z_j = r_j - y_j$, which implies that $dz_j = dr_j$, where r_j and $y_j \in B_{R+1}$, we get that

$$\begin{aligned}
&\int_{\Omega \cap B_R} \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \left[\int_0^1 \left| \nabla u_p \left(y_j + h(r_j - y_j) \right) \right| dh \right] dr_j \\
&\leq \int_0^1 \int_{B_2(R+1)} \omega_k \left(z_j - \frac{2}{k} n(z_j + y_j) \right) \left| \nabla u_p \left(y_j + h z_j \right) \right| dz_j dh \\
&= \int_0^1 \int_{B_2(R+1)} k^d \omega \left(k z_j - 2 n(z_j + y_j) \right) \left| \nabla u_p \left(y_j + h z_j \right) \right| dz_j dh.
\end{aligned}$$

Now, by the change of variable $\xi_j = k z_j$, which implies that $d\xi_j = k^d dz_j$ and $|\xi_j| \leq 1 + 2\|n\|_{L^\infty(\mathbb{R}^d)} := R'$ since $|\xi_j - 2n(\frac{\xi_j}{k} + y_j)| \leq 1$, we obtain

$$\begin{aligned} & \int_{\Omega \cap B_R} \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \left[\int_0^1 |\nabla u_p(y_j + h(r_j - y_j))| dh \right] dr_j \\ & \leq \int_0^1 \int_{B_{1+2\|n\|_{L^\infty(\mathbb{R}^d)}}} \omega \left(\xi_j - 2n \left(\frac{\xi_j}{k} + y_j \right) \right) |\nabla u_p \left(y_j + \frac{h}{k} \xi_j \right)| d\xi_j dh \\ & \leq \int_0^1 \int_{B_{R'}} |\nabla u_p \left(y_j + \frac{h}{k} \xi_j \right)| d\xi_j dh. \end{aligned}$$

Next, by performing the change of variable $s = y_j + \frac{h}{k} \xi_j$, which implies that $ds = (\frac{h}{k})^d d\xi_j$ and $s \in B(y_j, \frac{h}{k} R')$, we get

$$\begin{aligned} & \int_{\Omega \cap B_R} \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \left[\int_0^1 |\nabla u_p(y_j + h(r_j - y_j))| dh \right] dr_j \\ & \leq \int_0^1 \int_{B(y_j, \frac{h}{k} R')} |\nabla u_p(s)| \frac{ds}{(\frac{h}{k})^d} dh \\ & \leq C_{R'} \int_0^1 \int_{B(y_j, \frac{h}{k} R')} |\nabla u_p(s)| ds dh. \end{aligned}$$

Consider again the Hardy–Littlewood maximal function $\mathcal{M}_f(x) := \sup_{r>0} \int_{B(x,r)} |f(y)| dy$. We then have that

$$\int_{\Omega \cap B_R} \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \left[\int_0^1 |\nabla u_p(y_j + h(r_j - y_j))| dh \right] dr_j \leq C_{R'} \mathcal{M}_{|\nabla u_p|}(y_j, t). \quad (3.32)$$

Hence, by noting inequalities (3.31) and (3.32) in (3.30), we obtain

$$\|I_1^{2,p}\|_{L^1(\mathcal{D}_R)} \leq \frac{3 \cdot 2^J C_{R'} C}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \mathcal{M}_{|\nabla u_p|}(y_j, t) dy d\eta dt.$$

Using the Cauchy–Schwarz inequality and the boundedness of the maximal operator in

$L^2(\mathbb{R}^d)$, we get

$$\begin{aligned}
\|I_1^{2,p}\|_{L^1(\mathcal{D}_R)} &\leq \frac{3 \cdot 2^J C_{R'} C}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \|\varrho(\cdot, \cdot, t)\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1})} \|\mathcal{M}_{|\nabla u_p|}(y_j, t)\|_{L^2(\mathbb{R}^d)} dt \\
&\leq \frac{3 \cdot 2^J C_{R'} C (J+1)}{\epsilon} \int_0^T \|\varrho(\cdot, \cdot, t)\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1})} \|\nabla u_p(\cdot, t)\|_{L^2(\mathbb{R}^d)} dt \\
&\leq \frac{3 \cdot 2^J C_{R'} C (J+1)}{\epsilon} \|\varrho\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1} \times (0, T))} \|\nabla u_p\|_{L^2(0, T; L^2(\mathbb{R}^d))} \\
&\leq \frac{3 \cdot 2^J C_{R'} C (J+1)}{\epsilon} \|\varrho\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1} \times (0, T))} \|\nabla u\|_{L^2(0, T; L^2(\mathbb{R}^d))} \\
&\leq C (\|u_0\|_{L^2(\mathbb{R}^d)} + 1) \|\varrho\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1} \times (0, T))} \\
&\leq C \|\varrho\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1} \times (0, T))} \\
&= C \|\varrho\|_{L^2(\mathcal{D}_{R+1})}.
\end{aligned} \tag{3.33}$$

Now, since from (3.29) we have that

$$|I_1^2| \leq \liminf_{\delta_p \rightarrow 0} I_1^{2,p},$$

we deduce that

$$\begin{aligned}
\|I_1^2\|_{L^1(\mathcal{D}_R)} &= \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} |I_1^2| dv dr dt \\
&\leq \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \liminf_{\delta_p \rightarrow 0} I_1^{2,p} dv dr dt \\
&\leq \liminf_{\delta_p \rightarrow 0} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} I_1^{2,p} dv dr dt \\
&= \liminf_{\delta_p \rightarrow 0} \|I_1^{2,p}\|_{L^1(\mathcal{D}_R)}.
\end{aligned}$$

Finally, from (3.33), we get

$$\|I_1^2\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^2(\mathcal{D}_{R+1})},$$

as has been asserted.

Remark 3.1.8. We note in passing that in bounding $\|I_1^2\|_{L^1(\mathcal{D}_R)}$ we could have used a more direct argument. Indeed, by a standard property of the maximal function (cf., for example, Corollary 4.3 in [2]), we have that if $u \in W^{1,p}(\mathbb{R}^d)$, with $1 \leq p \leq \infty$ then there is a set E of measure zero such that the following inequality holds

$$|u(x) - u(y)| \leq c|x - y|(\mathcal{M}_{\nabla u}(x) + \mathcal{M}_{\nabla u}(y)), \tag{3.34}$$

for all $x, y \in \mathbb{R}^d \setminus E$. Since

$$I_1^2 := \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{1}{\epsilon} (u(r_j, t) - u(y_j, t)) \cdot \nabla \omega_k(v_j - \eta_j) dy d\eta,$$

we have that

$$\|I_1^2\|_{L^1(\mathcal{D}_R)} = \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \left| \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \left. \times \frac{1}{\epsilon} (u(r_j, t) - u(y_j, t)) \cdot \nabla \omega_k(v_j - \eta_j) dy d\eta \right| dv dr dt \\ \leq \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \left. \times \frac{1}{\epsilon} (u(r_j, t) - u(y_j, t)) \cdot \nabla \omega_k(v_j - \eta_j) \right| dy d\eta dv dr dt \\ = \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left| \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \left. \times \frac{1}{\epsilon} |u(r_j, t) - u(y_j, t)| |\nabla \omega_k(v_j - \eta_j)| dy d\eta dv dr dt. \right.$$

From inequality (3.34) it then directly follows that

$$\|I_1^2\|_{L^1(\mathcal{D}_R)} \leq \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\ \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \\ \times \frac{C}{\epsilon} |r_j - y_j| |\mathcal{M}_{\nabla u}(r_j, t) + \mathcal{M}_{\nabla u}(y_j, t)| |\nabla \omega_k(v_j - \eta_j)| dy d\eta dv dr dt$$

$$\begin{aligned}
& \times \left(\int_{\Omega \cap B_R} \left[\omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \right. \\
& \left. \left. \times \left(\mathcal{M}_{\nabla u}(r_j, t) + \mathcal{M}_{\nabla u}(y_j, t) \right) \right] dr_j \right) \left(\int_{B_R} |(\nabla \omega)_k(v_j - \eta_j)| dv_j \right) dy d\eta dt \\
& \leq \frac{3 \cdot 2^J c C}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \left(\int_{\Omega \cap B_R} \left[\omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \right. \\
& \left. \left. \times \left(\mathcal{M}_{\nabla u}(r_j, t) + \mathcal{M}_{\nabla u}(y_j, t) \right) \right] dr_j \right) dy d\eta dt \\
& = \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \left(\int_{\Omega \cap B_R} \left[\omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \right. \right. \\
& \left. \left. \times \mathcal{M}_{\nabla u}(r_j, t) \right] dr_j \right) dy d\eta dt \right. \\
& \left. + \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \left(\int_{\Omega \cap B_R} \omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) dr_j \right) \right. \\
& \left. \times \mathcal{M}_{\nabla u}(y_j, t) dy d\eta dt \right] \\
& \leq \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \left(\int_{\Omega \cap B_R} \left[\omega_k \left(r_j - \frac{2}{k} n(r_j) - y_j \right) \right. \right. \right. \\
& \left. \left. \times \mathcal{M}_{\nabla u}(r_j, t) \right] dr_j \right) dy d\eta dt \right. \\
& \left. + 2 \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \mathcal{M}_{\nabla u}(y_j, t) dy d\eta dt. \right]
\end{aligned}$$

Then, using changes of variable, we get

$$\begin{aligned}
& \|I_1^2\|_{L^1(\mathcal{D}_R)} \\
& \leq \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) \left(\int_{B_{2(R+1)}} \left[\omega_k \left(z_j - \frac{2}{k} n(z_j + y_j) \right) \right. \right. \right. \\
& \left. \left. \times \mathcal{M}_{\nabla u}(z_j + y_j, t) \right] dz_j \right) dy d\eta dt + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{B_{2(R+1)}} \varrho(y, \eta, t) \left(\int_{B_{2(R+1)}} \left[k^d \omega \left(k z_j - 2 n(z_j + y_j) \right) \right. \right. \right. \\
&\quad \left. \left. \left. \times \mathcal{M}_{\nabla u}(z_j + y_j, t) \right] dz_j \right) dy d\eta dt + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right] \\
&= \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{B_{R'}} \varrho(y, \eta, t) \left(\int_{B_{R'}} \left[\omega \left(\xi_j - 2 n \left(\frac{\xi_j}{k} + y_j \right) \right) \right. \right. \right. \\
&\quad \left. \left. \left. \times \mathcal{M}_{\nabla u} \left(\frac{1}{k} \xi_j + y_j, t \right) \right] d\xi_j \right) dy d\eta dt + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right] \\
&\leq \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int_{B_{R'}} \varrho(y, \eta, t) \left(\int_{B_{R'}} \left[\mathcal{M}_{\nabla u} \left(\frac{1}{k} \xi_j + y_j, t \right) \right] d\xi_j \right) dy d\eta dt \right. \\
&\quad \left. + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right].
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\|I_1^2\|_{L^1(\mathcal{D}_R)} &\leq \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int \varrho(y, \eta, t) \right. \\
&\quad \left. \times \left(\int_{B_{\left(y_j, \frac{1}{k} R'\right)}} \mathcal{M}_{\nabla u}(s, t) \frac{ds}{\frac{1}{k^d}} \right) dy d\eta dt + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right] \\
&\leq \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}(\Omega \cap B_{R+1})^{J+1}} \int \varrho(y, \eta, t) \mathcal{M}_{\mathcal{M}_{\nabla u}}(y_j, t) dy d\eta dt \right. \\
&\quad \left. + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right] \\
&\leq \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \|\varrho(\cdot, \cdot, t)\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1})} \|\mathcal{M}_{\mathcal{M}_{\nabla u}}(y_j, t)\|_{L^2(\mathbb{R}^d)} dt \right. \\
&\quad \left. + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{3 \cdot 2^J c C}{\epsilon} \left[\sum_{j=1}^{J+1} \int_0^T \|\varrho(\cdot, \cdot, t)\|_{L^2((\Omega \cap B_{R+1})^{J+1} \times B_{R+1}^{J+1})} \|\mathcal{M}_{\nabla u}(y_j, t)\|_{L^2(\mathbb{R}^d)} dt \right. \\
&\quad \left. + 2C \|\varrho\|_{L^2(\mathcal{D}_{R+1})} \right] \\
&\leq C \|\varrho\|_{L^2(\mathcal{D}_{R+1})}.
\end{aligned}$$

Therefore, we have that

$$\|I_1\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^2(\mathcal{D}_{R+1})},$$

Now, we shall show that $\|I_2\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}$. Indeed,

$$\begin{aligned}
\|I_2\|_{L^1(\mathcal{D}_R)} &= \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \left| \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{\eta_j} \cdot E_j(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\
&\quad \left. \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \varrho(y, \eta, t) dy d\eta \right| dv dr dt \\
&\leq \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} |\partial_{\eta_j} \cdot E_j(y, \eta, t)| \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \\
&\quad \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \varrho(y, \eta, t) dy d\eta dv dr dt \\
&\leq C 2^{J+1} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) |\partial_{\eta_j} \cdot E_j(y, \eta, t)| dy d\eta dt \\
&\leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}.
\end{aligned}$$

In conclusion, we have that

$$\|r_k^1(\varrho)\|_{L^1(\mathcal{D}_R)} \leq C \|\varrho\|_{L^2(\mathcal{D}_{R+1})},$$

as has been asserted. That completes the proof of the lemma. \square

Then, for $\varrho \in L^\infty(0, T; L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$ we argue again by density: we consider a sequence $(\varrho_\epsilon)_{\epsilon>0}$ of smooth functions, such that $\varrho_\epsilon \rightarrow \varrho$ in $L^2_{\text{loc}}(\overline{\mathcal{D}})$, and we write:

$$r_k^1(\varrho) = r_k^1(\varrho_\epsilon) + r_k^1(\varrho - \varrho_\epsilon),$$

which obviously converges to 0 in $L^1_{\text{loc}}(\overline{\mathcal{D}})$ as $k \rightarrow \infty$ and $\epsilon \rightarrow 0$.

Next, we consider the term $r_k^3(\varrho)$. We begin by observing that the following equalities hold:

$$\begin{aligned} r_k^3(\varrho) &= \left(\frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \partial_{v_j}^2 \varrho \right) \star_{r,k} \omega_k \star_v \omega_k - \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \partial_{v_j}^2 \varrho_k \\ &= \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ \partial_{\eta_j}^2 \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right. \\ &\quad \left. - \partial_{v_j}^2 \left[\varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right] \right\} dy d\eta, \end{aligned}$$

and hence, by using an integration by parts on the first integrand, we obtain

$$\begin{aligned} r_k^3(\varrho) &= \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ \varrho(y, \eta, t) \partial_{\eta_j}^2 [\omega_k(v_j - \eta_j)] \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ &\quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) - \varrho(y, \eta, t) \partial_{v_j}^2 [\omega_k(v_j - \eta_j)] \\ &\quad \times \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \left. \right\} dy d\eta, \\ &= \frac{\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ \varrho(y, \eta, t) (\Delta \omega)_k(v_j - \eta_j) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ &\quad \times \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) - \varrho(y, \eta, t) (\Delta \omega)_k(v_j - \eta_j) \\ &\quad \times \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1, m \neq j}^{J+1} \omega_k(v_m - \eta_m) \left. \right\} dy d\eta \\ &= 0. \end{aligned}$$

Having dealt with the terms $r_k^1(\varrho)$, $r_k^3(\varrho)$ and $r_k^4(\varrho)$, we are now left with the task of considering the remaining term, $r_k^2(\varrho)$. We begin by noting that

$$\begin{aligned} r_k^2(\varrho) &= \sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, t)) \varrho_k - \left(\sum_{j=1}^{J+1} (\partial_{v_j} \cdot E_j(r, v, t)) \varrho \right) \star_{r,k} \omega_k \star_v \omega_k \\ &= \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ (\partial_{v_j} \cdot E_j(r, v, t)) \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\ &\quad \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) - \partial_{\eta_j} \cdot E_j(y, \eta, t) \varrho(y, \eta, t) \\ &\quad \times \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \left. \right\} dy d\eta, \end{aligned}$$

which, as long as ϱ and E_j are sufficiently smooth, converges to 0 in L^1_{loc} as k tends to ∞ by standard results on convolutions. The general case then follows by using a density argument using the inequality which we shall next prove. For a constant $C > 0$, independent of k , we have that

$$\begin{aligned}
\|r_k^2(\varrho)\|_{L^1(\mathcal{D}_R)} &= \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \left| \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ [(\partial_{v_j} \cdot E_j(r, v, t)) \right. \right. \\
&\quad \left. \left. - \partial_{\eta_j} \cdot E_j(y, \eta, t)] \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \right. \\
&\quad \left. \left. \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right\} dy d\eta \right| dv dr dt \\
&\leq \frac{1}{\epsilon} \sum_{j=1}^{J+1} \int_0^T \int_{(\Omega \cap B_R)^{J+1}} \int_{B_R^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \left\{ [(\partial_{v_j} \cdot E_j(r, v, t)) \right. \\
&\quad \left. - \partial_{\eta_j} \cdot E_j(y, \eta, t)] \varrho(y, \eta, t) \prod_{l=1}^{J+1} \omega_k \left(r_l - \frac{2}{k} n(r_l) - y_l \right) \right. \\
&\quad \left. \times \prod_{m=1}^{J+1} \omega_k(v_m - \eta_m) \right\} dy d\eta \Big| dy d\eta dv dr dt \\
&\leq C 2^{J+1} \sum_{j=1}^{J+1} \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} |(\partial_{v_j} \cdot E_j(r, v, t)) \\
&\quad - \partial_{\eta_j} \cdot E_j(y, \eta, t)| \varrho(y, \eta, t) dy d\eta dt \\
&\leq C \sum_{j=1}^{J+1} \|\partial_{v_j} \cdot E_j\|_{L^\infty(0, T; L^\infty((\Omega \cap B_{R+1}) \times B_{R+1}))} \\
&\quad \times \int_0^T \int_{B_{R+1}^{J+1}} \int_{(\Omega \cap B_{R+1})^{J+1}} \varrho(y, \eta, t) dy d\eta dt \\
&\leq C \|\varrho\|_{L^1(\mathcal{D}_{R+1})}.
\end{aligned}$$

Then, for $\varrho \in L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$ we argue again by density: we consider a sequence $(\varrho_\epsilon)_\epsilon$ of smooth functions, such that $\varrho_\epsilon \rightarrow \varrho$ in $L^1_{\text{loc}}(\overline{\mathcal{D}})$, and we write:

$$r_k^2(\varrho) = r_k^2(\varrho_\epsilon) + r_k^2(\varrho - \varrho_\epsilon),$$

which obviously converges to 0 in $L^1_{\text{loc}}(\overline{\mathcal{D}})$ as $k \rightarrow \infty$ and $\epsilon \rightarrow 0$. That completes Step 2. \square

STEP 3: PASSING TO THE LIMIT. Thanks to (3.9) we have that $\varrho_k(\cdot, t)$ converges to $\varrho(\cdot, t)$ in $L^1_{\text{loc}}(\mathcal{O})$ for almost all $t \in [0, T]$ and we denote by t_0 such a time. For all $k, l \in \mathbb{N}$ the difference $\varrho_k - \varrho_l$ belongs to $W^{1,1}(0, T; W^{1,\infty}_{\text{loc}}(\mathcal{O}))$ and solves

$$\Lambda_{E_j}(\varrho_k - \varrho_l) = r_k - r_l \quad \text{in} \quad \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)).$$

The estimate (3.7) applied to $\varrho_k - \varrho_l$ and Lemma 3.1.2 imply that, for all compact sets $K \subset \mathcal{O}$, one has

$$\sup_{\tau \in [0, T]} \|(\varrho_k - \varrho_l)(\cdot, \tau)\|_{L^1(K)} \xrightarrow{k, l \rightarrow \infty} 0. \quad (3.35)$$

We then deduce from (3.35) that there exists for all $t \in [0, T]$ a function $\gamma_t \varrho$ such that $\varrho_k(\cdot, t)$ converges to $\gamma_t \varrho$ in $C([0, T]; L^1_{\text{loc}}(\mathcal{O}))$, and in particular from (3.9), by uniqueness of the limit we get

$$\varrho(r, v, t) = \gamma_t \varrho(r, v) \text{ for almost every } (r, v, t) \text{ in } \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T). \quad (3.36)$$

Moreover, for all $t \in [0, T]$ and $R > 0$ we have from (3.9) and (3.36), by lower semi-continuity and thanks to Lebesgue's dominated convergence theorem, since ϱ_k is bounded in $L^\infty(0, T; L^1_{\text{loc}}(\mathcal{O}))$, that

$$\|\gamma_t \varrho\|_{L^1_R} \leq \lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \|\varrho_k(\cdot, t)\|_{L^1_R} = \|\varrho\|_{L^\infty, 1}.$$

We have that $\varrho_k(\cdot, t) = (\gamma_t \varrho) \star_{r, k} \omega_k \star_v \omega_k$ a.e. in \mathcal{O} for all $k \in \mathbb{N}$ and $t \in [0, T]$, and since the two functions $\varrho_k(\cdot, t)$ and $(\gamma_t \varrho) \star_{r, k} \omega_k \star_v \omega_k$ are continuous, this holds everywhere in \mathcal{O} and thus $\varrho_k(\cdot, t) \rightarrow \gamma_t \varrho$ in $L^1_{\text{loc}}(\mathcal{O})$ for all $t \in [0, T]$. We note that $\gamma_t \varrho = \varrho(\cdot, t)$. From (3.35), we deduce that $\varrho \in C([0, T]; L^1_{\text{loc}}(\mathcal{O}))$.

The estimate (3.8) applied to $\varrho_k - \varrho_l$, Lemma 3.1.2 and the convergence (3.9) imply that for all compact subsets $K \subset \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d}$ one has

$$\int_0^T \int_K |\gamma \varrho_k - \gamma \varrho_l| \, d\mu_2(r, v, t) \xrightarrow{k, l \rightarrow \infty} 0. \quad (3.37)$$

We deduce from (3.37) the existence of a function $\gamma \varrho \in L^1_{\text{loc}}(\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times [0, T], d\mu_2)$, which is the limit of $(\gamma \varrho_k)$.

Finally, for a fixed $\varphi \in C_0^\infty(\overline{\Omega^{J+1}} \times \mathbb{R}^{(J+1)d} \times [0, T])$ there exists a constant $C > 0$ such that $|\varphi(r, v, t)| \leq C |n(r_j) \cdot v_j|$ on $\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T)$, to ensure that the integral

$$\sum_{j=1}^{J+1} \int_{t_0}^{t_1} \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} (v_j \cdot n(r_j)) \gamma \varrho \varphi \, dv \, ds(r) \, d\tau,$$

appearing on the right-hand side of (3.5), is finite (since $\gamma \varrho \in L^1_{\text{loc}}(\partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times [0, T], d\mu_2)$, where $d\mu_2 = |n(r_j) \cdot v_j|^2 \, dv \, ds(r) \, d\tau$). Therefore, the Green's formula (3.5) is established by writing it first for ϱ_k and then passing to the limit $k \rightarrow \infty$. Uniqueness of the trace follows from Green's formula. That completes the proof. \square

3.2 Fokker–Planck equation with specular reflection on the boundary

We show in this section that the specular boundary condition is attained in a strong sense by the solution of equation (3.1). In the previous section we showed that $\varrho \in$

$L^\infty(0, T; L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$ is a solution to the problem (3.1), (3.2) in the sense of distributions, i.e.,

$$\int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \varrho \Lambda_{E_j}^*(\varphi) \, dv \, dr \, d\tau = 0, \quad (3.38)$$

for all test functions $\varphi \in W_0^{1,1}(0, T; W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$ with $s > (J+1)d + 1$. Now, we want to prove that the solution ϱ satisfies the following *specular boundary condition* on $\partial\Omega^{(j)}$, $j = 1, \dots, J+1$:

$$\varrho(r, v, t) = \varrho(r, v_*^{(j)}, t) \quad \text{for all } (r, v, t) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T], \text{ with } v \cdot \nu^{(j)}(r) < 0, \quad (3.39)$$

where

$$v_*^{(j)} = v_*^{(j)}(r, v) := v - 2(v \cdot \nu^{(j)}(r)) \nu^{(j)}(r), \quad j = 1, \dots, J+1.$$

To do so, let introduce some notational conventions. We define the field Π_{r_j} of projection operators on the hyperplane, which is orthogonal to $\nu(r_j)$, in such a way that

$$v_j = (\nu(r_j) \cdot v_j) \nu(r_j) + \Pi_{r_j} v_j,$$

and

$$\nu(r_j) \cdot \Pi_{r_j} v_j = 0, \quad \text{for all } v_j \in \mathbb{R}^d.$$

Given three functions $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^{(J+1)d} \times (0, T))$, $\psi \in \mathcal{C}_0^\infty([0, \infty))$ with $\psi(0) = 0$, and $\Psi \in \mathcal{C}_0^\infty(\mathbb{R}^{d-1})$, we set

$$\varphi(r, v, t) = \phi(r, t) \psi((\nu(r_j) \cdot v_j)^2) \Psi(\Pi_{r_j} v_j), \quad (3.40)$$

and we define, following [50], the class \mathcal{RS} (standing for *réflexion spéculaire*) as the space of functions φ which can be expressed in the form (3.40). We now show that φ satisfies the specular boundary condition. By replacing v in (3.40) with $v_*^{(j)}$, we have

$$\varphi(r, v_*^{(j)}, t) = \phi(r, t) \psi((\nu(r_j) \cdot v_*^{(j)})^2) \Psi(\Pi_{r_j} v_*^{(j)}). \quad (3.41)$$

Since

$$\begin{aligned} v_*^{(j)} &= v_j - 2(\nu(r_j) \cdot v_j) \nu(r_j) \\ &= \Pi_{r_j} v_j - (\nu(r_j) \cdot v_j) \nu(r_j), \end{aligned}$$

we have that $\Pi_{r_j} v_*^{(j)} = \Pi_{r_j} v_j$ and $(\nu(r_j) \cdot v_*^{(j)})^2 = (\nu(r_j) \cdot v_j)^2 |\nu(r_j)|^2 = (\nu(r_j) \cdot v_j)^2$. In particular, we get

$$\varphi(r, v_*^{(j)}, t) = \varphi(r, v_j, t).$$

Therefore, thanks to (3.38) and the Green's formula (3.5), the trace $\gamma\varrho$ is well-defined and satisfies

$$\sum_{j=1}^{J+1} \int_0^T \int_{\partial\Omega^{(j)}} \int_{\mathbb{R}^{(J+1)d}} (v_j \cdot \nu(r_j)) \gamma\varrho(r, v, \tau) \varphi(r, v, \tau) \, dv \, ds(r) \, d\tau = 0 \quad \forall \varphi \in \mathcal{RS}. \quad (3.42)$$

Hence, for almost every $(r, t) \in \partial\Omega^{(j)} \times (0, T)$, for all $\tilde{\psi}$ odd, such that $|\tilde{\psi}(z)| \leq C z^2$, for all Ψ , by summing twice the same integral we have that

$$\sum_{j=1}^{J+1} \int_{v' \in \Pi_{r_j}(\mathbb{R}^d)} \int_{v'' \in \mathbb{R}_{\geq 0}} \left[\gamma \varrho(r, v' + v'' \nu(r_j), \tau) + \gamma \varrho(r, v' - v'' \nu(r_j), \tau) \right] \Psi(v') \tilde{\psi}(v'') \, dv'_j \, dv'' = 0. \quad (3.43)$$

Hence, by performing a change of variable in the second integral (v'' becomes $-v''$ and we use the fact that $\tilde{\psi}$ is an odd function), we get

$$\sum_{j=1}^{J+1} \int_{v' \in \Pi_{r_j}(\mathbb{R}^d)} \int_{v'' \in \mathbb{R}_{\geq 0}} \left[\gamma \varrho(r, v' + v'' \nu(r_j), \tau) - \gamma \varrho(r, v' - v'' \nu(r_j), \tau) \right] \Psi(v') \tilde{\psi}(v'') \, dv'_j \, dv'' = 0, \quad (3.44)$$

which is equivalent to $\gamma \varrho(r, v, t) = \gamma \varrho(r, v_*^{(j)}, t)$ for almost every $(r, v, t) \in \partial\Omega^{(j)} \times \mathbb{R}^{(J+1)d} \times (0, T]$, i.e., ϱ satisfies the specular reflection boundary condition (3.39).

Chapter 4

The small-mass limit and equilibration in momentum space

In the previous Chapter we showed the existence of functions $u = u_\epsilon$ and $\widehat{q} = \widehat{q}_\epsilon$, such that

$$u_\epsilon \in \mathcal{C}([0, T]; L^\sigma(\Omega)^d) \cap L^2(0, T; W_0^{1,\sigma}(\Omega)^d) \cap W^{1,2}(0, T; W^{-1,\sigma}(\Omega)^d),$$

with $\sigma = \min(\widehat{\sigma}, z) > d$, $\widehat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, is a weak solution to the Oseen system (1.3), and \widehat{q}_ϵ with

$$\mathcal{F}(\widehat{q}_\epsilon) \in L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

$$\nabla_v \sqrt{\widehat{q}_\epsilon} \in L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$

$$\nabla_v \widehat{q}_\epsilon \in L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{and} \quad M \partial_t \widehat{q}_\epsilon \in L^2(0, T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'),$$

$$s > (J+1)d + 1,$$

satisfies the following weak form of the Fokker–Planck equation: for all $t \in (0, T]$,

$$\begin{aligned} & \int_0^t \langle M \partial_\tau \widehat{q}_\epsilon(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{q}_\epsilon \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{q}_\epsilon \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u_\epsilon(r_j, \tau)) \widehat{q}_\epsilon \cdot \partial_{v_j} \varphi dv dr d\tau \right) = 0 \\ & \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ & s > (J+1)d + 1. \end{aligned} \tag{4.1}$$

Furthermore $\widehat{q}_\epsilon(\cdot, \cdot, 0) = \widehat{q}_0(\cdot, \cdot)$ in the sense of $\mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$, and

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M \widehat{q}_\epsilon(r, v, t) dr dv = \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M \widehat{q}_0(r, v) dr dv = 1 \quad \forall t \in (0, T]. \tag{4.2}$$

In addition, $\widehat{\varrho}_\epsilon$ satisfies the following energy inequality:

$$\begin{aligned} \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_\epsilon(t)) \, dv \, dr + \frac{\beta^2}{2\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_\epsilon|^2}{\widehat{\varrho}_\epsilon} \, dv \, dr \, d\tau \\ \leq C \left[1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \mathcal{F}(\widehat{\varrho}_0) \, dv \, dr \right], \end{aligned} \quad (4.3)$$

where $C = C(\|u_0\|_{W^{1-\frac{2}{\sigma}, \sigma}(\Omega)}, \|b\|_{L^\infty(0, T; L^\infty(\Omega))})$, $\sigma = \min(\widehat{\sigma}, z) > d$, $\widehat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0, 1)$, as in the previous section; in particular, C is independent of $\epsilon > 0$. Motivated by the ideas in [54], the aim of this section is to rigorously identify the small-mass limit of the system, corresponding to passage to the limit $\epsilon \rightarrow 0_+$.

We begin by noting that $(\mathcal{F}(\widehat{\varrho}_\epsilon))_{\epsilon > 0}$ is bounded in $L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$, and the sequence $(\nabla_v \sqrt{\widehat{\varrho}_\epsilon})_\epsilon$ is bounded in $L^2(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. Hence from equation (4.1) we have that the sequence $(M \partial_t \widehat{\varrho}_\epsilon)_{\epsilon > 0}$ is bounded in $L^2(0, T; (W^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})))'$, for $s > (J+1)d + 1$.

