

Classifying Spaces of Low-Dimensional Bordism Categories



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St John's College
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A thesis submitted for the degree of
Doctor of Philosophy

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The truncated bordism category Cob_d is the symmetric monoidal category whose objects are closed oriented $(d - 1)$ -manifolds and whose morphisms are diffeomorphism classes of compact oriented d -dimensional bordisms. In this thesis we study the classifying spaces of Cob_1 , Cob_2 , and several related categories.

In the first part we show that the conditions in Steimle's "additivity theorem for cobordism categories" [Ste18b] can be weakened to only require *locally* (co)Cartesian fibrations, making it applicable to a larger class of functors. As an application, we construct a fiber sequence that relates the classifying space of the $(2, 1)$ -category $\mathbb{C}\text{sp}$ of cospans of finite sets to the classifying space of its homotopy category $\text{Csp} = h(\mathbb{C}\text{sp})$, where isomorphic cospans are identified.

In the second part we compute the classifying space of Cob_1 by exhibiting it as a circle bundle over $\Omega^{\infty-2}\mathbb{C}\mathbb{P}_{-1}^{\infty}$. As part of our proof we construct a quotient $\text{Cob}_1 \rightarrow \text{Cob}_1^{\text{red}}$ where circles are deleted and we show that its classifying space is $\Omega^{\infty-2}\mathbb{C}\mathbb{P}_{-1}^{\infty}$. Based on this we construct a bordism model for the topological cyclic homology of simply connected spaces. We also explicitly describe an infinite loop space map $B(\text{Cob}_1^{\text{red}}) \rightarrow Q(\Sigma^2\mathbb{C}\mathbb{P}_+^{\infty})$ and use it to derive combinatorial formulas for rational cocycles on $\text{Cob}_1^{\text{red}}$ representing shifted Miller-Morita-Mumford classes $\bar{\kappa}_i \in H^{2i+2}(B(\text{Cob}_1); \mathbb{Q})$.

The third part concerns the classifying space of the surface category Cob_2 and its subcategory $\text{Cob}_2^{\chi \leq 0} \subset \text{Cob}_2$ that contains all bordisms without disks or spheres. We show that, after passing to a refinement $\text{Cob}_2 \rightarrow \text{Cob}_2$, the classifying space of the surface category is $B\text{Cob}_2 \simeq S^1$ – proving that a conjecture of Tillmann becomes true in this setting. We also compute the classifying space of $\text{Cob}_2^{\chi \leq 0}$ and show that its rational homotopy groups contain the homology of the tropical moduli spaces Δ_g , which is known to grow exponentially. The tools we develop can be applied to a large class of symmetric monoidal categories, which we call *labelled cospan categories*. In particular, we show that $B(\mathbb{C}\text{sp})$ is contractible, whereas the classifying space of its homotopy category is $B(\text{Csp}) \simeq \tau_{\geq 3}Q(S^2)$.

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Chapter 0

Introduction

Let us begin by motivating the concepts explored in this thesis. An overview of the content as well as a reader's guide can be found below.

0.1 Motivation

0.1.1 A brief history of bordism categories

The idea that d -dimensional bordisms can be assembled into a category goes back at least to Milnor's lecture notes on the h-cobordism theorem [Mil65]. He defines a category where objects are closed $(d - 1)$ -manifolds and morphisms are diffeomorphism classes of compact d -dimensional bordisms. Throughout this thesis we shall usually assume that all manifolds are oriented and we will denote the resulting category by Cob_d . For Milnor this category was useful as a book-keeping tool for how compact oriented d -manifolds can be glued along their boundaries.

In the late 80s the bordism category Cob_d found a new purpose. Atiyah's definition of topological quantum field theories [Ati88] can conveniently be packaged into the data of a symmetric monoidal functor $\mathcal{Z} : \text{Cob}_d \rightarrow \text{Vect}$. Even though this is not Atiyah's original formulation, it is the perspective most often taken in modern treatments. (E.g. Freed's survey [Fre13].) In fact, Atiyah's definition was based on Segal's definition of conformal field theories in dimension 2 [Seg04], which was explicitly phrased in terms of a conformal cobordism category. The functorial point of view also allows for topological field theories with values in a general symmetric monoidal category \mathcal{V} as symmetric monoidal functors $\text{Cob}_d \rightarrow \mathcal{V}$. For low dimensions one can give generators and relations for Cob_d – for example for $d = 2$ symmetric monoidal functors $\text{Cob}_d \rightarrow \mathcal{V}$ correspond to commutative Frobenius algebra objects in \mathcal{V} .

In the late 90s a new perspective on bordism categories emerged: based on the ideas of Segal's conformal bordism category [Seg04] a topological category \mathcal{C}_d was constructed in [MT01] and [GMTW09] whose Hom-spaces have the homotopy type of the moduli space of bordisms $\coprod_{[W]:M \rightarrow N} B\text{Diff}^\partial(W)$. This recovers the original bordism category as its homotopy category, i.e.

by passing to connected components on Hom-spaces: $\text{Cob}_d \simeq h(\mathcal{C}_d)$. These topological bordism categories proved to be an extremely useful way of packaging all the moduli spaces of manifolds of a given dimension into one algebraic object. Most notably they were used in [GMTW09] to give an alternative proof of the Madsen-Weiss theorem, which describes the stable cohomology of moduli spaces of complex curves.

From a modern perspective on topological field theories the topological category \mathcal{C}_d can be thought of as a specific model of the $(\infty, 1)$ -category Bord_d . One may then define a topological field theory with values in a symmetric monoidal $(\infty, 1)$ -category \mathcal{V} as a symmetric monoidal $(\infty, 1)$ -functor $\mathcal{Z} : \text{Bord}_d \rightarrow \mathcal{V}$. In fact, one usually also “extends down” to obtain an (∞, d) -category of bordisms $\text{Bord}_{(0\dots d)}$. This is the context in which Lurie’s proof sketch of the Baez-Dolan cobordism hypothesis [Lur09b] applies, see there for more details.

0.1.2 The role of classifying spaces

To every group G one can assign a classifying space BG such that maps $X \rightarrow BG$ classify principle G bundles on X . Cohomology classes of BG are characteristic classes for bundles with structure group G . The classifying space BG can be constructed as the geometric realisation $|N_\bullet G|$ where the nerve $N_\bullet G$ is defined as $N_n G := G^n$. This construction generalises to any (topological) category \mathcal{C} by defining $N_n \mathcal{C}$ as the space of functors $[n] = \{0 \leq \dots \leq n\} \rightarrow \mathcal{C}$ and setting $B\mathcal{C} := |N_\bullet \mathcal{C}|$. While it is generally difficult to say what exactly is classified by the classifying space $B\mathcal{C}$ (see [Wei05]), the construction turns out to nevertheless be extremely useful because of how it relates to the classifying spaces of certain groups.

For example, if $x \in \mathcal{C}$ is any object and $\text{End}_{\mathcal{C}}(x) := \text{Hom}_{\mathcal{C}}(x, x)$ denotes its monoid of endomorphisms, then there is a natural map $B\text{End}_{\mathcal{C}}(x) \rightarrow B\mathcal{C}$ and also $\text{End}_{\mathcal{C}}(x) \rightarrow \Omega B\mathcal{C}$. In the case of the topological bordism category \mathcal{C}_d this induces a map $B\text{Diff}^\partial(W) \rightarrow \Omega BC_d$. Therefore cohomology classes of ΩBC_d can be pulled back to cohomology classes on $B\text{Diff}^\partial(W)$. One can think of $H^*(\Omega BC_d)$ as the ring of universal characteristic classes for all smooth bundles of compact d -manifolds.

The classifying space of \mathcal{C}_d was computed in the celebrated [GMTW09] as $BC_d \simeq \Omega^{\infty-1} MTSO_d$ where $MTSO_d$ is a certain Thom-spectrum. This in particular allowed for a computation of $H^*(\Omega BC_d; \mathbb{Q})$ as a free polynomial ring on so-called generalised κ -classes. Work of [GRW14] has since shown that for highly-connected manifolds $W_{g,1}$ of dimension $d = 2n \geq 6$ the cohomology of $B\text{Diff}^\partial(W_{g,1})$ is (in a range) entirely given by generalised κ -classes.

The classifying spaces of bordism categories are also important from the perspective of topological field theories. The first cohomology group $H^1(B\text{Cob}_d; \mathbb{C}^\times)$ classifies *invertible d -dimensional TFTs*. For example, Tillmann in [Til96] showed that $B\text{Cob}_2 \simeq S^1 \times X$ for a simply connected infinite loop space X , and correspondingly invertible commutative Frobenius algebras in Vect are

classified by a single invertible number $\lambda \in \mathbb{C}^\times \cong H^1(\text{Cob}_2; \mathbb{C}^\times)$. It is worth noting that because the canonical functor $\mathcal{C}_d \rightarrow \text{Cob}_d$ induces a 2-connected map $BC_d \rightarrow BCob_d$, the distinction between \mathcal{C}_d and Cob_d is irrelevant for the purpose of computing the first cohomology group. The second cohomology group $H^2(BCob_d, \mathbb{Z})$ also admits a field-theoretic interpretation as it classifies central extensions of the bordism category, which are used to define anomalous/projective topological field theories. Concretely, a computation of¹ $H^2(BC_3; \mathbb{Z})$ was used in [Sch17] to resolve a question about anomaly-cancellation for 3-dimensional TFTs.

Beyond the first two cohomology groups, it has proven useful to consider the entire infinite loop space BC_d when studying invertible TFTs in higher categories. Most notably, Freed and Hopkins in [FH20] classified reflection-positive invertible topological phases with various structure groups by studying the classifying space $B(\text{Bord}_{(0\dots d)})$ from the perspective of C_2 -equivariant stable homotopy theory.

0.1.3 Cobordism categories and algebraic K-theory

In his seminal paper [Qui73] Quillen defines the higher algebraic K-theory of an exact category \mathcal{E} as the homotopy groups $K_*(\mathcal{E}) := \pi_{*+1}B(Q(\mathcal{E}))$ of the classifying space of a category $Q(\mathcal{E})$. In this paper Quillen pioneers the use of classifying spaces of categories and his Theorem A and Theorem B are still some of the most important tools in computing classifying spaces. The category $Q(\mathcal{E})$ that Quillen constructs is a certain subcategory of the category $\text{Span}(\mathcal{E})$, which has the same objects as \mathcal{E} and where morphisms are spans $A \leftarrow X \rightarrow B$ that are composed by taking pullbacks. This resembles the cobordism category: we can think of Cob_d as a subcategory of the cospan category of manifolds $\text{Csp}(\text{Man})$, though one has to be careful to ensure that when composing by taking a pushout the result is still a well-defined manifold.

This analogy between bordism categories and algebraic K-theory was given a concrete mathematical interpretation by Tillmann who in [Til99] constructed a map from $\Omega BC_2^{\partial+}$ to Waldhausen's $A(*)$, which is the algebraic K-theory of the sphere spectrum \mathbb{S} . From this point of view the map sends a bordism W to the \mathbb{S} -module $\Sigma_+^\infty W$.

Bökstedt and Madsen in [BM14] improved this to a map $\Omega BC_d^{\text{unor}} \rightarrow A(BO_d)$. Moreover, this map was interpreted as a map of cospan categories by Raptis and Steimle. In [RS19] they construct a cobordism model $\text{Cob}(\mathcal{W})$ for the algebraic K-theory of any Waldhausen category \mathcal{W} such that $K(\mathcal{W}) \simeq \Omega BCob(\mathcal{W})$ and they show that the map of Bökstedt and Madsen can be obtained from a functor $\mathcal{C}_d \rightarrow \text{Cob}(\mathcal{W})$ where \mathcal{W} is the category of retractive spaces over BO_d . In fact, this map is expected to lift to the Hermitian K-theory of \mathbb{S} : [CDH⁺20] define a cobordism category of

¹To be precise one should use the “spectrum cohomology” here, i.e. the group of homotopy classes of infinite loop space maps $BCob_3 \rightarrow K(\mathbb{Z}, 2)$. This is to ensure that the central extensions are themselves symmetric monoidal.

Poincare objects that models Hermitian K-theory, and sending W to $\Sigma_+^\infty W$ should define a functor from \mathcal{C}_d to their category.

0.1.4 The classifying spaces of truncated bordism categories

Despite the complete computation of BC_d in [GMTW09] very little is known about the classifying spaces $BCob_d$ of the truncated bordism categories Cob_d . This might be surprising at first since the discrete 1-category Cob_d is a much simpler object than the topological category \mathcal{C}_d or the ∞ -category $Bord_d$. However, the proofs of [GMTW09] rely on a variant of the Thom-construction, which cannot be applied to Cob_d as morphisms are abstract diffeomorphism classes of bordisms that cannot be canonically embedded into \mathbb{R}^N .

On a similar note, the classifying space $BCsp$ of the category of cospans of finite sets has not been studied at all. From the perspective taken in this introduction this is surprising as $Csp = Span(Fin^{op})$ is the most basic non-algebraic analogue of Quillen's $Q(\mathcal{E})$ that one could study.²

The program pursued in this thesis aims to give a more complete understanding of the classifying spaces of truncated bordism categories and other categories whose morphisms resemble cospans. We prove results about $B(Csp)$, $B(Cob_1)$, $B(Cob_2)$, $B(Cob_2^{X \leq 0})$, and provide a general framework for studying the classifying spaces of *labelled cospan categories*. In doing so we will show that there is a wealth of interesting structure in the seemingly simple combinatorics of how low-dimensional bordisms compose.

0.1.5 Higher category theory

In higher category theory, bordism categories parametrise various types of algebraic structures. For example, a symmetric monoidal functor $Cob_2 \rightarrow \mathcal{D}$ into a symmetric monoidal 1-category \mathcal{D} corresponds to a commutative Frobenius algebra object in \mathcal{D} . Another example is the one-dimensional cobordism hypothesis, which states that a symmetric monoidal ∞ -functor $Bord_1 \rightarrow \mathcal{C}$ to a symmetric monoidal ∞ -category \mathcal{C} corresponds to a choice of a dualisable object in \mathcal{C} .

In [Ste21b] we use the universal property of $Bord_1$ to compute the moduli space of *trace-like transformations* in terms of moduli spaces of labelled circles. As an application we give a characterisation of the trace for endomorphisms of dualisable objects in any symmetric monoidal ∞ -category. As this strays a bit far from the core of this thesis, it is not included here.

From the perspective of higher category theory, the classifying space of a 1-category or ∞ -category is its universal ∞ -groupoidification. The aforementioned computations of classifying spaces therefore allow us to study, for example, symmetric monoidal ∞ -functors $A : Csp \rightarrow \mathcal{C}$ from the $(2, 1)$ -category Csp of cospans or finite sets to a symmetric monoidal $(\infty, 1)$ -category \mathcal{C} .

²Note that $Span(Fin)$ has \emptyset as an initial object and hence $B(Span(Fin))$ is automatically contractible. We will see that the case of $BCsp$ is not as straight-forward.

We show that if $A(*) \simeq 1_{\mathcal{C}}$, then A is canonically equivalent to the constant functor. This means that every special commutative Frobenius algebra structure on the unit $1_{\mathcal{C}} \in \mathcal{C}$ is canonically trivialisable. On the other hand, the computations also show that the categories $\mathbb{C}\text{sp}$ and Cob_2 are not as well-behaved as their refinements $\mathbb{C}\text{sp}$ and Cob_2 : for example there are many non-trivial symmetric monoidal ∞ -functors $A : \mathbb{C}\text{sp} \rightarrow \mathcal{C}$ with $A(*) \simeq 1_{\mathcal{C}}$, which all become trivial after passing to $\mathbb{C}\text{sp}$.

0.2 Overview

This thesis consists of three self-contained parts. Each of the parts begins with an introductory chapter that motivates the approach taken and outlines how the part is structured. Here we provide an overview of the thesis as a whole, explaining how the parts interlink.

Part I: Locally (co)Cartesian fibrations as realisation fibrations

This part has appeared as a preprint [Ste19] and is given here in slightly improved form.

In part I we prove the “local additivity theorem for bordism categories”, which is a tool for studying classifying spaces through locally co/Cartesian functors that plays a crucial role in part II. Chapter 2 is concerned with the proof of this theorem. We generalise a theorem of Steimle by showing that his proof can be adapted to work with locally co/Cartesian functors. Just like the original paper, we prove the theorem for quasi-categories in section 2.1.1 and then translate it to complete Segal spaces and weakly unital topological categories.

Chapter 3 applies this new tool to study the difference between the $(2, 1)$ -category $\mathbb{C}\text{sp}$ where 2-morphisms are bijections between cospans and the 1-category $\text{Csp} = h(\mathbb{C}\text{sp})$ where isomorphic cospans are identified. This difference will play a key role in part III where we consider an analogous refinement $\text{Cob}_d \rightarrow \text{Cob}_d$. We construct a homotopy fiber sequence $\tau_{\geq 2}Q(S^1) \rightarrow B\mathbb{C}\text{sp} \rightarrow BC\text{sp}$, but the computation of $BC\text{sp}$ is only completed in part III, where we show that $BC\text{sp} \simeq *$. Moreover, we relate our computations for Csp to the cobordism model for algebraic K-theory of [RS19].

Part II: The one-dimensional bordism category

This part has been published as [Ste21a]. We have reproduced it here without major modifications.

In this part we study the truncated one-dimensional bordism category Cob_1 . Chapter 5 computes $BC\text{ob}_1$ as a circle bundle over $\Omega^{\infty-2}MTSO_2$ by comparing it to the topologically enriched bordism category \mathcal{C}_1 , for which $BC_1 \simeq QS^0$ is known by [GMTW09]. Based on the ideas of part I, we show that there is a “reduction fiber sequence” (Theorem F) that allows us to decompose Cob_d and \mathcal{C}_d into their closed and reduced components. What allows for the computation of $BC\text{ob}_1$ is that for $d = 1$ the moduli spaces of reduced bordisms are contractible because $\text{Diff}^{\partial}([0, 1]) \simeq *$.

We also show that the reduced one-dimension bordism category $\mathcal{C}_1^{\text{red}}(X)$ may be thought of as a cobordism model for the topological cyclic homology of a simply-connected space X .

Chapter 6 is concerned with gaining a better understanding of how the 1-category $\text{Cob}_1^{\text{red}}$ provides a combinatorial model for the infinite loop space $\Omega^{\infty-2}MTSO_2$. We give an explicit construction of the infinite loop space map $B\text{Cob}_1^{\text{red}} \rightarrow Q(\Sigma^2BSO_2)$ that continues the aforementioned reduction fiber sequence. To do so we introduce the *space of cuts* Cut_\bullet , which will also play a key role in part III. We are able to show that Cut_\bullet is in a sense freely generated by a copy of Connes' cyclic category Λ - in the language of part III, Λ is the factorisation category of Cob_1 . This description of the map $B\text{Cob}_1^{\text{red}} \rightarrow Q(\Sigma^2B\Lambda)$ allows us to pull back combinatorial formulas for cocycles on Cob_1 that represent the polynomial generators $\bar{\kappa}_i \in H^{2i+2}(B\text{Cob}_1; \mathbb{Q})$, which may be thought of as shifted version of the Miller-Morita-Mumford classes on \mathcal{C}_2 .

Part III: The surface category and tropical curves

In this part we address the question of computing $B\text{Cob}_d$ more generally. In particular we compute $B\text{Cob}_2$, $B\text{Csp}$ and relate a large subcategory of Cob_2 to the tropical moduli spaces Δ_g .

Chapter 8 provides a general framework for studying the classifying spaces of symmetric monoidal 1-categories that behave like truncated bordism categories. These so-called *labelled cospan categories* are characterised as symmetric monoidal functors $(\pi : \mathcal{C} \rightarrow \text{Csp})$ satisfying certain axioms. Examples include Cob_d , which served as a motivation, and also Csp itself. We prove a *Decomposition Theorem* 8.2.1 based on the ideas of part II, though with an entirely new proof, and a *Surgery Theorem* 8.3.1 based on the ideas of [GRW14] and [GMTW09].

Together these theorems allow us to describe the classifying space BC of many labelled cospan categories in terms of their positive boundary subcategory $\mathcal{C}^{\partial+} \subset \mathcal{C}$ and their factorisation categories $\mathcal{F}_g(\mathcal{C})$. In doing this it is important to replace \mathcal{C} by a 2-categorical refinement $\mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ that is constructed in analogy with the functor $\mathbb{C}\text{sp} \rightarrow \text{Csp}$ of chapter 3. We are able to show that the factorisation categories are often contractible, which allows us to compute $B\text{Cob}_2 \simeq S^1$ answering a question asked by Tillmann [Til96] (see [JT13, Conjecture 5.3]) and $B\text{Csp} \simeq *$ finishing the computation of part I. The latter can be interpreted as a statement about commutative Frobenius algebras in higher categories, as we explain in the introductory chapter 7.

Chapter 9 focuses on the subcategory $\text{Cob}_2^{\chi \leq 0} \subset \text{Cob}_2$ that contains all objects and those morphisms that do not contain disks or spheres. This can be thought of as the free symmetric monoidal category on a non-co/unital commutative Frobenius algebra. While the change from Cob_2 to $\text{Cob}_2^{\chi \leq 0}$ might seem minor, we show that it causes a vast difference in classifying spaces. With the disks removed the argument for the contractibility of $B\mathcal{F}_g(\text{Cob}_2)$ cannot be transferred to $\text{Cob}_2^{\chi \leq 0}$, and in fact we show that $B\mathcal{F}_g(\text{Cob}_2) \simeq BJ_g$ where J_g is the category of \mathbb{N} -weighted graphs that is used in the study of tropical moduli spaces. Together with the tools of chapter 8

this allows us to express $BCob_2^{\chi \leq 0}$ in terms of free loop spaces on $\Sigma^2 BJ_g$. Moreover, we show that BJ_g is rationally equivalent to the tropical moduli space Δ_g - a fact that is probably known to experts. This allows us to combine our computations with the work of [CGP18] to show that the rational homotopy groups $\pi_*^{\mathbb{Q}} BCob_2^{\chi \leq 0}$ are isomorphic to the homology of Kontsevich's commutative graph complex, plus an additional summand.

0.3 A reader's guide

Throughout this thesis we will be using basic results about classifying spaces of (weakly-unital, topological) categories and geometric realisations more generally. These tools are recalled in section 5.1 and chapter 2, though the reader who is unfamiliar with them might want to consult [ERW19a] for a more thorough treatment.

Each part of this thesis starts with an introductory chapter - these should provide a good idea of how to best approach the respective parts. Chapter 7 in particular provides a good overview, as it was written with the most hindsight.

Beyond the introductory chapters, I would advise the reader to start with section 8.1, which establishes the concept of labelled cospan categories and states two structure theorems. (Even though the results for $d = 1$ studied in part II are not formal consequences of this, they conceptually fit the same story.) The structure theorems are proved in chapter 8, but it will be useful to first read about the $d = 1$ case in chapters 5 and 6, which informed the general case.

The reader interested in learning about how tropical moduli spaces turn up in the surface category can skip directly to chapter 9, though some definitions from section 8.1 will be needed.

Part I

Locally (co)Cartesian fibrations as realisation fibrations

Chapter 1

Introduction

When trying to compute the homotopy type of the classifying space BC of some category \mathcal{C} it can be an effective strategy to first construct a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to another category and to then determine the homotopy fiber of the induced map on classifying spaces. Quillen’s [Qui73, Theorem B] states that, under certain conditions, the homotopy fiber of $BF : BC \rightarrow BD$ is computed by the classifying space of the comma category $(F \downarrow d)$ of the functor F . Here $(F \downarrow d)$ can be thought of as a categorical analogue of the path fiber in topology. However, there also is a much simpler variant of the fiber of a functor:

Definition 1.0.1. *The genuine fiber $P|_b = \{b\} \times_{\mathcal{B}} \mathcal{E}$ of a functor $P : \mathcal{E} \rightarrow \mathcal{B}$ at an object $b \in \mathcal{B}$ has as objects those $e \in \mathcal{E}$ with $P(e) = b$ and as morphisms those $(f : e \rightarrow e') \in \mathcal{E}$ with $P(f) = \text{id}_b$.*

We will study functors for which already the *genuine* fiber computes the homotopy fiber.

Definition 1.0.2 (Based on [Rez14]). *A functor $P : \mathcal{E} \rightarrow \mathcal{B}$ is a realisation fibration if, for any other functor $F : \mathcal{C} \rightarrow \mathcal{B}$, the following diagram is homotopy Cartesian:*

$$\begin{array}{ccc} B(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) & \longrightarrow & B\mathcal{E} \\ \downarrow & & \downarrow_{BP} \\ BC & \xrightarrow{BF} & B\mathcal{B}. \end{array}$$

Intuitively this property states that BP behaves like a fibration: pullbacks along P become homotopy pullbacks along BP . While this does not mean that BP is actually a fibration, it still implies that for any $b \in \mathcal{B}$ the genuine fiber $B\mathcal{E}|_b$ is the homotopy fiber of $B\mathcal{E} \rightarrow B\mathcal{B}$ at b .

In [Ste18a] Steimle proves an “additivity theorem for bordism categories”, which provides us with a supply of realisation fibrations by giving categorical criteria for a functor to be a realisation fibration. We generalise this theorem by showing that the criteria can be weakened. In the special case of discrete categories our main theorem is:

Theorem A. *If a functor is locally Cartesian and locally coCartesian, then it is a realisation fibration.*

Whether a functor $P : \mathcal{E} \rightarrow \mathcal{B}$ is locally (co)Cartesian can be checked as follows:

Definition 1.0.3. A morphism $f : x \rightarrow y$ is called locally P -Cartesian if for all $x' \in \mathcal{E}$ with $P(x') = P(x)$ post-composition with f induces a bijection

$$(f \circ _) : \mathcal{E}_{|P(y)}(x', x) = \{g : x' \rightarrow x \mid P(g) = \text{id}_{P(x)}\} \longrightarrow \{h : x' \rightarrow y \mid P(h) = P(f)\}.$$

The functor P is called locally Cartesian if for all morphisms $(k : a \rightarrow b) \in \mathcal{B}$ and all lifts $y \in \mathcal{E}$ with $P(y) = b$ there is $x \in \mathcal{E}$ and a locally P -Cartesian morphism $f : x \rightarrow y$ with $P(f) = k$.

We say that P is locally coCartesian if $P^{op} : \mathcal{E}^{op} \rightarrow \mathcal{B}^{op}$ is locally Cartesian.

In fact, with applications in mind, we will prove our main theorem in the more general context of weakly unital topological categories. It reads as follows:

Theorem B (See 2.4.8). Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a weakly unital functor of weakly unital topological categories such that \mathcal{B} is fibrant, P is a local fibration, P is locally Cartesian, and P is locally coCartesian. Then P is a realisation fibration.

This generalises [Ste18a, Theorem 2.3], where P was required to be Cartesian and coCartesian. Our proof will follow the same structure as in Steimle’s paper. In particular, we will also prove intermediate versions of the main theorem in the context of quasicategories and semi Segal spaces generalising [Ste18a, Theorem 2.11 and 2.14], respectively. These theorems are stated as 2.1.1 and 2.3.14 in the main text and might be of independent interest.

As an application of Theorem B we study the 2-category $\mathbb{C}\text{sp}$ of cospans of finite sets. The objects of this category are finite sets and the morphisms from A to B are cospans, that is diagrams of the shape $(A \rightarrow W \leftarrow B)$. The 2-morphisms $\alpha : (A \rightarrow W \leftarrow B) \Rightarrow (A \rightarrow V \leftarrow B)$ are compatible bijections $\alpha : W \rightarrow V$. As we recall in section 3.1 this defines a $(2, 1)$ -category. We write $\mathbb{C}\text{sp} = h(\mathbb{C}\text{sp})$ for the homotopy category, where two cospans are identified whenever there is a bijection between them. Some of what we do here is a special case of the more general discussion of labelled cospan categories in section 8.1 in part III.

Theorem B does not directly apply to the quotient functor $P : \mathbb{C}\text{sp} \rightarrow \mathbb{C}\text{sp}$. We therefore construct a further quotient $R : \mathbb{C}\text{sp} \rightarrow \mathbb{C}\text{sp}^{\text{red}}$. In this “reduced cospan” category $\mathbb{C}\text{sp}^{\text{red}}$ two cospans $[A \rightarrow W_i \leftarrow B]$ are identified if they become isomorphic after removing those points of W_i that do not lie in the image of A or B . We will see that a morphism $[A \rightarrow W \leftarrow B]$ in $\mathbb{C}\text{sp}$ is locally R -Cartesian if and only if it is *reduced*, i.e. if $A \amalg B \rightarrow W$ is surjective. This implies that the two functors R and $R \circ P$ are locally (co)Cartesian, but not (co)Cartesian. Theorem B therefore applies to R and $R \circ P$ (while the main theorem of [Ste18a] does not) and yields two homotopy fiber sequences:

Theorem C. *There is a commutative diagram of homotopy fiber sequences:*

$$\begin{array}{ccccc}
Q(S^1) & \longrightarrow & BC\text{Sp} & \xrightarrow{R \circ P} & BC\text{Sp}^{\text{red}} \\
\downarrow & & \downarrow P & & \parallel \\
S^1 & \longrightarrow & BC\text{Sp} & \xrightarrow{R} & BC\text{Sp}^{\text{red}}
\end{array}$$

where $Q(S^1) := \text{colim}_{n \rightarrow \infty} \Omega^n S^{n+1}$ is the free infinite loop space on S^1 . As a consequence the homotopy fiber of $P : BC\text{Sp} \rightarrow BC\text{Sp}$ is equivalent to the universal cover of $Q(S^1)$.

This cospan category is related to the cobordism model for K -theory constructed in [RS19]. There the authors construct a ‘‘cobordism category’’ $\text{Cob}^{\text{sym}}(\mathcal{C})$ for any unpointed Waldhausen category \mathcal{C} such that $\Omega B\text{Cob}^{\text{sym}}(\mathcal{C})$ is the algebraic K -theory of \mathcal{C} . In the case where \mathcal{C} is the category Fin of finite sets, their cobordism category is equivalent to the subcategory $\mathbb{C}\text{Sp}^{\text{inj}} \subset \mathbb{C}\text{Sp}$ of those cospans for which both legs are injections. By the same techniques as used for Theorem C we obtain a new proof of their result in the case $\mathcal{C} = \text{Fin}$:

Corollary D (Compare [RS19, Corollary 3.3]). *There is a commutative diagram of homotopy fiber sequences*

$$\begin{array}{ccccc}
Q(S^1) & \xrightarrow{F} & BC\text{Sp}^{\text{inj}} & \xrightarrow{Q} & BC\text{Sp}^{\text{red, inj}} \\
\parallel & & \downarrow I & & \downarrow I' \\
Q(S^1) & \longrightarrow & BC\text{Sp} & \xrightarrow{R \circ P} & BC\text{Sp}^{\text{red}}
\end{array}$$

Moreover, $BC\text{Sp}^{\text{red, inj}}$ is contractible and therefore F is an equivalence.

Structure of the paper

We begin by proving the main Theorem for quasicategories as 2.1.1 in section 2.1. This section also contains the Key-Lemma 2.1.14, which shows that the core idea of Steimle’s proof remains true under our weaker assumptions. Section 2.2 provides a cubical version of the pasting lemma for homotopy pullbacks. This is used in section 2.3 to prove the main theorem for semi-Segal spaces from its version for quasicategories. In doing so, we repair a proof of [Ste18b]. Finally, section 2.4 uses the results of the previous sections to prove the main Theorem 2.4.8/B for weakly unital topological categories.

The final section 3.1 applies these techniques to the category of cospans. We prove Theorem C by first constructing an explicit topological model for $\mathbb{C}\text{Sp}$ and then applying Theorem 2.4.8. Then, the same setup is used to prove Corollary D.

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Chapter 2

The local additivity theorem for bordism categories

2.1 Realisation fibrations for quasicategories

In proving the main theorem we will mostly follow Steimle’s strategy, but generalise all necessary steps to the locally (co)Cartesian situation. The first, and maybe most surprising step is the following generalisation of [Ste18a, Theorem 2.14], which might be of independent interest:

Theorem 2.1.1 (Main Theorem for quasicategories). *If a map $p : X \rightarrow Y$ of simplicial sets is a locally Cartesian and a locally coCartesian fibration, then it is a realisation fibration.*

This means that for any simplicial map $f : Z \rightarrow Y$ the following diagram is a homotopy pullback in the Quillen model structure on simplicial sets:

$$\begin{array}{ccc} Z \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow p \\ Z & \xrightarrow{f} & Y. \end{array}$$

We recall the necessary definitions related to quasicategories from [Lur09a] below.

Definition 2.1.2. *Recall that a map $p : X \rightarrow Y$ of simplicial sets is called a weak homotopy equivalence if its geometric realisation $p : |X| \rightarrow |Y|$ is a weak equivalence of spaces. These are the weak equivalences in the Quillen model structure on simplicial sets.*

Definition 2.1.3. *A map $p : X \rightarrow Y$ of simplicial sets is called a realisation fibration if for any simplicial map $f : Z \rightarrow Y$ the following is a homotopy pullback square of spaces:*

$$\begin{array}{ccc} |Z \times_Y X| & \longrightarrow & |X| \\ \downarrow & & \downarrow p \\ |Z| & \xrightarrow{f} & |Y|. \end{array}$$

REMARK 2.1.4. A map of simplicial sets p is a realisation fibration if the map $Z \times_Y X \rightarrow Z \times_Y^h X$ from the pullback to the homotopy pullback in the Quillen model structure is a weak homotopy equivalence. In particular every Kan-fibration is a realisation fibration.

We need some notation to denote fibers over simplices:

Definition 2.1.5. For any map $p : X \rightarrow Y$ and simplex $\sigma \in Y_n$ we write

$$p_{|\sigma} : (X_{|\sigma} := X \times_Y \Delta^n) \longrightarrow \Delta^n$$

to denote the pullback along $\sigma : \Delta^n \rightarrow Y$.

Just like in [Ste18a] the following lemma is essential for our proof. It characterizes realisation fibrations in terms of their fibers over simplices.

Lemma 2.1.6 ([Wal85, Lemma 1.4.B], see also [Ste18a] and [Rez14]). A map of simplicial sets $p : X \rightarrow Y$ is a realisation fibration if and only if for any n -simplex $\sigma : \Delta^n \rightarrow Y$ with first and last simplex $\sigma(0), \sigma(n) : \Delta^0 \rightarrow Y$ the following inclusions are weak homotopy equivalences:

$$X_{|\sigma(0)} \longrightarrow X_{|\sigma} \longleftarrow X_{|\sigma(n)}.$$

Proof. Waldhausen deduces this from Quillen’s [Qui73, Theorem B]. A direct proof can be found in [Ste18a, Lemma 3.4], or in the unpublished manuscript [Rez14], where a more general version for simplicial spaces is proven. Note that the “only if” direction is trivial. \square

In order to prove theorem 2.1.1 we will need to recall some machinery from Lurie’s “Higher Topos Theory”, in particular [Lur09a, section 2.4]. We will not attempt to give an introduction to the theory of quasicategories, but only establish the necessary notation.

Definition 2.1.7. A simplicial map $p : X \rightarrow Y$ is called an inner fibration if it satisfies the horn-lifting condition for all inner horns Λ_k^n , $0 < k < n$. The map p is a trivial Kan-fibration if it has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$, $n \geq 0$. A quasicategory is a simplicial set X such that $X \rightarrow *$ is an inner fibration.

Definition 2.1.8. For a simplicial set X and a simplex $\sigma \in X_n$ the under-category $X_{\sigma/}$ of X at σ has as set of l -simplices

$$(X_{\sigma/})_l := \{f : \Delta^{n+l+1} \rightarrow X \mid f_{|[0, \dots, n]} = \sigma\}.$$

The projection map $X_{\sigma/} \rightarrow X$ restricts f to $[n+1, \dots, n+l+1] \subset \Delta^{n+l+1}$.

REMARK 2.1.9. Of course the above does not quite define the simplicial set $X_{\sigma/}$: we still have to specify face and degeneracy maps. We refer the reader to [Lur09a, Proposition 1.2.9.2] for more details.

Consider the case when $X = NC$ is the nerve of an ordinary category and $\sigma : [n] \rightarrow \mathcal{C}$ represents some simplex $\sigma \in X_n$. Then the simplicial set $X_{\sigma/}$ is canonically isomorphic to the nerve of the ordinary under category $\mathcal{C}_{\sigma(n)/}$ with objects $(c \in \mathcal{C}, g : \sigma(n) \rightarrow c)$.

Definition 2.1.10. An edge $(f : x \rightarrow y) : \Delta^1 \rightarrow X$ is called p -coCartesian with respect to some $p : X \rightarrow Y$ if the following map is a trivial Kan fibration:

$$X_{f/} \rightarrow X_{x/} \times_{Y_{p(x)/}} Y_{p(f)/}.$$

An edge $f : \Delta^1 \rightarrow X$ is called locally p -coCartesian if it is a coCartesian edge for $p|_{\sigma} : X|_{\sigma} \rightarrow \Delta^1$ where $\sigma = p \circ f : \Delta^1 \rightarrow Y$.

We say that $p : X \rightarrow Y$ is a (locally) coCartesian fibration, if it is an inner fibration and for every edge $\text{cl}f : \text{cl}x \rightarrow \text{cl}y$ in Y and $x \in X$ with $p(x) = \text{cl}x$ there is a (locally) p -coCartesian $f : x \rightarrow y$ in X such that $p(y) = \text{cl}y$ and $p(f) = \text{cl}f$.

REMARK 2.1.11. A simplicial map $p : X \rightarrow Y$ is a locally coCartesian fibration if and only if it is an inner fibration and for every $\sigma : \Delta^1 \rightarrow Y$ the pullback $p|_{\sigma} : X|_{\sigma} \rightarrow \Delta^1$ is a coCartesian fibration.

2.1.1 Proof of Theorem 2.1.1

We will use Waldhausens's criterion (lemma 2.1.6) to prove theorem 2.1.1. Unlike [Ste18a], we will do not check the criteria directly, but rather use Quillen's theorem A for quasicategories. Our Key-Lemma 2.1.14 shows that when applying Quillen A to the fibers over a simplex, we in fact only use that the map in question is a *locally* (co)Cartesian fibration. This is the most essential step of our generalisation.

Theorem 2.1.12 (Quillen's theorem A for quasicategories, [Lur09a]). *If $p : X \rightarrow Y$ satisfies that $X \times_Y Y_y$ is weakly contractible for any $y \in Y$ and Y is a quasicategory, then p is a weak homotopy equivalence.*

Remark on the reference. [Lur09a, Theorem 4.1.3.1] states that under the above assumptions p has to be a cofinal map and [Lur09a, Proposition 4.1.1.3.(3)] states that cofinal maps are weak equivalences in the Quillen model structure. □

We now proceed to check that for coCartesian fibrations over Δ^n the inclusion of the fiber of the last vertex $n \in \Delta^n$ into the total space satisfies the conditions of Quillen's Theorem A.

Lemma 2.1.13. *For any coCartesian fibration $p : X \rightarrow \Delta^n$ and $x \in X$ the simplicial set $X|_n \times_X X_{x/}$ is weakly contractible.*

Proof. Using remark 2.1.9 note that $(\Delta^n)_{f/} \cong (\Delta^n)_{n/} \cong \{n\}$ holds for the unique morphism $f : p(x) \rightarrow n$ in Δ^n . We use this to write

$$X_{|n} = X \times_{\Delta^n} \{n\} \cong X \times_{\Delta^n} (\Delta^n)_{f/}.$$

Therefore

$$X_{x/} \times_X X_{|n} \cong X_{x/} \times_X (X \times_{\Delta^n} (\Delta^n)_{f/}) \cong X_{x/} \times_{\Delta^n} (\Delta^n)_{f/}.$$

Both of the maps $X_{x/} \rightarrow \Delta^n$ and $(\Delta^n)_{f/} \rightarrow \Delta^n$ factor through the inclusion $(\Delta^n)_{k/} \cong \{k, \dots, n\} \subset \Delta^n$ for $k = p(x)$, so we can further rewrite this as

$$X_{x/} \times_X X_{|n} \cong X_{x/} \times_{\Delta^n} (\Delta^n)_{f/} \cong X_{x/} \times_{(\Delta^n)_{k/}} (\Delta^n)_{f/}$$

Let now $\text{cl}f : x \rightarrow x'$ be a p -coCartesian lift of $f : k \rightarrow n$. Then, by definition of coCartesian lifts, the following map is a trivial Kan-fibration

$$X_{\text{cl}f/} \longrightarrow X_{x/} \times_{\Delta^n_{p(\text{cl}x)/}} (\Delta^n)_{p(\text{cl}f)/} \cong X_{x/} \times_X X_{|n}.$$

The left-hand-side is weakly contractible as it has the initial object $\text{cl}f$. But trivial Kan-fibrations are in particular weak homotopy equivalence, hence also the right-hand-side is weakly contractible. \square

In fact, the previous lemma holds in the more general case of *locally* coCartesian fibrations, as we shall now show. This is the key observation in generalising Steimle's theorem.

Lemma 2.1.14 (Key-Lemma). *For any locally coCartesian fibration $p : X \rightarrow \Delta^n$ and $x \in X$ the simplicial set $X_{|n} \times_X X_{x/}$ is weakly contractible.*

Proof. We will prove this using lemma 2.1.13 Write $k := p(x)$ and $\sigma : \Delta^1 \rightarrow \Delta^n$ for the edge $\{k, n\} \subset \Delta^n$. Since $p|_{\sigma} : X|_{\sigma} \rightarrow \Delta^1$ is a coCartesian fibration, the previous lemma tells us that

$$(X|_{\sigma})_{|1} \times_{X|_{\sigma}} (X|_{\sigma})_{(x,0)/}$$

is weakly contractible, where $(x, 0) \in X \times_{\Delta^n} \Delta^1 = X|_{\sigma}$. We will argue that the map

$$(X|_{\sigma})_{|1} \times_{X|_{\sigma}} (X|_{\sigma})_{(x,0)/} \rightarrow X_{|n} \times_X X_{x/}$$

is an isomorphism. Clearly $(X|_{\sigma})_{|1} \cong X_{|n}$.

More importantly, consider the simplicial map $s : (X|_{\sigma})_{(x,0)/} \rightarrow (X_{x/})_{|\sigma}$. We check “by hand” that it induces a bijection on l -simplices and hence is an isomorphism of simplicial sets:

$$\begin{aligned} ((X|_{\sigma})_{(x,0)/})_l &= \{a : \Delta^{l+1} \rightarrow X|_{\sigma} \mid a(0) = (x, 0)\} \\ &\cong \{b : \Delta^{l+1} \rightarrow X, c : \Delta^{l+1} \rightarrow \Delta^1 \mid b(0) = x, c(0) = 0, p \circ b = \sigma \circ c\} \\ &\cong \{b : \Delta^l \rightarrow X_{x/}, c : \Delta^l \rightarrow \Delta^1 \mid p \circ b = \sigma \circ c\} = ((X_{x/})_{|\sigma})_l. \end{aligned}$$

Using this we see that

$$\begin{aligned} (X|_{\sigma})|_1 \times_{X|_{\sigma}} (X|_{\sigma})_{(x,0)/} &\cong X|_n \times_{X|_{\sigma}} (X_{x/})|_{\sigma} \cong X|_n \times_{X|_{\sigma}} (X|_{\sigma} \times_X X_{x/}) \\ &\cong X|_n \times_X X_{x/} \end{aligned}$$

which is the desired isomorphism. As mentioned in the beginning of the proof the left-hand-side is contractible by lemma 2.1.13 and therefore so is the right-hand-side, proving the claim. \square

From the above main theorem for quasicategories follows immediately.

Proof of theorem 2.1.1. We need to verify the conditions of lemma 2.1.6. In fact it suffices to check the second equivalence, the first one then follows by applying the same argument to p^{op} . But this is exactly the combination of the key lemma 2.1.14 and Quillen's theorem A 2.1.12. \square

2.1.2 A characterisation of locally Cartesian fibrations

We conclude this section by providing a characterisation of locally (co)Cartesian fibrations that is needed in section 2.3. First recall the following characterisation of (co)Cartesian fibrations:

Lemma 2.1.15 ([Lur09a, Remark 2.4.1.4]). *Let $p : X \rightarrow Y$ be an inner fibration and $f : \Delta^1 \rightarrow X$ an edge, then f is p -Cartesian if (and only if) any diagram of simplicial sets*

$$\begin{array}{ccc} \Lambda_n^n & \xrightarrow{i} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

with $i([n-1, n]) = f$ admits a lift as indicated.

Corollary 2.1.16 (Local version of 2.1.15). *Let $p : X \rightarrow Y$ be inner fibration and $f : \Delta^1 \rightarrow X$ an edge, then f is locally p -Cartesian if¹ any diagram of simplicial sets*

$$\begin{array}{ccc} \Lambda_n^n & \xrightarrow{i} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{j} & Y \end{array}$$

with $i([n-1, n]) = f$ and $j([0, \dots, n-1])$ totally degenerate, admits a lift as indicated.

Proof. We need to check that $(f, [0, 1])$ is Cartesian in the restricted fibration $X|_{\sigma}$ where $\sigma := p \circ f : \Delta^1 \rightarrow Y$. This can be done using lemma 2.1.15 by solving all lifting problems

$$\begin{array}{ccccc} \Lambda_n^n & \xrightarrow{i} & X|_{\sigma} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p|_{\sigma} & \lrcorner & \downarrow p \\ \Delta^n & \xrightarrow{j} & \Delta^1 & \xrightarrow{\sigma} & Y. \end{array}$$

¹Indeed, this condition is not only sufficient, but also necessary. We will shall not give a proof of the converse here as it is irrelevant to our argument. This converse also remains true if we replace "degenerate" by the weaker condition "invertible".

with $i([n-1, n]) = f$ and consequently $j([0, \dots, n-1]) = [0]$ and $j([n-1, n]) = [0, 1]$. By definition of the restriction $X|_\sigma$ as a pullback we can alternatively solve the lifting problem

$$\begin{array}{ccc} \Lambda_n^n & \xrightarrow{i} & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \xrightarrow{\sigma \circ j} & Y. \end{array}$$

But j sends $\Delta^{n-1} \subset \Delta^n$ to $\Delta^0 \subset \Delta^1$, so $j([0, \dots, n-1])$ is totally degenerate and we have the desired lift by the assumption of this corollary. \square

2.2 A cubical pasting lemma for homotopy Cartesian squares

In the two upcoming sections we will work with homotopy pullbacks and squares that are homotopy Cartesian at certain point, a notion we define below in 2.2.2. In this section we establish a technical tool for manipulating such squares. First, recall the pasting law for pullbacks:

Lemma 2.2.1 (The (homotopy) pullback pasting law). *For any diagram of the shape*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

where the right hand square is homotopy Cartesian the following are equivalent:

- the left hand square is homotopy Cartesian.
- the big rectangle is homotopy Cartesian.

Proof. The pasting law is well-known for strict pullbacks in any category. (One can also verify the respective universal properties by hand.) To see that the lemma also holds for homotopy pullbacks in any model category, simply replace all maps in the diagram by fibrations. \square

While the definition of a Cartesian edge 2.4.6 asks for a certain diagram to be homotopy Cartesian, it's local counter-part only asks for two specific homotopy fibers to be equivalent. We formalise this notion:

Definition 2.2.2. *We say that a commutative diagram of spaces*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow p & & \downarrow q \\ C & \xrightarrow{j} & D \end{array}$$

is homotopy Cartesian at a point $c \in C$, if the induced map on homotopy fibers

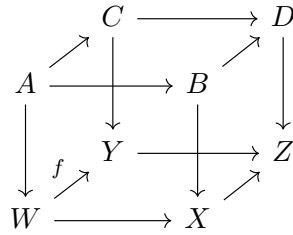
$$\text{hofib}_c(p) \xrightarrow{i} \text{hofib}_{j(c)}(q)$$

is a weak equivalence.

REMARK 2.2.3. If a square is homotopy Cartesian at x , and $y \in C$ is in the same path component as x , then the square is homotopy Cartesian at y . A square is homotopy Cartesian, if and only if it is homotopy Cartesian at all points, if and only if it is homotopy Cartesian at at least one point in every path component.

We will need the following variant of the pasting law, which we prove via a diagram chase.

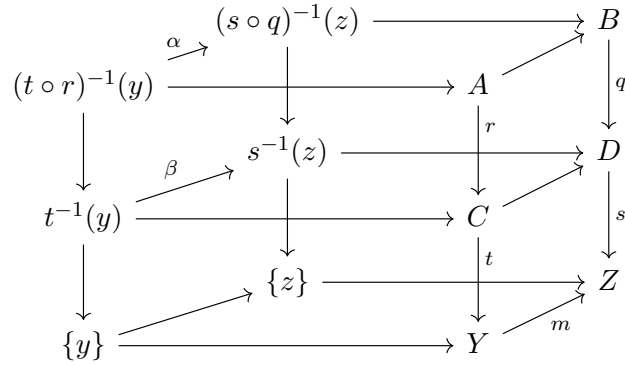
Lemma 2.2.4. Consider a diagram of cubical shape:



where the top and the bottom square are homotopy Cartesian. If the back square is homotopy Cartesian at $f(w) \in Y$, then the front square is homotopy Cartesian at $w \in W$.

Proof. In what follows we used the preimage notation to denote the homotopy fibers. For example $f^{-1}(y)$ denotes the homotopy fiber of $f : W \rightarrow Y$ at $y \in Y$. This can be made precise by replacing all maps by equivalent fibrations.

Consider the following diagram, where the elements y, z are chosen as $y = f(w)$ and $z = m(y)$.



By assumption $CDYZ$ is homotopy Cartesian at $y = f(w)$ and hence β is an equivalence. Also by assumption $ABCD$ is homotopy Cartesian and an application of the pullback pasting lemma shows that α also is an equivalence.

Next consider the diagram

$$\begin{array}{ccccccc}
& & k^{-1}(x) & \xrightarrow{\quad} & (l \circ k)^{-1}(z) & \xrightarrow{\quad} & B \\
& \nearrow \gamma & \downarrow & \searrow \alpha & \downarrow & & \downarrow k \\
g^{-1}(w) & \xrightarrow{\quad} & (f \circ g)^{-1}(y) & \xrightarrow{\quad} & A & \xrightarrow{\quad} & X \\
\downarrow & & \downarrow & & \downarrow g & & \downarrow l \\
& \nearrow & \{x\} & \xrightarrow{\quad} & l^{-1}(z) & \xrightarrow{\quad} & W \\
\{w\} & \xrightarrow{\quad} & f^{-1}(y) & \xrightarrow{\quad} & W & \xrightarrow{\quad} & Z \\
& & \downarrow & \searrow \delta & \downarrow f & & \downarrow m \\
& & \{y\} & \xrightarrow{\quad} & Y & \xrightarrow{\quad} & Z \\
& & & \nearrow & \{z\} & \xrightarrow{\quad} & Z
\end{array}$$

The map α is the same as in the previous diagram, it goes between the fibers of the rectangle $ABYZ$, in particular we have shown that it is an equivalence. By assumption $WXYZ$ is homotopy Cartesian, so δ is an equivalence. But this implies that γ is an equivalence, and hence $ABWX$ is homotopy Cartesian at w as claimed. \square

2.3 Realisation fibrations and semi-Segal spaces

Before we can prove the additivity theorem for weakly unital categories, we first need to establish an intermediate version for semi-Segal spaces. Most importantly, we will introduce a notion of locally Cartesian fibrations between semi-Segal spaces and check that it is compatible with the notions we have for quasicategories and weakly unital topological categories.

Definition 2.3.1. *The semi-simplicial category $\Delta^{inj} \subset \Delta$ has all objects, but only those maps that are injective. We write $s\Delta^n$ for the semi-simplicial set represented by $[n] \in \Delta^{inj}$. It is a sub-semi-simplicial set of Δ^n and contains precisely those simplices that are not degenerate.*

Definition 2.3.2. *For a semi-simplicial space $X : (\Delta^{inj})^{op} \rightarrow Top$ and a semi-simplicial set A we define*

$$X(A) = \operatorname{colim}_{s\Delta^n \rightarrow A} X_n$$

where the colimit is taken over all $n \geq 0$ and all n -simplices of A , ordered by inclusion. This definition is such that $X(s\Delta^n) \cong X_n$.

Moreover, if $x \in X_k$ we define X_n/x to be the homotopy fiber, at the point x , of the map $X_k \rightarrow X_n$ induced by $[k] \cong \{n-k, \dots, n\} \hookrightarrow \{0, \dots, n\} = [n]$.

Definition 2.3.3. *A map $f : X \rightarrow Y$ of semi-simplicial spaces is a Reedy-fibration, if for any inclusion $A \subset B$ of semi-simplicial sets the induced map*

$$X(A) \rightarrow X(B) \times_{Y(B)} Y(A).$$

is a trivial Kan-fibration.

Definition 2.3.4. A semi-simplicial space X is a semi-Segal space, if the natural map

$$X_n \rightarrow X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1$$

is a weak equivalence for all $n \geq 2$.

Definition 2.3.5. An edge $e \in X_1$ in a semi-Segal space X is an equivalence if the maps $d_1 : X_2/e \rightarrow X_1/d_0e$ and $e \setminus X_2 \rightarrow d_1e \setminus X_1$ are weak equivalences. Let $X^\simeq \subset X$ denote the sub semi-simplicial space of those simplices where all edges are equivalences.

REMARK 2.3.6. By the semi-Segal property the map $d_2 : X_2/e \rightarrow X_1/d_1e$ is a weak equivalence for all edges $e \in X_1$. Up to inverting this weak equivalence we therefore have a map

$$X_1/d_1e \xleftarrow{\simeq} X_2/e \longrightarrow X_1/d_0e.$$

Thinking of $e : a \rightarrow b$ as a morphism, this zig-zag $X_1/a \dashrightarrow X_1/b$ can be thought of as post-composition with e . In this sense e is an equivalence if and only if post- and pre-composition with it induce an equivalence.

If $e \in X_1$ is an equivalence and $e' \in X_1$ lies in the same path-component as e , then e' also is an equivalence. To see this note that as we move from e to e' the spaces X_2/e , X_1/d_0e , $e \setminus X_2$, and $d_1e \setminus X_1$ stay the same up to weak equivalence, as they are homotopy fibers. As a consequence the simplicial subspace $X^\simeq \subset X$ is level-wise a union of connected components in every simplicial degree.

Definition 2.3.7. A semi-Segal space X is weakly unital if the map $d_0 : X_1^\simeq \rightarrow X_0$ is surjective on connected components. A map of semi-Segal spaces $F : X \rightarrow Y$ is weakly unital if it sends equivalences to equivalences.

Definition 2.3.8. An edge $e \in X$ is p -Cartesian for $p : X \rightarrow Y$ a map of semi-simplicial spaces, if the following square is homotopy Cartesian:

$$\begin{array}{ccc} X_2/e & \xrightarrow{d_1} & X_1/d_0e \\ \downarrow p & & \downarrow p \\ Y_2/p(e) & \xrightarrow{d_1} & Y_1/d_0p(e) \end{array}$$

We say that e is locally p -Cartesian if the square is homotopy Cartesian at all $s \in Y_2/p(e)$ for which d_2s is an equivalence.

Write $\mathcal{C}_1^{p-(l)cart} \subset \mathcal{C}_1$ for the subset of those edges that are (locally) p -Cartesian. A map $p : \mathcal{C} \rightarrow \mathcal{D}$ of semi-Segal spaces is a (locally) Cartesian fibration, if the canonical map $\mathcal{C}_1^{p-(l)cart} \rightarrow \mathcal{C}_0 \times_{\mathcal{D}_0}^h \mathcal{D}_1$ is surjective on connected components. The coCartesian notions are defined dually.

To transfer the result about quasicategorical realisation fibrations to the setting of semi-Segal spaces we use the following construction:

Definition 2.3.9. For any semi-simplicial space X we will X^δ let denote the semi-simplicial set obtained by forgetting the topology of X .

Lemma 2.3.10 ([Ste18a, 3.11(i+iv)]). If $P : \mathcal{C} \rightarrow \mathcal{D}$ is a weakly unital Reedy-fibration of weakly unital Reedy-fibrant semi-Segal spaces, then \mathcal{C}^δ and \mathcal{D}^δ admit simplicial structures such that they are quasicategories and P^δ is an inner fibration of simplicial maps.

We now study how locally (co)Cartesian fibrations behave under the functor $(_)^\delta$.

Lemma 2.3.11. Let $P : \mathcal{C} \rightarrow \mathcal{D}$ be as in lemma 2.3.10. If $e \in \mathcal{C}_1$ is P -Cartesian in the sense of semi-Segal spaces (2.3.8), then $e \in \mathcal{C}_1^\delta$ is P^δ -Cartesian in the sense of quasicategories (2.1.10).

This lemma is equivalent to [Ste18b, Lemma 3.6]. A correct proof idea was given there, but the second diagram and the accompanying statement are stated incorrectly, making it difficult to understand the proof. Below we give a repaired version of the proof, as our proof of Lemma 2.3.12 builds on it.

Proof. Recall that we defined \mathcal{C}_{n+1}/e as a homotopy fiber. In the context of this lemma all relevant maps are fibrations, so the homotopy fiber and strict fiber are equivalent. For the purpose of this proof, we will let \mathcal{C}_{n+1}/e denote the strict fiber as it is easier to work with.

For an edge $e : x' \rightarrow x$ to be P^δ -Cartesian in the sense of quasicategories it suffices, by lemma 2.1.15, to check that the following map is surjective:

$$I : \mathcal{C}_{n+1}/e \rightarrow (\mathcal{C}(\Lambda_{n+1}^{n+1})/e) \times_{\mathcal{D}(\Lambda_{n+1}^{n+1})} \mathcal{D}_{n+1}.$$

This map is the restriction of $\mathcal{C}_{n+1} \rightarrow \mathcal{C}(\Lambda_{n+1}^{n+1}) \times_{\mathcal{D}(\Lambda_{n+1}^{n+1})} \mathcal{D}_{n+1}$, which is a fibration as P is assumed to be a Reedy-fibration. Therefore, if we can show that I is a weak equivalence, then it is an acyclic fibration and in particular surjective, showing that e is P^δ -Cartesian.

We begin by using the Segal property $\mathcal{C}_n \simeq \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_{n-1}$ to compute:

$$\mathcal{C}_n/x \simeq \{x\} \times_{\mathcal{C}_0}^h \mathcal{C}_n \simeq \{x\} \times_{\mathcal{C}_0}^h \mathcal{C}_1 \times_{\mathcal{C}_0}^h \mathcal{C}_{n-1} \simeq (\mathcal{C}_1/x) \times_{\mathcal{C}_0}^h \mathcal{C}_{n-1}$$

Together with the pullback pasting lemma 2.2.1 this implies that the following two squares are homotopy Cartesian:

$$\begin{array}{ccccc} \mathcal{C}_{n+1}/e & \xrightarrow{d_1} & \mathcal{C}_n/x & \longrightarrow & \mathcal{C}_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_2/e & \longrightarrow & \mathcal{C}_1/x & \longrightarrow & \mathcal{C}_0 \end{array} \tag{2.1}$$

Using a similar argument for \mathcal{D} we can now draw a commutative cube whose top and bottom square are homotopy Cartesian:

$$\begin{array}{ccccc}
& & \mathcal{C}_2/e & \xrightarrow{d_1} & \mathcal{C}_1/x \\
& \nearrow & \downarrow P & \nearrow & \downarrow P \\
\mathcal{C}_{n+1}/e & \xrightarrow{d_1} & \mathcal{C}_n/x & & \\
\downarrow P & & \downarrow P & & \\
& \nearrow & \mathcal{D}_2/P(e) & \xrightarrow{d_1} & \mathcal{D}_1/P(x) \\
& & \downarrow P & & \downarrow P \\
\mathcal{D}_{n+1}/P(e) & \xrightarrow{d_1} & \mathcal{D}_n/P(x) & &
\end{array}$$

Since the edge $e : x' \rightarrow x$ is Cartesian in the sense of semi Segal spaces, the back of the cube is homotopy Cartesian. Now corollary 2.2.4 applies to this cube and shows that its front square also has to be homotopy Cartesian.

The rest of the proof is completely analogous to the one given in [Ste18b]. We factor the newly obtained homotopy Cartesian square as:

$$\begin{array}{ccccc}
\mathcal{C}_{n+1}/e & \longrightarrow & \mathcal{C}(\Lambda_0^{n+1})/e & \xrightarrow{d_1} & \mathcal{C}_n/x \\
\downarrow P & & \downarrow P & & \downarrow P \\
\mathcal{D}_{n+1}/e & \longrightarrow & \mathcal{D}(\Lambda_0^{n+1})/p(e) & \xrightarrow{d_1} & \mathcal{D}_n/x
\end{array}$$

Our goal is to use the fact that the total rectangle is homotopy Cartesian to deduce that the left square is homotopy Cartesian. We will show this by induction over n . For $n = 0$ the map $d_1 : \Lambda_0^1 \cong \Delta^1$ is an isomorphism and so the right horizontal maps in the above diagram are isomorphisms. The claim follows.

For the induction step note that the inclusion $\Delta^n \subset \Lambda_0^{n+1}$ of the n th face can be obtained by a sequence of pushouts that fill horns $\Lambda_0^k \subset \Delta^k$ (with $k \leq n$) where the last edge is $(n, n+1)$. The right square in the above diagram hence factors as a sequence of squares each of which is a homotopy pullback by the induction hypothesis.

Therefore the right square in the above diagram is homotopy Cartesian and then the pasting lemma for pullbacks implies that the left square is a homotopy pullback. This concludes the proof. \square

Lemma 2.3.12. *Let $P : \mathcal{C} \rightarrow \mathcal{D}$ be as in lemma 2.3.10. If $e \in \mathcal{C}_1$ is locally P -Cartesian, then $e \in \mathcal{C}_1^\delta$ is locally P^δ -Cartesian.*

Proof. This is completely analogous to the proof of 2.3.11, except that we will use lemma 2.1.16 instead of 2.1.15 and that in every step we only show that the respective diagram is homotopy Cartesian at certain points.

Let $\mathcal{D}^\simeq \subset \mathcal{D}$ denote the sub-simplicial space consisting of those simplices for which every one-dimensional edge is an equivalence. This is a Segal space and in each simplicial degree it is a union of connected components of \mathcal{D}_n . We want to show that the following map is surjective:

$$I : (\mathcal{C}_{n+1}/e) \times_{\mathcal{D}_n} \mathcal{D}_n^\simeq \rightarrow (\mathcal{C}(\Lambda_{n+1}^{n+1})/e) \times_{\mathcal{D}(\Lambda_{n+1}^{n+1})} \mathcal{D}_{n+1} \times_{\mathcal{D}_n} \mathcal{D}_n^\simeq.$$

Once we know that I is surjective lemma 2.1.16 applies since degenerate 1-simplices are in particular equivalences. The map is a fibration since $\mathcal{C} \rightarrow \mathcal{D}$ was assumed to be a Reedy-fibration, so, if we can show that it is a weak equivalence, surjectivity follows.

Now the same proof as for 2.3.11 applies: we only need to check that the respective squares are homotopy Cartesian at those $\sigma \in \mathcal{D}_{n+1}$ for which $d_{n+1}\sigma \in \mathcal{D}_n$ lies in $\mathcal{D}_n^\simeq \subset \mathcal{D}_n$. Note that in particular the critical step where we apply corollary 2.2.4 works just as well for locally Cartesian edges. \square

We now recover [Ste18a, 3.11(iii)] plus a version for locally (co)Cartesian fibrations:

Corollary 2.3.13. *Let $P : \mathcal{C} \rightarrow \mathcal{D}$ be as in lemma 2.3.10.*

- *If P is (co)Cartesian, then $P^\delta : \mathcal{C}^\delta \rightarrow \mathcal{D}^\delta$ is a (co)Cartesian fibration of quasicategories.*
- *If P is locally (co)Cartesian, then $P^\delta : \mathcal{C}^\delta \rightarrow \mathcal{D}^\delta$ is a locally (co)Cartesian fibration of quasicategories.*

Proof. We show the first point. By definition of a Cartesian fibration between semi-Segal spaces we know that $\mathcal{C}_1^{P\text{-cart}} \rightarrow \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{D}_1$ is π_0 -surjective. It also is a fibration and hence has to be surjective, which remains true after applying $(_)^\delta$. Seeing as P -Cartesian implies P^δ -Cartesian by the first part of the proof this shows that for every edge $f \in \mathcal{D}_1$ and $y \in \mathcal{C}_0$ with $d_i f = P(y)$ there is a P^δ -Cartesian lift $e \in \mathcal{C}_1$ with $P(e) = f$ and $d_i e = y$. In other words, P^δ is a Cartesian fibration. \square

We can now deduce the additivity theorem for semi-Segal spaces from the version for quasicategories:

Theorem 2.3.14 (Main Theorem for semi Segal spaces). *Let*

$$\mathcal{B}' \xrightarrow{F} \mathcal{B} \xleftarrow{P} \mathcal{E}$$

be a diagram of weakly unital semi-Segal spaces and weakly unital maps. If P is a level-wise fibration, a locally Cartesian and a locally coCartesian fibration, then the (level-wise) pullback diagram

$$\begin{array}{ccc} \mathcal{B}' \times_{\mathcal{B}} \mathcal{E} & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow P \\ \mathcal{B}' & \xrightarrow{F} & \mathcal{B} \end{array}$$

realises to a homotopy pullback diagram of spaces.

Proof of 2.3.14 from 2.1.1. The notion of weakly unital functor and locally (co)Cartesian fibration are invariant under level-wise equivalence: For weakly unital functors this is shown in [Ste18a, Lemma 3.9]. The case for locally (co)Cartesian fibrations is very similar, one just has to check that definition 2.3.8 only uses “equivalence invariant notions”. Therefore the hypothesis of the theorem are invariant under weak equivalences and using [Ste18a, Lemma 3.8] we may replace \mathcal{B} by a Reedy-fibrant semi-Segal space and F and P by Reedy-fibrations.

Applying the (pullback preserving) functor $(_)^\delta$ to the diagram we obtain a square

$$\begin{array}{ccc} (\mathcal{B}')^\delta \times_{\mathcal{B}^\delta} \mathcal{E}^\delta & \longrightarrow & \mathcal{E}^\delta \\ \downarrow & & \downarrow P^\delta \\ \mathcal{B}' & \xrightarrow{F^\delta} & \mathcal{B}. \end{array}$$

Using 2.3.10 we can choose simplicial structures such that this is a diagram of quasicategories in which all maps are inner fibrations. In lemma 2.3.12 we showed that $P^\delta : \mathcal{E}^\delta \rightarrow \mathcal{B}^\delta$ is locally Cartesian and locally coCartesian, so the quasicategory version of the additivity theorem (theorem 2.1.1) applies and shows that the square realizes to a homotopy Cartesian square of spaces.

Now the natural transformation $(_)^\delta \Rightarrow \text{Id}$ induces a map from the above square to the square in the statement of the proposition. By [Ste18a, Lemma 3.11] this map becomes a weak equivalence in every entry after geometric realisation, and so it follows that the square in the statement of the theorem realizes to a homotopy Cartesian square as claimed. \square

2.4 Realisation fibrations and weakly unital topological categories

In this section we prove the main theorem for realisation fibrations between weakly unital topological categories, which is stated below as 2.4.8. We recall the necessary definitions, but refer the reader to [ERW19a] and [Ste18a] for a detailed introduction to non-unital or weakly unital topological categories.

Definition 2.4.1. A non-unital topological category \mathcal{C} is a space of objects \mathcal{C}_0 and a space of morphisms \mathcal{C}_1 together with source and target maps $s, t : \mathcal{C}_1 \rightarrow \mathcal{C}_0$ and a composition map $m : \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \rightarrow \mathcal{C}_1$ such that for every composable triple $(f, g, h) \in \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1$ we have

$$s(m(f, g)) = s(g) \quad t(m(f, g)) = t(g) \quad m(f, m(g, h)) = m(m(f, g), h).$$

A non-unital functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of continuous maps $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \rightarrow \mathcal{D}_1$ commuting with s, t , and m .

Definition 2.4.2. The classifying space BC of \mathcal{C} is the (fat) geometric realisation of its semi-simplicial nerve NC .

Definition 2.4.3. For two objects $x, y \in \mathcal{C}_0$ we define the hom-space as the pullback

$$\mathcal{C}(x, y) := \{x\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{y\}.$$

We say that a morphism $f \in \mathcal{C}(x, y)$ is an equivalence if both pre- and post-composition with f are weak equivalences for all objects $w, z \in \mathcal{C}_0$:

$$f_* : \mathcal{C}(w, x) \rightarrow \mathcal{C}(w, y) \quad \text{and} \quad f^* : \mathcal{C}(y, z) \rightarrow \mathcal{C}(x, z).$$

Definition 2.4.4. An endomorphism $u : x \rightarrow x$ is a weak unit if it is an equivalence and $u \circ u$ is homotopic to u in $\mathcal{C}(x, x)$. A weakly unital category is a non-unital topological category such that all objects have weak units. A weakly unital functor is a functor that sends weak units to weak units.

Definition 2.4.5. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ of weakly unital topological categories is a local fibration if the continuous maps $F_0 : \mathcal{C}_0 \rightarrow \mathcal{D}_0$ and $((s, t), F_1) : \mathcal{C}_1 \rightarrow (\mathcal{C}_0)^2 \times_{(\mathcal{D}_0)^2} \mathcal{D}_1$ are Serre fibrations.

A non-unital topological category \mathcal{C} is locally fibrant if the functor $\mathcal{C} \rightarrow *$ is a local fibration. Explicitly, this means that the “source-target map” $(s, t) : \mathcal{C}_1 \rightarrow (\mathcal{C}_0)^2$ is a Serre fibration.

Definition 2.4.6. Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a weakly unital functor:

- A morphism $f : e \rightarrow e'$ in \mathcal{E} is P -Cartesian if the following diagram is a homotopy pullback for all $t \in \mathcal{E}_0$:

$$\begin{array}{ccc} \mathcal{E}(t, e) & \xrightarrow{f \circ _} & \mathcal{E}(t, e') \\ \downarrow P_{(t, e)} & & \downarrow P_{(t, e')} \\ \mathcal{B}(P(t), P(e)) & \xrightarrow{P(f) \circ _} & \mathcal{B}(P(t), P(e')) \end{array}$$

- A morphism $f : e \rightarrow e'$ is locally P -Cartesian, if there is an equivalence $u : b \rightarrow P(e)$ such that for all $t \in \mathcal{E}_0$ with $P(t) = b$ the above square is homotopy Cartesian at $u \in \mathcal{B}(P(t), P(e))$. (In the sense of 2.2.2.) Concretely, this requires that the induced map

$$(f \circ _) : \text{hofib}_u(P_{(t, e)}) \xrightarrow{\cong} \text{hofib}_{f \circ u}(P_{(t, e')})$$

on homotopy fibers is a weak equivalence.

- We say that P is (locally) Cartesian if for $e \in \mathcal{E}$ and $g : P(e) \rightarrow b'$ there is a (locally) P -Cartesian $f : e \rightarrow e'$ with $P(f) = g$.
- We say that P is (locally) coCartesian if $P^{op} : \mathcal{E}^{op} \rightarrow \mathcal{B}^{op}$ is (locally) Cartesian.

REMARK 2.4.7. As we will see in lemma 2.4.9, the definition of locally P -Cartesian edges does not depend on the choice of b and u . Also note that if \mathcal{B} is unital we can choose $b = P(e)$ and $u = \text{id}_b$, recovering definition 1.0.3.

We now have all the definitions at hand to state Theorem B.

Theorem 2.4.8 (Main Theorem for weakly unital topological categories). *Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a weakly unital functor of weakly unital topological categories such that \mathcal{B} is fibrant, P is a local fibration, P is locally Cartesian, and P is locally coCartesian. Then P is a realisation fibration.*

Hence, for every weakly unital functor $F : \mathcal{C} \rightarrow \mathcal{B}$, the following is a homotopy pullback square:

$$\begin{array}{ccc} B(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) & \longrightarrow & B\mathcal{E} \\ \downarrow & & \downarrow P \\ B\mathcal{C} & \xrightarrow{F} & B\mathcal{B}. \end{array}$$

Before we prove this from Theorem 2.3.14, we first need to translate from weakly unital categories to semi-Segal spaces.

Lemma 2.4.9. *Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a weakly unital functor of weakly unital topological categories and let $f : e \rightarrow e'$ be a locally P -Cartesian edge in the sense of definition 2.4.6. More concretely, there is an equivalence $u : t \rightarrow e$ such that the following square is homotopy Cartesian at $P(u) \in \mathcal{B}(P(t), P(e))$:*

$$\begin{array}{ccc} \mathcal{E}(t, e) & \xrightarrow{f \circ -} & \mathcal{E}(t, e') \\ \downarrow P_{(t,e)} & & \downarrow P_{(t,e')} \\ \mathcal{B}(P(t), P(e)) & \xrightarrow{P(f) \circ -} & \mathcal{B}(P(t), P(e')). \end{array}$$

Then the square is also homotopy Cartesian at $P(u')$ for any other choice of an equivalence $u' : t' \rightarrow e$.

Proof. Given equivalences $u : t \rightarrow e$ and $u' : t' \rightarrow e$ we can find an equivalence $g : t \rightarrow t'$ such that $u \simeq u' \circ g$. To see this, note that $(u' \circ -) : \mathcal{E}(t, t') \rightarrow \mathcal{E}(t, e)$ is a weak equivalence, and so we can pick a $g \in \mathcal{E}(t, t')$ such that $u' \circ g$ is in the same path component as u . This g is then a weak equivalence by the 2-out-of-3 property.

Composition with g and $P(g)$ induces weak equivalences² on the mapping spaces and therefore the diagonal maps in the following commutative diagram are weak equivalences:

$$\begin{array}{ccccc} & & \mathcal{E}(t, e) & \xrightarrow{f \circ -} & \mathcal{E}(t, e') \\ & \nearrow_{- \circ g} & \downarrow P_{(t,e)} & & \downarrow P_{(t,e')} \\ \mathcal{E}(t', e) & \xrightarrow{f \circ -} & \mathcal{E}(t', e') & \xrightarrow{- \circ g} & \mathcal{E}(t, e') \\ \downarrow P_{(t',e)} & & \downarrow P_{(t',e')} & & \downarrow P_{(t,e')} \\ & \nearrow_{- \circ P(g)} & \mathcal{B}(P(t), P(e)) & \xrightarrow{P(f) \circ -} & \mathcal{B}(P(t), P(e')) \\ \mathcal{B}(P(t'), P(e)) & \xrightarrow{P(f) \circ -} & \mathcal{B}(P(t'), P(e')) & \xrightarrow{- \circ P(g)} & \mathcal{B}(P(t), P(e')) \end{array}$$

²Since P is weakly unital $P(g)$ also is a weak equivalence.

The back square is homotopy Cartesian at $P(u)$, and hence also at $P(u') \circ P(g) = P(u' \circ g)$ as it is in the same path component as $P(u)$. Since the front and back squares are weakly equivalent it follows that the front square is homotopy Cartesian at $P(u')$ as claimed. \square

Lemma 2.4.10. *If a weakly unital functor $P : \mathcal{E} \rightarrow \mathcal{B}$ of weakly unital, locally fibrant categories is a locally Cartesian fibration and a local fibration, then the induced map on nerves $NP : N\mathcal{E} \rightarrow N\mathcal{B}$ is a locally Cartesian fibration of semi-Segal spaces.*

Proof. By definition the hom-space $\mathcal{E}(t, e)$ is the pullback $\{t\} \times_{\mathcal{E}_0} \mathcal{E}_1 \times_{\mathcal{E}_0} \{e\}$, which could also be denoted as $t \backslash \mathcal{E}_1 / e$. Since \mathcal{E} is locally fibrant this pullback is equivalent to the homotopy pullback. For any edge $f : e \rightarrow e'$ we have the following diagram, in which both squares are homotopy Cartesian:

$$\begin{array}{ccccc} \mathcal{E}(t, e) & \xrightarrow{(f \circ _)} & \mathcal{E}(t, e') & \longrightarrow & \{t\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}_2/f & \xrightarrow{d_1} & \mathcal{E}_1/e' & \longrightarrow & \mathcal{E}_0. \end{array}$$

Using this we see that in the following diagram the top and the bottom square are homotopy Cartesian:

$$\begin{array}{ccccc} & & \mathcal{E}_2/f & \xrightarrow{d_1} & \mathcal{E}_1/e' \\ & \nearrow & \downarrow & & \downarrow P_1 \\ \mathcal{E}(t, e) & \xrightarrow{(f \circ _)} & \mathcal{E}(t, e') & \longrightarrow & \mathcal{E}_1/e' \\ & \downarrow P_2 & \downarrow & & \downarrow P_1 \\ & \mathcal{B}_2/P(f) & \xrightarrow{d_1} & \mathcal{B}_1/P(e') & \\ \downarrow & \nearrow i & \downarrow & \nearrow & \\ \mathcal{B}(P(t), P(e)) & \xrightarrow{(P(f) \circ _)} & \mathcal{B}(P(t), P(e')) & & \end{array}$$

If f is P -Cartesian, then by lemma 2.4.9 for any $t \in \mathcal{E}_0$ and any equivalence $u : t \rightarrow e$ the back square is homotopy Cartesian at $P(u) \in \mathcal{B}(t, e)$. By corollary 2.2.4 this implies that the front square is homotopy Cartesian at $i(P(u))$. This shows that f is NP -Cartesian because any $\sigma \in \mathcal{B}_2/P(f)$ with $d_2\sigma$ an equivalence is homotopic to $i(P(d_2\sigma))$ and the previous argument applies. \square

Proof of theorem 2.4.8. Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a weakly unital functor of weakly unital topological categories such that \mathcal{B} is fibrant, P is a fibration, P is locally Cartesian, and P is locally coCartesian. Then we need to show that P is a realisation fibration. In other words, for every weakly unital functor $F : \mathcal{C} \rightarrow \mathcal{B}$ we need to show that the following diagram is a homotopy pullback square:

$$\begin{array}{ccc} B(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) & \longrightarrow & B\mathcal{E} \\ \downarrow & & \downarrow P \\ BC & \xrightarrow{F} & B\mathcal{B}. \end{array}$$

We do this by applying theorem 2.3.14, the version of the additivity theorem for semi-Segal spaces, to the diagram of semi-Segal spaces

$$NC \xrightarrow{F} NB \xleftarrow{P} NE.$$

This indeed is a diagram of weakly unital maps between semi-Segal spaces by [Ste18a, Remark 2.10] and NP is locally Cartesian and locally coCartesian by lemma 2.4.10. The conclusion of 2.3.14 indeed proves our claim since the classifying space BC is exactly defined as the geometric realisation of NC . □

Chapter 3

The cospan category

3.1 An application: the cospan category

The classifying space of various cobordism categories have been computed via geometric methods, most notably in [GMTW09]. These cobordism categories are, in some sense, categories of cospans of manifolds. In this section we apply Theorem B to study the category of cospans of finite sets, which one might think of as a combinatorial bordism category. Note that this theorem can also be proven using the methods of part III of this thesis, where we prove the more general proposition 8.2.4.

3.1.1 The strict model for the cospan category

In order to study the classifying space of the cospan category $\mathbb{C}\text{sp}$ we need a strictly associative model for the composition of cospans that still keeps track of non-trivial isomorphism the cospans might have. We define this below in definition 3.1.7, but to do so we need to be extremely careful about how we construct pushouts. We therefore establish some notation. An alternative approach to this that also takes care of the Γ -space structure is given in section 8.1.2.1.

NOTATION 3.1.1. For any integer $n \geq 0$ we let \underline{n} denote the finite set

$$\underline{n} := \{1, \dots, n\} \subset \mathbb{Z}.$$

Given an equivalence relation R on \underline{n} we write xRy to denote that x is equivalent to y under R . The equivalence class of $i \in \underline{n}$ will be denoted by

$$[i]_R := \{j \in \underline{n} \mid iRj\}.$$

The quotient of \underline{n} by R is the set of equivalence classes

$$\underline{n}/R := \{[i]_R \mid i \in \underline{n}\} \subset \mathfrak{P}(\underline{n}),$$

which is technically a certain subset of the power-set of \underline{n} .

REMARK 3.1.2. Note that it is a property for a set X to be of the form \underline{n}/R . Concretely, if for two numbers $n, m \geq 0$ and two relations R and S the two sets \underline{n}/R and \underline{m}/S are equal, then $n = m$ and $R = S$. Of course we can still have $\underline{n}/R \cong \underline{m}/S$ for $n \neq m$.

Definition 3.1.3. For two finite sets A and B we let $\mathbb{C}\text{sp}(A, B)$ be the groupoid with objects cospans $(A \rightarrow W \leftarrow B)$ such that $W = \underline{n}/R$ for some $n \geq 0$ and some equivalence relation R on \underline{n} . A morphism in this groupoid is a bijection σ making the following diagram commute:

$$\begin{array}{ccc} & W & \\ A \xrightarrow{i} & & \xleftarrow{j} B \\ & \downarrow \sigma & \\ & W' & \\ A \xrightarrow{i'} & & \xleftarrow{j'} B \end{array}$$

Next, we choose an explicit representative of the pushout of two cospans:

Definition 3.1.4. For three finite sets A, B, C we define the functor

$$\mu : \mathbb{C}\text{sp}(B, C) \times \mathbb{C}\text{sp}(A, B) \longrightarrow \mathbb{C}\text{sp}(A, C)$$

by sending $(B \rightarrow \underline{m}/S \leftarrow C)$ and $(A \rightarrow \underline{n}/R \leftarrow B)$ to the cospan $(A \rightarrow (\underline{n+m})/Q \leftarrow C)$. Here the equivalence relation Q on $\underline{n+m}$ is generated by the following requirements:

$$\forall x, y \in \underline{n} : xRy \Rightarrow xQy \quad \forall z, w \in \underline{m} : zSw \Rightarrow (z+n)Q(w+n) \quad \forall b \in B : j(b)Ql(b).$$

In other words, Q is the unique equivalence relation on $\underline{n+m}$ such that the diamond in the following diagram is well defined and a pushout square:

$$\begin{array}{ccccc} & & \underline{n+m}/Q & & \\ & \nearrow & & \nwarrow & \\ A \xrightarrow{i} & \underline{n}/R & & \underline{m}/S & \xleftarrow{l} C \\ & \nwarrow & & \nearrow & \\ & B & & & \\ & \xrightarrow{k} & & & \end{array}$$

On morphisms μ is defined via the universal property of the pushout.

REMARK 3.1.5. It is straightforward to check that μ is strictly associative. The composite of the two cospans above with a third cospan $(C \rightarrow \underline{l}/U \leftarrow D)$ will be of the form $(A \rightarrow (\underline{n+m+l})/T \leftarrow D)$ for some equivalence relation T on $\underline{n+m+l}$. This equivalence relation is uniquely determined by a universal property and does not depend on which order we composed the cospans in.

REMARK 3.1.6. Usually the category of cospans is defined as a bicategory biCsp , see for example [Bén67]. The above defines a non-unital strict 2-category $\mathbb{C}\text{sp}^{\text{str}}$ with objects finite sets and hom-categories the groupoids $\mathbb{C}\text{sp}(A, B)$. Every bicategory can be strictified and $\mathbb{C}\text{sp}^{\text{str}}$ is a specific choice of a (non-unital) strictification of biCsp . Indeed the canonical map $\mathbb{C}\text{sp}^{\text{str}} \rightarrow \text{biCsp}$ is a biequivalence, in the sense that it is essentially surjective and an equivalence on hom-categories. To see this, observe that every finite set is in bijection with a set of the form \underline{n}/R , and therefore every cospan $(A \rightarrow W \leftarrow B)$ is isomorphic to a cospan of the form $(A \rightarrow \underline{n}/R \leftarrow B)$.

Definition 3.1.7. *The non-unital topologically enriched category $\mathbb{C}\text{sp}$ has as objects natural numbers $n \geq 0$ and as hom-spaces¹*

$$\text{hom}_{\mathbb{C}\text{sp}}(a, b) := B\mathbb{C}\text{sp}(\underline{a}, \underline{b}).$$

Composition is induced by the functor μ .

We define $\mathbb{C}\text{sp}$ as the ordinary category that is obtained from $\mathbb{C}\text{sp}$ by taking π_0 of the morphism spaces. We let $P : \mathbb{C}\text{sp} \rightarrow \mathbb{C}\text{sp}$ denote the quotient functor. This functor sends two cospans to the same morphism, whenever there is an isomorphism between the cospans.

Lemma 3.1.8. *The above defines a weakly unital topological category $\mathbb{C}\text{sp}$. Moreover, the hom-spaces in $\mathbb{C}\text{sp}$ decompose as*

$$\text{hom}_{\mathbb{C}\text{sp}}(A, B) \simeq \coprod_{[A \rightarrow W \leftarrow B] \in \mathbb{C}\text{sp}(A, B)} B\Sigma_k$$

where Σ_k is the symmetric group of order $k = |W \setminus \text{Im}(A \amalg B \rightarrow W)|$.

Proof. $\mathbb{C}\text{sp}$ defines a non-unital topologically enriched category: we observed earlier that μ is strictly associative, so $B\mu$ is strictly associative, too. It has weak units of the form $[A = A = A]$.

Fixing two finite sets A and B , we would like to understand the space $\text{hom}_{\mathbb{C}\text{sp}}(A, B)$, or equivalently, the groupoid $\mathbb{C}\text{sp}(A, B)$. It decomposes as a disjoint union of its isomorphism classes, which by definition are the elements of $\mathbb{C}\text{sp}(A, B)$. All that remains to show that the group of automorphisms of a cospan $(A \rightarrow W \leftarrow B)$ is indeed Σ_k for $k = |W \setminus \text{Im}(A \amalg B \rightarrow W)|$.

Let α be an automorphism of $(A \rightarrow W \leftarrow B)$. Then $\alpha : W \rightarrow W$ is a bijection, which has to fix both the image of $A \rightarrow W$ and the image of $B \rightarrow W$. So

$$\text{Aut}(A \rightarrow W \leftarrow B) \cong \text{Aut}(W \setminus \text{Im}(A \amalg B \rightarrow W)) \cong \Sigma_k$$

and the claim follows. □

3.1.2 Proof of theorem C

According to lemma 3.1.8 the difference between $\mathbb{C}\text{sp}$ and $\mathbb{C}\text{sp}$ lies in the fact that a cospan $(A \rightarrow W \leftarrow B)$ in $\mathbb{C}\text{sp}$ has non-trivial automorphisms defined by permuting points in $W \setminus \text{Im}(A \amalg B \rightarrow W)$. We now consider “reduced cospans”, for which this set is trivial:

Definition 3.1.9. *For a cospan $(A \rightarrow W \leftarrow B)$ we define its reduced and its closed part as*

$$r(W) := \text{Im}(A \amalg B \rightarrow W) \quad \text{and} \quad c(W) := W \setminus r(W).$$

Here we suppress the dependence of $r(W)$ and $c(W)$ on the maps $A, B \rightarrow W$ in the notation. We call the cospan reduced if $W = r(W)$ and closed if $W = c(W)$.

¹Here B is defined by taking the simplicial nerve and then the standard geometric realisation. In particular, B commutes with Cartesian products. This would not be the case if we took the “fat geometric realisation”.

Definition 3.1.10. The reduced cospan category $\mathbf{Csp}^{\text{red}}$ has as objects finite sets \underline{a} as morphisms isomorphism classes of reduced cospans. Composition is defined by

$$[\underline{b} \rightarrow V \leftarrow \underline{c}] \circ [\underline{a} \rightarrow W \leftarrow \underline{b}] := [\underline{a} \rightarrow r(W \cup_B V) \leftarrow \underline{c}].$$

We let R denote the reduction functor

$$R : \mathbf{Csp} \rightarrow \mathbf{Csp}^{\text{red}}, \quad [\underline{a} \rightarrow W \leftarrow \underline{b}] \mapsto [\underline{a} \rightarrow r(W) \leftarrow \underline{b}].$$

REMARK 3.1.11. Note that the composition of two reduced cospans is not always reduced:

$$(\underline{1} \xrightarrow{1} \underline{2} \xleftarrow{2} \underline{1}) \circ (\underline{0} \rightarrow \underline{1} \leftarrow \underline{1}) = (\underline{0} \rightarrow \underline{3}/(1 \sim 2) \xleftarrow{3} \underline{1}) \cong (\underline{0} \rightarrow \underline{2} \xleftarrow{2} \underline{1}).$$

(Recall that the order of composition in the cospan category is counterintuitive.) This shows that it is essential to define the composition in $\mathbf{Csp}^{\text{red}}$ as $r(W \cup_B V)$, since the pushout $W \cup_B V$ might not be reduced. It also shows that $\mathbf{Csp}^{\text{red}}$ is not a subcategory of \mathbf{Csp} , but rather a quotient category:

The reduction functor is a bijection on objects and surjective on morphisms. We can hence define $\mathbf{Csp}^{\text{red}}$ as the quotient of \mathbf{Csp} by the relation that identifies two cospans if their reduced parts are isomorphic.

We now wish to compute the homotopy fiber of the quotient functor $R : \mathbf{Csp} \rightarrow \mathbf{Csp}^{\text{red}}$. To apply theorem 2.4.8 to the functor R we need to check that it is locally Cartesian and coCartesian:

Lemma 3.1.12. The reduction functor

$$R : \mathbf{Csp} \rightarrow \mathbf{Csp}^{\text{red}}$$

is locally Cartesian and locally coCartesian. However, it is neither Cartesian nor coCartesian.

Proof. Let us first look at the fiber of the map

$$R : \mathbf{Csp}(A, B) \rightarrow \mathbf{Csp}^{\text{red}}(A, B)$$

at some reduced cospan $[A \rightarrow U \leftarrow B]$. Every cospan in the fiber is of the form

$$[A \rightarrow W \leftarrow B] = [A \rightarrow U \leftarrow B] \amalg [\emptyset \rightarrow c(W) \leftarrow \emptyset].$$

Hence the fiber is in bijection with \mathbb{N} by counting the elements of $c(W) = W \setminus \text{Im}(A \amalg B \rightarrow W)$.

Consider a cospan $[A \rightarrow W \leftarrow B] \in \mathbf{Csp}$; we will show that it is locally R -Cartesian precisely if it is reduced. By definition 1.0.3 the cospan W is locally R -Cartesian precisely if the top map in the following diagram is a bijection:

$$\begin{array}{ccc} R^{-1}([A = A = A]) & \xrightarrow{(_ \cup_A W)} & R^{-1}([A \rightarrow r(W) \leftarrow B]) \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{N} & \xrightarrow{(_ + |c(W)|)} & \mathbb{N} \end{array}$$

This diagram commutes, and hence $[A \rightarrow W \leftarrow B]$ is locally R -Cartesian precisely if $|c(W)| = 0$, i.e. if it is reduced.

The proof that $[A \rightarrow W \leftarrow B]$ is locally R -coCartesian precisely if it is reduced, is completely analogous. In fact, all variants of the cospans categories are isomorphic to their opposite via the functor that sends $(A \rightarrow W \leftarrow B)$ to $(B \rightarrow W \leftarrow A)$. Since R commutes with this involution, the claim that a morphism is locally R -coCartesian if and only if it is reduced follows formally.

To see that R is indeed locally Cartesian and locally coCartesian, we observe that any morphism in $\mathbb{Csp}^{\text{red}}$ has a reduced lift to \mathbb{Csp} .

Finally, we show that R is neither a Cartesian nor a coCartesian fibration. Say R was Cartesian, then by [Lur09a, Proposition 2.4.2.8] the composite of two locally R -Cartesian morphisms ought to be locally R -Cartesian. But as we saw in remark 3.1.11 the composite of two reduced cospans is not always reduced, leading to a contradiction. \square

Similarly, we have:

Lemma 3.1.13. *The reduction functor*

$$R \circ P : \mathbb{Csp} \rightarrow \mathbb{Csp}^{\text{red}}$$

is locally Cartesian and locally coCartesian.

Proof. Let Fin^{\simeq} denote the groupoid of finite sets and bijections. The fiber of

$$R \circ P : \mathbb{Csp}(A, B) \rightarrow \mathbb{Csp}^{\text{red}}(A, B)$$

at some cospan $[A \rightarrow W \leftarrow B]$ is equivalent to $BFin^{\simeq}$. The equivalence is the restriction of the functor

$$\mathbb{Csp}(A, B) \rightarrow Fin^{\simeq} \quad (A \rightarrow W \leftarrow B) \mapsto c(W) = W \setminus \text{Im}(A \amalg B \rightarrow W)$$

to the fiber. Just like in lemma 3.1.12 we see that for reduced cospans in $\mathbb{Csp}(A, B)$ the induced maps on the fiber over weak units is the identity on Fin^{\simeq} . Therefore reduced cospans satisfy the condition in point 2 of definition 2.4.6, which means that they are locally $(R \circ P)$ -Cartesian in the sense of weakly unital topological categories.

The rest of the proof follows like in lemma 3.1.12. \square

With the two previous lemmas at hand we can now apply theorem 2.4.8:

Proof of Theorem C. Let $*$ and \mathbb{N} denote the unital categories with one object and endomorphisms $\{\text{id}_*\}$ and \mathbb{N} , respectively. There is a pullback diagram of categories

$$\begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{C}\text{Sp} \\ \downarrow & & \downarrow R \\ * & \longrightarrow & \mathbb{C}\text{Sp}^{\text{red}} \end{array}$$

where the bottom functor sends $*$ to the object $\emptyset \in \mathbb{C}\text{Sp}^{\text{red}}$. Theorem 2.4.8 applies to this diagram: all categories involved are unital and discrete, hence weakly unital and fibrant, and lemma 3.1.12 verifies the only non-trivial condition. This shows that the following is a homotopy fiber sequence:

$$S^1 \simeq B\mathbb{N} \longrightarrow B\mathbb{C}\text{Sp} \xrightarrow{R} B\mathbb{C}\text{Sp}^{\text{red}}.$$

Similarly, we obtain the fiber sequence for $\mathbb{C}\text{Sp}$: let \mathcal{F} denote the full (topological) subcategory of $\mathbb{C}\text{Sp}$ on the object \emptyset . Then there is a pullback diagram of topologically enriched non-unital categories:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathbb{C}\text{Sp} \\ \downarrow & & \downarrow R \circ p \\ * & \longrightarrow & \mathbb{C}\text{Sp}^{\text{red}}. \end{array}$$

Theorem 2.4.8 applies to this diagram: lemma 3.1.13 checks that $R \circ p$ is locally Cartesian and locally coCartesian; moreover, $R \circ p$ is trivially a local fibration since $\mathbb{C}\text{Sp}^{\text{red}}$ and the space of objects of $\mathbb{C}\text{Sp}$ are discrete.

To obtain the desired homotopy fiber sequence recall from the proof of 3.1.13 that \mathcal{F} is the monoidal groupoid Fin^{\simeq} thought of as a bicategory with one object. Its classifying space is hence the classifying space of the topological monoid

$$B\text{Fin}^{\simeq} \simeq \prod_{k \geq 0} B\Sigma_k.$$

By the Barrat-Priddy-Quillen theorem we have that

$$\Omega B(B\text{Fin}^{\simeq}) \simeq Q(S^0)$$

and therefore $B\mathcal{F} \simeq B(B\text{Fin}^{\simeq}, \mathbb{I}) \simeq Q(S^1)$. Under this identification the map $B(B\text{Fin}^{\simeq}, \mathbb{I}) \rightarrow B\mathbb{N}$ is the infinite loop space map $QS^1 \rightarrow S^1$ which induces an isomorphism on π_1 . Its fiber is the universal covering of QS^1 and this proves the final claim in Theorem C. \square

REMARK 3.1.14. *We will see in part III of this thesis that the diagram in Theorem C is a diagram of infinite loop spaces. See proposition 8.2.4.*

3.1.3 Comparison to the cospan category of Raptis and Steimle

We now apply our techniques to the cobordism category $\text{Cob}(\mathcal{C})$ defined in [RS19]. In the special case where \mathcal{C} is the category of finite sets with injections as cofibrations their result says:

$$\Omega B\text{Cob}(\text{Fin}) \simeq K(\text{Fin}) \simeq Q(S^0).$$

In fact, this cobordism category of finite sets is equivalent to a certain subcategory of $\mathbb{C}\text{sp}$:

Definition 3.1.15. *We let $\mathbb{C}\text{sp}^{\text{inj}} \subset \mathbb{C}\text{sp}$ be the subcategory that contains all objects, but only those morphisms $(A \rightarrow W \leftarrow B)$ where the map $A \rightarrow W$ is injective. Let $\mathbb{C}\text{sp}^{\text{red, inj}} \subset \mathbb{C}\text{sp}^{\text{red}}$ denote the subcategory defined by the same condition.*

We can now provide a new proof of the main result of [RS19] in the example $\mathcal{C} = \text{Fin}$:

Proof of Corollary D. Consider the diagram of weakly unital topological categories:

$$\begin{array}{ccc} \mathbb{C}\text{sp}^{\text{inj}} & \xrightarrow{Q} & \mathbb{C}\text{sp}^{\text{red, inj}} \\ \downarrow I & & \downarrow I' \\ \mathbb{C}\text{sp} & \xrightarrow{R \circ P} & \mathbb{C}\text{sp}^{\text{red}}. \end{array}$$

It is a pullback diagram because a cospan $(A \rightarrow W \leftarrow B) \in \mathbb{C}\text{sp}$ lies in the subcategory $\mathbb{C}\text{sp}^{\text{inj}}$, precisely if the reduced cospan $(A \rightarrow r(W) \leftarrow B) \in \mathbb{C}\text{sp}^{\text{red}}$ lies in the subcategory $\mathbb{C}\text{sp}^{\text{red, inj}}$.

Using Theorem B we showed in 3.1.13 that $R \circ P$ is a realisation fibration. Therefore the above square becomes a homotopy pullback square after geometric realisation. The inclusion $\mathcal{F} \rightarrow \mathbb{C}\text{sp}$ of the full subcategory on \emptyset factors through the subcategory $\mathbb{C}\text{sp}^{\text{inj}}$ and we hence have the following commutative diagram:

$$\begin{array}{ccccc} B\mathcal{F} & \xrightarrow{F} & B\mathbb{C}\text{sp}^{\text{inj}} & \xrightarrow{Q} & B\mathbb{C}\text{sp}^{\text{red, inj}} \\ \parallel & & \downarrow I & & \downarrow I' \\ B\mathcal{F} & \longrightarrow & B\mathbb{C}\text{sp} & \xrightarrow{R \circ P} & B\mathbb{C}\text{sp}^{\text{red}} \end{array}$$

Both rows are fiber sequences and we have that $B\mathcal{F} \simeq Q(S^0)$ as in the proof of Theorem C.

It remains to check that $B\mathbb{C}\text{sp}^{\text{red, inj}}$ is contractible. First, note that $\mathbb{C}\text{sp}^{\text{red, inj}}$ is equivalent to an ordinary category: its hom-spaces have contractible components. In fact, it is equivalent to the category of finite sets and partially defined maps. For our purposes it suffices to note that the object $\emptyset = \underline{0}$ is terminal: any reduced and injective cospan $(A \rightarrow W \leftarrow \underline{0})$ is isomorphic to $(A = A \leftarrow \underline{0})$. This proves the final claim of the corollary as the classifying space of any category with a terminal object is contractible. \square

Part II

The one-dimensional bordism category

Chapter 4

Introduction

Bordism categories play an important role in organising geometry, mathematical physics, and algebra. Their most prominent use is probably as the main ingredient to Atiyah and Segal’s definition of topological and conformal field theory. More recently, cobordism categories have been a key tool in the study moduli spaces of manifolds, see for instance [GMTW09] and [GRW14]. This approach uses a more sophisticated version of the cobordism category: namely, a topological category \mathcal{C}_d where for a diffeomorphism class $[W]$ there is not just one, but “ $B\text{Diff}^\partial(W)$ -many” morphisms. In other words, one needs to consider the $(\infty, 1)$ -category of cobordisms $\text{Bord}_{(d-1,d)}^{\text{or}}$, for which \mathcal{C}_d is a specific model, rather than just its homotopy category $\text{Cob}_d^{\text{or}} \simeq h\mathcal{C}_d$. From the perspective of topological field theories \mathcal{C}_d has also proven to be the more natural object of study. For example, Lurie’s proof-sketch of the Baez-Dolan cobordism hypothesis [Lur09b], crucially relies on the presence of higher structure.

In each of the above settings the classifying space BC_d plays an important role. When studying moduli spaces, BC_d controls universal characteristic classes for manifolds ([MW07], [GRW17]), and when studying field theories BC_d classifies the invertible ones (e.g. [FH20]). Generally, the classifying space construction also allows one to build categorical models for interesting homotopy types. The homotopy type of BC_d was completely determined by Galatius, Madsen, Tillmann, and Weiss, who in [GMTW09] showed that it is the infinite loop space of a certain Thom-spectrum $MTSO_d$.

This raises the question for a similar computation of $B(h\mathcal{C}_d)$. One might expect this to be a simpler problem since the homotopy category $h\mathcal{C}_d$ is only an ordinary, rather than a topological category. However, very little is known about $B(h\mathcal{C}_d)$. Except for the simple observation $B(h\mathcal{C}_0) \simeq S^1 \times S^1$, the only result in this direction is Tillmann’s splitting [Til96] of the surface case: $B(h\mathcal{C}_2) \simeq S^1 \times X$ for some mysterious simply connected infinite loop space X .

Our main theorem is a complete computation of $B(h\mathcal{C}_1)$, which is already far more complex than $B(\mathcal{C}_1) \simeq QS^0$. We also prove a similar result for the unoriented case (Theorem 5.4.13).

Theorem E. *There is an equivalence of infinite loop spaces*

$$B(h\mathcal{C}_1) \simeq \Omega^{\infty-2}(\text{hofib}(MTSO_2 \rightarrow H\mathbb{Z}))$$

where $MTSO_2 \rightarrow H\mathbb{Z}$ is the map of spectra classifying the generator of $H^0(MTSO_2) \cong \mathbb{Z}$. Consequently, the rational cohomology of the identity component $B(h\mathcal{C}_1)_0 \subset B(h\mathcal{C}_1)$ is given by

$$H^*(B(h\mathcal{C}_1)_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \dots]$$

where the generators are of degree $|\bar{\kappa}_i| = 2i + 2$ and we have $\pi_0 B(h\mathcal{C}_1) = \mathbb{Z}$.

The reduction fiber sequence

In order to prove Theorem E we construct a fiber sequence, which we believe to be of independent interest. Just like Genauer's sequence [Gen12] for bordisms with horizontal boundary, this fiber sequence relates the classifying spaces of three bordism categories, which we now describe.

Every d -dimensional cobordism $W : M \rightarrow N$ decomposes canonically as a disjoint union $W = c(W) \amalg r(W)$ where $c(W)$ is a closed manifold and $r(W)$ is *reduced*, i.e. it has no closed components. We define two new topologically enriched bordism categories: the closed bordism category is the full subcategory $\mathcal{C}_d^{\text{cl}} \subset \mathcal{C}_d$ on the object \emptyset , and the reduced bordism category is the quotient category $\mathcal{C}_d \rightarrow \mathcal{C}_d^{\text{red}}$ where two bordisms W and V are identified if $r(W) = r(V)$. Every morphism in $\mathcal{C}_d^{\text{red}}$ can be uniquely represented by a reduced bordism. We refer the reader to definition 5.2.2 for a complete definition, and to Figure 4.1 for an illustration of the case $d = 1$.

Theorem F. *For any $d \geq 0$ there are two compatible homotopy fiber sequences of infinite loop spaces:*

$$\begin{array}{ccccc} BC_d^{\text{cl}} & \longrightarrow & BC_d & \xrightarrow{R} & BC_d^{\text{red}} \\ \downarrow & & \downarrow & & \downarrow \\ B(h\mathcal{C}_d^{\text{cl}}) & \longrightarrow & B(h\mathcal{C}_d) & \xrightarrow{R} & B(h\mathcal{C}_d^{\text{red}}). \end{array}$$

Here R is induced by the functor $\mathcal{C}_d \rightarrow \mathcal{C}_d^{\text{red}}$ that deletes closed components.

In theorem 5.3.1 we also prove Theorem F in the presence of tangential structures. Most of the technical work towards a proof of Theorem F has been completed in our previous paper [Ste19], where we generalise Steimle's *additivity theorem for bordism categories* [Ste18a] to a broader class of functors.

The crucial observation needed to now deduce Theorem E from Theorem F is that in dimension $d = 1$ we have that $\mathcal{C}_1^{\text{red}} \simeq h\mathcal{C}_1^{\text{red}}$ because the diffeomorphism group of an interval is contractible.

Topological cyclic homology of simply connected spaces

One of the more mysterious aspects of the cobordism category is its relation to Waldhausen K -theory. Recall that the Waldhausen K -theory $A(X)$ of a space X is the infinite loop space of the algebraic K -theory of the ring spectrum $\mathbb{S}[\Omega X] := \Sigma^\infty(\Omega X)_+$. An effective way of computing algebraic K -theory is via the cyclotomic trace to topological cyclic homology (TC). In the relevant case we have a map $\text{trc}_p : A(X) \rightarrow \Omega^\infty TC(\mathbb{S}[\Omega X]; p)$ for any prime p .

For $X = *$ there is, up to p completion, an equivalence $TC(\mathbb{S}; p) \simeq \mathbb{S} \vee \Sigma \mathbb{C}\mathbb{P}_{-1}^\infty$ by [BHM93]. Here $\mathbb{C}\mathbb{P}_{-1}^\infty$ is just another name for the spectrum $MTSO_2$, whose infinite loop space is ΩBC_2 . In light of this apparent coincidence Ib Madsen concludes his 2006 ICM plenary address [Mad07, 407] with:

Finally, and maybe most important, there are reasons to believe that the moduli space of Riemann surfaces is related to $TC(\mathbb{S}; p)$, possibly via field theories. The spectrum $\mathbb{C}\mathbb{P}_{-1}^\infty$ occurs in both theories. It is a challenge to understand why.

This curious coincidence has been known to experts for a while and, even though no concrete explanations have been suggested, the idea itself has inspired some intriguing research such as Bökstedt and Madsen's map from ΩBC_d to A -theory, see [BM14] and [RS14].

In order to understand Madsen's question, it makes sense to think of $TC(\mathbb{S}; p)$ and BC_2 as the values of the functors $TC(\mathbb{S}[\Omega X]; p)$ and $BC_2(X)$ for $X = *$. Here $\mathcal{C}_2(X)$ is a variant of the surface category where every object and bordism is equipped with a map to X . If there is a fundamental relation between surfaces and TC , one could hope for a relation between these functors. However, the analogy fails in this case: the main theorem of [GMTW09] implies $BC_2(X) \simeq \Omega^\infty(MTSO_2 \wedge X_+)$ and hence the functor $BC_2(X)$ is excisive in X , whereas $TC(\mathbb{S}[\Omega X])$ is not.

We will try to argue that instead, topological cyclic homology is more naturally related to the reduced 1-dimensional bordism category. As a consequence of Theorem F we have that BC_1^{red} is equivalent to $\Omega^{\infty-2}MTSO_2$, a delooping of BC_2 . For a space X we define $\mathcal{C}_1^{\text{red}}(X)$ to be the category of reduced bordisms with maps to X . A generalisation of Theorem F together with the main theorem of [BCC⁺96] implies:

Theorem G. *For any simply connected space X of finite type there is an equivalence of infinite loop spaces*

$$\Omega^\infty TC(\mathbb{S}[\Omega X]; p) \simeq \left(Q(X_+) \times \Omega BC_1^{\text{red}}(X) \right)_p^\wedge.$$

One way of interpreting this theorem is to say that $\mathcal{C}_1^{\text{red}}(X)$ is a *bordism model* for the topological cyclic homology of simply connected spaces. This is similar in spirit to Raptis and Steimle's cobordism model for Waldhausen K -theory. In [RS19] they construct, for every space X , $\mathcal{C}^{\text{sym}}(X)$

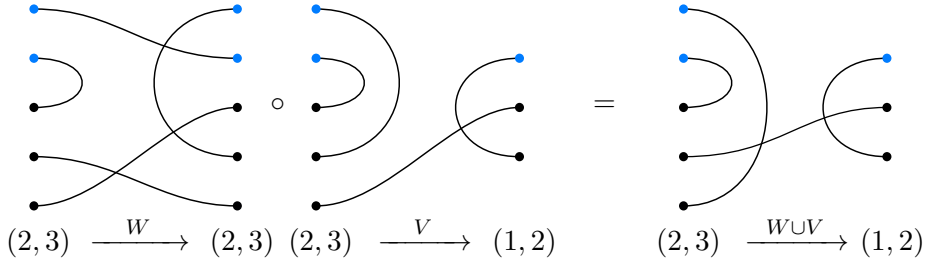


Figure 4.1: Two morphisms in $h\mathcal{C}_1^{\text{red}}$ and their composite.

such that $\Omega BC^{\text{sym}}(X) \simeq A(X)$. One could therefore try to understand the cyclotomic trace in terms of these models. Already for $X = *$ this is related to the difficult question of whether the cyclotomic traces $\text{trc}_p : A(*) \rightarrow (\Omega^{\infty-1} \mathbb{C}\mathbb{P}_{-1}^{\infty})_p^{\wedge}$ admit a common integral lift for all p .

Reduced bordisms as a combinatorial model for $\mathbb{C}\mathbb{P}_{-1}^{\infty} = MTSO_2$

As the previous section illustrates $\mathbb{C}\mathbb{P}_{-1}^{\infty}$ is an interesting stable homotopy type arising in many important situations, and it is helpful to have different models at hand. Theorem F implies that $\Omega^{\infty-2} \mathbb{C}\mathbb{P}_{-1}^{\infty}$ is, as an infinite loop space, equivalent to the classifying space of the symmetric monoidal category $h\mathcal{C}_1^{\text{red}} \simeq \mathcal{C}_1^{\text{red}}$. By the Madsen-Weiss theorem \mathcal{C}_2 also is a cobordism model for this infinite loop space, though there is a dimension shift:

$$\Omega^2 B(h\mathcal{C}_1^{\text{red}}) \simeq \Omega^{\infty} MTSO_2 \simeq \Omega B\mathcal{C}_2.$$

Even though $h\mathcal{C}_1^{\text{red}}$ and \mathcal{C}_2 both are cobordism models for $MTSO_2$, they are of very different flavours. The surface category \mathcal{C}_2 is of geometric nature as it is built from the diffeomorphism groups of surfaces. The reduced one-dimensional bordism category $h\mathcal{C}_1^{\text{red}}$, on the other hand, admits a completely combinatorial description. See figure 4.1 for an example of how morphisms are composed in $h\mathcal{C}_1^{\text{red}}$.

While Theorem F implies an equivalence $B(h\mathcal{C}_1^{\text{red}}) \simeq \Omega^{\infty-2} MTSO_2$, it does not explicitly construct a map. Chapter 6 aims to resolve this issue.

For this we construct an auxiliary simplicial space Cut_d , which we believe may be of independent interest. Cut_d is a quotient of the nerve $N(\mathcal{C}_d^{\text{red}})$ and we show that the geometric realisation $\|\text{Cut}_d\|$ is the free infinite loop space on $\Sigma^2 B\text{Diff}^+(W)_+$ for all closed connected d -manifolds W . We prove:

Theorem H. *The quotient map $N(\mathcal{C}_d^{\text{red}}) \rightarrow \text{Cut}_d$ continues the reduction fiber sequence of Theorem F in the sense that*

$$B(\mathcal{C}_d) \longrightarrow B(\mathcal{C}_d^{\text{red}}) \longrightarrow \|\text{Cut}_d\|$$

is a homotopy fiber sequence of infinite loop space maps.

In dimension 1 we have $\|\text{Cut}_1\| \simeq Q(\Sigma^2(\mathbb{C}\mathbb{P}^\infty)_+)$ and the quotient map $BC_1^{\text{red}} \rightarrow \|\text{Cut}_1\|$ is a rational equivalence on the basepoint component, see corollary 6.2.5. In the presence of a background space X this map has an interpretation in terms of TC , see corollary 6.2.6.

Application: cocycles for Miller-Morita-Mumford classes

The Miller-Morita-Mumford classes are characteristic classes for surface bundles, which, under the equivalence $BC_2 \simeq \Omega^{\infty-1}MTSO_2$, give rise to polynomial generators:

$$\kappa_i \in H^*(\Omega_0^\infty MTSO_2; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots].$$

From the equivalence $B(C_1^{\text{red}}) \simeq \Omega^{\infty-2}MTSO_2$ we can compute the rational cohomology of C_1^{red} as:

$$H^*(B(C_1^{\text{red}})_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_0, \bar{\kappa}_1, \dots].$$

Here $\bar{\kappa}_i$ is a double de-looping of κ_i and hence of degree $2i+2$. Also note $\pi_0 B(C_1^{\text{red}}) \cong \pi_0 B(C_1) \cong \mathbb{Z}$.

Generalising the notion of group cocycles, one can represent every cohomology class on the classifying space BD of some discrete category \mathcal{D} by a cocycle $\alpha : N_i \mathcal{D} \rightarrow \mathbb{Z}$ on the nerve. While this does not work for the topologically enriched category \mathcal{C}_2 , hC_1^{red} is discrete and it is hence possible to describe the deloopings $\bar{\kappa}_i$ of the κ -classes in terms of cocycles on hC_1^{red} .

To actually compute cocycles we crucially use that every $\bar{\kappa}_i$ can be obtained as a pullback along the map $BC_1^{\text{red}} \rightarrow \|\text{Cut}_1\|$ of Theorem H. We identify $\|\text{Cut}_1\|$ with $Q(\Sigma^2(B\Lambda)_+)$, where Λ is Connes' category of cyclic sets. Then we use Igusa's description [Igu04] of cocycles on Λ representing the powers of the first Chern class $(c_1)^i \in H^{2i}(B\Lambda; \mathbb{Q})$. Explicitly, we have:

Definition. For each $i \geq 0$ we construct a $(2i+2)$ -cocycle γ_i on hC_1^{red} and hC_1 in three steps.

(1) For a $(2i+1)$ -tuple of disjoint points $a_0, \dots, a_{2i} \in S^1$ the sign $\text{sign}(a_0, \dots, a_{2i}) \in \{\pm 1\}$ is defined to be equal to the sign of any permutation σ of $\{0, \dots, 2i\}$ such that the sequence $(a_{\sigma(0)}, \dots, a_{\sigma(2i)})$ is in cyclic order.

(2) For a $(2i+1)$ -tuple of disjoint finite subsets $A_0, \dots, A_{2i} \subset S^1$ the reduced sign $\overline{\text{sign}}_{2i}(A_0, \dots, A_{2i}) \in \mathbb{Q}$ is defined by an averaging procedure. Say that $(a_0, \dots, a_{2i}) \in \prod_i A_i$ contains no neighbours if for all $k < l$ there is a j such that the positively oriented arc $[a_k, a_l] \subset S^1$ from a_k to a_l intersects A_j in more than one point. Now define:

$$\overline{\text{sign}}_{2i}(A_0, \dots, A_{2i}) := \frac{1}{\prod_{j=0}^{2i} |A_j|} \sum_{\substack{(a_0, \dots, a_{2i}) \in \prod_j A_j \\ \text{contains no neighbours}}} \text{sign}(a_0, \dots, a_{2i}) \in \mathbb{Q}.$$

(3) For all $i \geq 1$ define a $(2i + 2)$ -cochain on the simplicial set $N(h\mathcal{C}_1^{\text{red}})$ by the formula

$$\gamma_i(M_0 \xrightarrow{W_1} M_1 \xrightarrow{W_2} \dots \xrightarrow{W_{2i+2}} M_{2i+2}) := \frac{(-1)^{i_i!}}{(2i)!} \sum_{[\iota: S^1 \hookrightarrow W]} \overline{\text{sign}}_{2i}(\iota^{-1}(M_1^+), \dots, \iota^{-1}(M_{2i+1}^+)).$$

Here we write W for the composition $W_1 \cup_{M_1} \dots \cup_{M_{2i+1}} W_{2i+2}$ and $M_i^+ \subset M_i$ for the set of positively oriented points. The sum $\sum_{[\iota: S^1 \hookrightarrow W]}$ runs over isotopy classes of oriented embeddings $\iota: S^1 \hookrightarrow W$ such that $\iota(S^1)$ intersects M_j for all $1 \leq j \leq 2i + 1$.

Theorem I. For $i \geq 1$ the cochain γ_i on $h\mathcal{C}_1^{\text{red}}$ defined above is a $(2i + 2)$ cocycle and the cohomology class $[-\gamma_i]$ corresponds to, possibly up to a sign $(-1)^i$, the generator $\bar{\kappa}_i$ under the isomorphism

$$H^*(B(h\mathcal{C}_1^{\text{red}})_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_0, \bar{\kappa}_1, \bar{\kappa}_2, \dots].$$

The same formula describes a cocycle for $\pm \bar{\kappa}_i$ on $h\mathcal{C}_1$.

Outline

This part is divided into two chapters, chapter 5 is centered around establishing the reduction fiber sequence. In section 5.1 we recall various standard facts and constructions for bordism categories and topological categories in general. We also prove a generalisation of the base-change theorem [ERW19a, Theorem 5.2], which allows one to change the space of objects of a topological category without changing the homotopy type of its classifying space. Section 5.2 introduces the closed and reduced bordisms categories and studies some of their basic properties. In section 5.3 we apply the main theorem of [Ste19] to prove the reduction fiber sequence Theorem F. Specialising to $d = 1$, in section 5.4 we observe that $\mathcal{C}_1^{\text{red}} \simeq h\mathcal{C}_1^{\text{red}}$ and use this to deduce Theorem E and Theorem G. Here we also consider the unoriented case $B(h\mathcal{C}_{1, \text{unor}})$.

Chapter 6 is focused on gaining a more concrete understanding of $\mathcal{C}_1^{\text{red}}$. We begin in section 6.1 where we, by elementary means, compute the 2-cocycle on $h\mathcal{C}_d^{\text{red}}$ that classifies the central extension $h\mathcal{C}_d \rightarrow h\mathcal{C}_d^{\text{red}}$. Section 6.2 introduces the simplicial space of cuts Cut_d and proves Theorem H, identifying the connecting homomorphism of the reduction fiber sequence. This is then used in section 6.3 to construct the cocycles for $\bar{\kappa}$ -classes on $h\mathcal{C}_1^{\text{red}}$.

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Chapter 5

The classifying space of the one-dimensional bordism category

5.1 Recollections on moduli spaces and cobordism categories

While our main object of interest is the homotopy category of the bordism category $h\mathcal{C}_1$, the proof of Theorem E crucially relies on comparing it to the topological category \mathcal{C}_1 and some of its variants. It is hence essential for us to have a good understanding of the embedded models for the cobordism category. In this section we recall various definitions and facts.

Although we introduce all the necessary tools, we can only do so concisely. The interested reader is referred to [GR20] for a discussion of moduli spaces for manifolds, and to [ERW19a] for an introduction to the world of non-unital topological categories and semisimplicial spaces.

REMARK 5.1.1. *The specific model for the cobordism category \mathcal{C}_d that we are using is essentially that of [GMTW09]. This is a concrete model for the $(\infty, 1)$ -category of bordisms, which is often denoted $\text{Bord}_{(d-1, d)}^{\text{or}}$. From it we will also derive the homotopy category $h\mathcal{C}_d$. This is the ordinary 1-category of bordisms, which is often denoted by Cob_d^{or} . (E.g. [Lur09b]).*

5.1.1 Tangential structures

Most of the manifolds we consider will be oriented. Recall that one way of defining an orientation on a manifold M is by giving an equivariant continuous map $l : \text{Fr}(TM) \rightarrow \{-1, +1\}$ from the total space of the frame bundle of M to the two-element set. Here equivariance is with respect to the group action of GL_d on the left by base-change and on the right by multiplication with the sign of the determinant. Tangential structures generalise this notion of orientation.

Definition 5.1.2. *A d -dimensional tangential structure θ is a space θ with GL_d -action. Given such a θ a θ -structure on a d -dimensional manifold W is a GL_d -equivariant map $l : \text{Fr}(TW) \rightarrow \theta$ from the frame bundle of W to θ . The space of θ -structures on M will be denoted by*

$$\text{Bun}^\theta(M) := \text{Map}_{\text{GL}_d}(\text{Fr}(TW), \theta).$$

In the case of the tangential structure for orientation $\theta^{\text{or}} := \{\pm 1\}$, we are often interested in the group of *orientation preserving* diffeomorphisms $\text{Diff}^+(W)$. For more general tangential structures θ the group of diffeomorphism that fix a specific θ -structure ‘on the nose’ is generally not well-behaved. Instead, we define a moduli space that acts as the classifying space of this hypothetical group.

Definition 5.1.3. For θ and W as above we define $B\text{Diff}^\theta(W)$ as the homotopy orbit space

$$B\text{Diff}^\theta(W) := \text{Bun}^\theta(W) // \text{Diff}(W) \stackrel{\text{def}}{=} (\text{Bun}^\theta(W) \times E\text{Diff}(W)) / \text{Diff}(W).$$

5.1.2 Spaces of manifolds and cobordisms

We recall the space of submanifolds of \mathbb{R}^N and how to use it to define a topological space of (embedded) cobordisms.

Definition 5.1.4. For $U \subset \mathbb{R}^N$ open let $\Psi_{d,\theta}(U)$ denote the set of pairs (M, l) where $M \subset U$ is a d -dimensional submanifold of \mathbb{R}^N that is closed as a subset of \mathbb{R}^N and $l : \text{Fr}(TM) \rightarrow \theta$ is a θ -structure on M . We let $\Psi_{d,\theta}$ denote the colimit of $\Psi_{d,\theta}((-1, 1)^N)$ as $N \rightarrow \infty$. For finite N we topologise this according to [GRW10, Definition 2.1] and for $N = \infty$ as the colimit over all finite N .

One can think of $\Psi_{d,\theta}$ as a concrete topological model for the ‘moduli space of θ -structured closed d -dimensional manifolds’. In [GR20, Section 2.2] this space is denoted \mathcal{M}^θ . The moduli space decomposes as a disjoint union over diffeomorphism types:

Fact 5.1.5. *There is a weak equivalence*

$$\Psi_{d,\theta} \simeq \coprod_{[W]} B\text{Diff}^\theta(W)$$

where W runs over a set of representatives of diffeomorphism classes of closed d -dimensional manifolds.

Definition 5.1.6. We say that $(W, l) \in \Psi_{d,\theta}(\mathbb{R} \times U)$ is cylindrical over some interval (a, b) if $W|_{(a,b)} := W \cap ((a, b) \times U)$ is equal to the product $(a, b) \times M$ for some $M \in \Psi_d(U)$ and the θ -structure $l|_{(a,b)} : \text{Fr}(TW|_{(a,b)}) \rightarrow \theta$ can be factored as

$$\text{Fr}(TW|_{(a,b)}) \cong \text{Fr}(T(a, b) \times TM) \xrightarrow{\text{pr}} \text{Fr}(\mathbb{R} \oplus TM) \cong (\text{GL}_d \times \text{Fr}(TM)) / \text{GL}_{d-1} \rightarrow \theta$$

where the last map is induced by some GL_{d-1} -equivariant map $l' : \text{Fr}(TM) \rightarrow \theta$.

Definition 5.1.7. For d and θ as above and $\varepsilon > 0$ we define $\Phi_{d,\theta}^\varepsilon \subset \Psi_{d,\theta}(\mathbb{R} \times (-1, 1)^\infty) \times \mathbb{R}_{>0}$ as the space of those $((W, l), t)$ that are cylindrical over $(-\infty, \varepsilon)$ and $(t - \varepsilon, \infty)$. We let the space of θ -structured d -dimensional cobordisms $\Phi_{d,\theta}$ be the colimit as $\varepsilon \rightarrow 0$.

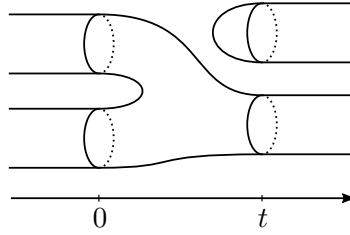


Figure 5.1: A point in the space of two-dimensional cobordisms $\Phi_{2,\theta}$.

We think of $t > 0$ as the length of the cobordism (W, l) . Indeed, the cylindricality condition implies that the part of the bordisms that does not lie over $[0, t]$ is superfluous. We still keep track of the cylinders over $(-\infty, 0)$ and (t, ∞) because it simplifies the definition of the topology. For example, we can use them to define the map that sends a cobordism to its boundary:

Definition 5.1.8. *We define a map*

$$(\partial_0, \partial_1) : \Phi_{d,\theta} \rightarrow \Psi_{d-1,\theta} \times \Psi_{d-1,\theta}$$

by sending $((W, l), t)$ to the unique tuple $((M_0, l_0), (M_1, l_1))$ such that $(-\infty, 0) \times M_0 \cup (t, \infty) \times M_1$ is a codimension 0 submanifold of W and l_0 and l_1 are the induced θ -structures.

5.1.3 The cobordism category

All cobordism categories we consider will be weakly unital¹ topological categories. We refer the reader to [ERW19a] for an excellent introduction to semisimplicial spaces, non-unital topological categories, and fibrancy conditions. By convention all our categories \mathcal{D} will be non-unital and $B\mathcal{D}$ will denote the “fat geometric realisation” of the semisimplicial space $N\mathcal{D}$, even if \mathcal{D} happens to have units.

Definition 5.1.9. *A non-unital topological category \mathcal{D} consists of the following data: a space of objects O , a space of morphisms M , source and target maps $s, t : M \rightarrow O$, and a composition map*

$$c : M^t \times_O^s M = \{(f, g) \in M^2 \mid t(f) = s(g)\} \longrightarrow M.$$

This data is subject to the axioms

$$s(c(f, g)) = s(f) \quad t(c(f, g)) = t(g) \quad c(c(f, g), h) = c(f, c(g, h))$$

for any three morphisms $f, g, h \in M$ with $t(f) = s(g)$ and $t(g) = s(h)$.

¹We will be using the definition of weakly unital given in [Ste19] as it is the one relevant for the local additivity theorem 5.3.3. Since the cobordism category is also fibrant this will imply that it is weakly left and right unital in the sense of [ERW19a]. (See [ERW19a, Remark 3.12].)

Definition 5.1.10. For any two objects $x, y \in \mathcal{O}$ the space of morphism from x to y is defined as the pullback

$$\mathrm{hom}_{\mathcal{D}}(x, y) := \{x\} \times_{\mathcal{O}}^s M \times_{\mathcal{O}}^t \{y\} = \{f \in M \mid s(f) = x \text{ and } t(f) = y\}.$$

We usually write $f : x \rightarrow y$ to say $f \in \mathrm{hom}_{\mathcal{D}}(x, y)$, and we also write $g \circ f$ for the composite $c(f, g)$.

For us the main example of a topological category is the cobordism category, which we now define using the space of cobordisms $\Psi_{d,\theta}$ constructed in the previous section as a space of morphisms.

Definition 5.1.11. Fix a dimension d and a tangential structure θ . The non-unital topological category $\mathcal{C}_{d,\theta}$ has $\Psi_{d-1,\theta}$ as space of objects and $\Phi_{d,\theta}$ as space of morphisms. The source and target maps are the maps $\partial_0, \partial_1 : \Phi_{d,\theta} \rightarrow \Psi_{d-1,\theta}$ from definition 5.1.8. Composition is defined by

$$((W', l'), t') \circ ((W, l), t) := ((W'', l''), t + t')$$

where

$$W'' = (W \cap (-\infty, t] \times (-1, 1)^\infty) \cup ((W' + t \cdot e_0) \cap ([t, \infty) \times (-1, 1)^\infty))$$

and l'' is induced by l and l' . Here $(_ + t \cdot e_0)$ denotes translation by t in the direction of $e_0 = (1, \underline{0}) \in \mathbb{R} \times (-1, 1)^\infty$.

REMARK 5.1.12. Note that $\mathcal{C}_{d,\theta}$ is well-behaved: it is fibrant in the sense of [ERW19a, Definition 3.5] by [ERW19b, Proposition 3.2.4(ii)]. Moreover, the cylinder bordisms $(M \times \mathbb{R}, 1) : M \rightarrow M$ define weak units in the sense of [Ste19].

In the case of the bordism category there is a geometric interpretation of the homotopy type of this hom space.

Fact 5.1.13. For any two objects (M, l) and (N, l') in $\mathcal{C}_{d,\theta}$ there is a weak equivalence

$$\mathrm{hom}_{\mathcal{C}_{d,\theta}}((M, l), (N, l')) \simeq \coprod_{[W, \varphi]} \mathrm{BDiff}^\theta(W \text{ rel } M \amalg N).$$

Here the coproduct runs over a set of representatives for compact d -manifolds W with boundary $M \amalg N$, under the equivalence relation defined by diffeomorphisms that fix the boundary. The space $\mathrm{BDiff}^\theta(W \text{ rel } M \amalg N)$ is defined as in definition 5.1.3 with the modification that diffeomorphisms are trivial near the boundary that the tangential structure agrees with l and l' near the boundary.

5.1.4 Homotopy categories

The homotopy category $h\mathcal{D}$ of a topological category \mathcal{D} is an ordinary category obtained from \mathcal{D} in two steps: first we need to pass to an enriched category $\delta\mathcal{D}$ and then we apply π_0 to the hom-spaces.

Definition 5.1.14. *For any non-unital topological category \mathcal{D} we define its discretification $\delta(\mathcal{D})$ as the non-unital topological category with object space $\text{Obj}(\delta(\mathcal{D}))$ the set $\text{Obj}(\mathcal{D})$, equipped with the discrete topology, and morphism space the disjoint union*

$$\text{Mor}(\delta(\mathcal{D})) := \coprod_{x,y \in \mathcal{D}} \text{hom}_{\mathcal{D}}(x, y).$$

The homotopy category $h(\mathcal{D})$ is the ordinary category with object set $\text{Obj}(\delta(\mathcal{D}))$ and hom-sets

$$\text{hom}_{h(\mathcal{D})}(x, y) := \pi_0 \text{hom}_{\mathcal{D}}(x, y).$$

There are canonical functors $\mathcal{D} \xleftarrow{\iota} \delta(\mathcal{D}) \xrightarrow{\pi} h(\mathcal{D})$ and they are natural in \mathcal{D} .

REMARK 5.1.15. *Recall that a (non-unital) topologically enriched category is the same datum as a (non-unital) topological category $\mathcal{D} = (O, M, s, t, c)$ where the space of objects O is discrete.*

In this sense $\delta(\mathcal{D})$ is always a topologically enriched category. The canonical functor $\iota : \delta(\mathcal{D}) \rightarrow \mathcal{D}$ is a continuous bijection on the object the morphism space, but it is not a homeomorphism unless $\text{Obj}(\mathcal{D})$ was already discrete. From the perspective of classifying spaces, however, $\delta(\mathcal{D})$ is equivalent to \mathcal{D} as long as \mathcal{D} is sufficiently well-behaved.

Lemma 5.1.16. *If \mathcal{D} is a fibrant non-unital topological category with weak left (or right) units, then ι induces a weak equivalence $B(\delta(\mathcal{D})) \rightarrow B(\mathcal{D})$.*

Proof. This is the basechange theorem [ERW19a, Theorem 5.2] in the case where X is the set $\text{Obj}(\mathcal{D})$ equipped with the discrete topology. We will give an independent proof of (a generalisation of) the basechange theorem in lemma 5.1.30. \square

We cannot generally expect the homotopy category $h\mathcal{D}$ to have a classifying space equivalent to that of \mathcal{D} . Nevertheless, the canonical map is always 2-connected:

Lemma 5.1.17. *The map $B(\delta\mathcal{D}) \rightarrow B(h\mathcal{D})$ is 2-connected for any non-unital topological category \mathcal{D} .*

Proof. This is well-known and for example follows from [ERW19a, Lemma 2.4] together with the observation that $\text{Mor}(\delta(\mathcal{D})) \rightarrow \pi_0 \text{Mor}(\delta(\mathcal{D})) = \text{Mor}(h(\mathcal{D}))$ is 1-connected. \square

Of course, our main category of interest is the cobordism category. The homotopy category of the bordism category also admits a more conceptual description that does not rely on the embeddings. For simplicity, we only spell this out in the oriented case.

Fact 5.1.18. *The homotopy category $h(\mathcal{C}_{d,\theta^{\text{or}}})$ is equivalent to the following category:*

- *objects are closed oriented $(d - 1)$ -dimensional manifolds,*
- *morphisms $W : M \rightarrow N$ are equivalence classes of compact oriented d -dimensional manifolds with $\partial W = M^- \amalg N$, where two such manifolds are equivalent if there is an orientation preserving diffeomorphism between them that fixes the boundary, and*
- *composition is defined by gluing cobordisms.*

5.1.5 Infinite loop space structures

All homotopy categories of cobordism categories have symmetric monoidal structures defined by the disjoint union operation for manifolds. This induces an infinite loop space structure on their classifying spaces. Many of our computations will rely on these infinite loops space structures and their compatibility with certain constructions. We will be keeping track of the infinite loop space structures using Segal's Γ -spaces [Seg74], which we recall here.

Definition 5.1.19. *Segal's category Γ^{op} has as objects natural numbers $n \geq 0$ and as morphisms from n to m maps of sets $\lambda : \{*, 1, \dots, n\} \rightarrow \{*, 1, \dots, m\}$ satisfying $\lambda(*) = *$. We let $\rho_n^i : n \rightarrow 1$ denote the morphism with $\rho_n^i(j) = *$ for $i \neq j$ and $\rho_n^i(i) = 1$.*

A Γ -space is a functor $X : \Gamma^{op} \rightarrow \text{Top}$. We call X special if the Segal map

$$X(n) \xrightarrow{(\rho_n^1, \dots, \rho_n^n)} (X(1))^n$$

is a weak equivalence for all $n \geq 0$. A special Γ -space is very special if the map

$$\pi_0 X(2) \xrightarrow{(\rho_2^1, \mu)} \pi_0 X(1) \times \pi_0 X(1)$$

is a bijection. Here $\mu : 2 \rightarrow 1$ is the morphism with $\mu(1) = \mu(2) = 1$.

REMARK 5.1.20. *For any special Γ -space X the set $\pi_0 X(1)$ has an abelian monoid structure with multiplication defined by*

$$m : \pi_0 X(1) \times \pi_0 X(1) \xleftarrow{\cong} \pi_0 X(2) \xrightarrow{X(\mu)} \pi_0 X(1).$$

This uses that the Segal map $X(2) \rightarrow X(1) \times X(1)$ is a bijection on π_0 . The special Γ -space is very special if and only if this abelian monoid happens to be an abelian group.

Following [Ngu17] we use the tangential structure to define a Γ structure on the cobordism category.

Definition 5.1.21. For any dimension d and tangential structure θ we define a Γ -object in non-unital categories by setting $\mathcal{C}_{d,\theta}(n) := \mathcal{C}_{d,n\theta}$ where $n\theta$ denotes $\{1, \dots, n\} \times \theta$. To a morphism $\lambda : n \rightarrow m$ in Γ^{op} we associate the functor

$$\lambda_* : \mathcal{C}_{d,n\theta} \rightarrow \mathcal{C}_{d,m\theta} \quad \text{with} \quad ((W, l), t) \mapsto ((W', (\lambda \times \text{id}_\theta) \circ l), t)$$

where $W' \subset W$ is the preimage of $\{1, \dots, m\}$ under the map $(\lambda \circ \text{pr}_{\{1, \dots, n\}} \circ l) : W \rightarrow \{*, 1, \dots, m\}$.

Lemma 5.1.22 ([Ngu17]). The Γ -space $BC_{d,\theta}$ is very special.

We also have infinite loop space structures on the classifying space of the homotopy categories.

Lemma 5.1.23. The Γ -space $Bh\mathcal{C}_{d,\theta}$ is very special.

Proof. Since $h\mathcal{C}_{d,\theta}$ is a symmetric monoidal category, it follows from Segal's original paper [Seg74, §2] that $Bh\mathcal{C}_{d,\theta}$ is special. Being very special is a condition on the monoid $\pi_0 Bh\mathcal{C}_{d,\theta}$ and follows by the considerations in lemma 5.1.22, seeing as $\pi_0 Bh\mathcal{C}_{d,\theta} \cong \pi_0 BC_{d,\theta}$. \square

5.1.6 The Galatius-Madsen-Tillmann-Weiss theorem

In [GMTW09] the authors determined the infinite loop space $B(\mathcal{C}_{d,\theta})$ in terms of modified Thom spectra. As our strategy is to compare $B(h\mathcal{C}_1)$ with $B(\mathcal{C}_1)$, we briefly recall their result.

Definition 5.1.24. Let $E(d, n)$ denote the space of linear embeddings of \mathbb{R}^d into \mathbb{R}^n , with the GL_d -action by precomposition, and let $Gr(d, n) := E(d, n)/\text{GL}_d$ be the Grassmannian. The canonical bundle γ_d on $Gr(d, n)$ is defined by

$$\gamma_{d,n} := (E(d, n) \times \mathbb{R}^d)/\text{GL}_d \longrightarrow Gr(d, n).$$

This is a subbundle of the trivial bundle $Gr(d, n) \times \mathbb{R}^n$ via the map $(e, v) \mapsto ([e], e(v))$. We let $\gamma_{d,n}^\perp$ denote the orthogonal complement of $\gamma_{d,n}$ in $Gr(d, n) \times \mathbb{R}^n$.

Definition 5.1.25. For any d and θ the θ -structured Madsen-Tillmann spectrum $MT\theta$ is the sequential spectrum with

$$(MT\theta)_n := \text{Th}(a^* \gamma_{d,n}^\perp)$$

for $a : (\theta \times E(d, n))/\text{GL}_d \rightarrow Gr(d, n)$ the projection. The structure maps $\Sigma(MT\theta)_n \rightarrow (MT\theta)_{n+1}$ are defined using the canonical bundle map $\gamma_{d,n}^\perp \oplus \underline{\mathbb{R}} \rightarrow \gamma_{d,n+1}^\perp$.

The theorem by Galatius, Madsen, Tillmann, and Weiss uses Segal's 'scanning' method to construct an equivalence from $BC_{d,\theta}$ to a loop space of $MT\theta$. For our purposes it will be sufficient to treat this as a black box and we refer the interested reader to the original paper.

Theorem 5.1.26 (Main theorem of [GMTW09]). *For any d and θ the scanning map defines a weak equivalence of infinite loop spaces*

$$BC_{d,\theta} \simeq \Omega^{\infty-1} MT\theta = \operatorname{colim}_{n \rightarrow \infty} \Omega^{n-1} (MT\theta)_n.$$

This theorem has been used to great effect in the study of diffeomorphism groups. The essential ingredient is a natural map from $B\operatorname{Diff}^\theta(W)$ to $\Omega BC_{d,\theta}$ for any closed d -manifold W . This map is defined by interpreting W as a morphism $W : \emptyset \rightarrow \emptyset$ in \mathcal{C}_d , which then yields a based loop in the classifying space.

Definition 5.1.27. *The inclusion $\Psi_{d,\theta} \hookrightarrow \Phi_{d,\theta} = N_1\mathcal{C}_{d,\theta}$ induces a map $\Psi_{d,\theta} \times |\Delta^1| \rightarrow BC_{d,\theta}$ such that $\Psi_{d,\theta} \times |\partial\Delta^1|$ is sent to the base point. Hence, it defines a map $\alpha' : \Sigma(\Psi_{d,\theta})_+ \rightarrow BC_{d,\theta}$ and by adjunction a map*

$$\alpha : \Psi_{d,\theta} \rightarrow \Omega BC_{d,\theta}.$$

Together with the equivalence from fact 5.1.5 this yields, for every closed d -manifold W , a map

$$\alpha_W : B\operatorname{Diff}^\theta(W) \rightarrow \Omega BC_{d,\theta}.$$

We conclude this section with the so-called ‘Genauer fiber sequence’. Recall that $Q(X)$ denotes the free infinite loop space on a based space X defined as the colimit $Q(X) := \operatorname{colim}_n \Omega^n(\Sigma^n X)$. It has the property that any map of spaces $X \rightarrow Y$ into an infinite loop space Y induces a map of infinite loop spaces $Q(X) \rightarrow Y$.

Lemma 5.1.28 ([Gal06]). *There is a homotopy fiber sequence of infinite loop spaces*

$$\Omega^{\infty-1} MTSO_d \rightarrow Q(\Sigma(BSO_d)_+) \rightarrow \Omega^{\infty-1} MTSO_{d-1}.$$

Moreover, the right-hand map can be described as the composite:

$$Q(\Sigma(BSO_d)_+) \xrightarrow{Q(\Sigma Bi)} Q(\Sigma B\operatorname{Diff}^+(S^{d-1})) \xrightarrow{\alpha_{S^{d-1}}} BC_{d-1}^{\text{or}} \xrightarrow{\simeq} \Omega^{\infty-1} MTSO_{d-1}$$

where $i : SO_d \rightarrow \operatorname{Diff}^+(S^{d-1})$ is the canonical inclusion and $\alpha_{S^{d-1}}$ is the map from definition 5.1.27.

References. This homotopy fiber sequence is well-known through the paper [GMTW09], but already appeared in Galatius’ paper [Gal06]. We refer the reader to [Ebe13, Proposition A.4] where the fiber sequence is explained in notation very similar to ours. The second part of the lemma, the identification of the infinite loop space map $Q(\Sigma(BSO_d)_+) \rightarrow \Omega^{\infty-1} MTSO_{d-1}$, is harder to extract from the literature. It will suffice to understand the map of spaces $BSO_d \rightarrow \Omega^\infty MTSO_{d-1}$. In [Ebe13, Proposition A.4] Ebert describes it as the Madsen-Tillmann-Weiss map for the sphere

bundle $BSO_{d-1} \rightarrow BSO_d$, and indeed Galatius in [Gal06, Lemma 2.1] constructs the relevant map as a parametrised Pontrjagin-Thom construction. This map factors as:

$$BSO_d \rightarrow B\text{Diff}^+(S^{d-1}) \rightarrow \Omega^\infty MTSO_{d-1}$$

where the left-hand map is the classifying map for the sphere bundle $BSO_{d-1} \rightarrow BSO_d$ and the right-hand map is the Pontrjagin-Thom construction for the universal oriented S^{d-1} -bundle. The sphere bundle is classified by $B(i) : BSO_d \rightarrow B\text{Diff}^+(S^{d-1})$ and so the claim follows. \square

5.1.7 A generalised change of base theorem

In this section we will prove a small generalisation of [ERW19a, Theorem 5.2] that will come in handy later. First, recall the notion of base-change from [ERW19a, section 5].

Definition 5.1.29. *Let \mathcal{D} be a topological category and let X be a space with a map $f : X \rightarrow \text{Obj}(\mathcal{D})$. The base-change of \mathcal{D} along f is the topological category, denoted by \mathcal{D}^f , with object space X and morphism space $X^2 \times_{\text{Obj}(\mathcal{D})^2} \text{Mor}(\mathcal{D})$. Composition is induced by \mathcal{D} .*

Lemma 5.1.30. *Let \mathcal{D} be a weakly unital topological category and consider a zig-zag of maps $g : Y \rightarrow Z \leftarrow \text{Obj}(\mathcal{D}) : p$. Let $f : Y \times_Z \text{Obj}(\mathcal{D}) \rightarrow \text{Obj}(\mathcal{D})$ be the projection. Then the canonical map*

$$B(\mathcal{D}^f) \longrightarrow B\mathcal{D}$$

is a weak equivalence, if the following conditions hold:

- *the map $g : Y \rightarrow Z$ is surjective on connected components,*
- *for all $n \geq 0$ the composite $p : N_n \mathcal{D} \rightarrow \text{Obj}(\mathcal{D})^{n+1} \xrightarrow{p^{n+1}} Z^{n+1}$ is a fibration.*

Note that in the case of $Z = \text{Obj}(\mathcal{D})$ and $p = \text{id}_{\text{Obj}(\mathcal{D})}$ the second condition is equivalent to asking that \mathcal{D} be fibrant; hence we recover [ERW19a, Theorem 5.2] as a special case.

Proof. For any space X we let $T(X)$ denote the trivial groupoid on X . Its space of object is X and its space of morphisms is $X \times X$. In other words, for every pair of objects (x, y) there is precisely one morphism $x \rightarrow y$. Composition is defined by $(y, z) \circ (x, y) = (x, z)$.

This is a fibrant topological category. For X non-empty we can pick a point y and define an extra degeneracy on the simplicial space $N(X)$ by $s_{-1}(x_0, \dots, x_n) = (y, x_0, \dots, x_n)$. This shows that $B(T(X))$ is contractible for X non-empty.

Note that the basechange \mathcal{D}^f of \mathcal{D} along a map $f : X \rightarrow \text{Obj}(\mathcal{D})$ can be written as a pullback $\mathcal{D}^f \cong T(X) \times_{T(\text{Obj}(\mathcal{D}))} \mathcal{D}$ where we use the canonical functor $\mathcal{D} \rightarrow T(\text{Obj}(\mathcal{D}))$, which is the identity on objects.

To prove the lemma, we first observe that we can replace the map $g : Y \rightarrow Z$ by a fibration. To do this, factor g as a composite $Y \xrightarrow{i} Y_0 \xrightarrow{g'} Z$ where i is a level-wise weak equivalence and g' is a level-wise fibration. Then the simplicial map

$$i : N(\mathcal{D}^f) \cong N(T(Y)) \times_{N(T(Z))} N(\mathcal{D}) \longrightarrow N(T(Y_0)) \times_{N(T(Z))} N(\mathcal{D})$$

is the pullback of a level-wise weak equivalence along a level-wise fibration, and hence a level-wise weak equivalence. In particular, it induces a weak equivalence on geometric realisations. It now suffices to prove the lemma for g' .

Assume from now on that g was already a fibration. This implies that the pullback $f : Y \times_Z \text{Obj}(\mathcal{D}) \rightarrow \text{Obj}(\mathcal{D})$ is a fibration, too. Since g is also surjective on connected components, this implies that g and f are surjective. We now claim that the additivity theorem for bordism categories 5.3.3 (see [Ste18a] and [Ste19]) applies to the pullback square:

$$\begin{array}{ccc} \mathcal{D}^f & \longrightarrow & T(Y \times_Z \text{Obj}(\mathcal{D})) \\ \downarrow & & \downarrow T(f) \\ \mathcal{D} & \longrightarrow & T(\text{Obj}(\mathcal{D})). \end{array}$$

Since f is a fibration, so is $T(f)$. The trivial groupoids $T(X)$ are always fibrant and unital, and the functors in the square are always unital. The only non-trivial condition to check is that $T(f)$ is indeed Cartesian and coCartesian. But this is easy to see since $T(f)$ is surjective on objects and there is only one morphism between every two objects in $T(X)$.

We may hence apply theorem 5.3.3 and obtain a homotopy pullback square

$$\begin{array}{ccc} B\mathcal{D}^f & \longrightarrow & B(T(Y \times_Z \text{Obj}(\mathcal{D}))) \\ \downarrow & \lrcorner^h & \downarrow B(T(f)) \\ B\mathcal{D} & \longrightarrow & B(T(\text{Obj}(\mathcal{D}))). \end{array}$$

As discussed in the beginning of the proof $B(T(X))$ is contractible for all X and so the right vertical map is an equivalence. Because the square is a homotopy pullback this implies that $B\mathcal{D}^f \rightarrow B\mathcal{D}$ is an equivalence. \square

5.2 Closed and reduced bordism categories

We now introduce the key idea of this paper: the decomposition of a cobordism into its closed and its reduced part.

Definition 5.2.1. *A bordism $W : M \rightarrow N$ is called closed if both M and N are empty and it is called reduced if the inclusion $M \amalg N \rightarrow W$ is surjective on connected components. We define subspaces*

$$\Phi_{d,\theta}^{\text{cl}} \subset \Phi_{d,\theta} \quad \text{and} \quad \Phi_{d,\theta}^{\text{red}} \subset \Phi_{d,\theta}$$

of closed and reduced bordisms. Define retractions $c : \Phi_{d,\theta} \rightarrow \Phi_{d,\theta}^{\text{cl}}$ and $r : \Phi_{d,\theta} \rightarrow \Phi_{d,\theta}^{\text{red}}$ by deleting those connected components of W that violate the respective condition. This is illustrated in figure 5.2.

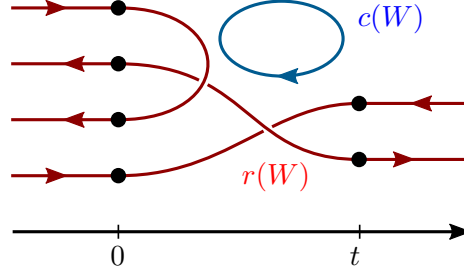


Figure 5.2: An oriented 1-dimensional cobordism decomposed into its closed and reduced components.

Every bordism $W \in \Phi_{d,\theta}$ has a canonical decomposition as $W = c(W) \amalg r(W)$. A homotopically meaningful version of this statement is proved in lemma 5.2.5. Our goal is to obtain a similar decomposition on the level of bordism categories. The first step is to define categories of closed and of reduced bordisms.

Definition 5.2.2. The closed bordism category $\mathcal{C}_{d,\theta}^{\text{cl}}$ is the full subcategory of $\mathcal{C}_{d,\theta}$ on the single object $\emptyset \in \mathcal{C}_{d,\theta}$. The reduced bordism category $\mathcal{C}_{d,\theta}^{\text{red}}$ has the same objects as $\mathcal{C}_{d,\theta}$, but has as morphisms only reduced bordisms. Composition is defined by composing in $\mathcal{C}_{d,\theta}$ and then applying the retraction $r : \Phi_{d,\theta} \rightarrow \Phi_{d,\theta}^{\text{red}}$.

Definition 5.2.3. The inclusion $\Phi_{d,\theta}^{\text{cl}} \hookrightarrow \Phi_{d,\theta}$ defines a functor denoted by

$$I : \mathcal{C}_{d,\theta}^{\text{cl}} \longrightarrow \mathcal{C}_{d,\theta}$$

and the retraction $r : \Phi_{d,\theta} \rightarrow \Phi_{d,\theta}^{\text{red}}$ defines a functor denoted by

$$R : \mathcal{C}_{d,\theta} \longrightarrow \mathcal{C}_{d,\theta}^{\text{red}}.$$

REMARK 5.2.4. The functors I and R allow us to think of $\mathcal{C}_{d,\theta}^{\text{cl}}$ as a subcategory and of $\mathcal{C}_{d,\theta}^{\text{red}}$ as a quotient category of $\mathcal{C}_{d,\theta}$. It is important to note that the opposite is not the case: the inclusion $\Phi_{d,\theta}^{\text{red}} \hookrightarrow \Phi_{d,\theta}$ does not induce a functor $\mathcal{C}_{d,\theta}^{\text{red}} \rightarrow \mathcal{C}_{d,\theta}$. The reason for this is that the composite of two reduced bordisms $W : M \rightarrow N$ and $V : N \rightarrow L$ is not necessarily reduced. We could for example set $M = L = \emptyset$, $N = S^{d-1}$, and let W and V both be the d -dimensional disc; once as a bordism $\emptyset \rightarrow S^{d-1}$ and once as a bordism $S^{d-1} \rightarrow \emptyset$. Then both W and V are reduced, but $W \cup_N V = D^d \cup_{S^{d-1}} D^d = S^d : \emptyset \rightarrow \emptyset$ is not. See figure 5.3 for another example showing that the composite of two reduced bordisms need not be reduced.

Locally the bordism category $\mathcal{C}_{d,\theta}$ decomposes as a product of $\mathcal{C}_{d,\theta}^{\text{cl}}$ and $\mathcal{C}_{d,\theta}^{\text{red}}$:

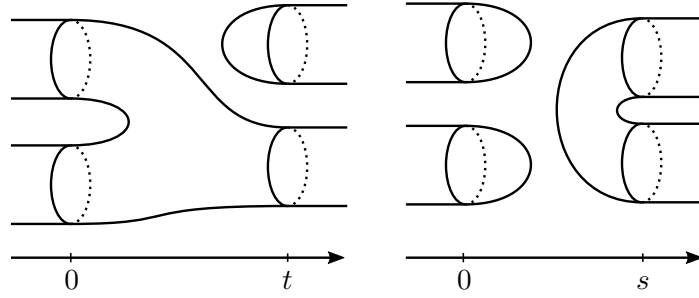


Figure 5.3: Two reduced bordisms whose composite is not reduced.

Lemma 5.2.5. *For any two objects $M, N \in \mathcal{C}_{d,\theta}$ the map*

$$(c, r) : \text{hom}_{\mathcal{C}_{d,\theta}}(M, N) \longrightarrow \text{hom}_{\mathcal{C}_{d,\theta}^{\text{cl}}}(\emptyset, \emptyset) \times \text{hom}_{\mathcal{C}_{d,\theta}^{\text{red}}}(M, N)$$

is a weak equivalence.

Proof. We will show that the map that decomposes a bordism into its closed and its reduced part

$$(c, r) : \Phi_{d,\theta} \longrightarrow \Phi_{d,\theta}^{\text{cl}} \times \Phi_{d,\theta}^{\text{red}} \quad (5.1)$$

is a weak equivalence.

Once this is established the lemma follows by observing that both sides of (5.1) are fibered over $\Phi_{d-1,\theta} \times \Phi_{d-1,\theta}$ via the boundary map $\partial = (\partial_0, \partial_1)$, which is a Serre fibration because $\mathcal{C}_{d,\theta}$ is fibrant, see [ERW19b, Proposition 3.2.4(ii)]. This implies the claim because a weak equivalence between Serre fibrations induces weak equivalences on the fibers.

We begin by observing that (c, r) is an open embedding. As a map of sets (c, r) is an injection with image those tuples $(W, V) \in \Phi_{d,\theta}^{\text{cl}} \times \Phi_{d,\theta}^{\text{red}}$ such that W and V are disjoint. That it is open and continuous follows by a close examination of the topology defined in [GRW10, Definition 2.1]. We leave the details to the reader.

To show that (c, r) is a weak equivalence we construct two subspaces $A, B \subset \Phi_{d,\theta}$ defined by:

- $(W, l) \in A$ if for all $x = (x_0, x_1, \dots) \in W$ we have $x_1 = 0$.
- $(W, l) \in B$ if for all $x \in c(W)$ we have $x_1 = \frac{1}{2}$ and for all $x \in r(W)$ we have $x_1 = -\frac{1}{2}$.

There is a map $s : A \rightarrow B$ that takes (W, l) and moves up $c(W)$ by $\frac{1}{2}$ and moves down $r(W)$ by $\frac{1}{2}$ in the first coordinate. This map and the inclusions fit into a homotopy commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & B \\ \downarrow i & \nearrow j & \downarrow k \\ \Phi_{d,\theta} & \xrightarrow{(c,r)} & \Phi_{d,\theta}^{\text{cl}} \times \Phi_{d,\theta}^{\text{red}} \end{array}$$

The map $k : B \rightarrow \Phi_{d,\theta}^{\text{cl}} \times \Phi_{d,\theta}^{\text{red}}$ is defined by restricting (c, r) , so the bottom triangle commutes by definition. To see that the top triangle commutes up to homotopy consider the map $s_\lambda : A \rightarrow \Phi_{d,\theta}$ defined for $\lambda \in [0, \frac{1}{2}]$ by moving up $c(W)$ by λ and moving down $r(W)$ by λ . This defines a homotopy between i and $j \circ s$.

Next, we observe that i and k are homotopy equivalences. In both cases this follows because the inclusion $\mathbb{R} \times \{0\} \times (-1, 1)^{\infty-1} \rightarrow \mathbb{R} \times (-1, 1)^\infty$ admits a deformation retraction via smooth embeddings preserving the x_0 coordinate. Given a homotopy commutative diagram of the shape as above where i and k are weak equivalences, it follows from the 2-out-of-6 property that all other maps are also weak equivalences. Concretely, we can apply the functor π_* and observe that because $\pi_*(i)$ and $\pi_*(k)$ are isomorphisms, $\pi_*((c, r))$ has to be an isomorphism, too. \square

Lemma 5.2.6. *The spaces $B(\mathcal{C}_{d,\theta}^{\text{cl}})$, $B(\mathcal{C}_{d,\theta}^{\text{red}})$, $B(h\mathcal{C}_{d,\theta}^{\text{cl}})$, and $B(h\mathcal{C}_{d,\theta}^{\text{red}})$ are very special Γ -spaces.*

Proof. This is proved just like lemma 5.1.22 and 5.1.23. To check that they are *very special*, note that $\pi_0 B(\mathcal{C}_{d,\theta}^{\text{cl}}) = \pi_0 B(h\mathcal{C}_{d,\theta}^{\text{cl}}) = *$ is trivial and that $\pi_0 B(\mathcal{C}_{d,\theta}^{\text{red}}) = \pi_0 B(h\mathcal{C}_{d,\theta}^{\text{red}}) = \pi_0 B(\mathcal{C}_{d,\theta})$ is the θ -structured bordism group. \square

We conclude this section by a lemma indicating that the functor R is homotopically well-behaved.

Lemma 5.2.7. *The retraction $r : \Phi_{d,\theta} \rightarrow \Phi_{d,\theta}^{\text{red}}$ is a Serre fibration and hence the functor $R : \mathcal{C}_{d,\theta} \rightarrow \mathcal{C}_{d,\theta}^{\text{red}}$ induces a level-wise fibration on the nerves.*

Proof. To simplify notation we will ignore the tangential structures in this proof, though they can easily be added.

For every map $W : [0, 1]^k \rightarrow \Phi_d^{\text{red}}$ and any lift $V : [0, 1]^{k-1} \rightarrow \Phi_d$ with $r(V(s_1, \dots, s_{k-1})) = W(s_1, \dots, s_{k-1}, 0)$ we need to find an extension $V' : [0, 1]^k \rightarrow \Phi_d$ such that $r(V'(s_1, \dots, s_k)) = W(s_1, \dots, s_k)$.

Recall that we defined Φ_d as a colimit over submanifolds of \mathbb{R}^N for $N \rightarrow \infty$. Since $[0, 1]^k$ is compact, the image of W and V is contained in a finite stage of the colimit. We can therefore find N such that $W(s_1, \dots, s_{k-1}), V(s_1, \dots, s_k) \subset \mathbb{R}^{N-1}$ for all s_i . Let now $T_\lambda : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ be the diffeomorphism defined by applying $x_N \mapsto \lambda + x_N$ to the N th coordinate. We use it to define

$$V'(s_1, \dots, s_k) := W(s_1, \dots, s_k) \cup T_{s_k/2}(c(V(s_1, \dots, s_{k-1}))).$$

The union of $W(\dots)$ and $T_{s_k/2}(\dots)$ is in fact a disjoint union: the first part has N th coordinate 0 and the second part has N th coordinate $s_k/2$. For $s_k = 0$ they are disjoint by assumption. Since the union is disjoint $V(s_1, \dots, s_k)$ is indeed a manifold and a well-defined point in Φ .

It remains to check that V is actually continuous. As remarked in the proof of lemma 5.2.5 the topology on Φ is equal to the subspace topology induced by the injection $(c, r) : \Phi \hookrightarrow \Phi^{\text{cl}} \times \Phi^{\text{red}}$.

It hence suffices to check that both $c(V)$ and $r(V)$ depend continuously on (s_1, \dots, s_k) , but this is obvious by construction. \square

5.3 The reduction fiber sequence

Our main technical theorem is what we call the “reduction fiber sequence”. It establishes a decomposition of the classifying space of the bordism category into its closed and reduced components.

Theorem 5.3.1. *For any dimension d and tangential structure θ the rows in the commutative diagram of infinite loop spaces*

$$\begin{array}{ccccc} BC_{d,\theta}^{\text{cl}} & \longrightarrow & BC_{d,\theta} & \xrightarrow{R} & BC_{d,\theta}^{\text{red}} \\ \downarrow & & \downarrow & & \downarrow \\ B(hC_{d,\theta}^{\text{cl}}) & \longrightarrow & B(hC_{d,\theta}) & \xrightarrow{R} & B(hC_{d,\theta}^{\text{red}}) \end{array}$$

are homotopy fiber sequences.

This is similar to lemma 5.2.5 where we observed a splitting of the morphism spaces. Note, however, that because c is not functorial there is no splitting as a product on the level of categories, but only a homotopy fiber sequence.

Warning 5.3.2. *The vertical maps in the theorem are in fact zig-zags $BC_{d,\theta} \leftarrow B(\delta C_{d,\theta}) \rightarrow B(hC_{d,\theta})$ as described in definition 5.1.14. We will usually omit the left-ward arrow as it depends naturally on C and is a weak equivalence by lemma 5.1.16.*

The essential technical ingredient for the proof of theorem 5.3.1 is the Steimle’s “additivity theorem for bordism categories” from [Ste18a]. As we will see in remark 5.3.7 the version proved by Steimle is not sufficient for establishing the reduction fiber sequence and we will instead need the following generalised “local” version, which we proved in [Ste19] to study the category of cospans.

Theorem 5.3.3 (Local additivity theorem for bordism categories, [Ste19, Theorem 5.8]). *Let $P : \mathcal{E} \rightarrow \mathcal{B}$ be a weakly unital functor of weakly unital topological categories such that \mathcal{B} is fibrant, P is a local fibration, P is locally Cartesian, and P is locally coCartesian.*

Then, for every weakly unital functor $F : \mathcal{C} \rightarrow \mathcal{B}$, the following is a homotopy pullback square:

$$\begin{array}{ccc} B(\mathcal{C} \times_{\mathcal{B}} \mathcal{E}) & \longrightarrow & B\mathcal{E} \\ \downarrow & & \downarrow P \\ B\mathcal{C} & \xrightarrow{F} & B\mathcal{B}. \end{array}$$

REMARK 5.3.4. *All non-unital categories in this paper have weak units in the form of cylinders and all functors considered will preserve these. We will recall the other technical terms as needed.*

The main step in verifying the conditions of the additivity theorem will be to check that the functor $R : \mathcal{C}_{d,\theta} \rightarrow \mathcal{C}_{d,\theta}^{\text{red}}$ is indeed locally (co)Cartesian. This is a consequence of lemma 5.2.5.