We now proceed analogously as in the paragraph following (2.51i). We consider the Maxwellian-weighted Orlicz space $L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, with Young's function $\Phi(r) = \mathcal{F}(1 + |r|)$ (cf. Kufner, John & Fučík [39], Sec. 3.18.2). This has a separable predual $E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, with Young's function $\Psi(r) = e^{|r|} - |r| - 1$; the space $E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ is defined as the closure of all bounded measurable functions in the norm of the Orlicz space $L_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$. As there exists a constant K such that $\mathcal{F}(1 + r) \leq K(1 + \mathcal{F}(r))$ for all $r \geq 0$, it follows from (2.51a) that the sequence $(\mathcal{F}(1 + \widehat{\varrho}_\epsilon))_{\epsilon > 0}$ is bounded in $L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))$. Hence, $\widehat{\varrho}_\epsilon$ is bounded in $L^\infty(0, T; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) = L^\infty(0, T; (E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})))'$. By the Banach–Alaoglu theorem, there exists a subsequence (not indicated) of the sequence $(\widehat{\varrho}_\epsilon)_{\epsilon > 0}$ and a

$$\widehat{\varrho}_{(0)} \in L^\infty(0, T; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$$

$$(\text{whereby also } \mathcal{F}(\widehat{\varrho}_{(0)}) \in L^\infty(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})))$$

(not to be confused with the initial datum $\widehat{\varrho}_0$) such that, as $\epsilon \rightarrow 0_+$,

$$\widehat{\varrho}_\epsilon \rightharpoonup \widehat{\varrho}_{(0)} \geq 0 \quad \text{weakly* in } L^\infty(0, T; L_M^\Phi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) = L^\infty(0, T; (E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})))'. \quad (4.4)$$

As, by definition, $L^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \subset E_M^\Psi(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$, it follows in particular that

$$\begin{aligned} \widehat{\varrho}_\epsilon \rightharpoonup \widehat{\varrho}_{(0)} \quad \text{weakly in } L^p(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \forall p \in [1, \infty), \\ M \partial_t \widehat{\varrho}_\epsilon \rightharpoonup M \partial_t \widehat{\varrho}_{(0)} \quad \text{weakly in } L^2(0, T; (W^{s, 2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})))', \quad s > (J+1)d + 1. \end{aligned} \quad (4.5)$$

$$v_j \widehat{\varrho}_\epsilon \rightharpoonup v_j \widehat{\varrho}_{(0)} \quad \text{weakly in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad j = 1, \dots, J+1,$$

After multiplying (2.48) by ϵ^2 , taking $t = T$, omitting the first and third term from the left-hand side, passing to the limit $\alpha \rightarrow 0_+$ for a fixed $\gamma \in (0, 1]$ using the weak lower-semicontinuity of the second term on the left-hand side, and then passing to the limit $\epsilon \rightarrow 0_+$, noting, as in (2.68), that

$$\|u_\epsilon\|_{L^2(0, T; W^{1, \sigma}(\Omega)) \cap W^{1, 2}(0, T; W^{-1, \sigma}(\Omega))} \leq C(1 + \|u_0\|_{W^{1-\frac{2}{\sigma}, \sigma}(\Omega)}), \quad (4.6)$$

with $\sigma > d$, whereby

$$\|u_\epsilon\|_{L^2(0,T;L^\infty(\Omega))} \leq C(1 + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}),$$

where C is a positive constant independent of ϵ , we have that

$$\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \frac{|\partial_{v_j} \widehat{\varrho}_{(0)}|^2}{\widehat{\varrho}_{(0)} + \gamma} dv dr d\tau \leq 0. \quad (4.7)$$

Hence, $\partial_{v_j} \widehat{\varrho}_{(0)} = 0$ a.e. in $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)$ for all $j \in \{1, \dots, J+1\}$. As $\widehat{\varrho}_{(0)}$ has vanishing weak derivatives with respect to all coordinates of v_j for all $j \in \{1, \dots, J+1\}$ it follows that $\widehat{\varrho}_{(0)}$ is constant with respect to all v_j , $j \in \{1, \dots, J+1\}$. In other words, $\widehat{\varrho}_{(0)}(r, v, t) = \eta(r, t)$ for a function $\eta \in L^\infty(0, T; L^1(\Omega^{J+1}))$, to be determined.

An identical argument to the one following Lemma 2.2.1 implies that

$$\widehat{\varrho}_{(0)} \in \mathcal{C}_w([0, T]; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})).$$

It then follows from (4.2) that

$$\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \widehat{\varrho}_{(0)}(r, v, t) dr dv = 1 \quad \forall t \in (0, T]. \quad (4.8)$$

We deduce from (4.6) that

$$\begin{aligned} u_\epsilon &\rightharpoonup u_{(0)} && \text{weakly in } L^2(0, T; W_0^{1,\sigma}(\Omega)^d) \text{ as } \epsilon \rightarrow 0_+, \quad \sigma > d, \\ u_\epsilon &\rightharpoonup u_{(0)} && \text{weakly in } W^{1,2}(0, T; W^{-1,\sigma}(\Omega)^d) \text{ as } \epsilon \rightarrow 0_+, \quad \sigma > d, \\ u_\epsilon &\rightarrow u_{(0)} && \text{strongly in } L^2(0, T; \mathcal{C}^{0,\gamma}(\overline{\Omega})^d) \text{ as } \epsilon \rightarrow 0_+, \quad 0 < \gamma < 1 - \frac{d}{\sigma}, \quad \sigma > d, \end{aligned} \quad (4.9)$$

where the last result follows, via the Aubin–Lions lemma, thanks to the compact embedding of the Sobolev space $W^{1,\sigma}(\Omega)^d$ into the Hölder space $\mathcal{C}^{0,\gamma}(\overline{\Omega})^d$ for $0 < \gamma < 1 - \frac{d}{\sigma}$, $\sigma > d$. Hence also

$$((\mathcal{L}r)_j + u_\epsilon(r_j, \tau)) \widehat{\varrho}_\epsilon \rightharpoonup ((\mathcal{L}r)_j + u_{(0)}(r_j, \tau)) \widehat{\varrho}_{(0)} \quad \text{weakly in } L^2(0, T; L_M^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad (4.10)$$

for each $j = 1, \dots, J+1$. Using (4.4), (4.5), (4.9) and (4.10) we can now pass to the limit $\epsilon \rightarrow 0_+$ in (4.1) to deduce that, for all $t \in (0, T]$,

$$\begin{aligned} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_{(0)} \cdot \partial_{v_j} \varphi dv dr d\tau &= 0 \\ \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s &> (J+1)d + 1. \end{aligned} \quad (4.11)$$

Thus,

$$\sum_{j=1}^{J+1} \partial_{v_j} \cdot (M(v) \partial_{v_j} \widehat{\varrho}_{(0)}) = 0 \quad \text{in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)). \quad (4.12)$$

By defining

$$\varrho_{(0)} := M \widehat{\varrho}_{(0)} = M \eta, \quad (4.13)$$

with $\eta \in L^\infty(0, T; L^1(\Omega^{J+1}))$, to be determined, it directly follows from (4.12) that

$$\mathcal{L}_0^* \varrho_{(0)} = 0 \quad \text{in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)).$$

Using (4.11), (4.1) can now be rewritten in the following equivalent form: for all $t \in (0, T]$,

$$\begin{aligned} & \epsilon \int_0^t \langle M \partial_\tau \widehat{\varrho}_\epsilon(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \rangle d\tau \\ & + \left(\beta^2 \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \left(\frac{\widehat{\varrho}_\epsilon - \widehat{\varrho}_{(0)}}{\epsilon} \right) \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\ & - \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\epsilon \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\ & - \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u_\epsilon(r_j, \tau)) \widehat{\varrho}_\epsilon \cdot \partial_{v_j} \varphi dv dr d\tau \right) = 0 \\ & \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d + 1. \end{aligned} \quad (4.14)$$

We now continue by performing some formal calculations, where the word ‘formal’ refers to the fact that all manipulations with limits with respect to $\epsilon \rightarrow 0_+$ that we shall encounter will be assumed to be meaningful, without rigorous justification. The purpose of these formal calculations is to illuminate why the partial differential equation satisfied by η is indeed the one that our subsequent rigorous, but less enlightening, argument will ultimately deliver.

First we let $\epsilon \rightarrow 0_+$ in (4.14) and note (4.5) and (4.10) to deduce that, for all $t \in (0, T]$,

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0_+} \left(\beta^2 \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \left(\frac{\widehat{\varrho}_\epsilon - \widehat{\varrho}_{(0)}}{\epsilon} \right) \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\ & - \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_{(0)} \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\ & - \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) ((\mathcal{L}r)_j + u_{(0)}(r_j, \tau)) \widehat{\varrho}_{(0)} \cdot \partial_{v_j} \varphi dv dr d\tau \right) = 0 \\ & \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W_*^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d + 1, \end{aligned} \quad (4.15)$$

and hence, also, for all test functions $\varphi \in \mathcal{C}_0^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$.

We define $\widehat{\varrho}_{(1)} \in \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$ by

$$\widehat{\varrho}_{(1)} := \lim_{\epsilon \rightarrow 0_+} \frac{\widehat{\varrho}_\epsilon - \widehat{\varrho}_{(0)}}{\epsilon},$$

with the limit understood in the sense of $\mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, and let

$$\varrho_{(1)} := M \widehat{\varrho}_{(1)}.$$

By taking $t = T$ in (4.16) passage to the limit $\epsilon \rightarrow 0_+$ yields

$$\mathcal{L}_0^* \varrho_{(1)} = -\mathcal{L}_1(u_{(0)})^* \varrho_{(0)} \quad \text{in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)).$$

Expanding the right-hand side of this equality we have that

$$\mathcal{L}_0^* \varrho_{(1)} = \sum_{j=1}^{J+1} M v_j \cdot \partial_{r_j} \eta + ((\mathcal{L}r)_j + u_{(0)}(r_j, t)) \cdot (\partial_{v_j} M) \eta \quad \text{in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)). \quad (4.17)$$

As $v_j M = -\beta \partial_{v_j} M$, we therefore have that

$$\mathcal{L}_0^* \varrho_{(1)} = - \sum_{j=1}^{J+1} \left(\beta \partial_{r_j} \eta - \eta ((\mathcal{L}r)_j + u_{(0)}(r_j, t)) \right) \cdot \partial_{v_j} M \quad \text{in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)). \quad (4.18)$$

By (1.7), $\mathcal{L}_{0,j}^* (\partial_{v_j} M) = -(\partial_{v_j} M)$, and upon taking the inner product of this d -component equality with the d -component vector field $\beta \partial_{r_j} \eta - \eta ((\mathcal{L}r)_j + u_{(0)}(r_j, t))$, which is, clearly, independent of v_j , and then summing through $j = 1, \dots, J+1$, we deduce that one solution of (4.18) is

$$\sum_{j=1}^{J+1} \left(\beta \partial_{r_j} \eta - \eta ((\mathcal{L}r)_j + u_{(0)}(r_j, t)) \right) \cdot \partial_{v_j} M. \quad (4.19)$$

Therefore the general solution of (4.18) is

$$\begin{aligned} \varrho_{(1)} &= \sum_{j=1}^{J+1} \left(\beta \partial_{r_j} \eta - \eta ((\mathcal{L}r)_j + u_{(0)}(r_j, t)) \right) \cdot \partial_{v_j} M + \eta_{(1)}(r, t) M \\ &= \frac{1}{\beta} \sum_{j=1}^{J+1} M \left(\eta ((\mathcal{L}r)_j + u_{(0)}(r_j, t)) - \beta \partial_{r_j} \eta \right) \cdot v_j + \eta_{(1)}(r, t) M, \end{aligned}$$

where $\eta_{(1)} \in \mathcal{D}'(\Omega^{J+1} \times (0, T))$ is arbitrary, because $\mathcal{L}_0^*(\eta_{(1)} M) = \eta_{(1)} \mathcal{L}_0^*(M) = 0$ thanks to $\mathcal{L}_0^*(M) = 0$. As it will transpire from the calculations that follow, the choice of $\eta_{(1)}$ does not affect η , and $\eta_{(1)}$ will be therefore, ultimately, set to 0. Since M and $\partial_{v_j} M$, $j = 1, \dots, J+1$, belong to the topological vector space \mathcal{S} of rapidly decreasing functions defined on $\mathbb{R}^{(J+1)d}$ (the test space for the Schwarz space \mathcal{S}' of tempered distributions), the structure of $\varrho_{(1)}$ implies that $\varrho_{(1)} \in \mathcal{D}'(\Omega^{J+1} \times (0, T)) \otimes \mathcal{S}$, where the latter is the linear space of all finite linear combinations of products of the form $a(r, t) b(v)$ with $a \in \mathcal{D}'(\Omega^{J+1} \times (0, T))$ and $b \in \mathcal{S}$.

We now define $\widehat{\varrho}_{(2)} \in \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$ by

$$\widehat{\varrho}_{(2)} := \lim_{\epsilon \rightarrow 0_+} \frac{\widehat{\varrho}_\epsilon - \widehat{\varrho}_{(0)} - \epsilon \widehat{\varrho}_{(1)}}{\epsilon^2},$$

with the limit understood to be in $\mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, and let

$$\varrho_{(2)} := M \widehat{\varrho}_{(2)}.$$

Next, we subtract the equality (4.16) from (4.14), divide the difference by ϵ , and use test functions $\varphi \in \mathcal{C}_0^\infty(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, so as to rewrite the resulting equality as one in $\mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, and then pass to the limit $\epsilon \rightarrow 0_+$, noting the definitions of $\widehat{\varrho}_{(0)}$, $\widehat{\varrho}_{(1)}$, $\widehat{\varrho}_{(2)}$, $u_{(0)}$, and defining

$$u_{(1)} := \lim_{\epsilon \rightarrow 0_+} \frac{u_\epsilon - u_{(0)}}{\epsilon} \quad \text{in } \mathcal{D}'(\Omega \times (0, T)).$$

Hence,

$$\begin{aligned} M \partial_t \widehat{\varrho}_{(0)} - \sum_{j=1}^{J+1} \partial_{v_j} \cdot (M \partial_{v_j} \widehat{\varrho}_{(2)}) + \sum_{j=1}^{J+1} M v_j \cdot \partial_{r_j} \widehat{\varrho}_{(1)} + \sum_{j=1}^{J+1} (\mathcal{L}r)_j \cdot \partial_{v_j} (M \widehat{\varrho}_{(1)}) \\ + \sum_{j=1}^{J+1} u_{(1)}(r_j, \cdot) \cdot \partial_{v_j} (M \widehat{\varrho}_{(0)}) + u_{(0)}(r_j, \cdot) \cdot \partial_{v_j} (M \widehat{\varrho}_{(1)}) = 0. \end{aligned}$$

Recalling that, by definition, $\varrho_{(i)} = M \widehat{\varrho}_{(i)}$, $i = 0, 1, 2$, we then have that

$$\begin{aligned} \partial_t \varrho_{(0)} - \mathcal{L}_0^* \varrho_{(2)} + \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho_{(1)} + \sum_{j=1}^{J+1} (\mathcal{L}r)_j \cdot \partial_{v_j} \varrho_{(1)} + \sum_{j=1}^{J+1} u_{(1)}(r_j, \cdot) \cdot \partial_{v_j} \varrho_{(0)} \\ + u_{(0)}(r_j, \cdot) \cdot \partial_{v_j} \varrho_{(1)} = 0. \end{aligned}$$

Equivalently,

$$\begin{aligned} \mathcal{L}_0^* \varrho_{(2)} = \partial_t \varrho_{(0)} + \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho_{(1)} + \sum_{j=1}^{J+1} ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \cdot \partial_{v_j} \varrho_{(1)} \\ + \sum_{j=1}^{J+1} u_{(1)}(r_j, \cdot) \cdot \partial_{v_j} \varrho_{(0)}. \end{aligned} \quad (4.20)$$

Since both $\varrho_{(0)}$ and $\varrho_{(1)}$ belong to $\mathcal{D}'(\Omega^{J+1} \times (0, T)) \otimes \mathcal{S}$, the same is true of the right-hand side of (4.20). It is therefore meaningful to test both sides of (4.20) with $\mathbb{I}(v)$ (considered as an element of \mathcal{S}'); upon noting that

$$\mathcal{S}' \langle \mathbb{I}(v), \mathcal{L}_0^* \varrho_{(2)} \rangle_{\mathcal{S}} = \mathcal{S}' \langle \mathcal{L}_0(\mathbb{I}(v)), \varrho_{(2)} \rangle_{\mathcal{S}} = \mathcal{S}' \langle 0, \varrho_{(2)} \rangle_{\mathcal{S}} = 0$$

because $\mathcal{L}_{0,j}(\mathbb{I}(v_j)) = 0$ for all $j = 1, \dots, J+1$, we arrive at

$$0 = \left\langle \mathbb{I}(v), \partial_t \varrho_{(0)} + \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho_{(1)} + \sum_{j=1}^{J+1} ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \cdot \partial_{v_j} \varrho_{(1)} + \sum_{j=1}^{J+1} u_{(1)}(r_j, \cdot) \cdot \partial_{v_j} \varrho_{(0)} \right\rangle_{\mathcal{S}},$$

as an equality in $\mathcal{D}'(\Omega^{J+1} \times (0, T))$, where ${}_{S'}\langle \cdot, \cdot \rangle_S$ denotes the duality pairing between S' and S . Hence,

$$\begin{aligned} 0 &= (\partial_t \eta) \left\langle \mathbb{I}(v), M \right\rangle_S + \left\langle \mathbb{I}(v), \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho_{(1)} \right\rangle_S \\ &+ \left\langle \mathbb{I}(v), \sum_{j=1}^{J+1} ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \cdot \partial_{v_j} \varrho_{(1)} \right\rangle_S + \left\langle \mathbb{I}(v), \sum_{j=1}^{J+1} u_{(1)}(r_j, \cdot) \cdot \partial_{v_j} \varrho_{(0)} \right\rangle_S, \end{aligned}$$

as an equality in $\mathcal{D}'(\Omega^{J+1} \times (0, T))$. Thanks to the definition of partial derivative of a tempered distribution the last two terms on the right-hand side vanish, while ${}_{S'}\langle \mathbb{I}(v), M \rangle_S = \int_{\mathbb{R}^{(J+1)d}} M(v) dv = 1$, resulting in

$$\partial_t \eta + \left\langle \mathbb{I}(v), \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho_{(1)} \right\rangle_S = 0.$$

In order to simplify the second term on the left-hand side, we consider

$$v_j \cdot \partial_{r_j} \varrho_{(1)} = v_j \cdot \partial_{r_j} \left(\sum_{k=1}^{J+1} \left(\beta \partial_{r_k} \eta - \eta((\mathcal{L}r)_k + u_{(0)}(r_k, \cdot)) \right) \cdot \partial_{v_k} M \right) + v_j \cdot \partial_{r_j} (\eta_{(1)} M).$$

As $v_j \cdot \partial_{r_j} (\eta_{(1)} M) = -\beta \partial_{v_j} (M(\partial_{r_j} \eta_{(1)}))$, we have that ${}_{S'}\langle \mathbb{I}(v), v_j \cdot \partial_{r_j} (\eta_{(1)} M) \rangle_S = 0$ for all $j = 1, \dots, J+1$. Consequently, the precise choice of $\eta_{(1)}$ is immaterial, to the extent that

$$\partial_t \eta + \left\langle \mathbb{I}(v), \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \left(\sum_{k=1}^{J+1} \left(\beta \partial_{r_k} \eta - \eta((\mathcal{L}r)_k + u_{(0)}(r_k, \cdot)) \right) \cdot \partial_{v_k} M \right) \right\rangle_S = 0, \quad (4.21)$$

regardless of the specific choice of $\eta_{(1)}$. Now (with the integral over $\mathbb{R}^{(J+1)d}$ considered below understood as a Gel'fand–Pettis integral of a function with values in a topological vector space, which is in our case $\mathcal{D}'(\Omega^{J+1} \times (0, T))$), we have that

$$\begin{aligned} &{}_{S'} \left\langle \mathbb{I}(v), \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \left(\sum_{k=1}^{J+1} \left(\beta \partial_{r_k} \eta - \eta((\mathcal{L}r)_k + u_{(0)}(r_k, \cdot)) \right) \cdot \partial_{v_k} M \right) \right\rangle_S \\ &= \int_{\mathbb{R}^{(J+1)d}} \sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \left(\sum_{k=1}^{J+1} \left(\beta \partial_{r_k} \eta - \eta((\mathcal{L}r)_k + u_{(0)}(r_k, \cdot)) \right) \cdot \partial_{v_k} M \right) dv \\ &= \sum_{j,k=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} v_j \cdot \partial_{r_j} \left(\left(\beta \partial_{r_k} \eta - \eta((\mathcal{L}r)_k + u_{(0)}(r_k, \cdot)) \right) \cdot \partial_{v_k} M \right) dv \\ &= - \sum_{j,k=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \partial_{v_k} \cdot (A_{jk} v_j) M dv = - \sum_{j=1}^{J+1} \int_{\mathbb{R}^{(J+1)d}} \text{tr}(A_{jj}) M dv = - \sum_{j=1}^{J+1} \text{tr}(A_{jj}), \end{aligned}$$

where

$$\begin{aligned} A_{jk} &:= \partial_{r_j} \left(\beta \partial_{r_k} \eta - \eta((\mathcal{L}r)_k + u_{(0)}(r_k, \cdot)) \right) \in [\mathcal{D}'(\Omega^{J+1} \times (0, T))]^{d \times d}, \\ & \quad j, k = 1, \dots, J+1. \end{aligned}$$

Thus, (4.21) yields the following partial differential equation satisfied by η :

$$\partial_t \eta - \sum_{j=1}^{J+1} \left(\beta \partial_{r_j}^2 \eta - \partial_{r_j} \cdot \left(\eta \left((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot) \right) \right) \right) = 0, \quad \text{in } \mathcal{D}'(\Omega^{J+1} \times (0, T)). \quad (4.22)$$

This is the nonlinear Fokker–Planck equation associated with the McKean–Vlasov diffusion

$$\dot{r}_j = (\mathcal{L}r)_j + u_{(0)}(r_j, t; \eta) + \sqrt{2\beta} \dot{W}_j, \quad (4.23)$$

where $u_{(0)}$ is the limit (cf. (4.9)) of the sequence $(u_\epsilon)_{\epsilon>0}$ defined above. We emphasize here that we are yet to show that $u_{(0)}$ is a solution of the Oseen equation, whose right-hand side is to be identified.

This concludes our formal calculations. The rest of the section is devoted to making the above formal passage to the small-mass limit $\epsilon \rightarrow 0_+$ rigorous, including the rigorous identification of the equation (4.22) satisfied by η .

We shall suppose henceforth that the initial datum ϱ_0 for the Fokker–Planck equation has the following factorized form: $\varrho_0(r, v) = M(v) \widehat{\varrho}_0(r)$, where $\widehat{\varrho}_0$ is a nonnegative function of r only, such that $\int_{\Omega^{J+1}} \widehat{\varrho}_0(r) dr = 1$, and

$$\widehat{\varrho}_0 \in L^2(\Omega^{J+1}; \mathbb{R}_{\geq 0}).$$

Under this hypothesis it directly follows that

$$\widehat{\varrho}_\epsilon \in L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})) \cap L^2(0, T; L^2(\Omega^{J+1}; W_M^{1,2}(\mathbb{R}^{(J+1)d})))$$

and

$$\partial_t \widehat{\varrho}_\epsilon \in L^2(0, T; (W_M^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))').$$

Consequently, by a density argument, (4.1) implies that

$$\begin{aligned} & \int_0^t \left\langle M \partial_\tau \widehat{\varrho}_\epsilon(\cdot, \cdot, \tau), \varphi(\cdot, \cdot, \tau) \right\rangle d\tau + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \partial_{v_j} \widehat{\varrho}_\epsilon \cdot \partial_{v_j} \varphi dv dr d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\epsilon \cdot \partial_{r_j} \varphi dv dr d\tau \right) \\ & - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \left((\mathcal{L}r)_j + u_\epsilon(r_j, \tau) \right) \widehat{\varrho}_\epsilon \cdot \partial_{v_j} \varphi dv dr d\tau \right) = 0 \\ & \forall \varphi \in L^2(0, T; W_{*,M}^{1,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad \forall t \in (0, T]. \end{aligned} \quad (4.24)$$

Further, $\widehat{\varrho}_\epsilon(\cdot, \cdot, 0) = \widehat{\varrho}_0(\cdot, \cdot)$ in the sense of $\mathcal{C}_w([0, T]; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}))$.

The next step in our rigorous passage to the limit $\epsilon \rightarrow 0_+$ in (4.24) is motivated by the proof of Lemma 2 on p.1374 in the work of Carrillo and Goudon [15]. First, we formulate

the ‘macroscopic’ equations satisfied by the moments of ϱ_ϵ . By taking $\varphi(r, v, t) = \phi(r, t)$ with $\phi \in L^2(0, T; W^{1,2}(\Omega^{J+1}))$ in (4.24) and defining

$$\bar{\rho}_\epsilon(r, t) := \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\epsilon(r, v, t) dv$$

and

$$\mathcal{J}_{\epsilon,j}(r, t) := \frac{1}{\epsilon} \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \widehat{\varrho}_\epsilon(r, v, t) dv, \quad j = 1, \dots, J+1,$$

we have that

$$\begin{aligned} & \int_0^t \langle \partial_\tau \bar{\rho}_\epsilon(\cdot, \tau), \phi(\cdot, \tau) \rangle d\tau - \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \mathcal{J}_{\epsilon,j} \cdot \partial_{r_j} \phi dr d\tau = 0 \\ & \forall \phi \in L^2(0, T; W^{1,2}(\Omega^{J+1})), \quad \forall t \in (0, T], \end{aligned} \quad (4.25)$$

subject to the initial condition $\bar{\rho}_\epsilon(\cdot, 0) = \bar{\rho}_0(\cdot) := \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\varrho}_\epsilon(r, v, 0) dv$.

Next, let $v_{i,\ell}$, $\ell = 1, \dots, d$, denote the components of the vector $v_i \in \mathbb{R}^d$, and consider the test functions $\varphi(r, v, t) = \phi(r, t) v_{i,\ell}$ in (4.1), for $i = 1, \dots, J+1$ and $\ell = 1, \dots, d$, and let $\mathcal{J}_{\epsilon,i,\ell}$, for $\ell = 1, \dots, d$, denote the components of the d -component vector-function $\mathcal{J}_{\epsilon,i}$ for $i = 1, \dots, J+1$. Hence, we are led, for $i = 1, \dots, J+1$ and $\ell = 1, \dots, d$, to

$$\begin{aligned} & \epsilon^2 \int_0^t \langle \partial_\tau \mathcal{J}_{\epsilon,i,\ell}, \phi \rangle d\tau + \beta^2 \int_0^t \int_{\Omega^{J+1}} \mathcal{J}_{\epsilon,i,\ell} \phi dr d\tau \\ & - \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \left(\int_{\mathbb{R}^{(J+1)d}} M(v) v_j v_{i,\ell} \widehat{\varrho}_\epsilon dv \right) \cdot (\partial_{r_j} \phi) dr d\tau \\ & - \int_0^t \int_{\Omega^{J+1}} \bar{\rho}_\epsilon ((\mathcal{L}r)_{i,\ell} + u_{\epsilon,\ell}(r_i, \tau)) \phi dr d\tau = 0 \quad \forall \phi \in L^2(0, T; C_0^\infty(\Omega^{J+1})), \\ & \forall t \in (0, T], \end{aligned}$$

where $(\mathcal{L}r)_{i,\ell}$, $\ell = 1, \dots, d$, are the components of the vector-function $(\mathcal{L}r)_i$, and $u_{\epsilon,\ell}$, $\ell = 1, \dots, d$, are the components of the vector-function u_ϵ .

We note that

$$\begin{aligned} \sum_{j=1}^{J+1} v_j v_{i,\ell} \cdot \partial_{r_j} \phi &= \sum_{j=1}^{J+1} \sum_{k=1}^{J+1} v_{j,k} v_{i,\ell} \cdot \partial_{r_{j,k}} \phi = [(v \otimes v) \partial_r \phi]_{i,\ell}, \quad i = 1, \dots, J+1, \\ & \ell = 1, \dots, d, \end{aligned}$$

where v and r are $(J+1)d$ -component vectors, whose components are denoted by $v_{j,k}$, $j = 1, \dots, J+1$, $k = 1, \dots, d$ (or $v_{i,\ell}$, $i = 1, \dots, J+1$, $\ell = 1, \dots, d$), and $r_{j,k}$, $j = 1, \dots, J+1$, $k = 1, \dots, d$, respectively. Thus, by defining the $\mathbb{R}^{(J+1)d} \times \mathbb{R}^{(J+1)d}$ -valued function \mathbb{P}_ϵ by

$$[\mathbb{P}_\epsilon(r, t)]_{i,\ell,j,k} := \int_{\mathbb{R}^{(J+1)d}} M(v) v_{i,\ell} v_{j,k} \widehat{\varrho}_\epsilon(r, v, t) dv,$$

for $i, j = 1, \dots, J + 1$ and $\ell, k = 1, \dots, d$, we have that

$$\begin{aligned}
& \epsilon^2 \int_0^t \langle \partial_\tau \mathcal{J}_{\epsilon, i, \ell}, \phi \rangle \, d\tau + \beta^2 \int_0^t \int_{\Omega^{J+1}} \mathcal{J}_{\epsilon, i, \ell} \phi \, dr \, d\tau \\
& - \int_0^t \int_{\Omega^{J+1}} [\mathbb{P}_\epsilon \partial_r \phi]_{i, \ell} \, dr \, d\tau \\
& - \int_0^t \int_{\Omega^{J+1}} \bar{\rho}_\epsilon ((\mathcal{L}r)_{i, \ell} + u_{\epsilon, \ell}(r_i, \tau)) \phi \, dr \, d\tau = 0 \\
& \forall \phi \in L^2(0, T; \mathcal{C}_0^\infty(\Omega^{J+1})), \forall t \in (0, T], \tag{4.26}
\end{aligned}$$

where $[\mathbb{P}_\epsilon \partial_r \phi]_{i, \ell} := \sum_{j=1}^{J+1} \sum_{k=1}^d [\mathbb{P}_\epsilon]_{i, \ell, j, k} \partial_{r_j, k} \phi$.

Lemma 4.0.1. *Let $0 < T < \infty$ and $0 < \epsilon < 1$; then, the following properties hold:*

- (i) *The sequence $(u_\epsilon)_{\epsilon > 0}$ is bounded in $L^2(0, T; \mathcal{C}^{0, \gamma}(\bar{\Omega})^d)$, with $0 < \gamma < 1 - \frac{d}{\sigma}$, $\sigma > d$, $d = 2, 3$, and therefore also in $L^2(0, T; L^\infty(\Omega)^d)$;*
- (ii) *$(\bar{\varrho}_\epsilon)_{\epsilon > 0}$ and $(\mathcal{L}r \bar{\varrho}_\epsilon)_{\epsilon > 0}$ are bounded in $L^\infty(0, T; L^2(\Omega^{J+1}))$ and $L^\infty(0, T; L^2(\Omega^{J+1})^{(J+1)d})$, respectively;*
- (iii) *Consider the $(J + 1)d$ -component vector-function $\bar{\rho}_\epsilon u_\epsilon$, whose components are*

$$\bar{\rho}_\epsilon(r_1, \dots, r_{J+1}, t) u_\epsilon(r_i, t), \quad \text{for } i = 1, \dots, J + 1.$$

Then, $(\bar{\varrho}_\epsilon u_\epsilon)_{\epsilon > 0}$ is bounded in $L^2(0, T; L^2(\Omega^{(J+1)d})^{(J+1)d})$;

- (iv) *The sequences of dissipation terms*

$$\left(\frac{1}{\sqrt{\epsilon M(v)}} \left(\frac{v_j}{\beta} \varrho_\epsilon - \sqrt{\epsilon} M(v) u_\epsilon(r_j, \cdot) + \partial_{v_j} \varrho_\epsilon \right) \right)_{\epsilon > 0}, \quad j = 1, \dots, J + 1,$$

are bounded in $L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))^d$;

- (v) *The sequences $(\mathcal{J}_{\epsilon, j})_{\epsilon > 0}$, $j = 1, \dots, J + 1$, are bounded in $L^2(\Omega^{J+1} \times (0, T))^d$;*

- (vi) *\mathbb{P}_ϵ can be expressed as $\mathbb{P}_\epsilon = \beta \bar{\varrho}_\epsilon \mathbb{I} + \sqrt{\epsilon} \mathbb{R}_\epsilon$, with $(\mathbb{R}_\epsilon)_{\epsilon > 0}$ bounded in $L^2(\Omega^{J+1} \times (0, T))^{(J+1)d \times (J+1)d}$, and $\mathbb{I}_{i, \ell, j, k} := \delta_{i, j} \delta_{\ell, k}$ for $i, j = 1, \dots, J + 1$ and $\ell, k = 1, \dots, d$.*

Proof. (i) In the previous section we showed that

$$u_\epsilon \in \mathcal{C}([0, T]; L^\sigma(\Omega)^d) \cap L^2(0, T; W_0^{1, \sigma}(\Omega)^d), \quad \text{with } \sigma > d,$$

and $(u_\epsilon)_{\epsilon > 0}$ is a bounded sequence in the norms of the function spaces appearing on the right-hand side of this inclusion. Hence, using Morrey's inequality, we readily deduce (i).