Lemma 5.3.5. *A morphism $W : M \rightarrow N$ in $\mathcal{C}_{d,\theta}$ is locally R -Cartesian in the sense of [Ste19, Definition 5.6], if and only if it is reduced.*

Proof. According to [Ste19, Definition 5.6] W is locally R -Cartesian if and only if for any (and by [Ste19, Lemma 5.9] equivalently all) equivalences $V : M' \rightarrow M$ in $\mathcal{C}_{d,\theta}^{\text{red}}$ the following diagram

$$\begin{array}{ccc} \text{hom}_{\mathcal{C}_{d,\theta}}(M', M) & \xrightarrow{(_ \cup_M W)} & \text{hom}_{\mathcal{C}_{d,\theta}}(M, N) \\ \downarrow R & & \downarrow R \\ \text{hom}_{\mathcal{C}_{d,\theta}^{\text{red}}}(M', M) & \xrightarrow{r(_ \cup_M r(W))} & \text{hom}_{\mathcal{C}_{d,\theta}^{\text{red}}}(M, N) \end{array}$$

induces a weak equivalence between the homotopy fiber of R at V and the homotopy fiber of R at $r(V \cup_M W) = r(V \cup_M r(W))$. The choice of the equivalence V does not matter, but to be concrete we can take $M' = M$ and $V : M \rightarrow M$ to be the cylinder of length one. Using lemma 5.2.5 we can identify the homotopy fibers with $\text{hom}_{\mathcal{C}_{d,\theta}^{\text{cl}}}(\emptyset, \emptyset)$. Note that in light of lemma 5.2.7 these homotopy fibers are equivalent to the genuine fibers. The induced map between the homotopy fibers is exactly composition with $c(V \cup_M W)$ in $\mathcal{C}_{d,\theta}^{\text{cl}}$:

$$(_ \cup_{\emptyset} c(V \cup_M W)) : \text{hom}_{\mathcal{C}_{d,\theta}^{\text{cl}}}(\emptyset, \emptyset) \longrightarrow \text{hom}_{\mathcal{C}_{d,\theta}^{\text{cl}}}(\emptyset, \emptyset).$$

This map is homotopic to the identity if $c(V \cup_M W) = (\emptyset, t)$ is the empty cobordism of some length $t > 0$, and since V is a cylinder this is the case if and only if W is reduced. Therefore W is R -Cartesian if it is reduced.

Conversely, if W is not reduced, then $c(V \cup_M W)$ is non-empty and the map $(_ \cup_{\emptyset} c(V \cup_M W))$ cannot be an equivalence as it does not hit the connected component of $(\emptyset, t) \in \text{hom}_{\mathcal{C}_{d,\theta}^{\text{cl}}}(\emptyset, \emptyset)$. \square

Lemma 5.3.6. *A morphism $[W] : M \rightarrow N$ in $h\mathcal{C}_{d,\theta}$ is locally (hR) -Cartesian in the sense of [Ste19, Definition 5.6], if and only if it is reduced.*

Proof. This is proved just like lemma 5.3.5, but using the bijection

$$\text{hom}_{h\mathcal{C}_{d,\theta}}(M, N) \cong \text{hom}_{h\mathcal{C}_{d,\theta}^{\text{cl}}}(\emptyset, \emptyset) \times \text{hom}_{h\mathcal{C}_{d,\theta}^{\text{red}}}(M, N).$$

It is important to note that, while the proofs are completely analogous, the statement of this lemma is not a formal consequence of the statement of lemma 5.3.5. See warning 5.3.8. \square

REMARK 5.3.7. *As we will see in the proof of theorem 5.3.1, this lemma implies that $R : \mathcal{C}_{d,\theta} \rightarrow \mathcal{C}_{d,\theta}^{\text{red}}$ is locally Cartesian and, by reversing bordisms, also locally coCartesian. Note, however, that usually R is neither Cartesian nor coCartesian. A morphism $W : M \rightarrow N$ in $\mathcal{C}_{d,\theta}$ is R -Cartesian if the square*

in lemma 5.3.5 is a homotopy pullback for all M' . In particular, the vertical homotopy fibers at V and $r(V \cup_M r(W))$ have to agree for all reduced $V : M' \rightarrow M$ and not just the equivalences. It follows from the proof of lemma 5.3.5 that for this to be the case $c(V \cup_M W)$ has to be empty. Therefore, if $W : M \rightarrow N$ and $V : M' \rightarrow M$ are reduced morphisms such that $V \cup_M W$ is not reduced, then there cannot be an R -Cartesian morphism $\widetilde{W} : M \rightarrow N$ with $r(\widetilde{W}) = W$.

Note that reduced bordisms W, V with $V \cup_M W$ not reduced exist in all bordism categories with $d > 0$ and $\theta \neq \emptyset$.² This shows that Steimle's original version of the additivity theorem for bordism categories [Ste18a] would not be sufficient for our purposes.

Proof of Theorem 5.3.1. Let \mathcal{R}_+ denote the topological semigroup $(\mathbb{R}_{>0}, +)$ thought of as a non-unital topological category with one object. For all variants of the bordism category there is a canonical functor $\mathcal{C} \rightarrow \mathcal{R}_+$ that records the length of a bordism. Picking any object $M \in \mathcal{C}$ we also have a functor $\text{Cyl}_M : \mathcal{R}_+ \rightarrow \mathcal{C}$ that sends t to $(t, \mathbb{R} \times M)$, the cylinder of length t on M . Using these functors we construct a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_{d,\theta}^{\text{cl}} & \longrightarrow & \mathcal{C}_{d,\theta} \\ \downarrow & & \downarrow R \\ \mathcal{R}_+ & \xrightarrow{\text{Cyl}_\theta} & \mathcal{C}_{d,\theta}^{\text{red}}. \end{array}$$

This is trivially a pullback diagram on the spaces of objects. By the proof of lemma 5.2.5 it also is a pullback diagram on the spaces of morphisms, and hence it is a pullback diagram of categories internal to topological spaces.

We now wish to apply the local additivity theorem for bordism categories of [Ste19] to this, which we recalled as theorem 5.3.3. The fibrancy condition on $\mathcal{C}_{d,\theta}^{\text{red}}$ follows from the fibrancy for $\mathcal{C}_{d,\theta}$, see [ERW19b, Proposition 3.2.4(ii)] and the fibrancy condition on R follows from lemma 5.2.7. The functor R is locally Cartesian since any morphism $W : M \rightarrow N$ in $\mathcal{C}_{d,\theta}^{\text{red}}$ has a lift to $\mathcal{C}_{d,\theta}$ given by W itself. This lift is reduced and hence by lemma 5.3.5 locally Cartesian. Turning around all bordisms involved we see by the same argument that R is locally coCartesian. Finally, we note that the categories involved are weakly unital with the weak units given by the cylinders $(t, \mathbb{R} \times M)$ described above. The functors in the diagram preserve the cylinders and are hence weakly unital. This checks all conditions of theorem 5.3.3.

²It suffices to construct a counterexample for $\theta = GL_d$ and all $d > 0$, then the others can be produced using functoriality of $\mathcal{C}_{d,\theta}$ in the GL_d -space θ . So, consider the bordism category of framed 1-manifolds \mathcal{C}_{1, GL_1} . Here a counterexample is given by $V = D^1 : \emptyset \rightarrow \{+, -\}$ and $W = D^1 : \{+, -\} \rightarrow \emptyset$ with the obvious framing. The counterexample in \mathcal{C}_{d, GL_d} is now obtained by taking the product with $(S^1)^{d-1}$ with the Lie-group framing.

As a result of the additivity theorem we see that

$$\begin{array}{ccc} BC_{d,\theta}^{\text{cl}} & \longrightarrow & BC_{d,\theta} \\ \downarrow & & \downarrow B(R) \\ B\mathcal{R}_+ & \longrightarrow & BC_{d,\theta}^{\text{red}} \end{array}$$

is a homotopy pullback diagram. The non-unital topological category \mathcal{R}_+ is weakly equivalent to the terminal unital category and hence $B\mathcal{R}_+$ is weakly contractible. This shows that

$$BC_{d,\theta}^{\text{cl}} \longrightarrow BC_{d,\theta} \longrightarrow BC_{d,\theta}^{\text{red}}$$

is a homotopy fiber sequence.

It remains to discuss the homotopy fiber sequence for the homotopy categories $h\mathcal{C}_{d,\theta}$. Basically all arguments can be copied from the topological case, (using lemma 5.3.6 instead of lemma 5.3.5) with the additional simplification that all morphism spaces are discrete and hence all fibrancy conditions are trivially satisfied. In particular this shows that the second sequence in the theorem is a homotopy fiber sequence.

The vertical maps in the theorem are induced by the canonical zig-zag of functors $\mathcal{D} \leftarrow \delta\mathcal{D} \rightarrow h\mathcal{D}$ that we have for any topological category and it is clear that the diagram commutes. We know from lemma 5.1.22, 5.1.23, and 5.2.6 that all six spaces are infinite loop spaces and that the maps respect this structure. Hence the diagram is a diagram of infinite loop spaces and in particular the two fiber sequences are fiber sequences of infinite loop spaces. \square

Warning 5.3.8. *There are topologically enriched functors $F : \mathcal{E} \rightarrow \mathcal{B}$ satisfying the conditions of the (local) additivity theorem for bordism categories such that $hF : h\mathcal{E} \rightarrow h\mathcal{B}$ does not satisfy the conditions of the theorem. The reason for this is that the property of being of (locally) (co)Cartesian is not preserved under taking homotopy categories. This for example fails when considering Genauer's sequence: Steimle's proof of the Genauer fiber sequence [Ste18a, Theorem 1.1] does not imply that there is an analogous homotopy fiber sequence*

$$B(h\mathcal{C}_d) \longrightarrow B(h\mathcal{C}_d^\partial) \xrightarrow{\partial} B(h\mathcal{C}_{d-1})$$

for the classifying spaces of the homotopy categories. In fact, the author believes that this cannot be a homotopy fiber sequence for any $d \geq 1$.

5.4 Computations for $d = 1$

5.4.1 The classifying space of the closed bordism category

The closed bordism category is a topological monoid and we think of it as the coherently commutative monoid freely generated by the connected closed manifolds. In lemma 5.4.1 we describe its

classifying space as the free infinite loop space generated by $B\text{Diff}^\theta(W)$ for all *connected* closed diffeomorphism types W . In dimension one, the only connected closed manifold is the circle and we can describe $BC_{1,\theta}^{\text{cl}}$ more explicitly in terms of the homotopy orbits of the space of free loops, see corollary 5.4.4.

Lemma 5.4.1. *The maps $\alpha_W : \Sigma B\text{Diff}^\theta(W) \rightarrow BC_{d,\theta}^{\text{cl}}$ from definition 5.1.27 induce a weak equivalence of infinite loop spaces:*

$$Q \left(\Sigma \left(\coprod_{[W] \text{ connected closed}} B\text{Diff}^\theta(W) \right)_+ \right) \xrightarrow{\simeq} BC_{d,\theta}^{\text{cl}}.$$

Proof. Let us first consider the case $d = 0$ with $\theta = X$ some space. Then a morphism in $\mathcal{C}_{0,X}^{\text{cl}}$ is a finite subset of $(0, t) \times (-1, 1)^\infty$, equipped with a map to X . In other words, the space of morphisms of $\mathcal{C}_{0,X}^{\text{cl}}$ is the X -labelled configuration space $\text{Conf}((-1, 1)^\infty; X)$ with the usual composition defined by putting configurations side by side. Segal's improved Barrat-Priddy-Quillen theorem [Seg74, Proposition 3.6] states that the group-completion of the special (but not very special) Γ -space $\text{Conf}((-1, 1)^\infty; X)$ is

$$\Omega B(\text{Conf}((-1, 1)^\infty; X)) \simeq Q(X_+).$$

The inverse map comes from the map $X \rightarrow \text{Conf}((-1, 1)^\infty; X)$ that sends x to the point 0 labelled by x . Since $\mathcal{C}_{0,X}^{\text{cl}}$ is this monoid, thought of as a category with one object, we conclude that $BC_{0,X}^{\text{cl}}$ is the delooping $\Omega^{-1}Q(X_+) \simeq Q(\Sigma X_+)$. Seeing as $X \cong \text{Bun}^\theta(*) \cong B\text{Diff}^\theta(*)$ we therefore have an equivalence $Q(\Sigma X_+) \simeq B\text{Conf}((-1, 1)^\infty; X) \simeq BC_{0,X}^{\text{cl}}$. Restricted to $\Sigma X \subset Q(\Sigma X_+)$ this equivalence agrees with the map $\alpha_{W=*} : \Sigma B\text{Diff}^\theta(*) \rightarrow BC_{0,X}^{\text{cl}}$ and therefore the equivalence we constructed agrees with the one in the claim. The lemma follows in the case $d = 0$.

To reduce the case $d > 0$ to the case $d = 0$ we define X as the space of closed connected θ -structured submanifolds of $(-1, 1)^\infty$ so that $X \simeq \coprod_{[W] \text{ con.}} B\text{Diff}^\theta(W)$. If we can show that $\mathcal{C}_{d,\theta}^{\text{cl}}$ and $\mathcal{C}_{0,X}^{\text{cl}}$ are weakly equivalent as non-unital topological Γ -categories, then the claim for $d > 0$ follows from the first part of the proof:

$$Q \left(\Sigma \left(\coprod_{[W] \text{ connected}} B\text{Diff}^\theta(W) \right)_+ \right) \simeq Q(\Sigma X_+) \simeq BC_{0,X}^{\text{cl}} \stackrel{?}{\simeq} BC_{d,\theta}^{\text{cl}}.$$

To obtain the weak equivalence $\mathcal{C}_{d,\theta}^{\text{cl}} \simeq \mathcal{C}_{0,X}^{\text{cl}}$ we first convince ourselves that the morphism spaces are abstractly equivalent. To see this take any closed d -dimensional manifold W and decompose it as $W = \coprod_{i=1}^n \coprod_{j=1}^{m_i} W_{i,j}$ where each $W_{i,j}$ is connected and $W_{i,j} \cong W_{i',j'}$ iff $i = i'$.

Then

$$\begin{aligned} B\text{Diff}^\theta(W) &= \text{Bun}^\theta(W) // \text{Diff}(W) \cong \left(\prod_{i=1}^n (\text{Bun}^\theta(W_{i,1}))^{m_i} \right) // \left(\prod_{i=1}^n \text{Diff}(W_{i,1}) \wr \Sigma_{m_i} \right) \\ &\simeq \prod_{i=1}^n \left((\text{Diff}^\theta(W_{i,1}))^{m_i} // \Sigma_{m_i} \right) \simeq \prod_{i=1}^n \left(\text{Conf}_{m_i}((-1, 1)^\infty; \text{Diff}^\theta(W_{i,1})) \right). \end{aligned}$$

This shows that the respective connected components of $\text{Mor}(\mathcal{C}_{d,\theta}^{\text{cl}})$ and $\text{Mor}(\mathcal{C}_{0,X}^{\text{cl}})$ are abstractly equivalent.

To obtain the desired equivalence of infinite loop spaces $BC_{d,\theta}^{\text{cl}} \simeq BC_{0,X}^{\text{cl}}$ we will construct a zigzag $\mathcal{C}_{d,\theta}^{\text{cl}} \leftarrow \mathcal{D} \rightarrow \mathcal{C}_{0,X}^{\text{cl}}$ of Γ -categories inducing the equivalence on the morphisms spaces. The new Γ -category \mathcal{D} also has one object. A morphism in \mathcal{D} is a tuple (W, i, j, l, t) where $t > 0$ is the length, W is a closed d -dimensional manifold, $i : W \hookrightarrow (-1, 1)^\infty \times (0, t)$ and $j : (\pi_0 W) \hookrightarrow (-1, 1)^\infty \times (0, t)$ are embeddings, and $l \in \text{Bun}^\theta(W)$ is a θ -structure on W . Such a tuple is identified with another tuple (W', i', j', l', t') if $t = t'$ and there is a diffeomorphism $\varphi : W \cong W'$ such that $i = i' \circ \varphi$, $j = j' \circ (\pi_0 \varphi)$, and $l = \varphi^* l'$. The Γ -structures are defined just like for $\mathcal{C}_{d,\theta}^{\text{cl}}$.

There is a projection map $p : \mathcal{D} \rightarrow \mathcal{C}_{d,\theta}^{\text{cl}}$ that forgets i . There also is a projection map $q : \mathcal{D} \rightarrow \mathcal{C}_{0,X}^{\text{cl}}$ that sends $[W, i, j, l]$ to the configuration $j(\pi_0 W) \subset (-1, 1)^\infty$, where each point $j([V])$ is labelled by the connected component $V \subset W$ with tangential structure $l|_V \in \text{Bun}^\theta(V)$. Both maps p and q are functors of Γ -categories by construction. It is not hard to see that they induce the abstract equivalence described above. \square

Definition 5.4.2. We will let \mathbb{T} denote the Lie group $SO(2)$. The free loop space $LX := \text{Map}(S^1, X)$ admits an action of \mathbb{T} by precomposition. We will denote the homotopy orbits of this action by

$$(LX)_{h\mathbb{T}} := LX // \mathbb{T} = (E\mathbb{T} \times LX) / \mathbb{T}.$$

Lemma 5.4.3. Consider the tangential structure $\theta = X \times \theta^{or}$ where $\theta^{or} = \{-1, 1\}$ with the non-trivial action of GL_1 . The moduli space of connected closed 1-manifolds with θ -structure is

$$B\text{Diff}^\theta(S^1) \simeq (LX)_{h\mathbb{T}}.$$

Proof. We compute

$$\begin{aligned} B\text{Diff}^\theta(S^1) &= \text{Bun}_\theta(S^1) // \text{Diff}(S^1) \simeq \text{Map}_{GL_1}(S^1 \times \{\pm 1\}, X \times \{\pm 1\}) // \text{Diff}(S^1) \\ &\simeq (\text{Map}(S^1, X) \times \text{Map}(S^1, \{\pm 1\})) // \text{Diff}(S^1). \end{aligned}$$

Inclusion of the subgroup $\mathbb{Z}/2 \rtimes \mathbb{T} \cong O(2) \subset \text{Diff}(S^1)$ is a weak equivalence and we may hence compute the homotopy orbits by first taking $(_)_{h\mathbb{T}}$ and then taking $(_)_{h\mathbb{Z}/2}$. As $(_)_{h\mathbb{T}}$ commutes with coproducts this results in

$$B\text{Diff}^\theta(S^1) \simeq ((LX)_{h\mathbb{T}} \times \{\pm 1\})_{h\mathbb{Z}/2} \simeq (LX)_{h\mathbb{T}}$$

as claimed. \square

Combining lemma 5.4.1 and 5.4.3 we recover a computation of $BC_{1,X \times \theta \text{or}}^{\text{cl}}$ that was stated in [Gia19, Proposition 5.1] without proof:

Corollary 5.4.4. *For all spaces X there is a weak equivalence of infinite loop spaces*

$$Q(\Sigma_+(LX)_{h\mathbb{T}}) \simeq BC_{1,X \times \theta \text{or}}^{\text{cl}}.$$

5.4.2 The homotopy category of the 1-dimensional bordism category

We now have all the tools ready to prove Theorem E. We will first compute the classifying space of the reduced bordism category $\mathcal{C}_1^{\text{red}}$, then compare it to $h\mathcal{C}_1^{\text{red}}$, and finally compute $B(h\mathcal{C}_1)$.

Theorem 5.4.5. *There is an equivalence of infinite loop spaces*

$$BC_1^{\text{red}} \simeq \Omega^{\infty-2}MTSO_2.$$

Proof. We will use the reduction fiber sequence of theorem 5.3.1 for \mathcal{C}_1 :

$$BC_1^{\text{cl}} \longrightarrow BC_1 \xrightarrow{R} BC_1^{\text{red}}.$$

Since R is surjective on connected components (in fact $R : \pi_0 BC_1 \rightarrow \pi_0 BC_1^{\text{red}}$ is a bijection) this remains a homotopy fiber sequence after we deloop each of the infinite loop spaces once:³

$$B(BC_1^{\text{cl}}) \longrightarrow B(BC_1) \longrightarrow B(BC_1^{\text{red}}).$$

We can therefore write $BC_1^{\text{red}} \simeq \Omega(B(BC_1^{\text{red}}))$ as the homotopy fiber of the left-hand map. We have a homotopy commutative diagram of infinite loop spaces as follows:

$$\begin{array}{ccccc} BC_1^{\text{red}} & \longrightarrow & B(BC_1^{\text{cl}}) & \longrightarrow & B(BC_1) \\ & & \uparrow \simeq & \nearrow \alpha_{S^1} & \downarrow \simeq \\ \Omega^{\infty-2}MTSO_2 & \longrightarrow & Q(\Sigma^2(BSO_2)_+) & \longrightarrow & \Omega^{\infty-2}MTSO_1 \end{array}$$

Here lemma 5.4.1 is our computation of BC_1^{cl} and theorem 5.1.26 is the main theorem of [GMTW09]. The bottom row is the Genauer fiber sequence from lemma 5.1.28. Since the top sequence is also a homotopy fiber sequence, we obtain a zig-zag of equivalences of infinite loop spaces:

$$BC_1^{\text{red}} \leftarrow \text{hofib} \left(Q(\Sigma^2(BSO_2)_+) \xrightarrow{\alpha_{S^1}} B(BC_1) \right) \rightarrow \Omega^{\infty-2}MTSO_2.$$

□

To obtain the desired computation of $Bh\mathcal{C}_1$ we first need to show that in dimension 1 the reduced bordism category is equivalent to its homotopy category.

³To see that this is indeed a homotopy fiber sequence, consider the canonical map $B(BC_1^{\text{cl}}) \rightarrow \text{hofib}(B(BC_1) \rightarrow B(BC_1^{\text{red}}))$. We know that it becomes an equivalence after applying Ω and we know that the left-hand space is connected. So all that is left to show is that the right-hand space is also connected, but this is a consequence of $\pi_0 R$ being surjective.

Lemma 5.4.6. *The natural functors $\mathcal{C}_1^{\text{red}} \leftarrow \delta(\mathcal{C}_1^{\text{red}}) \rightarrow h(\mathcal{C}_1^{\text{red}})$ induce equivalences on classifying spaces.*

Proof. For the left-ward pointing functor this is a consequence of lemma 5.1.16 and the fibrancy of $\mathcal{C}_{d,\theta}^{\text{red}}$, which in turn is a consequence of [ERW19a, Proposition 3.2.4(ii)].

The right-ward pointing functor $\delta(\mathcal{C}_1^{\text{red}}) \rightarrow h\mathcal{C}_1^{\text{red}}$ is the identity on the (discrete) space of objects. So to show that it is an equivalence, we will only have to show that for any two objects $M, N \in \mathcal{C}_1^{\text{red}}$ the projection

$$\text{hom}_{\mathcal{C}_1^{\text{red}}}(M, N) \longrightarrow \pi_0 \text{hom}_{\mathcal{C}_1^{\text{red}}}(M, N) = \text{hom}_{h\mathcal{C}_1^{\text{red}}}(M, N)$$

is a weak equivalence. To see this, recall from fact 5.1.13 that the left-hand-side can be written as

$$\text{hom}_{\mathcal{C}_1^{\text{red}}}(M, N) \simeq \coprod_{[W]} \text{BDiff}^+(W \text{ rel } \partial W)$$

where $[W]$ runs over diffeomorphism classes of reduced bordisms from M to N . By the classification of 1-manifolds, every such reduced bordism is the disjoint union of intervals: $W \cong \amalg^k [0, 1]$. A diffeomorphism of W relative to its boundary cannot permute the intervals and therefore the diffeomorphism group decomposes as a product. Since the diffeomorphism group of the interval relative to its boundary is contractible we have that

$$\text{BDiff}^+(W \text{ rel } \partial W) \cong \prod_{i=1}^k \text{BDiff}^+([0, 1] \text{ rel } \{0, 1\}) \simeq *.$$

Therefore the connected components of the hom spaces of $\delta\mathcal{C}_1^{\text{red}}$ are contractible, and the category is equivalent to its homotopy category. \square

Theorem 5.4.7. *There is a homotopy fiber sequence of infinite loop spaces*

$$S^1 \longrightarrow B(h\mathcal{C}_1) \longrightarrow \Omega^{\infty-2}MTSO_2.$$

The infinite loop space map $\Omega^{\infty-2}MTSO_2 \rightarrow K(\mathbb{Z}, 2)$ that continues this fiber sequence corresponds to the generator $\Sigma^2\kappa_0 \in H^2(\Sigma^2MTSO_2) \cong \mathbb{Z}$.

Proof. Consider the two compatible reduction fiber sequences of Theorem 5.3.1:

$$\begin{array}{ccccc} BC_1^{\text{cl}} & \longrightarrow & BC_1 & \xrightarrow{R} & BC_1^{\text{red}} \\ \downarrow & & \downarrow & & \downarrow \\ B(h\mathcal{C}_1^{\text{cl}}) & \longrightarrow & B(h\mathcal{C}_1) & \xrightarrow{R} & B(h\mathcal{C}_1^{\text{red}}) \end{array}$$

By lemma 5.4.6 the right vertical map is an equivalence and hence theorem 5.4.5 implies

$$B(h\mathcal{C}_1^{\text{red}}) \simeq BC_1^{\text{red}} \simeq \Omega^{\infty-2}MTSO_2.$$

The category $h\mathcal{C}_1^{\text{cl}}$ has one object, the endomorphisms of which are the natural numbers. Therefore its classifying space is $B(h\mathcal{C}_1^{\text{cl}}) = B\mathbb{N} \simeq S^1$. Therefore the bottom fiber sequence of the diagram now reads as

$$S^1 \longrightarrow B(h\mathcal{C}_1) \longrightarrow \Omega^{\infty-2}MTSO_2,$$

which proves the first claim of the theorem.

Continuing the fiber sequences once to the right we have

$$\begin{array}{ccccc} BC_1 & \xrightarrow{R} & BC_1^{\text{red}} & \xrightarrow{f} & Q(\Sigma^2(BSO_2)_+) \\ \downarrow & & \downarrow p_1 & & \downarrow p_2 \\ B(h\mathcal{C}_1) & \xrightarrow{R} & B(h\mathcal{C}_1^{\text{red}}) & \xrightarrow{g} & K(\mathbb{Z}, 2). \end{array}$$

The map f is an isomorphism for spectrum cohomology in positive degree, because its fiber QS^0 has spectrum cohomology concentrated in degree 0. The class $\kappa_0 \in H^0(MTSO_2)$ is defined as the pullback of the basepoint class in $H^0(\Sigma^\infty(BSO_2)_+)$. The map p_2 is 3-connected because it is the delooping of a 2-connected map, and as noted above p_1 is an equivalence. Therefore g pulls back the canonical class in $H^2(\Sigma^2 H\mathbb{Z})$ to κ_0 . \square

As a consequence of the computation of $B(h\mathcal{C}_1)$ we can compute its rational cohomology using the following standard fact:

Fact 5.4.8. *Let Y be a spectrum such that the spectrum cohomology $H^i(Y; \mathbb{Q})$ is finite dimensional for all $i \geq 0$. Then the rational cohomology of its infinite loop space is*

$$H^*(\Omega^\infty Y; \mathbb{Q}) \cong \prod_{\pi_0 Y} S[H^{*>0}(Y; \mathbb{Q})]$$

where $S[V]$ denotes the free symmetric algebra on a graded vector space V . The Hopf-algebra structure on the cohomology of the identity component $H^*(\Omega_0^\infty Y; \mathbb{Q}) \cong S[H^{*>0}(Y; \mathbb{Q})]$ is such that $H^{*>0}(Y; \mathbb{Q})$ consists of primitive elements.

Proof. This is well-known, but we provide a short sketch of proof. First, we have equivalences of spaces $\Omega^\infty Y \simeq \pi_0 Y \times \Omega_0^\infty Y \simeq \coprod_{\pi_0 Y} \Omega_0^\infty Y$ and so it will suffice to show that $H^*(\Omega_0^\infty Y; \mathbb{Q}) \cong S[H^{*>0}(Y; \mathbb{Q})]$. Because it has degree-wise finite dimensional homology $\tau_{\geq 1} Y$ is rationally equivalent to a direct sum of Eilenberg-Mac Lane spectra $\Sigma^n H\mathbb{Q}$, $n > 0$. We may hence assume $Y = \Sigma^n H\mathbb{Q}$, in which case we have $\Omega^{\infty-n} H\mathbb{Q} = K(\mathbb{Q}; n)$ and $H^*(K(\mathbb{Q}; n); \mathbb{Q}) = \mathbb{Q}[\beta]$ where $|\beta| = n$. \square

Corollary 5.4.9. *The rational cohomology rings of $h\mathcal{C}_1$ and $h\mathcal{C}_1^{\text{red}}$ are*

$$H^*(B(h\mathcal{C}_1)_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_1, \bar{\kappa}_2, \dots] \quad \text{and} \quad H^*(B(h\mathcal{C}_1^{\text{red}})_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_0, \bar{\kappa}_1, \bar{\kappa}_2, \dots]$$

where $|\bar{\kappa}_i| = 2i + 2$. Moreover, the $\bar{\kappa}_i$ are primitive with respect to the Hopf-algebra structure.

Proof. The spectrum cohomology of $\Sigma^2 MTSO_2$ is

$$H^*(\Sigma^2 MTSO_2; \mathbb{Q}) \cong \mathbb{Q}\langle \bar{\kappa}_{-1}, \bar{\kappa}_0, \bar{\kappa}_1, \bar{\kappa}_2, \dots \rangle$$

where $|\bar{\kappa}_i| = 2i + 2$. By fact 5.4.8 we have

$$H^*(\Omega_0^\infty \Sigma^2 MTSO_2; \mathbb{Q}) \cong S[\mathbb{Q}\langle \bar{\kappa}_0, \bar{\kappa}_1, \bar{\kappa}_2, \dots \rangle].$$

This implies the second claim seeing as $B(h\mathcal{C}_1^{\text{red}}) \simeq \Omega^{\infty-2} MTSO_2$ by theorem 5.4.5.

Theorem 5.4.7 states that $B(h\mathcal{C}_1)$ is a circle bundle over BC_1^{red} , so we can compute its cohomology using the Gysin sequence. The Euler class of the circle bundle is $\bar{\kappa}_0 \in H^2(B(h\mathcal{C}_1^{\text{red}}))$, so by the rational Gysin sequence

$$H^*(B(h\mathcal{C}_1)_0; \mathbb{Q}) \cong H^*(B(h\mathcal{C}_1^{\text{red}})_0; \mathbb{Q}) / \langle \bar{\kappa}_0 \rangle \cong \mathbb{Q}[\bar{\kappa}_1, \bar{\kappa}_2, \dots].$$

Alternatively we could have computed $H^*(\text{hofib}(\Sigma^2 MTSO_2 \rightarrow \Sigma^2 H\mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}\langle \alpha, \bar{\kappa}_1, \bar{\kappa}_2, \dots \rangle$ and applied fact 5.4.8 again. \square

5.4.3 The reduced 1-dimensional bordism category and topological cyclic homology of a simply connected spaces

We will now compute the homotopy type of $\mathcal{C}_{1,\theta}^{\text{red}}$ for more general tangential structures in terms of the so-called *circle transfer* map. For any space Y with \mathbb{T} -action the *circle transfer* is an infinite loop space map

$$\text{trf}_{\mathbb{T}} : Q(\Sigma(Y_{h\mathbb{T}})_+) \longrightarrow Q(Y_+)$$

natural with respect to \mathbb{T} -equivariant maps.

We will treat the circle transfer as a black box and refer the reader to [Gia19] for a more detailed discussion and pointers to the literature. The circle transfer for the free loop space $Y = LX$ turns up naturally as a map between classifying spaces of 1-dimensional bordism categories:

Theorem 5.4.10 ([Gia19, Proposition 6.1]). *Let $\theta = X \times \theta^{\text{or}}$ be the tangential structure as before, then the following diagram of infinite loop spaces commutes up to homotopy:*⁴

$$\begin{array}{ccc} BC_{1,\theta}^{\text{cl}} & \longrightarrow & BC_{1,\theta} \\ \simeq \uparrow & & \downarrow \simeq \\ Q(\Sigma_+(LX)_{h\mathbb{T}}) & \xrightarrow{\text{trf}_{\mathbb{T}}} & Q(LX_+) \xrightarrow{Q(\text{ev})} Q(X_+) \end{array}$$

Corollary 5.4.11. *For the tangential structure $\theta = X \times \theta^{\text{or}}$ there is a homotopy fiber sequence of infinite loop spaces:*

$$Q(\Sigma(LX)_{h\mathbb{T}}) \xrightarrow{Q(\text{ev}) \circ \text{trf}_{\mathbb{T}}} Q(X) \longrightarrow BC_{1,\theta}^{\text{red}}$$

⁴Note that there is a small misprint in [Gia19, Proposition 6.1]: The diagram should say $\text{ev} \circ \text{trf}$, not $\text{trf} \circ \text{ev}$, as is evident from the proof.

Proof. Theorem 5.3.1 gives us the reduction fiber sequence

$$BC_{1,\theta}^{\text{cl}} \longrightarrow BC_{1,\theta} \longrightarrow BC_{1,\theta}^{\text{red}}.$$

By corollary 5.4.4 $BC_{1,\theta}^{\text{cl}} \simeq Q(\Sigma_+(LX)_{h\mathbb{T}})$ and the main theorem of [GMTW09] implies $BC_{1,\theta} \simeq Q(X_+)$. Inserting these into the reduction fiber sequence we obtain a homotopy fiber sequence with the desired terms, and theorem 5.4.10 identifies the relevant map. \square

Corollary 5.4.12. *For any simply connected space X there is an equivalence*

$$TC(\mathbb{S}[\Omega X]; p) \simeq Q(X_+)_p^\wedge \times \left(\Omega BC_{1,X \times \theta^{\text{or}}}^{\text{red}} \right)_p^\wedge.$$

The left-hand-side denotes the topological cyclic homology of the ring spectrum $\mathbb{S}[\Omega X] := \Sigma^\infty(\Omega X)_+$.

Proof. According to [BCC⁺96, Proposition 3.9] there is a splitting

$$TC(\mathbb{S}[\Omega X]; p) \simeq Q(X_+)_p^\wedge \times \text{hofib} \left(Q(\Sigma(LX)_{h\mathbb{T}}) \xrightarrow{\text{ev} \circ \text{trf}_{\mathbb{T}}} Q(X) \right)_p^\wedge.$$

Using corollary 5.4.11 we can rewrite the second term as

$$\Omega \text{hofib} \left(Q(\Sigma^2(LX)_{h\mathbb{T}}) \xrightarrow{\text{evo} \circ \Omega^{-1} \text{trf}_{\mathbb{T}}} Q(\Sigma X) \right) \simeq \Omega BC_{d,X \times \theta^{\text{or}}}^{\text{red}}$$

and the claim follows. \square

5.4.4 The unoriented bordism category

In this section we consider the trivial tangential structure $\text{unor} := \{*\}$. We have results similar to the oriented case:

Theorem 5.4.13. *There is an equivalence of infinite loop spaces $BC_{1,\text{unor}}^{\text{red}} \simeq \Omega^{\infty-2}MTO_2$ and there is a homotopy fiber sequence of infinite loop spaces*

$$S^1 \longrightarrow B(h\mathcal{C}_{1,\text{unor}}) \longrightarrow \Omega^{\infty-2}MTO_2.$$

The rational cohomology ring of $h\mathcal{C}_{1,\text{unor}}$ is:

$$H^*(B(h\mathcal{C}_{1,\text{unor}})_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_2, \bar{\kappa}_4, \dots]$$

where $|\bar{\kappa}_i| = 2i + 2$, and $\pi_0 B(h\mathcal{C}_{1,\text{unor}}) \cong \mathbb{Z}/2$.

Proof. The first part is proved just like theorem 5.4.5 and theorem 5.4.7. We only need to observe that the equivalence of lemma 5.1.28 still holds in the unoriented case.

To compute the rational cohomology, we need to understand the map $f : \Omega^{\infty-2}MTO_2 \rightarrow K(\mathbb{Z}, 2)$ that continues the fiber sequence. Since $\pi_1 B(h\mathcal{C}_{1,\text{unor}}) = 0$ the map is necessarily surjective on π_2 . So, by the same arguments as in corollary 5.4.9 we compute the cohomology. Here we use that

$$H^*(MTO_2; \mathbb{Q}) = \mathbb{Q}\langle \kappa_0, \kappa_2, \kappa_4, \dots \rangle.$$

To determine $\pi_0 B(h\mathcal{C}_{1,\text{unor}})$ recall that the unoriented 0-dimensional bordism group is $\mathbb{Z}/2$. \square

Chapter 6

Reduced bordisms and cocycles

In the previous chapter we computed the homotopy types of $B(h\mathcal{C}_1)$ and $B(h\mathcal{C}_1^{\text{red}})$ and as a result showed that their rational cohomology rings are polynomial algebras on the generators $\bar{\kappa}_i \in H^{2i+2}(B(h\mathcal{C}_1); \mathbb{Q})$. This computation, however, was achieved abstractly and is perhaps unsatisfying in that it does not give us a concrete understanding of what the classes $\bar{\kappa}_i$ actually are. The purpose of this chapter is to gain a more concrete understanding of $B(h\mathcal{C}_1)$ and $B(h\mathcal{C}_1^{\text{red}})$, and in particular the $\bar{\kappa}$ -classes on them. All the $\bar{\kappa}_i$ can be obtained as a pullback along the connecting homomorphism of the reduction fiber sequence:

$$B(h\mathcal{C}_1^{\text{red}}) \xleftarrow{\simeq} BC_1^{\text{red}} \xrightarrow{f} Q(\Sigma^2(BSO_2)_+).$$

In section 6.1 give a hands-on construction of the 2-cocycle representing the class $\bar{\kappa}_0 \in H^2(B(h\mathcal{C}_1^{\text{red}}))$ coming from $H^2(f)$. In subsection 6.2.1 we construct a simplicial space of “cuts” Cut and show that the canonical quotient map $NC_1^{\text{red}} \rightarrow \text{Cut}$ is a geometric model for f . This is then used in the final section to give the cocycle formulas.

6.1 The 2-cocycle on the reduced bordism category

We can think of $h\mathcal{C}_{d,\theta}$ as a central extension of $h\mathcal{C}_{d,\theta}^{\text{red}}$ by the abelian monoid $h\mathcal{C}_{d,\theta}^{\text{cl}}$. In this section we construct the 2-cocycle α on $h\mathcal{C}_{d,\theta}^{\text{red}}$ corresponding to the central extension. This is the first step in understanding the cocycles representing the cohomology classes on $h\mathcal{C}_1$.

Definition 6.1.1. Let $\mathcal{A}_{d,\theta}$ denote the abelian group $\pi_1 BC_{d,\theta}^{\text{cl}}$.

REMARK 6.1.2. As a consequence of lemma 5.4.1 $\mathcal{A}_{d,\theta}$ is the free abelian group on the set of diffeomorphism classes of closed connected θ -structured d -manifolds. Recall that $\pi_0(\Phi_{d,\theta})$ is the commutative monoid of diffeomorphism classes of closed θ -structured d -manifolds under disjoint union. This monoid injects into $\mathcal{A}_{d,\theta}$, and $\mathcal{A}_{d,\theta}$ is the group completion $\pi_0(\Phi_{d,\theta})$. When $d = 1$ and $\theta = \{\pm 1\}$ is the tangential structure for orientation, then $\mathcal{A}_{1,\text{or}} \cong \mathbb{Z}$ with generator $[S^1]$. Similarly, $\mathcal{A}_{2,\text{or}} \cong \bigoplus_{g \geq 0} \mathbb{Z}$ where there is one summand for every genus $g \geq 0$.

Definition 6.1.3. We define a 2-chain $\alpha : N_2(h\mathcal{C}_{d,\theta}^{\text{red}}) \rightarrow \mathcal{A}_{d,\theta}$ by

$$\alpha : (M_0 \xrightarrow{W_0} M_1 \xrightarrow{W_1} M_2) \mapsto [c(W_0 \cup_{M_1} W_1)] \in \pi_0\Phi_{d,\theta} \subset \mathcal{A}_{d,\theta}.$$

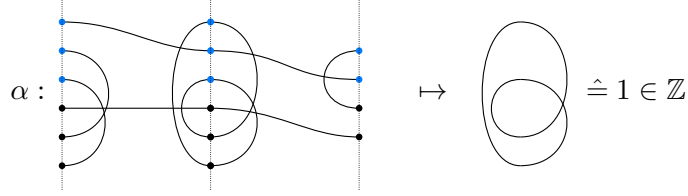


Figure 6.1: The 2-cochain α evaluated on a 2-simplex in $\text{Cut}_{d,\theta}$ for $d = 1$ and $\theta = \{\pm 1\}$.

Letting $d = 1$ and $\theta = \{\pm 1\}$ the 2-chain is \mathbb{Z} -valued and assigns to two reduced one-bordisms $W : M \rightarrow N$ and $V : N \rightarrow L$ the number of circles in the glued bordism $W \cup_N V$. See figure 6.1 for an illustration of this case. This measures the failure of the canonical section $h\mathcal{C}_1^{\text{red}} \rightarrow h\mathcal{C}_1$ to be functorial. In this sense it is reminiscent of the group 2-cocycle one usually assigns to a central extension of groups. We make this precise:

Lemma 6.1.4. The cochain α is a 2-cocycle and the cohomology class $[-\alpha] \in H^2(B(h\mathcal{C}_{d,\theta}^{\text{red}}); \mathcal{A}_{d,\theta})$ corresponds to the map

$$a : B(h\mathcal{C}_{d,\theta}^{\text{red}}) \longrightarrow K(\mathcal{A}_{d,\theta}, 2)$$

that continues the fiber sequence for homotopy categories of theorem 5.3.1. For $d = 1$ and $\theta = \pm 1$ we have $[-\alpha] = \bar{\kappa}_0$.

Proof. We first verify that α is indeed a cocycle. Consider a 3-simplex in the nerve of $h\mathcal{C}_{d,\theta}^{\text{red}}$: $x = (M_0 \xrightarrow{W_1} M_1 \xrightarrow{W_2} M_2 \xrightarrow{W_3} M_3)$. Evaluating α on $\partial x = \sum_i (-1)^i d_i x$ gives

$$[c(W_2 \cup_{M_2} W_3)] - [c(r(W_1 \cup_{M_1} W_2) \cup_{M_2} W_3)] + [c(W_1 \cup_{M_1} r(W_2 \cup_{M_2} W_3))] - [c(W_1 \cup_{M_1} W_2)].$$

To understand this, first consider $c(W_1 \cup_{M_1} W_2 \cup_{M_2} W_3)$. This is the submanifold of $W_1 \cup_{M_1} W_2 \cup_{M_2} W_3$ given by those components that do not intersect M_0 or M_3 . We can further decompose it as

$$c(W_1 \cup_{M_1} W_2 \cup_{M_2} W_3) = c(r(W_1 \cup_{M_1} W_2) \cup_{M_2} W_3) \amalg c(W_1 \cup_{M_1} W_2)$$

into those components that do, or do not, intersect M_2 . Similarly we have a decomposition

$$c(W_1 \cup_{M_1} W_2 \cup_{M_2} W_3) = c(W_1 \cup_{M_1} r(W_2 \cup_{M_2} W_3)) \amalg c(W_2 \cup_{M_2} W_3).$$

This shows that the two terms in $\alpha(\partial x)$ with a minus sign cancel the two terms with a plus sign. Hence α indeed is a cocycle.

For the second claim consider the map of homotopy fiber sequences

$$\begin{array}{ccccc} B(h\mathcal{C}_{d,\theta}^{\text{cl}}) & \longrightarrow & B(h\mathcal{C}_{d,\theta}) & \longrightarrow & B(h\mathcal{C}_{d,\theta}^{\text{red}}) \\ \downarrow \simeq & & \downarrow & & \downarrow a \\ K(\mathcal{A}_{d,\theta}, 1) & \longrightarrow & * & \longrightarrow & K(\mathcal{A}_{d,\theta}, 2). \end{array}$$

where the top row is the reduction fiber sequence from Theorem 5.3.1. The category $h\mathcal{C}_{d,\theta}^{\text{cl}}$ has one object and the morphisms form the monoid of diffeomorphism classes of closed manifolds under disjoint union. There is a canonical 1-cocycle $\beta \in H^1(h\mathcal{C}_{d,\theta}^{\text{cl}}; \mathcal{A}_{d,\theta})$, which sends a morphism $[W : \emptyset \rightarrow \emptyset]$ to $[W] \in \mathcal{A}_{d,\theta}$. This cocycle corresponds to the left vertical map in the above diagram. In the Serre spectral sequence for the bottom fiber sequence the canonical element $\beta \in H^1(K(\mathcal{A}_{d,\theta}, 1); \mathcal{A}_{d,\theta})$ transgresses to the canonical element $\beta' \in H^2(K(\mathcal{A}_{d,\theta}, 2); \mathcal{A}_{d,\theta})$. By naturality of the Serre spectral sequence this implies that $d_2[\beta] = a^*d_2[\beta] = a^*\beta'$. We will prove the lemma by showing that the d_2 -differential in the cohomological Serre spectral sequence

$$d_2 : H^1(h\mathcal{C}_{d,\theta}^{\text{cl}}; \mathcal{A}_{d,\theta}) \longrightarrow H^2(h\mathcal{C}_{d,\theta}^{\text{red}}; \mathcal{A}_{d,\theta})$$

sends $[\beta]$ to $-\alpha$.

Recall that the transgression $d_2[\beta]$ of $[\beta]$ can be uniquely characterised by requiring that in the following diagram we have $R^*(d_2[\beta]) = \delta([\beta])$.

$$\begin{array}{ccc} H^2(h\mathcal{C}_{d,\theta}^{\text{red}}, [\emptyset]; \mathcal{A}_{d,\theta}) & \xrightarrow{\cong} & H^2(h\mathcal{C}_{d,\theta}^{\text{red}}; \mathcal{A}_{d,\theta}) \\ \downarrow R^* & & \\ H^1(h\mathcal{C}_{d,\theta}^{\text{cl}}; \mathcal{A}_{d,\theta}) & \xrightarrow{\delta} & H^2(h\mathcal{C}_{d,\theta}, h\mathcal{C}_{d,\theta}^{\text{cl}}; \mathcal{A}_{d,\theta}) \end{array}$$

The coboundary operator δ is defined on β by choosing any extension $\widehat{\beta} : N_1(h\mathcal{C}_{d,\theta}) \rightarrow \mathcal{A}_{d,\theta}$ and setting $\delta(\beta) = \widehat{\beta} \circ \partial$. It will be convenient to choose $\widehat{\beta}(W) := [c(W)]$. The 2-cocycle $\delta(\beta) : N_2(h\mathcal{C}_{d,\theta}) \rightarrow \mathcal{A}_{d,\theta}$ is then

$$\delta(\beta)(M_0 \xrightarrow{W_1} M_1 \xrightarrow{W_2} M_2) = \widehat{\beta}(\partial(M_0 \xrightarrow{W_1} M_1 \xrightarrow{W_2} M_2)) = [c(W_2)] - [c(W_1 \cup_{M_1} W_2)] + [c(W_1)].$$

We can now compute $\delta(\beta) + R^*\alpha$ on some 2-simplex $x = (M_0 \xrightarrow{W_1} M_1 \xrightarrow{W_2} M_2)$:

$$\begin{aligned} (\delta(\beta) + R^*\alpha)(x) &= \widehat{\beta}(\partial x) + \alpha(R(x)) \\ &= \beta([W_2] - [W_1 \cup_M W_2] + [W_1]) + \alpha(M_0 \xrightarrow{r(W_1)} M_1 \xrightarrow{r(W_2)} M_2) \\ &= [c(W_2)] - [c(W_1 \cup_M W_2)] + [c(W_1)] + [c(r(W_1) \cup_{M_1} r(W_2))]. \end{aligned}$$

To see that this vanishes observe that the closed components of $(W_1 \cup_{M_1} W_2)$ can be decomposed as

$$c(W_1 \cup_{M_1} W_2) = c(r(W_1) \cup_{M_1} r(W_2)) \amalg c(W_1) \amalg c(W_2).$$

This shows that $\delta(\beta) = -R^*\alpha$ and hence $d_2[\beta] = -\alpha$. As we argued previously, this implies the second claim. \square

6.2 The continued reduction sequence and the space of cuts

6.2.1 The continued fiber sequence

In the case of homotopy categories the reduction fiber sequence can be thought of as a central extension that is classified by the 2-cocycle α constructed in the previous section. Similarly, the topologically enriched version of the reduction fiber sequence leads to a “central extension of infinite loop spaces” classified an infinite loop space map $BC_{d,\theta}^{\text{red}} \rightarrow Q(\Sigma^2(\coprod_{[W] \text{ con.}} B\text{Diff}^\theta(W))_+)$. In the remainder of this section we will explicitly describe this map, so that we can later use it to understand the higher cocycles on $h\mathcal{C}_1$.

It will be convenient to consider the following subspace, which in some sense freely generates $\Psi_{d,\theta}$:

Definition 6.2.1. Let $\Psi_{d,\theta}^{\text{con}} \subset \Psi_{d,\theta}$ be the subspace of those (W, l) where $W \subset (-1, 1)^\infty$ is connected or empty. We let $W = \emptyset$ be the base-point of this space. By fact 5.1.5 there is an equivalence:

$$\Psi_{d,\theta}^{\text{con}} \simeq \{\emptyset\} \amalg \coprod_{[W] \text{ connected}} B\text{Diff}^\theta(W).$$

To construct the map $BC_{d,\theta}^{\text{red}} \rightarrow Q(\Sigma^2\Psi_{d,\theta}^{\text{con}})$ we will need a new semi-simplicial space:

Definition 6.2.2. For any $n \geq 0$ we define a subspace

$$(\text{Cut}_{d,\theta})_n \subset N_n\mathcal{C}_{d,\theta} = \{(M_0 \xrightarrow{W_1} M_1 \rightarrow \cdots \rightarrow M_{n-1} \xrightarrow{W_n} M_n) \text{ in } \mathcal{C}_{d,\theta}\}$$

to contain those n -tuples where $M_0 = \emptyset$, $M_n = \emptyset$, and all the W_i are reduced.

There is a retraction $S_n : N_n\mathcal{C}_{d,\theta} \rightarrow (\text{Cut}_{d,\theta})_n$ for this inclusion, defined by deleting all connected components of $(W_1 \circ \cdots \circ W_n)$ that intersect $M_0 \amalg M_n$, or lie in $c(W_i)$ for some i . We define face operators for $\text{Cut}_{d,\theta}$ by $d_i^{\text{Cut}} := S_n \circ d_i^{\text{NC}}$.

REMARK 6.2.3. One needs to check that the d_i in fact satisfy all the simplicial relations. This follows immediately once one checks $S_n \circ d_i^{\text{NC}} = S_n \circ d_i^{\text{NC}} \circ S_n$ and $S_n \circ s_i^{\text{NC}} = s_i^{\text{NC}} \circ S_n$. See lemma 6.2.13 for more details. Conceptually, d_i^{Cut} acts by taking the face map as usual in the nerve and then deleting all connected components V that violate the condition $V \cap M_0 \amalg M_n = \emptyset$ or the condition $V \not\subset \coprod_i c(W_i)$. See figure 6.2 for an illustration.

Note that it follows from the definition that $S_\bullet : \text{NC}_{d,\theta} \rightarrow \text{Cut}_{d,\theta}$ is a map of semi-simplicial spaces and that it identifies $\text{Cut}_{d,\theta}$ as a quotient semi-simplicial space of $\text{NC}_{d,\theta}$.

Theorem 6.2.4. The natural quotient map $\text{NC}_{d,\theta}^{\text{red}} \rightarrow \text{Cut}_{d,\theta}$ induces, after geometric realisation, a continuation of the reduction fiber sequence from Theorem 5.3.1. More precisely, there is a map of

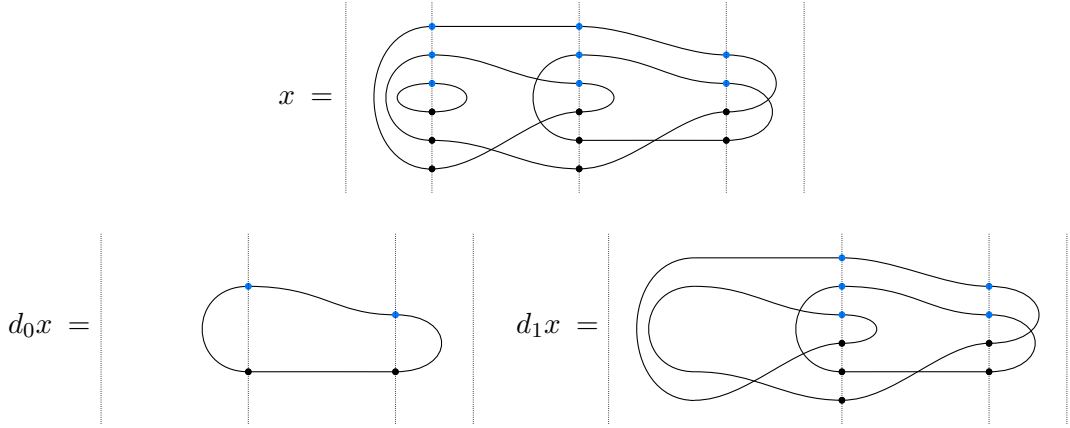


Figure 6.2: A 4-simplex in $\text{Cut}_{d,\theta}$ for $d = 1$ and $\theta = \{\pm 1\}$ and two of its simplicial faces.

homotopy fiber sequences of infinite loop spaces:

$$\begin{array}{ccccc}
 BC_{d,\theta}^{\text{cl}} & \longrightarrow & BC_{d,\theta} & \xrightarrow{R} & BC_{d,\theta}^{\text{red}} \\
 \parallel & & \downarrow & & \downarrow S \\
 BC_{d,\theta}^{\text{cl}} & \longrightarrow & * & \longrightarrow & \|\text{Cut}_{d,\theta}\|.
 \end{array}$$

In particular, there is an equivalence of infinite loop spaces:

$$\|\text{Cut}_{d,\theta}\| \simeq B(BC_{d,\theta}^{\text{cl}}) \simeq Q(\Sigma^2 \Psi_{d,\theta}^{\text{con}}).$$

In dimension $d = 1$ with $\theta = \{\pm 1\}$ we see that $\|\text{Cut}_{1,\text{or}}\| \simeq Q(\Sigma^2(\mathbb{C}\mathbb{P}^\infty)_+)$ and the map S comes from the cofiber sequence of spectra

$$\mathbb{S} \longrightarrow \Sigma^2 \mathbb{C}\mathbb{P}_{-1}^\infty \xrightarrow{s} \Sigma^{\infty+2}(\mathbb{C}\mathbb{P}^\infty)_+.$$

In particular, it is a rational equivalence on connective covers. This will be particularly useful in section 6.3 where we want to write the $\bar{\kappa}_i$ -classes as pullbacks along S . We record:

Corollary 6.2.5. For $d = 1$ and $\theta = \{\pm 1\}$ the map S induces a rational equivalence

$$BC_1^{\text{red}} \xrightarrow{(\pi_0, S)} \mathbb{Z} \times \|\text{Cut}_1\|.$$

If we introduce a background space X , then it follows from lemma 5.4.1 that $\Psi_{1,\{\pm 1\} \times X}^{\text{con}} \simeq ((LX)_{h\mathbb{T}})_+$. Let us denote $\text{Cut}_{1,\{\pm 1\} \times X}$ by $\text{Cut}_1(X)$. By theorem 6.2.4 we have $\|\text{Cut}_1(X)\| \simeq Q(\Sigma^2(LX_{h\mathbb{T}})_+)$. Recall that $\Sigma_+^\infty LX$ is also the topological Hochschild homology (THH) of the spherical group ring $\mathbb{S}[\Omega X]$. Indeed, we will see in section 6.3.1 that Cut_1 is closely related to Connes' category Λ and in the presence of a background space $\text{Cut}_1(X)$ should be thought of as a variant of the cyclic bar construction. This identification of Cut with THH is compatible with the relation between $\mathcal{C}_1^{\text{red}}$ and TC:

Corollary 6.2.6. *There is a homotopy commutative diagram of p -complete infinite loop spaces:*

$$\begin{array}{ccc} \Omega^\infty TC(\mathbb{S}[\Omega X]; p) & \xrightarrow{\alpha} & Q(\Sigma_+(LX)_{h\mathbb{T}})_p^\wedge \\ \downarrow b & & \downarrow \simeq \\ \Omega BC_1^{\text{red}}(X)_p^\wedge & \xrightarrow{\Omega S} & \Omega \|\text{Cut}_1(X)\|_p^\wedge \end{array}$$

where map α is the top map in [BHM93, Diagram (0.3)], see also [NS18, Theorem IV.3.6], and the map b is obtained by composing the equivalence from corollary 5.4.12 with the projection to the second factor.

Proof. As a consequence of the identifications made in corollary 5.4.11 and the continued fiber sequence in theorem 6.2.4 we have a homotopy commutative square:

$$\begin{array}{ccc} \text{hofib} \left(Q(\Sigma_+(LX)_{h\mathbb{T}}) \xrightarrow{\text{ev} \circ \text{trf}_{\mathbb{T}}} Q(X) \right) & \longrightarrow & Q(\Sigma_+(LX)_{h\mathbb{T}}) \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega BC_1^{\text{red}}(X) & \xrightarrow{\Omega S} & \Omega \|\text{Cut}_1(X)\|. \end{array}$$

Now we can, just like in corollary 5.4.12, use [BCC⁺96, Proposition 3.9] to obtain an equivalence

$$TC(\mathbb{S}[\Omega X]; p) \simeq Q(X_+)_p^\wedge \times \text{hofib} \left(Q(\Sigma(LX)_{h\mathbb{T}}) \xrightarrow{\text{ev} \circ \text{trf}_{\mathbb{T}}} Q(X) \right)_p^\wedge.$$

The square in the claim is now obtained by p -completing the above square and composing with the projection of TC onto the second factor. \square

The remainder of this section will be concerned with proving theorem 6.2.4.

6.2.2 The category of factorizations

Before we can prove theorem 6.2.4 we need a good understanding of how the space $\Psi_{d,\theta}^{\text{con}}$ relates to $\|\text{Cut}_{d,\theta}\|$ and $B(\mathcal{C}_{d,\theta}^{\text{cl}})$. This will make use of the following poset:

Definition 6.2.7. *The topological poset $F_{d,\theta}$ consists of tuples $((W, l), t)$ where $(W, l) \in \Psi_{d,\theta}^{\text{con}}$ is a closed d -dimensional submanifold of $(-1, 1)^\infty$ with W empty or connected and $t \in (-1, 1)$ is a regular value of $\text{pr}_1 : W \rightarrow (-1, 1)$ with $t \in \text{pr}_{\mathbb{R}}(W)$ or W empty. The relation is such that $((W, l), t) \leq ((W', l'), s)$ iff $(W, l) = (W', l')$ and $t \leq s$.*

One should think of $F_{d,\theta}$ as the category of possible factorizations of connected manifolds in $\mathcal{C}_{d,\theta}$. An object is a tuple of bordisms $W_0 : \emptyset \rightarrow M$ and $W_1 : M \rightarrow \emptyset$ such that $W_0 \cup_M W_1$ is connected and a morphism $(M, W_0, W_1) \rightarrow (N, V_0, V_1)$ is a bordism $X : M \rightarrow N$ such that $V_0 = W_0 \cup_M X$ and $W_1 = X \cup_N V_1$. In this sense $F_{d,\theta}$ is a full subcategory of the “over-and-under-category”; we will explore this perspective in section 6.3.1.

There is an augmentation map $F_{d,\theta} \rightarrow \Psi_{d,\theta}^{\text{con}}$ defined by forgetting the regular value t .

Proposition 6.2.8. *The augmentation induces an equivalence $B(F_{d,\theta}) \simeq \Psi_{d,\theta}^{\text{con}}$.*

We will prove this using the following technical lemma about classifying space of certain posets.

Lemma 6.2.9 ([GRW14]). *Let X be a space and $P \subset X \times \mathbb{R}$ an open subset such that $\text{pr}_X : P \rightarrow X$ is surjective. We equip P with the poset structure defined by*

$$(x, t) \leq (x', s) \Leftrightarrow (x = x' \text{ and } t \leq s).$$

Then the canonical augmentation $N_k P \rightarrow X$ induces a weak equivalence

$$B(P, \leq) \simeq X.$$

Proof. This is easily seen to be a special case of [GRW14, Theorem 6.2]. □

Proof of proposition 6.2.8. We want to apply the preceding lemma 6.2.9 to the poset $F_{d,\theta}$. The subset $F_{d,\theta} \subset \Psi_{d,\theta}^{\text{con}} \times \mathbb{R}$ is open because for t to be a regular value for $\text{pr}_1 : W \rightarrow \mathbb{R}$ is an open condition that continuously depends on W . The problem is that we also require $t \in \text{pr}_1(W)$; this leads to the projection map $F_{d,\theta} \rightarrow \Psi_{d,\theta}^{\text{con}}$ not being surjective. Lemma 6.2.9 therefore only shows that $B F_{d,\theta}$ is equivalent to the subspace $X \subset \Psi_{d,\theta}^{\text{con}}$ of those manifolds $(W, l) \in \Psi_{d,\theta}^{\text{con}}$ such that there is a $t \in \text{pr}_1(W)$ that is a regular value for $\text{pr}_1 : W \rightarrow \mathbb{R}$.

Since W is connected and the regular values of pr_1 are dense in \mathbb{R} , the only way for (W, l) not to lie in X is if $\text{pr}_1(W) = \{t\}$ is a single point. To prove the proposition we have to show that the inclusion $X \subset \Psi_{d,\theta}^{\text{con}}$ is a weak equivalence. We can write this inclusion as:

$$X \cong \coprod_{[W] \text{ con.}} \left(\text{Emb}(W, (-1, 1)^\infty)' \times \text{Bun}^\theta(W) \right) / \text{Diff}(W) \hookrightarrow \coprod_{[W] \text{ con.}} \left(\text{Emb}(W, (-1, 1)^\infty) \times \text{Bun}^\theta(W) \right) / \text{Diff}(W) \cong \Psi_{d,\theta}^{\text{con}}$$

Here $\text{Emb}(W, (-1, 1)^\infty)'$ is the subspace of those embeddings $\iota : W \hookrightarrow (-1, 1)^\infty$ where the first coordinate $\iota_0 : W \rightarrow (-1, 1)$ is not a constant map. Since each $\text{Emb}(W, (-1, 1)^\infty) \times \text{Bun}^\theta(W)$ is a $\text{Diff}(W)$ -principal bundle over the respective connected component of $\Psi_{d,\theta}$ it will suffice to check that each of the inclusions

$$\text{Emb}(W, (-1, 1)^\infty)' \times \text{Bun}^\theta(W) \hookrightarrow \text{Emb}(W, (-1, 1)^\infty) \times \text{Bun}^\theta(W)$$

is a weak equivalence. We will do so by showing that $\text{Emb}(W, (-1, 1)^\infty)'$ is contractible.

Let $\mathcal{C}^\infty(W, (-1, 1))$ be the space of smooth functions with the Whitney \mathcal{C}^∞ -topology and write $\mathcal{C}^\infty(W, (-1, 1))'$ for the subspace of non-constant smooth functions. Evaluating the first coordinate yields a map

$$E : \text{Emb}(W, (-1, 1)^\infty)' \longrightarrow \mathcal{C}^\infty(W, (-1, 1))'.$$

Pick a preferred embedding $j : W \hookrightarrow (-1, 1)^\infty$ and define an right-inverse J to E by setting $J(f)(w) = (f(w), j_1(w), j_2(w), \dots)$. It is clear that $E \circ J$ is the identity on $\mathcal{C}^\infty(W, (-1, 1))'$ and

$J \circ E$ is homotopic to the identity by the same standard argument that shows that $\text{Emb}(W, (-1, 1)^\infty)$ is homotopy equivalent to $\mathcal{C}^\infty(W, (-1, 1))'$.

We still need to check that $\mathcal{C}^\infty(W, (-1, 1))'$ is weakly contractible. Since $(-1, 1)$ is diffeomorphic to \mathbb{R} , the space $\mathcal{C}^\infty(W, (-1, 1))$ is homeomorphic to the vector space of smooth functions $\mathcal{C}^\infty(W, \mathbb{R})$. $\mathcal{C}^\infty(W, \mathbb{R})$ is a separable infinite dimensional Fréchet space and subspace of constant functions $\mathbb{R} \subset \mathcal{C}^\infty(W, \mathbb{R})$ is the union of countably many compacts. It therefore follows from [BP75, Theorem VI.5.2, Corollary V.6.2, and Theorem V.6.3] that $\mathcal{C}^\infty(W, (-1, 1))' \cong \mathcal{C}^\infty(W, \mathbb{R}) \setminus \mathbb{R}$ is homeomorphic to $\mathcal{C}^\infty(W, \mathbb{R})$ and in particular contractible. \square

6.2.3 Poset models

When working with bordism categories it is often convenient to replace the topological category \mathcal{C}_d by a topological poset PC_d with an equivalent classifying space. We learnt this trick from the work of Galatius–Randal-Williams (see e.g. [GRW10]). We recall PC_d and introduce four simplicial spaces, related to \mathcal{C}^{cl} , \mathcal{C}^{red} , Cut , and a new simplicial space D . Note that these will in fact be simplicial and not just semi-simplicial spaces since the poset PC_d (unlike $\mathcal{C}_d^{\text{cl}}, \mathcal{C}_d, \mathcal{C}_d^{\text{red}}$) does have identity morphisms. In what follows we will often suppress the data (d, θ) from the notation to make space for simplicial indices.

Definition 6.2.10. *The topological poset $PC_{d,\theta}$ has underlying space the space of tuples $((W, l), t)$ where $(W, l) \in \Psi_{d,\theta}(\mathbb{R} \times (-1, 1)^\infty)$ is a submanifold of the “tube” $\mathbb{R} \times (-1, 1)^\infty$ and $t \in \mathbb{R}$ is a regular value of the projection $W \rightarrow \mathbb{R}$. The poset structure is defined via*

$$((W, l), t) \leq ((W', l'), s) \Leftrightarrow (W, l) = (W', l') \text{ and } t \leq s.$$

Remark 6.2.11. *To compare this with $\mathcal{C}_{d,\theta}$ one defines another (non-unital) topological poset $P_{\text{cyl}}\mathcal{C}_{d,\theta}$ where for every object $((W, l), t)$ there is an $\varepsilon > 0$ such that (W, l) is cylindrical over $(t - \varepsilon, t + \varepsilon)$. Then there are functors*

$$\mathcal{C}_{d,\theta} \leftarrow P_{\text{cyl}}\mathcal{C}_{d,\theta} \rightarrow PC_{d,\theta}$$

and by [GRW10, Theorem 3.9] they both induce equivalences on the classifying spaces. All of these constructions are compatible with the Γ -space structures we described in section 5.1.5.

Let C denote the nerve of the poset $PC_{d,\theta}$. An n -simplex in C is a tuple $((W, l), t_0 \leq \dots \leq t_n)$ where $(W, l) \in \Psi_{d,\theta}(\mathbb{R} \times (-1, 1)^\infty)$ is a d -dimensional θ -structured manifold in the tube $\mathbb{R} \times (-1, 1)^\infty$ and the $t_i \in \mathbb{R}$ are regular values of $\text{pr}_{\mathbb{R}} : W \rightarrow \mathbb{R}$.

Definition 6.2.12. For all n we define subspaces $C_n^{\text{cl}}, C_n^{\text{red}}, \text{Cut}'_n, D_n \subset C_n$ to contain those n -simplices $w = ((W, l), t_0 \leq \dots \leq t_n) \in C_n$ satisfying certain conditions:

$$\begin{aligned} w \in C_n^{\text{cl}} &\Leftrightarrow \forall V \subset W : \text{pr}_{\mathbb{R}}(V) \cap \{t_0, \dots, t_n\} = \emptyset, \\ w \in C_n^{\text{red}} &\Leftrightarrow \forall V \subset W : \text{pr}_{\mathbb{R}}(V) \cap \{t_0, \dots, t_n\} \neq \emptyset, \\ w \in \text{Cut}'_n &\Leftrightarrow \forall V \subset W : \text{pr}_{\mathbb{R}}(V) \subset (t_0, t_n) \text{ and } \text{pr}_{\mathbb{R}}(V) \cap \{t_1, \dots, t_{n-1}\} \neq \emptyset, \\ w \in D_n &\Leftrightarrow \forall V \subset W : \text{pr}_{\mathbb{R}}(V) \subset (t_0, t_n). \end{aligned}$$

Here the \forall quantifier runs over all connected components $V \subset W$. When $n = 0$ or, more generally, when $t_0 = t_n$, the third and the fourth condition are interpreted as $W = \emptyset$.

Each of the subspaces $C_n^{\text{cl}}, C_n^{\text{red}}, \text{Cut}'_n, D_n \subset C_n$ is a union of connected components. The inclusion $D_n \subset C_n$ admits a canonical retraction $r : C_n \rightarrow D_n$ defined by sending $w = ((W, l), t_0 \leq \dots \leq t_n)$ to $((W', l|_{W'}), t_0 \leq \dots \leq t_n)$ where $W' \subset W$ is the union of those connected components $V \subset W$ satisfying $\text{pr}_{\mathbb{R}} \subset (t_0, t_n)$. Similarly, define $r : C_n \rightarrow C_n^{\text{cl}}$, $r : C_n \rightarrow C_n^{\text{red}}$, and $r : C_n \rightarrow \text{Cut}'_n$ by deleting the connected components that violate the relevant condition.

Define face operators d_i on $C_{\bullet}^{\text{cl}}, C_{\bullet}^{\text{red}}, \text{Cut}'_{\bullet}$, and D_{\bullet} by including to C_{\bullet} , applying the face operator of C_{\bullet} , and then applying the retraction. Degeneracy operators are defined similarly.

See figure 6.2 for an illustration of how face maps work in Cut'_{\bullet} and figure 6.3 for an example of a 4-simplex in D_{\bullet} .

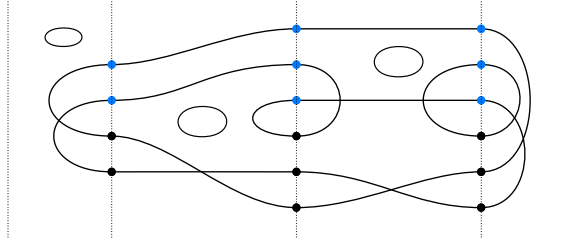


Figure 6.3: A 4-simplex in D for $d = 1$ and $\theta = \{\pm 1\}$.

Note that while the inclusion $C_{\bullet}^{\text{red}} \subset C_{\bullet}$ is not a simplicial map, the retraction $C_{\bullet} \rightarrow C_{\bullet}^{\text{red}}$ is:

Lemma 6.2.13. With the above definition the inclusion makes $C_{\bullet}^{\text{cl}} \subset C_{\bullet}$ a simplicial subspace and the retractions make $C_{\bullet}^{\text{red}}, \text{Cut}'_{\bullet}$, and D_{\bullet} into simplicial quotient spaces of C_{\bullet} . In each of the four cases the i th degeneracy operator simply repeats t_i .

Proof. Note that for any retraction $(i : A \rightarrow B, r : B \rightarrow A)$ of topological spaces the space A carries the subspace topology with respect to i and the quotient topology with respect to r . Therefore each of $C_n^{\text{red}}, \text{Cut}'_n$, and D_n is indeed a quotient space of C_n .

To check the first claim, we need to show that $C_{\bullet}^{\text{cl}} \subset C_{\bullet}$ is closed under face maps. But this is clear because $\text{pr}_{\mathbb{R}}(V) \cap \{t_0, \dots, t_n\} = \emptyset$ implies $\text{pr}_{\mathbb{R}}(V) \cap \{t_0, \dots, \hat{t}_i, \dots, t_n\} = \emptyset$ for any i .

For the second claim we need to check that $r \circ d_i = r \circ d_i \circ r$ holds for $r : C_\bullet \rightarrow C_\bullet^{\text{red}}$ the retraction and d_i a face operator on C_\bullet . Once this is shown, it follows that the induced face operators $d_i^{\text{red}} := r \circ d_i$ satisfy the simplicial identities and that $r : C_\bullet \rightarrow C_\bullet^{\text{red}}$ is a simplicial quotient map. The cases of Cut'_\bullet and D_\bullet are then shown similarly.

By definition the retraction deletes all connected components $V \subset W$ such that $V \cap \{t_0, \dots, t_n\} = \emptyset$. We need to make sure that if a component is deleted by r , then it is also deleted by $r \circ d_i$. But if $V \cap \{t_0, \dots, t_n\} = \emptyset$, then $V \cap \{t_0, \dots, \widehat{t_i}, \dots, t_n\} = \emptyset$, so the claim follows. As similar argument works in the other two cases because for each of them the conditions on $V \subset W$ get stricter as the set $\{t_0, \dots, t_n\}$ gets smaller. \square

Corollary 6.2.14. *There are zig-zags of level-wise weak equivalences of semi-simplicial spaces:*

$$N(\mathcal{C}_{d,\theta}^{\text{cl}}) \simeq C^{\text{cl}}, \quad N(\mathcal{C}_{d,\theta}) \simeq C, \quad N(\mathcal{C}_{d,\theta}^{\text{red}}) \simeq C^{\text{red}}, \quad \text{and} \quad \text{Cut}_{d,\theta} \simeq \text{Cut}'.$$

These are compatible with the canonical maps $C^{\text{cl}} \hookrightarrow C \twoheadrightarrow C^{\text{red}} \twoheadrightarrow \text{Cut}'$.

Proof. This is proved exactly as in [GRW10, Theorem 3.9]. Recall from remark 6.2.11 that we can define a (non-unital) topological poset $P_{\text{cyl}}\mathcal{C}$ that fits in a zig-zag

$$\mathcal{C} \leftarrow P_{\text{cyl}}\mathcal{C} \rightarrow PC.$$

In [GRW10, Theorem 3.9] the authors show that both functors induce level-wise weak equivalences on the nerves. This is exactly the second case. The other three cases follow from this case because level-wise each of C^{cl} , C^{red} , and Cut_n is a union of connected components of C_n and the weak equivalence restricts to a weak equivalence between these connected components. \square

6.2.4 Proof of theorem 6.2.4

Each of the simplicial spaces C'_\bullet has a Γ -structure defined using the tangential structure just like we did for $\mathcal{C}_{d,\theta}$ in section 5.1.5. We will continue to suppress d and θ from the notation and we will write \bullet as a placeholder for simplicial indices. By construction there is a commutative diagram of simplicial Γ -spaces

$$\begin{array}{ccccc} C_\bullet^{\text{cl}} & \longrightarrow & C_\bullet & \longrightarrow & C_\bullet^{\text{red}} \\ \parallel & & \downarrow & & \downarrow \\ C_\bullet^{\text{cl}} & \longrightarrow & D_\bullet & \longrightarrow & \text{Cut}'_\bullet \end{array}$$

where the composition of the two horizontal arrows in each row is canonically null-homotopic. By corollary 6.2.14 this diagram yields, after geometric realisation, a diagram of infinite loop spaces where the top row is the reduction fiber sequence of theorem 5.3.1. To prove theorem 6.2.4 it will therefore suffice to show that the bottom row is a homotopy fiber sequence after geometric realisation and that $\|D_\bullet\|$ is contractible.

We begin by simplifying the problem from understanding a homotopy fiber sequence of infinite loop spaces to understanding a homotopy cofiber sequence of spaces. To do so, we show that each of the three simplicial Γ -spaces C_{\bullet}^{cl} , D_{\bullet} , and Cut'_{\bullet} is freely generated by its subspace of connected manifolds, which we now define.

Definition 6.2.15. For $X = C_{\bullet}^{\text{cl}}$, D_{\bullet} , or Cut'_{\bullet} , let $X_{\bullet}^{\text{con}} \subset X_{\bullet}$ denote the sub-simplicial space of those tuples $((W, l), t_0 \leq \dots \leq t_n)$ where W is either connected or empty.

Observe that X_{\bullet}^{con} defines a subsimplicial space because face operators can only delete, but not create connected components. However, X_{\bullet}^{con} is not a special Γ -space anymore. This makes sense as the Γ -space structure is supposed to capture the operation defined by disjoint union of manifolds. The subspace $X_{\bullet}^{\text{con}} \subset X_{\bullet}$ should be thought of as the space of “indecomposables” for this operation and we will see that X_{\bullet} is in fact freely generated by them. This is a generalisation of lemma 5.4.1.

Lemma 6.2.16. For X any of the three spaces as above the inclusion $\|X_{\bullet}^{\text{con}}\| \rightarrow \|X_{\bullet}\|$ induces an equivalence of infinite loop spaces $Q(\|X_{\bullet}^{\text{con}}\|) \simeq \|X_{\bullet}\|$. These equivalences are compatible in the sense that there is a homotopy commutative diagram of infinite loop spaces

$$\begin{array}{ccccc} Q(\|(C_{\bullet}^{\text{cl}})^{\text{con}}\|) & \longrightarrow & Q(\|D^{\text{con}}\|) & \longrightarrow & Q(\|(\text{Cut}'_{\bullet})^{\text{con}}\|) \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \|C_{\bullet}^{\text{cl}}\| & \longrightarrow & \|D\| & \longrightarrow & \|\text{Cut}'_{\bullet}\|. \end{array}$$

Proof. For Y_{\bullet} a pointed simplicial space, let $\text{Conf}_*((-1, 1)^{\infty}; Y_{\bullet})$ be the simplicial space whose n -th level is the space of unordered configurations in $(-1, 1)^{\infty}$ with labels in Y_n , modulo the equivalence relation that deletes configuration points labelled by the basepoint. This is a simplicial Γ -space with the Γ -structure defined as usual for configuration spaces. There is a canonical simplicial inclusion $Y_{\bullet} \rightarrow \text{Conf}_*((-1, 1)^{\infty}; Y_{\bullet})$ defined by sending $y \in Y_n$ to the configuration $\{0\} \subset (-1, 1)^{\infty}$ labelled by y .

We construct for each X a zig-zag of equivalences of simplicial Γ -spaces:

$$X_{\bullet} \longleftarrow Z_{\bullet} \longrightarrow \text{Conf}_*((-1, 1)^{\infty}, X_{\bullet}^{\text{con}}).$$

Then the technical lemma 6.2.17 implies that $Q(\|X_{\bullet}^{\text{con}}\|) \simeq \|X_{\bullet}\|$ seeing as $\|X_{\bullet}^{\text{con}}\|$ is always connected. If we construct the zig-zags compatibly with the maps $C_{\bullet}^{\text{cl}} \rightarrow D_{\bullet} \rightarrow \text{Cut}'_{\bullet}$, then it also follows that the diagram of infinite loop spaces in the lemma is homotopy commutative.

In the case of $X = C^{\text{cl}}$ the relevant zig-zag was already constructed in lemma 5.4.1, though under a different name. We now recall the construction to see that it works in all three cases. A point in Z_n is represented by a tuple $(W, i, j, l, \underline{t})$ where W is a d -manifold with $l \in \text{Bun}^{\theta}(W)$ a θ -structure, $\underline{t} = (t_0 \leq \dots \leq t_n)$, $i : W \hookrightarrow \mathbb{R} \times (-1, 1)^{\infty}$ is an embedding and $j : \pi_0 W \hookrightarrow (-1, 1)^{\infty}$

is a configuration. Moreover, we require that $(i(W), i_*l, t_0 \leq \dots \leq t_n)$ is a well-defined point in X_n . (In particular, i has to be cylindrical in the appropriate places.) Z_n is the space of equivalence classes, where we identify $(W, i, j, l, \underline{t}) \sim (W', i \circ \varphi, j \circ (\pi_0 \varphi), \varphi^*(l), \underline{t})$ for any diffeomorphism $\varphi : W' \cong W$. The Γ -space is, as always, defined via the tangential structure θ .

The projection maps in the zig-zag are defined by

$$X_n \ni (i(W), i_*, t_0 \leq \dots \leq t_n) \longleftarrow [W, i, j, l, \underline{t}] \longmapsto (j(\pi_0 W), q) \in \text{Conf}_*((-1, 1)^\infty; X_n^{\text{con}})$$

where the labelling $q : j(\pi_0 W) \rightarrow X_n^{\text{con}}$ labels the configuration point $j([V])$ by $(i(V), i_*(l|_V), t_0 \leq \dots \leq t_n)$ for any connected component $V \subset W$.

It is not hard to see that Z is a simplicial Γ -space and that the two maps are compatible with this structure. The same arguments as in lemma 6.2.17 now show that they both are equivalences. \square

Lemma 6.2.17. *For Y_\bullet a pointed simplicial space, let $\text{Conf}_*((-1, 1)^\infty; Y_\bullet)$ be the simplicial Γ -space from the proof of lemma 6.2.16. If $\|Y\|$ is connected, then there is an equivalence of infinite loop spaces*

$$Q(\|Y\|) \xrightarrow{\simeq} \|\text{Conf}_*((-1, 1)^\infty; Y_\bullet)\|$$

compatible with the inclusion of $\|Y\|$ on both sides.

Proof. Consider the operad \mathcal{O} on the category of spaces where the space of n -ary operations is the set of n -tuples (i_1, \dots, i_n) of embeddings $i_j : (-1, 1)^\infty \rightarrow (-1, 1)^\infty$ with disjoint images, such that there are $r_j \in (0, 1)$ and $a_j \in (-1, 1)^\infty$ with $i_j(x) = r_j(x + a_j)$ for all $x \in (-1, 1)^\infty$. This is the *little ∞ -cubes* operad, and as all of the spaces $\mathcal{O}(n)$ are contractible, it is an E_∞ -operad. The associated monad $T_{\mathcal{O}}$ on the category of based spaces has a canonical map to the labelled configuration space:

$$T_{\mathcal{O}}(X) = \bigvee_{n \geq 1} (X^n \times \mathcal{O}(n))_{/\Sigma_n} \rightarrow \text{Conf}_*((-1, 1)^\infty; X), \quad [(x_1, \dots, x_n), (i_1, \dots, i_n)] \mapsto (\prod_j i_j(0), l)$$

where the labelling l is $l(i_j(0)) = x_j$. This map is clearly an equivalence for all based spaces X , as it only forgets the information of the $r_j \in (0, 1)$.