- (ii) The Cauchy–Schwarz inequality implies the following bound:

$$|\bar{\varrho}_\epsilon(r, t)|^2 \leq \left(\int_{\mathbb{R}^{(J+1)d}} M(v) \, dv \right) \left(\int_{\mathbb{R}^{(J+1)d}} |\widehat{\varrho}_\epsilon(r, v, t)|^2 M(v) \, dv \right),$$

which then implies (ii), since $\widehat{\varrho}_\epsilon = \varrho_\epsilon/M$ is bounded in the function space

$$L^\infty(0, T; L_M^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \cap L^2(0, T; L^2(\Omega^{J+1}; W_M^{1,2}(\mathbb{R}^{(J+1)d}))),$$

and $|\mathcal{L}r| \leq C$ for all $r \in \overline{\Omega^{J+1}}$, where C is a positive constant, independent of ϵ .

(iii) Finally, we have that

$$\begin{aligned} \int_0^T \int_{\Omega^{J+1}} |\overline{\varrho}_\epsilon u_\epsilon|^2 dr dt &\leq \int_0^T \|\overline{\varrho}_\epsilon(\cdot, t)\|_{L^2(\Omega^{J+1})}^2 \|u_\epsilon(\cdot, t)\|_{L^\infty(\Omega)}^2 dt \\ &\leq \|\overline{\varrho}_\epsilon\|_{L^\infty(0, T; L^2(\Omega^{J+1}))}^2 \int_0^T \|u_\epsilon(\cdot, t)\|_{L^\infty(\Omega)}^2 dt, \end{aligned}$$

which proves (iii) by using (i) and (ii).

(iv) Now, let us show that the sequence $(D_{\epsilon, j})_{\epsilon > 0}$ is bounded in $L^2(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))^d$ for each $j = 1, \dots, J+1$. On the one hand, we know that $(\widehat{\varrho}_\epsilon)_{\epsilon > 0}$ is a bounded sequence in the function space $L^\infty(0, T; L^2(\Omega^{J+1}; L_M^2(\mathbb{R}^{(J+1)d}))) \cap L^2(0, T; L^2(\Omega^{J+1}; W_M^{1,2}(\mathbb{R}^{(J+1)d})))$; in particular,

$$\begin{aligned} &\frac{\beta^2}{\epsilon^2} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left| \frac{\partial_{v_j} \varrho_\epsilon}{M(v)} + \frac{1}{\beta} \frac{v_j \varrho_\epsilon}{M(v)} \right|^2 M(v) dv dr dt \\ &= \frac{\beta^2}{\epsilon^2} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |\partial_{v_j} \widehat{\varrho}_\epsilon|^2 M(v) dv dr dt \leq C, \end{aligned} \quad (4.27)$$

where C is a positive constant, independent of ϵ .

On the other hand, we write

$$\begin{aligned} &\int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left| \frac{\partial_{v_j} \varrho_\epsilon}{M(v)} + \frac{1}{\beta} \frac{v_j \varrho_\epsilon}{M(v)} - \sqrt{\epsilon} u_\epsilon(r_j, \cdot) \right|^2 M(v) dv dr dt \\ &\leq 2 \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left| \frac{\partial_{v_j} \varrho_\epsilon}{M(v)} + \frac{1}{\beta} \frac{v_j \varrho_\epsilon}{M(v)} \right|^2 M(v) dv dr dt \\ &\quad + 2 \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |\sqrt{\epsilon} u_\epsilon(r_j, t)|^2 M(v) dv dr dt \\ &\leq C\epsilon^2 + 2|\Omega|^J \epsilon \int_0^T \int_{\Omega} |u_\epsilon(r_j, t)|^2 dr_j dt. \end{aligned}$$

Thus, using (i) and (4.27) it follows that, for each $j = 1, \dots, J+1$,

$$\int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |D_{\epsilon, j}|^2 dv dr dt \leq C\epsilon,$$

where C is a positive constant, independent of ϵ , which completes the proof of (iv).

(v) Next, we have that, since,

$$\int_{\mathbb{R}^{(J+1)d}} M(v) v_j \, dv = 0, \quad j = 1, \dots, J+1,$$

also

$$\int_{\mathbb{R}^{(J+1)d}} \bar{\varrho}_\epsilon(r, t) M(v) v_j \, dv = \bar{\varrho}_\epsilon(r, t) \int_{\mathbb{R}^{(J+1)d}} M(v) v_j \, dv = 0, \quad j = 1, \dots, J+1.$$

Therefore, by the Cauchy–Schwarz inequality and a Poincaré–Sobolev inequality with a Gaussian weight function,¹ we have that

$$\begin{aligned} |\mathcal{J}_{\epsilon,j}(r, t)|^2 &= \left| \int_{\mathbb{R}^{(J+1)d}} \frac{v_j}{\epsilon} \varrho_\epsilon(r, v, t) \, dv \right|^2 = \frac{1}{\epsilon^2} \left| \int_{\mathbb{R}^{(J+1)d}} v_j (\varrho_\epsilon(r, v, t) - \bar{\varrho}_\epsilon(r, t) M(v)) \, dv \right|^2 \\ &\leq \frac{\bar{\varrho}_\epsilon^2(r, t)}{\epsilon^2} \left(\int_{\mathbb{R}^{(J+1)d}} |v_j|^2 M(v) \, dv \right) \left(\int_{\mathbb{R}^{(J+1)d}} \left| \frac{\varrho_\epsilon(r, v, t)}{\bar{\varrho}_\epsilon(r, t) M(v)} - 1 \right|^2 M(v) \, dv \right) \\ &\leq \frac{\bar{\varrho}_\epsilon^2(r, t)}{\epsilon^2} \left(\int_{\mathbb{R}^{(J+1)d}} |v_j|^2 M(v) \, dv \right) \left(\int_{\mathbb{R}^{(J+1)d}} \left| \nabla_v \left(\frac{\varrho_\epsilon(r, v, t)}{\bar{\varrho}_\epsilon(r, t) M(v)} \right) \right|^2 M(v) \, dv \right) \\ &= \frac{1}{\epsilon^2} \left(\int_{\mathbb{R}^{(J+1)d}} |v_j|^2 M(v) \, dv \right) \left(\int_{\mathbb{R}^{(J+1)d}} \left| \nabla_v \left(\frac{\varrho_\epsilon(r, v, t)}{M(v)} \right) \right|^2 M(v) \, dv \right), \\ &\quad j = 1, \dots, J+1. \end{aligned}$$

Hence,

$$|\mathcal{J}_{\epsilon,j}(r, t)|^2 \leq \frac{C}{\epsilon^2} \int_{\mathbb{R}^{(J+1)d}} M |\nabla_v \hat{\varrho}_\epsilon(r, v, t)|^2 \, dv, \quad j = 1, \dots, J+1,$$

where C is a positive constant, independent of ϵ . Therefore,

$$\begin{aligned} \int_0^T \int_{\Omega^{J+1}} |\mathcal{J}_{\epsilon,j}(r, t)|^2 \, dr \, dt &\leq \frac{C}{\epsilon^2} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M |\nabla_v \hat{\varrho}_\epsilon(r, v, t)|^2 \, dv \, dr \, dt \leq C, \\ &\quad j = 1, \dots, J+1, \end{aligned}$$

where C is a positive constant, independent of ϵ . That completes the proof of (v).

(vi) We recall from part (iv) the definition of $D_{\epsilon,j}$, $j = 1, \dots, J+1$, and denote its k -th component by $D_{\epsilon,j,k}$, $k = 1, \dots, d$. Analogously, let $u_{\epsilon,k}$ denote the k -th component of u_ϵ ,

¹See p.941 in Nash [52], p.533 in Chernoff [16], and p.397 in Beckner [11].

$k = 1, \dots, d$. We then have that

$$\begin{aligned} [\mathbb{P}_\epsilon(r, t)]_{i, \ell, j, k} &= \int_{\mathbb{R}^{(J+1)d}} v_{i, \ell} v_{j, k} \varrho_\epsilon \, dv \\ &= \beta \sqrt{\epsilon} \int_{\mathbb{R}^{(J+1)d}} \left(v_{i, \ell} \sqrt{M(v)} \frac{D_{\epsilon, j, k}(r, v, t)}{\sqrt{\epsilon}} \right) \, dv \\ &\quad + \beta \sqrt{\epsilon} \int_{\mathbb{R}^{(J+1)d}} (v_{i, \ell} u_{\epsilon, k}(r_j, t)) M(v) \, dv - \beta \int_{\mathbb{R}^{(J+1)d}} v_{i, \ell} \partial_{v_{j, k}} \varrho_\epsilon \, dv. \end{aligned}$$

Focusing on the first two integrals, we define

$$[\mathbb{R}_\epsilon(r, t)]_{i, \ell, j, k} := \beta \int_{\mathbb{R}^{(J+1)d}} \left(v_{i, \ell} \sqrt{M(v)} \frac{D_{\epsilon, j, k}(r, v, t)}{\sqrt{\epsilon}} \right) \, dv + \beta \int_{\mathbb{R}^{(J+1)d}} (v_{i, \ell} u_{\epsilon, k}(r_j, t)) M(v) \, dv.$$

The last of the three integrals in the expression for \mathbb{P}_ϵ is equal to $\delta_{i, j} \delta_{\ell, k} \bar{\varrho}_\epsilon$ by partial integration. Hence,

$$[\mathbb{P}_\epsilon(r, t)]_{i, \ell, j, k} = \beta \delta_{i, j} \delta_{\ell, k} \bar{\varrho}_\epsilon + \sqrt{\epsilon} [\mathbb{R}_\epsilon(r, t)]_{i, \ell, j, k}.$$

To complete the proof of (vi) it therefore remains to establish a uniform (with respect to ϵ) bound on $[\mathbb{R}_\epsilon]_{i, \ell, j, k}$ in the norm of $L^2(\Omega^{J+1} \times (0, T))$, for $i, j = 1, \dots, J+1$ and $\ell, k = 1, \dots, d$.

We have that

$$\begin{aligned} &\frac{1}{\beta^2} \int_0^T \int_{\Omega^{J+1}} |[\mathbb{R}_\epsilon]_{i, \ell, j, k}|^2 \, dr \, dt \\ &= \int_0^T \int_{\Omega^{J+1}} \left| \int_{\mathbb{R}^{(J+1)d}} \left(v_{i, \ell} \sqrt{M(v)} \frac{D_{\epsilon, j, k}(r, v, t)}{\sqrt{\epsilon}} \right) \, dv \right. \\ &\quad \left. + \int_{\mathbb{R}^{(J+1)d}} (v_{i, \ell} u_{\epsilon, k}(r_j, t)) M(v) \, dv \right|^2 \, dr \, dt \\ &\leq 2 \int_0^T \int_{\Omega^{J+1}} \left| \int_{\mathbb{R}^{(J+1)d}} \left(v_{i, \ell} \sqrt{M(v)} \frac{D_{\epsilon, j, k}(r, v, t)}{\sqrt{\epsilon}} \right) \, dv \right|^2 \, dr \, dt \\ &\quad + 2 \int_0^T \int_{\Omega^{J+1}} \left| \int_{\mathbb{R}^{(J+1)d}} (v_{i, \ell} u_{\epsilon, k}(r_j, t)) M(v) \, dv \right|^2 \, dr \, dt \\ &\leq 2 \int_0^T \int_{\Omega^{J+1}} \left(\int_{\mathbb{R}^{(J+1)d}} |v_{i, \ell}|^2 M(v) \, dv \right) \left(\int_{\mathbb{R}^{(J+1)d}} \left| \frac{D_{\epsilon, j, k}(r, v, t)}{\sqrt{\epsilon}} \right|^2 \, dv \right) \, dr \, dt \\ &\quad + 2 \left(\int_{\mathbb{R}^{(J+1)d}} |v_{i, \ell}| M(v) \, dv \right)^2 \left(\int_0^T \int_{\Omega^{J+1}} |u_{\epsilon, k}(r_j, t)|^2 \, dr \, dt \right) \end{aligned}$$

$$\leq C \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left| \frac{D_{\epsilon,j,k}(r,v,t)}{\sqrt{\epsilon}} \right|^2 dv dr dt + C |\Omega|^J \int_0^T \int_{\Omega} |u_{\epsilon,k}(r_j,t)|^2 dr_j dt,$$

where C is a positive constant, since moments of any order of M are finite. Thus, the statement in part (vi) of the lemma follows from the assertions in parts (iv) and (i). \square

Using the equations (4.25), (4.26) together with the splitting of \mathbb{P}_ϵ introduced in part (vi) of Lemma 4.0.1, we arrive at the following system of moment equations:

$$\begin{cases} \partial_t \bar{\varrho}_\epsilon + \operatorname{div}_r \mathcal{J}_\epsilon = 0, \\ \beta \partial_r \bar{\varrho}_\epsilon = \sqrt{\epsilon} (-\epsilon \sqrt{\epsilon} \partial_t \mathcal{J}_\epsilon - \operatorname{div}_r \mathbb{R}_\epsilon) + \bar{\varrho}_\epsilon u_\epsilon - \beta \mathcal{J}_\epsilon + \mathcal{L}r \bar{\varrho}_\epsilon. \end{cases} \quad (4.28)$$

To proceed, we require the following Div-Curl Lemma.

Lemma 4.0.2. *Div-Curl Lemma.* Let \mathcal{S} be a bounded open Lipschitz domain in \mathbb{R}^N , and let

$$\begin{aligned} u_1^n, \dots, u_N^n &\rightharpoonup u_1, \dots, u_N && \text{weakly in } L^2(\mathcal{S}), \\ v_1^n, \dots, v_N^n &\rightharpoonup v_1, \dots, v_N && \text{weakly in } L^2(\mathcal{S}). \end{aligned}$$

If $\operatorname{div}(u^n)$ is bounded in $L^2(\mathcal{S})$ and if $\operatorname{curl}(v^n)$ is bounded in $W^{-1,2}(\mathcal{S})^{N^2}$ then

$$\sum_{i=1}^N u_i^n v_i^n \rightharpoonup \sum_{i=1}^N u_i v_i \quad \text{in the sense of distributions.}$$

Lemma 4.0.3. The sequence $(\bar{\varrho}_\epsilon)_\epsilon$ converges to $\bar{\varrho} = \eta$ weakly in the space $L^2(\Omega^{J+1} \times (0, T))$ and strongly in $L^p(\Omega^{J+1} \times (0, T))$ for all $p \in [1, 2)$. Furthermore, we have that

$$\lim_{\epsilon \rightarrow 0^+} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |\varrho_\epsilon - \bar{\varrho} M(v)| dv dr dt = 0.$$

Proof. We begin by focusing on the first equation in the system (4.28). We observe that the sequence $(\operatorname{div}_{(r,t)}(\mathcal{J}_\epsilon, \bar{\varrho}_\epsilon))_{\epsilon>0}$ (where $\operatorname{div}_{(r,t)}(\mathcal{J}_\epsilon, \bar{\varrho}_\epsilon)$ is the divergence with respect to the (r, t) variables of the vector field $(\mathcal{J}_\epsilon, \bar{\varrho}_\epsilon)$, defined as $(\operatorname{div}_r, \partial_t) \cdot (\mathcal{J}_\epsilon, \bar{\varrho}_\epsilon)$), is, thanks to (4.28)₁, the zero-sequence $(0)_{\epsilon>0}$, and it is therefore, trivially, precompact in $W^{-1,2}(\Omega^{J+1} \times (0, T))$.

Next, we focus on the second equation in the system (4.28), which we restate here for clarity:

$$\beta \partial_r \bar{\varrho}_\epsilon = \sqrt{\epsilon} (-\epsilon \sqrt{\epsilon} \partial_t \mathcal{J}_\epsilon - \operatorname{div}_r \mathbb{R}_\epsilon) + \bar{\varrho}_\epsilon u_\epsilon - \beta \mathcal{J}_\epsilon + \mathcal{L}r \bar{\varrho}_\epsilon. \quad (4.29)$$

Thanks to parts (iii), (v) and (ii) of Lemma 4.0.1 the sequence $(\bar{\varrho}_\epsilon u_\epsilon(r_j, \cdot) - \beta \mathcal{J}_{\epsilon,j} + (\mathcal{L}r)_j \bar{\varrho}_\epsilon)_{\epsilon>0}$ is bounded in the function space $L^2(\Omega^{J+1} \times (0, T))^d$, and therefore, thanks to the compact embedding of the space $L^2(\Omega^{J+1} \times (0, T))$ into $W^{-1,2}(\Omega^{J+1} \times (0, T))$, the sequence $(\bar{\varrho}_\epsilon u_\epsilon(r_j, \cdot) - \beta \mathcal{J}_{\epsilon,j} + (\mathcal{L}r)_j \bar{\varrho}_\epsilon)_{\epsilon>0}$ is a precompact set in the space $W^{-1,2}(\Omega^{J+1} \times (0, T))^d$, for each $j = 1, \dots, J+1$.

Furthermore, by parts (v) and (vi) of Lemma 4.0.1 we also have that the sequences $(\mathcal{J}_\epsilon)_{\epsilon>0}$ and $(\mathbb{R}_\epsilon)_{\epsilon>0}$ are bounded in the function spaces $L^2(\Omega^{J+1} \times (0, T))^{(J+1)d}$ and $W^{-1,2}(\Omega^{J+1} \times (0, T))^{(J+1)d \times (J+1)d}$, respectively; therefore, the sequence $(-\epsilon \sqrt{\epsilon} \partial_t \mathcal{J}_\epsilon -$

$\operatorname{div}_r \mathbb{R}_\epsilon)_{\epsilon>0}$ is bounded in $W^{-1,2}(\Omega^{J+1} \times (0, T))^{(J+1)d}$, whereby, upon multiplication by $\sqrt{\epsilon}$, we have that the sequence $(\sqrt{\epsilon}(-\epsilon\sqrt{\epsilon}\partial_t \mathcal{J}_\epsilon - \operatorname{div}_r \mathbb{R}_\epsilon))_{\epsilon>0}$ is precompact in the space $W^{-1,2}(\Omega^{J+1} \times (0, T))^{(J+1)d}$; more precisely, it converges to 0 in $W^{-1,2}(\Omega^{J+1} \times (0, T))^{(J+1)d}$, as $\epsilon \rightarrow 0_+$. Thus, since $\beta > 0$, we deduce from (4.29) that the sequence $(\partial_r \bar{\varrho}_\epsilon)_{\epsilon>0}$ is precompact in $W^{-1,2}(\Omega^{J+1} \times (0, T))^{(J+1)d}$. Hence, the sequence $(\operatorname{curl}_{(r,t)}(0, \bar{\varrho}_\epsilon))_{\epsilon>0}$ (where $\operatorname{curl}_{(r,t)}(0, \bar{\varrho}_\epsilon)$ is the curl with respect to the (r, t) variables, defined as $\partial_{(r,t)} - \partial_{(r,t)}^T$, of the $((J+1)d+1)$ -component vector field $(0, \bar{\varrho}_\epsilon)$, where 0 is a $(J+1)d$ -component zero-vector), is a precompact set in $W^{-1,2}(\Omega^{J+1} \times (0, T))^{((J+1)d+1) \times ((J+1)d+1)}$.

Hence, with $\mathcal{S} = \Omega^{J+1} \times (0, T)$, a direct application of the Div-Curl Lemma (cf. Lemma 4.0.2) yields that the weak limit of the scalar product of the sequences $((\mathcal{J}_\epsilon, \bar{\varrho}_\epsilon))_{\epsilon>0}$ and $((0, \bar{\varrho}_\epsilon))_{\epsilon>0}$ is equal to the scalar product of their weak limits; i.e.,

$$(\mathcal{J}_\epsilon, \bar{\varrho}_\epsilon) \cdot (0, \bar{\varrho}_\epsilon) = \bar{\varrho}_\epsilon^2 \rightharpoonup (\mathcal{J}, \bar{\varrho}) \cdot (0, \bar{\varrho}) = \bar{\varrho}^2 \quad \text{in } \mathcal{D}'(\Omega^{J+1} \times (0, T)).$$

Combining this with the weak convergence result $\bar{\varrho}_\epsilon \rightharpoonup \bar{\varrho}$ in $L^2(\Omega^{J+1} \times (0, T))$, we have that

$$\begin{aligned} & \int_{\Omega^{J+1} \times (0, T)} |\bar{\varrho}_\epsilon - \bar{\varrho}|^2 \phi \, dr \, dt \\ &= \int_{\Omega^{J+1} \times (0, T)} [\bar{\varrho}_\epsilon]^2 \phi \, dr \, dt + \int_{\Omega^{J+1} \times (0, T)} [\bar{\varrho}]^2 \phi \, dr \, dt - 2 \int_{\Omega^{J+1} \times (0, T)} \bar{\varrho}_\epsilon \bar{\varrho} \phi \, dr \, dt \\ &= \langle [\bar{\varrho}_\epsilon]^2, \phi \rangle + \langle [\bar{\varrho}]^2, \phi \rangle - 2 \langle \bar{\varrho}_\epsilon, \bar{\varrho} \phi \rangle \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0_+, \text{ for all } \phi \in \mathcal{C}_0^\infty(\Omega^{J+1} \times (0, T)). \end{aligned}$$

This proves the strong convergence of $\bar{\varrho}_\epsilon$ to $\bar{\varrho}$ in $L^2_{\text{loc}}(\Omega^{J+1} \times (0, T))$. Thus, for any compact subset \mathfrak{D} of $\Omega^{J+1} \times (0, T)$, we can extract a subsequence from the sequence $(\bar{\varrho}_\epsilon)_{\epsilon>0}$ that converges to $\bar{\varrho}$ a.e. on \mathfrak{D} . Hence, by considering a countable nested family of compact sets $\mathfrak{D}_j \subset \Omega^{J+1} \times (0, T)$ with $\cup_{j \geq 1} \mathfrak{D}_j = \Omega^{J+1} \times (0, T)$, by successive extraction of subsequences, there exists a subsequence of $(\bar{\varrho}_\epsilon)_{\epsilon>0}$ (not indicated), which converges to $\bar{\varrho}$ a.e. on $\Omega^{J+1} \times (0, T)$.

By combining the weak convergence $\bar{\varrho}_\epsilon \rightharpoonup \bar{\varrho}$ in $L^2(\Omega^{J+1} \times (0, T))$ (which implies the weak convergence $\bar{\varrho}_\epsilon \rightharpoonup \bar{\varrho}$ in $L^1(\Omega^{J+1} \times (0, T))$), and thereby, thanks to the Dunford–Pettis theorem (cf. Theorem 2.54 in [26]), equiintegrability of $(\bar{\varrho}_\epsilon)_{\epsilon>0}$ on $\Omega^{J+1} \times (0, T)$ and the a.e. convergence of $\bar{\varrho}_\epsilon$ to $\bar{\varrho}$, Vitali’s convergence theorem (cf. Theorem 2.24 in [26]) yields the strong convergence of $\bar{\varrho}_\epsilon$ to $\bar{\varrho}$ in $L^1(\Omega^{J+1} \times (0, T))$, and therefore, thanks to the boundedness of the sequence $\bar{\varrho}_\epsilon$ in $L^p(\Omega^{J+1} \times (0, T))$, $1 \leq p \leq 2$, we have strong convergence $\bar{\varrho}_\epsilon \rightarrow \bar{\varrho}$ in $L^p(\Omega^{J+1} \times (0, T))$ for all $p \in [1, 2)$.

Next, by the triangle inequality and noting that $\int_{\mathbb{R}^{(J+1)d}} M(v) \, dv = 1$, we have that

$$\begin{aligned} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |\varrho_\epsilon - M(v)\bar{\varrho}| \, dv \, dr \, dt &\leq \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |\varrho_\epsilon - M(v)\bar{\varrho}_\epsilon| \, dv \, dr \, dt \\ &\quad + \int_0^T \int_{\Omega^{J+1}} |\bar{\varrho} - \bar{\varrho}_\epsilon| \, dr \, dt. \end{aligned}$$

We have already shown that the second integral on the right-hand side of this inequality tends to 0 as ϵ tends to 0. For the first integral, using the Cauchy–Schwarz inequality and

a Poincaré–Sobolev inequality with a Gaussian weight function (cf. the proof of item (v) in Lemma 4.0.1), we obtain

$$\begin{aligned} \int_{\mathbb{R}^{(J+1)d}} |\varrho_\epsilon - M(v)\bar{\varrho}_\epsilon| dv &\leq \left(\int_{\mathbb{R}^{(J+1)d}} |\varrho_\epsilon - M(v)\bar{\varrho}_\epsilon|^2 \frac{1}{M(v)} dv \right)^{\frac{1}{2}} \\ &\leq \bar{\varrho}_\epsilon \left(\int_{\mathbb{R}^{(J+1)d}} \left| \frac{\varrho_\epsilon}{\bar{\varrho}_\epsilon M(v)} - 1 \right|^2 M(v) dv \right)^{\frac{1}{2}} \\ &\leq \left(\int_{\mathbb{R}^{(J+1)d}} \left| \nabla_v \left(\frac{\varrho_\epsilon}{M(v)} \right) \right|^2 M(v) dv \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\left(\int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} \left| \nabla_v \left(\frac{\varrho_\epsilon}{M(v)} \right) \right|^2 M(v) dv dr dt \right)^{\frac{1}{2}} \leq C\epsilon,$$

we deduce by the Cauchy–Schwarz inequality that

$$\int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |\varrho_\epsilon - M(v)\bar{\varrho}_\epsilon| dv dr dt \leq C\epsilon,$$

and therefore,

$$\lim_{\epsilon \rightarrow 0_+} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} |\varrho_\epsilon - M(v)\bar{\varrho}_\epsilon| dv dr dt = 0.$$

That completes the proof of the lemma. \square

Remark 4.0.4. *The strong convergence $\varrho_\epsilon \rightarrow M(v)\bar{\varrho} = M(v)\eta = \rho_{(0)}$ in $L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$ in the small-mass limit $\epsilon \rightarrow 0_+$, which we have rigorously proved above, is referred to in the chemical physics literature as equilibration in momentum space (cf. p.71 in [17]), in the sense that the limiting probability density function $\rho_{(0)}$ has the factorized form $M(v)\eta$, where $\eta = \eta(r, t)$ is completely independent of v , and satisfies a Fokker–Planck equation, which we shall carefully identify below; furthermore, by noting part (vi) of Lemma 4.0.1 and Lemma 4.0.3, we deduce that*

$$\lim_{\epsilon \rightarrow 0_+} \int_{\mathbb{R}^{(J+1)d}} v_{i,l} v_{j,k} \varrho_\epsilon dv = \beta \delta_{i,j} \delta_{\ell,k} \eta, \quad \text{where } \beta = kT\zeta,$$

strongly in $L^p(\Omega^{J+1} \times (0, T))$ for all $p \in [1, 2)$ and weakly in $L^2(\Omega^{J+1} \times (0, T))$, which is yet another manifestation of equilibration in momentum space, as a consequence of the small mass limit $\epsilon \rightarrow 0_+$. For further details in this direction, we point the reader to the paper of Schieber and Öttinger [57], and references therein.

Having shown the strong convergence $\varrho_\epsilon \rightarrow M(v)\bar{\varrho} = M(v)\eta = \rho_{(0)}$ in $L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, we are now ready to pass to the limit $\epsilon \rightarrow 0_+$ in the Oseen equation. All

that remains to be done in this respect is to identify the weak* limit $\mathbb{K}_{(0)}$ of the sequence $(\mathbb{K}_\epsilon)_{\epsilon>0}$ in terms of the limit η of the sequence $(\widehat{\varrho}_\epsilon)_{\epsilon>0}$, where

$$\mathbb{K}_\epsilon := \frac{\mathfrak{A}_\epsilon}{\mathfrak{B}_\epsilon}, \quad \epsilon > 0,$$

with

$$\begin{aligned} \mathfrak{A}_\epsilon &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} \sum_{j=1}^J (F(q_j) \otimes q_j) M \widehat{\varrho}_\epsilon(B(q, x), v, t) \, dq \, dv, \\ \mathfrak{B}_\epsilon &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} M \widehat{\varrho}_\epsilon(B(q, x), v, t) \, dq \, dv. \end{aligned}$$

The limit $\mathbb{K}_{(0)}$ is anticipated to be of the form

$$\frac{\mathfrak{A}_{(0)}}{\mathfrak{B}_{(0)}},$$

where

$$\begin{aligned} \mathfrak{A}_{(0)} &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} \sum_{j=1}^J (F(q_j) \otimes q_j) M \eta(B(q, x), t) \, dq \, dv \\ &= \int_{D^J} \sum_{j=1}^J (F(q_j) \otimes q_j) \eta(B(q, x), t) \, dq, \\ \mathfrak{B}_{(0)} &:= \int_{D^J \times \mathbb{R}^{(J+1)d}} M \eta(B(q, x), t) \, dq \, dv = \int_{D^J} \eta(B(q, x), t) \, dq. \end{aligned}$$

The proof of this is identical to the proof, presented in Section 2.7, that the weak* limit \mathbb{K} of the sequence $(\mathbb{K}^{(k)})_{k \geq 0}$, where $\mathbb{K}^{(k)} = \frac{\mathfrak{A}^{(k)}}{\mathfrak{B}^{(k)}}$, $k = 0, 1, \dots$, considered in terms of the limit $\widehat{\varrho}$ of the sequence $(\widehat{\varrho}^{(k)})_{k \geq 0}$, is of the form $\frac{\mathfrak{A}}{\mathfrak{B}}$, the key ingredient in the argument being the strong convergence $\varrho_\epsilon \rightarrow M(v) \bar{\varrho} = M(v) \eta = \rho_{(0)}$ in $L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T))$, guaranteed by Lemma 4.0.3. We do not repeat the proof, therefore.

We now return to (4.25), and perform partial integration in the first term on the left-hand side, yielding

$$\begin{aligned} &\int_{\Omega^{J+1}} \bar{\rho}_\epsilon(r, t) \phi(r, t) \, dr - \int_0^t \int_{\Omega^{J+1}} \bar{\rho}_\epsilon(r, \tau) \partial_\tau \phi(r, \tau) \, dr \, d\tau - \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \mathcal{J}_{\epsilon, j} \cdot \partial_{r_j} \phi \, dr \, d\tau \\ &= \int_{\Omega^{J+1}} \bar{\rho}_0(r) \phi(r, 0) \, dr \quad \forall \phi \in L^2(0, T; W^{1,2}(\Omega^{J+1})) \cap W^{1,2}(0, T; L^2(\Omega^{J+1})) \\ &\forall t \in (0, T], \end{aligned} \tag{4.30}$$

since $\bar{\rho}_\epsilon(\cdot, 0) = \bar{\rho}_0(\cdot) := \int_{\mathbb{R}^{(J+1)d}} M(v) \widehat{\rho}_\epsilon(r, v, 0) dv$. Passage to the limit $\epsilon \rightarrow 0_+$ then gives

$$\begin{aligned} & \int_{\Omega^{J+1}} \eta(r, t) \phi(r, t) dr - \int_0^t \int_{\Omega^{J+1}} \eta(r, \tau) \partial_\tau \phi(r, \tau) dr d\tau - \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \mathcal{J}_j \cdot \partial_{r_j} \phi dr d\tau \\ &= \int_{\Omega^{J+1}} \bar{\rho}_0(r) \phi(r, 0) dr \quad \forall \phi \in L^2(0, T; W^{1,2}(\Omega^{J+1})) \cap W^{1,2}(0, T; L^2(\Omega^{J+1})) \\ & \forall t \in (0, T], \end{aligned} \quad (4.31)$$

where $\mathcal{J}_j := -\beta \partial_{r_j} \eta + \eta((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot))$ for $j = 1, \dots, J+1$. To see that this is indeed the case, we recall from the proof of Lemma 4.0.3 that the sequence

$$(\sqrt{\epsilon}(-\epsilon \sqrt{\epsilon} \partial_t \mathcal{J}_\epsilon - \operatorname{div}_r \mathbb{K}_\epsilon))_{\epsilon > 0}$$

converges to 0 in $W^{-1,2}(\Omega^{J+1} \times (0, T))^d$ as $\epsilon \rightarrow 0_+$. It then follows from (4.29) that, for each $j = 1, \dots, J+1$,

$$\lim_{\epsilon \rightarrow 0_+} (-\beta \partial_{r_j} \bar{\rho}_\epsilon + \bar{\rho}_\epsilon((\mathcal{L}r)_j + u_\epsilon(r_j, \cdot))) = \mathcal{J}_j \quad \text{in } W^{-1,2}(\Omega^{J+1} \times (0, T))^d. \quad (4.32)$$

Thanks to (4.9)₃ and since $\bar{\rho}_\epsilon \rightharpoonup \bar{\rho} = \eta$ weakly* in $L^\infty(0, T; L^2(\Omega^{J+1}))$, it follows that, for each $j = 1, \dots, J+1$,

$$\bar{\rho}_\epsilon((\mathcal{L}r)_j + u_\epsilon(r_j, \cdot)) \rightharpoonup \eta((\mathcal{L}r)_j + u(r_j, \cdot)) \quad \text{weakly in } L^2(0, T; L^2(\Omega^{J+1})^d).$$

Also, $\beta \partial_{r_j} \bar{\rho}_\epsilon \rightharpoonup \beta \partial_{r_j} \eta$ weakly* in $L^\infty(0, T; W^{-1,2}(\Omega^{J+1})^d)$. Hence,

$$\mathcal{J}_j := -\beta \partial_{r_j} \eta + \eta((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)), \quad j = 1, \dots, J+1,$$

as an equality in $W^{-1,2}(\Omega^{J+1} \times (0, T))^d$.

Now, since $\mathcal{J}_j \in L^2(\Omega^{J+1} \times (0, T))^d$ and $\eta((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \in L^2(\Omega^{J+1} \times (0, T))^d$ for all $j = 1, \dots, J+1$, it follows that $\partial_{r_j} \eta \in L^2(\Omega^{J+1} \times (0, T))^d$ for all $j = 1, \dots, J+1$. Therefore,

$$\mathcal{J}_j := -\beta \partial_{r_j} \eta + \eta((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)), \quad j = 1, \dots, J+1, \quad (4.33)$$

as an equality in $L^2(\Omega^{J+1} \times (0, T))^d$.

To summarize the main result of this section, we have shown that the small-mass limit of the coupled Oseen–Fokker–Planck system under consideration satisfies the following coupled problem: the velocity-pressure pair $(u_{(0)}, \pi_{(0)})$ solves the Oseen system

$$\begin{aligned} \partial_t u_{(0)} + (b \cdot \nabla) u_{(0)} - \mu \Delta u_{(0)} + \nabla \pi_{(0)} &= \nabla \cdot \mathbb{K}_{(0)} & \text{for } (x, t) \in \Omega \times (0, T], \\ \nabla \cdot u_{(0)} &= 0 & \text{for } (x, t) \in \Omega \times (0, T], \\ u_{(0)}(x, t) &= 0 & \text{for } (x, t) \in \partial\Omega \times (0, T], \\ u_{(0)}(x, 0) &= u_0(x) & \text{for } x \in \Omega, \end{aligned} \quad (4.34)$$

with

$$\mathbb{K}_{(0)}(x, t) := \frac{\int_{D^J} \sum_{j=1}^J (F(q_j) \otimes q_j) \eta(B(q, x), t) dq}{\int_{D^J} \eta(B(q, x), t) dq} \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (4.35)$$

and the nonnegative function η , with $\int_{\Omega^{J+1}} \eta(r, t) \, dr = 1$ for all $t \in [0, T]$, solves the following parabolic initial-boundary-value problem:

$$\partial_t \eta = \sum_{j=1}^{J+1} \left(\beta \partial_{r_j}^2 \eta - \partial_{r_j} \cdot \left(\eta ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \right) \right) \quad \text{in } \Omega^{J+1} \times (0, T], \quad (4.36)$$

$$\eta(\cdot, 0) = \widehat{\varrho}_0 \in L^2(\Omega^{J+1}; \mathbb{R}_{\geq 0}), \quad (4.37)$$

subject to the weakly imposed boundary condition $\mathcal{J}_j \cdot \nu(r_j) = 0$ on $\partial\Omega^{(j)} \times (0, T]$ for $j = 1, \dots, J+1$ (implied by the third term on the left-hand side of the equation (4.31)); i.e., by recalling the identity (4.33), we have the following zero-normal-flux boundary condition on η :

$$\left(\beta \partial_{r_j} \eta - \eta ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \right) \cdot \nu(r_j) = 0 \quad \text{on } \partial\Omega^{(j)} \times (0, T] \text{ for } j = 1, \dots, J+1. \quad (4.38)$$

We note that the partial differential equation (4.36) is of the form $\text{div}_{(r,t)}(\mathcal{J}, \eta) = 0$, where $\text{div}_{(r,t)}$ is the space-time divergence of the $((J+1)d+1)$ -component vector-function (\mathcal{J}, η) defined on $\Omega^{J+1} \times (0, T)$, with $(\mathcal{J}, \eta) \in L^2(\Omega^{J+1} \times (0, T))^{(J+1)d} \times L^2(\Omega^{J+1} \times (0, T))$. Consequently, by a standard trace theorem for the function space $H(\text{div}, \mathfrak{D})$, with $\mathfrak{D} = \Omega^{J+1} \times (0, T)$, the vector-function (\mathcal{J}, η) has a well-defined normal trace on the boundary $\partial(\Omega^{J+1} \times (0, T))$ of the domain $\Omega^{J+1} \times (0, T)$, contained in $W^{-\frac{1}{2}, 2}(\partial(\Omega^{J+1} \times (0, T)))$; see, for example, Theorem 18.7 in [3]. Thus, the boundary condition (4.38) for (4.36) is meaningful, as an equality in $W^{-\frac{1}{2}, 2}(\partial\Omega^{(j)} \times (0, T))$ (the dual space of $W_{00}^{\frac{1}{2}, 2}(\partial\Omega^{(j)} \times (0, T))$, $j = 1, \dots, J+1$; cf., for example, Theorem 18.9 in [3]).

We complete this section by proving the existence of a unique solution to the parabolic initial-boundary-value problem satisfied by η . To this end, we introduce the real-valued function $\tilde{\eta}$ defined on $\Omega^{J+1} \times [0, T]$ by

$$\begin{aligned} \tilde{\eta}(r, t) &:= \eta(r, t) - \frac{1}{|\Omega|} \int_{\Omega^{J+1}} \eta(r, t) \, dr \\ &= \eta(r, t) - \frac{1}{|\Omega|}. \end{aligned}$$

Hence, we have that the function $\tilde{\eta}$, with $\int_{\Omega^{J+1}} \tilde{\eta}(r, t) \, dr = 0$ for all $t \in [0, T]$, solves the following parabolic initial-boundary-value problem:

$$\partial_t \tilde{\eta} = \sum_{j=1}^{J+1} \left(\beta \partial_{r_j}^2 \tilde{\eta} - \partial_{r_j} \cdot \left(\left(\tilde{\eta} + \frac{1}{|\Omega|} \right) ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \right) \right) \quad \text{in } \Omega^{J+1} \times (0, T], \quad (4.39)$$

$$\tilde{\eta}_0 := \tilde{\eta}(\cdot, 0) = \widehat{\varrho}_0 - \frac{1}{|\Omega|} \in L^2(\Omega^{J+1}; \mathbb{R}_{\geq 0}), \quad \int_{\Omega^{J+1}} \tilde{\eta}_0(r) \, dr = 0, \quad (4.40)$$

$$\begin{aligned} \left(\beta \partial_{r_j} \tilde{\eta} - \left(\tilde{\eta} + \frac{1}{|\Omega|} \right) ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \right) \cdot \nu(r_j) &= 0 \quad \text{on } \partial\Omega^{(j)} \times (0, T], \\ j &= 1, \dots, J+1. \end{aligned} \quad (4.41)$$

Let us introduce the Hilbert space

$$H_{\star}^1(\Omega^{J+1}) := \left\{ \varphi \in H^1(\Omega^{J+1}) : \int_{\Omega^{J+1}} \varphi(r) \, dr = 0 \right\}$$

equipped with the norm of $H^1(\Omega^{J+1})$, with an analogous definition of $L^2_\star(\Omega^{J+1})$ equipped with the norm of $L^2(\Omega^{J+1})$.

By (4.31), the weak formulation of the problem (4.39)–(4.41) therefore amounts to seeking a function

$$\tilde{\eta} \in \mathcal{C}([0, T]; L^2_\star(\Omega^{J+1})) \cap L^2(0, T; H^1_\star(\Omega^{J+1}))$$

with

$$\partial_t \tilde{\eta} \in L^2(0, T; H^1_\star(\Omega^{J+1})'),$$

such that $\tilde{\eta}(\cdot, 0) = \tilde{\eta}_0(\cdot)$, and

$$\begin{aligned} \langle \partial_t \tilde{\eta}, \varphi \rangle_{H^1_\star(\Omega^{J+1})' \times H^1_\star(\Omega^{J+1})} &+ \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} [\beta \partial_{r_j} \tilde{\eta} - \tilde{\eta} ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot))] \cdot \partial_{r_j} \varphi \, dr \\ &= \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \frac{1}{|\Omega|} ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \cdot \partial_{r_j} \varphi \, dr \quad \forall \varphi \in H^1_\star(\Omega^{J+1}). \end{aligned}$$

We consider the bilinear form $a(\cdot, \cdot)$ defined on $H^1_\star(\Omega^{J+1}) \times H^1_\star(\Omega^{J+1})$ by

$$a(\psi, \varphi) := \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} [\beta \partial_{r_j} \psi - \psi ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot))] \cdot \partial_{r_j} \varphi \, dr, \quad \psi, \varphi \in H^1_\star(\Omega^{J+1}),$$

and set

$$\ell(\varphi) := \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \frac{1}{|\Omega|} ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \cdot \partial_{r_j} \varphi \, dr, \quad \varphi \in H^1_\star(\Omega^{J+1}).$$

Because $u_{(0)} \in L^2(0, T; L^\infty(\Omega)^d)$, we have that $\ell \in L^2(0, T; H^1_\star(\Omega^{J+1})')$. The bilinear form $a(\cdot, \cdot)$ is obviously well-defined for every ψ, φ in $H^1_\star(\Omega^{J+1})$. Moreover, by the Cauchy–Schwarz inequality, $a(\cdot, \cdot)$ is bounded (and therefore continuous); i.e.,

$$|a(\psi, \varphi)| \leq C \|\psi\|_{H^1(\Omega^{J+1})} \|\varphi\|_{H^1(\Omega^{J+1})} \quad \forall \psi, \varphi \in H^1_\star(\Omega^{J+1}),$$

for some positive constant C , independent of $t \in [0, T]$. Furthermore, $a(\cdot, \cdot)$ satisfies a Gårding inequality; indeed, we have that

$$\begin{aligned} a(\psi, \psi) &= \beta \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} |\partial_{r_j} \psi|^2 - \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} \psi ((\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)) \cdot \partial_{r_j} \psi \, dr \\ &\geq \frac{\beta}{2} \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} |\partial_{r_j} \psi|^2 - \frac{1}{2\beta} \sum_{j=1}^{J+1} \int_{\Omega^{J+1}} |(\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)|^2 |\psi|^2 \, dr \\ &\geq \frac{\beta}{2} \|\psi\|_{H^1(\Omega^{J+1})}^2 - \frac{1}{2\beta} \left(\text{ess. sup}_{r \in \Omega^{J+1}} \sum_{j=1}^{J+1} |(\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)|^2 \right) \|\psi\|_{L^2(\Omega^{J+1})}^2 \\ &= \frac{\beta}{2} \|\psi\|_{H^1(\Omega^{J+1})}^2 - \left(\frac{\beta}{2} + \frac{1}{2\beta} \left(\text{ess. sup}_{r \in \Omega^{J+1}} \sum_{j=1}^{J+1} |(\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)|^2 \right) \right) \|\psi\|_{L^2(\Omega^{J+1})}^2 \end{aligned}$$

for all $\psi \in H_\star^1(\Omega^{J+1})$, which leads to

$$a(\psi, \psi) \geq \alpha \|\psi\|_{H^1(\Omega^{J+1})}^2 - C \|\psi\|_{L^2(\Omega^{J+1})}^2 \quad \forall \psi \in H_\star^1(\Omega^{J+1}),$$

where

$$\alpha := \beta/2 \quad \text{and} \quad C := \frac{\beta}{2} + \frac{1}{2\beta} \left(\text{ess. sup}_{r \in \Omega^{J+1}} \sum_{j=1}^{J+1} |(\mathcal{L}r)_j + u_{(0)}(r_j, \cdot)|^2 \right)$$

are positive constants.

A classical abstract result due to J.-L. Lions (cf. [13], Theorem 10.9) then implies that, for any initial datum $\tilde{\eta}_0 \in L_\star^2(\Omega^{J+1})$ (and $u_{(0)} \in L^2(0, T; W^{1, \sigma}(\Omega^d))$, with $\sigma > d$, fixed), there exists a unique function $\tilde{\eta}$ satisfying:

$$\tilde{\eta} \in \mathcal{C}([0, T]; L_\star^2(\Omega^{J+1})) \cap L^2(0, T; H_\star^1(\Omega^{J+1})), \quad \partial_t \tilde{\eta} \in L^2(0, T; H_\star^1(\Omega^{J+1})'),$$

$$\langle \partial_t \tilde{\eta}, \varphi \rangle_{H_\star^1(\Omega^{J+1})' \times H_\star^1(\Omega^{J+1})} + a(\tilde{\eta}, \varphi) = \ell(\varphi) \quad \text{for a.e. } t \in (0, T), \quad \forall \varphi \in H_\star^1(\Omega^{J+1}),$$

and

$$\tilde{\eta}(\cdot, 0) = \tilde{\eta}_0(\cdot).$$

That concludes the proof of the existence of a unique weak solution to the parabolic initial-boundary-value problem (4.39)–(4.41) satisfied by $\tilde{\eta}$, which therefore also establishes the existence of a unique weak solution to the parabolic initial-boundary-value problem satisfied by $\eta = \tilde{\eta} + 1/|\Omega|$, for $u_{(0)} \in L^2(0, T; W^{1, \sigma}(\Omega^d))$, with $\sigma > d$, fixed. Similarly, the Oseen system has, for a given fixed η , a unique weak solution pair $(u_{(0)}, \pi_{(0)})$ (with $\pi_{(0)}$ understood to be unique up to an additive constant). The uniqueness of a solution triple $(u_{(0)}, \pi_{(0)}, \eta)$ satisfying the coupled problem we have arrived at in the small-mass limit is of course not guaranteed, since $\mathbb{K}_{(0)}$ is a nonlinear function of η and $u_{(0)}$ enters into the evolution equation for η , so the coupled system for the small-mass limit $(u_{(0)}, \pi_{(0)}, \eta)$ is still very much nonlinear.

Chapter 5

The small mass limit problem and the classical Hookean bead-spring-chain model

Our aim in this chapter is to explore the connection between the small-mass-limit problem (4.34)–(4.38) and the classical Hookean bead-spring-chain model for dilute polymeric fluids. We begin by recalling that

$$x = \frac{1}{J+1}(r_1 + \cdots + r_{J+1}) \quad \text{and} \quad q_j = r_{j+1} - r_j \quad \text{for } j = 1, \dots, J,$$

and perform a change of variables in order to transform the partial derivatives in (4.36) with respect to the variables r_j , $j = 1, \dots, J+1$, into partial derivatives with respect to x and q_j , $j = 1, \dots, J$. To this end, note that

$$\begin{aligned} \partial_{r_1} &= -\partial_{q_1} + \frac{1}{J+1}\partial_x, \\ \partial_{r_{j+1}} &= \partial_{q_j} - \partial_{q_{j+1}} + \frac{1}{J+1}\partial_x, \quad j = 1, \dots, J-1, \\ \partial_{r_{J+1}} &= \partial_{q_J} + \frac{1}{J+1}\partial_x. \end{aligned}$$

Thus,

$$\partial_{r_1}^2 + \cdots + \partial_{r_{J+1}}^2 = (-\partial_{q_1})^2 + (\partial_{q_1} - \partial_{q_2})^2 + \cdots + (\partial_{q_{J-1}} - \partial_{q_J})^2 + (\partial_{q_J})^2 + \frac{1}{J+1}\partial_x^2.$$

Consider the matrix $\mathcal{B} \in \mathbb{R}^{(J+1)d \times Jd}$, called the *incidence matrix*, which is a $(J+1) \times J$ block matrix with $d \times d$ blocks, defined by

$$\mathcal{B} := \begin{pmatrix} -\mathbf{I} & \mathbf{O} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{I} & -\mathbf{I} & \mathbf{O} & \ddots & \mathbf{O} \\ \mathbf{O} & \mathbf{I} & -\mathbf{I} & \mathbf{O} & \mathbf{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{I} & -\mathbf{I} \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{O} & \mathbf{I} \end{pmatrix}.$$

The $d \times d$ block at position (i, j) in \mathcal{B} is equal $-\mathbb{I}$ if the j th spring starts at bead i , it is equal to \mathbb{I} if the j th spring ends at bead i , and it is equal to \mathbb{O} otherwise, for $i = 1, \dots, J+1$ and $j = 1, \dots, J$. Note that

$$\mathcal{R} := \mathcal{B}^T \mathcal{B} = \begin{pmatrix} 2\mathbb{I} & -\mathbb{I} & \mathbb{O} & \dots & \mathbb{O} \\ -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \ddots & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} & \mathbb{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \mathbb{O} & \dots & -\mathbb{I} & 2\mathbb{I} & -\mathbb{I} \\ \mathbb{O} & \dots & \mathbb{O} & -\mathbb{I} & 2\mathbb{I} \end{pmatrix}.$$

The symmetric positive definite block matrix $\mathcal{R} := \mathcal{B}^T \mathcal{B}$ of size $Jd \times Jd$ is referred to as the *Rouse matrix*. In terms of the Rouse matrix we have

$$\partial_{r_1}^2 + \dots + \partial_{r_{J+1}}^2 = \partial_q^T \mathcal{B}^T \mathcal{B} \partial_q + \frac{1}{J+1} \partial_x^2 = \partial_q^T \mathcal{R} \partial_q + \frac{1}{J+1} \partial_x^2, \quad (5.1)$$

where $\partial_q := (\partial_{q_1}^T, \dots, \partial_{q_J}^T)^T$.

Next, note that

$$\partial_{r_1} \cdot (\eta(\mathcal{L}r)_1) = (\partial_{r_1} \eta) \cdot (\mathcal{L}r)_1 + \eta \partial_{r_1} \cdot (\mathcal{L}r)_1 = (\partial_{r_1} \eta) \cdot q_1 - d\eta.$$

We define, with $r = B(q, x)$, where $q = (q_1^T, \dots, q_J^T)^T \in D^J$ and $x \in \Omega$,

$$\psi(x, q, t) := \eta(B(q, x), t) = \eta(r, t).$$

Hence,

$$\partial_{r_1} \cdot (\eta(\mathcal{L}r)_1) = \left(-\partial_{q_1} \psi + \frac{1}{J+1} \partial_x \psi \right) \cdot q_1 - d\psi.$$

Similarly,

$$\partial_{r_{j+1}} \cdot (\eta(\mathcal{L}r)_{j+1}) = \left(\partial_{q_j} \psi - \partial_{q_{j+1}} \psi + \frac{1}{J+1} \partial_x \psi \right) \cdot (q_{j+1} - q_j) - 2d\psi, \quad j = 1, \dots, J-1,$$

and

$$\partial_{r_{J+1}} \cdot (\eta(\mathcal{L}r)_{J+1}) = \left(\partial_{q_J} \psi + \frac{1}{J+1} \partial_x \psi \right) \cdot (-q_J) - d\psi.$$

Thus we have that

$$\begin{aligned} \sum_{j=1}^{J+1} \partial_{r_j} \cdot (\eta(\mathcal{L}r)_j) &= -[\mathcal{B} \partial_q \psi \cdot \mathcal{B}q + 2dJ\psi] = -[(\partial_q \psi)^T \mathcal{B}^T \mathcal{B}q + 2dJ\psi] \\ &= -[(\partial_q \psi)^T \mathcal{B}^T \mathcal{B}q + (\partial_q^T (\mathcal{B}^T \mathcal{B}q))\psi] = -\partial_q^T (\psi \mathcal{R} q). \end{aligned} \quad (5.2)$$

By combining (5.1) and (5.2) we deduce that

$$\begin{aligned} -\sum_{j=1}^{J+1} (\beta \partial_{r_j}^2 \eta - \partial_{r_j} \cdot (\eta(\mathcal{L}r)_j)) &= -[\beta \partial_q^T \mathcal{R} \partial_q \psi + \partial_q^T (\mathcal{R} q \psi)] - \frac{\beta}{J+1} \partial_x^2 \psi \\ &= -\beta \partial_q \cdot \left[\mathcal{R} \left(\partial_q \psi + \frac{1}{\beta} q \psi \right) \right] - \frac{\beta}{J+1} \partial_x^2 \psi. \end{aligned} \quad (5.3)$$

Let

$$\mathfrak{M}(q) := (2\pi\beta)^{-\frac{1}{2}Jd} \exp(-|q|^2/2\beta), \quad \text{where } q = (q_1^T, \dots, q_J^T)^T \in D^J.$$

Hence, (5.3) yields

$$-\sum_{j=1}^{J+1} (\beta \partial_{r_j}^2 \eta - \partial_{r_j} \cdot (\eta(\mathcal{L}r)_j)) = -\beta \partial_q \cdot \left[\mathcal{R} \mathfrak{M}(q) \partial_q \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \partial_x^2 \psi. \quad (5.4)$$

Next, observe that

$$\begin{aligned} \partial_{r_j} \cdot (\eta u_{(0)}(r_j, t)) &= u_{(0)}(r_j, t) \cdot \partial_{r_j} \eta \\ &= u_{(0)}(r_j, t) \cdot \begin{cases} -\partial_{q_1} \psi + \frac{1}{J+1} \partial_x \psi, & j = 1, \\ \partial_{q_{j-1}} \psi - \partial_{q_j} \psi + \frac{1}{J+1} \partial_x \psi, & j = 2, \dots, J, \\ \partial_{q_J} \psi + \frac{1}{J+1} \partial_x \psi, & j = J+1. \end{cases} \end{aligned}$$

Thus we have that

$$\begin{aligned} \sum_{j=1}^{J+1} \partial_{r_j} \cdot (\eta u_{(0)}(r_j, t)) &= \left(\frac{1}{J+1} \sum_{j=1}^{J+1} u_{(0)}(r_j, t) \right) \cdot \partial_x \psi + \sum_{j=1}^J (u_{(0)}(r_{j+1}, t) \\ &\quad - u_{(0)}(r_j, t)) \cdot \partial_{q_j} \psi. \end{aligned}$$

By performing the approximations

$$\left(\frac{1}{J+1} \sum_{j=1}^{J+1} u_{(0)}(r_j, t) \right) \approx u_{(0)}(x, t)$$

and

$$(u_{(0)}(r_{j+1}, t) - u_{(0)}(r_j, t)) \approx (\nabla u_{(0)}(x, t))(r_{j+1} - r_j) = (\nabla u_{(0)})(x, t) q_j,$$

we obtain

$$\begin{aligned} \sum_{j=1}^{J+1} \partial_{r_j} \cdot (\eta u_{(0)}(r_j, t)) &\approx u_{(0)}(x, t) \cdot \partial_x \psi + \sum_{j=1}^J (\nabla u_{(0)})(x, t) q_j \cdot \partial_{q_j} \psi \\ &= u_{(0)}(x, t) \cdot \partial_x \psi + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u_{(0)})(x, t) q_j \psi), \end{aligned} \quad (5.5)$$

where the last equality is a consequence of the fact that

$$\partial_{q_j} \cdot ((\nabla u_{(0)}) q_j) = \text{tr}(\nabla u_{(0)}) = \nabla \cdot u_{(0)} = 0.$$

By substituting (5.4) and (5.5) into (4.36) and writing ∇ instead of ∂_x and Δ instead of ∂_x^2 , we have that

$$\partial_t \psi + u_{(0)} \cdot \nabla \psi + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u_{(0)}) q_j \psi) - \beta \partial_q \cdot \left[\mathcal{R} \mathfrak{M}(q) \partial_q \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta \psi \approx 0,$$

which can also be written as

$$\begin{aligned} \partial_t \psi + u_{(0)} \cdot \nabla \psi + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u_{(0)})_{q_j} \psi) - \beta \sum_{i,j=1}^J \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] \\ - \frac{\beta}{J+1} \Delta \psi \approx 0. \end{aligned} \quad (5.6)$$

Upon replacing the approximate equality in (5.6) by equality we arrive at the Fokker–Planck equation for the classical Hookean bead-spring-chain model with centre-of-mass diffusion:

$$\begin{aligned} \partial_t \psi + u_{(0)} \cdot \nabla \psi + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u_{(0)})_{q_j} \psi) - \beta \sum_{i,j=1}^J \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] \\ - \frac{\beta}{J+1} \Delta \psi = 0. \end{aligned} \quad (5.7)$$

The equation (5.7) is supplemented by the initial condition

$$\psi(x, q, 0) = \psi_0(x, q), \quad (5.8)$$

where $\psi_0(x, q) := \hat{\rho}_0(B(q, x))$ (cf. (4.37)).

Since (5.7) is now posed on the domain $\Omega \times D^J \times (0, T]$ rather than on $\Omega^{J+1} \times (0, T]$, it is natural to replace the zero-normal-flux boundary condition (4.38) on $\partial\Omega^{(J+1)} \times (0, T]$ by zero-normal-flux boundary conditions on $\partial\Omega \times D^J \times (0, T]$ and $\Omega \times \partial D^J \times (0, T]$; i.e.,

$$\nabla \psi(x, q, t) \cdot n_x(x) = 0 \quad \text{for all } (x, q, t) \in \partial\Omega \times D^J \times (0, T], \quad (5.9)$$

where n_x is the unit outward normal vector to $\partial\Omega$, and

$$\sum_{i=1}^J \left[\beta \mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) - ((\nabla u_{(0)})_{q_j} \psi) \right] \cdot n_{q_j} = 0 \quad (5.10)$$

for all $(x, q, t) \in \Omega \times (D \times \cdots \times \partial D \times \cdots \times D) \times (0, T]$, $j = 1, \dots, J$, where n_{q_j} is the unit outward normal vector to ∂D for the j th copy of the domain D in the Cartesian product $D^J = D \times \cdots \times D$.

By integrating the Fokker–Planck equation (5.7) over D^J and using the boundary condition (5.10), and then integrating both the boundary condition (5.9) and the initial condition (5.8) over D^J , we obtain

$$\begin{aligned} \partial_t \left(\int_{D^J} \psi \, dq \right) + u_{(0)} \cdot \nabla \left(\int_{D^J} \psi \, dq \right) - \frac{\beta}{J+1} \Delta \left(\int_{D^J} \psi \, dq \right) = 0 \quad \text{in } \Omega \times (0, T], \\ \nabla \left(\int_{D^J} \psi \, dq \right) \cdot n_x = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (5.11) \\ \left(\int_{D^J} \psi \, dq \right) (\cdot, 0) = \left(\int_{D^J} \psi_0(\cdot, q) \, dq \right) \quad \text{in } \Omega. \end{aligned}$$

If the initial datum ψ_0 is such that, for some constant $n > 0$,

$$\int_{D^J} \psi_0(x, q) \, dq = n^{-1} \quad \text{for a.e. } x \in \Omega,$$

then, by uniqueness of the solution to the initial-boundary-value problem (5.11), it follows that

$$\int_{D^J} \psi(x, q, t) \, dq = n^{-1} \quad \text{for a.e. } (x, t) \in \Omega \times [0, T];$$

that is

$$\int_{D^J} \eta(B(q, x), t) \, dq = \int_{D^J} \psi(x, q, t) \, dq = n^{-1} \quad \text{for a.e. } (x, t) \in \Omega \times [0, T],$$

whereby the expression for the tensor $\mathbb{K}_{(0)}$ stated in (4.35) simplifies to

$$\mathbb{K}_{(0)} = n \int_{D^J} \sum_{j=1}^J (F(q_j) \otimes q_j) \psi(x, q, t) \, dq. \quad (5.12)$$

In this form, $\mathbb{K}_{(0)}$ is referred to as *Kramers' expression* for the polymeric extra stress tensor for the bead-spring-chain model with J springs. We highlight one small but relevant difference between the classical Kramers expression and (5.12): in the classical Kramers expression the integral in q is taken over the whole of \mathbb{R}^{Jd} , whereas in our case the integral in q is over $D^J \subset \mathbb{R}^{Jd}$, where $D := \Omega - \Omega$. In this respect the formula (5.12) is more consistent with the definition of the configuration vectors $q_j := r_{j+1} - r_j$, $j = 1, \dots, J$, than its classical counterpart; it also avoids the nonphysical feature of the classical Hookean model that springs in a linear bead-spring-chain are allowed to stretch out to infinity even though their endpoints are confined to a bounded flow domain Ω . In our case, in contrast, if Ω is bounded, then so is D^J . Of course, if Ω happens to be the whole of \mathbb{R}^d then $D^J = \mathbb{R}^{Jd}$, so (5.12) and its classical counterpart will then coincide.

The main obstacle in proving the existence of global weak solutions to the Hookean bead-spring-chain model (cf. [7], for example,) where integration in the Kramers expression is over \mathbb{R}^{Jd} , is lack of weak compactness of the sequence of approximating solutions to the Fokker–Planck equation in the $|q|^2$ -weighted L^1 space $L^1_{|q|^2}(\mathbb{R}^{Jd})$ (even though the sequence of approximating solutions is strongly convergent in $L^1_{\text{loc}}(\mathbb{R}^{Jd})$), which then obstructs passage to the limit in the classical Kramers expression precisely because integration with respect to the configuration spatial variable q there is over the whole of \mathbb{R}^{Jd} rather than a bounded subset of \mathbb{R}^{Jd} . This difficulty was ultimately overcome in [10] in the case of $d = 2$ through a rigorous proof of the fact that the macroscopic closure of the Hookean dumbbell model ($J = 1$) is the Oldroyd-B model, for which a global existence result is available (cf. [4]). The existence of global weak solutions to the Hookean dumbbell model in the case of $d = 3$, with the Kramers expression in its classical form (i.e. with integration over $q \in \mathbb{R}^{Jd}$) however remains an open problem. With the Kramers expression defined by (5.12) now, the situation is radically different: the technical difficulties caused by loss of compactness disappear, enabling completion of the proof of existence of global weak solutions to the Hookean bead-spring-chain model in both two and three space dimensions by replicating the proof contained in [7]. We shall address this task in the next chapter.

Chapter 6

Existence of global weak solutions to the Hookean bead-spring-chain model

6.1 Introduction

We show the existence of global weak solutions to the Hookean bead-spring-chain model in both two and three space dimensions by following the proof contained in [7]. For this purpose we need to prove existence of global weak solutions to the system of equations (5.7)-(5.10) together with the system (1.17), where the expression for the tensor is given in equation (5.12) and where the Oseen equation is now replaced by the Navier–Stokes equation. We thus consider the following system:

$$\begin{aligned} \partial_t \psi + u_{(0)} \cdot \nabla \psi + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u_{(0)}) q_j \psi) \\ - \beta \sum_{i,j=1}^J \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta \psi = 0, \end{aligned} \quad (6.1a)$$

$$\psi(x, q, 0) = \psi_0(x, q) := \hat{\rho}_0(B(q, x)), \quad (6.1b)$$

$$\nabla \psi(x, q, t) \cdot n_x(x) = 0 \quad \text{for all } (x, q, t) \in \partial\Omega \times D^J \times (0, T], \quad (6.1c)$$

$$\sum_{i=1}^J \left[\beta \mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) - ((\nabla u_{(0)}) q_j \psi) \right] \cdot n_{q_j} = 0, \quad (6.1d)$$

for all $(x, q, t) \in \Omega \times (D \times \cdots \times \partial D \times \cdots \times D) \times (0, T]$, $j = 1, \dots, J$, where n_{q_j} is the unit outward normal vector to ∂D for the j th copy of the domain D in the Cartesian product $D^J = D \times \cdots \times D$, coupled with the system

$$\partial_t u_{(0)} + (u_{(0)} \cdot \nabla) u_{(0)} - \mu \Delta u_{(0)} + \nabla \pi = \nabla \cdot \mathbb{K}_{(0)} \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (6.2a)$$

$$\nabla \cdot u_{(0)} = 0 \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (6.2b)$$

$$u_{(0)}(x, t) = 0 \quad \text{for } (x, t) \in \partial\Omega \times (0, T], \quad (6.2c)$$

$$u_{(0)}(x, 0) = u_0(x) \quad \text{for } x \in \Omega, \quad (6.2d)$$

where

$$\mathbb{K}_{(0)} = n \int_{D^J} \sum_{j=1}^J (F(q_j) \otimes q_j) \psi(x, q, t) \, dq, \quad (6.3)$$

with $\mathbb{K}_0 : \Omega \times (0, T] \rightarrow \mathbb{R}_{\text{symm}}^{d \times d}$ is the elastic extra stress tensor, which satisfies:

$$\|\mathbb{K}_{(0)}\|_{L^\infty(0, T; L^\infty(\Omega))} < \infty. \quad (6.4)$$

In this chapter we consider the Hookean model and therefore $F(q) = q$, with $q \in D^J$. In what follows we use a Galerkin method to construct a sequence of spatially semi-discrete approximations to the initial-boundary-value problem.