By [May72, Theorem 12.2] any monad associated to an operad commutes with geometric realisation up to homeomorphism, and so we have an equivalence

$$T_{\mathcal{O}}(\|Y_\bullet\|) \cong \|T_{\mathcal{O}}(Y_\bullet)\| \xrightarrow{\simeq} \text{Conf}_*((-1, 1)^\infty; \|Y_\bullet\|).$$

But \mathcal{O} is an E_∞ -operad and $\|Y_\bullet\|$ is connected, so by [May72, Corollary 6.3] $T_{\mathcal{O}}(\|Y_\bullet\|)$ is equivalent to $Q(\|Y_\bullet\|)$ and the claim follows. \square

In light of the previous lemma we can prove theorem 6.2.4 by showing that

$$\|(C_{\bullet}^{\text{cl}})^{\text{con}}\| \longrightarrow \|D_{\bullet}^{\text{con}}\| \longrightarrow \|(\text{Cut}'_{\bullet})^{\text{con}}\|$$

is a homotopy cofiber sequence and that $\|D^{\text{con}}\|$ is contractible. To see that this is a homotopy cofiber sequence¹ is not difficult: for each n we actually have $D_n^{\text{con}} \simeq (C_n^{\text{cl}})^{\text{con}} \vee (\text{Cut}'_n)^{\text{con}}$. The crucial step, however, is to check that $\|D^{\text{con}}\|$ is contractible, indeed we will show the following identification of homotopy cofiber sequences:

Lemma 6.2.18. *Let $F_{d,\theta} \rightarrow \Psi_{d,\theta}^{\text{con}}$ be the augmented topological poset from definition 6.2.7. Then there is an equivalence of homotopy cofiber sequences:*

$$\begin{array}{ccccc} \|(C_{\bullet}^{\text{cl}})^{\text{con}}\| & \longrightarrow & \|D_{\bullet}^{\text{con}}\| & \longrightarrow & \|(\text{Cut}'_{\bullet})^{\text{con}}\| \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ \Sigma(\Psi_{d,\theta}^{\text{con}}) & \longrightarrow & \Sigma(C(BF_{d,\theta} \rightarrow \Psi_{d,\theta}^{\text{con}})) & \longrightarrow & \Sigma^2(BF_{d,\theta}) \end{array}$$

where $C(BF_{d,\theta} \rightarrow \Psi_{d,\theta}^{\text{con}})$ denotes the cone of the augmentation map $BF_{d,\theta} \rightarrow \Psi_{d,\theta}^{\text{con}}$. In particular, since this map is a weak equivalence by proposition 6.2.8, the space $\|D^{\text{con}}\|$ is contractible.

The rest of this section will be concerned with proving this lemma.

Part 1 of the proof of lemma 6.2.18. The crucial observation for the proof of the lemma is that each of the simplicial spaces $(C_{\bullet}^{\text{cl}})^{\text{con}}$, D_{\bullet}^{con} , and $(\text{Cut}'_{\bullet})^{\text{con}}$ decomposes level-wise as a wedge of simpler spaces. For example in the first case we have a canonical equivalence:

$$(C_{\bullet}^{\text{cl}})^{\text{con}} \simeq \bigvee_{k=1}^n \Psi_{d,\theta}^{\text{con}} \times \{k\}$$

defined by sending an n -simplex $((W, l), t_0 \leq \dots \leq t_n)$ to the tuple $((W, l), k)$ where $k \in \{1, \dots, n\}$ is the unique number such that $\text{pr}_1(W) \subset (t_{k-1}, t_k)$. This makes sense since W is assumed to be connected and $\text{pr}_1(W)$ cannot contain any of the t_i . If $W = \emptyset$ we send $((\emptyset, \emptyset), t_0 \leq \dots \leq t_n)$ to the basepoint. This map is a fiber bundle with fiber the convex subset of \mathbb{R}^{n+1} containing the possible choices of t_i such that $t_0 \leq \dots \leq t_{k-1} \leq \min(\text{pr}_1(W))$ and $\max(\text{pr}_1(W)) \leq t_k \leq \dots \leq t_n$.

There are similar decompositions for D_n^{con} and $(\text{Cut}'_n)^{\text{con}}$, with the additional complication that we need to keep track of a tuple $(a \leq b)$ such that $\text{pr}_1(W) \subset (t_{a-1}, t_b)$. Indeed, using the nerve of the augmented topological poset² $F_{d,\theta} \rightarrow \Psi_{d,\theta}^{\text{con}}$ we can write

$$D_n^{\text{con}} \simeq \bigvee_{1 \leq a \leq b \leq n} N_{a-b-1} F_{d,\theta} \quad \text{and} \quad (\text{Cut}'_n)^{\text{con}} \simeq \bigvee_{1 \leq a < b \leq n} N_{a-b-1} F_{d,\theta}$$

¹For this to be a homotopy cofiber sequence we need to specify a nullhomotopy of the composite. The simplicial map $(C_{\bullet}^{\text{cl}})^{\text{con}} \rightarrow (\text{Cut}'_{\bullet})^{\text{con}}$ sends $((W, l), t_0 \leq \dots \leq t_n)$ to $((\emptyset, \emptyset), t_0 \leq \dots \leq t_n)$, so there is a canonical nullhomotopy given by contracting the t_i to 0.

²An augmented topological poset is a topological poset (P, \leq) together with a map $a : P \rightarrow X$ satisfying $a(x) = a(y)$ for all $x, y \in P$ with $x \leq y$. Its nerve is the augmented simplicial space NP with $N_n P = \{p \in P^{n+1} \mid p_0 \leq \dots \leq p_n\}$ for $n \geq 0$ and $N_{-1} P = X$. The map a is used as the face operator $f_0 : N_0 P = P \rightarrow X = N_{-1} P$.

where $N_{-1}F_{d,\theta} = \Psi_{d,\theta}^{con}$. We complete this proof below. \square

To complete the proof we need to properly understand how these wedges of space fit together to form a simplicial space. For this we introduce the notion of a simplicial (relative) cone:

Definition 6.2.19. *Let (X_\bullet, X_{-1}, x_0) be a pointed augmented simplicial space. Then the relative cone and the opposite relative cone are the pointed augmented simplicial spaces $C(X, X_{-1})$ and $C^{op}(X, X_{-1})$ defined as follows. In both case the space of n -simplices is*

$$C(X, X_{-1})_n := \bigvee_{k=-1}^n (X_k \times \{k\}) =: C^{op}(X, X_{-1})_n$$

The face and degeneracy operators are defined for the cone as:

$$d_i(x, k) = \begin{cases} (d_i x, k-1) & \text{if } i \leq k \\ (x, k) & \text{if } i > k \end{cases} \quad \text{and} \quad s_i(x, k) = \begin{cases} (s_i x, k+1) & \text{if } i \leq k \\ (x, k) & \text{if } i > k \end{cases}$$

and for the opposite cone as:

$$d_i^{op}(x, k) = \begin{cases} (d_{i-(n-k)} x, k-1) & \text{if } i \geq n-k \\ (x, k) & \text{if } i < n-k \end{cases} \quad \text{and} \\ s_i^{op}(x, k) = \begin{cases} (s_{i-(n-k)} x, k+1) & \text{if } i \geq n-k \\ (x, k) & \text{if } i < n-k. \end{cases}$$

These definitions are chosen such that $C^{op}(X, X_{-1}) = (C(X^{op}, X_{-1}))^{op}$.

We define the reduced cone $C_{red}^{(op)}(X, X_{-1})$ as the quotient of $C^{(op)}(X, X_{-1})$ by X , included in the obvious way. Moreover, we write $\Sigma^{(op)} X := C_{red}^{(op)}(X, *)$ when X has the trivial augmentation.

Lemma 6.2.20. *For any augmented pointed simplicial space (X, X_{-1}, x_0) there is a cofiber sequence of simplicial spaces*

$$\text{const}(X_{-1}) \longrightarrow C_{red}(X, X_{-1}) \longrightarrow C_{red}(X, *)$$

that realises to the cofiber sequence

$$X_{-1} \longrightarrow \text{Cone}(\|X\| \rightarrow X_{-1}) \longrightarrow \Sigma\|X\|.$$

Proof. We begin by showing that the augmented simplicial space $C^{op}(X, X_{-1})$ admits an extra degeneracy $s_{-1} : C^{op}(X, X_{-1})_n \rightarrow C^{op}(X, X_{-1})_{n+1}$ by $s_{-1}(x, k) = (x, k)$. This satisfies $d_0 s_{-1}(x, k) = (x, k)$ because $s_{-1}(x, k) = (x, k) \in C^{op}(X, X_{-1})_{n+1}$ falls under the second case of the definition of d_0 . (Since $0 < (n+1) - k$.) We also observe that $d_{i+1} s_{-1} = s_{-1} d_i$ and $s_{j+1} s_{-1} = s_{-1} s_j$ hold by construction. Hence s_{-1} is indeed an extra degeneracy and so the augmentation map $C^{op}(X, X_{-1}) \rightarrow X_{-1}$ induces a homotopy equivalence: $\|C^{op}(X, X_{-1})\| \simeq X_{-1}$. Taking opposites appropriately, we obtain the same result for $C(X, X_{-1})$. This also implies that $\|C(X, *)\|$ is contractible.

Next, we observe that the canonical inclusion $X \rightarrow C(X, X_{-1})$ defined by $(x \in X_n) \mapsto (x, n)$ is a level-wise cofibration and hence induces a cofibration $\|X\| \rightarrow \|C(X, X_{-1})\|$. Here we write X for the non-augmented simplicial space. By the first part of the proof $\|C(X, X_{-1})\|$ is equivalent to X_{-1} via the augmentation map and hence $\|C(X, X_{-1})/X\|$ is cone for the augmentation map. The inclusion $\text{const}(X_{-1}) \rightarrow C(X, X_{-1})/X$ is in each level the inclusion of a wedge summand and hence a cofibration. The quotient of this map is $C(X, X_{-1})/(X \vee \text{const}(X_{-1})) = C(X, *) / X = \Sigma X$. \square

Part 2 of the proof of lemma 6.2.18. We will use NF as a short-hand for the simplicial nerve $N(F_{d,\theta})$ and we let ψ_\bullet denote the constant simplicial space $\psi_n := \Psi_{d,\theta}^{\text{con}}$. We will construct simplicial maps

$$\begin{array}{ccccc} (C_\bullet^{\text{cl}})^{\text{con}} & \longrightarrow & D_\bullet^{\text{con}} & \longrightarrow & (\text{Cut}'_\bullet)^{\text{con}} \\ \downarrow f & & \downarrow g & & \downarrow h \\ \Sigma(\psi_\bullet) & \longrightarrow & \Sigma(C_{\text{red}}(NF, \psi)) & \longrightarrow & \Sigma(C_{\text{red}}(NF, *)) \end{array}$$

such that each of f , g , and h is a level-wise equivalence. The first map $f : (NC_{d,\theta}^{\text{cl}})^{\text{con}} \rightarrow \Sigma(\psi_\bullet)$ is defined by

$$f : ((W, l), t_0 \leq \dots \leq t_n) \mapsto ((W, l), k) \in (\psi \times \{k\}) \subset (\Sigma\psi)_n$$

where $k \in \{0, \dots, n-1\}$ is the unique number such that $\text{pr}_1(W) \subset (t_k, t_{k+1})$.

Before we define the second map $g : D^{\text{con}} \rightarrow \Sigma(C^{\text{op}}(NF, \psi)/NF)$, recall that an n -simplex in $\Sigma(C_{\text{red}}^{\text{op}}(NF, \psi))$ can be written as (x, l, k) where $k \in \{0, \dots, n-1\}$, $l \in \{-1, \dots, k-1\}$, and $x \in NF_l$. We can hence define the map as

$$g : ((W, l), t_0 \leq \dots \leq t_n) \mapsto (((W, l), t_{k-l} \leq \dots \leq t_k), l, k) \in (NF_l \times \{l\} \times \{k\})$$

where (k, l) are the smallest numbers such that $\text{pr}_1(W) \subset (t_{k-l-1}, t_{k+1})$. The face maps for $\Sigma(C_{\text{red}}^{\text{op}}(NF, \psi))$ can be derived from the formulas in definition 6.2.19 as³

$$d_i(x, l, k) = \begin{cases} (x, l, k) & \text{for } i > k \\ (x, l, k-1) & \text{for } i < k-l \\ (d_{i-(k-l)}x, l-1, k-1) & \text{for } k-l \leq i \leq k. \end{cases}$$

Here we implicitly identify an n -simplex (x, l, k) with the base-point $*$ if $l = k$ or $k = n$. Using these formulas it is not hard to check that g is indeed simplicial. The map h is defined by the same formula as g , which makes sense because $(\text{Cut}'_\bullet)^{\text{con}}$ is a quotient of D_\bullet^{con} . In other words h is induced by the fact that both rows in the diagram are level-wise cofiber sequences.

This concludes the construction of the diagram. The maps f , g , and h are level-wise homotopy equivalences by the observations in the first part of the proof. \square

³There is a typo here in the published version [Ste21a].

6.3 Identifying cocycles

By our main theorem the rational cohomology of $h\mathcal{C}_1$ is a polynomial algebra on a generator α in degree 0 and generators $\bar{\kappa}_i$ in degree $(2i + 2)$. We wish to give combinatorial formulas for these cocycles representing $\bar{\kappa}_i$.

Our strategy is as follows: The computations of section 5.4 imply that all the $\bar{\kappa}_i$ are pulled back from the space of cuts $\|\text{Cut}_1\|$ and in the previous section we gave an equivalence $\|\text{Cut}_1\| \simeq Q(\Sigma^2 BF_1)$. We will now show that the topological poset F_1 of ‘factorizations of circles’ has an equivalent classifying space to Connes’ category Λ of cyclicly ordered sets. It is well-known that $B\Lambda \simeq \mathbb{C}\mathbb{P}^\infty$ and Igusa has constructed cocycles $\gamma_k \in C^{2k}(\Lambda; \mathbb{Q})$ on Λ that represent $c^k \in \mathbb{Q}[c] = H^*(\mathbb{C}\mathbb{P}^\infty; \mathbb{Q})$. Once the necessary identifications are made it is only a matter of correctly pulling and pushing the cocycles through our equivalences in order to obtain the desired formulas for $\bar{\kappa}_k$.

6.3.1 The category of factorizations and Connes’ category Λ

In this section we compare the category of factorizations $F_{1, \{\pm 1\}}$ in dimension $d = 1$ with tangential structure $\theta^{\text{or}} = \{\pm 1\}$ to Connes’ category Λ of cyclic sets. Concretely, we will construct a zig-zag of topological functors

$$F_1 \xleftarrow{J} F_1^\delta \xrightarrow{D} \mathcal{F}_1 \xleftarrow{H} \Lambda$$

such that J is (almost) a continuous bijection that induces an equivalence on classifying spaces, D is a level-wise equivalence on nerves, and H is an equivalence of ordinary categories. One can show that after taking nerves each of these functor induces a weak equivalence in the complete Segal space model structure, so that F_1 and Λ represent equivalent $(\infty, 1)$ -categories. However, we will only show that they all induce equivalences on classifying spaces, as that is all we need.

Definition 6.3.1. *Connes’ category Λ has as objects natural numbers $n \geq 1$. A morphism $[f] : n \rightarrow m$ is represented by a weakly monotone map $[f] : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying $f(x + n) = f(x) + m$ for all $x \in \mathbb{Z}$. Two such maps $f, f' : \mathbb{Z} \rightarrow \mathbb{Z}$ represent the same morphism if and only if there is a $k \in \mathbb{Z}$ such that $f(x) = f'(x) + km$ for all $x \in \mathbb{Z}$.*

REMARK 6.3.2. *There also a non-skeletal version Λ^{big} of Λ such that $\Lambda \hookrightarrow \Lambda^{\text{big}}$ is an equivalence of categorie. A cyclic ordering on a finite set A is an equivalence class of bijections $\gamma : A \cong \{1, \dots, |A|\}$, where two such bijections are identified whenever they differ by a cyclic permutation of $\{1, \dots, |A|\}$. However, a cyclic morphism $(A, [\gamma]) \rightarrow (B, [\delta])$ has slightly more data than just cyclic order preserving maps $f : A \rightarrow B$. Namely, when $|f(A)| = 1$ one also has to keep track of a ‘minimal preimage’ $a_f \in A$. We will not go into detail here, but the reader may check themselves that the forgetful functor $\Lambda \rightarrow \text{Sets}$ is indeed not faithful.*

The poset $F_{d,\theta}$ is difficult to work with because it is not fibrant. There is, however, a map $F_{d,\theta} \rightarrow \Psi_{d-1,\theta}$ defined by sending $((W, l), t)$ to the preimage $\text{pr}_1^{-1}(t) \cap W$ equipped with the induced tangential structure. We can use this map to define a version of $F_{d,\theta}$ that is better behaved:

Definition 6.3.3. *The topological poset $F_{d,\theta}^\delta$ has as underlying space $F_{d,\theta} \times_{\Psi_{d-1,\theta}} \delta(\Psi_{d-1,\theta})$ where the latter denotes $\Psi_{d-1,\theta}$ with the discrete topology. The partial ordering on $F_{d,\theta}^\delta$ is the same as on $F_{d,\theta}$, except that we remove the identity morphisms.*

Definition 6.3.4. *For $W \in \Psi_{d,\theta}$ and $s < t \in \mathbb{R}$ regular values we write:*

$$W|_t := W \cap (\{t\} \times (-1, 1)^\infty) \quad \text{and} \quad W|_{[s,t]} := W \cap ([s, t] \times (-1, 1)^\infty).$$

Lemma 6.3.5. *The canonical map $J : F_{d,\theta}^\delta \rightarrow F_{d,\theta}$ induces a weak equivalence on classifying spaces.*

Proof. Let $F'_{d,\theta}$ denote the non-unital subcategory of $F_{d,\theta}$ containing those objects (W, t) where W is cylindrical over $(t - \varepsilon, t + \varepsilon)$ for some $\varepsilon > 0$ and those morphisms $(W, t \leq s)$ where $t < s$. Similarly, let $F_{d,\theta}^{\delta'} \subset F_{d,\theta}^\delta$ be the non-unital subcategory defined by the same conditions.

It follows from standard rescaling arguments (e.g. [GRW10, Proof of 3.9]) that the two inclusions $F'_{d,\theta} \hookrightarrow F_{d,\theta}$ and $F_{d,\theta}^{\delta'} \hookrightarrow F_{d,\theta}^\delta$ induce level-wise weak equivalences on nerves and hence weak equivalences on geometric realisations.

The functor $F_{d,\theta}^{\delta'} \rightarrow F'_{d,\theta}$ is a base-change in the sense of lemma 5.1.30. The relevant map

$$N(F'_{d,\theta})_n \longrightarrow (\Psi_{d,\theta})^n, \quad (W, t_0 < \dots < t_n) \mapsto (W|_{t_0}, \dots, W|_{t_n})$$

is a fibration by the same arguments as in the proof of [ERW19b, Proposition 3.2.4(ii)]. Therefore, by the base-change lemma 5.1.30, $F_{d,\theta}^{\delta'} \rightarrow F'_{d,\theta}$ is a weak equivalence on classifying spaces. By 2-out-of-3 this implies that $F_{d,\theta}^\delta \rightarrow F_{d,\theta}$ is also a weak equivalence on classifying spaces. \square

Specializing to dimension 1 we now introduce an ordinary category \mathcal{F}_1 that will interpolate between F_1^δ and Connes' Λ .

Definition 6.3.6. *The category \mathcal{F}_1 has as objects triples (M, W_0, W_1) where $(W_0 : \emptyset \rightarrow M, W_1 : M \rightarrow \emptyset) \in N_2(\text{hC}_1)$ is a composable tuple of morphisms in hC_1 such that M is non-empty and the composite $W_0 \cup_M W_1$ is a circle. A morphism $(M, W_0, W_1) \rightarrow (N, V_0, V_1)$ is a morphism $X : M \rightarrow N$ in hC_1 such that $W_0 \cup_M X = V_0$ and $W_1 = X \cup_N V_1$.*

Definition 6.3.7. *Define a functor $P : F_1^\delta \rightarrow \mathcal{F}_1$ by sending $((W, l), t) \in F_1^\delta$ to*

$$([W|_{t_0}, l], [W|_{[-1, t_0]}, l], [W|_{[t_0, 1]}, l]),$$

and on morphisms by sending $((W, l), t_0 < t_1)$ to $[W|_{[t_0, t_1]}, l]$.

Lemma 6.3.8. *The canonical functor $F_1^\delta \rightarrow \mathcal{F}_1$ induces a level-wise equivalence on nerves.*

Proof. For every k there is a (non-simplicial) map

$$N_k(F_1) \longrightarrow N_{k+2}(\mathcal{C}_1^{\text{red}}),$$

$$((W, l), t_0 < \dots < t_n) \mapsto (\emptyset \xrightarrow{W|_{[-1, t_0]}} W|_{t_0} \xrightarrow{W|_{[t_0, t_1]}} W|_{t_1} \rightarrow \dots \rightarrow W|_{t_n} \xrightarrow{W|_{[t_n, 1]}} \emptyset).$$

which is an equivalence onto the connected components it hits. This also defines a map for F_1^δ :

$$N_k(F_1^\delta) \longrightarrow N_{k+2}(\delta\mathcal{C}_1^{\text{red}}),$$

which again is an equivalence onto the connected components it hits.

To prove the comparison with \mathcal{F}_1 note that the nerve of \mathcal{F}_1 embeds level-wise a subset $N_k(\mathcal{F}_1) \hookrightarrow N_{k+2}(h\mathcal{C}_1^{\text{red}})$ in the same way that $N_k(F_1^\delta)$ embeds into $N_{k+2}(\delta\mathcal{C}_1^{\text{red}})$. Both maps in fact hit the same connected components; namely those $((W, l), t_0 \leq \dots \leq t_{k+2})$ where $W \cong S^1$ and $W|_{t_i} = \emptyset$ iff $i \in \{0, k+2\}$. But we already observed in lemma 5.4.6 that in dimension 1 with tangential structure $\theta = \{\pm 1\}$ the simplicial space $N(\delta\mathcal{C}_1^{\text{red}})$ is level-wise equivalent to the simplicial set $N(h\mathcal{C}_1^{\text{red}})$. This shows that $N_k(F_1^\delta)$ is indeed equivalent to the discrete space $N_k(\mathcal{F}_1)$. \square

Definition 6.3.9. For every oriented 0-manifold $(M, l : M \rightarrow \{\pm 1\})$ let $M^\pm := \{p \in M \mid l(p) = \pm 1\}$ denote the set of positively or negatively oriented points, respectively.

REMARK 6.3.10. In the following lemma we will construct an equivalence of categories $H : \mathcal{F}_1 \rightarrow \Lambda$. If we allow ourselves to use the non-skeletal $\Lambda^{\text{big}} \supset \Lambda$ from remark 6.3.2, the inverse equivalence $\tilde{H} : \mathcal{F}_1 \rightarrow \Lambda^{\text{big}}$ admits a conceptually easier description: it sends an object $(M, W_0, W_1) \in \mathcal{F}_1$ to the finite set $\pi_0(W_0)$, equipped with the cyclic ordering coming from the fact that W_0 is a disjoint union of intervals in an oriented circle $W_0 \subset W_0 \cup_M W_1 \cong S^1$. To a morphism $X : (M, W_0, W_1) \rightarrow (N, V_0, V_1)$ the functor \tilde{H} assigns the cyclic morphism $\pi_0(W_0) \rightarrow \pi_0(W_0 \cup_M X) \cong \pi_0(V_0)$. The problem with this description is that, as pointed out in remark 6.3.2, cyclic morphism have a subtle bit of extra data, which makes it hard to verify functoriality. Instead, we will explicitly construct an equivalence between the skeleta.

Lemma 6.3.11. There is an equivalence of ordinary categories $H : \Lambda \rightarrow \mathcal{F}_1$.

Proof. We begin by fixing some notation for this proof. For all $n \geq 1$ and $[k] \in \mathbb{Z}/n$ we choose two points in $(-1, 1)^\infty$, denoted by $[k]_n^+$ and $[k]_n^-$ such that all of these points are disjoint. Using these we define oriented 0-manifolds for all $n \geq 1$ by $M(n) := \{[1]_n^+, [1]_n^-, \dots, [n]_n^+, [n]_n^-\} \subset (-1, 1)^\infty$ with orientation $l([i]_n^\pm) := \pm 1$. Next, define diffeomorphism classes of bordisms $W(n) : \emptyset \rightarrow M(n)$ and $V(n) : M(n) \rightarrow \emptyset$ in $h\mathcal{C}_1$ by requiring that $W(n)$ is a disjoint union of n intervals with boundary $\{[i]_n^+, [i+1]_n^-\}$ and $V(n)$ is a disjoint union of n intervals with boundary $\{[i]_n^-, [i]_n^+\}$. By construction the glued manifold $W(n) \cup_{M(n)} V(n)$ is a circle and therefore $(M(n), W(n), V(n))$ defines an object of \mathcal{F}_1 . In fact, these objects define a skeleton for \mathcal{F}_1 .

Before we begin with the actual proof, we need to understand morphisms in the category $h\mathcal{C}_1^{\text{red}}$. For any $n, m \geq 0$ there is a canonical bijection

$$\sigma : \text{Hom}_{h\mathcal{C}_1^{\text{red}}}(M(n), M(m)) \cong \text{Hom}_{\text{Fin}^{\text{bij}}}(M(n)^+ \amalg M(m)^-, M(n)^- \amalg M(m)^+).$$

This map sends a bordism $X : M(n) \rightarrow M(m)$ to the bijection $\sigma_X : M(n)^+ \amalg M(m)^- \rightarrow M(n)^- \amalg M(m)^+$ with $\sigma_X(a) = b$ whenever there is an edge in X connecting a and b . It is possible to implicitly describe $\sigma_{Y \cup_{M(n)} X}$ in terms of σ_X and σ_Y , but we leave this to the reader.

We want to define a functor $H : \Lambda \rightarrow \mathcal{F}_1$ that sends the object n to the object $(M(n), W(n), V(n)) \in \mathcal{F}_1$. To do so, we need to give for every $[f] : n \rightarrow m$ in Λ a bordism $X : M(n) \rightarrow M(m)$ such that $W(n) \cup_{M(n)} X = W(m)$ and $V(n) = X \cup_{M(m)} V(m)$. Equivalently, we need to give a bijection $\sigma : M(n)^+ \amalg M(m)^- \cong M(n)^- \amalg M(m)^+$ satisfying certain conditions.

Fix a representative $f : \mathbb{Z} \rightarrow \mathbb{Z}$. We define X via σ_X as

$$\sigma_X([i]_n^+) = \begin{cases} [i]_n^- & \text{if } f(i) = f(i+1) \\ [f(i)]_m^+ & \text{if } f(i) \neq f(i+1) \end{cases}$$

$$\sigma_X([j]_m^-) = \begin{cases} [j+1]_m^+ & \text{if } j+1 \notin f(\mathbb{Z}) \\ [k]_n^+ & \text{if } k+1 = \min(f^{-1}(j+1)). \end{cases}$$

Since $f(x+n) = f(x) + m$ this is well defined on $[i] \in \mathbb{Z}/n$ and $[j] \in \mathbb{Z}/m$. Moreover, σ_X does not change if we replace f by $f + m$ and therefore X only depends on the equivalence class $[f] \in \text{Hom}_\Lambda(m, n)$. This construction is illustrated in figure 6.4. One checks by hand that X defines a morphism in \mathcal{F}_1 and that the construction is functorial.

As noted before, every object of \mathcal{F}_1 is isomorphic to one of the form $(M(n), W(n), V(n))$, and hence the functor H is essentially surjective.

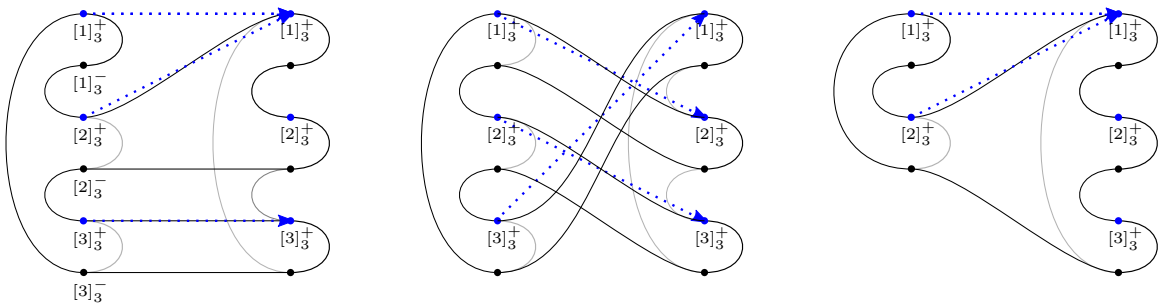


Figure 6.4: Three morphisms $X_i : (M(n_i), W(n_i), V(n_i)) \rightarrow (M(m_i), W(m_i), V(m_i))$ in \mathcal{F}_1 and their associated morphisms $f_i : n_i \rightarrow m_i$ in Λ . The bordisms $W(n_i), X_i, V(m_i)$ are drawn in black, the bordisms $V(n_i)$ and $W(m_i)$ are indicated in grey, and the maps $f_i : \mathbb{Z}/m_i \rightarrow \mathbb{Z}/n_i$ are shown in blue.

We still need to show that H is fully faithful. To do so, we will construct from a given $X : (M(n), W(n), V(n)) \rightarrow (M(m), W(m), V(m))$ a morphism $[f] : n \rightarrow m$ and show that this construction is inverse to the definition of H .

Let $A \subset \{1, \dots, n\} \cong M(n)^+$ be the (non-empty!) subset of those points where the relevant edge of X ends in $M(n)^+$. For the minimal element a_{\min} of A we pick any value $f(a_{\min}) = j \in \mathbb{Z}$ such that $\sigma_X[a_{\min}]_n^+ = [j]_m^+$. The set of such j is of the form $j + m\mathbb{Z}$. For all other $a \in A + n\mathbb{Z}$ we set $f(a) = j_a$ for the unique $j_a \in \mathbb{Z}$ with $\sigma_X([a]_n^+) = [j_a]_m^+$ and $f(a_{\min}) < j_a < f(a_{\min}) + m$. Then, for all $i \in \mathbb{Z} \setminus (A + n\mathbb{Z})$ we define $f(i) := f(i + 1)$, which makes sense recursively.

The map $f : \mathbb{Z} \rightarrow \mathbb{Z}$ we constructed satisfies $f(x + n) = f(x) + m$ by construction. That f is weakly monotone is enforced by the condition that the manifold $W(n) \cup_{M(n)} X \cup_{M(m)} W(m)$ is a circle. (See figure 6.4.) We have therefore constructed mutually inverse bijections between the relevant morphisms in \mathcal{F} and Λ . \square

6.3.2 Cocycles on the cyclic category

In this section we recall the description of Igusa's rational $2k$ -cocycles β_k on $N\Lambda$ that represent the powers of the first Chern class $c_1 \in H^*(B\Lambda; \mathbb{Q}) \cong \mathbb{Q}[c_1]$. It will be useful to first define the cocycles on a certain simplicial set \mathcal{U} , which admits compatible maps

$$\begin{array}{ccccccc} N(F_1) & \xleftarrow{J} & N(F_1^\delta) & \xrightarrow{D} & N(\mathcal{F}_1) & \xleftarrow{H} & N\Lambda \\ & & & & \downarrow s & & \\ & & & & \mathcal{U} & & \end{array}$$

By the previous section the realisations of the top row are all equivalent to $\mathbb{C}\mathbb{P}^\infty$ and the horizontal maps are equivalences. We think of \mathcal{U} as a rational model for $\mathbb{C}\mathbb{P}^\infty$ similar to Kontsevich's combinatorial $BU(1)^{\text{comb}}$, see [Kon92, section 2.2]. However, we will not actually show \mathcal{U} is rationally equivalent to $\mathbb{C}\mathbb{P}^\infty$. Instead we will only show that the maps $B\Lambda \rightarrow \|\mathcal{U}\|$ are rationally surjective. Concretely, we define cocycles $\overline{\text{sign}}_{2k}$ on \mathcal{U} and show that their pullback to the top row is a certain multiple of $(c_1)^k$. The advantage of working with \mathcal{U} is that it is much easier to construct the diagram as described above than it is to lift Igusa's cocycle against the maps H and J , see remark 6.3.14.

Definition 6.3.12. *The n -simplices in \mathcal{U} are represented by n -tuples (A_0, \dots, A_n) of finite non-empty disjoint subsets $A_i \subset S^1$. We identify two such n -tuples (A_0, \dots, A_n) and (B_0, \dots, B_n) if there is an orientation preserving diffeomorphism $\varphi : S^1 \cong S^1$ with $\varphi(A_i) = B_i$ for all i . The i th face map forgets A_i and the i th degeneracy map sends $[A_0, \dots, A_n]$ to $[A_0, \dots, A_i, A_i^\varepsilon, \dots, A_n]$ where A_i^ε is obtained from A_i by rotating by a sufficiently small angle $\varepsilon > 0$.*

Definition 6.3.13. *We define a simplicial map $s : N\mathcal{F}_1 \rightarrow \mathcal{U}$ as follows. A k -simplex of $N\mathcal{F}_1$ can be represented by a $(k + 2)$ -simplex $(M_0 \xrightarrow{[W_1]} M_1 \rightarrow \dots \rightarrow M_{k+2}) \in N_{k+2}h\mathcal{C}_1$ where $W_1 \cup_{M_1} \dots \cup_{M_{k+1}} W_{k+2}$ is diffeomorphic to S^1 . (In particular $M_0 = \emptyset = M_{k+2}$.) To define s , pick an orientation preserving diffeomorphism $\varphi : W_1 \cup_{M_1} \dots \cup_{M_{k+1}} W_{k+2} \cong S^1$ and set*

$$s(M_0 \xrightarrow{[W_1]} M_1 \rightarrow \dots \rightarrow M_{k+2}) := [\varphi(M_1^+), \dots, \varphi(M_{k+1}^+)]$$

where $M_i^+ \subset M_i$ denotes the subset of positively oriented points. The resulting k -simplex in \mathcal{U}_k does not depend on the choice of φ nor on the choice of representatives for morphisms $W_i : M_{i-1} \rightarrow M_i$.

REMARK 6.3.14. Note that given $s : N(\mathcal{F}_1) \rightarrow \mathcal{U}$ there is a unique map $s' : N(F_1) \rightarrow \mathcal{U}$ such that the diagram above commutes. Uniqueness holds since J is surjective except for identity morphisms, and to check existence we just observe that the same formula

$$s'(M_0 \xrightarrow{W_1} M_1 \rightarrow \cdots \rightarrow M_{k+2}) := [\varphi(M_1^+), \dots, \varphi(M_{k+1}^+)]$$

for some orientation preserving diffeomorphism $\varphi : W_1 \cup_{M_1} \cdots \cup_{M_{k+1}} W_{k+1}$ is still well-defined and independent of φ . In fact, one can check that $\pi_0 N_n(F_1) \rightarrow \mathcal{U}_n$ is a bijection for all n , and so all discrete cocycles on $N(F_1)$ have to come from \mathcal{U} . This is the crucial advantage of using \mathcal{U} : it is much easier to construct a simplicial map to \mathcal{U} , then a simplicial map $N(F_1) \rightarrow N\Lambda$. In fact, any such map has to factor through \mathcal{U} because $N\Lambda$ is level-wise discrete and there seem to be no interesting maps $\mathcal{U} \rightarrow N\Lambda$.

We now define the cocycles on \mathcal{U} .

Definition 6.3.15. The sign of a $(2k+1)$ -tuple of disjoint points $a_0, \dots, a_{2k} \in S^1$ is defined as the sign of any permutation σ of $\{0, \dots, 2k\}$ such that the sequence $(a_{\sigma(0)}, \dots, a_{\sigma(2k)})$ is in cyclic order. This is well-defined because cyclic permutations on a set with $(2k+1)$ elements have sign $+1$.

We extend this definition to disjoint finite subsets $A_0, \dots, A_{2k} \subset S^1$ by averaging:

$$\text{sign}_{2k}(A_0, \dots, A_{2k}) := \frac{1}{\prod_{i=0}^{2k} |A_i|} \sum_{a_0 \in A_0} \cdots \sum_{a_{2k} \in A_{2k}} \text{sign}(a_0, \dots, a_{2k}) \in \mathbb{Q}.$$

We will also need a reduced version of the averaged sign, where some summands are omitted:

Definition 6.3.16. Given (A_0, \dots, A_l) as above and $a_i \in A_i$ and $a_j \in A_j$, let $I \subset S^1$ be the positively oriented arc that starts at $a_{\min\{i,j\}}$ and ends at $a_{\max\{i,j\}}$. We say a_i and a_j are neighbours if $A_k \cap I$ has at most one element for all k .

We say an $(l+1)$ -tuple (a_0, \dots, a_l) contains neighbours if there are $i \neq j$ such that a_i and a_j are neighbours. Using this the reduced sign is defined as:

$$\overline{\text{sign}}_{2k}(A_0, \dots, A_{2k}) := \frac{1}{\prod_{i=0}^{2k} |A_i|} \sum_{\substack{(a_0, \dots, a_{2k}) \in \prod_i A_i \\ \text{contains no neighbours}}} \text{sign}(a_0, \dots, a_{2k}) \in \mathbb{Q}.$$

REMARK 6.3.17. Note that, given (A_0, \dots, A_n) as above the notion of being neighbours induces an equivalence relation on the disjoint union $\coprod_{i=0}^n A_i$.

Lemma 6.3.18. For all $k \geq 1$ the maps

$$\text{sign}_{2k} : \mathcal{U}_{2k} \rightarrow \mathbb{Q} \quad \text{and} \quad \overline{\text{sign}}_{2k} : \mathcal{U}_{2k} \rightarrow \mathbb{Q}$$

are well-defined $2k$ -cocycles.

Proof. Observe that $\text{sign}_{2k}(A_0, \dots, A_{2k})$ and $\overline{\text{sign}}_{2k}(A_0, \dots, A_{2k})$ do not change if we act on (A_0, \dots, A_{2k}) by an orientation preserving diffeomorphism, as they only depend on the relative cyclic ordering of the elements of $\coprod_{i=0}^{2k} A_i$. Therefore sign_{2k} and $\overline{\text{sign}}_{2k}$ are well-defined cochains on \mathcal{U} .

The rest of the proof will be concerned with proving that they satisfy the cocycle condition. First we will show that for any $(2k+2)$ -tuple (a_0, \dots, a_{2k+1}) of disjoint points in S^1 we have:

$$\sum_{i=0}^{2k+1} (-1)^i \text{sign}(a_0, \dots, \widehat{a_i} \dots, a_{2k+1}) = 0.$$

Indeed, if we exchange two consecutive a_j on the left-hand side, then this changes the sign of the overall sum. We therefore have, for any permutation $\sigma \in \Sigma_{2k+2}$:

$$\sum_{i=0}^{2k+1} (-1)^i \text{sign}(a_0, \dots, \widehat{a_i} \dots, a_{2k+1}) = \text{sign}(\sigma) \sum_{i=0}^{2k+1} (-1)^i \text{sign}(a_{\sigma(0)}, \dots, \widehat{a_{\sigma(i)}} \dots, a_{\sigma(2k+1)}).$$

We can choose σ such that the resulting tuple $(a_{\sigma(0)}, \dots, a_{\sigma(2k+1)})$ is in cyclic order, in which case it follows that

$$\text{sign}(\sigma) \sum_{i=0}^{2k+1} (-1)^i \text{sign}(a_{\sigma(0)}, \dots, \widehat{a_{\sigma(i)}} \dots, a_{\sigma(2k+1)}) = \text{sign}(\sigma) \sum_{i=0}^{2k+1} (-1)^i 1 = 0.$$

Let now $A = (A_0, \dots, A_{2k+1})$ represent a $(2k+1)$ -simplex in \mathcal{U} . We need to show that $\text{sign}_{2k}(\partial A) = 0$. Spelling out the definition we have

$$\begin{aligned} \text{sign}_{2k}(\partial A) &= \sum_{i=0}^{2k+1} (-1)^i \text{sign}_{2k}(A_0, \dots, \widehat{A_i} \dots, A_{2k+1}) \\ &= \sum_{i=0}^{2k+1} (-1)^i \frac{1}{\prod_{j=0, j \neq i}^{2k+1} |A_j|} \sum_{a_0 \in A_0} \dots \sum_{a_i \in A_i} \dots \sum_{a_{2k} \in A_{2k}} \text{sign}(a_0, \dots, a_{2k}) \\ &= \sum_{i=0}^{2k+1} (-1)^i \frac{1}{\prod_{j=0}^{2k+1} |A_j|} \sum_{a_0 \in A_0} \dots \sum_{a_{2k+1} \in A_{2k+1}} \text{sign}(a_0, \dots, \widehat{a_i} \dots, a_{2k+1}). \end{aligned}$$

In the last step we introduced a sum $\sum_{a_i \in A_i}$ even though the variable a_i is not used in the summand. This amounts to multiplication by $|A_i|$, cancelling the $\frac{1}{|A_i|}$ that was introduced before the sum. We can now rearrange the sum to

$$\text{sign}_{2k}(\partial A) = \frac{1}{\prod_{j=0}^{2k+1} |A_j|} \sum_{a_0 \in A_0} \dots \sum_{a_{2k+1} \in A_{2k+1}} \sum_{i=0}^{2k+1} (-1)^i \text{sign}(a_0, \dots, \widehat{a_i} \dots, a_{2k+1}).$$

By the first part of the proof the innermost sum is 0 and hence $\text{sign}_{2k}(\partial A) = 0$ as claimed.

It remains to check that the reduced sign $\overline{\text{sign}}_{2k}$ is a cocycle, too. We can attempt to run the same argument, leading us to the expression

$$\overline{\text{sign}}_{2k}(\partial A) = \sum_{i=0}^{2k+1} (-1)^i \frac{1}{\prod_{j=0}^{2k+1} |A_j|} \sum_{\substack{(a_0, \dots, a_{2k+1}) \in \prod_j A_j \\ \text{no neighbours except for } a_i}} \text{sign}(a_0, \dots, \widehat{a}_i, \dots, a_{2k+1}).$$

The problem with this is that we are also summing over the $(2k+2)$ -tuples (a_0, \dots, a_{2k+1}) where a_i is a neighbour of one of the other a_j . By the argument for sign_{2k} all other terms sum up to 0 and so we are left with:

$$\overline{\text{sign}}_{2k}(\partial A) = \frac{1}{\prod_{j=0}^{2k+1} |A_j|} \sum_{i=0}^{2k+1} (-1)^i \sum_{\substack{j \neq i \\ (a_0, \dots, a_{2k+1}) \in \prod_j A_j \\ a_i \text{ and } a_j \text{ neighbours}}} \text{sign}(a_0, \dots, \widehat{a}_i, \dots, a_{2k+1}). \quad (6.1)$$

Note that the a_j is uniquely determined since the tuple $(a_0, \dots, \widehat{a}_i, \dots, a_{2k+1})$ is not allowed to contain neighbours and being neighboured is an equivalence relation.

The lemma will now follow from the observation that whenever (a_i, a_j) is the unique tuple of neighbours in (a_0, \dots, a_{2k+1}) then

$$\text{sign}(a_0, \dots, \widehat{a}_i, \dots, a_{2k+1}) = (-1)^{j-i-1} \text{sign}(a_0, \dots, \widehat{a}_j, \dots, a_{2k+1}). \quad (6.2)$$

Indeed, once we establish this, one can see that every term in equation 6.1 appears exactly twice, with opposite sign. We only have to check equation 6.2 in the case $i < j$. Both expressions only depend on the cyclic ordering of the set $\{a_0, \dots, a_{2k+1}\} \subset S^1$. If we, in the left-hand side of the equation, move the entry a_j to the left by $(j-i-1)$, then this cancels the sign $(-1)^{j-i-1}$ on the right-hand side. Since we assumed that a_i and a_j are neighbours and that they are the only neighbours, there cannot be any a_l between a_i and a_j on S^1 . Therefore we have

$$\text{sign}(a_0, \dots, a_{i-1}, a_j, a_{i+1}, \dots, \widehat{a}_j, \dots, a_{2k+1}) = \text{sign}(a_0, \dots, a_{i-1}, a_i, a_{i+1}, \dots, \widehat{a}_j, \dots, a_{2k+1}).$$

This proves equation 6.2 and hence completes the proof. \square

Proposition 6.3.19. *For all $k \geq 1$ the cocycle*

$$\beta_k := \frac{(-1)^k k!}{(2k)!} \cdot s^* \overline{\text{sign}}_{2k}$$

represents, possibly up to a factor of $(-1)^k$, the cohomology class of $(c_1)^k \in H^(B\mathcal{F}_1; \mathbb{Q}) \cong \mathbb{Q}[c_1]$.*

Proof. Consider the subcategory $\Lambda^{inj} \subset \Lambda$ where we only allow morphisms represented by maps $\mathbb{Z} \rightarrow \mathbb{Z}$ that are injective. In [Igu04] Igusa provides combinatorial formulas for rational cocycles

$c_{\mathbb{Z}}^k$ that represent the powers of the first Chern class on $B(\Lambda^{inj}) \simeq \mathbb{C}\mathbb{P}^\infty$. He denotes Λ^{inj} by \mathcal{Z} . We will consider the composite

$$q : N(\Lambda^{inj}) \hookrightarrow N\Lambda \xrightarrow{N(H)} N\mathcal{F}_1 \xrightarrow{s} \mathcal{U}$$

and we will show that the pullback cocycles

$$(H^* \beta_k)_{|\Lambda^{inj}} = \frac{(-1)^k k!}{(2k)!} \cdot q^* \overline{\text{sign}}_{2k}$$

agree with the cocycle $c_{\mathbb{Z}}^k$ of in [Igu04, (1) on page 478]. Then the claim will follow if we can show that the composite $\Lambda^{inj} \rightarrow \Lambda \rightarrow \mathcal{F}_1$ induces an equivalence on classifying spaces. This works because every self-equivalence of $\mathbb{C}\mathbb{P}^\infty$ induces either the identity or $(c_1)^k \mapsto (-c_1)^k$ on cohomology.

We saw in lemma 6.3.11 that $\Lambda \rightarrow \mathcal{F}_1$ is an equivalence of categories and so we only need to show that $\Lambda^{inj} \hookrightarrow \Lambda$ is an equivalence on classifying spaces. For this, let Λ_∞ be the paracyclic category, which is defined just like Λ except that we do not quotient by the equivalence relation on hom sets. There is a free action of the simplicial abelian group $N\mathbb{Z}$ on $N\Lambda_\infty$ with quotient $N\Lambda$. In [NS18, Theorem B.3] the authors show that Λ_∞ has a contractible classifying space and conclude that $B\Lambda \simeq \mathbb{C}\mathbb{P}^\infty$. The same proof applies to show that Λ_∞^{inj} has a contractible classifying space and that hence $B\Lambda^{inj} \simeq \mathbb{C}\mathbb{P}^\infty$. Moreover, since the map $B\Lambda_\infty^{inj} \rightarrow B\Lambda_\infty$ is $B\mathbb{Z}$ -equivariant, we see that $B\Lambda^{inj} \rightarrow B\Lambda$ is indeed an equivalence.

Note that the category Λ^{inj} is easier to work with than Λ because the functor $F : \Lambda^{inj} \rightarrow \text{Set}$ that sends n to \mathbb{Z}/n and $[f] : n \rightarrow m$ to the induced map $[f] : \mathbb{Z}/n \rightarrow \mathbb{Z}/m$ is faithful. In other words, for an injective map between two cyclic sets being cyclic is a *property*, whereas in general it is a structure.⁴

The remainder of the proof is concerned with showing that the pullback cocycle $(H^* \beta_k)_{|\Lambda^{inj}}$ agrees with Igusa's cocycle $c_{\mathbb{Z}}^k$. Using our definition of the sign cocycle (definition 6.3.15) we can rewrite $c_{\mathbb{Z}}^k$ as

$$c_{\mathbb{Z}}^k(n_0 \xrightarrow{f_1} n_1 \rightarrow \dots \xrightarrow{f_{2k}} n_{2k}) := \frac{(-1)^k k!}{(2k)!} \cdot \frac{1}{\prod_{i=0}^{2k} |A_i|} \sum_{\substack{(a_0, \dots, a_{2k}) \in \prod_{i=0}^{2k} A_i \\ a_i \neq a_j \text{ for all } i \neq j}} \text{sign}_{2k}(a_0, \dots, a_{2k})$$

where A_i is the image of the composite map $(f_n \circ \dots \circ f_l) : \mathbb{Z}/n_l \rightarrow \mathbb{Z}/n_{2k}$. Here we think of \mathbb{Z}/n_{2k} as a subset S^1 in the usual way.

⁴The reason that a general morphism in Λ has more structure than just a map $f : \mathbb{Z}/n \rightarrow \mathbb{Z}/m$ is because it comes with total orderings of the fibers $f^{-1}([i])$ for all $[i] \in \mathbb{Z}/m$. These total orderings can be recovered from the cyclic ordering if $f(\mathbb{Z}/n) \subset \mathbb{Z}/m$ has more than one element. If, on the other hand, we fix some $f : \mathbb{Z}/n \rightarrow \mathbb{Z}/m$ with $f([x]) = [0]$ for all x , then there are n different cyclic maps $f_{(i)} : n \rightarrow m$ that induce f .

We would like to show that this is equal to (a multiple of) the pullback cocycle $q^*\overline{\text{sign}}_{2k}$. Indeed, spelling out the definition of the map $q = s \circ N(H) : N\Lambda^{inj} \rightarrow \mathcal{U}$, we see that in the notation of lemma 6.3.11

$$q^*\overline{\text{sign}}_{2k}(n_0 \xrightarrow{f_1} n_1 \rightarrow \dots \xrightarrow{f_{2k}} n_{2k}) = \frac{1}{\prod_{i=0}^{2k} n_i} \overline{\text{sign}}_{2k}(\varphi^{-1}(M(n_0)^+), \dots, \varphi^{-1}(M(n_{2k})^+))$$

Here the $M(n_i)^+$ are subsets of the 1-manifold

$$X := W(n_0) \cup_{M(n_0)} X_1 \cup_{M(n_1)} \dots \cup_{M(n_{2k})} X_{2k} \cup_{M(n_{2k})} V(n_{2k}) \stackrel{\varphi}{\cong} S^1$$

where $X_i : M(n_{i-1}) \rightarrow M(n_i)$ is the image of the morphism $f_i : n_{i-1} \rightarrow n_i$ under H . The map from $M(n_i)^+ \cong \mathbb{Z}/n_i$ to $M(n_{2k})^+ \cong \mathbb{Z}/n_{2k}$ sends a point $x \in M(n_i)^+$ to the unique $y \in M(n_{2k})^+$ such that there is a positively oriented arc $\iota : [0, 1] \hookrightarrow X$ with $\iota(0) = x$, $\iota(1) = y$, and $\iota^{-1}(M(n_{2k})^+) = \{y\}$. Since the maps $f_i : n_{i-1} \rightarrow n_i$ are all injective, this arc automatically satisfies $\iota^{-1}(M(n_i)^+) = \{x\}$ as well. (Every element of $\iota^{-1}(M(n_i)^+)$ would be mapped to y by the map $[f_{2k} \circ \dots \circ f_{i+1}] : \mathbb{Z}/n_i \rightarrow \mathbb{Z}/n_{2k}$, but by injectivity x is the only such element.)

Let $x' \in M(n_j)^+$ with $i < j$ then x and x' are neighboured in the sense of definition 6.3.16 with respect to the tuple $(\varphi^{-1}(M(n_0)^+), \dots, \varphi^{-1}(M(n_{2k})^+))$, if and only if x' lies on the arc $\iota : [0, 1] \hookrightarrow X$ that we considered above. In other words, x and x' are neighboured if and only if their images in $M(n_{2k})^+$ agree. In formulas this means

$$\sum_{\substack{(a_0, \dots, a_{2k}) \in \prod_{i=0}^{2k} A_i \\ a_i \neq a_j \text{ for all } i \neq j}} \text{sign}_{2k}(a_0, \dots, a_{2k}) = \sum_{\substack{(a_0, \dots, a_{2k}) \in \prod_{i=0}^{2k} \varphi^{-1}(M(n_i)^+) \\ \text{no neighbours}}} \text{sign}_{2k}(a_0, \dots, a_{2k})$$

where we let A_i denote the image of the map $M(n_i)^+ \rightarrow M(n_{2k})^+ \hookrightarrow S^1$ described above.

After multiplying with the correct coefficient this implies

$$c_{\mathbb{Z}}^k(n_0 \xrightarrow{f_1} \dots \xrightarrow{f_{2k}} n_{2k}) = q^*\overline{\text{sign}}_{2k}(n_0 \xrightarrow{f_1} n_1 \rightarrow \dots \xrightarrow{f_{2k}} n_{2k})$$

just claimed. □

REMARK 6.3.20. *The proposition proves that β_k in fact represents the integral cohomology class $(c_1)^k \in H^{2k}(B\mathcal{F}_1; \mathbb{Z})$, at least up to a sign. This implies that there is an integral cocycle that is rationally cohomologous to β_k . The reason we prefer to work with the rational cocycle β_k instead is that it has the useful property of being conjugation invariant: it satisfies for all i and isomorphisms $\sigma : n_i \cong n_i$ that*

$$\beta(n_0 \xrightarrow{f_1} \dots \xrightarrow{f_{2k}} n_{2k}) = \beta(n_0 \xrightarrow{f_1} \dots n_{i-1} \xrightarrow{\sigma \circ f_i} n_i \xrightarrow{f_i \circ \sigma^{-1}} n_{i+1} \dots \xrightarrow{f_{2k}} n_{2k}).$$

This property is extremely convenient when extending β along the zig-zag $N(F_1) \leftarrow N(F_1^\delta) \rightarrow N(\mathcal{F}_1)$, and one should not expect to be able find an integral cocycle with this property. Indeed, as noted in remark 6.3.14 any cocycle on $N(F_1)$ has to be come from a cocycle on \mathcal{U} and hence is necessarily conjugation invariant.

REMARK 6.3.21. It seems likely that the two cocycles sign_{2k} and $\overline{\text{sign}}_{2k}$ are in fact cohomologous for all $k \geq 1$. If this were true we could use sign_{2k} in what follows, yielding a description of the cocycles representing $\overline{\kappa}_k$ on $h\mathcal{C}_1$ without having to introduce the notion of a neighbour. It is however hard to 'by hand' guess a $(2k+1)$ -cochain on \mathcal{U} whose boundary is $\text{sign}_{2k} - \overline{\text{sign}}_{2k}$. Alternatively, one could try to find a $2k$ -cycle on Λ on which both sign_{2k} and $\overline{\text{sign}}_{2k}$ evaluate to the same non-zero number.

6.3.3 Cocycles on the cobordism category

Definition 6.3.22. For all $k \geq 0$ define a $(2k+2)$ -cochain on the simplicial space Cut_1 by the formula

$$\gamma_k(M_0 \xrightarrow{W_1} M_1 \xrightarrow{W_2} \dots \xrightarrow{W_{2k+2}} M_{2k+2}) := \frac{(-1)^k k!}{(2k)!} \sum_{[\iota: S^1 \hookrightarrow W]} \overline{\text{sign}}_{2k}(\iota^{-1}(M_1^+), \dots, \iota^{-1}(M_{2k+2}^+)).$$

Here we write W for the composition $W_1 \cup_{M_1} \dots \cup_{M_{2k+1}} W_{2k+2}$, and the sum runs over isotopy classes of oriented embeddings $\iota: S^1 \hookrightarrow W$ such that⁵ $\iota(S^1)$ intersects M_i for all $1 \leq i \leq 2k+1$.

The map $\gamma_k: (\text{Cut}_1)_k \rightarrow \mathbb{Q}$ is continuous and hence also well-defined on $\pi_0(\text{Cut}_1)_k$. We may therefore pull it back along the compatible maps

$$\begin{array}{ccccccc} N(h\mathcal{C}_1) & \longrightarrow & N(h\mathcal{C}_1^{\text{red}}) & \longleftarrow & N(\delta\mathcal{C}_1^{\text{red}}) & \longrightarrow & N(\mathcal{C}_1^{\text{red}}) & \longrightarrow & \text{Cut}_1 \\ & & & & & & & & \downarrow \\ & & & & & & & & \pi_0 \text{Cut}_1 \end{array}$$

to obtain cocycles on $\mathcal{C}_1^{\text{red}}$, $h\mathcal{C}_1^{\text{red}}$, and $h\mathcal{C}_1$, which we will also denote by γ_k .

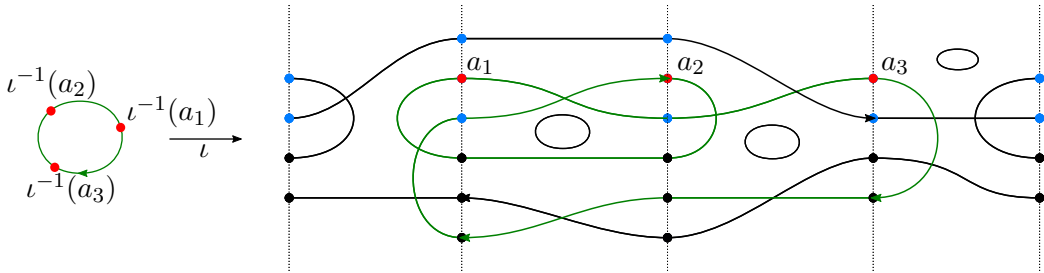


Figure 6.5: One of the summands in the definition of the 4-cocycle γ_1 . The figure depicts a choice of oriented embedding $\iota: S^1 \hookrightarrow W = W_1 \cup W_2 \cup W_3 \cup W_4$ and positively oriented points $a_i \in M_i^+ \cap \iota(S^1)$. In the case shown $\text{sign}(\iota^{-1}(a_1), \iota^{-1}(a_2), \iota^{-1}(a_3)) = -1$, because the three points are not in cyclic order on S^1 . In fact, the triple (a_1, a_2, a_3) depicted is the only one with no neighbours, and hence $\overline{\text{sign}}_2(\iota^{-1}(M_1^+), \iota^{-1}(M_2^+), \iota^{-1}(M_3^+)) = \frac{-1}{4}$. Moreover, the embedding shown is the only one that is allowed, and hence γ_1 evaluates on this 4-simplex as $\frac{(-1)^1 1!}{2!} \frac{-1}{4} = \frac{1}{4}$.

⁵In principle we could also omit the condition that $\iota(S^1) \cap M_i \neq \emptyset$, and just use the convention that $\overline{\text{sign}}_{2k}(A_1, \dots, A_{2k+1}) = 0$ whenever any of the A_i is empty.

When $k = 0$ the reduced sign $\overline{\text{sign}}_0(\iota^{-1}(M_1^+))$ is always equal to 1 and therefore $\bar{\kappa}_0$ simply counts the number of isotopy classes of oriented embeddings $\iota : S^1 \hookrightarrow W_0 \cup_{M_1} W_1$. Hence γ_0 is equal, as a cocycle on $h\mathcal{C}_1^{\text{red}}$, to the 2-cocycle α that we constructed in section 6.1.

We will now prove our second main theorem, which states that the cocycles γ_k indeed represent the $\bar{\kappa}$ -classes on $B(h\mathcal{C}_1)$.

Theorem 6.3.23. *For $k \geq 0$ the $(2k + 2)$ -cochains γ_k defined above are in fact cocycles and, the cohomology class $[-\gamma_k]$ is, possibly up to a sign $(-1)^k$, the generator $\bar{\kappa}_k$ in*

$$H^*(\|\text{Cut}_1\|; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_0, \bar{\kappa}_1, \bar{\kappa}_2, \dots].$$

As a consequence the same formula also yields well defined cocycles on $B(h\mathcal{C}_1^{\text{red}})$ and $B(h\mathcal{C}_1)$ where they still represent the classes $\bar{\kappa}_k$ in:

$$H^*(B(h\mathcal{C}_1^{\text{red}})_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_0, \bar{\kappa}_1, \bar{\kappa}_2, \dots], \quad \text{and} \quad H^*(B(h\mathcal{C}_1)_0; \mathbb{Q}) \cong \mathbb{Q}[\bar{\kappa}_1, \bar{\kappa}_2, \dots].$$

Before we prove this theorem, we first recall a simple lemma about primitive cocycles on H -spaces.

Lemma 6.3.24. *Let X be a simplicial set and $\mu : X \times X \rightarrow X$ a simplicial map that induces an associative H -space structure on $\|X\|$. Then every cocycle $\beta \in C^n(X)$ satisfying $\beta(s_i x) = 0$ and $\beta(\mu(x, y)) = \beta(x) + \beta(y)$ for all $x, y \in X_n$ and $i = 0, \dots, n$ represents a primitive element of $H^*(\|X\|)$.*

Proof. This is a direct consequence of the definition of the coproduct as the composite of the diagonal $\Delta^* : C^*(X) \rightarrow C^*(X \times X)$ with the dual of the Eilenberg-Zilber map $C_*(X) \otimes C_*(X) \rightarrow C_*(X \times X)$. \square

Proof of theorem 6.3.23. In lemma 6.3.18 we saw that $\overline{\text{sign}}_{2k}$ is a cocycle on \mathcal{U} and using this one can check the cochains γ_k are cocycles on Cut_1 .

We begin by showing that $[\gamma_k]$ is a rational multiple of $\bar{\kappa}_k$. To do so it will suffice to show that it is primitive with respect to Hopf algebra structure. By construction the cocycles γ_k are additive under disjoint union and hence satisfy the conditions of lemma 6.3.24 with respect to the multiplication $\Pi : \text{Cut}_1 \times \text{Cut}_1 \rightarrow \text{Cut}_1$. Therefore the classes $[\gamma_k] \in H^{2k+2}(\text{Cut}_1)$ are indeed primitive. By lemma 6.2.16 the infinite loop space $\|\text{Cut}_1\|$ is freely generated by $\|\text{Cut}_1^{\text{con}}\|$ and therefore the pullback along the inclusion induces an isomorphism

$$\text{Prim}(H^*(\|\text{Cut}_1\|; \mathbb{Q})) \cong H^*(\|\text{Cut}_1^{\text{con}}\|; \mathbb{Q}) = \mathbb{Q}\langle \bar{\kappa}_1, \bar{\kappa}_2, \dots \rangle$$

In order to prove the theorem it will hence suffice to show that the restriction of $[\gamma_k]$ to $\text{Cut}_1^{\text{con}} \subset \text{Cut}_1$ represents the class $-\bar{\kappa}_k$.

In lemma 6.2.18 and section 6.3.1 we established weak equivalences

$$\|\text{Cut}_1^{\text{con}}\| \simeq \Sigma^2 B(F_1) \quad \text{and} \quad B(F_1) \simeq B(F_1^\delta) \simeq B(\mathcal{F}_1) \simeq B(\Lambda) \simeq \mathbb{C}\mathbb{P}^\infty.$$

Each of the spaces on the right admits a map to the simplicial set \mathcal{U} from definition 6.3.12 and by proposition 6.3.19 the Chern class $(c_1)^k$ is represented by the pullback of the cocycle $\frac{(\pm 1)^k k!}{(2k)!} \overline{\text{sign}}_{2k}$. In fact, using the simplicial double suspension of definition 6.2.19, we have a simplicial map

$$q : \text{Cut}_1^{\text{con}} \rightarrow \Sigma^2 \mathcal{U}, \quad (M_0 \xrightarrow{W_1} M_1 \xrightarrow{W_2} \dots \xrightarrow{W_n} M_n) \mapsto ((\varphi^{-1}(M_{b-a}^+), \dots, \varphi^{-1}(M_b^+)), a, b)$$

where $\varphi : S^1 \cong W_0 \cup_{M_1} \dots \cup_{M_{2k+1}} W_{2k}$ is any orientation preserving diffeomorphism and $0 \leq a \leq b \leq n$ are maximal such that $\varphi(S^1)$ intersects M_{b-a} and M_a non-trivially. The geometric realisation of q is the double suspension of the map $B(F_1) \rightarrow \|\mathcal{U}\|$.

The pullback of (the double suspension of) $\frac{(\pm 1)^k k!}{(2k)!} \overline{\text{sign}}_{2k}$ from $\Sigma^2 \mathcal{U}$ to $\text{Cut}_1^{\text{con}}$ is exactly $(\gamma_k)|_{\text{Cut}_1^{\text{con}}}$. Therefore γ_k indeed represents the kappa class $\bar{\kappa}_k \in H^{2k+2}(\text{Cut}_1)$, possibly up to a sign.

We can't really hope to keep track of all the signs that were introduced by the equivalences we used. Instead, we observe that the relative sign of the $\bar{\kappa}_k$ is always the same (up to the $(-1)^k$ ambiguity from proposition 6.3.19). It therefore suffices to check the sign for one of the $\bar{\kappa}_k$. In lemma 6.1.4 we showed $-\alpha = \bar{\kappa}_0$, and since the 2-cocycles α and γ_0 are equal on $h\mathcal{C}_1^{\text{red}}$ this implies $-\gamma_0 = \bar{\kappa}_0$. In summary, we have $-\gamma_k = (\pm 1)^k \bar{\kappa}$ for a global choice of ± 1 . \square

Part III

The surface category and tropical curves

Chapter 7

Introduction

In the previous part we computed the classifying space $B(\text{Cob}_1)$ of the truncated bordism category in dimension $d = 1$. We showed that it is closely related to $MTSO_2$ and that its cohomology ring is polynomial on shifted kappa classes. This motivates the question:

Question 7.0.1. *What is the homotopy type of $B(\text{Cob}_d)$ for other values of d ?*

In dimensions $d \geq 2$ our strategy to work by comparison with [GMTW09] can no longer be employed. In dimension 1 we used that the functor $\mathcal{C}_1^{\text{red}} \rightarrow \text{Cob}_1^{\text{red}}$ is an equivalence because $\text{Diff}(W \text{ rel } \partial W)$ is contractible for any connected 1-manifold with non-empty boundary. This is of course no longer true for surfaces as they have non-trivial mapping class groups.

The surface category

In the case of the surface category Cob_2 Tillmann [Til96] computed the 1-type of $B(\text{Cob}_2)$ and showed that it can be split off from the rest of the homotopy type:

Theorem ([Til96]). *There is an equivalence of infinite loop spaces $B(\text{Cob}_2) \simeq S^1 \times X$ where X is some simply connected infinite loop space.*

The factor $S^1 \simeq B\mathbb{Z}$ in this theorem can be thought of as accounting for the genus of the surfaces. Seeing as connected surface bordisms between two fixed objects are uniquely determined by their genus, it is plausible to suspect that this is all there is and that X is contractible. It was hence conjectured [JT13, Conjecture 5.3] that the classifying spaces of Cob_2 and several related categories should be 1-types. One of our main theorems will be that this is *almost* true:

Theorem J. *There is an equivalence of infinite loop spaces $B(\mathbb{C}ob_2) \simeq S^1$.*

Here $\mathbb{C}ob_2$ is a $(2, 1)$ -category that we will think of as a refinement of the truncated bordism category in which closed components are counted “properly”. In general it is defined as follows:

Definition 7.0.2. For all $d \geq 0$ the 2-category \mathbb{Cob}_d has as objects closed oriented $(d-1)$ -manifolds and morphisms $W : M \rightarrow N$ are compact oriented bordisms from M to N . A 2-morphism $\alpha : W \Rightarrow W'$ is a bijection $\alpha : \pi_0(W) \cong \pi_0(W')$ such that there exists a diffeomorphism $\varphi : W \cong W'$ that is the identity on the boundary and satisfies $\pi_0(\varphi) = \alpha$.

It is important to note that the diffeomorphism φ is not part of the data of the 2-morphism $\alpha : W \Rightarrow W'$. An alternative construction of \mathbb{Cob}_d defines it as the quotient of the $(2, 1)$ -category $h_2(\text{Bord}_d)$, in which 2-morphisms are isotopy classes of diffeomorphisms $[\varphi]$, by the equivalence relation that identifies two 2-morphisms $[\varphi] \sim [\varphi']$ whenever they induce the same map on connected components: $\pi_0(\varphi) = \pi_0(\varphi')$. Hence this category fits into the following sequence of quotients of the $(\infty, 1)$ -category of bordisms:

$$\text{Bord}_d \longrightarrow h_2(\text{Bord}_d) \longrightarrow \mathbb{Cob}_d \longrightarrow \text{Cob}_d = h(\text{Bord}_d).$$

Note that for any given 1-morphism $W : M \rightarrow N$ in \mathbb{Cob}_d the group of 2-automorphisms $\alpha : W \rightarrow W$ is a product of symmetric groups, corresponding to the permutations of closed components, and in particular it is always finite. We will see that the functor $\mathbb{Cob}_d \rightarrow \text{Cob}_d$ is always a rational equivalence on classifying spaces and for $d = 2$ we show that there is an equivalence of infinite loop spaces:

$$B(\text{Cob}_2) \simeq B(\mathbb{Cob}_2) \times \tau_{\geq 3}Q\left(\bigvee_{g \geq 0} S^2\right).$$

Here $\tau_{\geq 3}Q(\bigvee_{g \geq 0} S^2)$ denotes the 2-connected cover of the free infinite loop space on $\bigvee_{g \geq 0} S^2$. In particular, the mysterious infinite loop space X in Tillmann's theorem can be described as the homotopy fiber of the map $Q(\bigvee_{g \geq 0} S^2) \rightarrow K(\bigoplus_{g \geq 0} \mathbb{Z}, 2)$. It is rationally trivial and can be thought as an "error-term" coming from the fact that in Cob_2 we are not counting the closed components "properly".

Tropical moduli spaces and the surface category without disks

The above result seems to suggest that – as long as we look at the slightly more sophisticated \mathbb{Cob}_2 – the homotopy type of surface category is far less interesting than what we discovered in the one-dimensional case.

In what follows we will see that the homotopical tranquility of $B(\mathbb{Cob}_2) \simeq S^1$ is easily disturbed by rather minor changes to the surface category. A crucial step in the proof of theorem J uses the fact that for any surface bordism $W : M \rightarrow N$ and point $p \in W$ one may decompose $W = W' \cup_{S^1} D^2$ where W' is obtained from W by removing a ball around p . To better understand the inner workings of Cob_2 we will consider a large subcategory where such a decomposition is not allowed.

Definition 7.0.3. *The subcategory $\text{Cob}_2^{\chi \leq 0} \subset \text{Cob}_2$ is defined to contain all objects, but only those morphisms $W : M \rightarrow N$ where every component $V \subset W$ has non-positive Euler characteristic $\chi(V) \leq 0$.*

The definition is formulated in terms of the Euler characteristic as this makes it easy to check that $\text{Cob}_2^{\chi \leq 0}$ is closed under both composition and disjoint union. It is worth noting, however, that the only morphisms of Cob_2 that do not lie in $\text{Cob}_2^{\chi \leq 0}$ are those that contain disks or spheres as components.¹ By passing to this subcategory we uncover the rich homotopical structure that was lurking behind the seemingly simple statement of theorem J:

Theorem K. *There is an equivalence of infinite loop spaces*

$$B(\text{Cob}_2^{\chi \leq 0}) \simeq S^1 \times Q(\Sigma^2 BO_2) \times Q\left(\bigvee_{g \geq 2} \Sigma^2(BJ_g)\right).$$

The categories J_g appear in the work of [CGP16] on tropical moduli spaces.² In fact, the tropical moduli space Δ_g can be defined as the colimit of a certain functor $\Delta^{E(_)} : J_g^{\text{op}} \rightarrow \text{Top}$. This description indicates that BJ_g agrees with Δ_g “up to finite graph automorphisms”. We make this precise by proving the following theorem, which we believe to be of independent interest.

Theorem L. *For any $g \geq 2$ the space BJ_g is simply connected and rationally equivalent to Δ_g .*

The rational homology of Δ_g was computed in [CGP18] in terms of Kontsevich’s commutative graph complex. Following their paper we will use the grading convention where a graph has grading $v - (g + 1)$ for v the number of vertices. (In this grading $H_*(\mathcal{G}_{(g)})$ is concentrated in $* \geq 0$.) Combining their results with ours we obtain:

Corollary M. *The rational homotopy groups of $B(\text{Cob}_2^{\chi \leq 0})$, or equivalently of $B(\text{Cob}_2^{\chi \leq 0})$, are given by:*

$$\pi_*^{\mathbb{Q}} B(\text{Cob}_2^{\chi \leq 0}) \cong \mathbb{Q}\langle \alpha \rangle \oplus \mathbb{Q}\langle \rho_1, \rho_2, \dots \rangle \oplus \bigoplus_{g \geq 2} \Sigma^{2g+1} H_*(\mathcal{G}_{(g)})$$

where $|\alpha| = 1$, $|\rho_i| = 4i + 2$, and $\mathcal{G}_{(g)}$ is the commutative graph complex.

Labelled cospan categories

The techniques used to prove the main theorems stated so far apply in much greater generality than just to Cob_2 and its subcategories. In fact, the main property of Cob_2 that we are using is

¹Note that one might also consider another, even larger subcategory $\text{Cob}_2^{\chi \leq 0} \subset \text{Cob}_2^{\text{no disks}} \subset \text{Cob}_2$ where spheres are allowed. This turns out not to be more interesting as one can split of the number of spheres as a separate factor $\text{Cob}_2^{\text{no disks}} \simeq \text{Cob}_2^{\chi \leq 0} \times \mathbb{N}$ and so $B\text{Cob}_2^{\text{no disks}} \simeq B(\text{Cob}_2^{\chi \leq 0}) \times S^1$.

²In the definition used by [CGP16] the category contains a terminal object, which we remove. See remark 9.1.11 for a comparison.

that we can talk about connected components of objects and morphisms. One way of formalising this is to say that π_0 defines a symmetric monoidal functor:

$$\pi_0 : \text{Cob}_2 \rightarrow \text{Csp}, \quad M \mapsto \pi_0(M), \quad ([W] : M \rightarrow N) \mapsto [\pi_0(M) \rightarrow \pi_0(W) \leftarrow \pi_0(N)]$$

from Cob_2 to the category Csp of cospans in finite sets. This category Csp has as objects finite sets A and as morphisms $A \rightarrow B$ it has isomorphism classes of cospans $[A \rightarrow X \leftarrow B]$. Two cospans are isomorphic if there is a bijection $\sigma : X \cong X'$ compatible with the maps from A and B .

Since connected surfaces are uniquely determined by their genus and number of boundary components, the functor $\pi_0 : \text{Cob}_2 \rightarrow \text{Csp}$ remembers a lot of information. To give a morphism $M \rightarrow N$ in Cob_2 amounts to the same data as giving a cospan $[\pi_0(M) \rightarrow X \leftarrow \pi_0(N)]$ and a labeling $g : X \rightarrow \mathbb{N}$ that encodes the genus of each component. It will be a very useful perspective to think of Cob_2 as a category of cospans labelled by natural numbers³ as indicated in figure 7.1.

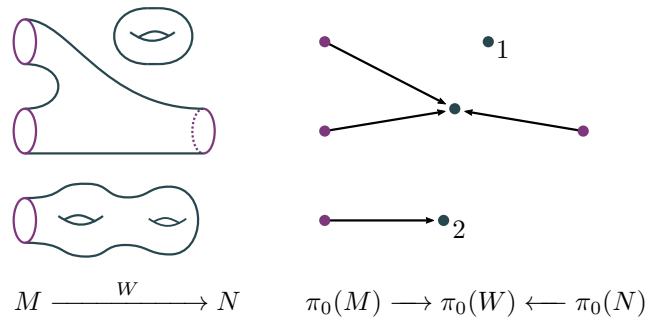


Figure 7.1: A morphism in Cob_2 can be thought of as a cospan of finite sets labelled in \mathbb{N} .

Formalising this leads to the definition of labelled cospan categories. The reader is referred to section 8.1, which serves as an introduction to this chapter from the perspective of labelled cospan categories.

Definition 7.0.4. A labelled cospan category is a symmetric monoidal category \mathcal{C} together with a symmetric monoidal functor $\pi : \mathcal{C} \rightarrow \text{Csp}$ satisfying four axioms, which ensure that every object and morphism in \mathcal{C} uniquely decomposes as the disjoint union of its connected components. (See definition 8.1.4 for details.)

In the study of topologically enriched bordism categories a key theorem is that the inclusion of the positive boundary subcategory $\mathcal{C}_d^{\partial^+} \subset \mathcal{C}_d$ induces an equivalence on classifying spaces. (See [GMTW09, Theorem 6.1] and [GRW14, Theorem 3.1].) Here the positive boundary subcategory contains all objects and all those bordisms $W : M \rightarrow N$ where each component $V \subset W$ intersects

³It is worth pointing out that describing the composition of morphisms from this perspective is a little more complicated than just adding the natural numbers, as gluing along multiple circles at once can increase the total genus. See definition 8.1.13 and lemma 8.1.14.

N non-trivially, i.e. has positive boundary. We can define a positive boundary subcategory $\mathcal{C}^{\partial+} \subset \mathcal{C}$ for any labelled cospan category $(\mathcal{C} \rightarrow \mathbb{C}\text{sp})$ by allowing those morphisms $f : x \rightarrow y$ where the right-hand arrow in the cospan $[\pi(x) \rightarrow \pi(f) \leftarrow \pi(y)]$ is surjective.

For truncated bordism categories the positive boundary category is often much easier to compute. For example we have $B(\text{Cob}_1^{\partial+}) \simeq QS^0$ by [Rap14, Corollary 4.3.2], $B(\text{Cob}_2^{\partial+}) \simeq S^1$ by [Til96], and $B(\text{Cob}_2^{\chi \leq 0, \partial+}) \simeq S^1$ and $B(\mathbb{C}\text{sp}^{\partial+}) \simeq *$ by proposition 8.3.27. However, we cannot generally expect the inclusion of the positive boundary category to be an equivalence. After all $B(\text{Cob}_1)$ is more complicated than QS^0 by theorem E and $B(\text{Cob}_2^{\chi \leq 0})$ is more complicated than S^1 by theorem K.

The key theorem about the classifying spaces of labelled cospan categories will describe the failure of $B(\mathcal{C}^{\partial+}) \rightarrow B(\mathcal{C})$ to be an equivalence. For this theorem to be stated in its most natural form we replace \mathcal{C} by a $(2, 1)$ -category $\mathbb{C}\text{sp}(\mathcal{C})$, in analogy with how we replaced Cob_2 by $\mathbb{C}\text{ob}_2$. This recovers the previous construction as $\mathbb{C}\text{ob}_2 \simeq \mathbb{C}\text{sp}(\text{Cob}_2)$ and just like before it does not change the classifying space up to rational equivalence: $B(\mathbb{C}\text{sp}(\mathcal{C})) \simeq_{\mathbb{Q}} B(\mathcal{C})$.

Theorem N (Decomposition and Surgery Theorem). *Let $(\mathcal{C} \rightarrow \mathbb{C}\text{sp})$ be a labelled cospan category that admits surgery (definition 8.1.42) and assume that $B(\mathcal{C})$ is group-complete. Then there is a fiber sequence of infinite loop spaces:*

$$B(\mathcal{C}^{\partial+}) \longrightarrow B(\mathbb{C}\text{sp}(\mathcal{C})) \longrightarrow Q\left(\bigvee_g \Sigma S(B\mathcal{F}_g(\mathcal{C}))\right).$$

Here the wedge runs over all connected morphisms $g : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$, $\mathcal{F}_g(\mathcal{C})$ denotes the category of non-trivial factorisations ($g : 1_{\mathcal{C}} \rightarrow x \rightarrow 1_{\mathcal{C}}$) as defined in definition 8.1.34, and S denotes the unreduced suspension.

This theorem applies to all truncated cobordism categories Cob_d as well as to $\mathbb{C}\text{sp}$ and various labelled cospan categories that one can construct by labelling cospans in an abelian monoid. The proof of this theorem takes up the majority of this part (section 8.2 and 8.3).

Once theorem N is established most other computations amount to determining the homotopy types of the factorisation categories $\mathcal{F}_g(\mathcal{C})$. In the case of $\mathcal{C} = \mathbb{C}\text{sp}$ we show that $B\mathcal{F}_*(\mathbb{C}\text{sp})$ is contractible and a similar argument applies to $B\mathcal{F}_{[W]}(\text{Cob}_d)$ for any closed d -manifold W . We therefore have:

Theorem O. *The classifying space of the $(2, 1)$ -category of cospans $\mathbb{C}\text{sp}$ is contractible and the classifying space of its homotopy category is $B(\mathbb{C}\text{sp}) \simeq \tau_{\geq 3}Q(S^2)$.*

Theorem P. *For any dimension $d \geq 2$ the classifying space of $\mathbb{C}\text{ob}_d$ is equivalent to the classifying space of its positive boundary subcategory: $B(\text{Cob}_d^{\partial+}) \simeq B(\mathbb{C}\text{ob}_d)$.*

The classifying spaces of factorisation categories are not always trivial, though. Our work in section 6.3 implies that $\mathcal{F}_{S^1}(\text{Cob}_1)$ is equivalent to Connes' cyclic category Λ . Even though Cob_1 does not admit surgery in the sense of definition 8.1.42, the conclusion of Theorem N still seems to apply and the map $B\mathbb{C}\text{sp}(\text{Cob}_1) \rightarrow Q(\Sigma S(B\mathcal{F}_{S^1}(\text{Cob}_1)))$ corresponds to the continuation of the reduction fiber sequence that we constructed in section 6.2.

The subcategory $\text{Cob}_2^{X \leq 0} \subset \text{Cob}_2$ is another example where the arguments for the contractibility of $\mathcal{F}_g(\text{Cob}_2)$ fail as we cannot use the disk morphism $D^2 : \emptyset \rightarrow S^1$. In chapter 9 we prove theorem K by showing that there is a zig-zag of functors inducing an equivalence of classifying spaces $B(\mathcal{F}_g(\text{Cob}_2^{X \leq 0})) \simeq B(J_g)$ for all $g \geq 2$. This in particular implies that the classifying space of $\mathcal{F}_g(\text{Cob}_2^{X \leq 0})$ is rationally equivalent to the tropical moduli space Δ_g .

Application: special commutative Frobenius algebras

We now present an interpretation of the aforementioned results from the perspective of higher category theory. This section will be self-contained and is not discussed further in the main text.

A classical theorem for topological field theories states that a symmetric monoidal functor $\text{Cob}_2 \rightarrow \mathcal{C}$ amounts to the same data as a commutative Frobenius algebra object in \mathcal{C} : i.e. an object $A = F(S^1)$ together with a multiplication $\mu : A \otimes A \rightarrow A$, comultiplication $\Delta : A \rightarrow A \otimes A$, unit $\nu : 1_{\mathcal{C}} \rightarrow A$, and counit $\varepsilon : A \rightarrow 1_{\mathcal{C}}$ satisfying certain axioms. From this it also follows that a symmetric monoidal functor $\mathbb{C}\text{sp} \rightarrow \mathcal{C}$ amounts to the same data as a *special* commutative Frobenius algebra object in \mathcal{C} , where one requires that $\mu \circ \Delta = \text{id}_A$. This is summarised by saying that Cob_2 is the free symmetric monoidal category on a commutative Frobenius algebra and $\mathbb{C}\text{sp}$ is the free symmetric monoidal category on a special commutative Frobenius algebra.

Motivated by this, one could define (special) commutative Frobenius algebras in a symmetric monoidal ∞ -category \mathcal{C} as symmetric monoidal ∞ -functors $\text{Cob}_2 \rightarrow \mathcal{C}$ and $\mathbb{C}\text{sp} \rightarrow \mathcal{C}$. However, we will argue below that such a definition would require too much higher coherence data, as it is usually difficult to map from a 1-category into an ∞ -category. The computation of the classifying spaces gives us a lower bound for the amount of coherence data, as we shall explain below. Based on how the classifying spaces of the refinements $\mathbb{C}\text{ob}_2 \rightarrow \text{Cob}_2$ and $\mathbb{C}\text{sp} \rightarrow \mathbb{C}\text{sp}$ are better behaved, we suggest the following definition:

Definition 7.0.5. *Let \mathcal{C} be a symmetric monoidal ∞ -category. A commutative Frobenius algebra in \mathcal{C} is a symmetric monoidal ∞ -functor $\text{Cob}_2 \rightarrow \mathcal{C}$. A special commutative Frobenius algebra in \mathcal{C} is a symmetric monoidal ∞ -functor $\mathbb{C}\text{sp} \rightarrow \mathcal{C}$.*

Our computations of the classifying spaces allow us to classify (special) commutative Frobenius algebras with invertible structure maps:

Corollary Q. *All special Frobenius algebra structures on the unit object $1_{\mathcal{C}}$ of a symmetric monoidal ∞ -category are canonically equivalent. In other words, every symmetric monoidal ∞ -functor $F : \mathbb{C}\text{sp} \rightarrow \mathcal{C}$ such that $F(*)$ is equivalent to $1_{\mathcal{C}}$ is canonically isomorphic to the constant functor at $1_{\mathcal{C}}$.*

Proof. We first show that the functor $h(F) : h(\mathbb{C}\text{sp}) \rightarrow h(\mathcal{C})$ inverts all morphisms. $h(F)$ induces both a commutative algebra structure and a cocommutative coalgebra structure on $h(F)(*) \in h\mathcal{C}$. Any (co)algebra structure on the unit of a symmetric monoidal 1-category has an invertible (co)unit and (co)multiplication morphism. All other morphisms of $h(\mathbb{C}\text{sp})$ can be generated from these morphisms by composing and taking disjoint unions. Hence $h(F)$ sends all morphisms to isomorphisms in $h(\mathcal{C})$ and therefore F lands in the maximal subgroupoid \mathcal{C}^{\simeq} .

Since \mathcal{C}^{\simeq} is a symmetric monoidal groupoid, all maps from $\mathbb{C}\text{sp}$ into it factor through the classifying space $\mathbb{C}\text{sp}[\mathbb{C}\text{sp}^{-1}] \simeq B(\mathbb{C}\text{sp})$. We showed in theorem O that this symmetric monoidal ∞ -groupoid is contractible and since F factors through $B(\mathbb{C}\text{sp})$ it is necessarily trivial. \square

In a similar vein one can show that commutative Frobenius structures on $1_{\mathcal{C}}$ are given by E_{∞} -maps $S^1 \rightarrow \mathcal{C}$, or equivalently by E_{∞} -maps $\mathbb{N} \rightarrow \text{Aut}_{\mathcal{C}}(1_{\mathcal{C}})$. This generalises the theorem [Til96, Theorem 12] that invertible commutative Frobenius algebras in a 1-category \mathcal{D} are classified by single invertible scalar $\lambda : 1_{\mathcal{D}} \rightarrow 1_{\mathcal{D}}$. However, it is worth noting that to give an E_{∞} -map $\mathbb{N} \rightarrow \text{Aut}_{\mathcal{C}}(1_{\mathcal{C}})$ one has to specify an element $\text{Aut}_{\mathcal{C}}(1_{\mathcal{C}})$ plus additional coherence data because \mathbb{N} is not the free E_{∞} algebra on one generator. This suggests that a better definition of commutative Frobenius algebra could be given by using the graph cobordism category of [Gal11], the homotopy category of which agrees with Cob_2 . (Note that we are not considering functors $F : \text{Bord}_2 \rightarrow \mathcal{C}$ here as $F(S^1)$ is only a E_2 -algebra and not an E_{∞} -algebra.)

To see why we choose $\mathbb{C}\text{sp}$ over Csp in definition 7.0.5, note that corollary Q fails for Csp : one can construct many non-trivial symmetric monoidal ∞ -functors $\text{Csp} \rightarrow \mathcal{C}$ that become trivial when precomposed with $\mathbb{C}\text{sp} \rightarrow \text{Csp}$. One can in fact make precise the idea that giving a symmetric monoidal ∞ -functor out of Csp or Cob_2 requires an infinite amount of coherence data. If a symmetric monoidal ∞ -category \mathcal{C} is finitely presented⁴ then it in particular has to be a compact object in the ∞ -category $\text{Cat}_{\infty}^{\otimes}$ of symmetric monoidal ∞ -categories, i.e. $\text{Hom}_{\text{Cat}_{\infty}^{\otimes}}(\mathcal{C}, _) : \text{Cat}_{\infty}^{\otimes} \rightarrow \mathcal{S}$ commutes with filtered colimits. Using the fact that S^1 is not a compact object in the ∞ -category of infinite loop spaces, one can show that none of the categories Csp , Cob_2 , $\text{Cob}_2^{\chi \leq 0}$, $\mathbb{C}\text{ob}_2$, or $\mathbb{C}\text{ob}_2^{\chi \leq 0}$ is a compact object in the ∞ -category of symmetric monoidal ∞ -categories.

⁴Let us call a symmetric monoidal ∞ -category finitely presented if it is the colimit of a diagram $\mathcal{D} : I \rightarrow \text{Cat}_{\infty}^{\otimes}$ where I is a finite ∞ -category and each $\mathcal{D}(i)$ is freely generated by a finite ∞ -category.

Outline

This part of the thesis is split into two chapters. Chapter 8 concerns the general theory of labelled cospan categories, and chapter 9 focuses on the case of $\text{Cob}_2^{X \leq 0}$ and relates it to tropical moduli spaces.

We begin in section 8.1 by defining labelled cospan categories, giving several general constructions, and stating the decomposition theorem and the surgery theorem (which together form theorem N). This section serves as an overview of chapter 8. Section 8.2 proves the decomposition theorem. First we establish a general fiber sequence criterion and a base-change theorem that will be useful throughout, these are then applied to various spaces of cuts similar to section 6.2. In the end of the section we also show that factorisation categories are often contractible. Section 8.3 proves the surgery theorem by applying ideas of [Gal11] and [GRW14] to labelled cospan categories. We also compute the positive boundary category of any weighted cospan category by generalising [Til96]. At this point theorems J, O, and P follow.

Chapter 9 is divided into two sections. Section 9.1 relates the factorisation category $\mathcal{F}_g(\text{Cob}_2^{X \leq 0})$ to the category of graphs J_g by constructing a category of filtered graphs $\int_J \text{Sub}'$ that admits functors to both. We then show that both functors induce equivalences by showing that the space of filtrations $B\text{Sub}'(X, w)$ on any given graph (X, w) is contractible. Together with the surgery and decomposition theorem this implies theorem K. Section 9.2 shows that BJ_g is rationally equivalent to the tropical moduli space Δ_g (theorem L) by constructing a rational homology isomorphism $BJ_g \simeq B\mathcal{T} \rightarrow \Delta_g$ and then verifying that $\pi_1(BJ_g) = *$ by direct computation.

Chapter 8

Classifying spaces of labelled cospan categories

8.1 Categories of labelled cospans

In this section we introduce the framework of labelled cospan categories, give several basic constructions that will be needed later, and state the two key theorems about classifying spaces of labelled cospan categories: the decomposition theorem 8.2.1 and the surgery theorem 8.3.1.

8.1.1 Labelled cospans

8.1.1.1 Definition

Definition 8.1.1. *The cospan category \mathcal{C}_{sp} has as objects finite sets A and as morphisms $A \rightarrow B$ isomorphism classes $[A \rightarrow X \leftarrow B]$ of cospans of finite sets. Here two cospans $f : A \rightarrow X \leftarrow B : g$ and $f' : A \rightarrow X' \leftarrow B : g'$ are called isomorphic if there is a bijection $\varphi : X \cong X'$ such that $f' = \varphi \circ f$ and $g' = \varphi \circ g$. The composite of two morphisms $[A \rightarrow X \leftarrow B]$ and $[B \rightarrow Y \leftarrow C]$ is defined by taking the pushout $[A \rightarrow X \cup_B Y \leftarrow C]$. This is a symmetric monoidal category under disjoint union.*

Example 8.1.2. *The key to the following definitions is that any bordism $W : M \rightarrow N$ yields a cospan of finite sets $\pi_0(M) \rightarrow \pi_0(W) \leftarrow \pi_0(N)$ in such a way that composition of bordisms corresponds to composition of cospans. In other words, taking connected components defines a symmetric monoidal functor $\pi_0 : \text{Cob}_d \rightarrow \mathcal{C}_{\text{sp}}$.*

Definition 8.1.3. *Consider a symmetric monoidal category $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ together with a symmetric monoidal functor $\pi : \mathcal{C} \rightarrow \mathcal{C}_{\text{sp}}$. We say that an object $M \in \mathcal{C}$ is connected with respect to π if $\pi(M)$ is a set with one element. Similarly, we say that a morphism $W : M \rightarrow N \in \mathcal{C}$ is connected with respect to π if the set $\pi(W)$ in the cospan $[\pi(M) \rightarrow \pi(W) \leftarrow \pi(N)]$ has a single element. We say that W is reduced if the cospan $[\pi(M) \rightarrow \pi(W) \leftarrow \pi(N)]$ has the property that $\pi(M) \amalg \pi(N) \rightarrow \pi(W)$*

is surjective. Let $\text{Hom}_{\mathcal{C}}^{\text{con}}(M, N) \subset \text{Hom}_{\mathcal{C}}(M, N)$ and $\text{Hom}_{\mathcal{C}}^{\text{red}}(M, N) \subset \text{Hom}_{\mathcal{C}}(M, N)$ denote the subset of connected and reduced morphisms, respectively.

The following definition encodes the idea that in Cob_d any object and bordism canonically decomposes into its set of connected components.

Definition 8.1.4. A labelled cospan category is a symmetric monoidal category $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ together with a symmetric monoidal functor $\pi : \mathcal{C} \rightarrow \text{Csp}$ satisfying the following axioms:

- (i) For any object $M \in \mathcal{C}$ such that $\pi(M)$ has n elements we can find connected objects $M_1, \dots, M_n \in \mathcal{C}$ such that $M \cong M_1 \otimes \dots \otimes M_n$. If $n = 0$, we require $M \cong 1_{\mathcal{C}}$.
- (ii) The abelian monoid $\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ is freely generated by the subset of connected morphisms.
- (iii) For any two objects $M, N \in \mathcal{C}$ the following map is a bijection

$$\otimes : \text{Hom}_{\mathcal{C}}^{\text{red}}(M, N) \times \text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \rightarrow \text{Hom}_{\mathcal{C}}(M, N).$$

- (iv) For any four objects $M_1, M_2, N_1, N_2 \in \mathcal{C}$ the following diagram is a pullback square:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}^{\text{red}}(M_1, N_1) \times \text{Hom}_{\mathcal{C}}^{\text{red}}(M_2, N_2) & \xrightarrow{\otimes} & \text{Hom}_{\mathcal{C}}^{\text{red}}(M_1 \otimes M_2, N_1 \otimes N_2) \\ \downarrow \pi & & \downarrow \pi \\ \text{Hom}_{\text{Csp}}^{\text{red}}(\pi(M_1), \pi(N_1)) \times \text{Hom}_{\text{Csp}}^{\text{red}}(\pi(M_2), \pi(N_2)) & \xrightarrow{\amalg} & \text{Hom}_{\text{Csp}}^{\text{red}}(\pi(M_1) \amalg \pi(M_2), \pi(N_1) \amalg \pi(N_2)) \end{array}$$

A functor of labelled cospan categories $F : (\pi_{\mathcal{C}} : \mathcal{C} \rightarrow \text{Csp}) \rightarrow (\pi_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Csp})$ is a symmetric monoidal functor $F : \mathcal{C} \rightarrow \mathcal{D}$ together with a symmetric monoidal natural isomorphism $\alpha : \pi_{\mathcal{D}} \circ F \cong \pi_{\mathcal{C}}$.

REMARK 8.1.5. We will show in lemma 8.1.28 that any labelled cospan category $(\mathcal{C} \rightarrow \text{Csp})$ is equivalent to another category $h\text{Csp}(\mathcal{C})$ where morphisms are indeed cospans of finite sets labelled by connected morphisms in \mathcal{C} . This justifies the name, but the list of axioms might still seem a little arbitrary. In definition 8.1.30 we provide a much simpler definition in the world of ∞ -categories, which conjecturally generalises the above and only imposes a single axiom. For now we shall motivate the definition by giving examples.

Example 8.1.6. For any dimension d and tangential structure θ the truncated θ -structured d -dimensional bordism category $\text{Cob}_{d,\theta} = h(\mathcal{C}_{d,\theta})$ is a labelled cospan category when equipped with the symmetric monoidal functor $\pi_0 : \text{Cob}_{d,\theta} \rightarrow \text{Csp}$ that sends a closed $(d-1)$ -manifold M to the finite set $\pi_0(M)$ and a d -dimensional bordism $W : M \rightarrow N$ to the cospan $[\pi_0(M) \rightarrow \pi_0(W) \leftarrow \pi_0(N)]$.

Example 8.1.7. Let $(\mathcal{C} \rightarrow \text{Csp})$ be some labelled cospan category and let $\mathcal{D} \subset \mathcal{C}$ be a symmetric monoidal subcategory such that:

(a) If $M \otimes N \in \mathcal{D}$ for some $M, N \in \mathcal{C}$, then $M, N \in \mathcal{D}$.

(b) If $f \otimes g \in \mathcal{D}$ for some $(f : M \rightarrow N), (g : M' \rightarrow N') \in \mathcal{C}$, then $f, g \in \mathcal{D}$.

Then $(\mathcal{D} \rightarrow \text{Csp})$ is a labelled cospan category. To see this we check the four properties:

(i) This follows because we had such a decomposition in \mathcal{C} and the connected objects appearing in the decomposition of an object of \mathcal{D} also need to lie in \mathcal{D} by (a).

(ii) This follows because if a submonoid $\text{Hom}_{\mathcal{D}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ of a free monoid $\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ satisfies the cancellation property (b), then it is necessarily freely generated on a subset of the generators of the original monoid.

(iii) The map is injective because it is the restriction of an injection and it is surjective by (b).

(iv) The map from the top-left corner to the pullback is injective because it is the restriction of an injection and it is surjective by (b).

Combining this with the previous example we see that $(\pi_0 : \text{Cob}_2^{X \leq 0} \rightarrow \text{Csp})$ is a labelled cospan category.

8.1.1.2 Decomposing labelled cospans

The following lemma explains how we can think of any morphism $W : M \rightarrow N$ as the cospan $\pi(M) \rightarrow \pi(W) \leftarrow \pi(N)$ together with a labelling of the components of $\pi(W)$ by connected morphisms in \mathcal{C} .

Lemma 8.1.8. *Consider two objects $M, N \in \mathcal{C}$ and decompose them into connected objects $M = M_1 \otimes \dots \otimes M_m$ and $N = N_1 \otimes \dots \otimes N_n$. The set of morphisms $M \rightarrow N$ can be described as:*

$$\text{Hom}_{\mathcal{C}}(M, N) \cong \text{Hom}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \times \coprod_{[X]:\pi(M) \rightarrow \pi(N)} \prod_{x \in X} \text{Hom}_{\mathcal{C}}^{\text{con}} \left(\bigotimes_{i \in f^{-1}(x)} M_i, \bigotimes_{j \in g^{-1}(x)} N_j \right)$$

where the coproduct runs over all isomorphism classes of reduced cospans $f : \pi(M) \rightarrow X \leftarrow \pi(N) : g$. This decomposition is natural with respect to functors of labelled cospan categories.

Proof. We have the decomposition $\text{Hom}_{\mathcal{C}}(M, N) \cong \text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \times \text{Hom}_{\mathcal{C}}^{\text{red}}(M, N)$ by condition (iii). The set of reduced morphisms maps to the set of reduced cospans, so we get a disjoint decomposition

$$\text{Hom}_{\mathcal{C}}^{\text{red}}(M, N) \cong \coprod_{[X]:\pi(M) \rightarrow \pi(N)} \text{Hom}_{\mathcal{C}}^{[X]}(M, N)$$

where $\text{Hom}_{\mathcal{C}}^{[X]}(M, N) = \{W : M \rightarrow N \mid \pi(W) = [X]\}$ is the set of morphisms with underlying cospan X . To understand this set of morphisms we decompose M and N according to the cospan X :

$$M \cong \bigotimes_{x \in X} M_x, \quad M_x := \bigotimes_{i \in f^{-1}(x)} M_i \quad \text{and} \quad N \cong \bigotimes_{x \in X} N_x, \quad N_x := \bigotimes_{j \in g^{-1}(x)} N_j$$

Here we identified $\pi(M)$ with $\{1, \dots, m\}$ according to the decomposition $M = M_1 \otimes \dots \otimes M_m$ and similarly for N . We can now apply condition (iv) multiple times and glue the pullback squares to obtain the following pullback square:

$$\begin{array}{ccc} \prod_{x \in X} \text{Hom}_{\mathcal{C}}^{\text{red}}(M_x, N_x) & \xrightarrow{\otimes} & \text{Hom}_{\mathcal{C}}^{\text{red}}(\bigotimes_{x \in X} M_x, \bigotimes_{x \in X} N_x) \\ \downarrow \pi & & \downarrow \pi \\ \prod_{x \in X} \text{Hom}_{\text{Csp}}^{\text{red}}(\pi(M_x), \pi(N_x)) & \xrightarrow{\amalg} & \text{Hom}_{\text{Csp}}^{\text{red}}(\amalg_{x \in X} \pi(M_x), \amalg_{x \in X} \pi(N_x)). \end{array}$$

By definition of M_x and N_x the cospan $[\pi(M) \rightarrow X \leftarrow \pi(N)]$ corresponds to the element of the product $\prod_{x \in X} \text{Hom}_{\text{Csp}}^{\text{red}}(\pi(M_x), \pi(N_x))$ that is given by the unique connected cospan $[\pi(M_x) \rightarrow \{x\} \leftarrow \pi(N_x)]$ in each entry. We can now compare the fibers of the vertical maps at the points corresponding to $[X]$. Since the diagram is a pullback square the fibers are isomorphic:

$$\prod_{x \in X} \{W \in \text{Hom}_{\mathcal{C}}^{\text{red}}(M_x, N_x) \mid \pi(W) = \{x\}\} \cong \{W \in \text{Hom}_{\mathcal{C}}^{\text{red}}(\bigotimes_{x \in X} M_x, \bigotimes_{x \in X} N_x) \mid \pi(W) = [X]\}.$$

The left-hand side is the product over $\text{Hom}_{\mathcal{C}}^{\text{con}}(M_x, N_x)$ and the right-hand side is $\text{Hom}_{\mathcal{C}}^{[X]}(M, N)$. Together with the first part this concludes the proof. \square

Corollary 8.1.9. *A functor $F : (\pi_{\mathcal{C}} : \mathcal{C} \rightarrow \text{Csp}) \rightarrow (\pi_{\mathcal{D}} : \mathcal{D} \rightarrow \text{Csp})$ of labelled cospan categories is an equivalence of categories if and only if it satisfies the following two conditions:*

- For any connected object $M_{\mathcal{D}} \in \mathcal{D}$ there is a connected object $M_{\mathcal{C}} \in \mathcal{C}$ such that $F(M_{\mathcal{C}}) \cong M_{\mathcal{D}}$.
- For any two objects $M, N \in \mathcal{C}$ the functor F induces a bijection

$$F : \text{Hom}_{\mathcal{C}}^{\text{con}}(M, N) \cong \text{Hom}_{\mathcal{D}}^{\text{con}}(F(M), F(N)).$$

Proof. The first condition implies that $F : \mathcal{C} \rightarrow \mathcal{D}$ is essentially surjective because by condition (i) every object in \mathcal{D} is the product of connected objects. The second condition applied to $M = 1_{\mathcal{C}} = N$ implies, together with condition (ii), that $\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \rightarrow \text{Hom}_{\mathcal{D}}(1_{\mathcal{D}}, 1_{\mathcal{D}})$ is a bijection. Then we can use the decomposition from lemma 8.1.8 to see that being a bijection on connected morphisms and endomorphisms of $1_{\mathcal{C}}$ implies that F is a bijection on all homs. \square

For any labelled cospan category we can define a reduced category in which all closed components are deleted.

Definition 8.1.10. *The reduced category \mathcal{C}^{red} has the same objects as \mathcal{C} and the morphisms are equivalence classes of morphisms in \mathcal{C} under the action of the abelian monoid $\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$. This indeed defines a category as for any $W : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$ the action $(W \amalg _) : \text{Hom}_{\mathcal{C}}(M, N) \rightarrow \text{Hom}_{\mathcal{C}}(M, N)$ commutes with pre- and post-composition with any morphism.*

Note that it follows from the axioms for labelled cospan categories that every morphism $[W] : M \rightarrow N$ in \mathcal{C}^{red} has a unique representative $W \in \text{Hom}_{\mathcal{C}}(M \rightarrow N)$ that is reduced. In other words, the composite

$$\text{Hom}_{\mathcal{C}}^{\text{red}}(M, N) \hookrightarrow \text{Hom}_{\mathcal{C}}(M, N) \rightarrow \text{Hom}_{\mathcal{C}^{\text{red}}}(M, N)$$

is a bijection. We could therefore equivalently define the category \mathcal{C}^{red} to have the same objects as \mathcal{C} and to have as morphisms the reduced morphisms in \mathcal{C} . Composition would then be defined by $W \circ_{\mathcal{C}^{\text{red}}} V = U$ where U is the unique reduced morphism such that there is $Q : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$ with $W \circ_{\mathcal{C}} V = U \amalg Q$.

Example 8.1.11. *There is a canonical bijection between the set of equivalence relations on $A \amalg B$ and $\text{Hom}_{\text{Csp}}^{\text{red}}(A, B)$ defined by sending an equivalence relation $R \subset (A \amalg B)^2$ to the cospan $[A \rightarrow (A \amalg B)/R \leftarrow B]$. This can be used to give an alternative description of Csp^{red} where morphisms $A \rightarrow B$ are equivalence relations on $A \amalg B$. In this category morphisms $R \subset (A \amalg B)^2$ and $S \subset (B \amalg C)^2$ are composed by restricting the equivalence relation on $A \amalg B \amalg C$ that is generated by R and S to the subset $A \amalg C$.*

8.1.1.3 Weighted cospans

Another class of examples can be constructed by labeling cospans with elements of an abelian monoid. This includes Csp , Cob_2 , and $\text{Cob}_2^{\chi \leq 0}$, and will provide a useful point of view in section 9.1. In principle it should be possible to label cospans with operations of any modular operad, or even a coloured properad (see conjecture 8.1.31). For the purpose of this thesis it will suffice to consider the case where the operations in the modular operad are elements of an abelian monoid A and pairing two inputs is given by addition with a fixed element α . We also allow to specify a subset $A_1 \subset A$, which can be thought of as a stability condition.

Definition 8.1.12. *A weighting monoid is a triple (A, A_1, α) where A is an abelian monoid, which we will denote additively, $A_1 \subset A$ is a subset satisfying $A_1 + A \subset A_1$, and $\alpha \in A_1$ is some element.*

The most important weighting monoid is $(\mathbb{N}, \mathbb{N}_{\geq 1}, 1)$, and we will usually think about this case.

Definition 8.1.13. For a monoid $(A, +)$ with a preferred element $\alpha \in A$ we define the (A, α) -weighted cospan category $\text{Csp}(A, \alpha)$ as follows. Objects are finite sets. Morphisms are equivalence classes of labelled cospans $[i : M \rightarrow X \leftarrow N : j, l : X \rightarrow A]$. Two morphisms are equivalent if there is a bijection $\varphi : X \cong X'$ satisfying $l' \circ \varphi = l$, $\varphi \circ i = i'$, and $\varphi \circ j = j'$. The composition of two morphisms $(i : M \rightarrow X \leftarrow N : j, l)$ and $(i' : N \rightarrow Y \leftarrow L : j', l')$ is defined by composing the cospans as usual and equipping the pushout with the labeling:

$$(l * l')([u] \in X \amalg_N Y) = \sum_{x \in X, x \in [u]} l(x) + \sum_{y \in Y, y \in [u]} l'(y) + \alpha^{b_1([u]; X, Y)}.$$

Here $b_1([u]; X, Y) \in \mathbb{N}$ is what one could call the first Betti number of the equivalence class $[u] \subset X \amalg Y$. Concretely, we define it as:

$$b_1([u]; X, Y) := |(N \rightarrow X \amalg_N Y)^{-1}([u])| - |(X \amalg Y \rightarrow X \amalg_N Y)^{-1}([u])| + 1.$$

Lemma 8.1.14. Definition 8.1.13 yields a well-defined category $\text{Csp}(A, \alpha)$ and disjoint union defines a symmetric monoidal structure on this category. Moreover, the forgetful functor $\text{Csp}(A, \alpha) \rightarrow \text{Csp}$ is symmetric monoidal and makes $\text{Csp}(A, \alpha)$ into a labelled cospan category.

Proof. To see that $\text{Csp}(A, \alpha)$ we need to check that composition is associative. Since we already know that the composition of cospans is associative up to canonical isomorphism, all that is left check is that this canonical isomorphism respects the labeling. Consider three composable morphisms $(i : M \rightarrow X \leftarrow N : j, l)$, $(i' : N \rightarrow Y \leftarrow L : j', l')$, and $(i'' : L \rightarrow Z \leftarrow O : j'', l'')$. To see that $(l * l') * l'' \equiv l * (l' * l'')$ it suffices to check the following identity on Betti numbers for every point $[u] \in X \amalg_N Y \amalg_L Z$:

$$\begin{aligned} b_1([u]; X \amalg_N Y, Z) + \sum_{[v] \in X \amalg_N Y, [v] \subset [u]} b_1([v]; X, Y) \\ \stackrel{?}{=} b_1([u]; X, Y \amalg_L Z) + \sum_{[w] \in Y \amalg_L Z, [w] \subset [u]} b_1([w]; Y, Z) \end{aligned}$$

(The point is represented by an equivalence class $[u] \subset X \amalg Y \amalg Z$.) Indeed one checks that both sides of the equation are equal to

$$|(N \amalg L \rightarrow (X \amalg_N Y \amalg_L Z))^{-1}([u])| - |(X \amalg Y \amalg Z \rightarrow (X \amalg_N Y \amalg_L Z))^{-1}([u])| + 1.$$

Hence the composition in $\text{Csp}(A, \alpha)$ is indeed associative.

The composition of (A, α) -labelled cospans is defined component-wise and hence disjoint union $\amalg : \text{Csp}(A, \alpha) \times \text{Csp}(A, \alpha) \rightarrow \text{Csp}(A, \alpha)$ is a functor. This defines a symmetric monoidal structure with the same structure isomorphisms as for Csp (now labelled by 0). By construction the forgetful functor $\text{Csp}(A, \alpha) \rightarrow \text{Csp}$ is symmetric monoidal. Moreover, one checks that this functor satisfies the conditions of definition 8.1.4 and hence defines a labelled cospan category. \square

Definition 8.1.15. For (A, α) as above and a subset $A_1 \subset A$ satisfying $\alpha \in A_1$ and $A_1 + A \subset A_1$ we define $\text{Csp}(A, A_1, \alpha) \subset \text{Csp}(A, \alpha)$ as the subcategory that contains all objects and those morphisms $(i : M \rightarrow X \leftarrow N : j, l)$ where for each $x \in X$ with $|i^{-1}(x) \amalg j^{-1}(x)| \leq 1$ we require $l(x) \in A_1$.

One checks that $\text{Csp}(A, A_1, \alpha) \subset \text{Csp}(A, \alpha)$ is indeed closed under composition by using $\alpha \in A_1$ and $A_1 + A \subset A_1$. It is also closed under disjoint union and further satisfies the condition of example 8.1.7. Hence $\text{Csp}(A, A_1, \alpha)$ is a labelled cospan category.

Lemma 8.1.16. *There are equivalences of labelled cospan categories:*

$$\text{Csp}(0, 0) \simeq \text{Csp}, \quad \text{Csp}(\mathbb{N}, 1) \simeq \text{Cob}_2, \quad \text{Csp}(\mathbb{N}, \mathbb{N}_{\geq 1}, 1) \simeq \text{Cob}_2^{\chi \leq 0}.$$

Proof. The first equivalence is clear since a cospan labelled in the trivial monoid 0 is the same data as just a monoid, so the forgetful map $\text{Csp}(0, 0) \rightarrow \text{Csp}$ is an isomorphism of categories.

For the second equivalence we define a functor $F : \text{Cob}_2 \rightarrow \text{Csp}(\mathbb{N}, 1)$ as the lift of the functor $\pi_0 : \text{Cob}_2 \rightarrow \text{Csp}$ by labelling the cospan $[\pi_0(M) \rightarrow \pi_0(W) \leftarrow \pi_0(N)]$ obtained from a morphism $W : M \rightarrow N$ with the function $g : \pi_0(W) \rightarrow \mathbb{N}$ that sends a component to its genus. We need to check that F is in fact functorial. This means that we have to verify for any two bordisms $W : M \rightarrow N$ and $V : N \rightarrow L$ that the genus of some component $U \subset W \cup_N V$ is given by:

$$\sum_{U_0 \in \pi_0(U \cap W)} g(U_0) + \sum_{U_1 \in \pi_0(U \cap V)} g(U_1) + (|\pi_0(U \cap N)| + 1 - |\pi_0(U \cap W) \amalg \pi_0(U \cap V)|).$$

Without loss of generality we may assume that $W \cup_N V$ is connected and hence that $U = W \cup_N V$. The Euler characteristic is additive under the composition of surface-bordisms and so we can compute:

$$\begin{aligned} \chi(W \cup_N V) &= \chi(W) + \chi(V) = \sum_{U_0 \subset W} (2 - 2g(U_0) - |\pi_0(\partial U_0)|) + \sum_{U_1 \subset V} (2 - 2g(U_1) - |\pi_0(\partial U_1)|) \\ &= -2 \left(\sum_{U_0 \subset W} g(U_0) + \sum_{U_1 \subset V} g(U_1) \right) + 2|\pi_0(W) \amalg \pi_0(V)| - 2|\pi_0(N)| - |\pi_0(M)| - |\pi_0(L)| \end{aligned}$$

Combining this with $\chi(W \cup_N V) = 2 - 2g(W \cup_N V) - |\pi_0(M)| - |\pi_0(L)|$ yields the desired formula.

The functor F is by construction symmetric monoidal and compatible with the projection to Csp . We may hence use corollary 8.1.9 to check that it is an equivalence of labelled cospan categories. Indeed, F is surjective on connected objects (it hits the one-point set) and F is fully faithful on connected morphisms since a connected surface with fixed boundary is uniquely determined by its genus.

The third equivalence is obtained by restricting the previous equivalence. The subcategory $\text{Cob}_2^{\chi \leq 0} \subset \text{Cob}_2$ corresponds exactly to $\text{Csp}^{\mathbb{N}_{\geq 1}}(\mathbb{N}, 1)$ since every connected bordism $W : M \rightarrow N$ satisfies:

$$\chi(W) \leq 0 \Leftrightarrow (|\pi_0(M) \amalg \pi_0(N)| \leq 1 \Rightarrow g(W) \in \mathbb{N}_{\geq 1}).$$

□

Example 8.1.17. For any $\gamma \in \mathbb{N}$ one can define a category $\text{Cob}_2^{g < \gamma}$ as the quotient of Cob_2 by the equivalence relation \sim_γ on morphisms that is generated as follows: whenever $W : M \rightarrow N$ is a bordism and $V \subset W$ is a component of genus at least g , then we set $W \sim_\gamma W \# (S^1 \times S^1)$ where the connected sum is taken at V . Under this equivalence relation two morphisms $W, W' : M \rightarrow N$ are identified if they become diffeomorphic after arbitrarily increasing the genus of every component that has genus at least g .

This category is equivalent to the weighted cospan category $\text{Csp}(\mathbb{N}/\gamma, 1)$ where \mathbb{N}/γ is the abelian monoid defined as $\mathbb{N}/(\mathbb{N} + \gamma)$. For $\gamma = 0$ we have that $\text{Cob}_2^{g < 0} \simeq \text{Csp}$.

One can also define a subcategory $\text{Cob}_2^{\chi \leq 0, g < \gamma} \subset \text{Cob}_2^{g < \gamma}$. (This will be the entire category if $\gamma = 0$.) This category is equivalent to $\text{Csp}^{\mathbb{N}_{\geq 1/\gamma}}(\mathbb{N}/\gamma, 1)$.

Example 8.1.18. For any $d \geq 2$ consider the full subcategory $\text{Cob}_d^{S^{d-1}} \subset \text{Cob}_d$ of the d -dimensional oriented bordism category on all the objects of the form $\coprod_{i=1}^k S^{d-1}$.¹ Let \mathcal{M}_d be the monoid of diffeomorphism classes of connected closed oriented d -manifolds with addition given by connected sum. For $d = 2$ we have $\mathcal{M}_2 \cong \mathbb{N}$ and for $d = 3$ Milnor [Mil62] showed that the monoid \mathcal{M}_3 is freely generated by the set of diffeomorphism classes of prime 3-manifolds. We define a functor

$$F : \text{Cob}_d^{S^{d-1}} \rightarrow \text{Csp}(\mathcal{M}_d, [S^1 \times S^{d-1}])$$

equipping the cospan $(\pi_0(M) \rightarrow \pi_0(W) \leftarrow \pi_0(N))$ with the labeling $l : \pi_0(W) \rightarrow \mathcal{M}_d$ that sends a connected component $(W' : M' \rightarrow N') \subset (W : M \rightarrow N)$ to the closed manifold

$$\widehat{W'} := (\amalg_{i=1}^k D^d) \cup_{\amalg S^{d-1}} W' \cup_{\amalg S^{d-1}} (\amalg_{j=1}^l D^d).$$

Here we use standard identifications $M' \cong \amalg_{i=1}^k S^{d-1}$ and $N' \cong \amalg_{j=1}^l S^{d-1}$, which are canonical up to permutation of components.

We need to show that F is functorial. Let $W : M \rightarrow N$ and $V : N \rightarrow L$ two morphisms in $\text{Cob}_d^{S^{d-1}}$ such that $W \cup_N V$ is connected. Then we need to check:

$$\widehat{W \cup_N V} \cong (\#_i \widehat{W}_i) \# (\#_j \widehat{V}_j) \# (S^1 \times S^{d-1})^{\#b_1}$$

¹Equivalently we could consider the full subcategory of Cob_d on all objects that are diffeomorphic to a disjoint union of spheres. However, without choosing identifications with S^{d-1} it is more difficult to define the functor F .

where $W_i \subset W$, $V_i \subset V$ denote the connected components and $b_1 \in \mathbb{N}$ is defined as $|\pi_0(N)| - |\pi_0(W \amalg V)| + 1$. Indeed, $\widehat{W \cup_N V}$ can be obtained from $\coprod_i \widehat{W}_i \amalg \coprod_j \widehat{V}_j$ by attaching a 1-handle for every component of N . Let us attach these 1-handles one after another. Whenever the 1-handle is attached at two disks that are not yet path-connected, attaching this handle is equivalent to taking a connected sum at these disks. This will happen $|\pi_0(W \amalg V)| - 1$ times. For each of the remaining 1-handles we attach the both endpoints to the same component. This is equivalent to taking the connected sum with $(S^1 \times S^{d-1})$ and will happen $|\pi_0(N)| - (|\pi_0(W \amalg V)| - 1) = b_1$ times. Hence the above formula holds and F is indeed a functor. It follows from the construction that F is symmetric monoidal and defines a functor of labelled cospan categories.

We would like to apply corollary 8.1.9 to show that F is an equivalence of labelled cospan categories. Since F is certainly essentially surjective we only need to check that for all $k, l \geq 0$ the map

$$\mathrm{Hom}_{\mathrm{Cob}_d}^{\mathrm{con}}(\amalg_{i=1}^k S^{d-1}, \amalg_{i=1}^l S^{d-1}) \longrightarrow \mathcal{M}_d, \quad [W] \mapsto [\widehat{W}]$$

is a bijection. To construct an inverse map choose for every connected closed oriented d -manifold V an orientation preserving embedding $\iota : (\amalg_{i=1}^k D^d) \amalg (\amalg_{j=1}^l D^d) \hookrightarrow V$. Then $V \cong \widehat{W}_\iota$ for $W_\iota = V \setminus \mathrm{Im}(\iota)^\circ$. For any other choice of orientation preserving embedding ι' there is a diffeomorphism $\psi : V \cong V$ with $\iota' \cong \psi \circ \iota$ (by [Hir94, Chapter 8, Theorem 3.2]). This restricts to a diffeomorphism $\psi|_{W_\iota} : W_\iota \cong W_{\iota'}$ that is compatible with the boundary identifications. So the construction $[V] \mapsto [W_\iota]$ is independent of ι and hence a well-defined map $\mathcal{M}_d \rightarrow \mathrm{Hom}_{\mathrm{Cob}_d}^{\mathrm{con}}(\amalg_{i=1}^k S^{d-1}, \amalg_{i=1}^l S^{d-1})$ that is inverse to $[W] \mapsto [\widehat{W}]$.

Note that the above example fails in the non-oriented case. The cylinder with flipped boundary identification defines a non-identity morphism $C : S^1 \rightarrow S^1$ in the unoriented surface category $\mathrm{Cob}_2^{\mathrm{unor}}$ such that $\widehat{C} = D^2 \cup_{S^1} C \cup_{S^1} D^2 \cong S^2$. Therefore the functor $F : \mathrm{Cob}_2^{\mathrm{unor}} \rightarrow \mathrm{Csp}(\mathcal{A}_2^{\mathrm{unor}}, [S^1 \times S^1])$ is not faithful.

8.1.2 The enhanced cospan category

Given a labelled cospan category $(\mathcal{C} \rightarrow \mathrm{Csp})$ we now want to build a refinement $\mathbb{C}\mathrm{sp}(\mathcal{C})$ that keeps track of permutations of closed components. We begin by defining the groupoid-enriched category $\mathbb{C}\mathrm{sp}$: the 2-category of finite sets, cospans of finite sets, and isomorphisms of cospans. To ensure that everything is a set let Ω denote some infinite ‘‘background set’’.

This subsection might be skipped at a first read. By theorem 8.2.4 the canonical functor $\mathbb{C}\mathrm{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ induces a rational equivalence on classifying spaces.

8.1.2.1 The unlabelled case

Definition 8.1.19. A presented set is a triple (l, X, R) where $l \in \{0, 1, \dots\}$, $X \subset \{1, \dots, l\}$, and $R \subset X \times X$, such that R is an equivalence relation on X . We let $\underline{X} := X/R$ denote the set of equivalence classes of X under R , and we will often write \underline{X} when we mean the entire datum (l, X, R) .

Two presented sets (l, X, R) and (k, Y, S) are called disjoint if $l = k$ and $X \cap Y = \emptyset$. In this case we write $(l, X, R) \perp (k, Y, S)$ and we define their disjoint union as $(l, X, R) \amalg (k, Y, S) := (l, X \cup Y, R \cup S)$.

For any two presented sets (l, X, R) and (k, Y, S) and maps $i : A \rightarrow X/R$ and $j : A \rightarrow Y/S$ from some set A we define the glueing as

$$(l, X, R) \cup_A (k, Y, S) := (l + k, X \cup (Y + l), R \cup_A (S + l))$$

where $Y + l := \{y + l \mid y \in Y\}$ and $R \cup_A (S + l)$ is the equivalence relation generated by $R, (S + l)$, and $i(a) \sim j(a)$ for all $a \in A$.

Note that the disjoint union of disjoint presented sets is strictly associative and commutative whenever it is defined. Moreover, the gluing operation $(\underline{X} \leftarrow A \rightarrow \underline{Y}) \rightarrow \underline{X} \cup_A \underline{Y}$ is strictly associative, which allows us to make the following definition:

Definition 8.1.20. The $(2, 1)$ -category \mathbb{C}_{SP} is defined as follows. Objects are finite subsets $A \subset \Omega$. Non-identity morphisms $A \rightarrow B$ are cospans $(f_X : A \rightarrow \underline{X} \leftarrow B : g_X)$ where \underline{X} is a presented set. We also have identity morphisms, which we denote as $(A = A = A)$. The composition of morphisms is defined as:

$$(f_Y : B \rightarrow \underline{Y} \leftarrow C : g_Y) \circ (f_X : A \rightarrow \underline{X} \leftarrow B : g_X) := (f_X : A \rightarrow \underline{X} \cup_B \underline{Y} \leftarrow C : g_Y).$$

2-morphisms $\varphi : (f_X : A \rightarrow \underline{X} \leftarrow B : g_X) \Rightarrow (f'_X : A \rightarrow \underline{X}' \leftarrow B : g'_X)$ are bijections $\varphi : \underline{X} \cong \underline{X}'$ such that $\varphi \circ f_X = f'_X$ and $\varphi \circ g_X = g'_X$. (Note that φ is not an isomorphism of presented sets, but just a bijection between the resulting quotient sets.) These are composed in the obvious way.

REMARK 8.1.21. The addition of identity morphisms in the above definition is a little ad-hoc. Let us briefly explain how they interact with the other morphisms. The horizontal composition is defined such that they act as identities: $(A = A = A) \cup_A (A \rightarrow \underline{X} \leftarrow B) := (A \rightarrow \underline{X} \leftarrow B)$, and similarly when composing on the other side. The 2-morphisms are defined by treating the middle A as if it were a presented set \underline{X} : a 2-morphism $\varphi : (A = A = A) \Rightarrow (f : A \rightarrow \underline{X} \leftarrow A : g)$ is a bijection $\varphi : A \rightarrow \underline{X}$ such that $\varphi \circ \text{id}_A = f$ and $\varphi \circ \text{id}_A = g$. This means that such a 2-morphism exists if and only if $f = g$ is a bijection and the 2-morphism is unique if it exists.

To describe the operation of disjoint union on $\mathbb{C}\text{sp}$ we will use the notion of partial commutative monoidal structures that we introduce in definition 8.2.5 and 8.2.11. See section 8.2.1 for an introduction to how we deal with symmetric monoidal structures.

Definition 8.1.22. *The partial commutative monoidal structure on the $(2, 1)$ -category $\mathbb{C}\text{sp}$ is defined by the relation \perp and the operation \amalg . On objects we set $A \perp B$ whenever $A \cap B = \emptyset$ and on morphisms we set $(\underline{X}, i, i') \perp (\underline{Y}, j, j')$ whenever $\underline{X} \perp \underline{Y}$ in the sense of 8.1.19. The operation \amalg is given by taking disjoint unions.*

Lemma 8.1.23. *The canonical functor $\mathbb{C}\text{sp} \rightarrow \text{Csp}$ that sends a cospan $(A \rightarrow \underline{X} \leftarrow B)$ to its isomorphism class induces an equivalence of categories $h(\mathbb{C}\text{sp}) \simeq \text{Csp}$.*

Proof. The 1-category $h(\mathbb{C}\text{sp})$ is defined by identifying all 1-morphisms in $\mathbb{C}\text{sp}$ that are connected by a 2-morphism. The functor $h(\mathbb{C}\text{sp}) \rightarrow \text{Csp}$ is hence faithful by definition: on both sides isomorphic cospans are identified. It is also full because every cospan $A \rightarrow Z \leftarrow B$ is isomorphic to a cospan $A \rightarrow \underline{X} \leftarrow B$ seeing as any finite set Z is in bijection with some presented set \underline{X} . The functor is also essentially surjective as any finite set A can be embedded $\iota : A \hookrightarrow \Omega$ and the cospan $(A \cong \iota(A) = \iota(A))$ is an isomorphism in Csp . \square

8.1.2.2 The general case

We now define a version of $\mathbb{C}\text{sp}$ where the finite sets and cospans are labelled by connected objects and morphisms in \mathcal{C} . In lemma 8.1.27 we check that this is indeed well-defined and that the 2-functors described below are indeed functorial.

Definition 8.1.24. *For any labelled cospan category $(\mathcal{C} \rightarrow \text{Csp})$ we define the enhanced cospan category $\mathbb{C}\text{sp}(\mathcal{C})$ of \mathcal{C} to be the following $(2, 1)$ -category.*

- *An objects is a tuple (A, c_A) where $A \subset \Omega$ is a finite set and $c_A : A \rightarrow \text{Obj}^{\text{con}}(\mathcal{C})$ is a labelling by connected objects of \mathcal{C} .*
- *A non-identity 1-morphism $(A, c_A) \rightarrow (B, c_B)$ is a tuple of a non-identity morphisms $(f : A \rightarrow \underline{X} \leftarrow B : g)$ in $\mathbb{C}\text{sp}$ and a labelling $o : \underline{X} \rightarrow \text{Mor}^{\text{con}}(\mathcal{C})$ such that for each $x \in \underline{X}$ the label $o(x)$ is a connected morphism*

$$\bigotimes_{a \in f^{-1}(x)} c_A(a) \rightarrow \bigotimes_{b \in g^{-1}(x)} c_B(b).$$

We also allow for identity morphisms $\text{id}_{(A, c_A)}$, which we think of as a cospan $(A = A = A)$ where the labelling $o : A \rightarrow \text{Mor}^{\text{con}}(\mathcal{C})$ is given by $o(a) := \text{id}_{c_A(a)}$.

- *A 2-morphism $(f : A \rightarrow \underline{X} \leftarrow B : g, o) \Rightarrow (f' : A \rightarrow \underline{Y} \leftarrow B : g', o')$ is a bijection $\varphi : \underline{X} \cong \underline{Y}$ such that $f' = \varphi \circ f$, $g' = \varphi \circ g$, and $o' \circ \varphi = o$.*

The composition of two morphisms $(f_X : A \rightarrow \underline{X} \leftarrow B : g_X, o_X)$ and $(f_Y : B \rightarrow \underline{Y} \leftarrow C : g_Y, o_Y)$ is defined as the glued cospan $(f_Z : A \rightarrow \underline{X} \cup_B \underline{Y} \leftarrow C : g_Z, o_X * o_Y)$ where the induced labeling $o_X * o_Y$ is defined as follows. For each $z \in \underline{X} \cup_B \underline{Y}$ we let $\underline{X}_z \subset \underline{X}$ and $\underline{Y}_z \subset \underline{Y}$ denote the preimages of z under $\underline{X} \amalg \underline{Y} \rightarrow \underline{X} \cup_B \underline{Y}$.

$$\begin{array}{ccc}
\bigotimes_{a \in f_Z^{-1}(z)} c_A(a) & \xrightarrow{(o_X * o_Y)(z)} & \bigotimes_{c \in g_Z^{-1}(z)} c_C(c) \\
\downarrow \cong & & \downarrow \cong \\
\bigotimes_{x \in X_z} \bigotimes_{a \in f_X^{-1}(x)} c_A(a) & \xrightarrow{\otimes_x o_X(x)} \bigotimes_{x \in X_z} \bigotimes_{b \in g_X^{-1}(x)} c_B(b) \cong \bigotimes_{y \in Y_z} \bigotimes_{b \in f_Y^{-1}(y)} c_B(b) & \xrightarrow{\otimes_y o_Y(y)} \bigotimes_{y \in Y_z} \bigotimes_{c \in g_Y^{-1}(y)} c_C(c)
\end{array}$$

Definition 8.1.25. There is a functor $\otimes_{\mathcal{C}} : \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ defined by sending (A, c_A) to $\bigotimes_{a \in A} c_A(a)$ and sending morphisms $(f_X : A \rightarrow \underline{X} \leftarrow B : g_X, o)$ to

$$\bigotimes_{a \in A} c_A(a) \cong \bigotimes_{x \in \underline{X}} \bigotimes_{a \in f_X^{-1}(x)} c_A(a) \xrightarrow{\otimes_{[x]} o([x])} \bigotimes_{x \in \underline{X}} \bigotimes_{b \in g_X^{-1}(x)} c_B(b) \cong \bigotimes_{b \in B} c_B(b).$$

Definition 8.1.26. We define a partial commutative monoidal structure (see definition 8.2.5) on $\mathbb{C}\text{sp}(\mathcal{C})$ by saying that two objects or morphisms are disjoint iff they are disjoint as objects or morphisms in $\mathbb{C}\text{sp}$ after forgetting labelings. The union of disjoint objects or morphisms is defined by taking the union in $\mathbb{C}\text{sp}$ and equipping it with the evident labeling.

Lemma 8.1.27. Horizontal composition of 1-morphisms in $\mathbb{C}\text{sp}(\mathcal{C})$ is associative and hence $\mathbb{C}\text{sp}(\mathcal{C})$ is a well-defined 2-category. Moreover, the following three constructions are 2-functors:

$$\text{forget} : \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathbb{C}\text{sp}, \quad \amalg : \mathbb{C}\text{sp}(\mathcal{C})^{2\perp} \rightarrow \mathbb{C}\text{sp}(\mathcal{C}), \quad \text{and} \quad \otimes_{\mathcal{C}} : \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}.$$

Proof. We will show that associativity holds when composing three composable labelled cospans: $(A \rightarrow \underline{X} \leftarrow B, o_X)$, $(B \rightarrow \underline{Y} \leftarrow C, o_Y)$, $(C \rightarrow \underline{Z} \leftarrow D, o_Z)$. The functoriality of “forget”, \amalg , and $\otimes_{\mathcal{C}}$ will follow from the proof. The underlying cospan is $\underline{X} \cup_B \underline{Y} \cup_C \underline{Z}$ and it does not matter how we parenthesise this expression as composition in $\mathbb{C}\text{sp}$ is strictly associative. We need to check that the two induced labellings on this composed cospan agree.

Before we show this, let us consider the case of a single composition $(\underline{X}, o_X) \cup_B (\underline{Y}, o_Y)$. For $p \in \underline{X} \cup_B \underline{Y}$ we let $\underline{X}_p \subset \underline{X}$, $\underline{Y}_p \subset \underline{Y}$, and $B_p \subset B$ denote the subsets that are sent to p under the quotient map. The label $(o_X * o_Y)(p)$ of p in the composite cospan is exactly the morphism $\otimes_{\mathcal{C}}(\underline{X}_p \cup_{B_p} \underline{Y}_p, o_X * o_Y)$ in \mathcal{C} . (If (\underline{W}, o_W) is a labelled cospan where $\underline{W} = \{p\}$ is a single point, then $\otimes_{\mathcal{C}}(\underline{W}, o_W) = o_W(p)$.) By inspecting the definition it follows that this composite label is

exactly $\otimes_{\mathcal{C}}(\underline{Y}_p, o_Y) \circ \otimes_{\mathcal{C}}(\underline{X}_p, o_X)$ as illustrated in the following diagram:

$$\begin{array}{ccccc}
\otimes_{a \in A} c_A(a) & \xrightarrow{(o_X * o_Y)(p) = \otimes_{\mathcal{C}}(\underline{X}_p \cup_{B_p} \underline{Y}_p, o_X * o_Y)} & \otimes_{c \in C} c_C(s) & & \\
\downarrow \cong & \searrow \otimes_{\mathcal{C}}(\underline{X}_p, o_X) & \nearrow \otimes_{\mathcal{C}}(\underline{Y}_p, o_Y) & & \\
\otimes_{x \in X_p} \otimes_{a \in f_X^{-1}(x)} c_A(a) & & \otimes_{b \in B_p} c_B(b) & & \otimes_{y \in Y_p} \otimes_{c \in g_Y^{-1}(y)} c_C(c) \\
& \searrow \otimes_x o_X(x) & \nearrow \cong & & \nearrow \otimes_y o_Y(y) \\
& & \otimes_{x \in X_p} \otimes_{b \in g_X^{-1}(x)} c_B(b) & \xrightarrow{\cong} & \otimes_{y \in Y_p} \otimes_{b \in f_Y^{-1}(y)} c_B(b)
\end{array}$$

This already implies that $\otimes_{\mathcal{C}} : \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ will be a 2-functor once we show that $\mathbb{C}\text{sp}(\mathcal{C})$ is a 2-category.

Returning to the case of a triple composition we can now see that a point $q \in \underline{X} \cup_B \underline{Y} \cup_C \underline{Z}$ is labelled by:

$$\begin{aligned}
((o_X * o_Y) * o_Z)(q) &= \otimes_{\mathcal{C}}(\underline{X}_q \cup_{B_q} \underline{Y}_q, o_X * o_Y) \circ \otimes_{\mathcal{C}}(\underline{Z}_q, o_Z) \\
&= \left(\otimes_{p \in \underline{X}_q \cup_{B_q} \underline{Y}_q} \otimes_{\mathcal{C}}(\underline{X}_p \cup_{B_p} \underline{Y}_p, o_X * o_Y) \right) \circ \otimes_{\mathcal{C}}(\underline{Z}_q, o_Z) \\
&= \left(\otimes_{p \in \underline{X}_q \cup_{B_q} \underline{Y}_q} \otimes_{\mathcal{C}}(\underline{X}_p, o_X) \circ \otimes_{\mathcal{C}}(\underline{Y}_p, o_Y) \right) \circ \otimes_{\mathcal{C}}(\underline{Z}_q, o_Z) \\
&= (\otimes_{\mathcal{C}}(\underline{X}_q, o_X) \circ \otimes_{\mathcal{C}}(\underline{Y}_q, o_Y)) \circ \otimes_{\mathcal{C}}(\underline{Z}_q, o_Z)
\end{aligned}$$

Since composition in \mathcal{C} is associative it follows that this agrees with $(o_X * (o_Y * o_Z))(q)$ and hence horizontal composition in $\mathbb{C}\text{sp}(\mathcal{C})$ is indeed associative. \square

Lemma 8.1.28. *The enhanced cospan category fits into a diagram:*

$$\begin{array}{ccccc}
\mathbb{C}\text{sp}(\mathcal{C}) & \longrightarrow & h\mathbb{C}\text{sp}(\mathcal{C}) & \xrightarrow{\cong} & \mathcal{C} \\
\downarrow \pi_0 & & \downarrow \pi_0 & & \downarrow \pi \\
\mathbb{C}\text{sp} & \longrightarrow & h\mathbb{C}\text{sp} & \xrightarrow{\cong} & \mathbb{C}\text{sp}
\end{array}$$

Moreover, for any two objects (A, c_A) and (B, c_B) there is a canonical equivalence of groupoids

$$\text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}((A, c_A), (B, c_B)) \longrightarrow \text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}(\emptyset, \emptyset) \times \text{Hom}_{\mathcal{C}}^{\text{red}}(\otimes(A, c_A), \otimes(B, c_B)).$$

and $\text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}(\emptyset, \emptyset)$ is the free symmetric monoidal groupoid on the set $\text{Hom}_{\mathcal{C}}^{\text{con}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$.

Proof. The existence of the commutative diagram is clear by construction. We need to check that $\otimes_{\mathcal{C}} : h(\mathbb{C}\text{sp}(\mathcal{C})) \rightarrow \mathcal{C}$ is an equivalence of categories. Note that $\pi_0 : h(\mathbb{C}\text{sp}(\mathcal{C})) \rightarrow h(\mathbb{C}\text{sp}) \simeq \mathbb{C}\text{sp}$

is a labelled cospan category by construction, so we can use corollary 8.1.9 to check that $\otimes_{\mathcal{C}}$ is an equivalence. Indeed, it is clear that $\otimes_{\mathcal{C}}$ is surjective on connected objects, and one also easily sees that $\otimes_{\mathcal{C}}$ is fully faithful on connected morphisms.

For the second claim we begin by defining a functor:

$$(c, r) : \text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}((A, c_A), (B, c_B)) \longrightarrow \text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}(\emptyset, \emptyset) \times \text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}^{\text{red}}((A, c_A), (B, c_B)).$$

Here $c(\underline{X}, o) = (\underline{Y}, o|_{\underline{Y}})$ where $\underline{Y} \subset \underline{X}$ is the subset containing those equivalence classes that are not in the image of $f \amalg g : A \amalg B \rightarrow \underline{X}$. Similarly we let $r(\underline{X}, o) = (\underline{Z}, o|_{\underline{Z}})$ where $\underline{Z} \subset \underline{X}$ is the image of $f \amalg g$. One now checks that (c, r) is indeed a functor and that it is an isomorphism between $\mathbb{C}\text{sp}((A, c_A), (B, c_B))$ and the subspace of the product on those $((\underline{Y}, o), (\underline{Z}, o'))$ such that \underline{Y} and \underline{Z} are disjoint. Hence (c, r) is fully faithful, and it is essentially surjective because any tuple is isomorphic to one that satisfies the disjointness. It remains to check that the functor \otimes induces an equivalence of groupoids:

$$\text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}^{\text{red}}((A, c_A), (B, c_B)) \longrightarrow \text{Hom}_{\mathcal{C}}^{\text{red}}(\otimes(A, c_A), \otimes(B, c_B)).$$

By lemma 8.1.8 this functor induces a bijection between the isomorphism classes of objects on the left and the elements on the right. Moreover, any object in $\text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}^{\text{red}}((A, c_A), (B, c_B))$ only has the identity automorphism as $\varphi : \underline{X} \rightarrow \underline{X}$ is uniquely determined by the requirement that it has to be compatible with the surjection $A \amalg B \rightarrow \underline{X}$. The right-hand side has only identity morphisms by definition and hence the two groupoids are equivalent.

Note also that $\text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})}(\emptyset, \emptyset)$ is equivalent to the groupoid of finite sets labelled in $\text{Hom}_{\mathcal{C}}^{\text{con}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$, which is indeed the free symmetric monoidal groupoid on this set. \square

8.1.2.3 Higher labelled cospan categories

While the list of conditions used to characterise labelled cospan categories in definition 8.1.4 is rather long and might seem a bit arbitrary, the enhanced version $\mathbb{C}\text{sp}(\mathcal{C})$ satisfies a much simpler condition:

Lemma 8.1.29. *For any labelled cospan category $(\pi : \mathcal{C} \rightarrow \mathbb{C}\text{sp})$ the following diagram is a pullback square of 2-categories:*

$$\begin{array}{ccc} \mathbb{C}\text{sp}(\mathcal{C})^{2\perp} & \xrightarrow{\amalg} & \mathbb{C}\text{sp}(\mathcal{C}) \\ \downarrow (\pi_0)^2 & & \downarrow \pi_0 \\ \mathbb{C}\text{sp}^{2\perp} & \xrightarrow{\amalg} & \mathbb{C}\text{sp}. \end{array}$$

Proof. On objects we have that

$$\begin{aligned} & \{A, B \subset \Omega, l : A, B \rightarrow \text{Obj}^{\text{con}}(\mathcal{C}) \mid A \cap B = \emptyset\} \\ & \cong \{A, B \subset \Omega \mid A \cap B = \emptyset\} \times_{\{C \subset \Omega\}} \{C \subset \Omega, l : C \rightarrow \text{Obj}^{\text{con}}(\mathcal{C})\} \end{aligned}$$

and a similar statement is true for 1-morphisms and 2-morphisms. \square

This motivates the following definition, which we will give in the language of ∞ -categories.

Definition 8.1.30. *A labelled ∞ -category of cospans is a symmetric monoidal ∞ -category \mathcal{C} together with a symmetric monoidal functor $\pi : \mathcal{C} \rightarrow \mathbb{C}\text{sp}$. This functor is subject to the condition that*

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{\otimes} & \mathcal{C} \\ \downarrow \pi \times \pi & & \downarrow \pi \\ \mathbb{C}\text{sp} \times \mathbb{C}\text{sp} & \xrightarrow{\amalg} & \mathbb{C}\text{sp}. \end{array}$$

is a pullback square of ∞ -categories.

We will not explore this definition in much more detail - it mainly serves as an aesthetically more pleasing analogue of definition 8.1.4. However, let us briefly note that one can show that for any ∞ -category of labelled cospans ($\mathcal{C} \rightarrow \mathbb{C}\text{sp}$) its homotopy category is a labelled cospan category ($h\mathcal{C} \rightarrow h\mathbb{C}\text{sp} \simeq \mathbb{C}\text{sp}$) in the sense of definition 8.1.4. Together with lemma 8.1.29 this almost proves that labelled cospan categories form a full reflective subcategory of the ∞ -category of labelled ∞ -categories of cospans.

Further, we suggest the following conjecture that relates the notions of labelled cospan category and labelled ∞ -category of cospans to previously established notions.

Conjecture 8.1.31. *There is a biequivalence between the 2-category of labelled cospan categories and the 2-category of coloured properads as described in [HRY15, Chapter 3].*

Let $\text{Cat}_{\infty}^{\otimes}$ be the ∞ -category of symmetric monoidal ∞ -categories and let $\mathcal{P} \subset (\text{Cat}_{\infty}^{\otimes})_{/\mathbb{C}\text{sp}}$ be the full subcategory of the comma category on those $(\mathcal{C} \rightarrow \mathbb{C}\text{sp})$ that satisfy the condition of definition 8.1.30. Then \mathcal{P} is equivalent to the ∞ -category of ∞ -properads described in [HRY15, Chapter 7].

8.1.3 The decomposition theorem for cospan categories

For the purpose of this section we fix a cospan category $(\mathcal{C} \rightarrow \mathbb{C}\text{sp})$ and we assume that $B\mathcal{C}$ is group-like. In lemma 8.1.40 we explain this assumption is often satisfied, and we note that in this case $B\mathcal{C}$ is equivalent to an infinite loop space. Moreover, we let $G := \text{Hom}_{\mathcal{C}}^{\text{con}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ denote the set of connected endomorphisms of the unit in \mathcal{C} .

We prove the following comparison result for $\mathbb{C}\text{sp}(\mathcal{C})$, \mathcal{C} , and \mathcal{C}^{red} , which is a generalisation of theorem C from part I of this thesis.

Proposition 8.1.32 (See 8.2.4). *There are homotopy fiber sequences of infinite loop spaces:*

$$\begin{array}{ccccc} Q(\bigvee_G S^1) & \longrightarrow & B(\mathbb{C}\text{sp}(\mathcal{C})) & \longrightarrow & B(\mathcal{C}^{\text{red}}) \\ \downarrow & & \downarrow & & \parallel \\ B(\bigoplus_G \mathbb{N}) & \longrightarrow & B(\mathcal{C}) & \longrightarrow & B(\mathcal{C}^{\text{red}}). \end{array}$$

In particular, the map $B(\mathbb{C}\text{sp}(\mathcal{C})) \rightarrow B(\mathcal{C})$ is a rational equivalence.

The next theorem will decompose the category \mathcal{C} into parts with no closed components and factorisations of closed components. We first make the necessary definitions:

Definition 8.1.33. *The simplicial set $N_{\bullet}^{\text{nc}}\mathcal{C}$ is the simplicial subset of the nerve $N_{\bullet}\mathcal{C}$ containing only those n -simplices $W : [n] \rightarrow \mathcal{C}$ where $W(0 \rightarrow n)$ is a reduced morphism.*

Definition 8.1.34. *For any connected endomorphism of the unit $s : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$ we define the factorisation category $\mathcal{F}_s(\mathcal{C})$, or \mathcal{F}_s for short, as follows. An object is a triple (M, W, V) where $M \not\cong 1_{\mathcal{C}}$ is a non-trivial object and $W : 1_{\mathcal{C}} \rightarrow M$ and $V : M \rightarrow 1_{\mathcal{C}}$ are morphisms such that $W \cup_M V = s$. A morphism $(M, W, V) \rightarrow (M', W', V')$ is a morphism $X : M \rightarrow M'$ in \mathcal{C} such that $W \cup_M X = W'$ and $V = X \cup_{M'} V'$.*

Theorem 8.1.35 (Decomposition Theorem, see 8.2.1). *For any labelled cospan category \mathcal{C} such that $B\mathcal{C}$ is group-like, there is a homotopy fiber sequence of infinite loop spaces:*

$$|N_{\bullet}^{\text{nc}}\mathcal{C}| \longrightarrow B(\mathbb{C}\text{sp}(\mathcal{C})) \longrightarrow Q \left(\bigvee_{g \in G} \Sigma S(B\mathcal{F}_g) \right).$$

Here the wedge runs over the set of connected morphisms $g : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$.

REMARK 8.1.36. *Note that as long as each \mathcal{F}_g is non-empty the unreduced suspension $S(B\mathcal{F}_g)$ is equivalent to the reduced suspension $\Sigma(B\mathcal{F}_g)$ for any choice of base-point.*

REMARK 8.1.37. *If $B\mathcal{C}$ is not group-like, then the above is still a homotopy cofiber sequence of E_{∞} -algebras. However, this case will not be as useful for computations. One of the particular advantages of the group-like case is that any homotopy cofiber sequence of infinite loop spaces is also a homotopy fiber sequence, and moreover induces a homotopy fiber sequence of underlying spaces. In the general case we still have a homotopy fiber sequence after passing to group-completions.*

Example 8.1.38. *In section 8.2.9 we will show that for $\mathcal{C} = \mathbb{C}\text{sp}$ and $\mathcal{C} = \text{Cob}_d$ with $d \geq 2$ all the factorisation categories $\mathcal{F}_g(\mathcal{C})$ have a contractible classifying space. In these cases it follows that $|N_{\bullet}^{\text{nc}}\mathcal{C}| \simeq B(\mathbb{C}\text{sp}(\mathcal{C}))$.*

Since $N_{\bullet}^{\text{nc}}\mathcal{C}$ is not the nerve of a category it is not a priori clear that its homotopy type is invariant under equivalences in \mathcal{C} .

Lemma 8.1.39. *Let $(\pi : \mathcal{C} \rightarrow \mathbb{C}\text{sp})$ and $(\pi' : \mathcal{D} \rightarrow \mathbb{C}\text{sp})$ be two labelled cospan categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be an equivalence of symmetric monoidal categories together with a symmetric monoidal natural isomorphism $F \circ \pi \cong \pi'$. Then F induces an equivalence:*

$$N_{\bullet}F : N_{\bullet}^{\text{nc}}\mathcal{C} \xrightarrow{\simeq} N_{\bullet}^{\text{nc}}\mathcal{D}.$$

Proof. Pick a homotopy-inverse functor $G : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $\alpha : F \circ G \cong \text{Id}_{\mathcal{D}}$ and $\beta : G \circ F \cong \text{Id}_{\mathcal{C}}$. Then we have maps $BF : BC \leftarrow BD : BG$ and α and β yield homotopies that witness that BF and BG are homotopy inverse. This homotopy equivalence restricts to a homotopy equivalence $|N_{\bullet}^{\text{nc}}\mathcal{C}| \simeq |N_{\bullet}^{\text{nc}}\mathcal{D}|$. To see this, note that if $a : [n] \rightarrow \mathcal{C}$ represents an n -simplex $a \in N_n\mathcal{C}$ that lies in $N_n^{\text{nc}}\mathcal{C}$ and $a' : [n] \rightarrow \mathcal{C}$ is naturally isomorphic to a , then a' lies in the no-closed components part as well: $a' \in N_n^{\text{nc}}\mathcal{C}$. \square

We briefly discuss some sufficient conditions for BC to be group-like.

Lemma 8.1.40. *Let $(\mathcal{C} \rightarrow \text{Csp})$ be a labelled cospan category satisfying that for all connected $M \in \mathcal{C}$ there is some object $N \in \mathcal{C}$ for which there exists a morphism $W : M \otimes N \rightarrow 1_{\mathcal{C}}$ or a morphism $V : 1_{\mathcal{C}} \rightarrow M \otimes N$. Then $\pi_0 BC$ is group-like and hence BC is equivalent to an infinite loop space.*

Proof. The connected components of the classifying space $\pi_0 B(\mathcal{C})$ can be computed as the commutative monoid of isomorphism classes of objects of \mathcal{C} modulo the relation $A \sim B$ whenever there is a morphism from A to B . By axiom (i) of definition 8.1.4 every object of \mathcal{C} can be written as a \otimes -product of connected objects. So the connected objects generate $\pi_0 B(\mathcal{C})$. The conditions of this lemma imply that for each connected object M there is some object N with $[M] \otimes [N] = [1_{\mathcal{C}}]$ in $\pi_0 B(\mathcal{C})$. This implies that all the connected objects have inverses and as these generate the monoid we can conclude that $\pi_0 B(\mathcal{C})$ is indeed a group under \otimes . It now follows from the methods of [Seg74], which we recall in section 8.2.5, that BC is equivalent to an infinite loop space. \square

8.1.4 The surgery theorem for cospan categories

In this section we try to understand the realisation of the simplicial set $N_{\bullet}^{\text{nc}}\mathcal{C}$, which is one of the two main components in the decomposition theorem. We will show that under favorable circumstances it is equivalent to the classifying space of the positive boundary subcategory $\mathcal{C}^{\partial+} \subset \mathcal{C}$, which will often be computable.

Definition 8.1.41. *For a labelled cospan category $(\mathcal{C} \rightarrow \text{Csp})$ we define the positive boundary category $\mathcal{C}^{\partial+}$ as the subcategory of \mathcal{C} that contains all objects, but only those morphisms $W : M \rightarrow N$ for which $\pi(N) \rightarrow \pi(W)$ is surjective.*

Definition 8.1.42. *We say that a labelled cospan category $(\mathcal{C} \rightarrow \text{Csp})$ admits surgery if one can pick the following data:*

- A connected object $O \in \mathcal{C}$ and a connected morphism $T : 1_{\mathcal{C}} \rightarrow O$.
- For any connected object $A \in \mathcal{C}$ a connected morphism $P_A : A \rightarrow O \otimes A$.

subject to the condition that for any two connected objects $A, B \in \mathcal{C}$ and any two arbitrary objects $M, N \in \mathcal{C}$ the following diagrams commute for all connected morphisms $U : A \otimes M \rightarrow B \otimes N$, $V : A \otimes B \otimes M \rightarrow N$, and $W : M \rightarrow A \otimes B \otimes N$:

$$\begin{array}{ccc}
A \otimes M & \xrightarrow{P_A \otimes \text{id}_M} & O \otimes A \otimes M & & A \otimes B \otimes M & \xrightarrow{P_A \otimes \text{id}_{B \otimes M}} & O \otimes A \otimes B \otimes M \\
\downarrow U & & \downarrow \text{id}_O \otimes U & & \downarrow (\beta_{O,A} \otimes \text{id}_{B \otimes M}) \circ (\text{id}_A \otimes P_B \otimes \text{id}_M) & & \downarrow \text{id}_O \otimes V \\
B \otimes N & \xrightarrow{P_B \otimes \text{id}_N} & O \otimes B \otimes N & & O \otimes A \otimes B \otimes M & \xrightarrow{\text{id}_O \otimes V} & O \otimes N
\end{array}$$

$$\begin{array}{ccc}
M & \xrightarrow{W} & A \otimes B \otimes N \\
\downarrow W & & \downarrow (\beta_{A,O} \otimes \text{id}_{B \otimes N}) \circ (\text{id}_A \otimes P_B \otimes \text{id}_N) \\
A \otimes B \otimes N & \xrightarrow{P_A \otimes \text{id}_{B \otimes N}} & O \otimes A \otimes B \otimes N.
\end{array}$$

See figure 8.1.4 for an illustration of these three equations.

REMARK 8.1.43. Note that the morphism T is not required to satisfy any equation. (In fact, we do not even need it to be connected.) Nevertheless, it is crucial that there exists a morphism from $1_{\mathcal{C}}$ to O as we use it to introduce new connected components during the surgery. This would for example be impossible in the negative boundary category $\text{Cob}_d^{\partial^-} = (\text{Cob}_d^{\partial^+})^{\text{op}}$. This category cannot satisfy the conclusion of the surgery theorem as

$$\pi_0 B((\text{Cob}_d^{\partial^-})^{\partial^+}) \longrightarrow \pi_0 |N_{\bullet}^{\text{nc}}(\text{Cob}_d^{\partial^+})| = \pi_0 B(\text{Cob}_d^{\partial^+}) \cong \Omega_{d-1}^{\text{or}}$$

is not injective because in $(\text{Cob}_d^{\partial^-})^{\partial^+}$ there are no morphisms to or from the empty set.

Theorem 8.1.44 (Surgery Theorem, see 8.3.1). *If the labelled cospan category $(\mathcal{C} \rightarrow \text{Csp})$ admits surgery, then the inclusion of simplicial sets $N_{\bullet} \mathcal{C}^{\partial^+} \subset N_{\bullet}^{\text{nc}} \mathcal{C}$ induces an equivalence:*

$$B(\mathcal{C}^{\partial^+}) \simeq |N_{\bullet}^{\text{nc}} \mathcal{C}|.$$

Combining this with the decomposition theorem we have the following immediate consequence:

Corollary 8.1.45 (Theorem N). *If the labelled cospan category $(\mathcal{C} \rightarrow \text{Csp})$ admits surgery and $B(\mathcal{C})$ is group-like, then there is a homotopy fiber sequence of infinite loop spaces.*

$$B(\mathcal{C}^{\partial^+}) \longrightarrow B(\text{Csp}(\mathcal{C})) \longrightarrow Q \left(\bigvee_{g \in G} \Sigma S(B\mathcal{F}_g) \right).$$

The condition of admitting surgery might seem unreasonably strong if one thinks about a general labelled cospan category $(\mathcal{C} \rightarrow \text{Csp})$. We are essentially requiring that $P_A : A \rightarrow O \otimes A$ commutes with all other operations. However, it turns out to be satisfied in the cases we are interested in.

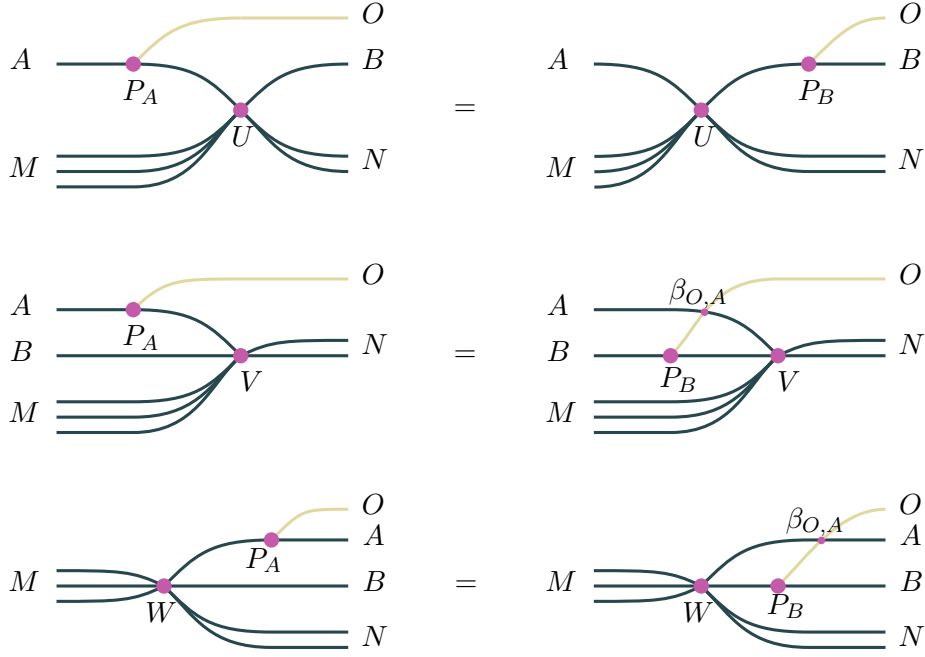


Figure 8.1: The three conditions that the morphism $P_A : A \rightarrow O \otimes A$ has to satisfy for all connected objects B , arbitrary objects M, N , and connected morphisms U, V, W .

Example 8.1.46. We will now show that many geometric examples admit surgery. Let, for example, $\mathcal{C} = \text{Cob}_d$ be the d -dimensional bordism category for $d \geq 2$. We can let $O := S^{d-1}$ be the sphere and $T := D^d : \emptyset \rightarrow S^{d-1}$ the disk. For the morphism $P_M : M \rightarrow S^{d-1} \amalg M$ we can choose the connected sum of the bordisms $D^d : \emptyset \rightarrow S^{d-1}$ and $M \times [0, 1] : M \rightarrow M$. Since both bordisms are oriented and connected it does not matter at which point we take the connected sum. We need to make sure that this makes the relevant diagrams commute. For example, for all connected morphisms $W : A \amalg M \rightarrow B \amalg N$ one checks that both ways of going around the diagram can be described as the connected sum of W with the bordism $D^{d-1} : \emptyset \rightarrow S^{d-1}$. Since W is connected any two such connected sums are diffeomorphic.

We can also replace the morphism $T : \emptyset \rightarrow S^{d-1}$ by any other connected manifold with boundary S^{d-1} . In particular for $d = 2$ we can choose a genus $g \geq 1$ surface with one boundary component, which implies that $\text{Cob}_2^{\chi \leq 0} \subset \text{Cob}_2$ also admits surgery.

Example 8.1.47. As similar argument as above also applies to the weighted cospan categories $\text{Csp}(A, A_1, \alpha)$ from definition 8.1.13. One can choose $O := *$ to be the singleton, $T : \emptyset \rightarrow *$ the cospan ($\emptyset \rightarrow * = *$) weighted by any element of A_1 (for example α), and $P : * \rightarrow * \amalg *$ the cospan ($* \rightarrow * \leftarrow * \amalg *$) weighted by the unit $0 \in A$. This makes all the relevant diagrams commute as the weighting monoid A is commutative.

By lemma 8.1.14, example 8.1.17, and example 8.1.18 this means that all of the following labelled

cospan categories admit surgery:

$$\text{Csp}, \quad \text{Cob}_2^{\chi \leq 0}, \quad \text{Cob}_d^{S^{d-1}}, \quad \text{Cob}_2^{g < \gamma}, \quad \text{and} \quad \text{Cob}_2^{\chi \leq 0, g < \gamma}.$$

Example 8.1.48. *If a labelled cospan category \mathcal{C} admits surgery, has a group-complete classifying space BC , and satisfies that $\mathcal{F}_g(\mathcal{C})$ is contractible for all $g : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$, then the inclusion of the positive boundary subcategory induces an equivalence:*

$$B(\mathcal{C}^{\partial+}) \simeq B(\text{Csp}(\mathcal{C})).$$

We will show that this is the case for Csp and Cob_d with $d \geq 2$ in section 8.2.9. This implies Theorem P and together with the computations $B\text{Csp}^{\partial+} \simeq *$ and $B\text{Cob}_2^{\partial+} \simeq S^1$ made in proposition 8.3.27 it implies Theorem O and Theorem J.

8.2 The decomposition theorem

In this section we decompose the classifying space of a labelled cospan category ($\mathcal{C} \rightarrow \text{Csp}$) into various smaller parts that are usually easier to compute. We will also show that the two variants $\text{Csp}(\mathcal{C})$ and \mathcal{C}^{red} are closely related to \mathcal{C} . Concretely we prove the following theorem:

Theorem 8.2.1 (Decomposition theorem). *Let $(\mathcal{C} \rightarrow \text{Csp})$ be a labelled cospan category such that BC is group-like and let $G = \text{Hom}_{\mathcal{C}}^{\text{con}}(1_{\mathcal{C}}, 1_{\mathcal{C}})$ be the set of connected endomorphisms of the unit. Then there is a homotopy fiber-sequence of infinite loop spaces:*

$$|N_{\bullet}^{\text{nc}}(\mathcal{C})| \longrightarrow B(\text{Csp}(\mathcal{C})) \longrightarrow Q\left(\bigvee_{g \in G} \Sigma S(B\mathcal{F}_g)\right)$$

REMARK 8.2.2. *Recall that we in 8.1.33 defined $N_{\bullet}^{\text{nc}}\mathcal{C} \subset N_{\bullet}\mathcal{C}$ as the simplicial subset on those n -simplices $W : [n] \rightarrow \mathcal{C}$ such that $W(0 \rightarrow n)$ has no closed components.*

Recall that SX denotes the unreduced suspension of an unpointed space. (See 8.2.37.) If we assume that each \mathcal{F}_g is non-empty, then we may choose any basepoint to form the reduced suspension and the result will be equivalent to the unreduced suspension: $\Sigma S(B\mathcal{F}_g) \simeq \Sigma^2 B\mathcal{F}_g$. However, it can happen that a connected endomorphism $g : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$ does not admit a factorisation. (For example if one adds spheres to $\text{Cob}_2^{\chi \leq 0}$.) In this case $B\mathcal{F}_g = \emptyset$ and by convention $S(\emptyset) = S^0$, so $\Sigma S(B\mathcal{F}_g) = S^1$.