The chapter is structured as follows. First, we shall introduce the necessary function spaces together with our assumptions on the data. Then, we shall state the main result of this chapter, concerning the existence of global weak solutions to the class of kinetic models under consideration. The rest of the chapter is then devoted to the proof of the theorem.

6.1.1 Function spaces and assumptions on the data

Proceeding as in [7], we define the following function spaces:

$$W_{n, \text{div}}^{1,2} := \overline{\left\{ v \in C^\infty(\overline{\Omega}) : v \cdot n = 0 \text{ on } \partial\Omega, \operatorname{div} v = 0 \text{ in } \Omega \right\}}^{\|\cdot\|_{W^{1,2}(\Omega)}},$$

$$W_{0, \text{div}}^{1,2} := \overline{\left\{ v \in C_0^\infty(\overline{\Omega}) : \operatorname{div} v = 0 \text{ in } \Omega \right\}}^{\|\cdot\|_{W^{1,2}(\Omega)}},$$

$$L_{0, \text{div}}^2 := \overline{W_{n, \text{div}}^{1,2}}^{\|\cdot\|_{L^2(\Omega)}},$$

$$W_{0, \text{div}}^{-1,2} := (W_{0, \text{div}}^{1,2})^*, \quad W_{n, \text{div}}^{-1,2} := (W_{n, \text{div}}^{1,2})^*.$$

We now state our assumptions on the initial conditions. For the initial velocity u_0 we assume that

$$u_0 \in L_{0, \text{div}}^2. \quad (6.5)$$

For $\hat{\psi}_0 := \frac{\psi_0}{\mathfrak{M}(q)}$, where ψ_0 is the initial value of the probability density function ψ , we assume that

$$\hat{\psi}_0 \geq 0 \text{ a.e. in } \Omega \times D^J, \quad \hat{\psi}_0 \ln \hat{\psi}_0 \in L_M^1(\Omega \times D^J), \quad (6.6)$$

and in addition we require that

$$\varrho_0 \in L^\infty(\Omega), \quad \text{where } \varrho_0(x) := \int_{D^J} \mathfrak{M}(q) \hat{\psi}_0(x, q) \, dq. \quad (6.7)$$

Remark 6.1.1. We note that, as $\mathbb{K}_{(0)}$ is defined in (6.3), with $n := n(x, t) = \frac{1}{\int_{D^J} \psi(x, q, t) \, dq} = c$, where c is a positive constant, then $\psi(x, q, 0)$ must be independent of x , and therefore $\hat{\psi}_0(x, q)$ in (6.7) is also independent of x , and therefore ϱ_0 is a positive constant. Hence, instead of ϱ_0 being constant, we are studying a more general problem here, with $\varrho_0 \in L^\infty(\Omega)$.

6.2 The main result

In this section, we show the existence of global weak solutions to the system in the case of a sufficiently large viscosity coefficient in the Navier–Stokes equation. In the next section, we will explore the possibility of removing the additional largeness assumption on the viscosity coefficient by considering boundary corrections of the polymeric extra stress tensor, to ensure the validity of a formal energy inequality for the coupled system. Before stating the main result, as in [7], using the boundary condition (6.2c), we give the weak formulation of the Navier–Stokes equation (6.2a). Assuming that the test function ω is smooth and that $\omega(\cdot, T) = 0$ and integrate by parts with respect to t , we obtain the following weak formulation for the Navier–Stokes equation

$$\begin{aligned} - \int_{\Omega} u_{(0)}(x, 0) \cdot \omega(x, 0) \, dx + \int_0^T \int_{\Omega} \left[-u_{(0)} \cdot \partial_t \omega + [(u_{(0)} \cdot \nabla) u_{(0)}] \cdot \omega \right. \\ \left. + \mu \nabla u_{(0)} : \nabla \omega \right] \, dx \, dt = - \int_0^T \mathbb{K}_{(0)} : \nabla \omega \, dx \, dt, \end{aligned} \quad (6.8)$$

for all $\omega \in C^1(0, T; W_{0,\text{div}}^{1,\infty}(\Omega))$.

In addition, using the boundary conditions (6.2c); (6.1c); (6.1d) and assuming that the test function φ is smooth and that $\varphi(\cdot, \cdot, T) = 0$ and integrate by parts with respect to t , we obtain the following weak formulation for the Fokker–Planck equation (6.1a)

$$\begin{aligned} \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \left[-\hat{\psi} \partial_t \varphi + \left[\frac{\beta}{J+1} \nabla \hat{\psi} - u_{(0)} \hat{\psi} \right] \nabla \varphi \right] \, dq \, dx \, dt \\ - \int_{\Omega \times D^J} \mathfrak{M}(q) \varphi(\cdot, \cdot, 0) \hat{\psi}(\cdot, \cdot, 0) \, dq \, dx \\ + \beta \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{i,j=1}^J \mathcal{R}_{ij} \partial_{q_j} \hat{\psi} \cdot \partial_{q_j} \varphi \, dq \, dx \, dt \\ - \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{j=1}^J \nabla u_{(0)} q_j \hat{\psi} \cdot \partial_{q_j} \varphi \, dq \, dx \, dt = 0, \end{aligned} \quad (6.9)$$

for all $\varphi \in C^1(0, T; W_0^{1,\infty}(\Omega \times D^J))$.

As in [7], the main result is the following

Theorem 6.2.1. *Let $J \in \mathbb{N}$ be arbitrary. Assume that the initial data $u_0, \hat{\psi}_0$ satisfy (6.5)–(6.7). Then, there exist $(u_{(0)}, \mathbb{K}_{(0)}, \hat{\psi})$ satisfying the weak formulations (6.8) and (6.9) such that*

$$u_{(0)} \in L^\infty(0, T; L_{0,\text{div}}^2(\Omega)^d) \cap L^2(0, T; W_0^{1,2}(\Omega)^d) \cap W^{1,2}(0, T; W_{0,\text{div}}^{-1,2}(\Omega)^d), \quad (6.10)$$

$$\mathbb{K}_{(0)} \in L^2(0, T; L^2(\Omega)^{d \times d}), \quad (6.11)$$

and

$$\begin{aligned}
\hat{\psi} &\in L^\infty(\Omega \times (0, T); L^1_{\mathfrak{M}(q)}(D^J)) \cap L^2(0, T; W^{1,1}_{\mathfrak{M}(q)}(\Omega \times D^J)), \\
\hat{\psi} &\geq 0 \text{ a.e. in } \Omega \times D^J \times (0, T), \\
\mathfrak{M}(q) \hat{\psi} &\in W^{1,1}(0, T; W^{-1,1}(\Omega \times D^J)), \\
\hat{\psi} \log \hat{\psi} &\in L^\infty(0, T; L^1_{\mathfrak{M}(q)}(\Omega \times D^J)).
\end{aligned} \tag{6.12}$$

Furthermore, the initial data are attained strongly in $L^2(\Omega) \times L^1_{\mathfrak{M}(q)}(\Omega \times D^J)$, i.e.

$$\lim_{t \rightarrow 0^+} \|u_{(0)}(\cdot, t) - u_0(\cdot)\|_{L^2(\Omega)}^2 + \|\hat{\psi}(\cdot, t) - \hat{\psi}_0(\cdot)\|_{L^1_{\mathfrak{M}(q)}(\Omega \times D^J)} = 0. \tag{6.13}$$

Moreover, for all $t \in (0, T)$ the following energy inequality holds in a weak sense:

$$\begin{aligned}
&\frac{d}{dt} \left(\int_{\Omega \times D^J} n \mathfrak{M}(q) \hat{\psi} \log \hat{\psi} \, dx \, dq + \frac{1}{2} \|u_{(0)}\|_{L^2(\Omega)}^2 \right) + \|\nabla u_{(0)}\|_{L^2(\Omega)}^2 \\
&\quad + 4n \left(\mathfrak{M}(q) \nabla \sqrt{\hat{\psi}}, \nabla \sqrt{\hat{\psi}} \right)_{\Omega \times D^J} \\
&\quad + 4n \left(\mathcal{R} \mathfrak{M}(q) \partial_q \sqrt{\hat{\psi}}, \partial_q \sqrt{\hat{\psi}} \right)_{\Omega \times D^J} \leq 0.
\end{aligned} \tag{6.14}$$

The purpose of the remaining sections is to provide a proof of this theorem, following a similar line of arguments as in [14], by constructing a sequence of approximating sequences for $u_{(0)}$ and $\hat{\psi}$ and passing to the respective limits of these.

6.2.1 Approximate problem

We introduce in this section an approximate problem for which the analysis of existence of solutions is relatively easy and can be performed by using the Galerkin method and suitable a priori entropy estimates. In what follows we mainly follow the paper [14]. In order to handle the momentum equation we truncate the convective term and $\mathbb{K}_{(0)}$. More precisely, we introduce a smooth nonnegative function $\Gamma \in \mathcal{D}(-2, 2)$, such that $\Gamma(s) = 1$ for all $s \in [-1, 1]$ and for an arbitrary $l \in \mathbb{N}$ we define $\Gamma_l(s) := \Gamma(\frac{s}{l})$. The primitive function to Γ_l is denoted by

$$T_l(s) := \int_0^s \Gamma_l(r) \, dr. \tag{6.15}$$

Next, the l -th approximation of $\mathbb{K}_{(0)}$ is defined by

$$\mathbb{K}_{(0)}^l(x, t) = n \int_{D^J} \sum_{j=1}^J \mathfrak{M}(q) T_l(\hat{\psi}(x, q, t)) q_j \otimes q_j \, dq. \tag{6.16}$$

We then define the l -approximation of (6.2a) as follows

$$\partial_t u_{(0)}^l + \nabla \cdot \left(\Gamma_l(|u_{(0)}^l|^2) u_{(0)}^l \otimes u_{(0)}^l \right) - \mu \Delta u_{(0)}^l + \nabla \pi^l = \nabla \cdot \mathbb{K}_{(0)}^l \quad \text{in } \Omega \times (0, T], \tag{6.17}$$

with boundary and initial data given by (6.2c)–(6.2d) with $u_{(0)}$ replaced by $u_{(0)}^l$ on the left-hand sides of the equalities (6.2c)–(6.2d). In order to preserve the energy identity under this truncation process, we shall also modify (6.1a). We define the l -th approximation of (6.1a) as

$$\begin{aligned} & \partial_t \psi^l + u_{(0)}^l \cdot \nabla \psi^l + \sum_{j=1}^J \partial_{q_j} \cdot ((\nabla u_{(0)}^l)_{q_j} \psi^l) \\ & - \beta \sum_{i,j=1}^J \partial_{q_j} \cdot [\mathcal{R}_{ij} \mathfrak{M}(q) \partial_{q_i} \hat{\psi}^l] - \frac{\beta}{J+1} \Delta \psi^l = 0 \end{aligned} \quad (6.18)$$

in $\Omega \times D^J \times (0, T]$,

supplemented by the Neumann boundary conditions corresponding to (6.1c)–(6.1d). We also truncate the initial condition for $\hat{\psi}^l$ as follows

$$\hat{\psi}^l(x, q, 0) = T_l(\hat{\psi}_0(x, q)). \quad (6.19)$$

For such an approximation, we are in the same setting as in the paper [14] and the main theorem in their subsection Approximate Problem still holds here. First, we set

$$\Lambda_l(s) := s \Gamma_l(s), \quad (6.20)$$

then we have

Theorem 6.2.2. *Let the initial data $(u_0, \hat{\psi}_0)$ satisfy (6.5)–(6.7). Then, for any $l \in \mathbb{N}$, there exists $(u_{(0)}^l, \mathbb{K}_{(0)}^l, \hat{\psi}^l)$ such that*

$$\begin{aligned} & u_{(0)}^l \in L^\infty(0, T; L_{0,\text{div}}^2(\Omega)^d) \cap L^2(0, T; W^{1,2}(\Omega)^d) \cap W^{1,2}(0, T; W_{0,\text{div}}^{-1,2}), \\ & \mathbb{K}_{(0)}^l \in L^\infty(0, T; L^\infty(\Omega)^{d \times d}), \\ & \hat{\psi}^l \in L^\infty(\Omega \times (0, T); L_{\mathfrak{M}(q)}^1(D^J)) \cap L^2(0, T; W_{\mathfrak{M}(q)}^{1,1}(\Omega \times D^J)), \\ & \hat{\psi}^l \geq 0 \text{ a.e. in } \Omega \times D^J \times (0, T), \\ & \mathfrak{M} \hat{\psi}^l \in W^{1,1}(0, T; W^{-1,1}(\Omega \times D^J)), \end{aligned} \quad (6.21)$$

satisfying the following system of equations

$$\begin{aligned} & - \int_{\Omega} u_{(0)}^l(x, 0) \cdot \omega(x, 0) \, dx \\ & + \int_0^T \int_{\Omega} \left[-u_{(0)}^l \cdot \partial_t \omega - [\Gamma_l(|u_{(0)}^l|^2) u_{(0)}^l \otimes u_{(0)}^l] \cdot \nabla \omega + \mu \nabla u_{(0)}^l : \nabla \omega \right] \, dx \, dt \\ & = - \int_0^T \mathbb{K}_{(0)}^l : \nabla \omega \, dx \, dt, \end{aligned} \quad (6.22)$$

for all $\omega \in L^2(0, T; W_{0,\text{div}}^{1,2}(\Omega))$.

$$\mathbb{K}_{(0)}^l(x, t) = n \int_{D^J} \sum_{j=1}^J \mathfrak{M}(q) T_l(\hat{\psi}(x, q, t)) q_j \otimes q_j \, dq. \quad (6.23)$$

$$\begin{aligned}
& \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \left[-\hat{\psi}^l \partial_t \varphi + \left[\frac{\beta}{J+1} \nabla \hat{\psi}^l - u_{(0)}^l \hat{\psi}^l \right] \nabla \varphi \right] dq dx dt \\
& - \int_{\Omega \times D^J} \mathfrak{M}(q) \varphi(\cdot, \cdot, 0) \hat{\psi}^l(\cdot, \cdot, 0) dq dx \\
& + \beta \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{i,j=1}^J \mathcal{R}_{ij} \partial_{q_j} \hat{\psi}^l \cdot \partial_{q_j} \varphi dq dx dt \\
& - \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{j=1}^J \nabla u_{(0)}^l q_j \Lambda_l(\hat{\psi}^l) \cdot \partial_{q_j} \varphi dq dx dt = 0,
\end{aligned} \tag{6.24}$$

for all $\varphi \in C^1(0, T; W^{1, \infty}(\Omega \times D^J))$, attaining the initial conditions in the following sense

$$\lim_{t \rightarrow 0^+} \|u_{(0)}^l(\cdot, t) - u_0\|_{L^2(\Omega)}^2 + \|\hat{\psi}^l(\cdot, t) - T_l(\hat{\psi}_0(\cdot))\|_{L^1_{\mathfrak{M}(q)}(\Omega \times D^J)} = 0, \tag{6.25}$$

satisfying, for all $t \in (0, T)$, the energy inequality

$$\begin{aligned}
& \int_{\Omega \times D^J} \mathfrak{M}(q) G(\hat{\psi}^l(\cdot, t)) dx dq + \frac{1}{2} \int_{\Omega} |u_{(0)}^l(\cdot, t)|^2 dx \\
& + 4C_{\mathcal{R}} \int_0^t \int_{\Omega \times D^J} \mathfrak{M}(q) \left| \nabla_{x,q} \sqrt{\hat{\psi}^l} \right|^2 dx dq dt + \mu \int_0^t \int_{\Omega} |\nabla u_{(0)}^l|^2 dx dt \\
& \leq \int_0^t \mathfrak{M}(q) G(T_l(\hat{\psi}_0)) dx dq + \frac{1}{2} \int_{\Omega} |u_0(\cdot)|^2 dx,
\end{aligned} \tag{6.26}$$

where, as in [14], G is defined as

$$G(s) := s \log s + e^{-1},$$

and in particular the following uniform a priori estimate holds:

$$\begin{aligned}
& \sup_{t \in (0, T)} \left(\|u_{(0)}^l(\cdot, t)\|_{L^2(\Omega)}^2 + \|\hat{\psi}^l(\cdot, t) \log \hat{\psi}^l(\cdot, t)\|_{L^1_{\mathfrak{M}(q)}(\Omega \times D^J)} + \|\varrho^l(\cdot, t)\|_{L^\infty(\Omega)} \right) \\
& \int_0^T \left(\|u_{(0)}^l\|_{W^{1,2}(\Omega)}^2 + \|\sqrt{\hat{\psi}^l}\|_{W^{1,2}_{\mathfrak{M}(q)}(\Omega \times D^J)}^2 + \|\mathbb{K}_{(0)}^l\|_{L^2(\Omega)}^2 + \|\nabla \varrho^l\|_{L^2(\Omega)}^2 \right) dt \\
& \leq C(n, \Omega, D^J, T, \mathcal{R}, \mathfrak{M}(q), u_0, \hat{\psi}_0),
\end{aligned} \tag{6.27}$$

where

$$\varrho^l(x, t) := \int_{D^J} \mathfrak{M}(q) \hat{\psi}^l(x, q, t) dq. \tag{6.28}$$

The rest of this section is devoted to the proof of Theorem 6.2.2. In order to simplify the presentation we shall take without loss of generality the constant n in the Kramers expression (6.16) to be 1, and we shall write $u_{(0)}$ and $\hat{\phi}$ instead of $u_{(0)}^l$ and $\hat{\phi}^l$, and similarly for all other analogous quantities; the omitted superscript l will be reinstated later on in the paper when we consider the question of passing to the limit $l \rightarrow \infty$.

6.2.2 Galerkin approximation

Proceeding as in [14], we introduce a Galerkin approximation of (6.22)–(6.24). We need the following abstract lemma from [25], which state standard results (essentially, the Hilbert–Schmidt theorem and some of its corollaries).

Lemma 6.2.3. *Let H and V be separable infinite-dimensional Hilbert spaces, with $V \subset H$ and $\bar{V} = H$ in the norm of H . Let $a : V \times V \rightarrow \mathbb{R}$ be a nonzero, symmetric, bounded and elliptic bilinear form. Then, there exist sequences of real numbers $(\lambda_n : n \in \mathbb{N})$ and unit H -norm members of V $(e_n : n \in \mathbb{N})$, which solve the following problem: Find $\lambda \in \mathbb{R}$ and $e \in H \setminus \{0\}$ such that*

$$a(e, v) = \lambda \langle e, v \rangle_H \quad \forall v \in V.$$

The λ_n , which can be assumed to be in increasing order with respect to n , are positive, bounded from below away from 0, and $\lim_{n \rightarrow \infty} \lambda_n = \infty$. Additionally, the e_n form an H -orthonormal system whose H -closed span is H and the rescaling $\frac{e_n}{\sqrt{\lambda_n}}$ gives rise to an a -orthonormal system whose a -closed span is V .

The Hilbert space $W_{0,\text{div}}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d$, equipped with the inner product of $W^{d+1,2}(\Omega)^d$ is compactly and densely embedded in the Hilbert space $L_{0,\text{div}}^2(\Omega)^d$. Therefore, using the Hilbert–Schmidt theorem with $V = W_{0,\text{div}}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d$ and $H = L_{0,\text{div}}^2(\Omega)^d$ and $a(\cdot, \cdot)$ taken to be the inner product of $W^{d+1,2}(\Omega)^d$, there exists a countable set $\{\omega_i\}_{i \in \mathbb{N}}$ of eigenfunctions in $W_{0,\text{div}}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d$ whose linear span is dense in $L_{0,\text{div}}^2(\Omega)^d$, s.t. the ω_i , $i = 1, 2, \dots$ are orthogonal in the inner product $W^{d+1,2}(\Omega)^d$ and orthonormal in the inner product of $L^2(\Omega)^d$.

Similarly, for each $m \in \mathbb{N}$ we find a countable set $\{\varphi_i^m\}_{i \in \mathbb{N}}$ of eigenfunctions in $W^{1,2}(\Omega \times D^J)$ that are orthogonal in $W_{\mathfrak{M}(q)}^{1,2}(\Omega \times D^J)$ and orthonormal in $L_{\mathfrak{M}(q)}^2(\Omega \times D^J)$.

Finally, we fix $m, n \in \mathbb{N}$ and look for $(u^{m,n}, \hat{\psi}^{m,n})$ given by

$$u^{m,n}(x, t) := \sum_{i=1}^m c_i^{m,n}(t) \omega_i(x), \quad (6.29)$$

$$\hat{\psi}^{m,n}(x, q, t) := \sum_{i=1}^n d_i^{m,n}(t) \varphi_i^m(x, q), \quad (6.30)$$

that solve

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left[\partial_t u^{m,n} \cdot \omega_i - \Gamma_l(|u^{m,n}|^2) u^{m,n} \otimes u^{m,n} \cdot \omega_i + \mu \nabla u^{m,n} : \nabla \omega_i \right] dx dt \\
&= - \int_0^T \mathbb{K}_{(0)}^{m,n} : \nabla \omega_i dx dt,
\end{aligned} \tag{6.31}$$

for all $i = 1, \dots, m$ and a.e. $t \in (0, T)$, where

$$\mathbb{K}_{(0)}^{m,n} := \int_{D^J} \sum_{j=1}^J \mathfrak{M}(q) T_l(\psi^{m,n}) q_j \otimes q_j dq, \quad \text{a.e. in } \Omega \times (0, T), \tag{6.32}$$

$$\begin{aligned}
& \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \left[\varphi_i^m \partial_t \hat{\psi}^{m,n} + \left[\frac{\beta}{J+1} \nabla \hat{\psi}^{m,n} - u^{m,n} \hat{\psi}^{m,n} \right] \nabla \varphi_i^m \right] dq dx dt \\
&+ \beta \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{i,j=1}^J \mathcal{R}_{ij} \partial_{q_j} \hat{\psi}^{m,n} \cdot \partial_{q_j} \varphi_i^m dq dx dt \\
&- \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{j=1}^J \nabla u^{m,n} q_j \Lambda_l(\hat{\psi}^{m,n}) \cdot \partial_{q_j} \varphi_i^m dq dx dt = 0,
\end{aligned} \tag{6.33}$$

for all $i = 1, \dots, n$ and a.e. $t \in (0, T)$, with initial data given by

$$u^{m,n}(x, 0) := \sum_{i=1}^m (u_0, \omega_i) \omega_i(x),$$

$$\hat{\psi}^{m,n}(x, q, 0) = \hat{\psi}_0^{m,n}(x, q) := \sum_{i=1}^n (T_l(\hat{\psi}_0), \varphi_i^m)_{\Omega \times D^J} \varphi_i^m(x, q).$$

The local in time existence of the functions $u^{m,n}$ and $\hat{\psi}^{m,n}$ for fixed m, n follows from Carathéodory's theorem (see Theorem 5.1. in [31]). Moreover, using the estimates established below we can extend the solution onto the whole time interval $(0, T)$.

6.2.3 n -Independent a priori estimates

Our objective, as in [14], is to derive estimates that do not depend on n . We have the following result.

Proposition 6.2.4. *We have that $u^{m,n}$ and $\hat{\psi}^{m,n}$ satisfy the following energy inequality*

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} |u^{m,n}|^2 dx + \int_{\Omega \times D^J} \mathfrak{M}(q) \mathcal{F}(\hat{\psi}^{m,n}) dq dx \right\} \\ & + \left[\mu \int_{\Omega} |\nabla u^{m,n}|^2 + 4\epsilon \int_{\Omega \times D^J} \mathfrak{M}(q) |\nabla_x \sqrt{\hat{\psi}^{m,n}}|^2 dq dx \right. \\ & \left. + 4C_{\mathcal{R}} \int_{\Omega \times D} \mathfrak{M}(q) \sum_{i=1}^J |\nabla_q \sqrt{\hat{\psi}^{m,n}}|^2 dq dx \right] \leq 0. \end{aligned} \quad (6.34)$$

Proof. We first note that, since $\mathfrak{M}(q)$ is bounded and because of the presence of the cut-off function T_l in (6.32), we have that

$$|\mathbb{K}_{(0)}^{m,n}| \leq Cl. \quad (6.35)$$

Next, we multiply the i -th equation in (6.8) by $c_i^{m,n}(t)$ and sum with respect to $i = 1, \dots, m$, to deduce that

$$\frac{d}{dt} \left\{ \frac{1}{2} \int_{\Omega} |u^{m,n}|^2 dx \right\} + \mu \int_{\Omega} |\nabla u^{m,n}|^2 dx = - \int_{\Omega} \mathbb{K}_{(0)}^{m,n} : \nabla u^{m,n} dx. \quad (6.36)$$

Remark 6.2.5. *Note that the convective term vanishes since $\nabla \cdot u^{m,n} = 0$.*

We obtain the following estimates (see [14])

$$\sup_{t \in (0, T)} \|u^{m,n}\|_{L^2(\Omega)}^2 + \int_0^T \|u^{m,n}\|_{W^{1,2}(\Omega)}^2 dt \leq C(l, \mathfrak{M}(q), u_0), \quad (6.37)$$

and

$$\sup_{t \in (0, T); i=1, \dots, m} |c_i^{m,n}(t)| + \left| \frac{dc_i^{m,n}(t)}{dt} \right| \leq C(m, l, u_0, \mathfrak{M}(q)). \quad (6.38)$$

Similarly, multiplying the i -th equation in (6.9) by $d_i^{m,n}(t)$ and summing with respect to $i = 1, \dots, n$, we find that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\hat{\psi}^{m,n}\|_{L^2_{\mathfrak{M}(q)}(\Omega \times D^J)}^2 + \frac{\beta}{J+1} \|\nabla \hat{\psi}^{m,n}\|_{L^2_{\mathfrak{M}(q)}(\Omega \times D^J)}^2 + \beta \langle \mathfrak{M}(q) \mathcal{R} \partial_q \hat{\psi}^{m,n}, \partial_q \hat{\psi}^{m,n} \rangle_{\Omega \times D^J} \\ & = \langle \mathfrak{M}(q) \Lambda_l(\hat{\psi}^{m,n})(\nabla u^{m,n} q, \partial_q \hat{\psi}^{m,n}) \rangle_{\Omega \times D^J}. \end{aligned} \quad (6.39)$$

Next, noting the definitions of Λ_l and thanks to the boundedness of $\mathfrak{M}(q)$ together with the Young and Hölder inequalities we get that

$$\begin{aligned} \langle \mathfrak{M}(q) \Lambda_l(\hat{\psi}^{m,n})(\nabla u^{m,n} q, \partial_q \hat{\psi}^{m,n}) \rangle_{\Omega \times D^J} & \leq \frac{1}{2} C \int_{\Omega \times D^J} \mathfrak{M}(q) |\partial_q \hat{\psi}^{m,n}|^2 dx dq \\ & + C(l) \|\nabla u^{m,n}\|_{L^\infty(\Omega)}^2 \|\hat{\psi}^{m,n}\|_{L^2_{\mathfrak{M}(q)}(\Omega \times D^J)}. \end{aligned} \quad (6.40)$$

Further, using the smoothness of the basis (note that $W^{d+1,2} \hookrightarrow W^{1,\infty}$) and the estimate (6.38), we get that

$$\|\nabla u^{m,n}\|_{L^\infty(\Omega)}^2 \leq C(m, l). \quad (6.41)$$

Therefore, as in [14], inserting (6.40) into (6.39), we deduce that

$$\frac{d}{dt} \|\hat{\psi}^{m,n}\|_{L^2_{\mathfrak{M}(q)}(\Omega \times D^J)}^2 + \int_{\Omega \times D^J} \mathfrak{M}(q) |\nabla_{x,q} \hat{\psi}^{m,n}|^2 dx dq \leq C(l, m) \|\hat{\psi}^{m,n}\|_{L^2_{\mathfrak{M}(q)}(\Omega \times D^J)}^2. \quad (6.42)$$

Thus, the application of Gronwall's lemma, the fact that $\mathfrak{M}(q)$ is bounded and the definition of $\hat{\psi}_0^{m,n}$ imply that

$$\sup_{t \in (0, T)} \|\hat{\psi}^{m,n}\|_{L^2(\Omega \times D^J)}^2 + \int_0^T \|\hat{\psi}^{m,n}\|_{W^{1,2}(\Omega \times D^J)}^2 dt \leq C(l, m). \quad (6.43)$$

In addition, since $\mathfrak{M}(q)$ is Lipschitz continuous, it is easy to deduce from (6.43) that

$$\int_0^T \|\mathfrak{M}(q) \hat{\psi}^{m,n}\|_{W^{1,2}(\Omega \times D^J)}^2 dt \leq C(l, m). \quad (6.44)$$

Finally, it follows from (6.38), (6.43) and (6.33) that

$$\int_0^T \|\partial_t(\mathfrak{M}(q) \hat{\psi}^{m,n})\|_{W^{-1,2}(\Omega \times D^J)}^2 dt \leq C(l, m). \quad (6.45)$$

□

6.2.4 The limit $n \rightarrow \infty$

Here, proceeding as in [14], we let $n \rightarrow \infty$ in (6.31)–(6.33), (6.43). Thus, from the n -independent estimates (6.38), (6.43), (6.44), (6.45) and by using the Aubin–Lions Lemma we see that there exist subsequences, which we do not relabel, such that

$$c_i^{m,n} \rightharpoonup^* c_i^m \quad \text{weak}^* \text{ in } W^{1,\infty}(0, T), \quad (6.46)$$

$$c_i^{m,n} \rightarrow c_i^m \quad \text{strongly in } \mathcal{C}[0, T], \quad (6.47)$$

$$u^{m,n} \rightarrow c_i^m \quad \text{strongly in } \mathcal{C}(0, T; W_{0,div}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d), \quad (6.48)$$

$$\hat{\psi}^{m,n} \rightharpoonup \hat{\psi}^m \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega \times D^J)), \quad (6.49)$$

$$\partial_t(\mathfrak{M}(q) \hat{\psi}^{m,n}) \rightharpoonup \partial_t(\mathfrak{M}(q) \hat{\psi}^m) \quad \text{weakly in } L^2(0, T; W^{-1,2}(\Omega \times D^J)), \quad (6.50)$$

$$\hat{\psi}^{m,n} \rightarrow \hat{\psi}^m \quad \text{strongly in } L^2(0, T; L^2(\Omega \times D^J)), \quad (6.51)$$

$$\mathbb{K}_{(0)}^{m,n} \rightharpoonup^* \mathbb{K}_{(0)}^m \quad \text{weak}^* \text{ in } L^\infty(0, T; L^\infty(\Omega)). \quad (6.52)$$

In the light of these convergence results, as in [14], it is now standard to pass to the limit $n \rightarrow \infty$ in (6.31) and (6.33) to deduce that

$$\int_0^T \int_{\Omega} \left[\partial_t u^m \cdot \omega_i - \Gamma_l(|u^m|^2) u^m \otimes u^m \cdot \omega_i + \mu \nabla u^m : \nabla \omega_i \right] dx dt = - \int_0^T \mathbb{K}_{(0)}^m : \nabla \omega_i dx dt, \quad (6.53)$$

for all $i = 1, \dots, m$ and a.e. $t \in (0, T)$,

$$\begin{aligned} & \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \left[\varphi \partial_t \hat{\psi}^m + \left[\frac{\beta}{J+1} \nabla \hat{\psi}^m - u^m \hat{\psi}^m \right] \nabla \varphi \right] dq dx dt \\ & + \beta \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{i,j=1}^J \mathcal{R}_{ij} \partial_{q_j} \hat{\psi}^m \cdot \partial_{q_j} \varphi dq dx dt \\ & - \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{j=1}^J \nabla u^m q_j \Lambda_l(\hat{\psi}^m) \cdot \partial_{q_j} \varphi dq dx dt = 0, \end{aligned} \quad (6.54)$$

for all $\varphi \in W^{1,2}(\Omega \times D^J)$ and a.e. $t \in (0, T)$. Moreover, it is obvious that $u^m(x, 0) = u_0^m(x)$ and it is completely standard to show that

$$\lim_{t \rightarrow 0_+} \|\hat{\psi}^m(\cdot, t) - T_l(\hat{\psi}_0^m(\cdot))\|_{L^2(\Omega \times D^J)} = 0. \quad (6.55)$$

Then, using (6.51) and the Lebesgue Dominated Convergence Theorem, we can take the limit in (6.32) and deduce that

$$\mathbb{K}_{(0)}^m := \int_{D^J} \sum_{j=1}^J \mathfrak{M}(q) T_l(\psi^m) q_j \otimes q_j dq, \quad \text{a.e. in } \Omega \times (0, T), \quad (6.56)$$

6.2.5 Uniform m -independent estimates

This subsection is devoted to deriving a priori estimates that are independent of m . As a matter of fact, most of the estimates will also be independent of l , and this will be clearly highlighted in the text. Since we are in the same setting as in the paper [14], we deduce, using the same procedure, the following estimates.

First, using a minimum principle for $\hat{\psi}^{m,n}$, the nonnegativity of $\hat{\psi}^{m,n}$

$$\hat{\psi}^{m,n} \geq 0 \quad \text{a.e. in } \Omega \times D^J \times (0, T). \quad (6.57)$$

Then, we set $\varphi \equiv 1$ in (6.54) to deduce that

$$\int_{\Omega \times D^J} \mathfrak{M}(q) |\hat{\psi}^{m,n}| dx dq = \int_{\Omega \times D^J} \mathfrak{M}(q) |T_l(\hat{\psi}_0^{m,n})| dx dq \quad (6.58)$$

$$\leq \int_{\Omega \times D^J} \mathfrak{M}(q) |\hat{\psi}_0^m| dx dq \quad (6.59)$$

$$= \int_{\Omega \times D^J} \mathfrak{M}(q) |\hat{\psi}_0| dx dq \quad (6.60)$$

$$\leq C. \quad (6.61)$$

Next, as in [14], setting $\varphi(x, q, t) := \bar{\varphi}$ in (6.54) and defining

$$\varrho^m(x, t) := \int_{D^J} \mathfrak{M}(q) \hat{\psi}^m(x, q, t) dq, \quad (6.62)$$

we see that

$$\langle \partial_t \varrho^m, \bar{\varphi} \rangle - \langle u^m \varrho^m, \nabla \bar{\varphi} \rangle + \frac{\beta}{J+1} \langle \nabla \varrho^m, \nabla \bar{\varphi} \rangle = 0 \quad (6.63)$$

$$\text{for all } \bar{\varphi} \in W^{1,2}(\Omega) \text{ and a.e. } t \in (0, T), \quad (6.64)$$

supplemented by the initial condition $\varrho^m(\cdot, 0) = \varrho_0^m$, where

$$0 \leq \varrho_0^m(x) := \int_{D^J} \mathfrak{M}(q) T_l(\hat{\psi}_0^m(x, q)) dq \leq \int_{D^J} \mathfrak{M}(q) \hat{\psi}_0(x, q) dq = \varrho_0(x). \quad (6.65)$$

Consequently, since $\nabla \cdot u^m = 0$ we can use the maximum principle and (6.7) to deduce that

$$\begin{aligned} \|\varrho^m\|_{L^\infty(\Omega \times (0, T))} &\leq \|\varrho_0\|_{L^\infty(\Omega)} \leq C, \\ \int_0^T \|\nabla \varrho^m\|_{L^2(\Omega \times (0, T))}^2 dt &\leq \frac{1}{2} \|\varrho_0\|_{L^2(\Omega)}^2 \leq C. \end{aligned} \quad (6.66)$$

Finally, to obtain m -independent estimates for the velocity field u^m and for the x - and q -gradients of $\hat{\psi}^m$, we set $\varphi := \log(\hat{\psi}^m + \delta) + 1$ in (6.54), where $\delta > 0$ is arbitrary. Note that such a choice is legitimate. Hence, by defining

$$\begin{aligned} G_\delta(s) &:= (s + \delta) \log(s + \delta) + e^{-1}, & G(s) &:= s \log s + e^{-1}, \\ T_{\delta,l}(s) &:= \int_0^s \frac{\Lambda_l(t)}{t + \delta} dt = \int_0^s \frac{t \Gamma_l(t)}{t + \delta} dt, \end{aligned}$$

(note that $G_\delta \geq 0$, and $T_{\delta,l} \rightarrow_{\delta \rightarrow 0^+} T_l$ in $\mathcal{C}([0, \infty))$), we obtain from equation (6.54) with $\varphi := \log(\hat{\psi}^m + \delta) + 1$ the following identity

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega \times D^J} \mathfrak{M}(q) G_\delta(\hat{\psi}^m) dx dq - \left(\mathfrak{M}(q) u^m, \nabla G_\delta(\hat{\psi}^m) \right)_{\Omega \times D^J} \\ &+ \frac{\beta}{J+1} \left(\frac{\mathfrak{M}(q)}{\hat{\psi}^m + \delta} \nabla \hat{\psi}^m, \nabla \hat{\psi}^m \right)_{\Omega \times D^J} + \beta \left(\frac{\mathfrak{M}(q)}{\hat{\psi}^m + \delta} \mathcal{R} \partial_q \hat{\psi}^m, \partial_q \hat{\psi}^m \right)_{\Omega \times D^J} \\ &= \left(\mathfrak{M}(q) \nabla u^m q, \partial_q T_{\delta,l}(\hat{\psi}^m) \right)_{\Omega \times D^J}. \end{aligned} \quad (6.67)$$

We begin by observing as in [14] that the second term on the left–hand side vanishes thanks to the divergence–free constraint on u^m . Next, integrating (6.67) with respect to time over $(0, t)$ and using the assumed properties of the Rouse matrix \mathcal{R} , we get that

$$\begin{aligned} & \int_{\Omega \times D^J} \mathfrak{M}(q) G_\delta(\hat{\psi}^m(\cdot, t)) \, dx \, dq + C \int_0^t \int_{\Omega \times D^J} \frac{\mathfrak{M}(q)}{\hat{\psi}^m + \delta} |\nabla_{x,q} \hat{\psi}^m|^2 \, dx \, dq \, d\tau \\ & \leq \int_{\Omega \times D^J} \mathfrak{M}(q) G_\delta(T_l(\hat{\psi}_0^m)) \, dx \, dq + \int_0^t \left(\mathfrak{M}(q) \nabla u^m q, \partial_q T_{\delta,l}(\hat{\psi}^m) \right)_{\Omega \times D^J} \, d\tau. \end{aligned} \quad (6.68)$$

Now we let $\delta \rightarrow 0+$ in (6.68). It is easy to identify the limit in the first term on the left-hand side and the first term on the right-hand side. Moreover, using the Monotone Convergence Theorem, we also easily identify the limit in the second term on the left-hand side. We thus obtain

$$\begin{aligned} & \int_{\Omega \times D^J} \mathfrak{M}(q) G(\hat{\psi}^m(\cdot, t)) \, dx \, dq + 4C_{\mathcal{R}} \int_0^t \int_{\Omega \times D^J} \mathfrak{M}(q) \left| \nabla_{x,q} \sqrt{\hat{\psi}^m} \right|^2 \, dx \, dq \, d\tau \\ & \leq \int_{\Omega \times D^J} \mathfrak{M}(q) G(T_l(\hat{\psi}_0^m)) \, dx \, dq + \limsup_{\delta \rightarrow 0+} \int_0^t \left(\mathfrak{M}(q) \nabla u^m q, \partial_q T_{\delta,l}(\hat{\psi}^m) \right)_{\Omega \times D^J} \, d\tau. \end{aligned} \quad (6.69)$$

Finally, we focus on the last term on the right–hand side. Using integration by parts, we find, on noting the property $\nabla \cdot u^m = 0$, that

$$\begin{aligned} & \int_0^t \left(\mathfrak{M}(q) (\nabla u^m) q, \partial_q T_{\delta,l}(\hat{\psi}^m) \right)_{\Omega \times D^J} \, d\tau \\ & = - \int_0^t \left(\partial_q \cdot (\mathfrak{M}(q) (\nabla u^m) q), T_{\delta,l}(\hat{\psi}^m) \right)_{\Omega \times D^J} \, d\tau \\ & \quad + \sum_{j=1}^J \int_0^t \int_{\Omega \times \partial D} \mathfrak{M}(q) (\nabla u^m) \cdot n(q_j) q_j T_{\delta,l}(\hat{\psi}^m) \, dx \, ds(q) \, d\tau \\ & = - \sum_{j=1}^J \int_0^t \left(\nabla u^m, T_{\delta,l}(\hat{\psi}^m) \partial_{q_j} \mathfrak{M}(q) \otimes q_j \right)_{\Omega \times D^J} \, d\tau \\ & \quad + \sum_{j=1}^J \int_0^t \int_{\Omega \times \partial D} \mathfrak{M}(q) (\nabla u^m) \cdot n(q_j) q_j T_{\delta,l}(\hat{\psi}^m) \, dx \, ds(q) \, d\tau \\ & = \sum_{j=1}^J \int_0^t \left(\nabla u^m, T_{\delta,l}(\hat{\psi}^m) \mathfrak{M}(q) q_j \otimes q_j \right)_{\Omega \times D^J} \, d\tau \\ & \quad + \sum_{j=1}^J \int_0^t \int_{\Omega \times \partial D} \mathfrak{M}(q) (\nabla u^m) \cdot n(q_j) q_j T_{\delta,l}(\hat{\psi}^m) \, dx \, ds(q) \, d\tau. \end{aligned}$$

Thus, since $T_{\delta,l}$ converges to T_l in $\mathcal{C}([0, \infty))$, we can easily pass to the limit, as in [14], in the last integral of (6.69) to obtain

$$\begin{aligned}
& \int_{\Omega \times D^J} \mathfrak{M}(q) G(\hat{\psi}^m(\cdot, t)) \, dx \, dq + 4 C_{\mathcal{R}} \int_0^t \int_{\Omega \times D^J} \mathfrak{M}(q) \left| \nabla_{x,q} \sqrt{\hat{\psi}^m} \right|^2 \, dx \, dq \, d\tau \\
& \leq \int_{\Omega \times D^J} \mathfrak{M}(q) G(T_l(\hat{\psi}_0^m)) \, dx \, dq + \sum_{j=1}^J \int_0^t \left(\nabla u^m, T_l(\hat{\psi}^m) \mathfrak{M}(q) q_j \otimes q_j \right)_{\Omega \times D^J} \, d\tau \quad (6.70) \\
& + \sum_{j=1}^J \int_0^t \int_{\Omega \times \partial D} \mathfrak{M}(q) (\nabla u^m) \cdot n(q_j) q_j T_l(\hat{\psi}^m) \, dx \, ds(q) \, d\tau.
\end{aligned}$$

Finally, we multiply the i -th equation in (6.53) by $c_i^m(t)$ to deduce the following energy identity (note that the convective term vanishes)

$$\frac{1}{2} \frac{d}{dt} \|u^m\|_{L^2(\Omega)}^2 + \mu \|\nabla u^m\|_{L^2(\Omega)}^2 = -(\mathbb{K}_{(0)}^m, \nabla u^m). \quad (6.71)$$

Using $\nabla \cdot u^m = 0$ and the definition of $\mathbb{K}_{(0)}^m$ (cf. (6.56)) we deduce that

$$(\mathbb{K}_{(0)}^m, \nabla u^m) = \sum_{j=1}^J \left(\nabla u^m, \mathfrak{M}(q) T_l(\hat{\psi}^m) q_j \otimes q_j \right)_{\Omega \times D^J}. \quad (6.72)$$

Hence, as in [14], using this in (6.71), integrating over $(0, t)$ and adding the result to (6.70), we get

$$\begin{aligned}
& \int_{\Omega \times D^J} \mathfrak{M}(q) G(\hat{\psi}^m(\cdot, t)) \, dx + \frac{1}{2} \int_{\Omega} |u^m(\cdot, t)|^2 \, dx \, dq \\
& + 4 C_{\mathcal{R}} \int_0^t \int_{\Omega \times D^J} \mathfrak{M}(q) \left| \nabla_{x,q} \sqrt{\hat{\psi}^m} \right|^2 \, dx \, dq \, d\tau + \mu \int_0^t \|\nabla u^m\|_{L^2(\Omega)}^2 \, d\tau \\
& \leq \int_{\Omega \times D^J} \mathfrak{M}(q) G(T_l(\hat{\psi}_0^m)) \, dx \, dq + \frac{1}{2} \int_{\Omega} |u_0^m(\cdot)|^2 \, dx \quad (6.73) \\
& + \sum_{j=1}^J \int_0^t \int_{\Omega \times \partial D} \mathfrak{M}(q) (\nabla u^m) \cdot n(q_j) q_j T_l(\hat{\psi}^m) \, dx \, ds(q) \, d\tau.
\end{aligned}$$

Then, using Young's inequality together with the trace theorem, we get

$$\begin{aligned}
& \int_{\Omega \times D^J} \mathfrak{M}(q) G(\hat{\psi}^m(\cdot, t)) \, dx + \frac{1}{2} \int_{\Omega} |u^m(\cdot, t)|^2 \, dx \, dq \\
& \quad + 4C_{\mathcal{R}} \int_0^t \int_{\Omega \times D^J} \mathfrak{M}(q) \left| \nabla_{x,q} \sqrt{\hat{\psi}^m} \right|^2 \, dx \, dq \, d\tau + \mu \int_0^t \|\nabla u^m\|_{L^2(\Omega)}^2 \, d\tau \\
& \leq \int_{\Omega \times D^J} \mathfrak{M}(q) G(T_l(\hat{\psi}_0^m)) \, dx \, dq + \frac{1}{2} \int_{\Omega} |u_0^m(\cdot)|^2 \, dx \\
& \quad + \sum_{j=1}^J \int_0^t \left(\int_{\Omega} |\nabla u^m|^2 \, dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \|\hat{\psi}^m\|_{L^1(\partial D)}^2 \, dx \right)^{\frac{1}{2}} \, d\tau \\
& \leq \int_{\Omega \times D^J} \mathfrak{M}(q) G(T_l(\hat{\psi}_0^m)) \, dx \, dq + \frac{1}{2} \int_{\Omega} |u_0^m(\cdot)|^2 \, dx \tag{6.74} \\
& \quad + \epsilon \sum_{j=1}^J \int_0^t \int_{\Omega} |\nabla u^m|^2 \, dx \, d\tau + \frac{1}{4\epsilon} \sum_{j=1}^J \int_0^t \int_{\Omega} \|\hat{\psi}^m\|_{L^1(\partial D)}^2 \, dx \, d\tau \\
& \leq \int_{\Omega \times D^J} \mathfrak{M}(q) G(T_l(\hat{\psi}_0^m)) \, dx \, dq + \frac{1}{2} \int_{\Omega} |u_0^m(\cdot)|^2 \, dx \\
& \quad + \frac{\mu}{2} \sum_{j=1}^J \int_0^t \int_{\Omega} |\nabla u^m|^2 \, dx \, d\tau + \frac{C}{2\mu} \sum_{j=1}^J \int_0^t \int_{\Omega} \int_D \mathfrak{M}(q) \left| \nabla_q \sqrt{\hat{\psi}^m} \right|^2 \, dq \, dx \, d\tau \\
& \quad + \frac{C}{2\mu} \sum_{j=1}^J \int_0^t \int_{\Omega} \int_D \mathfrak{M}(q) \left| \sqrt{\hat{\psi}^m} \right|^2 \, dq \, dx \, d\tau.
\end{aligned}$$

Thus, assuming that $\mu \geq \frac{C}{2C_{\mathcal{R}}}$ and using Gronwall's inequality, we arrive at the following estimate that is uniform with respect to m :

$$\begin{aligned}
& \sup_{t \in (0, T)} \left(\|\mathfrak{M}(q) G(\hat{\psi}^m(\cdot, t))\|_{L^1(\Omega \times D^J)} + \|u^m(\cdot, t)\|_{L^2(\Omega)}^2 \right) \\
& \quad + \int_0^t \left\| \sqrt{\mathfrak{M}(q)} \nabla_{x,q} \sqrt{\hat{\psi}^m} \right\|^2 \, d\tau + \mu \int_0^t \|u^m\|_{W^{1,2}(\Omega)}^2 \, d\tau \tag{6.75} \\
& \leq C \left(\|\mathfrak{M}(q) G(T_l(\hat{\psi}_0^m))\|_{L^1(\Omega \times D^J)} + \|u_0^m(\cdot)\|_{L^2(\Omega)}^2 \right) \\
& \leq C(l).
\end{aligned}$$

Standard interpolation inequalities and (6.75) then yield the estimate

$$\int_0^t \|u^m\|_{L^{\frac{2(d+2)}{d}}(\Omega)}^{\frac{2(d+2)}{d}} \, d\tau \leq C(l). \tag{6.76}$$

It is evident from the definition (6.56) of $\mathbb{K}_{(0)}^m$ and from the boundedness of $\mathfrak{M}(q)$ that

$$\|\mathbb{K}_{(0)}^m\| \leq Cl. \quad (6.77)$$

Consequently, as in [14], thanks to the presence of Γ_l in the convective term, it directly follows from (6.75), (6.77) and (6.53) that

$$\int_0^t \|\partial_t u^m\|_{W_{0,div}^{-1,2}}^2 d\tau \leq C(l). \quad (6.78)$$

6.2.6 The limit $m \rightarrow \infty$

In this subsection, proceeding as in [14], we let $m \rightarrow \infty$ to establish the existence of a weak solution stated in Theorem 6.2.2. First, using (6.75)–(6.78) and the Aubin–Lions Lemma we deduce the existence of a subsequence that we do not relabel, and $(u, \mathbb{K}_{(0)})$, such that

$$u^m \rightharpoonup^* u \quad \text{weak}^* \text{ in } L^\infty(0, T; L_{0,div}^2), \quad (6.79)$$

$$u^m \rightharpoonup u \quad \text{weakly in } L^2(0, T; W_{0,div}^{1,2}), \quad (6.80)$$

$$\partial_t u^m \rightharpoonup \partial_t u \quad \text{weakly in } L^2(0, T; W_{0,div}^{-1,2}), \quad (6.81)$$

$$u^m \rightharpoonup u \quad \text{weakly in } L^{\frac{2(d+2)}{d}}(0, T; L^{\frac{2(d+2)}{d}}(\Omega)^d), \quad (6.82)$$

$$u^m \rightarrow u \quad \text{strongly in } L^1(0, T; L^1(\Omega)^d), \quad (6.83)$$

$$\mathbb{K}_{(0)}^m \rightharpoonup^* \mathbb{K}_{(0)} \quad \text{weak}^* \text{ in } L^\infty(0, T; L^\infty(\Omega)^{d \times d}). \quad (6.84)$$

With these convergence results it is then standard to let $m \rightarrow \infty$ in (6.53) to deduce (6.22). Also, as in [14], one can show the attainment of the initial condition for the velocity (6.25).

Further, in order to identify all limits in (6.54) we focus on the convergence properties of $\hat{\psi}^m$. First, we define $\psi^m := \mathfrak{M}(q) \hat{\psi}^m$; using the definition of G we then deduce from (6.75) and the boundedness of $\mathfrak{M}(q)$ that

$$\sup_{t \in (0, T)} \int_{\Omega \times D^J} \psi^m(x, q, t) \ln(1 + \psi^m(x, q, t)) dx dq \leq C(l). \quad (6.85)$$

Since (6.85) implies the uniform equiintegrability of the sequence ψ^m , it directly follows from the characterization of weakly compact sets in L^1 that there exists a $\psi \in L^1(\Omega \times D^J \times (0, T))$ and a subsequence that we do not relabel such that

$$\hat{\psi}^m \rightharpoonup \hat{\psi} \quad \text{weakly in } L^1(\Omega \times D^J \times (0, T)). \quad (6.86)$$

Next, as in [14], we show that there is a subsequence (again not relabelled) such that

$$\hat{\psi}^m \rightarrow \hat{\psi} \quad \text{a.e. in } \Omega \times D^J \times (0, T). \quad (6.87)$$

Hence, let $\mathcal{O}_0 \subset \bar{\mathcal{O}}_0 \subset \mathcal{O} := \Omega \times D^J$ be an arbitrary Lipschitz domain. It then follows from (6.75) and from the properties of $\mathfrak{M}(q)$ that

$$\sup_{t \in (0, T)} \left\| \sqrt{\hat{\psi}^m(\cdot, t)} \right\|_{L^2(\mathcal{O}_0)}^2 + \int_0^T \left\| \sqrt{\hat{\psi}^m} \right\|_{W^{1,2}(\mathcal{O}_0)}^2 dt \leq C(\mathcal{O}_0). \quad (6.88)$$

Using standard interpolation inequalities we then deduce from (6.88) that

$$\int_0^T \int_{\mathcal{O}_0} |\hat{\psi}^m|^{\frac{(J+1)d+2}{d(J+1)}} dx dq dt = \int_0^T \int_{\mathcal{O}_0} |\sqrt{\hat{\psi}^m}|^{\frac{2((J+1)d+2)}{d(J+1)}} dx dq dt \leq C(\mathcal{O}_0). \quad (6.89)$$

One can further interpolate using (6.88)–(6.89) and the Hölder inequality to obtain, for any $p \in [1, 2)$, that

$$\begin{aligned} \int_0^T \int_{\mathcal{O}_0} |\partial_{x,q} \hat{\psi}^m|^p dx dq dt &= C \int_0^T \int_{\mathcal{O}_0} |\partial_{x,q} \sqrt{\hat{\psi}^m}|^p |\sqrt{\hat{\psi}^m}|^p dx dq dt \\ &\leq C \left(\int_0^T \int_{\mathcal{O}_0} |\partial_{x,q} \sqrt{\hat{\psi}^m}|^2 dx dq dt \right)^{\frac{p}{2}} \left(\int_0^T \int_{\mathcal{O}_0} |\sqrt{\hat{\psi}^m}|^{\frac{2p}{2-p}} dx dq dt \right)^{\frac{2-p}{p}} \leq C(\mathcal{O}_0). \end{aligned} \quad (6.90)$$

provided that

$$\frac{2p}{2-p} \leq \frac{2((K+1)d+2)}{d(K+1)}.$$

Thus, by selecting the 'optimal' value $p := \frac{(K+1)d+2}{(K+1)d+1}$ that maximizes the power p on the left-hand side of the last inequality, we finally obtain

$$\int_0^T \int_{\mathcal{O}_0} |\partial_{x,q} \hat{\psi}^m|^{\frac{2((K+1)d+2)}{d(K+1)}} dx dq dt \leq C(\mathcal{O}_0). \quad (6.91)$$

The final improvement, the integrability of $\hat{\psi}^m$, will follow from the estimate on ϱ^m . Indeed, as in [14], it obviously follows from (6.62) and (6.66) that

$$\|\psi^m\|_{L^\infty(\Omega \times (0,T); L^1(D^J))} \leq C. \quad (6.92)$$

Thus interpolating between this and (6.89) and using the boundedness of $\mathfrak{M}(q)$, we see that for any $q_1 \in (1, \infty)$ there exists a $q_2 > 1$ such that

$$\|\psi^m\|_{L^{q_1}(\Omega_0 \times (0,T); L^{q_2}(D_0^J))} \leq C(\mathcal{O}_0), \quad (6.93)$$

where we have used the notation $\mathcal{O}_0 := \Omega_0 \times D_0$. Consequently, using (6.76) and Hölder's inequality, there exists a $\delta > 0$ such that

$$\|u^m \psi^m\|_{L^{1+\delta}(\mathcal{O}_0 \times (0,T))^d} \leq C(\mathcal{O}_0). \quad (6.94)$$

Next, we use all of the above auxiliary estimates over \mathcal{O}_0 to deduce pointwise convergence of $\hat{\psi}^m$ by means of the Div–Curl Lemma. To this end, for some $\alpha \in (0, \frac{1}{2})$ that will be specified later, we define two $(1+d+Jd)$ -component vector fields (now vector means vector in all variables x, q, t) as follows:

$$\begin{aligned} H^m &:= (\mathfrak{M}(q) \hat{\psi}^m, \mathfrak{M}(q) \hat{\psi}^m u^m + \beta \mathfrak{M}(q) \nabla \hat{\psi}^m, \mathfrak{M}(q) \Lambda_l(\hat{\psi}^m) u^m q + \frac{\beta}{J+1} \mathfrak{M}(q) \partial_q \hat{\psi}^m), \\ Q^m &:= ((1 + \hat{\psi}^m)^\alpha, \mathbb{O}_{\mathbb{R}^{d+Jd}}). \end{aligned}$$

Consequently, using (6.76), (6.91) and (6.94), we deduce the existence of a subsequence (not relabelled) such that

$$\begin{aligned} H^m &\rightharpoonup H \text{ weakly in } L^{1+\delta}(\mathcal{O}_0 \times (0, T))^{1+d+Jd}, \\ Q^m &\rightharpoonup Q \text{ weakly in } L^{\frac{1}{\alpha}}(\mathcal{O}_0 \times (0, T))^{1+d+Jd}, \end{aligned}$$

where (we use strong convergence of u^m)

$$\begin{aligned} H &:= (\mathfrak{M}(q) \hat{\psi}, \mathfrak{M}(q) \hat{\psi} u + \beta \mathfrak{M}(q) \nabla \hat{\psi}, \mathfrak{M}(q) \overline{\Lambda_l(\hat{\psi})} u q + \frac{\beta}{J+1} \mathfrak{M}(q) \partial_q \hat{\psi}), \\ Q &:= \overline{((1 + \hat{\psi})^\alpha, \mathbb{O}_{\mathbb{R}^{d+Jd}})}. \end{aligned}$$

It remains to check the assumptions of the Div–Curl Lemma. First, it follows from (6.54) that

$$\nabla_{t,x,q} \cdot H^m = 0 \text{ in } \mathcal{O}_0 \times (0, T).$$

Moreover, as in [14], we get by using (6.88) and the fact that $\alpha \in (0, \frac{1}{2})$ that

$$\begin{aligned} &\int_{\mathcal{O}_0 \times (0, T)} |\nabla_{t,x,q} Q^m - (\nabla_{t,x,q} Q^m)^T|^2 dx dq dt \\ &\leq C \int_{\mathcal{O}_0 \times (0, T)} |\nabla_{x,q} (1 + \hat{\psi}^m)^\alpha|^2 dx dq dt \\ &\leq C \int_{\mathcal{O}_0 \times (0, T)} |\nabla_{x,q} \sqrt{\hat{\psi}^m}|^2 dx dq dt \leq C(\mathcal{O}_0). \end{aligned} \tag{6.95}$$

Hence, the divergence of H^m is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0, T))$ and the curl of Q^m is precompact in $W^{-1,2}(\mathcal{O}_0 \times (0, T))$. Consequently, by choosing $\alpha < \frac{\delta}{1+\delta}$, we deduce that all the assumptions of the Div–Curl Lemma are satisfied, and therefore

$$H^m \cdot Q^m \rightharpoonup H \cdot Q \text{ weakly in } L^1(\mathcal{O}_0 \times (0, T)). \tag{6.96}$$

In particular, we have that

$$\mathfrak{M}(q) \hat{\psi}^m (1 + \hat{\psi}^m)^\alpha \rightharpoonup \mathfrak{M}(q) \hat{\psi} \overline{(1 + \hat{\psi})^\alpha};$$

This then implies that

$$(1 + \hat{\psi}^m)^{\alpha+1} \rightharpoonup (1 + \hat{\psi}) \overline{(1 + \hat{\psi})^\alpha};$$

Thanks to the convexity of the function $s \in [0, \infty) \mapsto s^{\alpha+1} \in [0, \infty)$ it follows that

$$(1 + \hat{\psi})^{\alpha+1} \leq (1 + \hat{\psi}) \overline{(1 + \hat{\psi})^\alpha},$$

and therefore,

$$(1 + \hat{\psi})^\alpha \leq \overline{(1 + \hat{\psi})^\alpha}.$$

On the other hand, the function $s \in [0, \infty) \mapsto s^\alpha \in [0, \infty)$ is concave, and therefore we immediately have that

$$(1 + \hat{\psi})^\alpha = \overline{(1 + \hat{\psi})^\alpha},$$

hence, we have identified the limit, and consequently, since $s \in [0, \infty) \mapsto s^\alpha \in [0, \infty)$ is strictly concave, thanks to Theorem 11.27 in [24] there exists a subsequence (not relabelled) such that

$$\hat{\psi}^m \rightarrow \hat{\psi} \text{ a.e. in } \mathcal{O}_0 \times (0, T). \quad (6.97)$$

Hence,

$$\psi^m \rightarrow \psi \text{ a.e. in } \mathcal{O}_0 \times (0, T). \quad (6.98)$$

Finally, as in [14], we select a nondecreasing sequence of nested sets $\{\mathcal{O}_0^k\}_{k \in \mathbb{N}}$ such that $\bigcup_{k=1}^\infty \mathcal{O}_0^k = \mathcal{O}$ and for each k we deduce pointwise convergence on \mathcal{O}_0^k . Thus using a diagonal procedure, we finally find a subsequence (not relabelled) such that (6.87) holds. Hence, by combining the uniform-integrability of the sequence ψ^m and (6.87) and recalling Vitali's Convergence Theorem we obtain that

$$\psi^m \rightarrow \psi \text{ strongly in } L^1(0, T; L^1(\mathcal{O})). \quad (6.99)$$

We now focus on passing to the limit $m \rightarrow \infty$ in (6.54). First, by interpolating between (6.66) and (6.99) we get that

$$\psi^m \rightarrow \psi \text{ strongly in } L^q(\Omega \times (0, T); L^1(D^J)), \text{ for all } q \in [1, \infty). \quad (6.100)$$

Next, for any measurable $U \subset (\Omega \times (0, T) \times D^J)$ we use Hölder's inequality to deduce that

$$\begin{aligned} \int_U \mathfrak{M}(q) |\nabla_{x,q} \hat{\psi}^m| dx dq dt &= 2 \int_U \mathfrak{M}(q) |\nabla_{x,q} \sqrt{\hat{\psi}^m}| \sqrt{\hat{\psi}^m} dx dq dt \\ &\leq 2 \left(\int_U \mathfrak{M}(q) |\nabla_{x,q} \sqrt{\hat{\psi}^m}|^2 dx dq dt \right)^{\frac{1}{2}} \left(\int_U \hat{\psi}^m dx dq dt \right)^{\frac{1}{2}} \\ &\leq C \epsilon^{\frac{1}{2}}, \end{aligned} \quad (6.101)$$

by using (6.75) and the equiintegrability of the sequence $\hat{\psi}^m$, provided that $|U| \leq \delta$. This then implies that we can extract a subsequence such that

$$\mathfrak{M}(q) \nabla_{x,q} \hat{\psi}^m \rightharpoonup \mathfrak{M}(q) \nabla_{x,q} \hat{\psi} \text{ weakly in } L^1(\Omega \times D^J \times (0, T))^{d(J+1)}, \quad (6.102)$$

where for the identification of the weak limit we used the fact that $\nabla_{x,q} \hat{\psi}^m$ converges weakly locally in L^1 , which follows from (6.91). In addition, by noting (6.100) it also follows from (6.75) that

$$\sqrt{\mathfrak{M}(q)} \nabla_{x,q} \sqrt{\hat{\psi}^m} \rightharpoonup \sqrt{\mathfrak{M}(q)} \nabla_{x,q} \sqrt{\hat{\psi}} \text{ weakly in } L^2(\Omega \times (0, T) \times D^J)^{d(J+1)}. \quad (6.103)$$

By using the same procedure as in (6.101) we also see that

$$\begin{aligned} & \int_{\Omega \times (0, T)} \left(\int_{D^J} \mathfrak{M}(q) |\nabla_{x, q} \hat{\psi}^m| \, dq \right)^2 \, dx \, dt \\ & \leq C \int_{\Omega \times (0, T)} \|\sqrt{\mathfrak{M}(q)} \nabla_{x, q} \sqrt{\hat{\psi}^m}\|_{L^2(D^J)}^2 \|\hat{\psi}^m\|_{L^1(D^J)}^2 \, dx \, dt \leq C, \end{aligned} \quad (6.104)$$

by using (6.66) and (6.75) and we can then strengthen (6.102) as follows

$$\mathfrak{M}(q) \nabla_{x, q} \hat{\psi}^m \rightharpoonup \mathfrak{M}(q) \nabla_{x, q} \hat{\psi} \text{ weakly in } L^2(\Omega \times (0, T); L^1(D^J)^{d(J+1)}). \quad (6.105)$$

Finally, as in [14], using (6.80), (6.82), (6.83) and (6.100) we deduce that, for all $q \in [1, \frac{r(d+2)}{d})$,

$$\mathfrak{M}(q) \hat{\psi}^m u^m \rightarrow \psi u \text{ strongly in } L^q(\Omega \times (0, T); L^1(D^J)^{d(J+1)}), \quad (6.106)$$

$$\Lambda_l(\hat{\psi}^m) \nabla u^m \rightharpoonup \Lambda_l(\hat{\psi}) \nabla u \text{ weakly in } L^2(\Omega \times D^J \times (0, T))^{d \times d}. \quad (6.107)$$

Consequently, using (6.54), and the convergence results (6.105)–(6.107) it follows that

$$\partial_t(\mathfrak{M}(q) \hat{\psi}^m) \rightharpoonup \partial_t \psi \text{ weakly in } L^1(\Omega \times (0, T); W^{-1,1}(D^J)^{d(J+1)}). \quad (6.108)$$

Thus, using the properties of the Rouse matrix, it is easy to let $m \rightarrow \infty$ in (6.54) to deduce (6.24). Moreover, one can also show (6.25)₂ by using standard arguments. Finally, to derive (6.26) and (6.27) we let $m \rightarrow \infty$ in (6.66), (6.74) and (6.75). To pass to the limit in all terms on the left-hand side, we use either weak lower semi-continuity of norms or Fatou's Lemma, and for the critical term on the right-hand side of (6.75) we have, by using (6.6), that

$$\|\mathfrak{M}(q) G(T_l(\hat{\psi}_0^m))\|_{L^1(\mathcal{O})} \rightarrow^{m \rightarrow \infty} \|\mathfrak{M}(q) G(T_l(\hat{\psi}_0))\|_{L^1(\mathcal{O})} \leq C.$$

Thus, the proof of Theorem 6.2.2 is complete.