REMARK 8.2.3. *One might think of the term $Q(\bigvee_{g \in G} \Sigma S(B\mathcal{F}_g))$ as an “error-term” that obstructs $B\text{Csp}(\mathcal{C})$ from being equivalent to the subspace with no closed components. In the analogy where \mathcal{C} is a topologically enriched bordism category this term always vanishes: it is a theorem by [GRW10, Definition 4.3 and lemma 4.7] that in these cases BC is equivalent to the subspace with no closed components. Note however, that this is only an analogy as our setting of labelled cospan categories does not actually accommodate for topologically enriched bordism categories, and moreover the definition of D_{\bullet}^{nc} is different in the topological case.*

We also prove the following comparison result for $\mathbb{C}\text{sp}(\mathcal{C})$, \mathcal{C} , and \mathcal{C}^{red} :

Proposition 8.2.4. *There are homotopy fiber sequences of infinite loop spaces:*

$$\begin{array}{ccccc} Q(\bigvee_G S^1) & \longrightarrow & B(\mathbb{C}\text{sp}(\mathcal{C})) & \longrightarrow & B(\mathcal{C}^{\text{red}}) \\ \downarrow & & \downarrow & & \parallel \\ B(\bigoplus_G \mathbb{N}) & \longrightarrow & B(\mathcal{C}) & \longrightarrow & B(\mathcal{C}^{\text{red}}). \end{array}$$

In particular the map $B(\mathbb{C}\text{sp}(\mathcal{C})) \rightarrow B(\mathcal{C})$ is a rational equivalence.

8.2.1 Partial commutative monoidal categories

Throughout this paper we will work with infinite loop spaces that arise as the classifying space of a symmetric monoidal categories. The symmetric monoidal categories that we consider will often be like embedded bordism categories in the sense that we can make sense of a notion of disjointness $M \cap N = \emptyset$ for objects M, N and that for disjoint objects taking their union $(M, N) \mapsto M \cup N$ is a strictly associative and commutative operation. To extract a symmetric monoidal structure from this we need to fix a functorial replacement $(M, N) \mapsto (M', N')$ such that $M' \cap N' = \emptyset$. This choice is always somewhat arbitrary, which would make it difficult to construct the simplicial objects in symmetric monoidal categories that we need later. Instead of making a choice of a replacement we will work with the notion of a partial commutative monoidal category, which formalises the idea of having a notion of disjointness and disjoint union.

Definition 8.2.5. *A partial commutative monoidal structure on a category \mathcal{C} is a choice of a relation \perp on the set of objects of \mathcal{C} , an object $\emptyset \in \mathcal{C}$, and a functor $\amalg : \mathcal{C}^{2\perp} \rightarrow \mathcal{C}$ where $\mathcal{C}^{2\perp} \subset \mathcal{C}^2$ is the full subcategory on those (x, y) where $x \perp y$. This data is subject to the following axioms:*

- (i) *For all $x \in \mathcal{C}$ we have $\emptyset \perp x$ and the functor $\mathcal{C} \rightarrow \mathcal{C}$ defined by $x \mapsto \emptyset \amalg x$ is the identity.*
- (ii) *Let $\mathcal{C}^{3\perp} \subset \mathcal{C}^3$ denote the full subcategory on triples (x, y, z) satisfying $x \perp y$, $x \perp z$, and $y \perp z$. Then for each such triple we have $x \perp (y \amalg z)$ and $(x \amalg y) \perp z$, and the two functors $\mathcal{C}^{3\perp} \rightarrow \mathcal{C}$ defined by $(x, y, z) \mapsto (x \amalg y) \amalg z$ and $(x, y, z) \mapsto x \amalg (y \amalg z)$ are equal.*
- (iii) *The relation \perp is symmetric and the two functors $\mathcal{C}^{2\perp} \rightarrow \mathcal{C}$ defined by $(x, y) \mapsto x \amalg y$ and $(x, y) \mapsto y \amalg x$ are equal.*
- (iv) *For any n -tuple $(x_1, \dots, x_n) \in \mathcal{C}^n$ one can find $(x'_1, \dots, x'_n) \in \mathcal{C}^n$ such that $x_i \cong x'_i$ for all i and $x'_i \perp x'_j$ for all $i \neq j$.*

Given two such partial commutative monoidal categories - or PCM categories for short - $(\mathcal{C}, \perp_{\mathcal{C}}, \emptyset_{\mathcal{C}}, \amalg_{\mathcal{C}})$ and $(\mathcal{D}, \perp_{\mathcal{D}}, \emptyset_{\mathcal{D}}, \amalg_{\mathcal{D}})$, we say that a strictly symmetric monoidal functor between them is a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ satisfying the properties: $F(\emptyset_{\mathcal{C}}) = \emptyset_{\mathcal{D}}$, $x \perp y \Rightarrow F(x) \perp F(y)$, and $\amalg_{\mathcal{D}} \circ (F, F) = F \circ \amalg_{\mathcal{C}}$ as functors $\mathcal{C}^{2\perp} \rightarrow \mathcal{D}$.

REMARK 8.2.6. One can think of a given PCM category as a category internal to the category of partial commutative monoids. Conversely, a category \mathcal{C} internal to partial commutative monoids represents a PCM category if it satisfies the following conditions:

- For any two morphisms $f : x \rightarrow y$ and $g : x' \rightarrow y'$ we have that $f \perp g$ if and only if $x \perp x'$ and $y \perp y'$.
- For any n -tuples of objects (x_1, \dots, x_n) one can find $(x'_1, \dots, x'_n) \in \mathcal{C}^n$ such that $x_i \cong x'_i$ for all i and $x'_i \perp x'_j$ for all $i \neq j$.

REMARK 8.2.7. The key advantage of this definition is that it is a property for a functor F to be commutatively monoidal. This will make it much easier to construct simplicial objects in PCMC than it would be to construct them in the category of symmetric monoidal categories.

To obtain infinite loop spaces from PCM categories we will use Segal's concept of Γ -spaces, which we now recall.

Definition 8.2.8. Let Γ^{op} denote the full subcategory of the category of finite pointed sets Fin_* on the objects $\langle n \rangle = \{*, 1, \dots, n\}$. Concretely, the morphisms $\langle n \rangle \rightarrow \langle m \rangle$ in Γ^{op} are maps $f : \{*, 1, \dots, n\} \rightarrow \{*, 1, \dots, m\}$ satisfying $f(*) = *$. We let $\rho_n^i : \langle n \rangle \rightarrow \langle 1 \rangle$ denote the unique morphism with $(\rho_n^i)^{-1}(1) = \{i\}$.

A Γ -space is a functor $X : \Gamma^{\text{op}} \rightarrow \text{Top}$ and we denote the value on $\langle n \rangle$ by $X^{\langle n \rangle}$. We say that X is a special Γ -space if for each n the following map is a weak equivalence:

$$(\rho_n^1, \dots, \rho_n^n) : X^{\langle n \rangle} \rightarrow X^{\langle 1 \rangle} \times \dots \times X^{\langle 1 \rangle}.$$

Our main source of special Γ -spaces will be of the form $BC^{\langle \bullet \rangle}$ for $\mathcal{C}^{\langle \bullet \rangle}$ is a special Γ -category.

Definition 8.2.9. A Γ -category is a functor $\mathcal{C}^{\langle \bullet \rangle} : \Gamma^{\text{op}} \rightarrow \text{Cat}$. We say that $\mathcal{C}^{\langle \bullet \rangle}$ is a special if for each n the map $(\rho_n^1, \dots, \rho_n^n) : \mathcal{C}^{\langle n \rangle} \rightarrow \mathcal{C}^{\langle 1 \rangle} \times \dots \times \mathcal{C}^{\langle 1 \rangle}$ is an equivalence of categories.

Lemma 8.2.10. For any partial commutative monoidal category $(\mathcal{C}, \perp, \emptyset, \amalg)$ we can define a special Γ -category by letting $\mathcal{C}^{\langle n \rangle} := \mathcal{C}^{n\perp}$ be the full subcategory of \mathcal{C}^n on those tuples (x_1, \dots, x_n) satisfying $x_i \perp x_j$ for all $i \neq j$.

Proof. It follows from axioms (i-iii) that for any finite family of objects $(x_i \in \mathcal{C})_{i \in I}$ that are disjoint in the sense that $i \neq j \Rightarrow x_i \perp x_j$ we can define:

$$\coprod_{i \in I} x_i := (\dots((x_{i_1} \amalg x_{i_2}) \amalg x_{i_3}) \dots \amalg x_{i_n})$$

and that this is independent of the enumeration (i_1, \dots, i_n) one chooses of I . Here we set $\coprod_{i \in \emptyset} x_i := \emptyset_{\mathcal{C}}$. We may therefore define for each $\lambda : \langle n \rangle \rightarrow \langle m \rangle$ a functor

$$\lambda_* : \mathcal{C}^{n\perp} \rightarrow \mathcal{C}^{m\perp}, \quad (x_1, \dots, x_n) \mapsto (y_1, \dots, y_m), \text{ where } y_j = \coprod_{i \in \lambda^{-1}(j)} x_i.$$

This makes $\langle n \rangle \mapsto \mathcal{C}^{n\perp}$ into a Γ -category. It remains to check that this is special, i.e. that the inclusion $\mathcal{C}^{n\perp} \rightarrow (\mathcal{C}^{1\perp})^n = \mathcal{C}^n$ is an equivalence of categories. This inclusion is fully faithful by definition and it is essentially surjective by axiom (iv). \square

We also briefly describe how to generalise this to the setting of 2-categories.

Definition 8.2.11. For a 2-category \mathcal{C} let $\text{Mor}(\mathcal{C})$ be the 1-category where objects are 1-morphisms $f : x \rightarrow y$ in \mathcal{C} and morphisms $f \rightarrow g$ are 2-morphisms $\alpha : f \Rightarrow g$. In particular there are no morphisms between f and g unless they have the same source and the same target.

A partial commutative monoidal structure on a 2-category \mathcal{C} is a partial commutative monoid structure $(\text{id}_{\emptyset}, \perp, \text{II})$ on $\text{Mor}(\mathcal{C})$ – except that instead of condition (iv) we require:

- (iv') For any n -tuple $(x_1, \dots, x_n) \in \mathcal{C}^n$ one can find an n -tuple $(x'_1, \dots, x'_n) \in \mathcal{C}^n$ such that x_i is equivalent to x'_i for all i and $\text{id}_{x'_i} \perp \text{id}_{x'_j}$ for all $i \neq j$.
- (iv'') For any n -tuple $(f_1 : x_1 \rightarrow y_1, \dots, f_n : x_n \rightarrow y_n) \in \text{Mor}(\mathcal{C})^n$ satisfying $x_i \perp x_j$ and $y_i \perp y_j$ for all $i \neq j$ one can find an n -tuple $(f'_1 : x_1 \rightarrow y_1, \dots, f'_n : x_n \rightarrow y_n) \in \text{Mor}(\mathcal{C})^n$ such that $f_i \cong f'_i$ for all i and $f_i \perp f_j$ for all $i \neq j$.
- (v) Composition defines a PCM functor $\circ : \text{Mor}(\mathcal{C}) \times_{\text{Obj}(\mathcal{C})} \text{Mor}(\mathcal{C}) \rightarrow \text{Mor}(\mathcal{C})$.

We will also write $x \perp y$ for two objects $x, y \in \mathcal{C}$ to mean $\text{id}_x \perp \text{id}_y$.

Corollary 8.2.12. For any PCM 2-category \mathcal{C} the classifying space BC naturally has the structure of a special Γ -space.

Proof. We would like to mimic the proof of lemma 8.2.10. For all n let $\mathcal{C}^{\perp n} \subset \mathcal{C}^n$ be the following sub 2-category. It contains an object (x_1, \dots, x_n) if $\text{id}_{x_i} \perp \text{id}_{x_j}$ holds for all $i \neq j$, and it contains a 1-morphism (f_1, \dots, f_n) between two such objects if $f_i \perp f_j$ holds for all $i \neq j$. It contains all 2-morphism between any 1-morphism it contains. By condition (v) this is indeed a sub 2-category. In the same way as in lemma 8.2.10 these 2-categories assemble into a Γ -object in 2-categories, and therefore $\langle n \rangle \mapsto BC^{\perp n}$ defines a Γ -space.

We need to show that the Segal map $BC^{\perp n} \rightarrow BC^n$ is a weak equivalence. The functor $\mathcal{C}^{\perp n} \rightarrow \mathcal{C}^n$ is surjective up to equivalence condition (iv') and by condition (iv'') it is an equivalence on Hom-categories. Now recall that the classifying space of a 2-category is defined by first taking the nerve of Hom-categories to obtain a simplicially enriched category and then taking the usual classifying space. By the above, the simplicially enriched functor obtained from the 2-functor $\mathcal{C}^{\perp n} \rightarrow \mathcal{C}^n$ is a Dwyer-Kan equivalence, and hence it induces an equivalence on classifying spaces. \square

8.2.2 A general fiber sequence

We now introduce the abstract technique that we will use to show that certain sequences of simplicial symmetric monoidal groupoids yield fiber sequences of infinite loop spaces.

Theorem 8.2.13. *Consider three simplicial special Γ -spaces $A_\bullet, B_\bullet, C_\bullet : \Delta^{\text{op}} \times \Gamma^{\text{op}} \rightarrow \text{Top}$ with two simplicial Γ -maps:*

$$A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{g} C_\bullet$$

and a homotopy of simplicial Γ -maps $\alpha : g \circ f \simeq 1_C$ to the unit in C_\bullet . Assume further that for every n there is a Γ -map $s_n : B_n \rightarrow A_n$ such that the induced maps

$$(s_n, g_n) : B_n^{(1)} \rightarrow A_n^{(1)} \times C_n^{(1)} \quad \text{and} \quad s_n \circ f_n : A_n^{(1)} \rightarrow A_n^{(1)}$$

are weak equivalences of spaces. Then the following is a homotopy cofiber sequence of special Γ -spaces:

$$\|A_\bullet\| \xrightarrow{f} \|B_\bullet\| \xrightarrow{g} \|C_\bullet\|$$

If moreover each of these spaces is group-like, then this is a homotopy fiber sequence.

Proof. We will base this proof on the framework of [GGN15]. Let \mathcal{S} denote the ∞ -category of spaces, which receives a functor $\text{Top} \rightarrow \mathcal{S}$. The category $\text{Mon}_{E_\infty}(\mathcal{S}) \subset \text{Fun}(\Gamma^{\text{op}}, \mathcal{S})$ is defined as the full subcategory on those functors that satisfy a Segal-condition. The homotopy category of special Γ -spaces is pre-additive, i.e. it has a 0-object $*$ and the canonical map $x \amalg y \rightarrow x \times y$ from the coproduct to the product is an isomorphism for all x and y . This allows us to apply lemma 8.2.14 and conclude from the existence of s_n that for each n the sequence:

$$A_n \longrightarrow B_n \longrightarrow C_n$$

is a homotopy cofiber sequence of special Γ -spaces. Since homotopy colimits commute, taking the homotopy colimit over Δ^{op} yields the following cofiber sequence in $\text{Mon}_{E_\infty}(\mathcal{S})$:

$$\text{hocolim}_{\Delta^{\text{op}}} A_\bullet \longrightarrow \text{hocolim}_{\Delta^{\text{op}}} B_\bullet \longrightarrow \text{hocolim}_{\Delta^{\text{op}}} C_\bullet.$$

To compute this homotopy colimit note that the functor $\text{Mon}_{E_\infty}(\mathcal{S}) \rightarrow \mathcal{S}$ that sends a special Γ -space A to the underlying space $A^{(1)}$ is conservative and commutes with sifted colimits. Therefore $(\text{hocolim}_{\Delta^{\text{op}}} A_\bullet)^{(1)} \simeq \text{hocolim}_{\Delta^{\text{op}}} A_\bullet^{(1)}$ where the latter is computed as a homotopy colimit in \mathcal{S} . Given a simplicial topological space $X_\bullet : \Delta^{\text{op}} \rightarrow \text{Top}$ the homotopy colimit of $X_\bullet : \Delta^{\text{op}} \rightarrow \text{Top} \rightarrow \mathcal{S}$ is modelled by the fat geometric realisation $\|X_\bullet\|$. Similarly, for a simplicial special Γ -space $A_\bullet^{(\bullet)} : \Delta^{\text{op}} \times \Gamma^{\text{op}} \rightarrow \text{Top}$ we can construct a Γ -space $\langle n \rangle \mapsto |A_\bullet^{(n)}|$ by taking the level-wise realisation. Note that this Γ -space is still special since $\|_-\|$ commutes with products up to weak equivalence (see [Seg74, Proposition A.1(iii)] or [ERW19a, Theorem 7.2]). It follows by our

previous comments that $\|A_\bullet\|$ is in fact a model for the homotopy colimit in special Γ -spaces. This shows that

$$\|A_\bullet\| \xrightarrow{f} \|B_\bullet\| \xrightarrow{g} \|C_\bullet\|$$

is a homotopy cofiber sequence in $\text{Mon}_{E_\infty}(\mathcal{S})$.

Finally we would like to show that $\|A_\bullet\| \rightarrow \|B_\bullet\| \rightarrow \|C_\bullet\|$ is homotopy fiber sequence in \mathcal{S} if we assume that each of the spaces is group-like. Indeed, for any cofiber sequence $X \rightarrow Y \rightarrow Z$ in $\text{Mon}_{E_\infty}(\mathcal{S})$, the group completion $X^{gp} \rightarrow Y^{gp} \rightarrow Z^{gp}$ is a fiber sequence. If we assume that each of $\|A_\bullet\|$, $\|B_\bullet\|$, and $\|C_\bullet\|$ is already group-like, then it follows that $\|A_\bullet\| \rightarrow \|B_\bullet\| \rightarrow \|C_\bullet\|$ is a homotopy fiber sequence. \square

The following lemma holds in any pre-additive ∞ -category \mathcal{C} , i.e. in every ∞ -category \mathcal{C} such that the homotopy category $h\mathcal{C}$ has a 0-object and such that coproducts and products agree via the canonical map. The same argument applies in any model category whose homotopy category is pre-additive.

Lemma 8.2.14. *Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two morphisms in a pre-additive ∞ -category \mathcal{C} such that $g \circ f$ is null-homotopic. Assume further that there is a map $s : B \rightarrow A$ such that the maps $(s, g) : B \rightarrow A \times C$ and $s \circ f : A \rightarrow A$ are equivalences. Then any choice of a null-homotopy for $g \circ f$ makes $A \rightarrow B \rightarrow C$ into a cofiber sequence in \mathcal{C} .*

Proof. Pick a morphism $r : C \rightarrow B$ such that $s \circ r = 0 : C \rightarrow A$ and $g \circ r = \text{id}_C$. This can be done by lifting $(0, \text{id}_C) : C \rightarrow A \times C$ against the equivalence $(s, g) : B \rightarrow A \times C$. We claim that the maps f and r exhibit B as a coproduct of A and C . So we need to check that $f \amalg r : A \amalg C \rightarrow B$ is an equivalence. Since (s, g) is an equivalence it will suffice to check that $(s, g) \circ (f \amalg r)$ is an equivalence. Using that $h\mathcal{C}$ is pre-additive we can represent this morphism by a 2×2 matrix with entries $s \circ f$, $s \circ r$, $g \circ f$, and $g \circ r$. By construction $s \circ r$ and $g \circ f$ are 0-maps and $s \circ f$ and $g \circ r$ are equivalences. Hence $(s, g) \circ (f \amalg r)$ is the direct sum of two equivalences and therefore itself an equivalence.

Consider the following diagram in the homotopy category $h\mathcal{C}$:

$$\begin{array}{ccccc} * & \longrightarrow & A & \longrightarrow & * \\ \downarrow & & \downarrow f & & \downarrow \\ C & \xrightarrow{r} & B & \xrightarrow{g} & C \end{array}$$

This diagram can be lifted to a diagram $\Delta^2 \times \Delta^1 \rightarrow \mathcal{C}$ in the ∞ -category by choosing any null-homotopy of $g \circ f$ in the right square and the trivial homotopy in the left square. The left-hand square is a pushout square because we checked that f and r exhibit B as the coproduct of A and C . The entire rectangle is a pushout square (independent of which homotopies we chose!) because the top horizontal arrow $* \rightarrow *$ and the bottom horizontal arrow $g \circ r : C \rightarrow C$ both are equivalences.

It therefore follows from the pushout pasting lemma [Lur09a, Lemma 4.4.2.1.] that the right-hand square is a pushout square, which is exactly what we claimed. \square

8.2.3 A base-change theorem for simplicial spaces

We prove a theorem that allows us to change the space of 0-simplices X_0 of a simplicial space X_\bullet without affecting the homotopy type of $\|X_\bullet\|$. This is motivated by the basic idea in category theory that one can change the number of objects of each isomorphism type without changing the category, up to equivalence. In this sense, the following definition can be thought of as a generalisation of the notion of a fully faithful functor:

Definition 8.2.15. *For any semi-simplicial object X_\bullet let $\partial_n^X : X_n \rightarrow (X_0)^{n+1}$ denote the map that sends an n -simplex to its vertices. We call a map of semi-simplicial spaces $f : X_\bullet \rightarrow Y_\bullet$ a homotopy base-change if the map*

$$(\partial_n^X, f_n) : X_n \longrightarrow (X_0)^{n+1} \times_{(Y_0)^{n+1}}^h Y_n$$

is an equivalence for all $n \geq 1$.

The following theorem generalises lemma 5.1.30 and [ERW19a, Theorem 5.2] to arbitrary simplicial spaces that are not necessarily the nerve of a topological category. Note, however, that [ERW19a, Theorem 5.2] has the advantage of applying to weakly unital categories \mathcal{C} , whereas we would need \mathcal{C} to have strict units in order to apply the theorem to $N\mathcal{C}$.

Theorem 8.2.16 (Base-change). *Let $f : X_\bullet \rightarrow Y_\bullet$ be a map of simplicial spaces that is a homotopy base-change in the above sense and satisfies that $\pi_0(f_0) : \pi_0(X_0) \rightarrow \pi_0(Y_0)$ is surjective. Then $\|f\| : \|X_\bullet\| \rightarrow \|Y_\bullet\|$ is a weak equivalence.*

Proof. For any space A let $T(A)_\bullet$ be the simplicial space defined by $T(A)_n := A^{n+1}$ with face and degeneracy operators given by forgetting and repeating. A simplicial map $Z_\bullet \rightarrow T(B)_\bullet$ amounts to the same data as a map $Z_0 \rightarrow B$. We can factor the map $X_\bullet \rightarrow Y_\bullet$ in the theorem as:

$$X_\bullet \longrightarrow T(X_0)_\bullet \times_{T(Y_0)_\bullet}^h Y_\bullet \longrightarrow Y_\bullet$$

The first map is a level-wise equivalence because $X_\bullet \rightarrow Y_\bullet$ is a base-change, and in particular it induces an equivalence on geometric realisations. To prove the theorem we need to show that the second map also induces an equivalence on geometric realisations.

For the purpose of this proof let us interpret “space” to mean an object of the category of simplicial sets $\mathcal{S} := sSet_Q$, which we equip with the Quillen model structure. A simplicial space is hence a bisimplicial set and we equip $\mathcal{S}^{\Delta^{\text{op}}} := \text{Fun}(\Delta^{\text{op}}, sSet_Q)$ with the projective model structure. This proof could also be translated into any other model, or it could be given in the

∞ -category of simplicial spaces.² Since the statement of the theorem is invariant under level-wise weak equivalences we may assume that $X_0 \rightarrow Y_0$ is a fibration, and in this case we may replace the homotopy pullback by the strict pullback.

In fact, we will show: for any surjective fibration of spaces (i.e. Kan-fibration of simplicial sets) $f : A \rightarrow B$ and for any simplicial space Z_\bullet with map $p : Z_0 \rightarrow B$ the natural map

$$\alpha_{(Z_\bullet, p)} : \|T(A)_\bullet \times_{T(B)_\bullet} Z_\bullet\| \longrightarrow \|Z_\bullet\|$$

is an equivalence. This map defines a natural transformation of functors $\mathcal{S}_{/T(B)}^{\Delta^{\text{op}}} \rightarrow \mathcal{S}$. We will prove the theorem by showing:

- (i) The map $\alpha_{(Z_\bullet, p)}$ is an equivalence if Z_\bullet is the discrete simplicial space Δ^n for some n .
- (ii) The category $\mathcal{S}_{/T(B)}^{\Delta^{\text{op}}}$ is generated under homotopy colimits and weak equivalences by objects of the form (Δ^n, p) .
- (iii) The source and target functor of α both commute with homotopy colimits.

Claim (i): For $Z_\bullet = \Delta^n$ the map $p : Z_\bullet \rightarrow T(B)_\bullet$ picks out $(n+1)$ points $p(0), \dots, p(n) \in B$. Since $f : A \rightarrow B$ is surjective we can find $a \in A$ with $f(a) = p(0)$, which we can use to define an extra degeneracy:

$$\begin{aligned} s_{-1} : A^{k+1} \times_{B^{k+1}} (\Delta^n)_k &\longrightarrow A^{k+2} \times_{B^{k+2}} (\Delta^n)_{k+1} \\ ((a_0, \dots, a_n), (l_0, \dots, l_n)) &\longmapsto ((a, a_0, \dots, a_n), (0, l_0, \dots, l_n)) \end{aligned}$$

Hence $\|T(A)_\bullet \times_{T(B)_\bullet} \Delta^n\|$ is contractible and since $\|\Delta^n\|$ is also contractible it follows that $\alpha_{(\Delta^n, p)}$ is an equivalence.

Claim (ii): Let $(p : Z_\bullet \rightarrow T(B)_\bullet) \in \mathcal{S}_{/T(B)}^{\Delta^{\text{op}}}$ be any object in the over category. The category $\mathcal{S}^{\Delta^{\text{op}}}$ is generated by the representables Δ^n under homotopy colimits. Concretely, [Dug01, Proposition 2.9] shows that we can find a diagram $F : I \rightarrow \mathcal{S}^{\Delta^{\text{op}}}$ such that each $F(i)$ is level-wise equivalent to some Δ^{n_i} , the colimit is $\text{colim}_{i \in I} F(i) \cong Z_\bullet$, and the map from the homotopy colimit $\text{hocolim}_{i \in I} F(i) \rightarrow Z_\bullet$ is a weak equivalence.

Using the map $p : Z_\bullet \rightarrow T(B)_\bullet$ we can lift F to a diagram $F : I \rightarrow \mathcal{S}_{/T(B)}^{\Delta^{\text{op}}}$. The homotopy colimit of this diagram is $\text{hocolim}_{i \in I} (F(i) \rightarrow T(B)_\bullet) \simeq (Z_\bullet \rightarrow T(B)_\bullet)$ since the forgetful functor $\mathcal{S}_{/T(B)}^{\Delta^{\text{op}}} \rightarrow \mathcal{S}^{\Delta^{\text{op}}}$ commutes with colimits and preserves weak equivalences.

Claim (iii): The geometric realisation functor $\|_-\| : \mathcal{S}^{\Delta^{\text{op}}} \rightarrow \mathcal{S}$ commutes with homotopy colimits and so does the forgetful functor $\mathcal{S}_{/T(B)}^{\Delta^{\text{op}}} \rightarrow \mathcal{S}^{\Delta^{\text{op}}}$. We need to show that the functor

$$\mathcal{S}_{/T(B)}^{\Delta^{\text{op}}} \rightarrow \mathcal{S}^{\Delta^{\text{op}}}, \quad (Z_\bullet \rightarrow T(B)_\bullet) \mapsto (T(A)_\bullet \times_{T(B)_\bullet} Z_\bullet)$$

²Let us briefly comment on how to this proof would go in ∞ -land. Claim (i) is proven in the same way. Claim (ii) uses the fact that the presheaf category (of Δ) is generated by representables under small colimits, see [Lur09a, Corollary 5.1.5.8]. Claim (iii) follows because in \mathcal{S} colimits are stable under pullback: [Lur09a, Lemma 6.1.3.14].

commutes with homotopy colimits. This functor preserves level-wise weak equivalences (because we assumed that $A \rightarrow B$ is a fibration) so it will suffice to show that it commutes with strict colimits. Strict colimits of bisimplicial sets (over $T(B)_\bullet$) can be computed point-wise, so the claim follows from the fact that in the category of sets the functor $(U \times_V _): \text{Set}/_V \rightarrow \text{Set}$ commutes with colimits. (This functor has a right-adjoint given by $W \mapsto \coprod_{v \in V} \text{Map}(U \times_V \{v\}, W)$.) \square

REMARK 8.2.17. *Note it is crucial for theorem 8.2.16 and corollary 8.2.20 that X_\bullet and Y_\bullet are simplicial spaces. The analogous statements for semi-simplicial spaces are generally not true. Consider for example the semi-simplicial space Y_\bullet with $Y_0 = *$ and $Y_i = \emptyset$ for $i > 0$. The semi-simplicial map $Y_\bullet \amalg Y_\bullet \rightarrow Y_\bullet$ is a base-change and it is surjective on Y_0 , but on geometric realisations it is $* \amalg * \rightarrow *$.*

To see how this is used in the proof note that in the above counter-example $Y_\bullet = \Delta_s^0$ is the semisimplicial 0-simplex, i.e. the representable object on $[0] \in \Delta_{\text{inj}}^{\text{op}}$ in $\mathcal{S}^{\Delta_{\text{inj}}^{\text{op}}}$. This means that already the analogue of claim (i), where we show the theorem for the case that Z_\bullet is representable, fails. Of course one could still prove claim (i) for Δ^n thought of as a semisimplicial space, but the Δ^n do not generate $\mathcal{S}^{\Delta_{\text{inj}}^{\text{op}}}$ under homotopy-colimits.

REMARK 8.2.18. *The theorem also holds with the fat geometric realisations $\|X_\bullet\|$ replaced by the standard one $|X_\bullet|$, as long as both simplicial spaces X_\bullet and Y_\bullet have the property that the quotient maps $\|X_\bullet\| \rightarrow |X_\bullet|$ and $\|Y_\bullet\| \rightarrow |Y_\bullet|$ are weak equivalences. This is the case if X_\bullet and Y_\bullet are good [Seg74, Proposition A.1.(iv)], which is the case if they are the level-wise realisation of a bi-simplicial set. As we will basically only be encountering simplicial spaces defined as the level-wise classifying space of a simplicial groupoid, this will always be the case.*

We now state a special case of the theorem where X_\bullet and Y_\bullet are both obtained as level-wise classifying spaces of simplicial groupoids. To do so we quickly recall the following definition:

Definition 8.2.19. *A functor $P : \mathcal{E} \rightarrow \mathcal{B}$ is an iso-fibration if for any object $e \in \mathcal{E}$ and isomorphism $f : P(e) \cong b$ there is an isomorphism $g : e \rightarrow \bar{b}$ in \mathcal{E} such that $P(\bar{b}) = b$ and $P(g) = f$.*

Corollary 8.2.20. *Let $F_\bullet : \mathcal{X}_\bullet \rightarrow \mathcal{Y}_\bullet$ be a functor between two simplicial groupoids such that*

- $F_0 : \mathcal{X}_0 \rightarrow \mathcal{Y}_0$ is essentially surjective,
- $(\partial_n^{\mathcal{X}}, F_n) : \mathcal{X}_n \rightarrow (\mathcal{X}_0)^{n+1} \times_{(\mathcal{Y}_0)^{n+1}} \mathcal{Y}_n$ is an equivalence of categories, and
- For all n at least one of the functors $\partial_n^{\mathcal{Y}} : \mathcal{Y}_n \rightarrow (\mathcal{Y}_0)^{n+1}$ and $F_n : \mathcal{X}_n \rightarrow \mathcal{Y}_n$ is an iso-fibration.

Then $|BF| : |B\mathcal{X}_\bullet| \rightarrow |B\mathcal{Y}_\bullet|$ is an equivalence.

Proof. The first condition implies the surjectivity of $\pi_0(BF_0) : B\mathcal{X}_0 \rightarrow B\mathcal{Y}_0$. The second condition implies that BF_\bullet is a homotopy base-change, if we can show that the map

$$B((\mathcal{X}_0)^{n+1} \times_{(\mathcal{Y}_0)^{n+1}} \mathcal{Y}_n) \longrightarrow (B\mathcal{X}_0)^{n+1} \times_{(B\mathcal{Y}_0)^{n+1}}^h B\mathcal{Y}_n$$

is an equivalence. By the third condition at least one of the functors involved in the pullback is an iso-fibration. Taking the pullback of classifying spaces along an iso-fibration of groupoids always yields the homotopy pullback - this can be shown by elementary means or by noting that iso-fibrations of groupoids are both Cartesian and coCartesian, so that theorem B may be applied. \square

To illustrate the use of the base-change theorem we quickly prove that the realisation of Rezk's relative nerve is always equivalent to the classifying space of the category. This will be useful later. First recall:

Definition 8.2.21. For any category \mathcal{C} its relative nerve³ is the simplicial groupoid $N_\bullet^{\text{rel}}\mathcal{C}$ defined for each n as the groupoid of functors $[n] \rightarrow \mathcal{C}$ with morphisms being natural isomorphisms. This contains the usual nerve as a simplicial subgroupoid $N_\bullet\mathcal{C} \subset N_\bullet^{\text{rel}}\mathcal{C}$ with only identity morphisms.

Lemma 8.2.22. The inclusion $N_\bullet\mathcal{C} \subset N_\bullet^{\text{rel}}\mathcal{C}$ always induces an equivalence:

$$B\mathcal{C} \simeq \|N_\bullet\mathcal{C}\| \simeq \|B(N_\bullet^{\text{rel}}\mathcal{C})\|.$$

Proof. We need to verify the conditions of the base-change theorem for simplicial groupoids 8.2.20. The functor $N_0\mathcal{C} \rightarrow N_0^{\text{rel}}\mathcal{C}$ is a bijection on objects and in particular essentially surjective. For any n the functor

$$N_n\mathcal{C} \rightarrow (N_0\mathcal{C})^{n+1} \times_{(N_0^{\text{rel}}\mathcal{C})^{n+1}} N_n^{\text{rel}}\mathcal{C}$$

is an isomorphism of groupoids, as both sides can be identified with the subcategory of $N_n^{\text{rel}}\mathcal{C}$ containing only identity morphisms. Finally, we claim that $\partial_n : N_n^{\text{rel}}\mathcal{C} \rightarrow (N_0^{\text{rel}}\mathcal{C})^{n+1}$ is an iso-fibration for all n . Indeed, given a functor $X : [n] \rightarrow \mathcal{C}$ and isomorphisms $\alpha_i : X(i) \cong X'(i)$ there is a (unique) functor $X' : [n] \rightarrow \mathcal{C}$ extending the $X'(i)$ such that $\alpha : X \cong X'$ is a natural isomorphism. (This functor X' is given by $X'(i \leq j) = \alpha_j \circ X(i \leq j) \circ \alpha_i^{-1}$.) \square

8.2.4 The simplicial groupoid $D(\mathcal{C})$

Now that all the tools are in place we define the simplicial groupoid $D_\bullet(\mathcal{C})$ that serves as a more convenient replacement for $NC\text{sp}(\mathcal{C})$.⁴ One can think of $D_\bullet(\mathcal{C})$ as Rezk's *relative nerve* construction applied to the $(2, 1)$ -category $\mathbb{C}\text{sp}(\mathcal{C})$. We will see that the realisation of $D_\bullet(\mathcal{C})$ is equivalent to the classifying space of $\mathbb{C}\text{sp}(\mathcal{C})$. The advantage of $D_\bullet(\mathcal{C})$ is that for each n the groupoid $D_n(\mathcal{C})$

³This is what Rezk calls the classifying diagram of the category, see [Rez01, Section 3.5].

⁴The name D is based on the topological poset $D_\theta^{\mathcal{C}}$ from [GRW14, Definition 2.13] that is used as a convenient replacement of the topological bordism category \mathcal{C}_d in their work.

inherits a PCM structure from \mathcal{C} (which would not be the case for $N_\bullet \mathbb{C}\text{sp}(\mathcal{C})$) and so we can think of $BD_\bullet(\mathcal{C})$ as a simplicial Γ -space.

Definition 8.2.23. *The simplicial groupoid $D_\bullet(\mathcal{C})$ is defined as follows. An object of D_n consists of n composable morphisms in $\mathbb{C}\text{sp}(\mathcal{C})$, i.e. a datum $((A_\bullet, c_\bullet), (\underline{X}_\bullet, o_\bullet))$ where $A_0, \dots, A_n \subset \Omega$ are finite sets with labellings by connected objects $c_i : A_i \rightarrow \text{Obj}^{\text{con}}(\mathcal{C})$ and cospans $(f_i : A_{i-1} \rightarrow \underline{X}_i \leftarrow A_i : g_i)$ with labellings by connected morphisms $o_i : \underline{X}_i \rightarrow \text{Mor}^{\text{con}}(\mathcal{C})$. A morphism of n -simplices $((A_\bullet, c_\bullet), (\underline{X}_\bullet, o_\bullet)) \rightarrow ((A'_\bullet, c'_\bullet), (\underline{X}'_\bullet, o'_\bullet))$ is a family of bijections $\alpha_i : A_i \cong A'_i$ and $\varphi_i : \underline{X}_i \cong \underline{X}'_i$ satisfying:*

$$c_i \circ \alpha_i = c'_i, \quad o_i \circ \varphi_i = o'_i, \quad \varphi_i \circ f_i = f'_i \circ \alpha_{i-1} \quad \text{and} \quad \varphi_i \circ g_i = g'_i \circ \alpha_i.$$

The face operator d_i is defined by composing the i th and $(i+1)$ st morphism according to the composition in $\mathbb{C}\text{sp}(\mathcal{C})$. The degeneracy operator s_i is defined by inserting identity morphisms.

REMARK 8.2.24. *In what follows we will often simply write D_\bullet for $D_\bullet(\mathcal{C})$ when the labelled cospan category $(\mathcal{C} \rightarrow \mathbb{C}\text{sp})$ is clear from the context. In a similar vein we often denote objects of $D_n(\mathcal{C})$ as tuples $(A_\bullet, \underline{X}_\bullet)$ where the labelings c_i and o_j are left implicit. For $0 \leq k < l \leq n$ we also use the notation*

$$\underline{X}_{k \rightarrow l} := \underline{X}_{k+1} \cup_{A_k} \cdots \cup_{A_{l-1}} \underline{X}_l$$

for the composite cospan and we write $o_{k \rightarrow l} : \underline{X}_{k \rightarrow l} \rightarrow \text{Mor}^{\text{con}}(\mathcal{C})$ for the labeling induced by composition. Note that in this notation $\underline{X}_k = \underline{X}_{k-1 \rightarrow k}$.

REMARK 8.2.25. *Note that there is an inclusion of simplicial groupoids:*

$$N_\bullet \mathbb{C}\text{sp}(\mathcal{C}) \longrightarrow D_\bullet$$

which in level n identifies $N_n \mathbb{C}\text{sp}(\mathcal{C})$ with the subcategory of D_n that contains all objects and all those morphisms $(\alpha_\bullet, \varphi_\bullet)$ where $\alpha_i = \text{id}_{A_i}$ holds for all i .

We can also define a variant with no closed components:

Definition 8.2.26. *Let $D_\bullet^{\text{nc}} \subset D_\bullet$ be the simplicial subgroupoid defined in each level n as the full subgroupoid $D_n^{\text{nc}} \subset D_n$ on those $(A_\bullet, \underline{X}_\bullet)$ such that the composite cospan $(A_0 \rightarrow \underline{X}_{0 \rightarrow n} \leftarrow A_n)$ is reduced.*

Lemma 8.2.27. *The inclusion of simplicial groupoids $N_\bullet \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow D_\bullet(\mathcal{C})$ induces equivalences:*

$$\begin{array}{ccc} |BN_\bullet^{\text{nc}} \mathbb{C}\text{sp}(\mathcal{C})| & \xrightarrow{\cong} & |BD_\bullet^{\text{nc}}(\mathcal{C})| \\ \downarrow & & \downarrow \\ |BN_\bullet \mathbb{C}\text{sp}(\mathcal{C})| & \xrightarrow{\cong} & |BD_\bullet(\mathcal{C})| \end{array}$$

Proof. Both equivalences can be checked by applying the base-change theorem for simplicial groupoids 8.2.20. We will verify the conditions for the bottom equivalence, the top map is similar. Both are entirely analogous to lemma 8.2.22.

To begin, note that $N_n \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow D_n$ is an inclusion of groupoids that is a bijection on objects. So $N_0 \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow D_0$ is certainly essentially surjective. Moreover, for all n both sides of

$$N_n \mathbb{C}\text{sp}(\mathcal{C}) \longrightarrow (N_0 \mathbb{C}\text{sp}(\mathcal{C}))^{n+1} \times_{(D_0)^{n+1}} D_n$$

can be thought of the subcategory of D_n that contains all objects and all those morphisms $(\alpha_\bullet, \varphi_\bullet)$ where $\alpha_i = \text{id}_{A_i}$ for all i . So this functor is an isomorphism. It remains to check that $\partial_D^n : D_n \rightarrow (D_0)^{n+1}$ is an isofibration. This means that given some n -simplex $(A_\bullet, \underline{X}_\bullet, c_\bullet, o_\bullet)$, an $(n+1)$ tuple (B_\bullet, c'_\bullet) , and $(n+1)$ bijections $\alpha_i : A_i \cong B_i$ such that $c'_i \circ \alpha_i = c_i$, we need to find an n -simplex $(B_\bullet, \underline{X}'_\bullet, c'_\bullet, o'_\bullet)$ that is isomorphic to the original n -simplex via some isomorphism of the form $(\alpha_\bullet, \varphi_\bullet)$. This can be achieved by simply setting $\underline{X}'_i := \underline{X}_i$ and then using the α_i to define the structure maps of the cospans as $B_{i-1} \cong A_{i-1} \rightarrow \underline{X}_i \leftarrow A_i \cong B_i$. The labeling o_\bullet stays the same. \square

In the case of the no closed component space we can also compare this to the much simpler simplicial set $N_\bullet^{\text{nc}} \mathcal{C} \subset N_\bullet \mathcal{C}$.

Lemma 8.2.28. *The canonical projection $N_\bullet^{\text{nc}} \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow N_\bullet^{\text{nc}} \mathcal{C}$ is an equivalence on realisations.*

Proof. The functor $\otimes_{\mathcal{C}} : \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ is essentially surjective, but not necessarily surjective on the nose. Let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory on those objects that are in the image, i.e. those that are \otimes -products of connected objects. The inclusion $N_\bullet^{\text{nc}} \mathcal{C}' \subset N_\bullet^{\text{nc}} \mathcal{C}$ is an equivalence on realisations by lemma 8.1.39.

We will show that $N_\bullet^{\text{nc}} \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow N_\bullet^{\text{nc}} \mathcal{C}'$ satisfies the condition of corollary 8.2.20. By construction $N_0^{\text{nc}} \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow N_0^{\text{nc}} \mathcal{C}'$ is surjective. The boundary map $\partial_{\mathcal{C}'}^n : N_n^{\text{nc}} \mathcal{C}' \rightarrow (N_0^{\text{nc}} \mathcal{C}')^{n+1}$ is an isofibration because it is a map between groupoids that only have identity morphisms. It remains to check that for all n the map

$$N_n^{\text{nc}} \mathbb{C}\text{sp}(\mathcal{C}) \longrightarrow (N_0^{\text{nc}} \mathbb{C}\text{sp}(\mathcal{C}))^{n+1} \times_{N_0^{\text{nc}} \mathcal{C}'} N_n \mathcal{C}'$$

is an equivalence of groupoids. This follows from the equivalence

$$\text{Hom}_{\mathbb{C}\text{sp}}^{\text{red}}((A, c_A), (B, c_B)) \simeq \text{Hom}_{\mathcal{C}}^{\text{red}}(\otimes_{\mathcal{C}}(A, c_A), \otimes_{\mathcal{C}}(B, c_B))$$

shown in lemma 8.1.28 since every morphism involved in $N_n^{\text{nc}} \mathbb{C}\text{sp}(\mathcal{C})$ is necessarily reduced. \square

8.2.5 The groupoid of cuts

We now proceed to construct a simplicial groupoid Cut_\bullet that represents the complement of D_\bullet^{nc} in D_\bullet . This is an immediate generalisation of the ideas of chapter 6, with the key difference being that the morphisms involved in Cut_\bullet are allowed to have closed components whereas the space of cuts in section 6.2.1 only used reduced morphisms.

Definition 8.2.29. *The simplicial groupoid Cut_\bullet is defined in each level as the full subgroupoid $\text{Cut}_n \subset D_n$ on those objects $(A_\bullet, \underline{X}_\bullet, c_\bullet, o_\bullet)$ where A_0 and A_n are both empty.*

There is a projection functor $R_n : D_n \rightarrow \text{Cut}_n$ that discards all elements of A_i and \underline{X}_i that correspond to an element of $\underline{X}_{0 \rightarrow n}$ that lies in the image of the map $A_0 \amalg A_n \rightarrow \underline{X}_{0 \rightarrow n}$.

We define the semi-simplicial structure on Cut_n by letting $\lambda : [n] \rightarrow [m]$ act as

$$\lambda_{\text{Cut}}^* : \text{Cut}_m \subset D_m \xrightarrow{\lambda^*} D_n \xrightarrow{R_n} \text{Cut}_n.$$

We equip each Cut_n with the PCM structure restricted from D_n .

Lemma 8.2.30. *The above definition yields a well-defined simplicial PCM groupoid Cut_\bullet such that the functors R_n induce a morphism of simplicial PCM groupoids: $R_\bullet : D_\bullet \rightarrow \text{Cut}_\bullet$.*

Proof. We need to check that the operations $\lambda_{\text{Cut}}^* : \text{Cut}_m \rightarrow \text{Cut}_n$ assemble into a simplicial object. Since we already know that D_\bullet is a simplicial object it will suffice to show that $R_n \circ \lambda^* \circ R_m = R_n \circ \lambda^*$, which will also imply that $R_\bullet : D_\bullet \rightarrow \text{Cut}_\bullet$ is a simplicial functor.

It will suffice to consider $\lambda = \delta^i : [n] \rightarrow [n+1]$, the unique injective morphism that does not hit i . (The degeneracy operators preserve the subgroupoid $\text{Cut}_n \subset D_n$.) Then $\lambda^* = d_i$ is the i -th face map, which composes the two adjacent morphisms if $0 < i < n$, forgets the first morphism if $i = 0$ and the last morphism if $i = n$. In the case $0 < i < n$ the total composite $\underline{X}_{0 \rightarrow n}$ stays the same before and after applying d_i . Therefore deleting anything that corresponds to an element of the image of $A_0 \amalg A_n \rightarrow \underline{X}_{0 \rightarrow n}$ does the same whether we do it before or after applying d_i . In formulas this means $\lambda^* \circ R_{n-1} = R_n \circ \lambda^*$. Since R_n is idempotent when thought of as an endofunctor of D_n , this in particular implies $R_{n-1} \circ \lambda^* \circ R_n = R_{n-1} \circ \lambda^*$, which is what we claimed.

We now check the case $i = 0$, the other remaining case $i = n$ is similar. When applying $R_{n-1} \circ d_0 \circ R_n$ to an n -simplex we first delete anything in the image of $A_0 \amalg A_n \rightarrow \underline{X}_{0 \rightarrow n}$, then we forget the first cospan, and then we delete anything in the image of $A_1 \amalg A_n \rightarrow \underline{X}_{1 \rightarrow n}$. It follows from inspection that anything we delete in the first step because it lies in the image of A_0 will either be in the first cospan or lie in the image of A_1 later, so it will be deleted anyway. This implies that the first R_n was redundant and so we have $R_{n-1} \circ d_0 \circ R_n = R_{n-1} \circ d_0$.

Since $\Delta_{\text{inj}}^{\text{op}}$ is generated by the δ^i it follows that Cut_\bullet is a well-defined simplicial object in the category of groupoids and that $R_\bullet : D_\bullet \rightarrow \text{Cut}_\bullet$ is simplicial. For the partial commutative

monoidal structure we simply note that $\text{Cut}_n \subset D_n$ is closed under disjoint union and isomorphism, so the structure can be restricted. Since we defined R_n in terms of components it commutes with disjoint unions. (As pointed out in remark 8.2.7 it is a property, and not a structure, for a functor between two partial commutative monoidal categories to preserve the structure.) It follows that R_\bullet is a morphism of simplicial PCM groupoids. \square

We have constructed Cut in such a way that every object $W \in D_n$ canonically decomposes as a disjoint union $W = W^{\text{nc}} \amalg W^{\text{Cut}}$ where $W^{\text{nc}} \in D_n^{\text{nc}}$ and $W^{\text{Cut}} \in \text{Cut}_n$. This means that we can apply the fiber sequence criterion 8.2.13 to show:

Lemma 8.2.31. *Assume that BC is group-like. Then the inclusion $I : D_\bullet^{\text{nc}} \subset D_\bullet$ and the simplicial functor R_\bullet yield a fiber sequence of infinite loop spaces:*

$$|BD_\bullet^{\text{nc}}| \longrightarrow |BD_\bullet| \xrightarrow{R} |BCut_\bullet|.$$

Proof. To check the criteria of theorem 8.2.13 we need to find for each n a PCM-functor $S_n : D_n \rightarrow D_n^{\text{nc}}$ such that the functors $S_n \circ I : D_n^{\text{nc}} \rightarrow D_n^{\text{nc}}$ and $(S_n, R_n) : D_n \rightarrow D_n^{\text{nc}} \times \text{Cut}_n$ are equivalences of groupoids.

In analogy to the definition of R_n we let $S_n(W)$ be defined by deleting all components of A_i or X_i that *do not* lie in the image of $A_0 \amalg A_n \rightarrow \underline{X}_{0 \rightarrow n}$. Note that this functor does not satisfy the conditions we checked in lemma 8.2.30 and so S_\bullet does not define a semi-simplicial map $D_\bullet \rightarrow D_\bullet^{\text{nc}}$. This is not a problem since theorem 8.2.13 does not require the sections S_n to be related in any way. The definition of S_n plays well with disjoint unions and hence defines a PCM-functor.

The composition $S_n \circ I$ is in fact the identity functor $\text{Id}_{D_n^{\text{nc}}}$ by construction, so we only need to check that (S_n, R_n) is an equivalence. We begin with essential surjectivity. The image of (S_n, R_n) in $D_n^{\text{nc}} \times \text{Cut}_n$ consists precisely of those tuples $(\underline{X}_\bullet, \underline{Y}_\bullet)$ such that $\underline{X}_i \perp \underline{Y}_i$ for all i . By standard arguments this disjointness may always be achieved, up to isomorphism, and so (S_n, R_n) is essentially surjective. We also need to check fully faithfulness. Let $(A_\bullet, \underline{X}_\bullet), (C_\bullet, \underline{Z}_\bullet) \in D_n$ be any two objects. (We suppress the data of the labelings c_{A_i} and o_{X_i} .) Then we need to show that isomorphism $(\alpha_\bullet, \varphi_\bullet) : (A_\bullet, \underline{X}_\bullet) \cong (C_\bullet, \underline{Z}_\bullet)$ is exactly given by a tuple of isomorphisms $S_n(A_\bullet, \underline{X}_\bullet) \cong S_n(C_\bullet, \underline{Z}_\bullet)$ and $R_n(A_\bullet, \underline{X}_\bullet) \cong R_n(C_\bullet, \underline{Z}_\bullet)$. But note that any such isomorphism induces compatible isomorphisms:

$$\begin{array}{ccc} A_0 \amalg A_n & \longrightarrow & \underline{X}_{0 \rightarrow n} = \underline{X}_1 \cup_{A_1} \cdots \cup_{A_{n-1}} \underline{X}_n \\ \alpha_0 \amalg \alpha_n \downarrow \cong & & \downarrow \cong \\ C_0 \amalg C_n & \longrightarrow & \underline{Z}_{0 \rightarrow n} = \underline{Z}_1 \cup_{C_1} \cdots \cup_{C_{n-1}} \underline{Z}_n. \end{array}$$

So the isomorphism $(\alpha_\bullet, \varphi_\bullet)$ preserves the subsets picked out by S_n and R_n , respectively, and it necessarily restricts to an isomorphism on both of those subsets. This shows that (S_n, R_n) is an equivalence of categories and so we may apply the fiber sequence criterion 8.2.13.

All that remains to complete the proof is to check that all three spaces are group-like. The groupoid Cut_0 is trivial by definition and hence $|\text{BCut}_\bullet|$ is connected and in particular group-like. For the middle space we observe that $D_0 \simeq N_0^{\text{rel}}\mathcal{C}$ and $D_1 \rightarrow N_1^{\text{rel}}\mathcal{C}$ is essentially surjective. Hence $\pi_0|BD_\bullet| \cong \pi_0BC$, which we assumed to be group-like. In the case of D_\bullet^{nc} it is still true that $D_0^{\text{nc}} \simeq N_0^{\text{rel}}\mathcal{C}$ and even though $D_1^{\text{nc}} \rightarrow N_1^{\text{rel}}\mathcal{C}$ is no longer essentially surjective, it is still true that for any morphism $W : M \rightarrow N$ in \mathcal{C} the 1-cell $S(W) \in D_1^{\text{nc}}$ connects M and N . Hence $\pi_0|BD_\bullet^{\text{nc}}| \cong \pi_0|BC|$ is group-like. \square

8.2.6 Decomposing the space of cuts

So far we have seen that $|BD_\bullet|$ fits into a fiber sequence between $|BD_\bullet^{\text{nc}}|$ and $|\text{BCut}_\bullet|$. The purpose of this section is to show that $|\text{BCut}_\bullet|$ is a free infinite loop space. We will do so by first showing that each of the Cut_n is a free symmetric monoidal groupoid on the connected cuts.

Definition 8.2.32. Let $\text{Cut}_\bullet^{\text{con}} \subset \text{Cut}_\bullet$ denote the simplicial subgroupoid defined in each level as the full subgroupoid on those $(A_\bullet, W_\bullet, c_\bullet, o_\bullet) \in \text{Cut}_n$ where $\underline{X}_{0 \rightarrow n}$ is connected or empty.

We need to allow $\underline{X}_{0 \rightarrow n}$ to be empty in order for $\text{Cut}_\bullet^{\text{con}}$ to be closed under those face maps that delete closed components. We take $\emptyset \in \text{Cut}_0^{\text{con}}$ as the base-point.

Lemma 8.2.33. The inclusion $\text{Cut}_\bullet^{\text{con}} \subset \text{Cut}_\bullet$ induces an equivalence of Γ -spaces $Q(|\text{BCut}_\bullet^{\text{con}}|) \simeq |\text{BCut}_\bullet|$.

We recall an explicit construction of the free Γ -category based on Segal's work [Seg74].

Definition 8.2.34. Let \mathcal{C} be a category. Then the category $\Sigma_{\mathcal{C}}^{\langle n \rangle}$ has as objects triples (T, α, x) where T is a finite set, $\alpha : T \rightarrow \{1, \dots, n\}$ a map, and $x : T \rightarrow \text{Obj}(\mathcal{C})$ an assignment of objects that we denote by $t \mapsto x_t$. Morphisms $(T, \alpha, x) \rightarrow (S, \beta, y)$ are pairs of a bijection $\varphi : T \cong S$ satisfying $\beta \circ \varphi = \alpha$ together with a collection of morphisms $f_t : x_t \rightarrow y_{\varphi(t)}$ indexed by $t \in T$. The composition of morphisms is given by $(\psi, g) \circ (\varphi, f) = (\psi \circ \varphi, h)$ where $h_t = g_{\varphi(t)} \circ f_t$. For a map $\lambda : \langle n \rangle \rightarrow \langle m \rangle$ in Γ^{op} we define a functor $\lambda_* : \Sigma_{\mathcal{C}}^{\langle n \rangle} \rightarrow \Sigma_{\mathcal{C}}^{\langle m \rangle}$ by sending (T, α, x) to $(T', \lambda \circ \alpha, x|_{T'})$ where $T' = \{t \in T \mid \lambda(\alpha(t)) \neq *\}$.

Lemma 8.2.35. Let \mathcal{A}_\bullet be a simplicial category such that $\mathcal{A}_0 = *$ is the trivial category and such that for all n the degeneracy operator $(s_0)^n : \mathcal{A}_0 \rightarrow \mathcal{A}_n$ induces a disjoint decomposition $\mathcal{A}_n = (s_0^n \mathcal{A}_0) \amalg \mathcal{A}'_n$. Define a semi-simplicial Γ -category \mathcal{D} by setting $\mathcal{D}_n^{\langle m \rangle} = \Sigma_{\mathcal{A}'_n}^{\langle m \rangle}$ with the aforementioned Γ -structure and face-maps defined by $d_i(T, \alpha, x) := (T', \alpha|_{T'}, y)$ where $T' = \{t \in T \mid d_i x_t \in \mathcal{A}'_n\}$ and $y_t := d_i x_t \in \mathcal{A}'_n$.

Then the inclusion $B(\mathcal{A}_n) \rightarrow B(\mathcal{D}_n^{\langle 1 \rangle})$ induces an equivalence of Γ -spaces:

$$Q(|B\mathcal{A}_\bullet|) \simeq |B(\mathcal{D}_\bullet)|.$$

Proof. The proof of this is almost entirely formal. We begin by noting that for any category \mathcal{A} the Γ -space $\langle n \rangle \mapsto B(\Sigma_{\mathcal{A}}^{\langle n \rangle})$ is isomorphic to the Γ -space that is denoted by Σ_X for $X = B\mathcal{A}$ in [Seg74, p. 299]. Let us denote by $(_)^{\text{gp}}$ denote the construction that sends a special Γ -space to its group completion, which we assume to come with the data of an infinite loop space. In particular, there is a canonical infinite loop space map from the free infinite loop space $Q(Y^{\langle 1 \rangle}) = \text{colim } \Omega^N \Sigma^N Y^{\langle 1 \rangle}$ to the group completion Y^{gp} for any special Γ -space $Y^{\langle \bullet \rangle}$.

Segal shows that the spectrum associated to Σ_X is the suspension spectrum of $X_+ = X \amalg \{*\}$. It follows that the inclusion $X \subset \Sigma_X^{\langle 1 \rangle}$ induces an equivalence:

$$Q(X_+) \rightarrow Q(\Sigma_X^{\langle 1 \rangle}) \rightarrow (\Sigma_X)^{\text{gp}}$$

Letting $X := B\mathcal{A}'_n$ we conclude that $Q((B\mathcal{A}'_n)_+) \simeq (B\Sigma_{\mathcal{A}'_n})^{\text{gp}} = (B\mathcal{D}_n)^{\text{gp}}$ for all n . Note that we assumed that \mathcal{A}_n is obtained by taking the disjoint union of \mathcal{A}'_n with the trivial category $s_0^n \mathcal{A}_0$. Hence $(B\mathcal{A}'_n)_+ = B\mathcal{A}_n$ and since all face maps preserve the basepoint $s_0^n \mathcal{A}_0 \subset \mathcal{A}_n$ the aforementioned equivalence is in fact an equivalence of simplicial infinite loop spaces:

$$Q(B\mathcal{A}_\bullet) \simeq (B\mathcal{D}_\bullet)^{\text{gp}}.$$

To proceed, recall that both $Q(_)$ and $(_)^{\text{gp}}$ commute with fat geometric realisations up to weak equivalence. This follows from [May72, Proposition 12.1 and Theorem 12.3], see also [BBP⁺17] for a more detailed account for the case of the group-completion functor. So we have:

$$Q(|B\mathcal{A}_\bullet|) \simeq (|B\mathcal{D}_\bullet|)^{\text{gp}}.$$

The lemma now follows from the observation that $|B\mathcal{D}_\bullet| \rightarrow (|B\mathcal{D}_\bullet|)^{\text{gp}}$ is an equivalence: we assumed that $\mathcal{A}_0 = *$, which implies $\mathcal{D}_0 = *$ and so $|B\mathcal{D}_\bullet|$ is connected and in particular grouplike. \square

Now that we have this general setting we can prove lemma 8.2.33 by comparing Cut_n to the free Γ -category on $\text{Cut}_n^{\text{con}}$.

Proof of lemma 8.2.33. The category of connected cuts decomposes as $\text{Cut}_n^{\text{con}} = \{\emptyset\} \amalg \text{Cut}_n^{\text{con}, \neq \emptyset}$ and the face operators all preserve \emptyset . We can therefore construct a semi-simplicial Γ -category $\mathcal{D}_\bullet^{\langle \bullet \rangle}$ as before by $\mathcal{D}_n^{\langle m \rangle} = \Sigma_{\text{Cut}_n^{\text{con}, \neq \emptyset}}^{\langle m \rangle}$. To prove the lemma it will suffice to construct a zig-zag of semi-simplicial Γ -functors $\mathcal{D}_\bullet \leftarrow \mathcal{E}_\bullet \rightarrow \text{Cut}_\bullet$ that induce equivalences

$$Q(|\text{Cut}_\bullet^{\text{con}}|) \stackrel{8.2.35}{\simeq} |B\mathcal{D}_\bullet| \simeq |B\mathcal{E}_\bullet| \simeq |B\text{Cut}_\bullet|.$$

For fixed n and m we let $\mathcal{E}_n^{\langle m \rangle} \subset \mathcal{D}_n^{\langle m \rangle}$ be the full subcategory on those $(T, \alpha, (A_\bullet^t, \underline{X}_\bullet^t)_{t \in T})$ where the $\underline{X}_i^t \in \text{Cut}_n^{\text{con}, \neq \emptyset}$ are pairwise disjoint as $t \in T$ varies. One checks that this is closed under face operators, and also under functoriality in $\langle m \rangle \in \Gamma$. As usual this hits all isomorphism classes and

hence the inclusion $\mathcal{E}_\bullet \rightarrow \mathcal{D}_\bullet$ is a level-wise equivalence of simplicial Γ -categories. In particular it realises to an equivalence $|\mathcal{B}\mathcal{E}_\bullet| \simeq |\mathcal{B}\mathcal{D}_\bullet|$.

The advantage of this subcategory is that the disjointness condition allows us to define a functor $\Pi : \mathcal{E}_n^{\langle m \rangle} \rightarrow \text{Cut}_n^{\langle m \rangle}$ by $(T, \alpha, (A_\bullet^t, \underline{X}_\bullet^t)_{t \in T}) \mapsto ((\coprod_{t \in T} A_\bullet^t, \coprod_{t \in T} \underline{X}_\bullet^t), \underline{X}_{0 \rightarrow n} \rightarrow T \xrightarrow{\alpha} \langle m \rangle)$. This is well-defined because Cut_n is a partial commutative monoidal category, and moreover it is natural in n and $\langle m \rangle$. To complete the proof we need to check that $\Pi : \mathcal{E}_n^{\langle 1 \rangle} \rightarrow \text{Cut}_n^{\langle 1 \rangle}$ is an equivalence of categories. To show essential surjectivity let $(A_\bullet, \underline{X}_\bullet) \in \text{Cut}_n$ be some object. Set $T := \underline{X}_{0 \rightarrow n}$ and let $B_i^t \subset A_i$ and $\underline{Y}_i^t \subset \underline{X}_i$ be the subsets that map to $t \in \underline{X}_{0 \rightarrow n}$. Then

$$\Pi(T, \alpha, (B_\bullet^t, \underline{Y}_\bullet^t)_{t \in T}) = \coprod_{t \in \underline{X}_{0 \rightarrow n}} (B_\bullet^t, \underline{Y}_\bullet^t) = (A_\bullet, \underline{X}_\bullet),$$

so Π is surjective. Next, we check fully faithfulness. For two objects $(T, \alpha, (A_\bullet, \underline{X}_\bullet))$ and $(S, \beta, (B_\bullet, \underline{Y}_\bullet))$ a morphism between them is a bijection $\varphi : T \cong S$ together with isomorphisms $(\beta_\bullet^t, \varphi_\bullet^t) : (A_\bullet^t, \underline{X}_\bullet^t) \cong (B_\bullet^{\varphi(t)}, \underline{Y}_\bullet^{\varphi(t)})$ for all $t \in T$. For any t the set $\underline{X}_{0 \rightarrow n}^t$ has a single element, so any isomorphism $(\beta_\bullet, \psi_\bullet) : \coprod_{t \in T} (A_\bullet^t, \underline{X}_\bullet^t) \cong \coprod_{s \in S} (B_\bullet^s, \underline{Y}_\bullet^s)$ can uniquely be written as $(\varphi, (\gamma_\bullet^t, \chi_\bullet^t)_{t \in T})$ where

$$\varphi := \psi_{1 \rightarrow n} : T = \underline{X}_{0 \rightarrow n} \rightarrow \underline{Y}_{0 \rightarrow n} = S$$

and $(\gamma_\bullet^t, \chi_\bullet^t)$ is defined as the restriction $(\beta_\bullet, \psi_\bullet)|_{(A_\bullet^t, \underline{X}_\bullet^t)}$. \square

8.2.7 From cuts to factorisations

In this section we compute the realisation of the space of connected cuts in terms of the category of factorisations. We will denote the set of connected endomorphism of 1_C by:

$$G := \text{Hom}_C^{\text{con}}(1_C, 1_C).$$

Definition 8.2.36. Define a simplicial groupoid Cut_\bullet^F as follows. For each n let Cut_n^F be the full subgroupoid of $\text{Cut}_{n+2}^{\text{con}}$ on those $(A_\bullet, \underline{X}_\bullet) \in D_{n+2}$ such that A_1 and A_{n+1} are non-empty. (Recall that A_0 and A_{n+2} are already required to be empty and $\underline{X}_{0 \rightarrow n+1}$ is required to be connected or empty.) The i th face map on Cut_n^F is defined as the restriction of the $(i+1)$ st face map on Cut_n . There is an augmentation $\text{Cut}_\bullet^F \rightarrow G_+$ defined by sending any non-empty $(A_\bullet, \underline{X}_\bullet, c_\bullet, o_\bullet)$ to the label $o_{1 \rightarrow n+1}(x) \in \text{Mor}^{\text{con}}(\mathcal{C})$ of $\{x\} = \underline{X}_{0 \rightarrow n+1}$. We send the unique $\emptyset \in \text{Cut}_n^F$ to the additional basepoint of $G_+ = G \amalg \{*\}$.

REMARK 8.2.37. The augmentation yields a disjoint decomposition:

$$\text{Cut}_n^F \cong \coprod_{g \in G_+} \text{Cut}_n^{F,(g)}$$

where we let $\text{Cut}_n^{F,(g)}$ be the fiber of $\text{Cut}_n^F \rightarrow G_+$ at g . In particular $\text{Cut}_n^{F,(\ast)} = \{\emptyset\}$ is the trivial category for $\ast \in G_+$ the additional base-point. The disjoint decomposition described above is compatible with face maps and hence induces a decomposition:

$$|BCut_{\bullet}^F| \cong \{\emptyset\} \amalg \coprod_{g \in G} |BCut_n^{F,(g)}|.$$

With this in mind the reduced cone of the based map $|BCut_{\bullet}^F| \rightarrow G_+$ that will appear in the following lemma may be rewritten as:

$$\text{cone}(|BCut_{\bullet}^F| \rightarrow G_+) = \bigvee_{g \in G_+} S|BCut_{\bullet}^{F,(g)}| \simeq \bigvee_{g \in G} S|BCut_{\bullet}^{F,(g)}|.$$

Here $SX = \ast \amalg_{X \times \{0\}} (X \times [0, 1]) \amalg_{X \times \{1\}} \ast$ denotes the unreduced suspension of an un-pointed space X . This can also be written as the reduced cone $SX \cong \text{cone}(X_+ \rightarrow S^0)$. Recall that for any choice of a basepoint in x the canonical map $SX \rightarrow \Sigma X$ is an equivalence, but if X is empty then $S(\emptyset) = \{0, 1\}$ whereas $\Sigma(\emptyset)$ is of course not defined.

Lemma 8.2.38. *The simplicial space $BCut_{\bullet}^{\text{con}}$ is isomorphic to the simplicial suspension of the simplicial cone of the map $BCut_{\bullet}^F \rightarrow G_+$. As a consequence there is a homotopy equivalence:*

$$|BCut_{\bullet}^{\text{con}}| \simeq \Sigma(\text{cone}(|BCut_{\bullet}^F| \rightarrow G_+)) \simeq \bigvee_{g \in G} \Sigma S|BCut_{\bullet}^{F,(g)}|.$$

Proof. We take the formulas for the simplicial suspension and the simplicial cone from 6.2.19. For any pointed augmented simplicial space $(d_{-1} : X_{\bullet} \rightarrow X_{-1})$ we construct a simplicial space Y as follows:

$$Y_n = \bigvee_{-1 \leq l < k < n} X_l \times \{l\} \times \{k\}.$$

The face-maps and degeneracy maps are defined as follows:

$$d_i(x, l, k) = \begin{cases} (x, l, k-1) & \text{for } i < k-l \\ (d_{i-(k-l)}x, l-1, k-1) & \text{for } k-l \leq i \leq k \\ (x, l, k) & \text{for } i > k. \end{cases} \quad \text{and}$$

$$s_i(x, l, k) = \begin{cases} (x, l, k+1) & \text{if } i < k-l \\ (s_{i-(k-l)}x, l+1, k+1) & \text{if } k-l \leq i \leq k \\ (x, l, k) & \text{if } i > k \end{cases}$$

It was shown in 6.2.20 and second part of proof of Lemma 6.2.18 that the geometric realisation of Y_{\bullet} is equivalent to the reduced suspension of the reduced mapping cone of the augmentation map $|X_{\bullet}| \rightarrow X_{-1}$:

$$|Y_{\bullet}| \simeq \Sigma(\text{cone}(|X_{\bullet}| \rightarrow X_{-1})).$$

Therefore all we need to show is that if we start out with $X_\bullet = BCut_\bullet^F$ pointed at \emptyset and $X_{-1} = G_+$, then the resulting Y_\bullet is equivalent to $BCut_\bullet^{\text{con}}$. Indeed, for each n we have a functor:

$$H_n : Cut_n^{\text{con}} \longrightarrow \{\emptyset\} \amalg \coprod_{-1 \leq l < k < n} Cut_l^F.$$

Defined by sending $(A_\bullet, \underline{X}_\bullet) \in D_n$ to $((B_\bullet, \underline{Y}_\bullet), l, k)$ where

$$k = \max\{i \in \{0, \dots, n-1\} \mid A_i \neq \emptyset\}, \quad l = \max\{j \in \{-1, \dots, k-1\} \mid A_{k-j} \neq \emptyset\},$$

$B_i := A_{i+k-l}$, and $\underline{Y}_j := \underline{X}_{j+k-l}$. It follows by inspection that H_n is an equivalence of categories and that the H_\bullet are compatible with the face and degeneracy maps we described above. This concludes the proof of the first equivalence and the second equivalence follows from remark 8.2.37. \square

Now we show that $|BCut_\bullet^F|$ is in fact the classifying space of the category of factorisations.

Definition 8.2.39. *An object in the category \mathcal{F} is a triple (M, W, W') where $M \in \mathcal{C}$ is a non-empty object and $W : 1_{\mathcal{C}} \rightarrow M$ and $W' : M \rightarrow 1_{\mathcal{C}}$ are two morphisms such that $W \cup_M W'$ is a connected morphism. A morphism $(M, W, W') \rightarrow (N, V, V')$ is a morphism $X : M \rightarrow N$ such that $W \cup_M X = W'$ and $V = X \cup_{M'} V'$.*

Remark 8.2.40. *In definition 8.1.34 we had defined factorisation categories \mathcal{F}_g for specific endomorphisms $g : 1_{\mathcal{C}} \rightarrow 1_{\mathcal{C}}$. Composing the two morphisms of a factorisation yields a functor $\mathcal{F} \rightarrow G$ where the latter is thought of as a category with only identity morphisms. We therefore have a disjoint decomposition:*

$$\mathcal{F} \cong \coprod_{g \in G} \mathcal{F}_g$$

compatible with the decomposition of Cut_\bullet^F into $Cut_\bullet^{F,(g)}$.

Lemma 8.2.41. *Let $N_\bullet^{\text{rel}} \mathcal{F}$ be Rezk's relative nerve as in definition 8.2.21. Then the canonical maps:*

$$Cut_\bullet^F \longrightarrow (N_\bullet^{\text{rel}} \mathcal{F})_+ \longleftarrow (N_\bullet \mathcal{F})_+$$

induce equivalences $\|BCut_\bullet^F\| \simeq \|BN_\bullet^{\text{rel}}\|_+ \simeq (B\mathcal{F})_+$.

Proof. Let us begin by considering the double-comma category $\mathcal{K} = \mathcal{C}_{/1_{\mathcal{C}}}$ where objects are tuples of morphisms $(W : 1_{\mathcal{C}} \rightarrow M, W' : M \rightarrow 1_{\mathcal{C}})$ and a morphism $(M, W, W') \rightarrow (N, V, V')$ is a morphism $X : M \rightarrow N$ such that $W \cup_M X = W'$ and $V = X \cup_{M'} V'$. By definition \mathcal{F} is a full subcategory of \mathcal{K} on those objects (M, W, W') where M is non-empty and $W \cup_M W'$ is connected. The nerve of \mathcal{K} can be described as follows:

$$N_n \mathcal{K} \cong \{W : [n+2] \rightarrow \mathcal{C} \mid W(0) = 1_{\mathcal{C}} = W(n+2)\} \subset N_{n+2} \mathcal{C}$$

where the face and degeneracy operators are given by $d_i^K := d_{i+1}^C$ and $s_i^K := s_{i+1}^C$. Note that because the isomorphisms in \mathcal{K} are isomorphisms \mathcal{C} that make the relevant diagrams commute, we also have an equivalence

$$N_n^{\text{rel}}\mathcal{K} \simeq \{(1_C, 1_C)\} \times_{(N_0^{\text{rel}}(\mathcal{C}))^2} N_{n+2}^{\text{rel}}(\mathcal{C})$$

where the functor $N_{n+2}^{\text{rel}}(\mathcal{C}) \rightarrow (N_0^{\text{rel}}(\mathcal{C}))^2$ is the one that picks out the first and last object.

Just like $\otimes_{\mathcal{C}} : \mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ induces a simplicial map $N_{\bullet}\mathbb{C}\text{sp}(\mathcal{C}) \rightarrow N_{\bullet}\mathcal{C}$ it also induces a simplicial map $D_{\bullet}(\mathcal{C}) \rightarrow N_{\bullet}^{\text{rel}}\mathcal{C}$. In defining this on morphisms we use that any bijection $\alpha : (A, c_A) \cong (A', c_{A'})$ with $c_{A'} \circ \alpha = c_A$ induces an isomorphism $\otimes_{\mathcal{C}}(A, c_A) \cong \otimes_{\mathcal{C}}(A', c_{A'})$. Restricting this map we obtain a map of simplicial groupoids $\text{Cut}_{\bullet}^F \rightarrow N_{\bullet}^{\text{rel}}\mathcal{K}$, and it is straightforward to check that it in fact lands in the relative nerve of the full subcategory $\mathcal{F}_+ = \mathcal{F} \amalg \{\emptyset\} \subset \mathcal{K}$. We hence have the following zig-zag of simplicial groupoids:

$$\text{Cut}_{\bullet}^F \longrightarrow N_{\bullet}^{\text{rel}}\mathcal{F}_+ \longleftarrow N_{\bullet}\mathcal{F}_+.$$

By lemma 8.2.22 the second map induces an equivalence on geometric realisations. So to finish the proof we will check that the first map satisfies the condition of the base-change corollary 8.2.20. We have already checked in lemma 8.2.22 that in the relative nerve of any category the boundary map $\partial_n : N_n^{\text{rel}}\mathcal{F} \rightarrow (N_0^{\text{rel}}\mathcal{F})^{n+1}$ is an iso-fibration, which verifies the third condition.

The first condition states that $\text{Cut}_0^F \rightarrow N_0^{\text{rel}}\mathcal{F}_+ = (\mathcal{F}_+)^{\text{iso}}$ is essentially surjective. Indeed, any factorisation $(W : 1_C \rightarrow M, W' : M \rightarrow 1_C)$ is isomorphic to one where M is the product of connected objects. In that case we can write $M = \otimes_{\mathcal{C}}(A, c_A)$ for $(A, c_A) \in \mathbb{C}\text{sp}(\mathcal{C})$. Since $\otimes_{\mathcal{C}} : h\mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ is an equivalence of categories by lemma 8.1.28, we can find $(\underline{X}, o) : \emptyset \rightarrow (A, c_A)$ and $(\underline{X}', o') : (A, c_A) \rightarrow \emptyset$ that are mapped to W and W' , respectively. The tuple $((\underline{X}, o), (\underline{X}', o'))$ defines a 2-simplex in D_{\bullet} that is an object in Cut_0^F and it is mapped to $(M, W, W') \in \mathcal{F}$.

The second condition is the base-change condition. We would like to show that

$$(\partial_n, \otimes_{\mathcal{C}}) : \text{Cut}_n^F \longrightarrow (\text{Cut}_0^F)^{n+1} \times_{(N_0^{\text{rel}}\mathcal{F}_+)^{n+1}} N_n^{\text{rel}}\mathcal{F}_+$$

is an equivalence of groupoids. By lemma 8.1.28 the n the following functor is essentially surjective and fully faithful when restricted to reduced morphisms:

$$(\partial_n, \otimes_{\mathcal{C}}) : D_{n+2} \longrightarrow (D_0)^{n+3} \times_{(N_0^{\text{rel}}\mathcal{C})^{n+3}} N_{n+2}^{\text{rel}}\mathcal{C}$$

By fixing the first and last object to be $\emptyset \in D_0$ and $1_C \in N_0^{\text{rel}}\mathcal{C}$, respectively, we obtain:

$$\{\emptyset, \emptyset\} \times_{(D_0)^2} D_{n+2} \longrightarrow (D_0)^{n+1} \times_{(N_0^{\text{rel}}\mathcal{C})^{n+1}} N_n^{\text{rel}}\mathcal{K}$$

Restricting to full subcategories we obtain the following equivalence:

$$\text{Cut}_n^F \longrightarrow (D_0)^{n+1} \times_{(N_0^{\text{rel}}\mathcal{C})^{n+1}} N_n^{\text{rel}}\mathcal{F}_+$$

Using the simplicial object $T(X)_n := X^{n+1}$ from the proof of theorem 8.2.16 we can write this as a level-wise equivalence of simplicial groupoids: $\text{Cut}_\bullet^F \simeq T(D_0) \times_{T(N_0^{\text{rel}}\mathcal{C})} N_\bullet^{\text{rel}}\mathcal{F}_+$. We may rewrite this further as:

$$\begin{aligned} \text{Cut}_\bullet^F &\simeq T(D_0) \times_{T(N_0^{\text{rel}}\mathcal{C})} N_\bullet^{\text{rel}}\mathcal{F}_+ \cong T(D_0 \times_{N_0^{\text{rel}}\mathcal{C}} N_0^{\text{rel}}\mathcal{F}_+) \times_{T(N_0^{\text{rel}}\mathcal{F}_+)} N_\bullet^{\text{rel}}\mathcal{F}_+ \\ &\simeq T(\text{Cut}_0^F) \times_{T(N_0^{\text{rel}}\mathcal{F}_+)} N_\bullet^{\text{rel}}\mathcal{F}_+. \end{aligned}$$

This is the desired equivalence that verifies the remaining condition in corollary 8.2.20. \square

To summarise the previous sections we record:

Corollary 8.2.42. *There are equivalences of Γ -spaces:*

$$\begin{aligned} |BCut_\bullet| &\simeq Q(|BCut_\bullet^{\text{con}}|) \simeq Q(\Sigma \text{cone}(|BCut_\bullet^F| \rightarrow G_+)) \\ &\simeq Q(\Sigma \text{cone}(B\mathcal{F}_+ \rightarrow G_+)) \simeq Q\left(\bigvee_{g \in G} \Sigma S(B\mathcal{F}_g)\right). \end{aligned}$$

Proof. Combine lemma 8.2.33, lemma 8.2.38, and lemma 8.2.41. \square

8.2.8 The reduced category

We now prove the comparison result proposition 8.2.4 between $\mathbb{C}\text{sp}(\mathcal{C})$, \mathcal{C} and \mathcal{C}^{red} .

Definition 8.2.43. *Let $\mathbb{C}\text{sp}(\mathcal{C})^{\text{cl}} \subset \mathbb{C}\text{sp}(\mathcal{C})$ denote the full subcategory on the object \emptyset and let $\mathcal{C}^{\text{cl}} \subset \mathcal{C}$ denote the full subcategory on the object $1_{\mathcal{C}}$.*

Lemma 8.2.44. *There are canonical equivalences of infinite loop spaces:*

$$B(\mathcal{C}^{\text{cl}}) \simeq B\left(\bigoplus_G \mathbb{N}\right) \quad \text{and} \quad B(\mathbb{C}\text{sp}(\mathcal{C})^{\text{cl}}) \simeq Q\left(\bigvee_G S^1\right).$$

Proof. By definition \mathcal{C}^{cl} is a category with a single object $1_{\mathcal{C}}$ and the endomorphisms of this object are $\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}}) \cong \bigoplus_G \mathbb{N}$ by axiom (ii) of definition 8.1.4. This implies the first equivalence.