6.3 Existence of global weak solutions for any viscosity

In the previous subsection, we showed the existence of global weak solutions to the system in the case of a sufficiently large viscosity coefficient in the Navier–Stokes equation, in this section, we are now exploring the possibility of removing the additional largeness assumption on the viscosity coefficient by considering boundary corrections of the polymeric extra stress tensor, to ensure the validity of a formal energy inequality for the coupled system. Therefore, we consider the following approximation of the polymeric extra stress tensor

$$\begin{aligned} \mathbb{K}^l_{(0)} &= n \int_{D^J} \mathfrak{M}(q) \sum_{j=1}^J (F(q_j) \otimes q_j) T_{\delta, l}(\hat{\psi}) \, dq \\ &+ \sum_{j=1}^J \int_{\partial D} \mathfrak{M}(q) \cdot n(q_j) q_j T_{\delta, l}(\hat{\psi}) \, ds(q). \end{aligned} \quad (6.109)$$

We use the same procedure as in the previous section.

6.3.1 Galerkin approximation for any viscosity

As in [14], we introduce a Galerkin approximation of (6.22)–(6.24).

The Hilbert space $W_{0,\text{div}}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d$, equipped with the inner product of $W^{d+1,2}(\Omega)^d$ is compactly and densely embedded in the Hilbert space $L_{0,\text{div}}^2(\Omega)^d$. Therefore, using the Hilbert–Schmidt theorem with $V = W_{0,\text{div}}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d$ and $H = L_{0,\text{div}}^2(\Omega)^d$ and $a(\cdot, \cdot)$ taken to be the inner product of $W^{d+1,2}(\Omega)^d$, there exists a countable set $\{\omega_i\}_{i \in \mathbb{N}}$ of eigenfunctions in $W_{0,\text{div}}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d$ whose linear span is dense in $L_{0,\text{div}}^2(\Omega)^d$, s.t. the ω_i , $i = 1, 2, \dots$ are orthogonal in the inner product $W^{d+1,2}(\Omega)^d$ and orthonormal in the inner product of $L^2(\Omega)^d$.

Similarly, for each $m \in \mathbb{N}$ we find a countable set $\{\varphi_i^m\}_{i \in \mathbb{N}}$ of eigenfunctions in $W^{1,2}(\Omega \times D^J)$ that are orthogonal in $W_{\mathfrak{M}(q)}^{1,2}(\Omega \times D^J)$ and orthonormal in $L_{\mathfrak{M}(q)}^2(\Omega \times D^J)$.

Finally, we fix $m, n \in \mathbb{N}$ and look for $(u^{m,n}, \hat{\psi}^{m,n})$ given by

$$u^{m,n}(x, t) := \sum_{i=1}^m c_i^{m,n}(t) \omega_i(x), \quad (6.110)$$

$$\hat{\psi}^{m,n}(x, q, t) := \sum_{i=1}^n d_i^{m,n}(t) \varphi_i^m(x, q), \quad (6.111)$$

that solve

$$\begin{aligned} & \int_0^T \int_{\Omega} \left[\partial_t u^{m,n} \cdot \omega_i - \Gamma_l(|u^{m,n}|^2) u^{m,n} \otimes u^{m,n} \cdot \omega_i + \mu \nabla u^{m,n} : \nabla \omega_i \right] dx dt \\ & = -n \int_0^T \mathbb{K}_{(0)}^{m,n} : \nabla \omega_i dx dt, \end{aligned} \quad (6.112)$$

for all $i = 1, \dots, m$ and a.e. $t \in (0, T)$, where

$$\begin{aligned} \mathbb{K}_{(0)}^{m,n} & := n \int_{D^J} \sum_{j=1}^J \mathfrak{M}(q) T_l(\hat{\psi}^{m,n}) q_j \otimes q_j dq \\ & + \sum_{j=1}^J \int_{\partial D} \mathfrak{M}(q) \cdot n(q_j) q_j T_l(\hat{\psi}^{m,n}) ds(q) \quad \text{a.e. in } \Omega \times (0, T), \end{aligned} \quad (6.113)$$

$$\begin{aligned} & \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \left[\varphi_i^m \partial_t \hat{\psi}^{m,n} + \left[\frac{\beta}{J+1} \nabla \hat{\psi}^{m,n} - u^{m,n} \hat{\psi}^{m,n} \right] \nabla \varphi_i^m \right] dq dx dt \\ & + \beta \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{i,j=1}^J \mathcal{R}_{ij} \partial_{q_j} \hat{\psi}^{m,n} \cdot \partial_{q_j} \varphi_i^m dq dx dt \\ & \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{j=1}^J \nabla u^{m,n} q_j \Lambda_l(\hat{\psi}^{m,n}) \cdot \partial_{q_j} \varphi_i^m dq dx dt = 0, \end{aligned} \quad (6.114)$$

for all $i = 1, \dots, n$ and a.e. $t \in (0, T)$, with initial data given by

$$u^{m,n}(x, 0) := \sum_{i=1}^m (u_0, \omega_i) \omega_i(x),$$

$$\hat{\psi}^{m,n}(x, q, 0) = \hat{\psi}_0^{m,n}(x, q) := \sum_{i=1}^n (T_l(\hat{\psi}_0), \varphi_i^m)_{\Omega \times D^J} \varphi_i^m(x, q).$$

The local in time existence of $u^{m,n}$ and $\hat{\psi}^{m,n}$ for fixed m, n follows from Carathéodory's theorem. Moreover, using the estimates established below we can extend the solution onto the whole time interval $(0, T)$.

6.3.2 The limit $n \rightarrow \infty$ for any viscosity

Here, proceeding as in [14], we let $n \rightarrow \infty$ in (6.31)–(6.33), (6.43). Thus, from the n -independent estimates (6.38), (6.43), (6.44), (6.45) and by using the Aubin–Lions Lemma we see that there exist subsequences, which we do not relabel, such that

$$c_i^{m,n} \rightharpoonup^* c_i^m \quad \text{weak}^* \text{ in } W^{1,\infty}(0, T), \quad (6.115)$$

$$c_i^{m,n} \rightarrow c_i^m \quad \text{strongly in } \mathcal{C}[0, T], \quad (6.116)$$

$$u^{m,n} \rightarrow c_i^m \quad \text{strongly in } \mathcal{C}(0, T; W_{0,\text{div}}^{1,2}(\Omega)^d \cap W^{d+1,2}(\Omega)^d), \quad (6.117)$$

$$\hat{\psi}^{m,n} \rightharpoonup \hat{\psi}^m \quad \text{weakly in } L^2(0, T; W^{1,2}(\Omega \times D^J)), \quad (6.118)$$

$$\partial_t(\mathfrak{M}(q) \hat{\psi}^{m,n}) \rightharpoonup \partial_t(\mathfrak{M}(q) \hat{\psi}^m) \quad \text{weakly in } L^2(0, T; W^{-1,2}(\Omega \times D^J)), \quad (6.119)$$

$$\hat{\psi}^{m,n} \rightarrow \hat{\psi}^m \quad \text{strongly in } L^2(0, T; L^2(\Omega \times D^J)), \quad (6.120)$$

$$\mathbb{K}_{(0)}^{m,n} \rightharpoonup^* \mathbb{K}_{(0)}^m \quad \text{weak}^* \text{ in } L^\infty(0, T; L^\infty(\Omega)). \quad (6.121)$$

In the light of these convergence results it is now standard to pass to the limit $n \rightarrow \infty$ in (6.31) and (6.33) to deduce that

$$\int_0^T \int_\Omega \left[\partial_t u^m \cdot \omega_i - \Gamma_l(|u^m|^2) u^m \otimes u^m \cdot \omega_i + \mu \nabla u^m : \nabla \omega_i \right] dx dt = -n \int_0^T \mathbb{K}_{(0)}^m : \nabla \omega_i dx dt, \quad (6.122)$$

for all $i = 1, \dots, m$ and a.e. $t \in (0, T)$,

$$\begin{aligned} & \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \left[\varphi \partial_t \hat{\psi}^m + \left[\frac{\beta}{J+1} \nabla \hat{\psi}^m - u^m \hat{\psi}^m \right] \nabla \varphi \right] dq dx dt \\ & + \beta \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{i,j=1}^J \mathcal{R}_{ij} \partial_{q_j} \hat{\psi}^m \cdot \partial_{q_j} \varphi dq dx dt \\ & \int_0^T \int_{\Omega \times D^J} \mathfrak{M}(q) \sum_{j=1}^J \nabla u^m q_j \Lambda_l(\hat{\psi}^m) \cdot \partial_{q_j} \varphi dq dx dt = 0, \end{aligned} \quad (6.123)$$

for all $\varphi \in W^{1,2}(\Omega \times D^J)$ and a.e. $t \in (0, T)$, Moreover, it is obvious that $u^m(x, 0) = u_0^m(x)$ and it is completely standard to show that

$$\lim_{t \rightarrow 0^+} \|\hat{\psi}^m(\cdot, t) - T_l(\hat{\psi}_0^m(\cdot))\|_{L^2(\Omega \times D^J)} = 0. \quad (6.124)$$

Then, as in [14], using (6.51) for the first term and (6.49) together with compactness of the trace operator (see Theorem 6.2 page 103 of [53]) and the Lebesgue Dominated Convergence Theorem for the second term, we can take the limit in (6.109) and deduce that

$$\begin{aligned} \mathbb{K}_{(0)}^m &:= n \int_{D^J} \sum_{j=1}^J \mathfrak{M}(q) T_l(\hat{\psi}^m) q_j \otimes q_j \, dq \\ &+ \sum_{j=1}^J \int_{\partial D} \mathfrak{M}(q) \cdot n(q_j) q_j T_l(\hat{\psi}^m) \, ds(q), \quad \text{a.e. in } \Omega \times (0, T), \end{aligned} \quad (6.125)$$

6.4 Uniform m -independent estimates for any viscosity

Here, the energy identity (6.71), becomes

$$\frac{1}{2} \frac{d}{dt} \|u^m\|_{L^2(\Omega)}^2 + \mu \|\nabla u^m\|_{L^2(\Omega)}^2 = -(\mathbb{K}_{(0)}^m, \nabla u^m). \quad (6.126)$$

where

$$\begin{aligned} (\mathbb{K}_{(0)}^m, \nabla u^m) &= \sum_{j=1}^J \left(\nabla u^m, \mathfrak{M}(q) T_l(\hat{\psi}^m) q_j \otimes q_j \right)_{\Omega \times D^J} \\ &+ \sum_{j=1}^J \int_{\Omega} \int_{\partial D} \nabla u^m \mathfrak{M}(q) \cdot n(q_j) q_j T_l(\hat{\psi}^m) \, ds(q) \, dx. \end{aligned} \quad (6.127)$$

Hence the inequality (6.73) becomes

$$\begin{aligned} &\int_{\Omega \times D^J} \mathfrak{M}(q) G(\hat{\psi}^m(\cdot, t)) \, dx + \frac{1}{2} \int_{\Omega} |u^m(\cdot, t)|^2 \, dx \, dq \\ &+ 4C_{\mathcal{R}} \int_0^t \int_{\Omega \times D^J} \mathfrak{M}(q) \left| \nabla_{x,q} \sqrt{\hat{\psi}^m} \right|^2 \, dx \, dq \, d\tau + \mu \int_0^t \|\nabla u^m\|_{L^2(\Omega)}^2 \, d\tau \\ &\leq \int_{\Omega \times D^J} \mathfrak{M}(q) G(T_l(\hat{\psi}_0^m)) \, dx \, dq + \frac{1}{2} \int_{\Omega} |u_0^m(\cdot)|^2 \, dx. \end{aligned} \quad (6.128)$$

Thus, without assuming anything on the viscosity other than its positivity, we arrive at the following energy inequality that is uniform with respect to m :

$$\begin{aligned} &\sup_{t \in (0, T)} \left(\|\mathfrak{M}(q) G(\hat{\psi}^m(\cdot, t))\|_{L^1(\Omega \times D^J)} + \|u^m(\cdot, t)\|_{L^2(\Omega)}^2 \right) \\ &\leq C(l). \end{aligned} \quad (6.129)$$

6.4.1 The limit $m \rightarrow \infty$

Then, using the same argument as in the previous section, by passing to the limit $m \rightarrow \infty$, we establish the existence of a weak solution stated in Theorem 6.2.2 where

$$\begin{aligned} \mathbb{K}_{(0)}^l(x, t) &= n \int_{D^J} \sum_{j=1}^J \mathfrak{M}(q) T_l(\hat{\psi}^l(x, q, t)) q_j \otimes q_j \, dq \\ &+ \sum_{j=1}^J \int_{\partial D} \mathfrak{M}(q) \cdot n(q_j) q_j T_l(\hat{\psi}^l(x, q, t)) \, ds(q). \end{aligned} \quad (6.130)$$

6.5 Proof of the main theorem

The final section is devoted to the proof of Theorem 6.2.1. As in [14], we use the sequence of approximate solutions $(u_{(0)}^l, \mathbb{K}_{(0)}^l, \hat{\psi}^l)$ constructed in Theorem 6.2.2 and let $l \rightarrow \infty$.

6.5.1 Weak/strong convergence results for $u_{(0)}^l$

First, we recall the uniform estimate (6.27) (where C is a generic positive constant that is independent of l)

$$\begin{aligned} &\sup_{t \in (0, T)} \left(\|u_{(0)}^l(\cdot, t)\|_{L^2(\Omega)}^2 + \|\hat{\psi}^l(\cdot, t) \log \psi^l(\cdot, t)\|_{L^1_{\mathfrak{M}(q)}(\Omega \times D^J)} + \|\varrho^l(\cdot, t)\|_{L^\infty(\Omega)} \right) \\ &\int_0^T \left(\|u_{(0)}^l\|_{W^{1,2}(\Omega)}^2 + \|\sqrt{\hat{\psi}^l}\|_{W^{1,2}_{\mathfrak{M}(q)}(\Omega \times D^J)}^2 + \|\mathbb{K}_{(0)}^l\|_{L^2(\Omega)}^2 + \|\nabla \varrho^l\|_{L^2(\Omega)}^2 \right) dt \\ &\leq C, \end{aligned} \quad (6.131)$$

where

$$\varrho^l(x, t) := \int_{D^J} \mathfrak{M}(q) \hat{\psi}^l(x, q, t) \, dq. \quad (6.132)$$

Next, as in [14], using function space interpolation one can deduce from (6.131) that

$$\int_0^T \|u_{(0)}^l\|_{L^{\frac{2(d+2)}{d}}}^2 \, d\tau \leq C. \quad (6.133)$$

Hence, using (6.131), (6.133) and (6.22)

$$\int_0^T \|\partial_t u_{(0)}^l\|_{W_{0, \text{div}}^{-1,2}}^2 \, d\tau \leq C. \quad (6.134)$$

Consequently, using the uniform estimates above in conjunction with the Aubin–Lions Lemma we deduce the existence of subsequences, which we do not relabel, and an associated couple $(u_{(0)}, \mathbb{K}_{(0)})$ such that

$$u_{(0)}^l \rightharpoonup^* u_{(0)} \quad \text{weak}^* \text{ in } L^\infty(0, T; L_{0,\text{div}}^2), \quad (6.135)$$

$$u_{(0)}^l \rightharpoonup u_{(0)} \quad \text{weakly in } L^2(0, T; W_{0,\text{div}}^{1,2}), \quad (6.136)$$

$$\partial_t u_{(0)}^l \rightharpoonup \partial_t u_{(0)} \quad \text{weakly in } L^2(0, T; W_{0,\text{div}}^{-1,2}), \quad (6.137)$$

$$u_{(0)}^l \rightharpoonup u_{(0)} \quad \text{weakly in } L^{\frac{2(d+2)}{d}}(0, T; L^{\frac{2(d+2)}{d}}(\Omega)^d), \quad (6.138)$$

$$u_{(0)}^l \rightarrow u_{(0)} \quad \text{strongly in } L^2(0, T; L^2(\Omega)^d), \quad (6.139)$$

$$\mathbb{K}_{(0)}^l \rightharpoonup \mathbb{K}_{(0)} \quad \text{weakly in } L^2(0, T; L^2(\Omega)^{d \times d}). \quad (6.140)$$

As, unlike the model considered in [14], where the solvent is supposed to be a non-Newtonian fluid with a nonlinear stress-strain relationship, in our case the situation is simpler, in that the solvent is supposed to be a Newtonian fluid. Therefore in our case only the passage to the limit $l \rightarrow \infty$ that we have obtained above is needed, in contrast with the more complicated limiting procedure considered in [14]. Using the above convergence results, it is now standard to let $l \rightarrow \infty$ in (6.22) to deduce (6.8) and (6.9).

Conclusions

In this work, we have reformulated a general class of classical bead-spring-chain models for dilute polymeric fluids, with Hookean spring potentials, as McKean–Vlasov diffusion. This resulted in a coupled system of partial differential equations involving the unsteady incompressible linearized Navier–Stokes equations, referred to as the Oseen system, for the velocity and the pressure of the fluid, with a source term which is a nonlinear function of the probability density function, and a second-order degenerate parabolic Fokker–Planck equation, whose transport terms depend on the velocity field, for the probability density function. We showed that the coupled Oseen–Fokker–Planck system has a large-data global weak solution. We have then performed a rigorous passage to the limit as the masses of the beads in the bead-spring-chain converge to zero, which was shown in particular to result in equilibration in momentum space. The limiting problem was then used to perform a rigorous derivation of the Hookean bead–spring–chain model for dilute polymeric fluids. We then extended the results obtained to the coupling of the incompressible Navier–Stokes equations with a Fokker–Planck equation, and dealt with the usual technical difficulties associated with the presence of the nonlinear convection term in the Navier–Stokes equation, and we showed the existence of global weak solutions to the Hookean bead–spring–chain model in both two and three space dimensions. The conclusion is that now it is clear that the configuration space domain is bounded and that before passing to the small–mass limit, the Fokker–Planck equation is degenerate parabolic and after passing to the small mass limit, the Fokker–Planck equation is parabolic. As for further direction of research, one could also consider more complex models with interacting polymer chains and with a more complex boundary condition than the reflection boundary condition. Another interesting question one could consider is letting the number of beads J tend to infinity.

Appendix A

Proof of Theorem 1.4.1

Theorem 1.4.1. *The probability density function denoted by $\varrho = \varrho(r, v, t)$ solves the non-linear partial differential equation*

$$\partial_t \varrho - \frac{\beta}{\epsilon^2} \left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (v_j \varrho) + \beta \partial_{v_j}^2 \varrho \right) + \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} v_j \cdot \partial_{r_j} \varrho + ((\mathcal{L}r)_j + u(r_j, t)) \cdot \partial_{v_j} \varrho \right) = 0, \quad (\text{A.1})$$

$$\begin{aligned} & \text{for all } (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T], \\ \varrho(r, v, 0) &= \varrho_0(r, v) \quad \text{for all } (r, v) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d}. \end{aligned} \quad (\text{A.2})$$

Proof. of Theorem 1.4.1. The proof is a straightforward consequence of Itô's formula, which says that, if $\varphi : \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T] \rightarrow \mathbb{R}$ is smooth with compact support, then

$$\begin{aligned} \varphi(r_T, v_T, T) &= \varphi(r_0, v_0, 0) + \int_0^T \partial_\tau \varphi(r_\tau, v_\tau, \tau) d\tau + \int_0^T \partial_r \varphi \cdot dr_\tau \\ &\quad + \int_0^T \partial_v \varphi \cdot dv_\tau + \frac{1}{2} \int_0^T \nabla_{r,v}^2 \varphi : d \begin{pmatrix} \langle r, r \rangle_\tau & \langle r, v \rangle_\tau \\ \langle v, r \rangle_\tau & \langle v, v \rangle_\tau \end{pmatrix} \\ &= \varphi(r_0, v_0, 0) + \int_0^T \partial_\tau \varphi(r_\tau, v_\tau, \tau) d\tau + \int_0^T \partial_r \varphi \cdot dr_\tau \\ &\quad + \int_0^T \partial_v \varphi \cdot dv_\tau + \frac{1}{2} \int_0^T \nabla_{r,v}^2 \varphi : d \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ \mathbb{O} & 2\epsilon^{-2} \beta \tau \mathbb{I} \end{pmatrix} \\ &= \varphi(r_0, v_0, 0) + \int_0^T \partial_\tau \varphi(r_\tau, v_\tau, \tau) d\tau + \int_0^T \partial_r \varphi \cdot dr_\tau \\ &\quad + \int_0^T \partial_v \varphi \cdot dv_\tau + \epsilon^{-2} \beta \int_0^T \partial_{v,v}^2 \varphi d\tau. \end{aligned}$$

Where in the above, $\langle r, v \rangle_\tau$ denotes the cross-variation of the process (r, v) (see [36], Definitions 5.3 and 5.5 on page 31). Replacing v_t and r_t , we obtain

$$\begin{aligned}\varphi(r_T, v_T, T) &= \varphi(r_0, v_0, 0) + \int_0^T \partial_\tau \varphi(r_\tau, v_\tau, \tau) \, d\tau + \epsilon^{-1} \int_0^T \partial_r \varphi \cdot v_\tau \, d\tau \\ &\quad + \epsilon^{-1} \int_0^T \partial_v \varphi \cdot (\mathcal{L}r + \mathcal{U}(r, t; \varrho) - \epsilon^{-1} v_\tau) \, d\tau \\ &\quad + \epsilon^{-1} \sqrt{2\beta} \int_0^T \partial_v \varphi \cdot dW_\tau + \epsilon^{-2} \beta \int_0^T \partial_{v,v}^2 \varphi \, d\tau.\end{aligned}$$

Taking the expectation on both sides, we get

$$\begin{aligned}\int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varphi(r, v, T) \varrho \, dr \, dv &= \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \varphi(r, v, 0) \varrho_0 \, dr \, dv \\ &\quad + \int_0^T \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_\tau \varphi(r_\tau, v_\tau, \tau) \varrho \, dr \, dv \, d\tau \\ &\quad + \epsilon^{-1} \int_0^T \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_r \varphi \cdot v_\tau \varrho \, dr \, dv \, d\tau \\ &\quad + \epsilon^{-1} \int_0^T \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_v \varphi \cdot (\mathcal{L}r + \mathcal{U}(r, t; \varrho) - \epsilon^{-1} v_\tau) \varrho \, dr \, dv \, d\tau \\ &\quad + \epsilon^{-2} \beta \int_0^T \int_{\mathbb{R}^{(J+1)d}} \int_{\Omega^{J+1}} \partial_{v,v}^2 \varphi \varrho \, dr \, dv \, d\tau,\end{aligned}$$

which was the stated result. □

Appendix B

Extension of strong solutions for a coupled Navier–Stokes–Fokker–Planck system

B.1 Introduction

In this appendix, our aim is to extend the results obtained in the previous chapters to the coupling of the incompressible Navier–Stokes equations with a Fokker–Planck equation, and to deal with the usual technical difficulties associated with the presence of the nonlinear convection term in the Navier–Stokes equation. More precisely, we shall consider the following unsteady Navier–Stokes system, with a source term which is a nonlinear function, on the space-time domain $\bar{\Omega} \times [0, T]$, where Ω is a bounded open convex domain in \mathbb{R}^d , $d \in \{2, 3\}$, with a $C^{2,1}$ boundary, and $T > 0$:

$$\partial_t u + (u \cdot \nabla)u - \mu \Delta u + \nabla \pi = \nabla \cdot \mathbb{K} \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (\text{B.1a})$$

$$\nabla \cdot u = 0 \quad \text{for } (x, t) \in \Omega \times (0, T], \quad (\text{B.1b})$$

$$u(x, t) = 0 \quad \text{for } (x, t) \in \partial\Omega \times (0, T], \quad (\text{B.1c})$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \Omega, \quad (\text{B.1d})$$

where $\mathbb{K} : \Omega \times (0, T] \rightarrow \mathbb{R}_{\text{symm}}^{d \times d}$ is the elastic extra stress tensor which satisfies:

$$\|\mathbb{K}\|_{L^\infty(0, T; L^\infty(\Omega))} < \infty. \quad (\text{B.2})$$

We shall assume, without loss of generality, that $\bar{\Omega}$ is contained in $[0, L]^d$ for a certain $L > 0$.

In the equations (B.1), $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}^d$ denotes the velocity field, and $\pi : \Omega \times (0, T] \rightarrow \mathbb{R}$ is the pressure; $u_0 \in L^2(\Omega)^d$ is a divergence-free (in the sense of distributions on Ω) initial velocity field; $\mu > 0$ is the viscosity coefficient.

B.1.1 Preliminary results

In what follows we consider $d = 3$. The exposition follows the arguments and notational conventions of the paper by [27].

B.1.2 The Stokes operator A

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{2,1}$. Let $[0, T)$, $0 < T \leq \infty$, be a time interval and let $1 < q < \infty$.

Following [27], we use the space

$$L_\sigma^q(\Omega)^3 = \overline{C_{0,\sigma}^\infty(\Omega)^3}^{\|\cdot\|_{L^q(\Omega)}},$$

where

$$C_{0,\sigma}^\infty(\Omega)^3 = \{v \in C_0^\infty(\Omega)^3 : \operatorname{div} v = 0\}.$$

Then we denote by $P_q : L^q(\Omega)^3 \rightarrow L_\sigma^q(\Omega)^3$ the Helmholtz projection, and we define the Stokes operator:

$$A_q := -P_q \Delta : D(A_q) \rightarrow L_\sigma^q(\Omega)^3,$$

with domain $D(A_q) = W^{2,q}(\Omega)^3 \cap W_0^{1,q}(\Omega)^3 \cap L_\sigma^q(\Omega)^3$ and range $R(A_q) = L_\sigma^q(\Omega)^3$. In what follows, we will use the notation $P := P_2$ and $A := A_2$ if $q = 2$. We now introduce fractional powers of the Stokes operator:

$$A_q^\alpha : D(A_q^\alpha) \rightarrow L_\sigma^q(\Omega)^3, \quad D(A_q^\alpha) \subset L_\sigma^q(\Omega)^3,$$

with $-1 \leq \alpha \leq 1$, and the semigroup operator:

$$S(t) := e^{-tA}, \quad t \geq 0.$$

The following embedding properties play a key role in what follows:

$$\|v\|_{L^q(\Omega)} \leq c \|A_\gamma^\alpha v\|_{L^\gamma(\Omega)}, \quad v \in D(A_\gamma^\alpha), \quad 1 < \gamma \leq q, \quad \frac{2}{\alpha} + \frac{3}{q} = \frac{3}{\gamma},$$

$$0 < \alpha \leq 1, \quad (\text{B.3})$$

and we also have the following estimate

$$\left\| A_q^\alpha e^{-tA_q} v \right\|_{L^q(\Omega)} \leq c t^{-\alpha} e^{-\delta t} \|v\|_{L^q(\Omega)}, \quad v \in L_\sigma^q(\Omega)^3, \quad 0 < \alpha \leq 1, \quad t > 0, \quad (\text{B.4})$$

where $c = c(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$.

We have that $D(A_q^{1/2}) = W_0^{1,q}(\Omega)^3 \cap L_\sigma^q(\Omega)^3$ and that the following two norms are equivalent

$$\left\| A_q^{1/2} v \right\|_{L^q(\Omega)} \approx \|\nabla v\|_{L^q(\Omega)}, \quad v \in D(A_q^{1/2}), \quad (\text{B.5})$$

in particular, for $q = 2$,

$$\left\| A_q^{1/2} v \right\|_{L^2(\Omega)} \approx \|\nabla v\|_{L^2(\Omega)}, \quad v \in D(A_q^{1/2}). \quad (\text{B.6})$$

Set $g = \operatorname{div} G$ where $G = (G_{ij})_{i,j=1}^3 \in L^q(\Omega)^3$. Then, an approximation argument shows that $A^{-\frac{1}{2}} P_q \operatorname{div} G \in L_\sigma^q(\Omega)^3$ is well-defined by the identity

$$\langle A_q^{-\frac{1}{2}} P_q \operatorname{div} G, \varphi \rangle = \langle G, \nabla A_{q'}^{-\frac{1}{2}} \varphi \rangle, \quad \varphi \in L_\sigma^{q'}(\Omega)^3,$$

$\frac{1}{q} + \frac{1}{q'} = 1$, and that

$$\left\| A_q^{-\frac{1}{2}} P_q \operatorname{div} G \right\|_{L^q(\Omega)} \leq c \|G\|_{L^q(\Omega)}, \quad (\text{B.7})$$

with $c = c(\Omega, q) > 0$.

B.1.3 The nonstationary Stokes equation

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, and let $0 < T \leq \infty$. Then we consider in $[0, T) \times \Omega$ the Stokes and the Navier–Stokes equations. We are interested in the notions of weak and strong solution defined as follows.

Definition B.1.1 *Let $u_0 \in L^2_\sigma(\Omega)^3$ be the initial value and let $f = \operatorname{div} F$ with $F = (F_{ij})_{i,j=1}^3 \in L^2(0, T; L^2(\Omega)^{3 \times 3})$ be the external force. A vector field*

$$u \in L^\infty(0, T; L^2_\sigma(\Omega)) \cap L^2(0, T; W_0^{1,2}(\Omega)^3) \quad (\text{B.8})$$

is a weak solution in the sense of Leray–Hopf of the Navier–Stokes system

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla \pi &= f, \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0, \end{aligned} \quad (\text{B.9})$$

with data u_0, f , if for every test function $w \in C_0^\infty([0, T); C_{0,\sigma}^\infty(\Omega)^3)$, we have the relation

$$-\langle u, w_t \rangle + \langle \nabla u, \nabla w \rangle - \langle u \otimes u, \nabla w \rangle = \langle u_0, w(0) \rangle - \langle F, \nabla w \rangle, \quad (\text{B.10})$$

and if the followig energy inequality is satisfied for $0 \leq t < T$:

$$\frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla u\|_{L^2(\Omega)}^2 \, d\tau \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 - \int_0^t \langle F, \nabla u \rangle \, d\tau. \quad (\text{B.11})$$

A weak solution u of the system (B.9) is called a strong solution in the sense of Serrin if there are exponents $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} \leq 1$ such that additionally Serrin's condition

$$u \in L^s(0, T; L^q(\Omega)^3), \quad (\text{B.12})$$

is satisfied.