The second equivalence can easily be proven from the perspective of [Seg74], but for the readers convenience we will prove it as an application of lemma 8.2.35. (Note, however, that this lemma is also based on [Seg74].) Indeed, the groupoid $\text{Hom}_{\mathbb{C}\text{sp}(\mathcal{C})^{\text{cl}}}(\emptyset, \emptyset)$ is freely generated by its connected objects in the sense of the lemma. Let K_\bullet be the level-wise full simplicial subgroupoid of the nerve $N_\bullet\mathbb{C}\text{sp}(\mathcal{C})$ containing all those objects $\underline{X} : [n] \rightarrow \mathbb{C}\text{sp}(\mathcal{C})$ where $\underline{X}_{0 \rightarrow n} : \emptyset \rightarrow \emptyset$ is connected or empty. Then lemma 8.2.35 applies to the tuple $(K_\bullet \subset N_\bullet\mathbb{C}\text{sp}(\mathcal{C}))$ and hence we have:

$$B(\mathbb{C}\text{sp}(\mathcal{C})^{\text{cl}}) \simeq Q(\|BK_\bullet\|).$$

We will show that BK_\bullet is level-wise equivalent to $Y_\bullet := (\Delta^1 \times G)/(\partial\Delta^1 \times G)$, the geometric realisation of which is a wedge of circles, one for each element of G . The n -simplices of $\Delta^1 \times G$ can be thought of as tuples $(i, g) \in \{0, \dots, n+1\} \times G$ where i represents the unique map $[n] \rightarrow [1]$ with $i \mapsto 0$ and $(i+1) \mapsto 1$. The simplicial set Y_\bullet is obtained by identifying all tuples (i, g) with $i \in \{0, n+1\}$. Indeed, there is a level-wise equivalence of simplicial groupoids $K_\bullet \simeq Y_\bullet$ defined by sending $(\underline{X} : [n] \rightarrow \mathbb{C}\text{sp}(\mathcal{C})) \in K_n$ to the basepoint if $\underline{X}_{0 \rightarrow n}$ is empty and otherwise to $(i, [\underline{X}_{0 \rightarrow n}])$ where $i \in \{1, \dots, n\}$ is the unique i such that $\underline{X}_{i-1 \rightarrow i}$ is non-empty. From this it follows that

$$\|BK_\bullet\| \simeq \|(\Delta^1 \times G)/(\partial\Delta^1 \times G)\| \simeq \bigvee_{g \in G} S^1.$$

□

Proposition 8.2.45. *There are homotopy fiber sequences of infinite loop spaces:*

$$\begin{array}{ccccc} Q(\bigvee_G S^1) & \longrightarrow & B(\mathbb{C}\text{sp}(\mathcal{C})) & \longrightarrow & B(\mathcal{C}^{\text{red}}) \\ \downarrow & & \downarrow & & \parallel \\ B(\bigoplus_G \mathbb{N}) & \longrightarrow & B(\mathcal{C}) & \longrightarrow & B(\mathcal{C}^{\text{red}}). \end{array}$$

In particular, the map $B(\mathbb{C}\text{sp}(\mathcal{C})) \rightarrow B(\mathcal{C})$ is a rational equivalence.

Proof. For the purpose of the proof we will assume that \mathcal{C} has a PCM structure, rather than a symmetric monoidal structure, and that the functor $\mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ is a PCM functor. Since $h\mathbb{C}\text{sp}(\mathcal{C}) \rightarrow \mathcal{C}$ is an equivalence and $\mathbb{C}\text{sp}(\mathcal{C}) \rightarrow h\mathbb{C}\text{sp}(\mathcal{C})$ is a PCM functor, this can always be achieved by replacing \mathcal{C} with $h\mathbb{C}\text{sp}(\mathcal{C})$. Consider the following diagram of simplicial PCM groupoids:

$$\begin{array}{ccccc} D_\bullet^{\text{cl}} & \longrightarrow & D_\bullet & \longrightarrow & D_\bullet^{\text{red}} \\ \downarrow & & \downarrow & & \downarrow \\ N_\bullet \mathcal{C}^{\text{cl}} & \longrightarrow & N_\bullet^{\text{rel}} \mathcal{C} & \longrightarrow & N_\bullet^{\text{rel}} \mathcal{C}^{\text{red}}. \end{array}$$

Here $D_\bullet^{\text{cl}} \subset D_\bullet$ is the level-wise full subgroupoid on those $(\underline{X}_\bullet, A_\bullet)$ with $A_i = \emptyset$ for all i . D_\bullet^{red} is defined level-wise as the full subgroupoid $D_n^{\text{red}} \subset D_n$ on those n -simplices that represent n -tuples of composable reduced cospans. Similar to the definition of Cut_\bullet we define the face maps in D_\bullet^{red} by taking the face map in D_\bullet and then forgetting closed components. In the bottom row $N_\bullet \mathcal{C}^{\text{cl}}$ is simply the simplicial commutative monoid where each $N_n \mathcal{C}^{\text{cl}}$ is isomorphic to $(\mathbb{N}\langle G \rangle)^n$.

It follows from lemma 8.2.44 and our previous results that the realisation of this diagram is a commutative diagram of Γ -spaces that is equivalent to the diagram stated in the claim of this proposition. Moreover, each of the terms is group-like, and so we have a diagram of infinite loop spaces.

To prove the proposition we apply the general fiber sequence criterion theorem 8.2.13 to the two sequences of simplicial PCM groupoids. For the first one, consider the PCM functor:

$$S_n : D_n \rightarrow D_n^{\text{cl}}, \quad (A_\bullet, \underline{X}_\bullet, c_\bullet, o_\bullet) \mapsto (\emptyset, \underline{Y}_\bullet, \emptyset, (o_\bullet)|_{\underline{Y}_\bullet})$$

where each $\underline{Y}_i \subset \underline{X}_i$ is defined as the complement of the image of $A_{i-1} \amalg A_i \rightarrow \underline{X}_i$. This map is a section to the inclusion $D_n^{\text{cl}} \rightarrow D_n$, so to apply the fiber sequence theorem we need to check that $(S_n, R_n) : D_n \rightarrow D_n^{\text{cl}} \times D_n^{\text{red}}$ is an equivalence. This functor is given by

$$(A_\bullet, \underline{X}_\bullet, c_\bullet, o_\bullet) \mapsto ((\emptyset, \underline{Y}_\bullet, \emptyset, (o_\bullet)|_{\underline{Y}_\bullet}), (A_\bullet, \underline{Z}_\bullet, c_\bullet, (o_\bullet)|_{\underline{Z}_\bullet}))$$

where \underline{Y}_i is as above and $\underline{Z}_i \subset \underline{X}_i$ is the image of $A_{i-1} \amalg A_i$, i.e. the complement of \underline{Y}_i . Just as in the proof of 8.2.31 we see that this functor is an isomorphism onto the subcategory of disjoint tuples and that hence it is an equivalence.

To conclude that the first row is a homotopy fiber sequence we note that $\pi_0|BD_\bullet^{\text{cl}}| \cong *$, $\pi_0|BD_\bullet| \cong \pi_0BC$, and $\pi_0|BD_\bullet^{\text{red}}| \cong \pi_0BC^{\text{red}}$ are all group-like.

For the second row the proof is similar. The section is given by

$$S_n : N_n^{\text{rel}}\mathcal{C} \rightarrow N_n\mathcal{C}^{\text{cl}}, \quad (W : [n] \rightarrow \mathcal{C}) \mapsto (V_1, \dots, V_n) \in (\text{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, 1_{\mathcal{C}}))^n = N_n\mathcal{C}^{\text{cl}}$$

where $V_i \subset W(i-1 \rightarrow i)$ is the union of the closed components. This is a PCM functor where the commutative monoid $N_n\mathcal{C}^{\text{cl}}$ is thought of as a PCM category with only identity morphisms and where all objects are disjoint. Since this is indeed a section to the inclusion $N_n\mathcal{C}^{\text{cl}} \rightarrow N_n^{\text{rel}}\mathcal{C}$ we again need to check that the functor $(S_n, R_n) : N_n^{\text{rel}}\mathcal{C} \rightarrow N_n\mathcal{C}^{\text{cl}} \times N_n\mathcal{C}^{\text{red}}$ is an equivalence. This follows from axiom (iii) of definition 8.1.4. This completes the proof that both rows are homotopy fiber sequences.

Finally, we want to check that $B(\mathbb{C}\text{sp}(\mathcal{C})) \rightarrow B(\mathcal{C})$ is a rational equivalence. By comparing the homotopy fibers of the first two vertical maps between the two homotopy fiber sequence we obtain the following homotopy fiber sequence:

$$\tau_{\geq 2}Q\left(\bigvee_{g \in G} S^1\right) \longrightarrow B(\mathbb{C}\text{sp}(\mathcal{C})) \longrightarrow B(\mathcal{C}).$$

The homotopy groups of the left-hand space are $\pi_k \tau_{\geq 2}Q\left(\bigvee_{g \in G} S^1\right) = \bigoplus_{g \in G} \pi_{k-1}^{\text{st}}(S^0)$ for $k \geq 2$. This is rationally trivial because the stable homotopy groups of spheres $\pi_*^{\text{st}}(S^0)$ are finite for $* \geq 1$. Hence the map $B(\mathbb{C}\text{sp}(\mathcal{C})) \rightarrow B(\mathcal{C})$ is an isomorphism on rational homotopy groups, as claimed. \square

8.2.9 Contracting the factorisation category

In this section we show that the factorisation categories for Cob_d have contractible classifying spaces for all $d \leq 2$. The argument is based on an argument showing $B\mathcal{F}(\text{Csp}) \simeq *$ and crucially uses the existence of the disk morphism $D^d : \emptyset \rightarrow S^{d-1}$. In particular, it will not apply to $\text{Cob}_2^{X \leq 0}$.

Definition 8.2.46. For a non-empty set X let $\text{Fin}_{/X}^{\neq \emptyset, \text{inj}}$ denote the category where objects are tuples (A, a) of a non-empty finite set A together with a map $a : A \rightarrow X$, and morphisms $(A, a) \rightarrow (B, b)$ are injections $i : A \rightarrow B$ such that $a = b \circ i$.

Lemma 8.2.47. For every non-empty set X the classifying space $B\text{Fin}_{/X}^{\neq \emptyset, \text{inj}}$ is contractible.

Proof. Pick any $x_0 \in X$ and consider the functor

$$F : \text{Fin}_{/X}^{\neq \emptyset, \text{inj}} \rightarrow \text{Fin}_{/X}^{\neq \emptyset, \text{inj}} \quad (A, a) \mapsto (A \amalg \{x_0\}, a \amalg \text{id}_{\{x_0\}})$$

that adds a new point to A and labels it by x_0 . There is a natural transformation $\alpha : \text{Id}_{\text{Fin}_{/X}^{\neq \emptyset, \text{inj}}} \Rightarrow F$ coming from the canonical inclusion $\alpha_A : A \hookrightarrow A \amalg \{x_0\}$. There also is a natural transformation $\beta : G \Rightarrow F$ from the constant functor $G(A, a) := (\{x_0\}, \text{id}_{\{x_0\}})$ coming from the other inclusion $\beta_A : \{x_0\} \hookrightarrow A \amalg \{x_0\}$. Both α and β induce homotopies after taking classifying spaces and together they imply that the identity on $B\text{Fin}_{/X}^{\neq \emptyset, \text{inj}}$ is homotopic to BG , which is constant. \square

Lemma 8.2.48. The factorisation category $\mathcal{F}(\text{Csp})$ has a contractible classifying space.

Proof. Consider the functor $\partial_0 : \mathcal{F}(\text{Csp}) \rightarrow \text{Fin}$ that sends the factorisation $(W : 1_C \rightarrow M, W' : M \rightarrow 1_C)$ to the underlying set $\pi(W)$ of the first morphism. To a morphism $X : M \rightarrow N$ from (M, W, W') to (N, V, V') this assigns the composite $\pi(W) \rightarrow \pi(W) \cup_{\pi(M)} \pi(X) \cong \pi(V)$. A short explanation is due to argue why we can talk about $\pi(W)$ as a well-defined set. A priori the morphism W only yields an isomorphism class of cospans $[\emptyset \rightarrow \pi(W) \leftarrow \pi(M)]$, but not a well-defined set $\pi(W)$. But because this is part of a factorisation we know that $\pi(M) \rightarrow \pi(W)$ is surjective. In particular we can define $\partial_0(M, W, W')$ as the quotient of $\pi(M)$ by the relation induced by $\pi(M) \rightarrow \pi(W)$.

We define a new category \mathcal{F}' as the Grothendieck construction of the functor $\mathcal{F} \rightarrow \text{Cat}$ that sends (M, W, W') to $\text{Fin}_{/\partial_0(M, W, W')}^{\neq \emptyset, \text{inj}, \text{op}}$. Spelling out the construction we see that an object in \mathcal{F}' is a tuple $((M, W, W'), (A, a))$ where $(M, W, W') \in \mathcal{F}(\text{Csp})$, A is a non-empty finite set and $a : A \rightarrow \partial_0(M, W, W')$. A morphism $((M, W, W'), (A, a)) \rightarrow ((N, V, V'), (B, b))$ is a tuple of a morphism $X : M \rightarrow N$ in \mathcal{F} and an injection $i : A \rightarrow B$ such that $\partial_0(X) \circ a = b \circ i$. Consider the forgetful functor $F : \mathcal{F}' \rightarrow \mathcal{F}(\text{Csp})$ that sends $((M, W, W'), (A, a))$ to (M, W, W') .

It is a consequence of Quillen's theorem A that, given any functor $\Phi : \mathcal{C} \rightarrow \text{Cat}$ where each $B(\Phi(c))$ is contractible, the forgetful functor $\int_{\mathcal{C}} \Phi \rightarrow \mathcal{C}$ is an equivalence on classifying spaces.

(One can also think of this as an instance of Thomason's homotopy colimit theorem [Tho79].)

In lemma 8.2.47 we showed that each $Fin^{\neq\emptyset, \text{inj}, \text{op}} / \partial_0(M, W, W')$ has a contractible classifying space, hence $F : \mathcal{F}' \rightarrow \mathcal{F}(\text{Csp})$ is a homotopy equivalence. We will prove the lemma by showing that the homotopy equivalence $B\mathcal{F}' \rightarrow B\mathcal{F}(\text{Csp})$ is null-homotopic.

Define a functor:

$$G : \mathcal{F}' \rightarrow \mathcal{F}, \quad ((M, [\emptyset \rightarrow W \leftarrow M], [M \rightarrow V \leftarrow \emptyset]), (a : A \rightarrow W)) \mapsto (A, [\emptyset \rightarrow A \xleftarrow{=} A], [A \rightarrow * \leftarrow \emptyset])$$

to a morphism $(X, i) : ((M, W, W'), (A, a)) \rightarrow ((N, V, V'), (B, b))$ this assigns the morphism

$$[A \xrightarrow{i} B \xleftarrow{=} B].$$

This functor by definition factors through $\mathcal{F}' \rightarrow Fin^{\neq\emptyset, \text{inj}}$. Since the classifying space $BFin^{\neq\emptyset, \text{inj}}$ is contractible (set $X = *$ in lemma 8.2.47), we can conclude that $BG : \mathcal{F}' \rightarrow \mathcal{F}$ is null-homotopic.

We construct a natural transformation $\alpha : G \Rightarrow F$ by assigning to each $((M, W, W'), (A, a))$ the morphism

$$\alpha_{((M, W, V), (A, a))} := [A \xrightarrow{a} W \leftarrow M] : (A, [\emptyset \rightarrow A \xleftarrow{=} A], [A \rightarrow * \leftarrow \emptyset]) \rightarrow (M, W, V).$$

To see that this is a well-defined morphism in the factorisation category we check

$$\begin{aligned} [A \rightarrow W \leftarrow M] \cup_M [M \rightarrow W' \leftarrow \emptyset] &= [A \rightarrow * \leftarrow \emptyset] \\ [\emptyset \rightarrow A \leftarrow A] \cup_A [A \rightarrow W \leftarrow M] &= [\emptyset \rightarrow W \leftarrow M]. \end{aligned}$$

Here the first equation used that $W \cup_M W' = *$ holds for all objects $(M, W, W') \in \mathcal{F}(\text{Csp})$. We also need to check that α is natural. To see this consider a morphism $(X, i) : ((M, W, W'), (A, a)) \rightarrow ((N, V, V'), (B, b))$ where $X = [M \rightarrow X \leftarrow N] : (M, W, W') \rightarrow (N, V, V')$ and $i : A \hookrightarrow B$. For any such morphism we have that:

$$[A \xrightarrow{i} B \xleftarrow{=} B] \cup_B [B \xrightarrow{b} V \leftarrow N] = [A \xrightarrow{a} W \leftarrow M] \cup_M [M \rightarrow X \leftarrow N].$$

since $V = W \cup_M X$ and $b \circ i = \partial_0(X) \circ a$.

In summary, we have shown that the forgetful functor induces a homotopy equivalence $BF : B\mathcal{F}' \rightarrow B\mathcal{F}(\text{Csp})$, that the map $BG : B\mathcal{F}' \rightarrow B\mathcal{F}(\text{Csp})$ factors through the contractible space $BFin^{\neq\emptyset, \text{inj}}$, and that α induces a homotopy between BF and BG . This implies that $B\mathcal{F}(\text{Csp})$ is contractible as claimed. \square

The same argument applies to $B\mathcal{F}_a(\text{Csp}(A, A_1, \alpha)) \simeq *$ for all $a \in A$ as long as $0 \in A_1$, i.e. as long as $A_1 = A$.

Lemma 8.2.49. *For all $d \geq 2$ and all diffeomorphism types $[Q]$ of closed d -manifolds the factorisation category $\mathcal{F}_{[Q]}(\text{Cob}_d)$ has a contractible classifying space.*

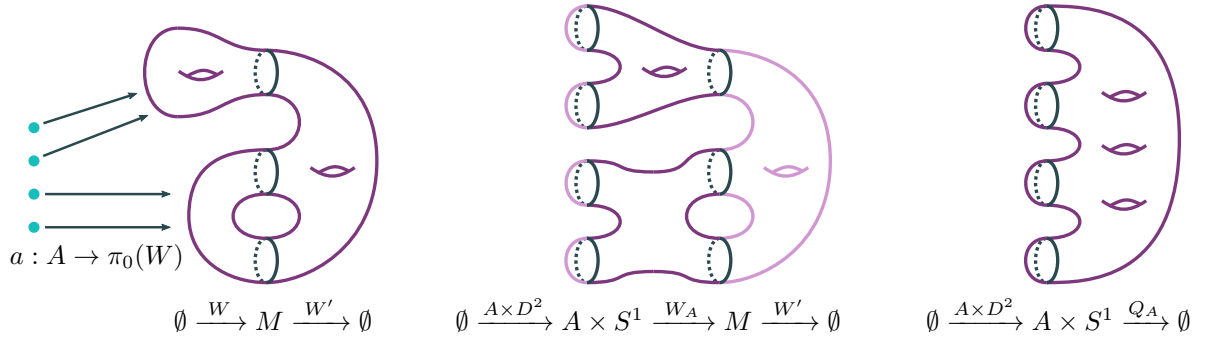


Figure 8.2: Left: an object $((M, W, W'), (A, a)) \in \mathcal{F}'_{g=3}(\text{Cob}_2)$. Middle: the value of the natural transformation $\alpha : G \Rightarrow \text{Id}$ at this object. Right: the value the functor G on this object.

Proof. An object of $\mathcal{F}_{[Q]}(\text{Cob}_d)$ is a triple $(M, [W], [W'])$ where M is a closed oriented $(d-1)$ -manifold and $[W]$ and $[W']$ are diffeomorphism classes of d -dimensional bordisms $W : \emptyset \rightarrow M$ and $W' : M \rightarrow \emptyset$ such that their composite $W \cup_M W'$ is diffeomorphic to Q . A morphism $(M, [W], [W']) \rightarrow (N, [V], [V'])$ is a diffeomorphism class of bordisms $[X] : M \rightarrow N$ satisfying $W \cup_M X \cong V$ and $W' \cong X \cup_N V'$.

We will choose explicit representatives for $[W]$ and $[W']$ throughout. This does not change the category \mathcal{F} , as long as we still define morphisms as above.

As in the proof of lemma 8.2.48 we construct a category \mathcal{F}' as follows. Objects are tuples $((M, W, W'), (a : A \rightarrow \pi_0(W)))$ where (M, W, W') is an object as above, A is a finite non-empty set, and $a : A \rightarrow \pi_0(W)$ a map. By the same argument as in lemma 8.2.48 the forgetful functor $F : \mathcal{F}' \rightarrow \mathcal{F}_{[Q]}(\text{Cob}_d)$ is an equivalence.

The following constructions are illustrated in figure 8.2. Consider the functor $G : \text{Fin}^{\neq \emptyset, \text{inj}} \rightarrow \mathcal{F}_{[Q]}(\text{Cob}_d)$ that sends a finite set A to the factorisation

$$G(A) = (A \times S^{d-1}, A \times D^d : \emptyset \rightarrow A \times S^{d-1}, Q_A : A \times S^{d-1} \rightarrow \emptyset)$$

where the bordism $Q_A : A \times S^{d-1} \rightarrow \emptyset$ is obtained by picking an injection $A \rightarrow Q$, removing a neighbourhood of its image, and identifying its boundary with $A \times S^{d-1}$ in the canonical orientation preserving way. Since Q is connected and of dimension $d \geq 2$ any choice of injection leads to the same diffeomorphism class of bordism. To injections $i : A \rightarrow B$ the functor G assigns the bordism:

$$G(i : A \rightarrow B) := [(A \times S^{d-1} \times [0, 1]) \amalg ((B \setminus i(A)) \times D^d)] : A \times S^{d-1} \rightarrow B \times S^{d-1}.$$

This bordism connects the sphere $\{a\} \times S^{d-1}$ with $\{i(a)\} \times S^{d-1}$ via a cylinder and caps off each of the spheres $\{b\} \times S^{d-1}$ with $b \notin i(A)$ with a disk. This defines a morphism in the factorisation category because closing off the $(B \setminus i(A))$ -spheres of Q_A yields a manifold diffeomorphic to Q_B :

$$G(i : A \rightarrow B) \cup_{B \times S^{d-1}} Q_B \cong Q_A \quad \text{and} \quad (A \times D^d) \cup_{A \times S^{d-1}} G(i : A \rightarrow B) \cong (B \times D^d).$$

Let us denote the composite of G with the forgetful functor $\mathcal{F}' \rightarrow \text{Fin}^{\neq \emptyset, \text{inj}}$ also by G .

To conclude the proof in the same manner as for lemma 8.2.48 we would like to construct a natural transformation $\alpha : G \Rightarrow F$ of functors $\mathcal{F}' \rightarrow \mathcal{F}$. For an object $((M, W, W'), (a : A \rightarrow \pi_0(W))) \in \mathcal{F}'$ we let $W_A : A \times S^{d-1} \rightarrow M$ be the bordism obtained by lifting the map $a : A \rightarrow \pi_0(W)$ to an embedding $a : A \hookrightarrow W$ and then removing a small neighbourhood of its image. We define the components of α as:

$$\alpha_{((M, W, W'), (a : A \rightarrow \pi_0(W)))} := W_A : (A \times S^{d-1}, A \times D^d, Q_A) \rightarrow (M, W, W').$$

To check that this is a well-defined morphism in the factorisation category we note:

$$(A \times D^d) \cup_{A \times S^{d-1}} W_A \cong W \quad \text{and} \quad W_A \cup_M V \cong Q_A.$$

To check naturality let $([X], i) : ((M, W, W'), (A, a)) \rightarrow ((N, V, V'), (B, b))$ be a morphism in \mathcal{F}' , i.e. $X : M \rightarrow N$ a bordism and $i : A \rightarrow B$ and injection, both satisfying certain conditions. Then we have:

$$G(i : A \rightarrow B) \cup_{B \times S^{d-1}} V_B \cong V_A \cong W_A \cup_M X.$$

Here the first diffeomorphism comes from the fact that both sides are obtained from $V : \emptyset \rightarrow N$ by removing disks according to $(\partial_0(X) \circ a : A \rightarrow \pi_0(W) \rightarrow \pi_0(V)) = (b \circ i : A \rightarrow B \rightarrow \pi_0(V))$, and the second diffeomorphism comes from the fact that we can arrange for the diffeomorphism $V \cong W \cup_M X$ to fix the disks corresponding to A .

This shows that $BG, BF : B\mathcal{F}' \rightarrow B\mathcal{F}_{[Q]}(\text{Cob}_d)$ are homotopic. Since BF is a homotopy equivalence and BG is null-homotopic, this implies that $B\mathcal{F}_{[Q]}(\text{Cob}_d)$ is contractible, as claimed. \square

Corollary 8.2.50. *For $d \geq 2$ there are equivalences:*

$$B\mathbb{C}\text{sp} \simeq |N_{\bullet}^{\text{nc}}\mathbb{C}\text{sp}| \quad \text{and} \quad B\text{Cob}_d \simeq |N_{\bullet}^{\text{nc}}\text{Cob}_d|.$$

Proof. Combine the decomposition theorem 8.2.1 with lemma 8.2.49 and lemma 8.2.48. \square

8.3 The surgery theorem

The purpose of this section is to prove the surgery theorem that relates space of “no compact components” with the positive boundary subcategory.

Theorem 8.3.1 (Surgery theorem). *For any labelled cospan category \mathcal{C} that admits surgery the inclusion $N\mathcal{C}^{\partial_+} \rightarrow N_{\bullet}^{\text{nc}}\mathcal{C}$ of simplicial sets induces an equivalence*

$$B(\mathcal{C}^{\partial_+}) \simeq |N_{\bullet}^{\text{nc}}\mathcal{C}|.$$

The proof of this theorem will be given by implementing a surgery similar to the one described in [Gal11, section 4.2, figure 3] using language similar to that of [GRW14, section 3 and 6]. For each $W \in C_n^{\text{nc}}$ and point $p \in W(i \leq i+1)$ there is either a path connecting it to $W(0)$ or to $W(n)$. We will use this path to do a surgery that connects p to $W(0)$ or $W(n)$. One of the key problems will be that there is not always a unique or canonical surgery, but rather a contractible space of choices. We will have make sure that any combination of surgeries commutes.

Definition 8.3.2. *For the rest of this section, we fix a labelled cospan category $(\pi : \mathcal{C} \rightarrow \text{Csp})$ together with the following data:*

- *a connected object $O \in \mathcal{C}$ and a connected morphism $T : 1_{\mathcal{C}} \rightarrow O$ and*
- *for any connected object $A \in \mathcal{C}$ a connected morphism $P_A : A \rightarrow O \otimes A$,*

subject to the conditions in 8.1.42. (See also figure 8.1.4.)

To construct the homotopies that are crucial to the proof of theorem 8.3.1 it will be extremely useful to work with a variant of \mathcal{C} where objects are simply subsets of some big background set Ω .

Definition 8.3.3. *Fix a set Ω and a map $F : \Omega \rightarrow \text{Obj}^{\text{con}}(\mathcal{C})$ such that every connected object $M \in \mathcal{C}$ has infinitely many preimages in Ω . For any finite subset $A \subset \Omega$ we let⁵ $F(A) := \bigotimes_{a \in A} F(a)$.*

The category \mathcal{C}^{Ω} has as objects finite subset of Ω and a morphism $W : A \rightarrow B$ is defined as a morphism $W : F(A) \rightarrow F(B)$ in \mathcal{C} .

In what follows we will often use n -simplices in the nerve of \mathcal{C}^{Ω} so it will be useful to fix some notation. An element of $N_n \mathcal{C}^{\Omega}$ is a functor $W : [n] \rightarrow \mathcal{C}^{\Omega}$ sending any $0 \leq i \leq n$ to an object $W(i) \in \mathcal{C}^{\Omega}$, i.e. a finite subset $W(i) \subset \Omega$, and any $0 \leq i \leq j \leq n$ to a morphism $W(i \leq j) : W(i) \rightarrow W(j)$. We will denote the underlying cospan of $W(i \leq j)$ by $W(i) \rightarrow W_{\pi}(i \leq j) \leftarrow W(j)$.

Definition 8.3.4. *We let $C_{\bullet}^{\text{nc}} \subset N_{\bullet} \mathcal{C}^{\Omega}$ be the simplicial subset containing those n -simplices $W : [n] \rightarrow \mathcal{C}^{\Omega}$ where $W(0 \leq n)$ has no closed components, i.e. those W for which $W(0) \amalg W(n) \rightarrow W_{\pi}(0 \leq n)$ is surjective.*

Moreover, we let $C_{\bullet}^{\partial+} \subset N_{\bullet} \mathcal{C}^{\Omega}$ be the simplicial subset containing those $W : [n] \rightarrow \mathcal{C}^{\Omega}$ where $W(k \leq k+1)$ is positive boundary for all k , i.e. where $W(k+1) \rightarrow W_{\pi}(k \leq k+1)$ is surjective for all k .

⁵To be precise we pick a total ordering on Ω and define $F(A) = (\dots (a_1 \otimes a_2) \otimes a_2) \otimes \dots \otimes a_n$ for $A = \{a_1 < \dots < a_n\}$. By the coherence theorem for symmetric monoidal categories there is a preferred isomorphism $F(A) \cong F(A_1) \otimes F(A_2)$ built from associators and braidings whenever $A = A_1 \cup A_2$ with $A_1 \cap A_2$. We will be suppressing these coherence isomorphisms unless they become relevant.

Lemma 8.3.5. *With the usual notion of disjointnes and disjoint union \mathcal{C}^Ω is a partial commutative monoidal category and the functor $F : \mathcal{C}^\Omega \rightarrow \mathcal{C}$ is an equivalence of categories. Moreover, F induces equivalences:*

$$|C_{\bullet}^{\partial+}| \simeq BC^{\partial+} \quad \text{and} \quad |C_{\bullet}^{\text{nc}}| \simeq |N_{\bullet}^{\text{nc}}\mathcal{C}|.$$

Proof. Since the objects of \mathcal{C}^Ω are simply finite subsets of Ω it is clear that the notion of disjointnes and disjoint union behave as expected. For \mathcal{C}^Ω to satisfy definition 8.2.5 we need to check that for any finite sequence of objects $A_1, \dots, A_n \subset \Omega$ one can find isomorphic replacements $A'_1, \dots, A'_n \subset \Omega$ that are pairwise disjoint. We required $F : \Omega \rightarrow \text{Obj}^{\text{con}}(\mathcal{C})$ to have infinite preimages for this exact purpose. Starting with A_2 we can find for each $a \in A_i$ an element $a' \in \Omega$ with $F(a) = F(a')$ and a' disjoint to all previous elements. Then a and a' are isomorphic via the morphism $\text{id}_{F(a)} : a \rightarrow a'$. Moreover, they are all disjoint by construction.

The functor F is fully faithful by construction and it is essentially surjective because every object in the labelled cospan category \mathcal{C} can be written as a product of connected objects. Therefore F is an equivalence of categories and it also restricts to an equivalence between $(\mathcal{C}^\Omega)^{\partial+}$ and $\mathcal{C}^{\partial+}$. Since $C_{\bullet}^{\partial+} \cong (\mathcal{C}^\Omega)^{\partial+}$ we see that $|C_{\bullet}^{\partial+}| \simeq BC^{\partial+}$. The equivalence $|C_{\bullet}^{\text{nc}}| \simeq |N_{\bullet}^{\text{nc}}\mathcal{C}|$ follows from lemma 8.1.39. \square

REMARK 8.3.6. *We have so far treated $W_\pi(k \leq l)$ as if it were a well-defined set, but one has to be careful with this. In general, for a morphism $W(k \leq l)$ we only have an isomorphism class of cospans $[W(k) \rightarrow W_\pi(k \leq l) \leftarrow W(l)]$. It does therefore not really make sense to talk about elements of $W_\pi(k \leq l)$. However, if the n -simplex $W : [n] \rightarrow \mathcal{C}^\Omega$ lies in $C_{\bullet}^{\text{nc}} \subset N_{\bullet}\mathcal{C}^\Omega$, then for each $k \leq l$ the map $W(k) \amalg W(l) \rightarrow W_\pi(k \leq l)$ is surjective and we may define the set $W_\pi(k \leq l)$ as the quotient of $W(k) \amalg W(l)$ by the relation induced by the cospan. As this is the only case that features in this section we will confidently speak about elements of $W_\pi(k \leq l)$.*

8.3.1 Surgery data

Given some n -simplex $W \in C_n^{\text{nc}}$ that is not yet in $C_n^{\partial+}$ there must be parts of W that are not positive boundary. Concretely, we can find an element $u \in W_\pi(i \leq i+1)$ that is not in the image of $W(i+1) \rightarrow W_\pi(i \leq i+1)$. We would like to find a homotopy that homotops this to a simplex in $C_n^{\partial+}$. To do so we first choose a “path” from u to $W(0)$ or $W(n)$ along which we can introduce a morphism that connects U to a positive boundary. This is the datum of a surgery path. To make sense of the idea of a path we introduce the following notion of representing space:

Definition 8.3.7. *For an n -simplex $W \in C_n^{\text{nc}}$ we define the representing space $|W|$ as*

$$|W| = (\{(t, k, \omega) \in [0, 1] \times \{0, \dots, n\} \times \Omega \mid \omega \in W(k)\} \amalg \{\perp, \top\}) / \sim$$

The equivalence relation \sim is defined as follows: for $k \in \{0, \dots, n\}$ it identifies two elements of $\{(t, k, \omega) \in |W| \mid t + k = m\} \cong W(k-1) \amalg W(k)$ whenever they are mapped to the same element under $W(k-1) \amalg W(k) \rightarrow W_\pi(k-1 \leq k)$. Moreover, we set $(0, 0, \omega_0) \sim \perp$ and $(1, n, \omega_n) \sim \top$ for all $\omega_0 \in W_\pi(0)$ and $\omega_n \in W_\pi(n)$.

This space naturally comes with a continuous projection $\text{pr} : |W| \rightarrow [0, n+1]$ sending $[t, k, \omega]$ to $t+k$, \perp to 0, and \top to $n+1$. We let $|W|_0 \subset |W|$ be the finite subset $\text{pr}^{-1}(\{1, \dots, n\})$.

REMARK 8.3.8. We can also think of $|W|$ as a long sequence of pushouts:

$$\begin{array}{ccccccc}
 & & W(0) \times [0, 1] & & W(1) \times [1, 2] & & W(n) \times [0, n+1] \\
 & \nearrow & & \nwarrow & \nearrow & & \nwarrow \\
 * & & & & W_\pi(0 \leq 1) \times \{1\} & & \dots & & * \\
 & \nwarrow & & \nearrow & & & & & \\
 W(0) \times \{0\} & & W(0) \times \{1\} & & W(1) \times \{1\} & & \dots & & W(n) \times \{n+1\}
 \end{array}$$

The finite subset $|W|_0 \subset |W|$ is exactly the image of the sets $W_\pi(i-1 \leq i) \times \{i\}$ for $1 \leq i \leq n$. We can therefore define a map $g : |W|_0 \rightarrow \text{Mor}^{\text{con}}(\mathcal{C})$ that sends each (i, k, ω) with $k+i \in \{1, \dots, n\}$ to the connected morphism in \mathcal{C} that represents the relevant component of $W_\pi(k+i-1 \leq k+i)$. This is exactly the labeling of the cospan in the enhanced $\mathbb{C}\text{sp}(\mathcal{C})$ -model. The datum of W together with g in fact uniquely encodes the n -simplex $W \in C_n^{\text{nc}}$. Motivated by this we will often be drawing pictures of $(|W|, g : |W|_0 \rightarrow \text{Mor}^{\text{con}}(\mathcal{C}))$ as in figure 8.3.

Definition 8.3.9. For $W \in C_n^{\text{nc}}$ a surgery path is a continuous path $p : [0, 1] \rightarrow |W|$ such that the composite $p_{\mathbb{R}} : [0, 1] \rightarrow |W| \rightarrow [0, n+1]$ is piece-wise linear, $p_{\mathbb{R}}(0) \in \{0, n+1\}$, and $p_{\mathbb{R}}(1) \in \{0, 1, \dots, n+1\}$.

If the integer $i := p_{\mathbb{R}}(1)$ is in $\{1, \dots, n\}$ then the end-point of the path corresponds to some element $p_{\text{end}} \in W_\pi(i-1 \leq i)$. We also denote by $p_\Omega(1) \in \Omega$ the last well-defined value of $(\text{pr}_\Omega \circ p)(r)$ as $r \rightarrow 1$, this is an element of either $W(i-1)$ or $W(i)$ that is sent to p_{end} in the cospan $W(i-1) \rightarrow W_\pi(i-1 \leq i) \leftarrow W(i)$.

Of course one surgery path by itself is not sufficient to make some W be positive boundary. We need several surgery paths at once, which we collect in a surgery datum:

Definition 8.3.10. A surgery datum for $W \in C_n^{\text{nc}}$ is a finite subset $A \subset \Omega_O := F^{-1}(O)$. disjoint to $\cup_{i=0}^n W(k) \subset \Omega$ together with the choice of a surgery path $p^\alpha : [0, 1] \rightarrow |W|$ for all $\alpha \in A$ such that they satisfy the following connectivity condition:

For any $i \in \{1, \dots, n\}$ and $u \in W_\pi(i-1 \leq i)$ either u is in the image of $W(i) \rightarrow W_\pi(i-1 \leq i)$, or there is an $\alpha \in A$ such that the path p^α ends at $u = p_{\text{end}}^\alpha$.

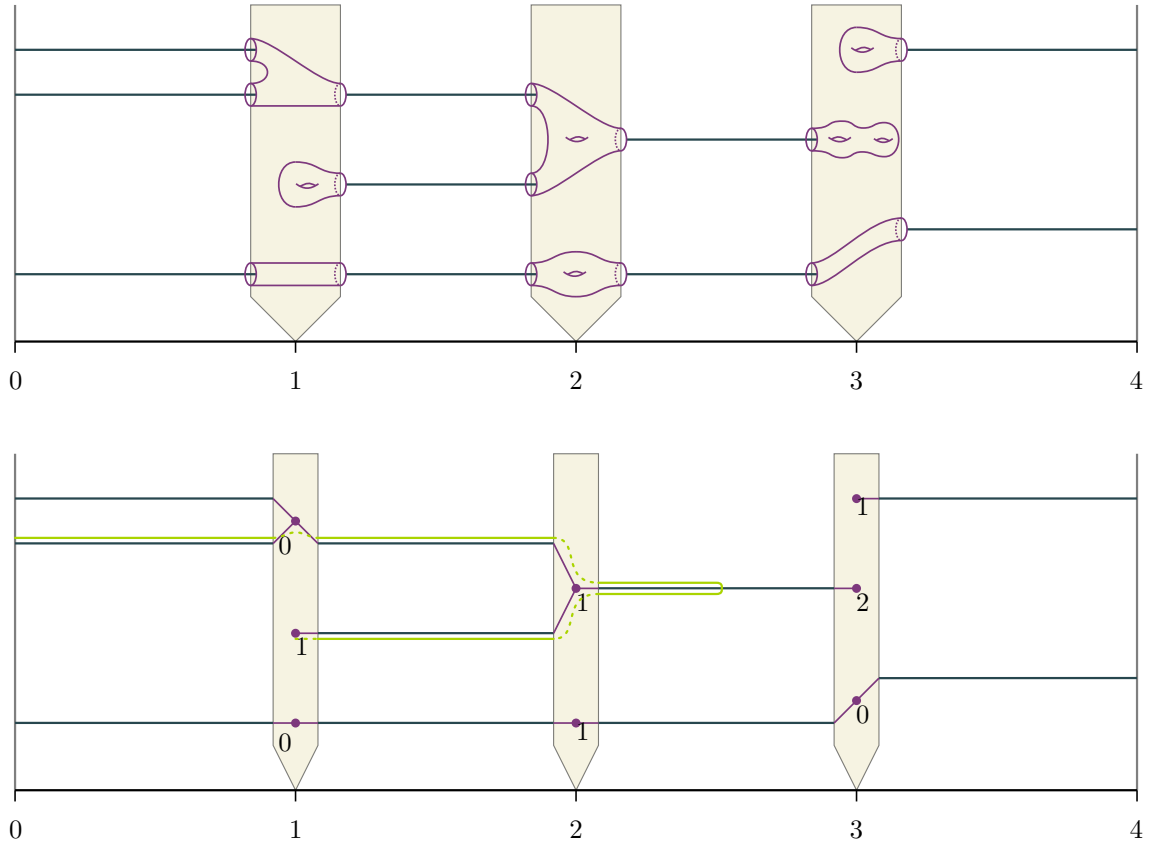


Figure 8.3: On top: a 3-simplex $W \in C_3^{\text{nc}}$ for the labelled cospan category $\mathcal{C} = \text{Cob}_2$. Below: The representing space $|W|$ with labels in \mathbb{N} recording the genus and a possible surgery path.

For a surgery datum $(p^\alpha)_{\alpha \in A}$ we let $A^{\text{in}} := \{\alpha \in A \mid p_{\mathbb{R}}^\alpha(0) = 0\} \subset A$ denote the subset of those α where the surgery path starts at the incoming boundary.

Note that the empty collection is a surgery datum for W if and only if W already lies in $C_n^{\partial+}$. In that case we do not need to do any surgery. We now describe how the surgery data behaves with respect to the simplicial structure:

Definition 8.3.11. For any injective morphism $\lambda : [n] \rightarrow [m]$ in Δ we define a map

$$\lambda^* : \{0, \dots, m+1\} \rightarrow \{0, \dots, n+1\} \quad k \mapsto \min\{j \mid \lambda(j) \geq k\}$$

where we set $\lambda^*(0) = 0$ and $\lambda^*(m+1) = n+1$ by convention. Extending affine linearly we obtain the piece-wise linear map $\lambda^\bullet : [0, m+1] \rightarrow [0, n+1]$. This defines a functor $\Delta_{\text{inj}}^{\text{op}} \rightarrow \text{Top}$.

For an n -simplex $W \in C_n^{\text{nc}}$ we let $\lambda_W : |W| \rightarrow |\lambda^*W|$ denote the unique continuous map that satisfies $\lambda_W([t, \lambda(k), \omega]) = [t, k, \omega]$ for all $[t, k, \omega] \in |\lambda^*W|$, $\lambda_W(\perp) = \perp$, $\lambda_W(\top) = \top$, and makes

the following diagram commute:

$$\begin{array}{ccc} |W| & \xrightarrow{\lambda_W} & |\lambda^*W| \\ \downarrow \text{pr} & & \downarrow \text{pr} \\ [0, n+1] & \xrightarrow{\lambda^\bullet} & [0, m+1] \end{array}$$

Definition 8.3.12. The semisimplicial set C_\bullet^σ has as n -simplices pairs of an n -simplex $W \in C_n^{\text{nc}}$ with a surgery datum $(p^\alpha)_{\alpha \in A}$ on W . The semisimplicial structure is defined by

$$\lambda^*(W, (p^\alpha)_{\alpha \in A}) := (\lambda^*W, (\lambda_W \circ p^\alpha)_{\alpha \in A}).$$

Lemma 8.3.13. If $(p^\alpha)_{\alpha \in A}$ is a surgery datum for W then $(\lambda_W \circ p^\alpha)_{\alpha \in A}$ is a surgery datum for λ^*W . Therefore C_\bullet^σ is a well-defined semi-simplicial set.

Proof. Since $\lambda_W : |W| \rightarrow |\lambda^*W|$ is continuous the new path $\lambda_W \circ p : [0, 1] \rightarrow |W| \rightarrow |\lambda^*W|$ is also continuous. Moreover, $\lambda^\bullet : [0, n+1] \rightarrow [0, m+1]$ is a piece-wise linear map and hence $(\lambda_W \circ p)_\mathbb{R} = \lambda^\bullet \circ p_\mathbb{R}$ is piece-wise linear, too. The condition $p_\mathbb{R}(0) \subset \{0, n+1\}$ is preserved because $\lambda^\bullet : [0, n+1] \rightarrow [0, m+1]$ preserves the minimal and the maximal element. The condition $p_\mathbb{R}(1) \in \{0, \dots, n+1\}$ implies $(\lambda^\bullet \circ p_\mathbb{R})(1) \in \{0, \dots, m+1\}$ since λ^\bullet sends integers to integers. Therefore $\lambda_W \circ p$ is indeed still a surgery path.

Next we need to check that $(\lambda_W \circ p^\alpha)_{\alpha \in A}$ is a surgery datum for λ^*W . It is clear that A is disjoint to λ^*W as we can only forget elements of Ω when passing from W to λ^*W . To check the connectivity condition let us assume that λ is the unique morphism $\delta^i : [n] \rightarrow [n+1]$ whose image does not contain i . Write $d_iW := (\delta^i)^*W$. The general case follows as the category Δ_{inj} is generated by the δ^i . Consider some $u \in (d_iW)_\pi(j-1 \leq j)$ that is not in the image of $(d_iW)_\pi(j)$. If $j \neq i$ it is clear from the connectivity condition for W that there is an $\alpha \in A$ such that p^α ends in $p_{\text{end}}^\alpha = u$. We may therefore assume that $j = i$, in which case $(d_iW)(i-1 \leq i)$ is the composite of $W(i-1 \leq i)$ and $W(i \leq i+1)$ and the set $(d_iW)_\pi(i-1 \leq i)$ is the pushout $W_\pi(i-1 \leq i) \amalg_{W(i)} W_\pi(i \leq i+1)$.

There are two cases: either $u \in (d_iW)_\pi(i-1 \leq i)$ is not hit by an element of $W_\pi(i \leq i+1)$ or it is. In the former case u is represented by an element $v \in W_\pi(i-1 \leq i)$ that is not in the image of $W(i)$. The connectivity condition for $(p^\alpha)_{\alpha \in A}$ implies that there is $\alpha \in A$ such that the surgery path p^α ends at $p_{\text{end}}^\alpha = v$. Consequently $\delta_W^i \circ p^\alpha$ ends at $(\delta_W^i \circ p^\alpha)_{\text{end}} = \delta_W^i(p_{\text{end}}^\alpha) = u$ and we are done. In the other case we can represent u by an element $v' \in W_\pi(i \leq i+1)$. This v' cannot lie in the image of $W(i+1) \rightarrow W_\pi(i \leq i+1)$, as otherwise u would be in the image of $(d_iW)(i) \rightarrow (d_iW)_\pi(i-1 \leq i)$, which we assumed not to be the case. This means that v does not have positive boundary as part of W and so we can find $\alpha \in A$ such that p^α ends in $p_{\text{end}}^\alpha = v$, and then $(\delta_W^i \circ p^\alpha)_{\text{end}} = \delta_W^i(p_{\text{end}}^\alpha) = u$.

This shows that $(\lambda_W \circ p^\alpha)_{\alpha \in A}$ is a surgery datum for λ^*W and hence the face operators λ^* yield well-defined elements of C_m^σ . Since the $\lambda \mapsto \lambda^\bullet$ construction is functorial it follows that this assembles into a well-defined functor $C_\bullet^\sigma : \Delta_{\text{inj}}^{\text{op}} \rightarrow \text{Set}$. \square

8.3.2 The basic surgery

We now want to describe how to do the surgery given a surgery datum. The problem of choosing this surgery datum will be dealt with in the next subsection. In formulas this means that we want to construct a homotopy from the forgetful map $|C_\bullet^\sigma| \rightarrow |C_\bullet^{\text{nc}}|$ to another map that lands in the subspace $|C_\bullet^{\partial+}| \subset |C_\bullet^{\text{nc}}|$. Moreover, we would like this homotopy to be constant in the case of an empty surgery datum. This is summarized in the following diagram:

$$\begin{array}{ccc} |C_\bullet^{\partial+}| & \longrightarrow & |C_\bullet^\sigma| \\ \parallel & \searrow \mathcal{S}_1 & \downarrow \mathcal{S}_0 \\ |C_\bullet^{\partial+}| & \longrightarrow & |C_\bullet^{\text{nc}}| \end{array}$$

Here $\mathcal{S} : |C_\bullet^\sigma| \times [0, 1] \rightarrow |C_\bullet^{\text{nc}}|$ is a homotopy, which we think of as a continuous family of maps \mathcal{S}_r indexed by $r \in [0, 1]$ such that \mathcal{S}_0 is the forgetful map and \mathcal{S}_1 is a retraction onto the subspace $|C_\bullet^{\partial+}|$.

The homotopy \mathcal{S} will be obtained by concatenating two homotopies ρ and K . With ρ we introduce a disjoint copy of id_α for all $\alpha \in A^{\text{in}}$ as pictured in figure 8.3.2. This is based on the idea of how the morphism $T : 1_C \rightarrow O$ yields a natural transformation $\text{Id}_C \Rightarrow O \otimes \text{Id}_C$, which in turn induces a homotopy of maps $BC \rightarrow BC$. The second homotopy K will be more complicated as we have to follow a surgery path to move a copy of the morphism $P_M : M \rightarrow O \otimes M$ from $W(0)$ or $W(n)$ to morphism at the end-point of the surgery path, in order to make that morphism positive boundary; see figure 8.6.

Definition 8.3.14. We let $\Omega_O := F^{-1}(O) \subset \Omega$. For any $\alpha \in \Omega_O$ and $\omega \in \Omega$ with $\omega \neq \alpha$ we define morphisms $T_\alpha : \emptyset \rightarrow \{\alpha\}$ and $P_{\omega, \alpha} : \{\omega\} \rightarrow \{\omega, \alpha\}$ in \mathcal{C}^Ω by:

$$T_\alpha := (F(\emptyset) = 1_C \xrightarrow{T} O = F(\alpha)) \quad \text{and} \quad P_{\omega, \alpha} := (F(\omega) \xrightarrow{P_{F(\omega)}} O \otimes F(\omega) \cong F(\{\omega, \alpha\}))$$

The basic surgery will not be a semi-simplicial map, rather it is only defined on geometric realisations. Recall that the space $|C_\bullet^{\text{nc}}|$ is a quotient of the disjoint union

$$\coprod_{n \geq 0} C_n^{\text{nc}} \times |\Delta^n| \longrightarrow |C_\bullet^{\text{nc}}|.$$

We will parametrise the space $|\Delta^n|$ as $\{\mathbf{t} = (t_0, \dots, t_{n+1}) \in [0, 1]^{n+2} \mid 0 = t_0 \leq t_1 \leq \dots \leq t_n \leq t_{n+1} = 1\}$. A point in $|C_\bullet^{\text{nc}}|$ can be represented by (W, \mathbf{t}) where $W : [n] \rightarrow \mathcal{C}^\Omega$ and \mathbf{t} is as above.

We begin by defining a homotopy from the forgetful map $|C_\bullet^\sigma| \rightarrow |C_\bullet^{\text{nc}}|$ to a map that sends (W, \mathbf{t}) to $(W \amalg \text{id}_A, \mathbf{t})$ for A some finite subset of Ω_O disjoint to W .

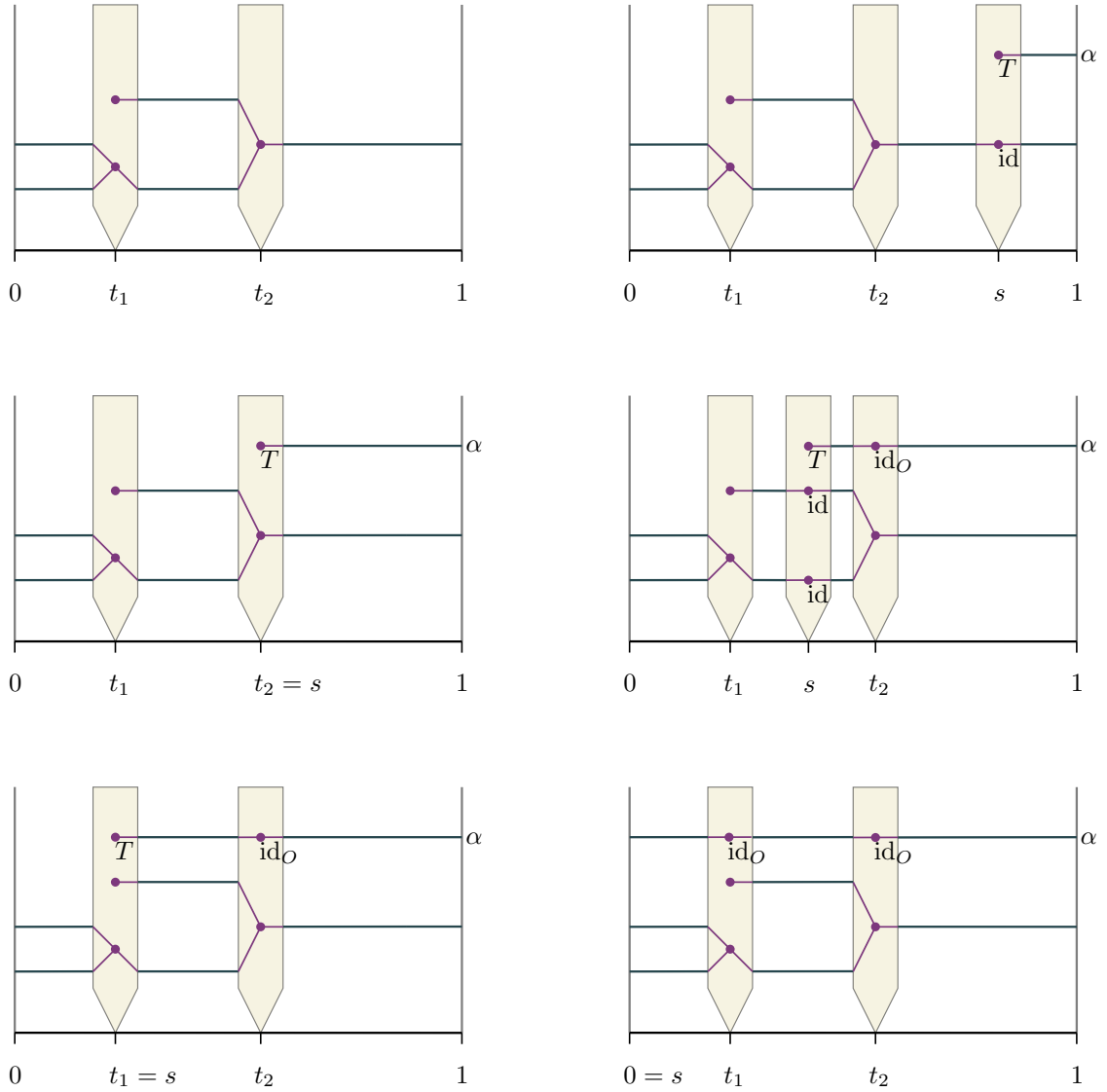


Figure 8.4: A slide show of the homotopy $\rho_\alpha^s(W, \mathbf{t})$ for a 2-simplex W as s moves from 1 to 0.

Definition 8.3.15. Given a point $(W, \mathbf{t}) \in |C_\bullet^{\text{nc}}|$, $\alpha \in \Omega_O$ disjoint to W and $s \in [t_i, t_{i+1}]$ we define

$$\rho_\alpha^s(W, \mathbf{t}) := (W', \mathbf{t}')$$

where \mathbf{t}' is $(t_0, \dots, t_i, s, t_{i+1}, \dots, t_n)$ and the $(n+1)$ -simplex W' is defined as

$$W'(j) = \begin{cases} W(j) & \text{for } j \leq i \\ W(j-1) \amalg \{\alpha\} & \text{for } j \geq i+1 \end{cases} \quad \text{and}$$

$$W'(j \leq j+1) = \begin{cases} W(j \leq j+1) & \text{for } j \leq i \\ \text{id}_{W(j+1)} \amalg T_\alpha & \text{for } j = i+1 \\ W(j-1 \leq j) \amalg \text{id}_\alpha & \text{for } j \geq i+2. \end{cases}$$

Given a finite set $A = \{\alpha_1, \dots, \alpha_N\} \subset \Omega$ that is disjoint to W we define:

$$\rho_A^r(W, \mathbf{t}) = (\rho_{\alpha_1}^r \circ \dots \circ \rho_{\alpha_N}^r)(W, \mathbf{t}).$$

Lemma 8.3.16. *The construction in definition 8.3.15 yields a continuous map*

$$\rho : |C_\bullet^\sigma| \times [0, 1] \longrightarrow |C_\bullet^{\text{nc}}|, \quad ((W, (p^\alpha)_{\alpha \in A}, \mathbf{t}), r) \mapsto \rho_{A^{\text{in}}}^r(W, \mathbf{t})$$

and restricted to $r = 1$ this map is the standard projection $|C_\bullet^\sigma| \rightarrow |C_\bullet^{\text{nc}}|$. Here as before $A^{\text{in}} \subset A$ denotes the subset corresponding to those paths that start at 0.

Proof. The map clearly only depends on the finite subset $A^{\text{in}} \subset \Omega$ and not on the rest of the surgery path $(p^\alpha)_{\alpha \in A}$, which we will only use later for the second part of the homotopy. For simplicity we will consider the case of a single element $A^{\text{in}} = \{\alpha\}$, the general case follows as the operations ρ_α^r and $\rho_{\alpha'}^r$ commute when $\alpha \neq \alpha'$.

Definition 8.3.15 is ambiguous when $r = t_i$ as we can think of it as both $r \in [t_i, t_{i+1}]$ and $r \in [t_{i-1}, t_i]$. In either case the tuple (W', \mathbf{t}') that is specified in the definition represents the same point as (W'', \mathbf{t}) in the geometric realisation $|C_\bullet^{\text{nc}}|$ where:

$$W''(j) = \begin{cases} W(j) & \text{for } j \leq i \\ W(j) \amalg \{\alpha\} & \text{for } j > i \end{cases} \quad \text{and}$$

$$W''(j \leq j+1) = \begin{cases} W(j \leq j+1) & \text{for } j \leq i-1 \\ W(i \leq i+1) \amalg T_\alpha & \text{for } j = i \\ W(j \leq j+1) \amalg \text{id}_\alpha & \text{for } j \geq i+1. \end{cases}$$

This shows that the definition of ρ on the closed subsets $U_i = \{(\mathbf{t}, r) \in |\Delta^n| \times [0, 1] \mid r \in [t_i, t_{i+1}]\}$ fits together on the overlaps. Since ρ is continuous on each of the U_i we get for every n -simplex $W \in C_n^\sigma$ a well-defined continuous map:

$$\rho_{A^{\text{in}}}^\bullet(W, _) : |\Delta^n| \times [0, 1] \rightarrow |C_\bullet^{\text{nc}}|, \quad (\mathbf{t}, r) \mapsto \rho_{A^{\text{in}}}^r(W, \mathbf{t}).$$

To obtain a well-defined map $|C_\bullet^\sigma| \times [0, 1] \rightarrow |C_\bullet^{\text{nc}}|$ it remains to check that these maps are compatible with taking face maps in the sense that $\rho_{A^{\text{in}}}^r(d_i W, \mathbf{t}) = \rho_{A^{\text{in}}}^r(W, \delta^i \mathbf{t})$. Here $\delta^i : |\Delta^{n-1}| \rightarrow |\Delta^n|$ is defined by repeating the i th entry. The equality $\rho_{A^{\text{in}}}^r(d_i W, \mathbf{t}) = \rho_{A^{\text{in}}}^r(W, \delta^i \mathbf{t})$ is clear when $r \neq t_i$ and in the case that $r = t_i$ it holds because of

$$W(i \leq i+2) \amalg T_\alpha = (W(i \leq i+1) \amalg T_\alpha) \cup (W(i+1 \leq i+2) \amalg \text{id}_\alpha).$$

Therefore ρ descends to a well-defined and continuous map $|C_\bullet^\sigma| \times [0, 1] \rightarrow |C_\bullet^{\text{nc}}|$ as claimed. \square

Now that we have the homotopy that introduces the cylinder id_α we proceed to describe a similar, but more complicated homotopy that homotops this cylinder along a surgery path to make a certain part of W positive boundary.

Definition 8.3.17. Given a point $(W, \mathbf{t}) \in |C_{\bullet}^{\text{nc}}|$, $\omega \in W_i$ for some i , $\alpha \in \Omega_O$ disjoint to W and $s \in [t_i, t_{i+1}]$ we define

$$\sigma_{\omega, \alpha}^s(W, \mathbf{t}) := (W', \mathbf{t}')$$

where \mathbf{t}' is $(t_0, \dots, t_i, s, t_{i+1}, \dots, t_n)$ and the $(n+1)$ -simplex W' is defined as

$$W'(j) = \begin{cases} W(j) & \text{for } j \leq i \\ W(j-1) \amalg \{\alpha\} & \text{for } j \geq i+1 \end{cases} \quad \text{and}$$

$$W'(j \leq j+1) = \begin{cases} W(j \leq j+1) & \text{for } j \leq i+1 \\ \text{id}_{W(i+1) \setminus \{\omega\}} \amalg P_{\omega, \alpha} & \text{for } j = i+1 \\ W(j-1 \leq j) \amalg \text{id}_{\alpha} & \text{for } j \geq i+2. \end{cases}$$

Lemma 8.3.18. If $\sigma_{\omega, \alpha}^s$ and $\sigma_{\omega', \alpha'}^r$ are both defined on (W, t) and $\alpha \neq \alpha'$, then the operations commute:

$$\sigma_{\omega, \alpha}^s(\sigma_{\omega', \alpha'}^r(W, t)) = \sigma_{\omega', \alpha'}^r(\sigma_{\omega, \alpha}^s(W, t)).$$

Proof. After applying both operations we obtain a point (W'', \mathbf{t}'') where $\mathbf{t}'' = (t_0, \dots, t_i, s, r, t_{i+1}, \dots, t_n)$, if $s \leq r$. If $s \neq r$, then it is a straight-forward check that the order does not matter. In the case $s = r$ there is a repetition in \mathbf{t}'' and we may use the simplicial relation to delete one of them by composing the relevant morphisms. To see that the operations commute we hence need to check that:

$$(\text{id}_{W(i) \cup \{\alpha\} \setminus \{\omega'\}} \amalg P_{\omega', \alpha'}) \circ (\text{id}_{W(i) \setminus \{\omega\}} \amalg P_{\omega, \alpha}) = (\text{id}_{W(i) \cup \{\alpha'\} \setminus \{\omega\}} \amalg P_{\omega, \alpha}) \circ (\text{id}_{W(i) \setminus \{\omega'\}} \amalg P_{\omega', \alpha'}).$$

For $\omega \neq \omega'$ this follows from the general fact that morphisms with disjoint support commute. In the more interesting case $\omega = \omega'$ we need to show that

$$(\text{id}_{\alpha} \amalg P_{\omega, \alpha'}) \circ P_{\omega, \alpha} = (\text{id}_{\alpha'} \amalg P_{\omega, \alpha}) \circ P_{\omega, \alpha'} : \{\omega\} \rightarrow \{\omega, \alpha, \alpha'\}.$$

Letting $A := F(\omega)$ both of these morphisms correspond to $(\text{id}_O \otimes P_A) \circ P_A : A \rightarrow O \otimes O \otimes A$ with the only difference being in how we identify $F(\{\alpha, \alpha'\})$ with $O \otimes O$. To prove the lemma we need to show that the morphism is symmetric in the two O s in the sense that:

$$(\text{id}_O \otimes P_A) \circ P_A = (\beta_{O, O} \otimes \text{id}_A) \circ (\text{id}_O \otimes P_A) \circ P_A.$$

As illustrated in figure 8.3.2 this follows from the first and third naturality condition we imposed on the morphisms $P_A : A \rightarrow O \otimes A$ in \mathcal{C} . \square

Definition 8.3.19. Consider a point $(W, \mathbf{t}) \in |C_{\bullet}^{\text{nc}}|$, a surgery path p for W , an element $\alpha \in \Omega_O$ disjoint to W , and a parameter $r \in [0, 1]$. Then we define the basic surgery of (W, \mathbf{t}) along the path p to be

$$K_{(p, \alpha)}^r(W, \mathbf{t}) = \sigma_{p_{\Omega(r)}, \alpha}^{P_{\mathbb{R}}(r)}(W, \mathbf{t}).$$

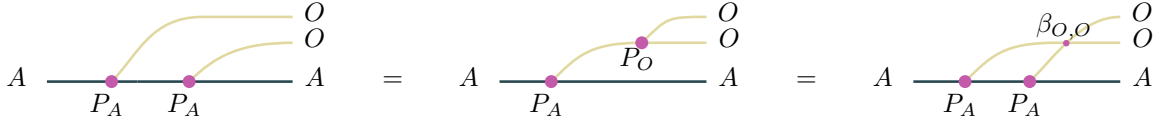


Figure 8.5: When applying P_A twice the order does not matter.

Whenever $p_\Omega(r)$ is not defined because $p_\mathbb{R}(r) \in \{0, \dots, n+1\}$ we choose either the most recent well-defined value $p_\Omega^-(r)$ or the next possible well-defined value $p_\Omega^+(r)$. In lemma 8.3.21 we show that this choice does not matter. For a surgery datum $(p^\alpha)_{\alpha \in A}$ on W we define

$$K_A^r(W, \mathbf{t}) = (K_{(p^{\alpha_1}, \alpha_1)}^r \circ \dots \circ K_{(p^{\alpha_a}, \alpha_a)}^r)(W, \mathbf{t})$$

where we choose any enumeration of $A = \{\alpha_1, \dots, \alpha_a\}$.

REMARK 8.3.20. Figure 8.6 illustrates the homotopy $K_{l,p,\alpha}^s(W, \mathbf{t})$ in the case $\alpha \in A^{\text{in}}$. The morphisms in the simplex W are $f : \omega_2 \otimes \omega_3 \rightarrow \omega_3$, $g : 1_C \rightarrow \omega_1$, and $h : \omega_1 \otimes \omega_3 \rightarrow \omega_2$. Throughout the homotopy we see other morphisms $f' = P_{\omega_3, \alpha} \circ f = (\text{id}_\alpha \otimes f) \circ (P_{\alpha, \omega_2} \otimes \text{id}_{\omega_3})$, $g' = P_{\omega_1, \alpha} \circ g$, and $h' = (\text{id}_\alpha \otimes h) \circ (\text{id}_{\omega_1} \otimes P_{\alpha, \omega_3}) = (\text{id}_\alpha \otimes h) \circ (P_{\alpha, \omega_1} \otimes \text{id}_{\omega_3})$. The key step of lemma 8.3.21 is to show that the two different expressions we have for f' and for h' yield the same morphisms.

See also figure 8.7, for an illustration of the case $\alpha \notin A^{\text{in}}$, where the surgery path starts at $p_\mathbb{R}(0) = 1$. Note that in this case we do not need to start out with a disjoint copy of id_α on top of the diagram. This difference will play a role when gluing the homotopies in proposition 8.3.23.

Lemma 8.3.21. Definition 8.3.19 defines a continuous map

$$K : |C_\bullet^\sigma| \times [0, 1] \rightarrow |C_\bullet^{\text{nc}}|, \quad ((W, (p^\alpha)_{\alpha \in A}, \mathbf{t}), r) \mapsto K_A^r(W, \mathbf{t}).$$

Proof. We will proceed similarly to lemma 8.3.16. If we fix an n -simplex $W \in C_\bullet^{\text{nc}}$ and a single surgery path (p, α) for W , then definition 8.3.17 gives us continuous maps:

$$U_i := \{(\mathbf{t}, r) \in |\Delta^n| \times [0, 1] \mid p_\mathbb{R}(r) \in [t_i, t_{i+1}]\} \rightarrow |C_\bullet^{\text{nc}}|, \quad (\mathbf{t}, r) \mapsto \sigma_{p_\Omega(r), \alpha}^{p_\mathbb{R}(r)}(W, \mathbf{t}).$$

The sets U_0, \dots, U_n are finite polyhedra that cover $|\Delta^n| \times [0, 1]$ and the key step of the proof will be to show that the map is consistent on overlaps. Let us consider those points where $p_\mathbb{R}(r) = t_i$, i.e. $V_i = U_{i-1} \cap U_i$. For such an $(\mathbf{t}, r) \in V_i$ the value of $K_{(l,p,\alpha)}^r(W, \mathbf{t})$ is ambiguous as we can either use $\sigma_{p_\Omega^+(r), \alpha}^{p_\mathbb{R}(r)}$ or $\sigma_{p_\Omega^-(r), \alpha}^{p_\mathbb{R}(r)}$.

For either choice the resulting tuple (W', \mathbf{t}') has a repeated value in \mathbf{t}' and we may use the relation $(W', \delta^i \mathbf{t}) \sim (d_i W, \mathbf{t})$ to rewrite this point of the geometric realisation $|C_\bullet^{\text{nc}}|$. The result is:

$$W''(j) = \begin{cases} W(j) & \text{for } j \leq i \\ W(j) \amalg \{\alpha\} & \text{for } j \geq i \end{cases} \quad \text{and} \quad W''(j \leq j+1) = \begin{cases} W(j \leq j+1) & \text{for } j \leq i-1 \\ X_\pm & \text{for } j = i \\ W(j \leq j+1) \amalg \text{id}_\alpha & \text{for } j \geq i+1. \end{cases}$$

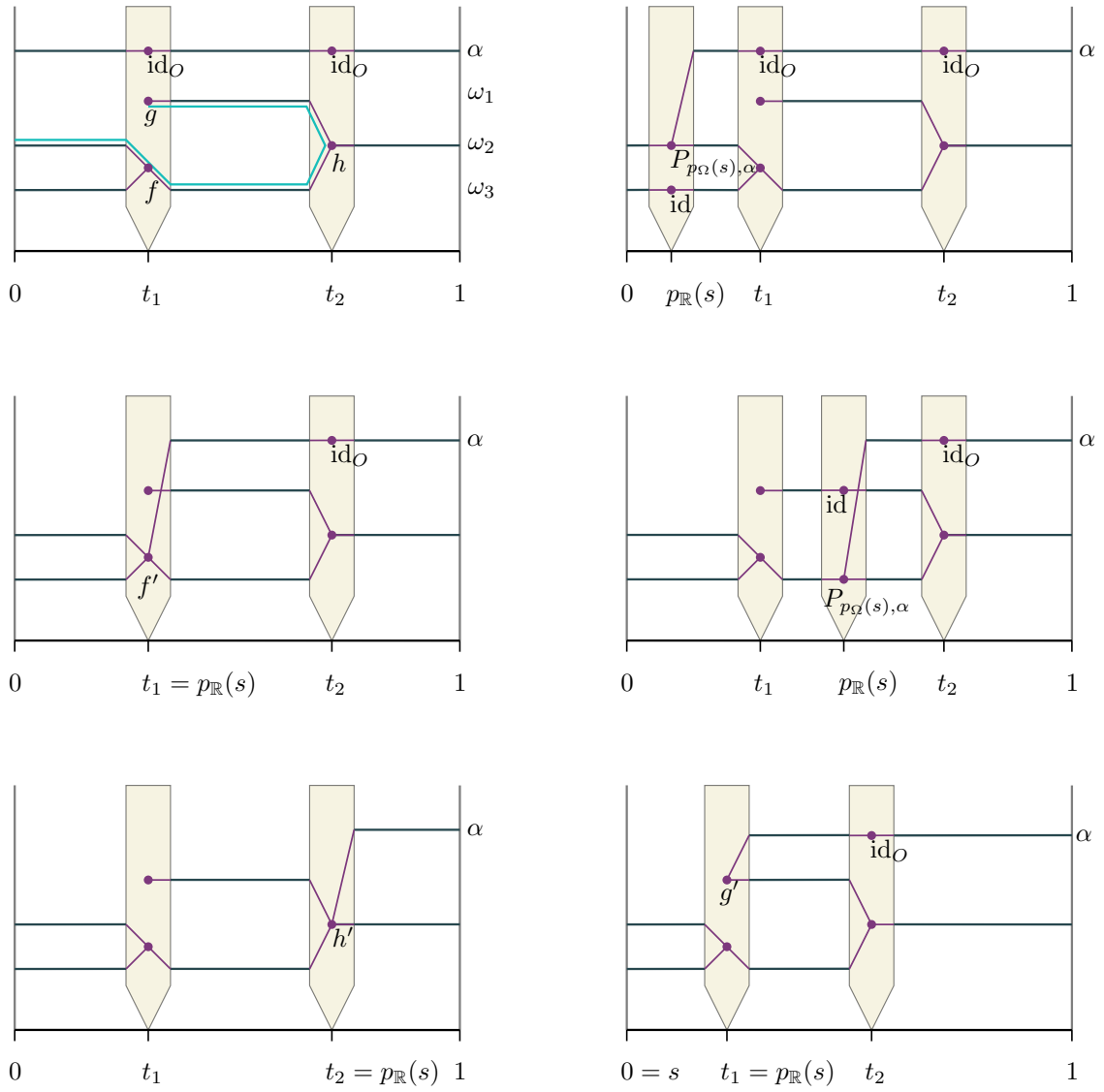


Figure 8.6: A slide-show depiction of the homotopy $K_{p,\alpha}^s(W, \mathbf{t})$ along the surgery path p indicated in the first picture, as s goes from 0 to 1. See remark 8.3.20.

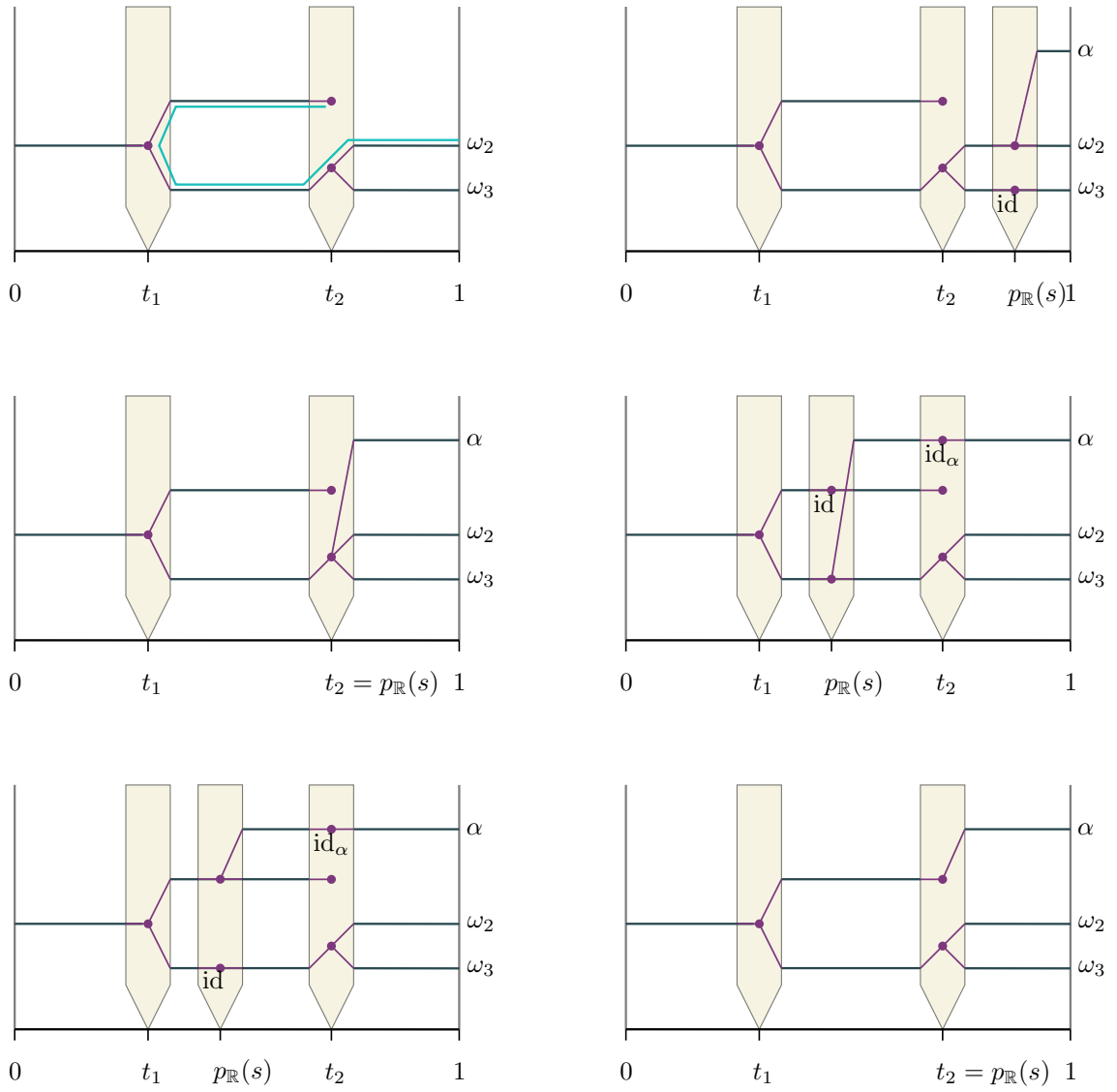


Figure 8.7: A slide-show depiction of the homotopy $K_{p,\alpha}^s(W, t)$ along a surgery path p that starts at 1.

Here the bordism X_{\pm} depends on whether we use $p_{\Omega}^{+}(r)$ or $p_{\Omega}^{-}(r)$ and we need to show that $X_{+} = X_{-}$. There are four possible cases one needs to check, depending on whether the values $p_{\Omega}^{\pm}(r)$ correspond to incoming or outgoing boundary components of the cospan $[W(i-1) \rightarrow W_{\pi}(i-1 \leq i) \leftarrow W(i)]$.

Each of the cases follows from the conditions on the morphisms $P_A : A \rightarrow O \otimes A$ as stated in definition 8.1.42. Let us for example consider the case where $p_{\Omega}^{+}(r) \in W(i+1)$ and $p_{\Omega}^{-}(r) \in W(i)$. Then

$$X_{+} = W(i \leq i+1) \cup_{W(i+1)} (\text{id}_{W(i+1)} \amalg P_{p_{\Omega}^{+}(r), \alpha}) \quad X_{-} = (\text{id}_{W(i)} \amalg P_{p_{\Omega}^{-}(r), \alpha}) \cup_{W(i)} (W(i \leq i+1) \amalg \text{id}_{\alpha}).$$

To check that $X_{-} = X_{+}$ we only need to consider the connected component of the morphism $W(i \leq i+1) : W(i) \rightarrow W(i+1)$ corresponding to $p(1)$. Without loss of generality we may therefore assume that $W(i \leq i+1)$ is connected. Write $M := F(W(i) \setminus \{p_{\Omega}^{-}(r)\})$, $N := F(W(i+1) \setminus \{p_{\Omega}^{+}(r)\})$, $A^{\pm} := F(p_{\Omega}^{\pm}(r))$, and $U := W(i \leq i+1) : A^{-} \otimes M \rightarrow A^{+} \otimes N$. By definition 8.1.42 we have a commutative diagram:

$$\begin{array}{ccc} A^{-} \otimes M & \xrightarrow{P_{A^{-} \otimes \text{id}_M}} & O \otimes A^{-} \otimes M \\ \downarrow U & & \downarrow \text{id}_O \otimes U \\ A^{+} \otimes N & \xrightarrow{P_{A^{+} \otimes \text{id}_N}} & O \otimes A^{+} \otimes N \end{array}$$

In this diagram the two ways of going from the top left to the bottom right are exactly the morphisms $F(W(i)) = A^{-} \otimes M \rightarrow O \otimes A^{+} \otimes N = F(W(i+1) \amalg \{\alpha\})$ that represent X_{-} and X_{+} . Hence it indeed follows from definition 8.1.42 that $X_{-} = X_{+}$. Similar arguments apply in the other three cases. (Two of the cases agree after swapping $p_{\Omega}^{+}(r)$ and $p_{\Omega}^{-}(r)$.)

We have now shown that the continuous maps on the U_i fit together on the overlaps and yield a continuous map:

$$|\Delta^n| \times [0, 1] \rightarrow |C_{\bullet}^{\text{nc}}|, \quad (\mathbf{t}, r) \mapsto \sigma_{p_{\Omega}^{\pm}(r), \alpha}^{p_{\mathbb{R}}(r)}(W, \mathbf{t}).$$

Similar to lemma 8.3.16 we see that this is compatible with face maps. Hence it yields a well-defined map $|C_{\bullet}^s| \times [0, 1] \rightarrow |C_{\bullet}^{\text{nc}}|$ for C_{\bullet}^s the semisimplicial set where each W comes equipped with a single surgery path.

Given a collection of surgery paths $(p^{\alpha})_{\alpha \in A}$ it follows from lemma 8.3.18 that the operations $\sigma_{p_{\Omega}^{\pm}(r), \alpha}^{p_{\mathbb{R}}^{\alpha}(r)}(_)$ commute for varying α . Therefore we can pick any order on A and compute $K : |C_{\bullet}^{\sigma}| \times [0, 1] \rightarrow |C_{\bullet}^{\text{nc}}|$ by applying the map $|C_{\bullet}^s| \times [0, 1] \rightarrow |C_{\bullet}^{\text{nc}}|$ multiple times. It follows that K is well-defined and continuous. \square

We can now concatenate the two homotopies to obtain the basic surgery.

Definition 8.3.22. *The homotopy \mathcal{S} is defined as follows:*

$$\mathcal{S} : |C_{\bullet}^{\sigma}| \times [0, 1] \rightarrow |C_{\bullet}^{\text{nc}}|, \quad ((W, (p^{\alpha})_{\alpha \in A}, \mathbf{t}), r) \mapsto \mathcal{S}_A^r(W, \mathbf{t}) := \begin{cases} \rho_{A^{\text{in}}}^{1-2r}(W, \mathbf{t}) & \text{for } 0 \leq r \leq \frac{1}{2}, \\ K_A^{2r-1}(W, \mathbf{t}) & \text{for } \frac{1}{2} \leq r \leq 1. \end{cases}$$

To conclude this section we record the fact that the homotopy \mathcal{S} we constructed has all the desired properties.