A vector field u satisfying (B.8) is a weak solution of the (linear) Stokes system

$$\begin{aligned} u_t - \Delta u + \nabla \pi &= f, \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0, \end{aligned} \quad (\text{B.13})$$

with data u_0, f , if for every test function $w \in C_0^\infty([0, T); C_{0,\sigma}^\infty(\Omega)^3)$, we have the relation

$$-\langle u, w_t \rangle + \langle \nabla u, \nabla w \rangle = \langle u_0, w(0) \rangle - \langle F, \nabla w \rangle, \quad (\text{B.14})$$

and if the followig energy equality is satisfied for $0 \leq t < T$:

$$\frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla u\|_{L^2(\Omega)}^2 \, d\tau = \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 - \int_0^t \langle F, \nabla u \rangle \, d\tau. \quad (\text{B.15})$$

We may assume in the following, without loss of generality, that every weak solution $u : [0, T) \rightarrow L^2_\sigma(\Omega)$ is weakly continuous. Therefore $u(0) = u_0$ is well-defined.

Let $1 < q < \infty$, $1 < s < \infty$. Then, for a given $f \in L^s(0, T; L^q_\sigma(\Omega)^3)$, there exists a unique solution $v \in C^0([0, T]; L^q_\sigma(\Omega)^3)$ of the nonstationary Stokes equation

$$v_t + A_q v = f, \quad v(0) = 0, \quad (\text{B.16})$$

satisfying $v_t \in L^s(0, T; L^q_\sigma(\Omega)^3)$, $A_q v \in L^s(0, T; L^q_\sigma(\Omega)^3)$, and the maximal regularity estimate

$$\|v_t\|_{L^s(0, T; L^q_\sigma(\Omega))} + \|A_q v\|_{L^s(0, T; L^q_\sigma(\Omega))} \leq \|f\|_{L^s(0, T; L^q_\sigma(\Omega))} \quad (\text{B.17})$$

with $c = c(\Omega, q, s) > 0$. The representation of this solution is

$$v(t) = \int_0^t e^{-(t-\tau)A_q} f(\tau) d\tau, \quad 0 \leq t < T. \quad (\text{B.18})$$

Using estimate (B.4), we get, for $0 < \alpha < 1$,

$$\|A_q^\alpha v(t)\|_{L^q(\Omega)} \leq c \int_0^t (t-\tau)^{-\alpha} \|f(\tau)\|_{L^q(\Omega)} d\tau, \quad 0 \leq t < T, \quad (\text{B.19})$$

with $c = c(\Omega, q) > 0$. Then, the Hardy–Littlewood estimate implies

$$\|A_q^\alpha v\|_{L^s(0, T; L^q_\sigma(\Omega))} \leq c \|f\|_{L^\gamma(0, T; L^q_\sigma(\Omega))}, \quad (\text{B.20})$$

where $1 < \gamma < s < \infty$, $1 - \alpha + \frac{1}{s} = \frac{1}{\gamma}$ and $c = c(\Omega, \alpha, q, s) > 0$ is independent of T .

Next we consider $f = \operatorname{div}F$ with $F \in L^r(0, T; L^2(\Omega)^{3 \times 3})$, $1 < r < \infty$. We have that

$$E(t) = e^{-tA} u_0 + \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div}F d\tau, \quad 0 \leq t < T, \quad (\text{B.21})$$

is well-defined with $u_0 \in L^2_\sigma(\Omega)^3$, we have that

$$E \in L^1_{\text{loc}}([0, T]; W^{1,2}_{0,\sigma}(\Omega)^3), \quad (\text{B.22})$$

and $u = E$ satisfies the relation (B.14). Inequality (B.7) gives

$$A^{-\frac{1}{2}} P \operatorname{div}F \in L^r(0, T; L^2(\Omega)^3). \quad (\text{B.23})$$

If $r = 2$ we obtain that

$$\begin{aligned} E \text{ defined by (B.21) with } r = 2 \text{ satisfies the energy equality (B.15)} \\ \text{and the condition (B.8).} \end{aligned} \quad (\text{B.24})$$

Therefore, E defined in (B.24) is a weak solution of the Stokes system (B.13) in the sense of Definition B.1.1. In particular, setting $E_0(t) = e^{-tA} u_0$, $u_0 \in L^2_\sigma(\Omega)^3$, (B.21) with $F = 0$ and (B.24) gives

$$\frac{1}{2} \|E_0(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla E_0\|_{L^2(\Omega)}^2 d\tau \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2, \quad 0 \leq t < T. \quad (\text{B.25})$$

If $u_0 \in L^2_\sigma(\Omega)^3$ and $F \in L^2(0, T; L^2(\Omega)^{3 \times 3})$, E in (B.24) satisfies

$$\frac{1}{2} \|E(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla E\|_{L^2(\Omega)}^2 d\tau \leq c \left(\|u_0\|_{L^2(\Omega)}^2 + \int_0^t \|F\|_{L^2(\Omega)}^2 d\tau \right), \quad 0 \leq t < T, \quad (\text{B.26})$$

with some constant $c > 0$ independent of t .

Set

$$E_1(t) = \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div} F d\tau, \quad 0 < t < T. \quad (\text{B.27})$$

Using (B.21) with $F \in L^2(0, T; L^2(\Omega)^{3 \times 3}) \cap L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega)^{3 \times 3})$, $\frac{2}{s} + \frac{3}{q} = 1$ we get

$$A^{-\frac{1}{2}} E_1(t) = \int_0^t e^{-(t-\tau)A} A^{-\frac{1}{2}} P \operatorname{div} F d\tau, \quad 0 \leq t < T. \quad (\text{B.28})$$

We note that (B.28) can be estimated in the same way as (B.16) with $v = A^{-\frac{1}{2}} E_1$, so we obtain with (B.17) and (B.7) that

$$\left\| \left(A^{-\frac{1}{2}} E_1 \right)_t \right\|_{L^2(0, T; L^2(\Omega))} + \left\| A^{\frac{1}{2}} E_1 \right\|_{L^2(0, T; L^2(\Omega))} \leq c \|F\|_{L^2(0, T; L^2(\Omega))}, \quad (\text{B.29})$$

and

$$\left\| \left(A^{-\frac{1}{2}} E_1 \right)_t \right\|_{L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega))} + \left\| A^{\frac{1}{2}} E_1 \right\|_{L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega))} \leq c \|F\|_{L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega))}, \quad (\text{B.30})$$

where $c = c(\Omega, q) > 0$.

Using estimate (B.3) for (B.27) with $2\alpha + \frac{3}{q} = \frac{3}{q/2}$, i.e., $\alpha = \frac{3}{2q}$, then (B.19) with $\alpha = \frac{9}{14} + \frac{3}{2q}$, i.e., $1 - \alpha + \frac{1}{s} = \frac{1}{s/2}$, and finally (B.20), we get

$$\|E_1\|_{L^s(0, T; L^q(\Omega))} \leq c \|F\|_{L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega))}, \quad (\text{B.31})$$

with $c = c(\Omega, q) > 0$. Also, using (B.16) and (B.18), we obtain the equality

$$A^{-\frac{1}{2}} E_1(t) = \int_0^t e^{-(t-\tau)A} \left((A^{-\frac{1}{2}} E_1)_\tau + A^{\frac{1}{2}} E_1 \right) d\tau, \quad (\text{B.32})$$

which implies that

$$E_1(t) = \int_0^t A^{\frac{1}{2}} e^{-(t-\tau)A} \left((A^{-\frac{1}{2}} E_1)_\tau + A^{\frac{1}{2}} E_1 \right) d\tau. \quad (\text{B.33})$$

In the same way as for (B.31) there exists a constant $c = c(\Omega, q) > 0$ such that

$$\|E_1\|_{L^s(0, T; L^q(\Omega))} \leq c \left(\left\| \left(A^{\frac{-1}{2}} E_1 \right)_t \right\|_{L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega))} + \left\| A^{\frac{1}{2}} E_1 \right\|_{L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega))} \right). \quad (\text{B.34})$$

B.1.4 Solvability and estimates for the Stokes system

We assume that $f = \operatorname{div} F$ in (B.13).

Lemma B.1.2 [27] *Let $\Omega \in \mathcal{C}^{2,\lambda}$, $0 < \lambda \leq 1$, for the problem (B.13), $1 < p, q < \infty$ and $F \in L^q(0, T, L^p(\Omega)^{3 \times 3})$. Then, there exists a unique weak solution to (B.13) with $f = \operatorname{div} F$ and $u(0) = u_0$ and it satisfies*

$$\|\nabla u\|_{L^q(0, T, L^p(\Omega))} \leq C \|F\|_{L^q(0, T, L^p(\Omega))}. \quad (\text{B.35})$$

The constant C does not depend on T .

Proof. To prove the result stated we use the following Corollary 4.2.a in [27]:

Corollary B.1.3 *Suppose Ω is as above, $u_0 = 0$ and $0 < T < \infty$, $1 < r, q < \infty$. Let $f = \sum_{\nu=1}^n \partial_\nu F_\nu$, where $F_\nu \in X = L^r((0, T); L^q_\sigma(\Omega)^{3 \times 3})$, $\nu = 1, 2, 3$. Then $A_q^{-\frac{1}{2}} f \in X$, $f \in L^r((0, T); \hat{D}(A_q^{\frac{1}{2}}))$. There exists a unique weak solution $u \in L^r(0, T; D(A_q^{\frac{1}{2}}))$ to (B.13) satisfying the following inequality*

$$\int_0^T \left\| \left(\frac{d}{dt} \right)^{\frac{1}{2}} u \right\|_{L^q(\Omega)}^r dt + \int_0^T \|A_q^{\frac{1}{2}} u\|_{L^q(\Omega)}^r dt \leq C \sum_{\nu=1}^3 \int_0^T \|F_\nu\|_{L^q(\Omega)}^r dt, \quad (\text{B.36})$$

with $C = C(\Omega, q, r)$ independent of f and T .

In particular, since

$$\|\nabla u\|_{L^q(\Omega)} \leq C \|A_q^{\frac{1}{2}} u\|_{L^q(\Omega)},$$

we conclude from Corollary B.1.3 that

$$\int_0^T \|\nabla u\|_{L^q(\Omega)}^r dt \leq C \sum_{\nu=1}^n \int_0^T \|F_\nu\|_{L^q(\Omega)}^r dt. \quad (\text{B.37})$$

Now we write

$$\begin{aligned} f = \operatorname{div} F &= \sum_{\nu=1}^3 \partial_\nu F_\nu \\ &= \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3 \\ &= \partial_1 \begin{pmatrix} F_{11} \\ F_{21} \\ F_{31} \end{pmatrix} + \partial_2 \begin{pmatrix} F_{12} \\ F_{22} \\ F_{32} \end{pmatrix} + \partial_3 \begin{pmatrix} F_{13} \\ F_{23} \\ F_{33} \end{pmatrix} \\ &= \begin{pmatrix} F_{11,1} \\ F_{21,1} \\ F_{31,1} \end{pmatrix} + \begin{pmatrix} F_{12,2} \\ F_{22,2} \\ F_{32,2} \end{pmatrix} + \begin{pmatrix} F_{13,3} \\ F_{23,3} \\ F_{33,3} \end{pmatrix}, \end{aligned}$$

where

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}.$$

Using the Helmholtz decomposition, we get

$$\begin{aligned} f &= \operatorname{div} F = \sum_{\nu=1}^3 \partial_{\nu} F_{\nu} \\ &= \sum_{\nu=1}^3 \partial_{\nu} F_{\nu}^1 - \sum_{\nu=1}^3 \nabla \partial_{\nu} F_{\nu}^2, \end{aligned}$$

where F_1^1, F_2^1, F_3^1 are three-component vector functions such that

$$\nabla \cdot F_{\nu}^1 = 0, \quad \text{and} \quad F_{\nu}^1 \cdot n|_{\partial\Omega} = 0,$$

and F_1^2, F_2^2, F_3^2 are scalar-valued functions. Then, we rewrite the system (B.13) as follows

$$\begin{aligned} u_t - \Delta u + \nabla \left(\pi + \sum_{\nu=1}^3 \partial_{\nu} F_{\nu}^2 \right) &= \sum_{\nu=1}^3 \partial_{\nu} F_{\nu}^1, \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0, \end{aligned} \tag{B.38}$$

where

$$\nabla \cdot F_{\nu}^1 = 0, \quad \text{and} \quad F_{\nu}^1 \cdot n|_{\partial\Omega} = 0.$$

Setting $\pi' = \pi + \sum_{\nu=1}^3 \partial_{\nu} F_{\nu}^2$, we finally get

$$\begin{aligned} u_t - \Delta u + \nabla \pi' &= \sum_{\nu=1}^n \partial_{\nu} F_{\nu}^1, \\ \operatorname{div} u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0, \end{aligned}$$

where

$$\nabla \cdot F_{\nu}^1 = 0, \quad \text{and} \quad F_{\nu}^1 \cdot n|_{\partial\Omega} = 0.$$

Therefore, we are in the setting of Corollary B.1.3 and we obtain

$$\int_0^T \|\nabla u\|_q^r dt \leq C \sum_{\nu=1}^n \int_0^T \|F_{\nu}^1\|_q^r dt. \tag{B.39}$$

Hence, also

$$\int_0^T \|\nabla u\|_q^r dt \leq C \sum_{\nu=1}^n \int_0^T \|F\|_q^r dt, \tag{B.40}$$

by properties of the Helmholtz decomposition of F . □

Now for the case $u_0 \neq 0$, on the one hand we consider the following problem

$$\begin{aligned}
\partial_t u - \Delta u + \nabla \pi &= 0 & \text{for } (x, t) \in \Omega \times (0, T], \\
\nabla \cdot u &= 0 & \text{for } (x, t) \in \Omega \times (0, T], \\
u(x, t) &= 0 & \text{for } (x, t) \in \partial\Omega \times (0, T], \\
u(x, 0) &= u_0(x) & \text{for } x \in \Omega.
\end{aligned} \tag{B.41}$$

From Theorem 1.1 in the paper [61], we have that if $u_0 \in B_{p,q}^{2-\frac{2}{q}}(\Omega)^3$, then $u \in L^q(0, T; W^{1,p}(\Omega))$. On the other hand, we consider the problem

$$\begin{aligned}
\partial_t u - \Delta u + \nabla \pi &= \nabla \cdot \mathbb{K} & \text{for } (x, t) \in \Omega \times (0, T], \\
\nabla \cdot u &= 0 & \text{for } (x, t) \in \Omega \times (0, T], \\
u(x, t) &= 0 & \text{for } (x, t) \in \partial\Omega \times (0, T], \\
u(x, 0) &= 0 & \text{for } x \in \Omega.
\end{aligned} \tag{B.42}$$

Since $\mathbb{K} \in L^p(0, T; L^q(\Omega)^{3 \times 3})$, that implies from Lemma B.1.2 that $u \in W_{p,q}^{1,\frac{1}{2}}(Q_T)^3$. Here we have used the following notion

$$W_{p,q}^{1,\frac{1}{2}}(Q_T) = \left\{ v \in L^q(0, T; W^{1,p}(\Omega)^3), \left(\frac{d}{dt} \right)^{\frac{1}{2}} v \in L^q(0, T; L^p(\Omega)^3) \right\}.$$

Hence, we superpose the two solutions of the problems (B.41) and (B.42), since these are linear problems with unique solutions. That implies that the solution u to our problem belongs to $L^q(0, T; W^{1,p}(\Omega)^3)$.

B.2 Existence of a solution to the Navier–Stokes system

We define $(u^{(k+1)}, \pi^{(k+1)})$, with $u^{(k+1)} \in L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; W_0^{1,2}(\Omega)^3)$, and $\pi^{(k+1)} \in \mathcal{D}'(0, T; L^2(\Omega)/\mathbb{R})$ as the weak solution of the unsteady Navier–Stokes system:

$$\begin{aligned}
\partial_t u^{(k+1)} + (u^{(k+1)} \cdot \nabla) u^{(k+1)} - \mu \Delta u^{(k+1)} + \nabla \pi^{(k+1)} &= \nabla \cdot \mathbb{K}^{(k)} & \text{for } (x, t) \in \Omega \times (0, T], \\
\nabla \cdot u^{(k+1)} &= 0 & \text{for } (x, t) \in \Omega \times (0, T], \\
u^{(k+1)}(x, t) &= 0 & \text{for } (x, t) \in \partial\Omega \times (0, T], \\
u^{(k+1)}(x, 0) &= u_0(x) & \text{for } x \in \Omega,
\end{aligned} \tag{B.43}$$

where $u_0 \in L^2(\Omega)^3 \cap B_{p,q}^{2-\frac{2}{q}}(\Omega)^3$, $1 < p, q < \infty$ is divergence-free, and thanks to (B.2),

$$\|\mathbb{K}^{(k)}\|_{L^\infty(0, T; L^\infty(\Omega)^{3 \times 3})} \leq C, \tag{B.44}$$

where C is a positive constant, independent of k . Thus, there exists a $\mathbb{K} \in L^\infty(0, T; L^\infty(\Omega)_{\text{symm}}^{3 \times 3})$ (to be identified), and a subsequence, not indicated, such that

$$\mathbb{K}^{(k)} \rightarrow \mathbb{K} \quad \text{weak* in } L^\infty(0, T; L^\infty(\Omega)_{\text{symm}}^{3 \times 3}) \text{ as } k \rightarrow \infty. \tag{B.45}$$

As $u_0 \in L^2(\Omega)^3$, by standard arguments from the analysis of the incompressible Navier–Stokes equations (cf., for example, [64], Chpt. III) we deduce from (B.44) that there exists a unique weak solution $(u^{(k+1)}, \pi^{(k+1)})$ to the system with $u^{(k+1)} \in L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; W_0^{1,2}(\Omega)^3)$, and

$$\|u^{(k+1)}\|_{L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega))} \leq C(1 + \|u_0\|_{L^2(\Omega)}),$$

where C is independent of k . We want to show the existence of a time $T^* > 0$ such that the weak solution $u^{(k+1)}$ satisfies Serrin’s condition:

$$u^{(k+1)} \in L^2(0, T^*; L^\infty(\Omega)^3).$$

Since $\mathbb{K} \in L^\infty(0, T; L^\infty(\Omega)^{3 \times 3})$, in particular $\mathbb{K} \in L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega)^{3 \times 3})$ where $2 < s < \infty$, $3 < q < \infty$, $\frac{2}{s} + \frac{3}{q} = 1$, are the so-called Serrin’s exponents. Therefore, we are in the same setting as in the paper [23]. We now state the main result in [23], Theorem 1.1,

Theorem B.2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, let $0 < T \leq \infty$, $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$, and let $\mathbb{K} \in L^2(0, T; L^2(\Omega)_{\text{symm}}^{3 \times 3}) \cap L^{\frac{s}{2}}(0, T; L^{\frac{q}{2}}(\Omega)_{\text{symm}}^{3 \times 3})$. Then there exists a constant $\epsilon^* = \epsilon^*(\Omega, q) > 0$ and a time $T > 0$ with the following property:*

If

$$\left(\int_0^T \|e^{-tA} u_0\|_{L^q(\Omega)}^s dt \right)^{\frac{1}{s}} + \left(\int_0^T \|\mathbb{K}\|_{L^{\frac{q}{2}}(\Omega)}^{\frac{s}{2}} dt \right)^{\frac{2}{s}} \leq \epsilon^*,$$

then the Navier–Stokes system (B.1) has a unique strong solution $u \in L^s(0, T; L^q(\Omega)^3)$ with data u_0, \mathbb{K} .

Following a similar line of argument as in [23], Theorem 1.1, we have the following theorem:

Theorem B.2.2 *Let $0 < T \leq \infty$, $2 < s < \infty$, $3 < q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$, and let $\mathbb{K} \in L^\infty(0, T; L^\infty(\Omega)_{\text{symm}}^{3 \times 3})$. Then there exists a constant $\epsilon^* = \epsilon^*(\Omega, q) > 0$ and a time $T^* > 0$ with the following property:*

If

$$\left(\int_0^{T^*} \|e^{-tA} u_0\|_{L^q(\Omega)}^s dt \right)^{\frac{1}{s}} + \|\mathbb{K}\|_{L^\infty(0, T; L^\infty(\Omega))} T^{*\frac{2}{s}} \leq \epsilon^*,$$

then the Navier–Stokes system (B.1) has a unique strong solution $u \in L^s(0, T^; L^q(\Omega)^3)$ with data u_0, \mathbb{K} .*

Proof. The proof is the same as in [23], Theorem 1.1. The assumption in that theorem is the following

$$\left(\int_0^{T^*} \|e^{-tA} u_0\|_{L^q(\Omega)}^s dt \right)^{\frac{1}{s}} + \left(\int_0^{T^*} \|\mathbb{K}\|_{L^{\frac{q}{2}}(\Omega)}^{\frac{s}{2}} dt \right)^{\frac{2}{s}} \leq \epsilon^*.$$

In our case, since $\mathbb{K} \in L^\infty(0, T; L^\infty(\Omega)_{\text{symm}}^{3 \times 3})$, we therefore have that

$$\left(\int_0^{T^*} \|\mathbb{K}\|_{L^{\frac{q}{2}}(\Omega)}^{\frac{s}{2}} dt \right)^{\frac{2}{s}} \leq \|\mathbb{K}\|_{L^\infty(0, T; L^\infty(\Omega))} T^{*\frac{2}{s}}.$$

Hence, we want to require that

$$\|\mathbb{K}\|_{L^\infty(0,T;L^\infty(\Omega))} T^{*\frac{2}{s}} \leq \epsilon^*,$$

i.e.

$$T^* \leq \left(\frac{\epsilon^*}{\|\mathbb{K}\|_{L^\infty(0,T;L^\infty(\Omega))}} \right)^{\frac{s}{2}}. \quad (\text{B.46})$$

□

In particular, from the previous theorem, we have that the Navier–Stokes system (B.1) has a unique strong solution $u \in L^8(0, T^*; L^4(\Omega)^3)$. We first move the convective term in the Navier–Stokes equation to the right-hand side of the equation, resulting in the following unsteady Stokes system with source term $\nabla \cdot (\mathbb{K} - u \otimes u)$:

$$\partial_t u - \mu \Delta u + \nabla \pi = \nabla \cdot (\mathbb{K} - u \otimes u). \quad (\text{B.47})$$

Hence,

$$u \otimes u \in L^4(0, T^*; L^2(\Omega)_{\text{symm}}^{3 \times 3}),$$

which implies

$$\mathbb{K} - u \otimes u \in L^4(0, T^*; L^2(\Omega)_{\text{symm}}^{3 \times 3}),$$

then by standard arguments from the analysis of the Stokes equations, we get, from equation (B.47), that:

$$u \in L^4(0, T^*; W^{1,2}(\Omega)^3).$$

We therefore have by Sobolev embedding

$$u \in L^4(0, T^*; L^6(\Omega)^3).$$

Again by the same arguments we get

$$\mathbb{K} - u \otimes u \in L^2(0, T^*; L^3(\Omega)_{\text{symm}}^{3 \times 3}),$$

which implies that

$$u \in L^2(0, T^*; W^{1,3}(\Omega)^3),$$

and therefore,

$$u \in L^2(0, T^*; L^q(\Omega)^3),$$

for all $1 < q < \infty$.

Theorem B.2.3 *The Navier–Stokes system (B.1) has a unique strong solution u contained in $L^2(0, T^*; L^\infty(\Omega)^3)$ with data u_0 , \mathbb{K} and T^* as in (B.46).*

Proof. Our strategy to show that $u \in L^2(0, T^*; L^q(\Omega)^3)$ will be the following:

1. First, we show by interpolation that there exist $\epsilon > 0$ and $\delta > 3$ such that

$$u \in L^{2+\epsilon}(0, T^*; L^{3+\delta}(\Omega)^3),$$

so that

$$u \otimes u \in L^{\frac{2+\epsilon}{2}}(0, T^*; L^{\frac{3+\delta}{2}}(\Omega)_{\text{symm}}^{3 \times 3}),$$

then

$$u \in L^{\frac{2+\epsilon}{2}}(0, T^*; W^{1, \frac{3+\delta}{2}}(\Omega)^3). \quad (\text{B.48})$$

2. Then, we prove that the following interpolation inequality holds: for $p > d$ and $u \in W_0^{1,p}(\Omega)$, there is a constant $C := C(d, p)$ such that

$$\|u\|_{L^\infty(\Omega)} \leq C(n, p) [u]_{C^{0,\alpha}(\Omega)}^{\frac{d}{p}} \|u\|_{L^p(\Omega)}^{1-\frac{d}{p}}. \quad (\text{B.49})$$

3. Finally, since by (B.48), we have that $u \in L^{\frac{2+\epsilon}{2}}(0, T^*; C^{0,\alpha}(\Omega)^3)$, $0 < \alpha < 1$, and since $u \in L^8(0, T^*; L^4(\Omega)^3)$, by interpolation, using the inequality (B.49), we show that $u \in L^2(0, T^*; L^\infty(\Omega)^3)$.

We start by proving the statements in Step 1 above.

1. We have that $u \in L^4(0, T^*; L^6(\Omega)^3)$ and $u \in L^2(0, T^*; L^q(\Omega)^3)$, for all $1 < q < \infty$. Therefore by interpolation, we get

$$\|u\|_{L^r(\Omega)} \leq C \|u\|_{L^6(\Omega)}^\alpha \|u\|_{L^q(\Omega)}^{1-\alpha},$$

where $\frac{1}{r} = \frac{\alpha}{6} + \frac{1-\alpha}{q}$ and $0 < \alpha \leq 1$; then choosing $\alpha = \frac{1}{4}$, we get

$$\|u\|_{L^r(\Omega)} \leq C \|u\|_{L^6(\Omega)}^{\frac{1}{4}} \|u\|_{L^q(\Omega)}^{\frac{3}{4}},$$

which implies that

$$\|u\|_{L^r(\Omega)}^l \leq C \|u\|_{L^6(\Omega)}^{\frac{1}{4}l} \|u\|_{L^q(\Omega)}^{\frac{3}{4}l},$$

with $l = \frac{16}{7} > 2$. Indeed,

$$\|u\|_{L^6(\Omega)}^{\frac{1}{4}} \in L^{16}(0, T),$$

and

$$\|u\|_{L^q(\Omega)}^{\frac{3}{4}} \in L^{\frac{8}{3}}(0, T).$$

Hence

$$\frac{1}{l} = \frac{3}{8} + \frac{1}{16} = \frac{7}{16}.$$

We also have that:

$$r := r(q) = \frac{1}{\frac{1}{24} + \frac{3}{4q}} = \frac{96q}{4q + 72},$$

and

$$r'(q) = \frac{96 \cdot 72}{(4q + 72)^2} > 0,$$

which implies that r is a strictly increasing function and we have that

$$\lim_{q \rightarrow \infty} r(q) = 24.$$

Therefore, choosing $r = 22$, we finally get:

$$u \in L^{\frac{16}{7}}(0, T^*; L^{22}(\Omega)^3),$$

that was the result stated. Hence

$$u \otimes u \in L^{\frac{16}{14}}(0, T^*; L^{11}(\Omega)^{3 \times 3}),$$

$$u \in L^{\frac{16}{14}}(0, T^*; W^{1,11}(\Omega)^3),$$

by Sobolev embedding, we get

$$u \in L^{\frac{16}{14}}(0, T^*; C^{0,\alpha}(\Omega)^3),$$

where $0 < \alpha \leq 1$.

2. Now, we want to prove the inequality (B.49). We compute for $x_0 \in \Omega$ and $R > 0$ the following quantity:

$$\begin{aligned} |u(x_0) - [u]_{B(x_0,R)}| &= \left| u(x_0) - \frac{1}{|B_R|} \int_{B(x_0,R)} u(y) \, dy \right| \\ &\leq \frac{1}{|B_R|} \int_{B(x_0,R)} |u(x_0) - u(y)| \, dy \\ &\leq \frac{1}{|B_R|} \int_{B(x_0,R)} [u]_{C^{0,\alpha}(\Omega)} |x_0 - y|^\alpha \, dy \\ &\leq R^\alpha [u]_{C^{0,\alpha}(\Omega)}. \end{aligned}$$

This implies that

$$\begin{aligned} |u(x_0)| &\leq |[u]_{B(x_0,R)}| + R^\alpha [u]_{C^{0,\alpha}(\Omega)} \\ &\leq \frac{1}{|B_R|} \int_{B(x_0,R)} |u| \, dy + R^\alpha [u]_{C^{0,\alpha}(\Omega)} \\ &\leq \left(\frac{1}{|B_R|} \int_{B(x_0,R)} |u|^p \, dy \right)^{\frac{1}{p}} + R^\alpha [u]_{C^{0,\alpha}(\Omega)} \\ &\leq \frac{C}{R^{\frac{3}{p}}} \|u\|_{L^p(\Omega)} + R^\alpha [u]_{C^{0,\alpha}(\Omega)}, \end{aligned}$$

for a constant $C > 0$ and for all $R > 0$. Set

$$f(R) := \frac{C}{R^{\frac{3}{p}}} \|u\|_{L^p(\Omega)} + R^\alpha [u]_{C^{0,\alpha}(\Omega)}$$

Hence,

$$f'(R) := \frac{3C R^{-\frac{3}{p}-1}}{p} \|u\|_{L^p(\Omega)} + \alpha R^{\alpha-1} [u]_{C^{0,\alpha}(\Omega)},$$

where $\alpha = 1 - \frac{3}{p}$ by Morrey's inequality. We want to find where the function f reaches its maximum, i.e. we want to find R_0 such that:

$$f'(R_0) = 0,$$

which is equivalent to

$$\frac{(1 - \frac{3}{p}) [u]_{C^{0,\alpha}(\Omega)}}{R_0^{\frac{3}{p}+1}} = \frac{C^{\frac{3}{p}} \|u\|_{L^p(\Omega)}}{R_0^{\frac{3}{p}}},$$

and we obtain for a positive constant $C := C(3, p)$ that

$$\begin{aligned} R_0 &= \frac{C^{\frac{3}{p}}}{1 - \frac{3}{p}} \cdot \frac{\|u\|_{L^p(\Omega)}}{[u]_{C^{0,\alpha}(\Omega)}} \\ &= C \frac{\|u\|_{L^p(\Omega)}}{[u]_{C^{0,\alpha}(\Omega)}}. \end{aligned}$$

Hence,

$$\begin{aligned} |u(x_0)| &\leq f(R_0) \\ &\leq C \|u\|_{L^p(\Omega)}^{1 - \frac{3}{p}} [u]_{C^{0,\alpha}(\Omega)}^{\frac{3}{p}}, \end{aligned}$$

for all $x_0 \in \Omega$. Then,

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^p(\Omega)}^{1 - \frac{3}{p}} [u]_{C^{0,\alpha}(\Omega)}^{\frac{3}{p}},$$

that is the conclusion stated.

3. Using the previous step, we interpolate between the two spaces where the solution u belongs i.e. we interpolate between $L^{\frac{16}{7}}(0, T; L^{22}(\Omega)^3)$ and $L^{\frac{16}{14}}(0, T; C^{0,\alpha}(\Omega)^3)$ where $0 < \alpha < 1$. We have indeed that

$$\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^{22}(\Omega)}^{1 - \frac{3}{22}} [u]_{C^{0,\alpha}(\Omega)}^{\frac{3}{22}},$$

that implies

$$\|u\|_{L^\infty(\Omega)}^l \leq C \|u\|_{L^{22}(\Omega)}^{\frac{19}{22}l} [u]_{C^{0,\alpha}(\Omega)}^{\frac{3}{22}l},$$

with $l = \frac{352}{175} \approx 2.01 > 2$. Indeed,

$$\|u\|_{L^{22}(\Omega)}^{\frac{19}{22}l} \in L^{\frac{22}{19} \times \frac{16}{7}}(0, T) = L^{\frac{352}{133}}(0, T),$$

and

$$[u]_{C^{0,\alpha}(\Omega)}^{\frac{3}{22}l} \in L^{\frac{22}{3} \times \frac{16}{14}}(0, T) = L^{\frac{176}{21}}(0, T).$$

Hence

$$\frac{1}{l} = \frac{133}{352} + \frac{21}{176} = \frac{175}{352},$$

we finally get:

$$u \in L^2(0, T^*; L^\infty(\Omega)^3),$$

that was the result stated.

Note that the uniqueness of u is a consequence of the Serrin condition, see, e. g., [60], V. Theorem 1.5.1. \square

We have therefore showed in this appendix that the results obtained in the previous chapters still hold when coupling the incompressible Navier–Stokes equations with a Fokker–Planck equation.

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