Proposition 8.3.23. *The above defines a continuous family of maps $\mathcal{S}^r : |C_{\bullet}^{\sigma}| \rightarrow |C_{\bullet}^{\text{nc}}|$ satisfying:*

- (i) $\mathcal{S}^0 : |C_{\bullet}^{\sigma}| \rightarrow |C_{\bullet}^{\text{nc}}|$ is the realisation of the map $C_{\bullet}^{\sigma} \rightarrow C_{\bullet}^{\text{nc}}$ that forgets the surgery data.
- (ii) $\mathcal{S}^1 : |C_{\bullet}^{\sigma}| \rightarrow |C_{\bullet}^{\text{nc}}|$ factors through the subspace $|C_{\bullet}^{\partial+}| \subset |C_{\bullet}^{\text{nc}}|$.
- (iii) Precomposed with the inclusion $|C_{\bullet}^{\partial+}| \rightarrow |C_{\bullet}^{\sigma}|$ that equips the empty surgery data, the family \mathcal{S}^r is the standard inclusion $|C_{\bullet}^{\partial+}| \rightarrow |C_{\bullet}^{\text{nc}}|$ for all r .

Proof. We have seen in lemma 8.3.16 and 8.3.21 that the two homotopies ρ_A^s and $K_A^{s'}$ are well-defined and continuous. So to check that \mathcal{S} is well-defined and continuous we only need to verify that $\rho_{A^{\text{in}}}^0(W, \mathbf{t}) = K_A^0(W, \mathbf{t})$ holds for all A and (W, \mathbf{t}) . Indeed both sides can be described as $(W \amalg \text{id}_{A^{\text{in}}}, \mathbf{t})$. To see this note that property (ii) for surgery paths says that $p_{\mathbb{R}}^{\alpha}(0) = 0$ or $n + 1$. For those α where $p_{\mathbb{R}}^{\alpha}(1) = n + 1$ the operation $\sigma_{p_{\Omega}^{\alpha}(1), \alpha}^{p_{\mathbb{R}}^{\alpha}(1)}$ does not do anything and in the case $p_{\Omega}^{\alpha}(1) = 0$ we have $\sigma_{p_{\Omega}^{\alpha}(1), \alpha}^{p_{\Omega}^{\alpha}(1)}(W, \mathbf{t}) = (W \amalg \text{id}_{\alpha}, \mathbf{t})$.

In claim (i) the map \mathcal{S}^0 is by definition $\rho_{A^{\text{in}}}^1$, which can easily be seen to map (W, \mathbf{t}) to (W, \mathbf{t}) . So all it does is forgetting the surgery data. Claim (iii) is similarly straightforward: if the surgery data is empty then neither ρ_{\emptyset} nor K_{\emptyset} change anything about (W, \mathbf{t}) .

Claim (ii) is the most interesting one. Before reading the proof it is useful to look at figure 8.6 and think about why the final picture is a simplex in the nerve of the positive boundary category. The following proof is just a formalisation of that visual verification. We need to show that $(W', \mathbf{t}') = K_A^1(W, \mathbf{t})$ is positive boundary for any n simplex W and surgery datum A . Since $p_{\mathbb{R}}^{\alpha}(1)$ is some integer i_{α} for any surgery path p^{α} we have that $\mathbf{t}' = \mathbf{t}$ and the simplex W' after the surgery can be described as:

$$W'(j) = W(j) \amalg \{\alpha \in A \mid i_{\alpha} \leq j\} \quad \text{and} \quad W'(j-1 \leq j) = P_A^j \cup W(j-1 \leq j) \amalg \{\text{id}_{\alpha} \mid i_{\alpha} < j\}$$

Here P_A^j is the morphism $W(j-1) \amalg \{\alpha \mid i_{\alpha} \leq j-1\} \rightarrow W(j-1) \amalg \{\alpha \mid i_{\alpha} \leq j\}$ obtained by suitably gluing all those $P_{p_{\text{end}}, \alpha}^{\alpha} : \{p_{\text{end}}^{\alpha}\} \rightarrow \{p_{\text{end}}^{\alpha}, \alpha\}$ where $i_{\alpha} = j-1$. (Note that the order of glueing does not matter by lemma 8.3.18.)

For W' to be positive boundary we need to check that for any j and $u \in W'_{\pi}(j-1 \rightarrow j)$ there is some $x \in W'(j)$ that maps to u . Any such component u is either an identity id_{α} that we added, or a component of $P_A^j \cup W(j-1 \rightarrow j)$. In the first case it is clearly positive boundary. In the second case consider the corresponding $u' \in W_{\pi}(j-1 \leq j)$. If this was already in the image of $W(j)$, then so is u and we are done. If u' is not positive boundary, then the connectivity condition that we imposed on surgery data implies that there is an $\beta \in A$ for which $p_{\text{end}}^{\beta-} = u'$ ends at u' . In the glued morphism $P_A^j \cup W(j-1 \leq j)$ the element u' is equivalent to β coming

from $P_{p_{\text{end}}, \beta}^\beta$, since the morphism $P_{p_{\text{end}}, \beta}^\beta$ is connected. So $u' \in W'_\pi(j-1 \leq j)$ is the image of $\beta \in W'(j) = W(j) \amalg \{\alpha \in A \mid i_\alpha \leq j\}$. So $W'(j-1 \leq j)$ is indeed positive boundary. This proves claim (ii). \square

8.3.3 Contractible surgery data

One problem we have not addressed yet is how to choose the surgery data $(p^\alpha)_{\alpha \in A}$ for a given point (W, t) in $|C_{\bullet, \bullet}^{\text{nc}}|$. Concretely, the problem is that the forgetful map $C_{\bullet, \bullet}^\sigma \rightarrow C_{\bullet, \bullet}^{\text{nc}}$ will not induce an equivalence on realisations. To resolve this we need to interpolate between different choices of surgery data. This is not really a problem as the operators K_A^λ and $K_A^{\lambda'}$ commute.

The following definition allows for multiple pieces of surgery data.

Definition 8.3.24. *The bi-semi-simplicial set $C_{\bullet, \bullet}^\sigma$ has as (n, m) -simplices tuples $(W, ((p_0^\alpha)_{\alpha \in A_0}, \dots, (p_m^\alpha)_{\alpha \in A_m}))$ where $W \in C_n^{\text{nc}}$ and each $(p_i^\alpha)_{\alpha \in A_i}$ is a surgery datum for W , such that A_i and A_j are disjoint for $i \neq j$. The i -th face operators in the second direction is defined by forgetting $(p_i^\alpha)_{\alpha \in A_i}$.*

We can always find surgery data that is disjoint to all previous surgery, which makes it very easy to show contractibility.

Lemma 8.3.25. *For any n the augmentation of $C_{n, \bullet}^\sigma$ induces an equivalence $|C_{n, \bullet}^\sigma| \simeq C_n^{\text{nc}}$.*

Proof. We will derive this from [GRW14, Theorem 6.2] though it should be remarked that the statement we are proving here is much simpler than the full generality to which their theorem applies. The augmented semi-simplicial set $C_{n, \bullet}^\sigma \rightarrow C_n^{\text{nc}}$ is essentially by definition an augmented topological flag complex in the sense of [GRW14, Definition 6.1]. Since it is discrete, the first condition of the theorem is trivially satisfied. The the third condition is also easy to see: if $(p_0^\alpha)_{\alpha \in A_0}, \dots, (p_m^\alpha)_{\alpha \in A_m}$ is a sequence of surgery data we can define a new surgery datum that is disjoint to all these by simply copying the first one. Concretely, we choose a subset $A' \subset \Omega$ that is disjoint to W and all of the A_i and pick a bijection $\varphi : A' \cong A_0$. Then $p_{n+1}^\alpha := p_0^{\varphi(\alpha)}$ defines a surgery datum indexed by $\alpha \in A'$ and is disjoint to all the others by construction.

Finally, the second condition simply states that every n -simplex $W \in C_n^{\text{nc}}$ admits at least one surgery datum. We will show this by constructing one. Let $A \subset \Omega_O$ be a subset disjoint to W and big enough so that we may choose a surjection $f : A \rightarrow \coprod_{i=1}^n W_\pi(i-1 \leq i) \subset |W|$. Since $W \in C_{\bullet, \bullet}^{\text{nc}}$ every connected component of $|W|$ is connected to either $\perp \sim [W(0) \times \{0\}]$ or $\top \sim [W(n) \times \{n+1\}]$. We can therefore find piece-wise linear paths $p^\alpha : [0, 1] \rightarrow |W|$ such that $p^\alpha(0) \in \{\perp, \top\}$ and $p^\alpha(1) = f(\alpha)$. This trivially satisfies the connectivity condition because it hits all components of morphisms, in particular those that are not positive boundary. Therefore $(p^\alpha)_{\alpha \in A}$ is the desired surgery datum for W . \square

We now need to construct the surgery homotopy \mathcal{S}_λ dependent on a weighted collection of surgery data. To do so let $\|C_{\bullet, \bullet}^\sigma\|$ denote the geometric realisation of the semisimplicial space $[n] \mapsto |C_{n, \bullet}^\sigma|$.

Definition 8.3.26. *We define a continuous map*

$$\mathcal{S} : \|C_{\bullet, \bullet}^\sigma\| \times [0, 1] \longrightarrow |C_{\bullet, \bullet}^{\text{nc}}|.$$

A point on the left is represented by a tuple $((W, \mathbf{t}), ((A_0, \dots, A_m), (s_0, \dots, s_m)), r)$ where the $s_i \in [0, 1]$ are such that $\sum_i s_i = 1$. We normalise them as $\bar{s}_i := s_i / \max(s_0, \dots, s_m)$ and define:

$$\mathcal{S}^r((W, \mathbf{t}), (A_0, \dots, A_m), (s_0, \dots, s_m)) := (\mathcal{S}_{A_0}^{r \cdot \bar{s}_0} \circ \dots \circ \mathcal{S}_{A_m}^{r \cdot \bar{s}_m})(W, \mathbf{t}).$$

Proof of theorem 8.3.1. We begin by noting that the map \mathcal{S} defined above is indeed well-defined. This was mostly covered in the previous section (see proposition 8.3.23, lemma 8.3.16, and lemma 8.3.21), but we also need to check that it is well-defined with respect to face maps in the new semi-simplicial direction. Concretely this means that for $s_i = 0$ we need to be able to forget A_i without changing the value of \mathcal{S} . Indeed this is the case as \mathcal{S}^0 acts as the identity by part (i) of proposition 8.3.23.

Now that we know that \mathcal{S}^r is a continuous family of continuous maps $\|C_{\bullet, \bullet}^\sigma\| \rightarrow |C_{\bullet, \bullet}^{\text{nc}}|$ we can fit it into the following diagram:

$$\begin{array}{ccc} |C_{\bullet, \bullet}^{\partial+}| & \xrightarrow{I} & \|C_{\bullet, \bullet}^\sigma\| \\ \parallel & \swarrow \mathcal{S}^1 & \downarrow \mathcal{S}^0 \\ |C_{\bullet, \bullet}^{\partial+}| & \xrightarrow{\quad} & |C_{\bullet, \bullet}^{\text{nc}}| \end{array}$$

The map I is defined by equipping (W, t) with the empty surgery $A_0 = \{\emptyset\}$. This is possible because (W, t) is already positive boundary in this case. By part (ii) of 8.3.23 \mathcal{S}^1 indeed lands in the subspace $|C_{\bullet, \bullet}^{\partial+}| \subset |C_{\bullet, \bullet}^{\text{nc}}|$, so the dashed map exists. For the empty surgery the map \mathcal{S}^1 does not do anything by part (iii) of 8.3.23 and so the top-left triangle commutes. By construction \mathcal{S}^λ is a homotopy for the bottom-right triangle in the diagram. Moreover, part (i) of 8.3.23 tells us that the map \mathcal{S}^0 simply forgets the surgery data and we saw in lemma 8.3.25 that this map is an equivalence.

In summary the above diagram is homotopy commutative and the right-hand vertical map is a homotopy equivalence. From this it follows formally from the 2-out-of-6 property that all other maps in the diagram are weak equivalences. We can check this by hand: Pick any base-points and apply π_k to the diagram. Considering the top-left triangle we see that the diagonal map $\pi_k(\mathcal{S}^1)$ has to be surjective because the identity on $\pi_k|C_{\bullet, \bullet}^{\partial+}|$ factors through it. Similarly, the homotopy commutativity of the bottom-right triangle and the fact that $\pi_k(\mathcal{S}^0)$ is an isomorphism implies that $\pi_k(\mathcal{S}^1)$ is injective. Hence we have shown that $\pi_k(\mathcal{S}^1)$ is an isomorphism and it follows quickly that all the other maps are, too. \square

8.3.4 The positive boundary subcategory

In this section we compute the classifying space of the positive boundary subcategory $\mathcal{C}^{\partial+} \subset \mathcal{C}$ for all weighted cospan categories. This includes the three cases Csp , Cob_2 , and $\text{Cob}_2^{\chi \leq 0}$ that appear in our main theorems.

Proposition 8.3.27. *For any weighting monoid (A, A_1, α) the classifying space of the positive boundary subcategory $\text{Csp}(A, A_1, \alpha)^{\partial+} \subset \text{Csp}(A, A_1, \alpha)$ is*

$$B(\text{Csp}(A, A_1, \alpha)^{\partial+}) \simeq BA$$

and the inclusion $BA \simeq B(\text{Csp}(A, A_1, \alpha)^{\partial+}) \rightarrow B(\text{Csp}(A, A_1, \alpha))$ admits a splitting as an infinite loop space map.

Proof. In the case of Cob_2 the equivalence $B\text{Cob}_2^{\partial+} \simeq S^1$ was shown in [Til96, Proposition 6] and the splitting constructed in [Til96, Theorem 10]. We begin by recalling the constructions made there so we can modify them for our purposes.

There is a functor $\Phi : \text{Cob}_2^{\partial+} \rightarrow \text{Cob}_2^{\partial+}$ that sends any object to S^1 and that sends a bordism $W : M \rightarrow N$ to the unique bordism $\Phi(W) : S^1 \rightarrow S^1$ that is connected and has genus:

$$g(\Phi(W)) = \frac{1}{2}(|\pi_0(N)| - |\pi_0(M)| + \chi(W)) = g(W) + |\pi_0(N)| - |\pi_0(W)|.$$

This is always a non-negative integer as $\pi_0(N) \rightarrow \pi_0(W)$ is always a surjection for W in the positive boundary subcategory. Using that the Euler characteristic is additive under glueing of even-dimensional manifolds one checks that this is functorial. There is a natural transformation $\rho : \text{Id}_{\text{Cob}_2^{\partial+}} \Rightarrow \Phi$ defined by letting $\rho_M : M \rightarrow S^1$ be the unique connected genus 0 bordism.

Now we can consider the full subcategory $\mathcal{C}_1 \subset \text{Cob}_2^{\partial+}$ on the object S^1 . By recording the genus as a natural number \mathcal{C}_1 is isomorphic to \mathbb{N} as a category with one object and hence $B\mathcal{C}_1 \simeq S^1$. The functor Φ lands in \mathcal{C}_1 and defines a retraction $B\Phi : B\text{Cob}_2^{\partial+} \rightarrow B\mathcal{C}_1$ to the inclusion. The natural transformation ρ gives a homotopy between the identity and $B\Phi : B\text{Cob}_2^{\partial+} \rightarrow B\mathcal{C}_1 \subset B\text{Cob}_2^{\partial+}$ and hence the inclusion and $B\Phi$ form a homotopy equivalence.

We will now generalise this argument to the weighted cospan category $\mathcal{C} := \text{Csp}(A, A_1, \alpha)$. Let $\mathcal{C}_1 \subset \mathcal{C}^{\partial+}$ denote the full subcategory on the object $*$. This category has a single object and morphisms A , so $B\mathcal{C}_1 = BA$. The functor $\Phi : \mathcal{C}^{\partial+} \rightarrow \mathcal{C}_1$ again sends all objects to the single object $*$, and on morphisms it is defined by sending $(M \rightarrow W \leftarrow N, a : W \rightarrow A)$ to the cospan $(* \rightarrow * \leftarrow *)$ weighted by:

$$\Phi(W) = \sum_{x \in W} a(W) + \alpha \cdot (|N| - |W|).$$

The number $|N| - |W|$ is again non-negative as $N \rightarrow W$ is surjective in the positive boundary category. One checks that this is functorial by using the definition of the composite weighting in definition 8.1.13.

To define the natural transformation $\rho : \text{Id} \Rightarrow \Phi$ we let $\rho_M : (M \rightarrow * \leftarrow *, a : * \rightarrow A)$ where the labeling is $a(*) = \alpha$. (We cannot set the labelling to be 0 like we did for the surface category case, as this might not be an allowed cospan for $M = \emptyset$.) Naturality is checked by inserting the definitions. By the same argument as above it now follows that $BA = BC_1 \rightarrow BC^{\partial+}$ is a homotopy equivalence.

To construct the splitting of $BC^{\partial+} \rightarrow BC$ let A^{gp} be the group completion of A . Define a functor $\Phi' : \mathcal{C} \rightarrow A^{\text{gp}}$ by sending a morphism $W : M \rightarrow N$ to

$$\Phi'(W) = \sum_{x \in W} a(W) + \alpha \cdot (|N| - |W|).$$

The number $|N| - |W|$ can now be negative, but this is fine because we have a formal inverse $-\alpha$ for α in A^{gp} . Φ' clearly extends Φ and it is functorial for the same reason that Φ was. This yields the desired splitting: the composite $BA = BC_1 \rightarrow BC^{\partial+} \rightarrow BC \xrightarrow{\Phi'} B(A^{\text{gp}})$ is the standard map $BA \rightarrow B(A^{\text{gp}})$, which is an equivalence. \square

Corollary 8.3.28. *The classifying space of a weighted cospan category is:*

$$B(\text{Csp}(A, A_1, \alpha)) \simeq BA \times Q \left(\bigvee_{a \in A} \Sigma SB(\mathcal{F}_a(\text{Csp}(A, A_1, \alpha))) \right).$$

Proof. The surgery theorem applies and by proposition 8.3.27 the left-hand map in theorem N (corollary 8.1.45) admits a splitting as infinite loop space map. Therefore the middle term decomposes as a product of $B(\text{Csp}(A, A_1, \alpha)^{\partial+})$ and the free infinite loop spaces on the $\Sigma S(B\mathcal{F}_a)$. \square

Chapter 9

Computations for the $(\chi \leq 0)$ surface category

Our work in the previous chapter shows that computing the classifying space $BCob_2^{\chi \leq 0}$ of the $(\chi \leq 0)$ -surface category essentially reduces to understanding its factorisation categories, which we will denote by $\mathcal{F}_g^{\chi \leq 0} := \mathcal{F}_{\Sigma_g}(\text{Cob}_2^{\chi \leq 0})$. In section 9.1 we show that on classifying spaces $\mathcal{F}_g^{\chi \leq 0}$ is equivalent to a certain finite category J_g of weighted graphs of genus g , which was introduced in [CGP16]¹ and plays an important role in studying the moduli spaces Δ_g of tropical curves. In section 9.2 we will show that BJ_g is rationally equivalent to Δ_g .

9.1 Factorisation categories and filtered graphs

In this section we prove:

Theorem 9.1.1. *For $g \geq 2$ there is an equivalence $B\mathcal{F}_g^{\chi \leq 0} \simeq BJ_g$ and for $g = 1$ we have $B\mathcal{F}_1^{\chi \leq 0} \simeq BO(2)$.*

In the case $g = 1$ we expect that $\mathcal{F}_1^{\chi \leq 0}$ is equivalent to $\mathcal{F}_{[S^1]}(\text{Cob}_1^{\text{unor}})$. Just like we showed that $\mathcal{F}_{[S^1]}(\text{Cob}_1) \simeq \Lambda$ is Connes' cyclic category Λ in section 6.3, it should be possible to show that $\mathcal{F}_{[S^1]}(\text{Cob}_1^{\text{unor}})$ is the dihedral version of Connes' cyclic category. This dihedral category is known to have classifying space $BO(2)$, see [Lod87, Proposition 3.11].

However, the approach we take to prove the theorem will be to treat the $g = 1$ and $g \geq 2$ computation of $B\mathcal{F}_g^{\chi \leq 0}$ in the same framework by considering a category of topological graphs \mathcal{J}_g such that $\mathcal{J}_g \simeq J_g$ for $g \geq 2$ and $\mathcal{J}_1 \simeq O(2)$. In fact we prove the following more general theorem, which describes the factorisation category of the weighted cospan category $\text{Csp}(A, A_1, \alpha)$ in the sense of 8.1.13. This recovers theorem 9.1.1 because $\text{Cob}_2^{\chi \leq 0} \simeq \text{Csp}(\mathbb{N}, \mathbb{N}_{\geq 1}, 1)$.

¹Note that as discussed in remark 9.1.11 our version J_g differs from their version in that we remove the terminal object \bullet_g .

Theorem 9.1.2. *For any weighting monoid (A, A_1, α) and $a \in A$ there is an equivalence:*

$$B\mathcal{F}_a(\text{Csp}(A, A_1, \alpha)) \simeq B\mathcal{J}_a^{(A, A_1, \alpha)}.$$

Note that this will only be interesting for $0 \notin A_1$ as otherwise an argument similar to lemma 8.2.48 shows that $B\mathcal{F}_a(\text{Csp}(A, A, \alpha)) \simeq *$.

9.1.1 Categories of graphs

9.1.1.1 Topological graphs

We begin by defining a category of weighted topological graphs. Throughout this section we fix a weighting monoid (A, A_1, α) . Recall that this means that A is an abelian monoid, $A_1 \subset A$ is a subset closed under $A_1 + A \subset A_1$, and $\alpha \in A$ is an element. The most relevant case is that of $(A, A_1, \alpha) = (\mathbb{N}, \mathbb{N}_{\geq 1}, 1)$ and will suffice to think of this case.

Definition 9.1.3. *An (A, A_1, α) -weighted topological graph is a tuple (X, w) of a topological space X and a (non-continuous) map $w : X \rightarrow A$ such that*

- *X is homeomorphic to a connected 1-dimensional CW complex with finitely many cells and is not a point.*
- *The set $X_w := \{x \in X \mid w(x) \neq 0\}$ is finite.*
- *For any $x \in X$ with $\nu(x) = 1$ we have $w(x) \in A_1$.*

Here we define the valence $\nu(x) \in \mathbb{N}$ to be $\nu(x) := |\pi_0(U \setminus x)|$ where $U \subset X$ is a small contractible neighbourhood of x .

The singular set $X_{\text{sing}} \subset X$ is defined as the set of points of valence $\nu(x) \neq 2$.

The genus of (X, w) is defined as:

$$g(X, w) := \sum_{x \in X_w} w(x) + b_1(X) \cdot \alpha \in A.$$

A subgraph of (X, w) is a tuple (Y, w') such that $Y \subset X$ is a closed subset with $\partial Y \subset X_w \cup X_{\text{sing}}$ and $w' = w|_Y$.

Definition 9.1.4. *The topologically enriched category $\mathcal{J}^{(A, A_1, \alpha)}$ has as objects (A, A_1, α) -weighted topological graphs and as morphisms $(X, w) \rightarrow (Y, w')$ continuous maps $f : X \rightarrow Y$ such that for all $y \in Y$ the preimage $f^{-1}(y)$ is either a single point of the same weight as y or a connected subgraph of (X, w) with genus $g(f^{-1}(y), w|_{f^{-1}(y)}) = w'(y)$. We topologise the space of morphisms with the compact-open topology.*

Example 9.1.5. *The only (A, A_1, α) -weighted topological graph (X, w) with $X_w = \emptyset = X_{\text{sing}}$ is the circle $(S^1, 0)$. Since this cannot have any subgraphs the fibers of morphisms $f : (S^1, 0) \rightarrow (S^1, 0)$ have to be single points. It follows that f is in fact a homeomorphism. So $\text{Hom}_{\mathcal{J}}((S^1, 0), (S^1, 0)) = \text{Homeo}(S^1) \simeq O(2)$. We will see that this case is the exception and that in general the spaces $\text{Hom}_{\mathcal{J}}((X, w), (Y, w'))$ have contractible components.*

Lemma 9.1.6. *For any two topological graphs (X, w) and (Y, w') with $(Y, w') \not\cong (S^1, 0)$ the space of morphisms $\text{Hom}_{\mathcal{J}}((X, w), (Y, w'))$ has contractible components. If $(Y, w') \cong (S^1, 0)$, then the components are equivalent to S^1 .*

Proof. Since each map $f : (X, w) \rightarrow (Y, w')$ is a surjective map between compact Hausdorff spaces, it is also a quotient map. We can therefore factor it as $X \rightarrow (X/\sim_f) \cong Y$ where \sim_f is the equivalence relation defined by $(x \sim_f x') \Leftrightarrow (f(x) = f(x'))$ and the second map is a homeomorphism. Let us call an equivalence relation $\sim \subset X^2$ admissible if each of its equivalence classes is either a single point or a subgraph of (X, w) , and let $\text{Eq}(X, w)$ denote the set of admissible equivalence relations. For such an equivalence relation, let $w_{\sim} : X/\sim \rightarrow A$ be the function that sends and equivalence class $[x]$ to the genus $g([x])$ of the subgraph.

We will show that sending a map f to the equivalence relation \sim_f and the isomorphism $[f] : (X/\sim, w_{\sim}) \cong (Y, w')$ induces a homeomorphism:

$$\text{Hom}_{\mathcal{J}}((X, w), (Y, w')) \cong \coprod_{\sim \in \text{Eq}(X, w)} \text{Hom}_{\mathcal{J}}^{\cong}((X/\sim, w_{\sim}), (Y, w')).$$

We begin by noting that the set of admissible equivalence relations is finite because the set of subgraphs of (X, w) is finite. One can check, using the definition of the compact-open topology, that the assignment $f \mapsto \sim_f$ is continuous with respect to the discrete topology on $\text{Eq}(X, w)$. Similarly, one checks that for each $\sim \in \text{Eq}(X, w)$ the assignment $f \mapsto [f]$ is continuous. Since there is a continuous inverse given by $(\sim, [f]) \mapsto (X \rightarrow X/\sim \rightarrow Y)$ it follows that the map is indeed a homeomorphism.

To determine the homotopy type of the connected components it therefore suffices to study the group of automorphisms $\text{Hom}_{\mathcal{J}}^{\cong}((Y, w'), (Y, w'))$. This is the subgroup of the homeomorphism group $\text{Homeo}(Y)$ containing precisely those homeomorphism $f : Y \cong Y$ with $w' \circ f = w'$. If $(Y, w') = (S^1, 0)$, then this is exactly the group of self-homeomorphism of the circle, which is homotopy equivalent to $O(2)$. In the case where $(Y, w') \not\cong (S^1, 0)$, we can find some $y \in Y$ with either $w(y) \neq 0$ or $\nu(y) \neq 2$. Consider the group $\text{Homeo}(Y, Y_w \cup Y_{\text{sing}})$ of homeomorphisms of Y that send the subset $Y_w \cup Y_{\text{sing}}$ to itself, possibly permuting it. This contains the automorphism group we are interested in as a union of connected components. Since $Y_w \cup Y_{\text{sing}}$ is non-empty, the components of $\text{Homeo}(Y, Y_w \cup Y_{\text{sing}})$ are all homeomorphic to $\prod_i \text{Homeo}([0, 1] \text{ rel } \{0, 1\})$, which is contractible. \square

9.1.1.2 Combinatorial graphs

We now give a combinatorial version of the notion of an (A, A_1, α) -weighted graph. This is based on the notation of [CGP16, section 2.2] and recovers their definitions in the case of $(\mathbb{N}, \mathbb{N}_1, 1)$.

Definition 9.1.7. A graph G is a finite set X together with maps $s : X \rightarrow X$ and $r : X \rightarrow X$ such that $s \circ s = \text{id}_X$, $r \circ r = r$, and satisfying that for all $x \in X$ we have $s(x) = x$ if and only if $r(x) = x$. The set X decomposes as the disjoint union of the set of vertices $V_G := \{x \in X \mid r(x) = x\}$ and the set of half-edges $H_G = \{x \in X \mid r(x) \neq x\}$. The map r sends a half-edge to its root and the map s sends a half-edge to its partner half-edge; both send vertices to themselves. We define the set of edges as $E_G = \{\{h, s(h)\} \mid h \in H_G\}$.

A graph G is connected if the equivalence relation on X generated by $x \sim s(x)$ and $x \sim r(x)$ has a single equivalence class. We define the valence of a vertex v in a graph to be the number of incident half-edges: $\text{val}(v) := |r^{-1}(v) \setminus \{v\}|$.

Example 9.1.8. The graph with one vertex and one loop \circlearrowleft can be defined in the above context as $X = \{v, h, h'\}$ with $s(h) = h'$, $s(h') = h$, $r(h) = r(h') = v$, $s(v) = r(v) = v$. This has one vertex v , two half-edges h and h' , and a single edge $\{h, h'\}$.

We can also consider combinatorial graphs weighted by some (A, A_1, α) . The most important weighting monoid is $(\mathbb{N}, \mathbb{N}_{\geq 1}, 1)$, and we will usually think about this case.

Definition 9.1.9. An (A, A_1, α) -weighted graph is a graph G together with a function $w : V_G \rightarrow A$ such that for any vertex $v \in V_G$ of valence 1 we have $w(v) \in A_1$ and for any vertex $v' \in V_G$ of valence 2 we have $w(v') \neq 0$. We define the genus of a connected weighted graph (G, w) to be

$$g(G, w) = (|E_G| - |V_G| + 1) \cdot \alpha + \sum_{v \in V_G} w(v) \in A.$$

We define the relevant notion of morphisms between weighted graphs:

Definition 9.1.10. Let $(G, w) = ((X, s, r), w)$ and $(G', w') = ((X', s', r'), w')$ be two (A, A_1, α) -weighted graphs. A morphism of (A, A_1, α) -weighted graphs $q : (G, w) \rightarrow (G', w')$ is a map $q : X \rightarrow X'$ satisfying $q \circ s = s' \circ q$ and $q \circ r = r' \circ q$, such that for each half-edge $h \in H_{G'}$ the preimage $q^{-1}(h)$ consists of a single half-edge of G and for each vertex $v \in V_{G'}$ the preimage $q^{-1}(v)$ is a connected subgraph of G with genus $g(q^{-1}(v), w|_{q^{-1}(v)}) = w'(v)$.

We let $J^{(A, A_1, \alpha)}$ denote the category of those (A, A_1, α) -weighted graphs that are connected and have at least one edge, with morphisms defined as above. For any $g \in A$ and we let $J_g^{(A, A_1, \alpha)} \subset J^{(A, A_1, \alpha)}$ denote the full subcategory on the graphs of genus g . We will often suppress the weighting monoid from the notation.

REMARK 9.1.11. In [CGP16, section 2.2] the authors define a category called J_g , on which our definition of the category $J_g^{(A, A_1, \alpha)}$ is based. Let us denote their category by J_g^{CGP} for the purpose of comparing it to ours. One key difference is that J_g^{CGP} contains the graph \bullet_g with one vertex and no edges. This is a terminal object, which makes the classifying space $BJ_g^{\text{CGP}} \simeq *$ contractible. If we remove this terminal object then $J_g^{\text{CGP}} \setminus \{\bullet_g\}$ is equivalent to our $J_g^{(\mathbb{N}, \mathbb{N}_{>0}, 1)}$. We will argue in remark 9.2.3 that our version of J_g still leads to the same tropical moduli space Δ_g as theirs.

REMARK 9.1.12. The isomorphisms in this category $\mathcal{J}^{(A, A_1, \alpha)}$ are exactly those $q : (G, w) \rightarrow (G', w')$ where the map $q : X \rightarrow X'$ is a bijection. Moreover, if two graphs have different genus, then there are no morphisms between them and so we can split $J^{(A, A_1, \alpha)}$ as a disjoint union of categories

$$J^{(A, A_1, \alpha)} = \coprod_{g \in A} J_g^{(A, A_1, \alpha)}.$$

REMARK 9.1.13. Note that for general weighting monoids (A, A_1, α) the category $J^{(A, A_1, \alpha)}$ is not as well-behaved as we would like. For example in the case of $(\mathbb{Z}, \mathbb{Z}, 1)$ -weighted graphs we cannot collapse an edge between two valence two vertices $v, v' \in V_G$ if they are labelled by $w(v) = k$ and $w(v') = -k$ as the resulting vertex would be a valence 2 vertex with label 0. Similarly, we cannot collapse an edge between two vertices $u, u' \in V_G$ where u has valence 1, u' has valence 3 and $w(u) = 0 = w(u')$. The latter example also works in the case of $(0, 0, 0)$ -weighted graphs, that is relevant for $\text{Csp} = \text{Csp}(0, 0, 0)$. This is why we have to impose the conditions in the following lemma.

Lemma 9.1.14. Let (A, A_1, α) be a weighting monoid where the only invertible element of A is 0 and $0 \notin A_1$. Let $g \in A$ be some element that is neither 0 nor α . Then there is a functor of topologically enriched categories: $\mathcal{J}_g^{(A, A_1, \alpha)} \rightarrow J_g^{(A, A_1, \alpha)}$ which is essentially surjective and an equivalence on Hom-spaces.

Proof. For a topological graph (X, w) we define its set of vertices as $V_{(X, w)} := X_w \cup X_{\text{sing}}$ and, by abuse of notation, let $w : V_{(X, w)} \rightarrow A$ denote the restriction of w . Unless the topological graph is $(X, w) \cong (S^1, 0)$ this set of vertices is non-empty. Note that the case $(S^1, 0)$ will not come up in what follows as it has total genus $g(S^1, 0) = \alpha$, which we excluded. The set of half-edges $H_{(X, w)}$ is defined as the set of maps $h : [0, 1] \rightarrow X$ such that $h|_{(0, 1)}$ is an embedding and $h^{-1}(V_{(X, w)}) = \{0, 1\}$, up to isotopies respecting these conditions. The involution $s : H_{(X, w)} \rightarrow H_{(X, w)}$ is given by $[h] \mapsto [h']$ where $h'(t) = h(1 - t)$ and the root map $r : H_{(X, w)} \rightarrow V_{(X, w)}$ is $[h] \mapsto h(0)$.

We define the functor $F : \mathcal{J}_g^{(A, A_1, \alpha)} \rightarrow J_g^{(A, A_1, \alpha)}$ on objects by sending (X, w) to the combinatorial graph $(H_{(X, w)} \amalg V_{(X, w)}, s, r, w)$. To a morphism $f : (X, w) \rightarrow (Y, w')$ we assign the map

$$F(f) : H_{(X,w)} \amalg V_{(X,w)} \rightarrow H_{(Y,w')} \amalg V_{(Y,w')}$$

$$F(f)(v \in V_{(X,w)}) = f(v) \in V_{(Y,w')}, \quad \text{and}$$

$$F(f)([h : [0, 1] \rightarrow X]) = \begin{cases} [f \circ h] \in H_{(Y,w')} & \text{if } (f \circ h)|_{(0,1)} \text{ injective} \\ f(h(0)) \in V_{(Y,w')} & \text{if } f \circ h \text{ constant.} \end{cases}$$

For this to be well-defined we need to check for each $x \in V_{(X,w)} = X_w \cup X_{\text{sing}}$ the image $f(x)$ lies in $V_{(Y,w)} = Y_w \cup Y_{\text{sing}}$.

Write $y := f(x)$ and consider the preimage $f^{-1}(y) \subset X$. If it consists of single point then f is a homeomorphism near x and so x and y agree in valence and weight, which already implies $y \in Y_w \cup Y_{\text{sing}}$. We may therefore assume that f^{-1} is a subgraph of (X, w) that contains at least one edge. If there is any $x' \in f^{-1}(y)$ with $w(x') \neq 0$, then $w'(y) = w(x') + \dots$ is non-zero because we assumed that A does not have any invertible elements but 0. Similarly, if $f^{-1}(y)$ is not contractible, then $b_1(f^{-1}(y)) > 0$ and $w'(y) = b_1(f^{-1}(y)) \cdot \alpha + \dots$ is non-zero because α is not invertible. So it suffices to consider the case where the subgraph $f^{-1}(y) \subset (X, w)$ has trivial weight and is contractible. This means that the subgraph is a finite tree with at least one edge, so it contains at least two leaves $l, l' \in \partial f^{-1}(y)$ whose valence within $f^{-1}(y)$ is 1. But we have already established that $w(l) = 0 = w(l')$, so l and l' cannot be valence 1 or 2 in (X, w) as we assumed $0 \notin A_1$. Hence there are at least four edges of (X, w) that do not lie in $f^{-1}(y)$, but end in l and l' . This implies that $y \in (Y, w')$ has valence at least 4 and hence lies in Y_{sing} .

To see that the functor F is essentially surjective we build, given a combinatorial graph $(G = (X, s, r), w)$, its geometric realisation as:

$$|G| := (X \times [0, 1]) / \sim, \quad \text{where} \quad (x, 1-t) \sim (s(x), t) \quad \text{and} \quad (x, 0) \sim (r(x), t).$$

The set of vertices includes into this as $v \mapsto [v, 0] = [v, t]$ for any t , and we can define $w : |G| \rightarrow A$ by extending it to 0 on edges. The space $|G|$ is a one-dimensional CW complex by construction and it is finite and connected because G is. Moreover, since vertices retain their valence and weight, all valence 1 vertices are still labelled in A_1 and hence $(|G|, w)$ is an (A, A_1, α) -weighted topological graph. In fact, this construction defines a functor:

$$|_|_ : \mathcal{J}_g^{(A, A_1, \alpha)} \longrightarrow \mathcal{J}_g^{(A, A_1, \alpha)}$$

and it follows by inspection that $F \circ |_|_$ is naturally isomorphic to the identity functor.

While the composite functor $|_|_ \circ F$ is not on-the-nose isomorphic to the identity functor, it is still isotopic to it in some sense. Concretely, we can pick isomorphisms $\alpha_{(X,w)} : (X, w) \cong (|F(X, w)|, w)$. If we require them to be the identity on vertices and to preserve edges, then the only choice we have is how to identify the edges relative to their boundary. There is no canonical choice because the edges of (X, w) are not identified with $[0, 1]$, whereas the edges of $|F(X, w)|$

are (modulo reflection). However, any two choices are isotopic because $\text{Homeo}([0, 1] \text{ rel } \{0, 1\})$ is contractible. From this one can show that, no matter which choice we make, α defines a natural transformation from the identity to $|_|\circ F$ as functors to the homotopy category $h\mathcal{J}_g^{(A, A_1, \alpha)}$. Therefore, $\mathcal{J}_g^{(A, A_1, \alpha)}$ is equivalent to the homotopy category $h\mathcal{J}_g^{(A, A_1, \alpha)}$.

To conclude the proof we show that $\mathcal{J}_g^{(A, A_1, \alpha)} \rightarrow h\mathcal{J}_g^{(A, A_1, \alpha)}$ is an equivalence on Hom-spaces. As noted before, our assumption that $g \neq \alpha$ implies that none of the graphs involved are isomorphic to $(S^1, 0)$. Hence it follows from lemma 9.1.6 that the components of the Hom-spaces in $\mathcal{J}_g^{(A, A_1, \alpha)}$ are contractible, which is what we claimed. \square

9.1.2 From factorisations to graphs

The purpose of this section is to prove theorem 9.1.2, i.e. to show that there is an equivalence on classifying spaces between the category of factorisation $\mathcal{F}(\text{Csp}(A, A_1, \alpha))$ and the category of graphs $\mathcal{J}^{(A, A_1, \alpha)}$. We fix the weighting monoid (A, A_1, α) throughout and suppress it in notation. The equivalence we will construct does not come from a functor in either direction. Rather, we define a third category $\int_{\mathcal{J}} \text{Sub}'$ where objects are topological graphs (X, w) together with an “admissible” subset $U \subset X$ and an ordering on ∂U . There are two functors:

$$\mathcal{J} \longleftarrow \int_{\mathcal{J}} \text{Sub}' \xrightarrow{S} \mathcal{F}$$

The one to the left simply forgets the subset U , and the one to the right takes (X, w, U) and thinks of it as a factorisation $X = U \cup_{\partial U} (X \setminus U^\circ)$. We will show that both functors induce equivalences on classifying spaces.

Definition 9.1.15. *A subset $U \subset X$ of a weighted topological graph (X, w) is called admissible if*

- U is closed and has finitely many components,
- U is neither empty nor all of X ,
- if $x \in U \cap (X_w \cup X_{\text{sing}})$ then U is a neighbourhood of x .

A boundary ordering for U is an injection $\alpha : \{1, \dots, u\} \rightarrow X$ for some $u \in \mathbb{N}$ such that $\alpha(\{1, \dots, u\}) = \partial U$.

We let $\text{Sub}(X, w)$ be the poset where objects are admissible subset $U \subset X$ and morphisms are $U \leq V$ if $U \subset V$. Similarly, we let $\text{Sub}'(X, w)$ be the poset where objects are tuples (U, α) of an admissible subset with boundary ordering α and morphisms are $(U, \alpha) \leq (V, \beta)$ if $U \subset V$.

We topologise $\text{Sub}'(X, w)$ with the subspace topology under the inclusion

$$\text{Sub}'(X, w) \hookrightarrow \coprod_{u \geq 2} \text{Map}^{\text{inj}}(\{1, \dots, u\}, X \setminus (X_w \cup X_{\text{sing}})) \times \text{Map}(X_w \cup X_{\text{sing}}, \{\perp, \top\})$$

that sends (U, α) to (α, χ) where $(\chi(x) = \top) \Leftrightarrow (x \in U)$. This inclusion in fact identifies $\text{Sub}'(X, w)$ with a union of certain connected components of the space on the right. We topologise $\text{Sub}(X, w)$ with the quotient topology under the map $\text{Sub}'(X, w) \rightarrow \text{Sub}(X, w)$ that forgets the boundary ordering.

Both Sub and Sub' define functors $\text{Sub}, \text{Sub}' : \mathcal{J}^{\text{op}} \rightarrow \text{TopCat}$

$$\text{Sub}(f)(U) := f^{-1}(U) \quad \text{and} \quad \text{Sub}'(f)(U, \alpha) := (f^{-1}(U), f^{-1} \circ \alpha)$$

to the category of topological categories.

REMARK 9.1.16. Let us briefly recall the Grothendieck construction, in the example of Sub' . An object of $\int_{\mathcal{J}} \text{Sub}'$ is a tuple of an object $(X, w) \in \mathcal{J}$ and an object $(U, \alpha) \in \text{Sub}'(X, w)$. It is important to keep in mind that the space of objects of $\int_{\mathcal{J}} \text{Sub}'$ is no longer discrete, even though it was for \mathcal{J} . A morphism $((X, w), (U, \alpha)) \rightarrow ((Y, w'), (V, \beta))$ is a morphism $f : (X, w) \rightarrow (Y, w')$ together with a morphism $(U, \alpha) \rightarrow \text{Sub}'(f)(V, \beta)$. Since Sub' is a poset we can simply say that a morphism $((X, w), (U, \alpha)) \rightarrow ((Y, w'), (V, \beta))$ is a morphism $f : (X, w) \rightarrow (Y, w')$ such that $U \subset f^{-1}(V)$, or equivalently such that $f(U) \subset V$. See the left-hand side of figure 9.1 for an example of such a morphism.

Definition 9.1.17. We define a functor $S : \int_{\mathcal{J}} \text{Sub}' \rightarrow \mathcal{F}$ by sending (X, w, U, α) to the factorisation

$$S((X, w), (U, \alpha)) := \left(\emptyset \rightarrow [\pi_0(U), w_U] \xleftarrow{\alpha} \{1, \dots, u\} \xrightarrow{\alpha} [\pi_0(X \setminus U^\circ), w_{X \setminus U}] \leftarrow \emptyset \right)$$

Here the weights $w_U : \pi_0(U) \rightarrow A$ and $w_{X \setminus U} : \pi_0(X \setminus U^\circ) \rightarrow A$ are defined by sending a component to its total genus.

To a morphism $f : (X, w) \rightarrow (Y, w')$ with $f(U) \subset V$ we assign the cospan

$$S(f) = \left(\{1, \dots, u\} \xrightarrow{\alpha} [\pi_0(f^{-1}(V) \setminus U^\circ), w_{V, U}] \xleftarrow{f^{-1} \circ \beta} \{1, \dots, v\} \right)$$

where $w_{V, U}$ sends each component of $V \setminus f(U)^\circ$ to its total genus. See figure 9.1 for an illustration of how this functor is applied to two objects and a morphism.

Lemma 9.1.18. The construction described above indeed describes a continuous functor $S : \int_{\mathcal{J}} \text{Sub}' \rightarrow \mathcal{F}$.

Proof. We begin by noting that S is continuous on objects and morphisms. Indeed, the object $S((X, w), (U, \alpha)) \in \mathcal{F}$ does not change as we continuously vary the admissible subset $U \subset X$. (This is the reason why we need the boundary ordering α and cannot take ∂U instead of $\{1, \dots, u\}$: it would not be continuous.) Similarly, if we continuously vary a morphism $f : (X, w) \rightarrow (Y, w')$

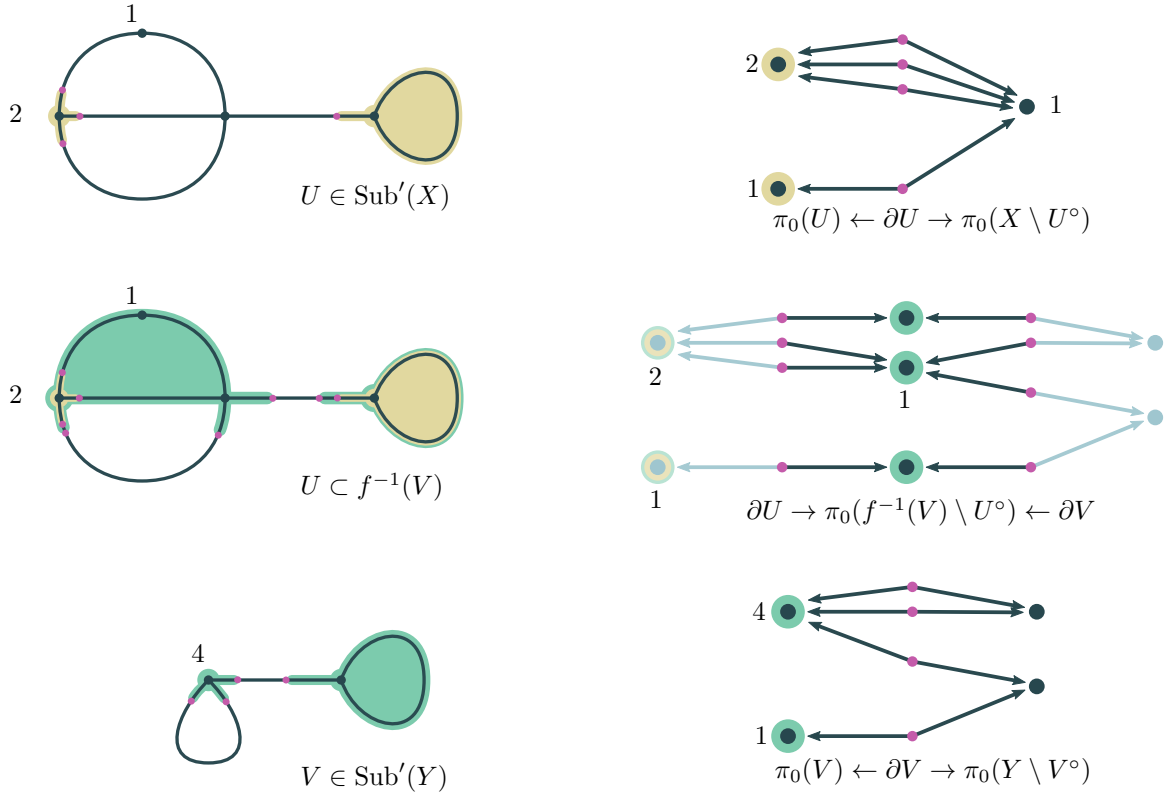


Figure 9.1: The functor S evaluated on a morphism $f : (X, U) \rightarrow (Y, V)$. The identifications $\partial U \cong \{1, 2, 3, 4\}$ and $\partial V \cong \{1, 2, 3, 4\}$ are left implicit.

and an admissible subset $(U, \alpha) \in \text{Sub}'(Y, u)$, the space $V \setminus f^{-1}(U^\circ)$ does not change up to label-preserving homotopy equivalence and hence the resulting cospan $S(f)$ remains the same. (If we moreover required that V is a neighbourhood of $f(U)$, then $V \setminus f(U)^\circ$ would even remain the same up to homeomorphism. We do not do this as it would remove the identity morphisms from $\text{Sub}'(X, w)$.)

It remains to check that S is functorial. Consider two morphisms $f : (X, w) \rightarrow (Y, w')$, $h : (Y, w') \rightarrow (Z, w'')$ and admissible opens $(U, \alpha) \in \text{Sub}'(X, w)$, $(V, \beta) \in \text{Sub}'(Y, w')$, and $(W, \gamma) \in \text{Sub}'(Z, w'')$ such that $U \subset f^{-1}(V)$ and $V \subset h^{-1}(W)$. We would like to show that the weighted cospan $S(h \circ f)$ agrees with the composite $S(f) \cup_{\{1, \dots, v\}} S(h)$. On underlying sets one can describe this pushout as:

$$\begin{aligned} \pi_0(f^{-1}(V) \setminus U^\circ) \cup_{\{1, \dots, v\}} \pi_0(h^{-1}(W) \setminus V^\circ) &\cong \pi_0(f^{-1}(V) \setminus U^\circ) \cup_{\partial V} \pi_0((h \circ f)^{-1}(W) \setminus h^{-1}(V^\circ)) \\ &\cong \pi_0((h \circ f)^{-1}(W) \setminus U^\circ) \end{aligned}$$

To conclude $S(f) \cup_{\{1, \dots, v\}} S(h) = S(h \circ f)$ we need to check that both sides are weighted in A in the same way. To show this let $C \subset (h \circ f)^{-1}(W) \setminus U^\circ$ some component. As a point in $S(h \circ f)$ this is labelled by its total genus. We have a decomposition of C into $C_0 = C \cap (f^{-1}(V) \setminus U^\circ)$

and $C_1 = C \cap ((h \circ f)^{-1}(W) \setminus f^{-1}(V)^\circ)$. The total genus of each component of C_1 agrees with that of the relevant component in $f(C_1)$, which is a union of components of $h^{-1}(W) \setminus V^\circ$. One can express the total genus of C in terms of the genus of the components of C_0 and C_1 as follows:

$$g(C) = \sum_{x \in C} w(x) + \alpha \cdot b_1(C) = \sum_{x \in C_0} w(x) + \sum_{y \in C_1} w(y) + \alpha \cdot b_1(C)$$

The first Betti number of C can be written as:

$$b_1(C) = \sum_{[x] \in \pi_0(C_0)} b_1([x]) + \sum_{[y] \in \pi_0(C_1)} b_1([y]) + (|C_0 \cap C_1| + 1 - |\pi_0(C_0) \amalg \pi_0(C_1)|).$$

(This follows from $\chi(C) = \chi(C_0) + \chi(C_1) - \chi(C_0 \cap C_1)$.) From this it follows that:

$$g(C) = \sum_{[x] \in \pi_0(C_0)} g([x]) + \sum_{[y] \in \pi_0(C_1)} g([y]) + (|\partial V \cap C| + 1 - |\pi_0(C_0) \amalg \pi_0(C_1)|).$$

This is the same formula as in definition 8.1.13 where we defined the weighting for composite cospans. Hence $S(g \circ f)$ indeed agrees with $S(f) \cup_{\{1, \dots, v\}} S(g)$ as a weighted cospan. \square

Proposition 9.1.19. *Let $\mathcal{F}' \subset \mathcal{F}$ be the full subcategory on all those $(M, [\emptyset \rightarrow W \leftarrow M], [M \rightarrow W' \leftarrow \emptyset])$ where $M = \{1, \dots, u\}$ for some $u \in \mathbb{N}$. For all n the map*

$$B \int_{\mathcal{J}} N_n \text{Sub}' \xrightarrow{S_n} N_n \mathcal{F}' \subset N_n \mathcal{F}$$

is a homotopy equivalence.

Proof. Let $\mathcal{C}_n := \int_{\mathcal{J}} N_n \text{Sub}'$ be the topological category defined as the Grothendieck construction. An object in this category is a topological graph (X, w) together with a filtration $(U_0 \subset \dots \subset U_n \subset X)$ by admissible subsets and bijections $\alpha_i : \partial U_i \cong \{1, \dots, u_i\}$. A morphism $(X, w, U_\bullet, \alpha_\bullet) \rightarrow (Y, w', V_\bullet, \beta_\bullet)$ is a morphism $f : (X, w) \rightarrow (Y, w')$ in \mathcal{J} with $f^{-1}(V_i) = U_i$ for all i and such that $\alpha_i \circ f|_{\partial U_i} = \beta_i$.

Let us call a filtration $(U_\bullet, \alpha_\bullet)$ on (X, w) *maximal* if every edge of (X, w) intersects $\coprod_{i=0}^n \partial U_i$. Let $\mathcal{M}_n \subset \mathcal{C}_n$ be the full subcategory on these objects. Note the objects of \mathcal{M}_n are a union of connected components in the space of objects of \mathcal{C}_n .

Claim 1: The inclusion $B\mathcal{M}_n \rightarrow B\mathcal{C}_n$ is a homotopy equivalence.

Consider the topological functor $Q : \mathcal{C}_n \rightarrow \mathcal{C}_n$ that sends $(X, w, U_\bullet, \alpha_\bullet)$ to the quotient $(X/\sim, w', (U_\bullet)_{/\sim}, \alpha_\bullet)$ where we collapse all edges that do not intersect $\coprod_{i=0}^n \partial U_i$. Since each of the collapsed edges was either contained entirely in U_i or its complement, the quotient $(U_i)_{/\sim} \subset X_{/\sim}$ is still an admissible subset. By construction the functor Q lands in the subcategory $\mathcal{M}_n \subset \mathcal{C}_n$. Since we constructed Q by collapsing edges there is a natural transformation $\eta : \text{Id}_{\mathcal{C}_n} \Rightarrow Q$ defined by the projections $(X, w) \rightarrow (X_{/\sim}, w')$. This natural transformation is continuous, in fact the

morphism does not change as we continuously vary the filtration. On classifying spaces Q is a map $BQ : BC_n \rightarrow B\mathcal{M}_n$ and η is a homotopy between id_{BC_n} and BQ postcomposed with the inclusion BI . Moreover, η restricts to an isomorphism between $\text{Id}_{\mathcal{M}_n}$ and $Q|_{\mathcal{M}_n}$, which yields a homotopy between $BI \circ BQ$ and $\text{id}_{B\mathcal{M}_n}$. This shows that BC_n and $B\mathcal{M}_n$ are homotopy equivalent.

Claim 2: \mathcal{M}_n is a topological groupoid.

By lemma 9.1.6 any morphism $f : (X, w) \rightarrow (Y, w')$ in \mathcal{J} can be factored as $(X, w) \rightarrow (X/\sim_f, w_f) \rightarrow (Y, w')$ where the first map collapses some edges and the second map is an isomorphism. If f collapses some edge $e : [0, 1] \rightarrow X$ and $(V_\bullet, \beta_\bullet)$ some filtration on (Y, w') , then $e([0, 1])$ never intersects $\partial f^{-1}(V_i)$, and hence $f^{-1}(V_\bullet)$ is not maximal. Therefore any morphism between objects in \mathcal{M}_n is necessarily invertible.

Claim 3: The components of $B\mathcal{M}_n$ are contractible.

We begin by showing that \mathcal{M}_n is a fibrant topological category in the sense that the source and target map $(s, t) : \text{Mor}(\mathcal{M}_n) \rightarrow \text{Obj}(\mathcal{M}_n)^2$ is a Serre fibration. The space of morphisms is by definition the space of morphisms $f : (X, w) \rightarrow (Y, w')$ together with a filtration $(V_\bullet, \beta_\bullet)$ on Y . The target map projects this to $(Y, w', V_\bullet, \beta_\bullet)$ and the source map to $(X, w, f^{-1}(U_\bullet), (f^{-1} \circ \beta_\bullet))$. We can fix (X, w) and (Y, w') as \mathcal{J} has a discrete space of objects. So we need to show that sending $(f, (V_\bullet, \beta_\bullet))$ to $((f^{-1}(V_\bullet), f^{-1} \circ \beta_\bullet), (V_\bullet, \beta_\bullet))$ is a Serre fibration. Moreover, since V_i is, up to passing to complements, uniquely determined by $\beta_i : \{1, \dots, v_i\} \hookrightarrow Y \setminus (Y_w \cup Y_{\text{sing}})$, it will suffice to show that $(f, \beta_\bullet) \mapsto (f^{-1} \circ \beta_\bullet, \beta_\bullet)$ is a Serre fibration. This follows from the fact that inverting and evaluating a homeomorphism f on a finite subset $A \subset Y \setminus (Y_w \cup Y_{\text{sing}})$ induces a Serre fibration $\text{Homeo}(Y \text{ rel } (Y_w \cup Y_{\text{sing}})) \rightarrow \text{Emb}(A, Y \setminus (Y_w \cup Y_{\text{sing}}))$.

Since \mathcal{M}_n is fibrant the base-change theorem [ERW19a, Theorem 5.1] (or equivalently theorem 8.2.16) implies that the inclusion of a skeleton $\mathcal{M}_n^{\text{sk}} \subset \mathcal{M}_n$ induces an equivalence on classifying spaces. Therefore, the components of $B\mathcal{M}_n$ are equivalent to $B \text{Aut}_{\mathcal{M}_n}(X, w, U_\bullet, \alpha_\bullet)$. Such an automorphism is a homeomorphism $f : X \cong X$ that respects the weight, the U_i and satisfies $f \circ \alpha_i = \alpha_i$. The latter means that it has to fix all elements in the image of the α_i . Since the filtration is maximal each edge contains such an element and hence f has to preserve each edge as a set. In fact, it also has to preserve the orientation of the edge as either it contains two points of some ∂U_i or one end lies in U_i and the other does not. This implies that f has to fix all points in $X_w \cup X_{\text{sing}}$. In summary, this group is exactly the group of homeomorphism that fix all vertices and all elements of ∂U_i . Therefore, the group is equivalent to a product $\prod_j \text{Homeo}^\partial([0, 1])$, which is contractible.

Claim 4: S_n induces a bijection between the isomorphism classes of \mathcal{M}_n and the n -simplices of $N_\bullet \mathcal{F}'$.

Given an n -simplex in $N_\bullet \mathcal{F}'$ we will construct a preimage under S_n . Such an n -simplex can be thought of as an $(n+2)$ -simplex in $N_\bullet \text{Csp}(A, A_1, \alpha)$. Using the notation from section 8.3 we write this as $W : [n+2] \rightarrow \text{Csp}(A, A_1, \alpha)$ where:

$$(\emptyset = W(0) \rightarrow W_\pi(0 \leq 1) \leftarrow W(1) \rightarrow \cdots \leftarrow W(n+1) \rightarrow W_\pi(n+1 \leq n+2) \leftarrow W(n+2) = \emptyset).$$

Moreover, we have that $W(i) = \{1, \dots, u_{i-1}\}$ for all $i = 1, \dots, n+1$. Each of the $W(i \leq i+1) : W(i) \rightarrow W(i+1)$ is a cospan of finite sets $W(i) \rightarrow W_\pi(i \leq i+1) \leftarrow W(i+1)$ together with a labeling $o_i : W_\pi(i \leq i+1) \rightarrow A$.

As discussed in definition 8.3.7 we can construct a representing space $|W|$ with a map $\text{pr} : |W| \rightarrow [0, n+3]$ and a labeling $o : |W|_0 = \text{pr}^{-1}(\{1, \dots, n+1\}) \rightarrow A$ coming from the o_i . Since for the simplex at hand $W(0) = \emptyset = W(n+2)$ are both empty the space $|W|$ will come with two discrete points $\perp, \top \in |W|$. Let us remove these and denote the result by $|W|' := |W| \setminus \{\perp, \top\}$. By construction $|W|'$ is a 1-dimensional CW complex and since $W(0 \leq n+2)$ is a connected morphism $|W|'$ is a connected space. Hence $(|W|', o)$ is a well-defined object of \mathcal{J} . Moreover, we have a filtration on $|W|'$ defined by $U_i := \text{pr}^{-1}((0, i+1))$ and a bijection $\alpha_i : \{1, \dots, u_i\} \cong W(i+1) \cong \partial U_i$. The tuple $((|W|', o), (U_\bullet, \alpha_\bullet))$ hence defines an object in \mathcal{M}_n .

One can now verify that for $W = S_n((X, w), (V_\bullet, \beta_\bullet))$ the resulting $((|W|', o), (U_\bullet, \alpha_\bullet))$ is isomorphic to $((X, w), (V_\bullet, \beta_\bullet))$, and conversely, $S_n((|W|', o), (U_\bullet, \alpha_\bullet))$ always recovers W . This shows that S_n indeed induces a bijection between the isomorphism classes of \mathcal{M}_n and the elements of $N_n \mathcal{F}$.

Conclusion:

By claim 2 \mathcal{M}_n is a groupoid and since isotopic objects are isomorphic in this topological groupoid, $\pi_0 B\mathcal{M}_n$ is the set of isomorphism classes of objects in \mathcal{M}_n . By claim 4 the functor S_n hence induces a bijection $\pi_0 B\mathcal{M}_n \cong N_n \mathcal{F}'$. By claim 1 and 3 the maps $\pi_0 B\mathcal{M}_n \leftarrow B\mathcal{M}_n \rightarrow BC_n$ are homotopy equivalences, and hence $BS_n : BC_n \rightarrow N_n \mathcal{F}'$ is indeed an equivalence. \square

Assuming the contractibility of $B\text{Sub}'(X, w)$, which we prove in the next section, we can now show that both functors $\mathcal{J} \leftarrow \int_{\mathcal{J}} \text{Sub}' \rightarrow \mathcal{F}$ are equivalence on classifying spaces.

Proof of theorem 9.1.2. Thomason showed that the classifying space of a Grothendieck construction $\int_{\mathcal{I}} F$ is equivalent to the homotopy colimit of the classifying space $BF(i)$, see [Tho79]. While Thomason only considered discrete categories, the analogous statement is true for topologically enriched categories, see [Rap14, Proposition 2.1.1]. By corollary 9.1.23 the classifying space $B\text{Sub}'(G, w)$ is contractible for all $(G, w) \in \mathcal{J}$. Therefore if we replace $\text{Sub}'(G, w)$ by the trivial category $*$, then this induces an equivalence on the homotopy colimits:

$$B \int_{\mathcal{J}} \text{Sub}' \simeq \text{hocolim}_{(G,w) \in \mathcal{J}^{\text{op}}} B\text{Sub}'(G, w) \xrightarrow{\simeq} \text{hocolim}_{(G,w) \in \mathcal{J}^{\text{op}}} * \simeq B \int_{\mathcal{J}} * \cong B\mathcal{J}.$$

This shows that the left-wards functor $\mathcal{J} \leftarrow \int_{\mathcal{J}} \text{Sub}'$ is an equivalence on classifying spaces.

For the other functor we note that, because geometric realisation commutes with homotopy colimits, we may also compute the classifying space of the Grothendieck construction as:

$$\left| B \int_{\mathcal{J}} N_{\bullet} \text{Sub}' \right| \simeq \left| \text{hocolim}_{(X,w) \in \mathcal{J}^{\text{op}}} N_{\bullet} \text{Sub}'(X, w) \right| \simeq \text{hocolim}_{(X,w) \in \mathcal{J}^{\text{op}}} B \text{Sub}'(X, w) \simeq B \left(\int_{\mathcal{J}} \text{Sub}' \right)$$

We have shown in proposition 9.1.19 that for all n the map

$$S_n : B \left(\int_{\mathcal{J}} N_n \text{Sub}' \right) \longrightarrow N_n \mathcal{F}'$$

is an equivalence. Hence, after passing to geometric realisations, we see that $BS : B \int_{\mathcal{J}} \text{Sub}' \rightarrow B\mathcal{F}'$ is an equivalence. Moreover $B\mathcal{F}' \simeq B\mathcal{F}$ since \mathcal{F}' is a full subcategory that contains at least one object per isomorphism class. \square

9.1.3 Height functions for graphs

In this section show that, in some sense, the choice of a filtration on a graph is a contractible choice. Concretely, we will show in proposition 9.1.22 that $B(\text{Sub}(X, w))$ is contractible for any topological graph (X, w) . To prove this we first introduce height functions $h : X \rightarrow \mathbb{R}$ and then we obtain filtrations as preimages of $(\infty, t]$ for regular values t .

For the purpose of this section we will assume that all topological graphs (X, w) come with a metric d . In particular this induces the structure of a Riemannian 1-manifold on $X \setminus X_{\text{sing}}$.

Definition 9.1.20. *A height function for a graph (X, w, d) is continuous map $h : X \rightarrow \mathbb{R}$ such that $h|_{X \setminus X_{\text{sing}}}$ is smooth. Let $C^{\infty}(X)$ denote the vector space of height functions on a graph G . We topologise this as a subvector space of $C^{\infty}(X \setminus X_{\text{sing}}, \mathbb{R})$ equipped with the Whitney C^{∞} -topology.*

Given a graph (X, w, d) with height-function $h : X \rightarrow \mathbb{R}$ we say that $t \in \mathbb{R}$ is a regular value for h if for any vertex $v \in (X_w \cup X_{\text{sing}})$ we have that $h(v) \neq t$ and if t is a regular value of $h|_{X \setminus X_{\text{sing}}}$.

In order to use the height-function to cut G non-trivially we need to require the height-function to not be constant. Homotopically, this is not a problem because of the following fact:

Fact 9.1.21 ([BP75, Theorem VI.5.2, Corollary V.6.2, and Theorem V.6.3]). *If V is an infinite-dimensional Frechét space and $U \subset V$ is a finite-dimensional subspace, then the complement $V \setminus U$ is contractible.*

Proposition 9.1.22. *For any $(X, w) \in \mathcal{J}$ the classifying space of the topological poset $\text{Sub}(X, w)$ is contractible.*

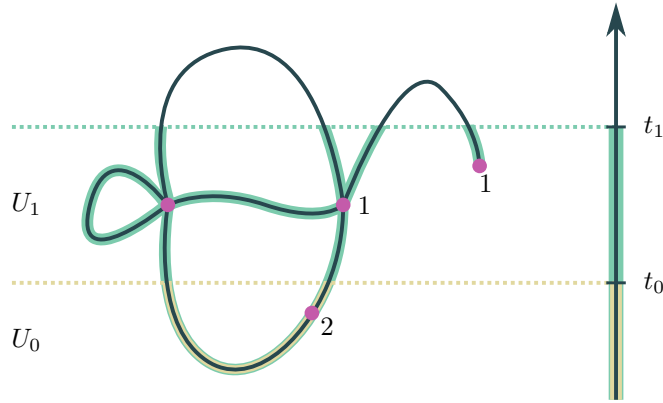


Figure 9.2: A filtration induced by a height function $h : G \rightarrow \mathbb{R}$ and regular values $t_0 \leq t_1$.

Proof. Fix (X, w) and pick some metric d . Consider the topological poset \mathcal{P} where an object is a tuple $(h, t) \in \mathcal{C}^\infty(X) \times \mathbb{R}$ such that t is a regular value for h and $t \in h(X)$. We define the relation of \mathcal{P} to be $(h, t) \leq (h', t')$ whenever $h = h'$ and $t \leq t'$. The nerve $N_\bullet \mathcal{P}$ admits an augmentation where we let $N_{-1} \mathcal{P} \subset \mathcal{C}^\infty(X)$ be the subspace of functions h such that there exists a regular value $t \in \mathbb{R}$ with $t \in h(X)$.

The augmented (semi-)simplicial space $N_\bullet \mathcal{P}$ is a topological flag complex in the sense of [GRW14, Definition 6.1]. Moreover, $\mathcal{P} \subset \mathcal{C}^\infty(X) \times \mathbb{R}$ is open, and therefore [GRW14, Theorem 6.2] tells us that the augmentation map $B\mathcal{P} \rightarrow N_{-1} \mathcal{P}$ is a weak equivalence. (See lemma 6.2.9 for a formulation of the special case we are using.)

By definition $N_{-1} \mathcal{P} \subset \mathcal{C}^\infty(X)$ is the space of those height functions h that admit a regular value t that lies in the image of h . By Sard's theorem the set of regular values of a given $h : X \rightarrow \mathbb{R}$ is open and dense in \mathbb{R} . Therefore we can always find such a regular value, unless the function h is constant. This means that $N_{-1} \mathcal{P}$ is the complement $\mathcal{C}^\infty(X) \setminus \mathbb{R}$ of the one-dimensional subspace of constant functions. By fact 9.1.21 this space is contractible and so we conclude that $B\mathcal{P} \simeq N_{-1} \mathcal{P} \simeq *$ is contractible.

To complete the proof we will construct a level-wise equivalence between $N_\bullet \mathcal{P}$ and $N_\bullet \text{Sub}(X, w)$. Consider the map $P : \mathcal{P} \rightarrow \text{Sub}(X, w)$ defined by sending (h, t) to the preimage $U := h^{-1}(-\infty, t]$ as illustrated in figure 9.2. This subset is always admissible because t is regular. It follows from inspection that this map is continuous and it preserves the relation as $h^{-1}(-\infty, t] \subset h^{-1}(-\infty, t']$ when $t \leq t'$. We therefore get a well-defined map of simplicial spaces $P_\bullet : N_\bullet \mathcal{P} \rightarrow N_\bullet \text{Sub}(X, w)$.

Fix some n . We would like to show that P_n is an equivalence. Consider the subspace $N_n \text{Sub}(X, w)^{<} \subset N_n \text{Sub}(X, w)$ that only contains those filtrations $U_0 \subset \cdots \subset U_n$ where ∂U_i is disjoint from U_{i-1} for all i . Similarly let $N_n \mathcal{P}^{<} \subset N_n \mathcal{P}$ be the subspace of those tuples (h, t_\bullet) where $t_{i-1} < t_i$ for all i . In both cases the inclusion of this subspace is a homotopy equivalence. We will construct a map $q : N_n \text{Sub}(X, w)^{<} \rightarrow N_n \mathcal{P}^{<}$ that is homotopy inverse to the restriction of P_n to $N_n \mathcal{P}^{<}$.

Fix some smooth function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\sigma(x) = 0$ for $x \leq 0$, $\sigma(x) = 1$ for $x \geq 1$, and $\sigma'(x) > 0$ for $x \in (0, 1)$. Given some length $(n + 1)$ -filtration $U_0 \subset \cdots \subset U_n \subset X$ we define a height function $h_{U_\bullet} : X \rightarrow \mathbb{R}$ as follows:

$$h_{U_\bullet}(x) = \begin{cases} -\sigma(d(x, \partial U_0)) & \text{if } x \in U_0 \\ i + \sigma\left(\frac{d(x, \partial U_i)}{d(x, \partial U_{i+1}) + d(x, \partial U_i)}\right) & \text{if } x \in U_{i+1} \setminus U_i^\circ \\ n + \sigma(d(x, \partial U_n)) & \text{if } x \in |G| \setminus U_n^\circ \end{cases}$$

One checks that h is indeed a smooth function with image in $[-1, n + 1]$ and that h depends smoothly on the filtration $U_\bullet \in N_n \text{Sub}(X, w)^{<}$. Any $i = 0, \dots, n$ is a regular value of h and we have that $h_{U_\bullet}^{-1}(-\infty, i] = U_i$ by construction. We define $q : N_n \text{Sub}(X, w)^{<} \rightarrow N_n \mathcal{P}^{<}$ by sending U_\bullet to $(h_{U_\bullet}, t_\bullet)$ with $t_i := i$. By construction $P_n \circ q$ is the identity on $N_n \text{Sub}(X, w)^{<}$, so q is a section of $(P_n)_{|N_n \mathcal{P}^{<}}$. To construct a homotopy between $q \circ (P_n)_{|N_n \mathcal{P}^{<}}$ and the identity note that given (h, t_\bullet) and $(h', t'_\bullet) \in N_n \mathcal{P}$ that induce the same filtration the function $(1 - \lambda)h + \lambda h'$ is a height function with regular values $(1 - \lambda)t_\bullet + \lambda t'_\bullet$ for any $\lambda \in [0, 1]$. We can therefore simply write down an affine linear homotopy between $q \circ P_n$ and $\text{id}_{N_n \mathcal{P}}$.

This completes the proof that $P : N_\bullet \mathcal{P} \rightarrow N_\bullet \text{Sub}(X, w)$ is a level-wise weak equivalence. After taking geometric realisations have weak equivalences

$$B\text{Sub}(X, w) \simeq B\mathcal{P} \simeq N_{-1}\mathcal{P} \simeq *.$$

□

Corollary 9.1.23. *For any topological graph (X, w) the classifying space $B\text{Sub}'(X, w)$ is contractible.*

Proof. This will follow from the above together with the base-change theorem 8.2.16 if we can show that $N_\bullet \text{Sub}'(X, w) \rightarrow N_\bullet \text{Sub}(X, w)$ is a base-change. This map is clearly surjective on 0-simplices so we need to check that the following diagram is a homotopy pullback square:

$$\begin{array}{ccc} N_n \text{Sub}'(X, w) & \longrightarrow & (N_0 \text{Sub}'(X, w))^{n+1} \\ \downarrow & & \downarrow \\ N_n \text{Sub}(X, w) & \longrightarrow & (N_0 \text{Sub}(X, w))^{n+1}. \end{array}$$

The map $\text{Sub}'(X, w) \rightarrow \text{Sub}(X, w)$ is a covering and the fiber at some $U \in \text{Sub}(X, w)$ is the discrete set of bijections $\{1, \dots, u\} \cong \partial U$. (This covering is actually trivial unless $(X, w) = (S^1, 0)$.) Hence $(\text{Sub}'(X, w))^{n+1} \rightarrow (\text{Sub}(X, w))^{n+1}$ is a covering and the fiber is the set of $(n + 1)$ -tuples of bijections $\alpha_i : \{1, \dots, u_i\} \cong \partial U_i$. Similar to the above $N_n \text{Sub}'(X, w) \rightarrow N_n \text{Sub}(X, w)$ is a covering with fiber the set of $(n + 1)$ -tuples of bijection α_i . Since both vertical maps are coverings and their fibers agree it follows that the square is indeed a homotopy pullback square. □

9.2 Comparison to Δ_g

In this section we relate the category $J_g = J_g^{(\mathbb{N}, \mathbb{N}_{\geq 1}, 1)}$ whose classifying space is equivalent to $B\mathcal{F}_g(\text{Cob}_2^{\chi \leq 0})$ to the tropical moduli space Δ_g . We will argue that BJ_g should be thought of as the homotopy type of the moduli stack of tropical curves – a version of Δ_g that “keeps track of finite automorphisms”. Indeed, there is a zig-zag $BJ_g \leftarrow B\mathcal{T} \rightarrow \Delta_g$ such that the left-hand map is an equivalence and such that the fiber of the right-hand map at some tropical curve $[\Gamma, w, d] \in \Delta_g$ is a $K(\pi, 1)$ for $\pi = \text{Aut}(\Gamma, w, d)$ the (finite) isometry group of (Γ, w, d) . Together with a computation showing $\pi_1(BJ_g) \cong *$ this will imply the following theorem:

Theorem 9.2.1. *For $g \geq 2$ BJ_g is simply connected and there is a rational equivalence $BJ_g \rightarrow \Delta_g$.*

We begin by recalling the definition of Δ_g as a colimit indexed by J_g following [CGP16]

Definition 9.2.2. *Consider the functor $\Delta^E : J^{\text{op}} \rightarrow \text{Top}$ that sends (G, w) to the space*

$$\Delta^{EG} = \{d : E_G \rightarrow [0, 1] \mid \sum_{e \in E_G} d(e) = 1\}.$$

We define Δ as the colimit $\text{colim}_{(G, w) \in J^{\text{op}}} \Delta^E(G, w)$. We let $\Delta_g \subset \Delta$ be the connected component defined by taking the colimit only over $J_g \subset J$.

REMARK 9.2.3. *In [CGP16] the authors first define a moduli space M_g^{trop} as the colimit over the functor $\sigma : J_g^{\text{CGP}} \rightarrow \text{Top}$ defined by sending G to $\sigma(G) = \mathbb{R}_{\geq 0}^{E_G} = \{l : E_G \rightarrow \mathbb{R}_{\geq 0}\}$. This colimit has a natural map to $\mathbb{R}_{\geq 0}$ defined by taking the sum of all edge-lengths: this is the volume of the tropical curve. The moduli space Δ_g is then defined subspace of M_g^{trop} containing the tropical curves of volume 1. This is canonically homeomorphic to our definition via:*

$$\begin{aligned} \Delta_g &\stackrel{[\text{CGP16}]}{=} \{1\} \times_{\mathbb{R}_{\geq 0}} \left(\text{colim}_{G \in J_g^{\text{CGP}}} \mathbb{R}_{\geq 0}^{E_G} \right) \cong \text{colim}_{G \in J_g^{\text{CGP}}} \left(\{1\} \times_{\mathbb{R}_{\geq 0}} \mathbb{R}_{\geq 0}^{E_G} \right) \\ &\cong \text{colim}_{G \in J_g^{\text{CGP}}} \Delta^{EG} \cong \text{colim}_{G \in J_g} \Delta^{EG}. \end{aligned}$$

Here the last homeomorphism uses that $J_g \simeq J_g^{\text{CGP}} \setminus \{\bullet_g\}$ (see remark 9.1.11) and $\Delta^E(\bullet_g) = \emptyset$.

Definition 9.2.4. *Let \mathcal{T} be the topological category obtained as the Grothendieck construction $\int_{\mathcal{J}} \Delta^E$. This comes with canonical functors:*

$$J \leftarrow \mathcal{T} = \int_{\mathcal{J}} \Delta^E \longrightarrow \text{colim}_{(G, w) \in J} \Delta^{EG}.$$

Note that because each Δ^{EG} is contractible the leftwards functor $\mathcal{T} \rightarrow J$ is a level-wise equivalence on nerves and hence $B\mathcal{T} \simeq BJ$ via this map. Since the Grothendieck-construction is a

model for the homotopy colimit the rightwards functor can be understood as the comparison map between homotopy colimit and ordinary colimit:

$$B\mathcal{T} = \operatorname{hocolim}_{(G,w) \in J} \Delta^{E_G} \longrightarrow \operatorname{colim}_{(G,w) \in J} \Delta^{E_G} = \prod_{g \geq 2} \Delta_g$$

The rest of the section will be concerned with showing that this map is a rational equivalence. The key-idea is that the automorphism group of any object in J is finite. To do so we will use the following consequence of the Vietoris-Begle theorem:

Lemma 9.2.5. *Let $(r : X_\bullet \rightarrow X_{-1})$ be an augmented simplicial space such that each X_n is a finite CW complex. Assume further that for each $x \in X_{-1}$ the fiber $Y_\bullet := r^{-1}(x)$ has trivial rational homology $H_*(|Y_\bullet|; \mathbb{Q}) = 0$. Then the augmentation $|X_\bullet| \rightarrow X_{-1}$ is an isomorphism in rational homology.*

Proof. For some $N \gg 0$ consider the N -skeleton $X_\bullet^{(N)} \subset X_\bullet$. Then $|X^{(N)}|$ is a compact Hausdorff space and $|X^{(N)}| \rightarrow |X|$ is N -connected. The fiber of the augmentation $|X^{(N)}| \rightarrow X_{-1}$ at some x is the N -skeleton $|Y^{(N)}|$, which has trivial rational homology $H_*(|Y^{(N)}|; \mathbb{Q})$ for $* < N$. We may therefore apply the Vietoris-Begle theorem to the restricted map $r : |X^{(N)}| \rightarrow X_{-1}$ and conclude that it is an isomorphism on rational homology in degrees $* < N$. This implies $r : |X_\bullet| \rightarrow X_{-1}$ is a rational homology isomorphism in this range. Since N was arbitrary the claim follows. \square

Lemma 9.2.6. *The canonical map $B\mathcal{T} \rightarrow \prod_{g \geq 2} \Delta_g$ is an isomorphism in rational homology.*

Proof. For the purpose of this proof let us assume that we have replaced the category J by a skeleton so that each of the J_g is finite. This implies that the space $N_n \mathcal{T}_g$ is a finite CW-complex for all n and g . We may therefore apply our variant of the Vietoris-Begle theorem 9.2.5 to the augmented simplicial space $(N_\bullet \mathcal{T}_g \rightarrow \Delta_g)$.

An element in the colimit Δ_g is represented by a triple (G, w, d) where $(G, w) \in J_g$ and $d \in \Delta^{E_G}$ is a metric on (G, w) . By collapsing those edges with $d_e = 0$ we can always find a representative where every edge has non-zero length. To apply the Vietoris-Begle theorem we need to consider the fibers $r^{-1}([G, w, d])$ of $r : B\mathcal{T}_g \rightarrow \Delta_g$ at one of these points. This fiber can be written as the classifying space of the full subcategory $\mathcal{C} \subset \mathcal{T} = \int_J \Delta^E$ on those objects (G', w', d') such that (G', w', d') is isomorphic to (G, w, d) as a metric weighted graph. Let $\mathcal{C}_0 \subset \mathcal{C}$ denote the full subcategory on those (G', w', d') where we also require that all edges $e \in E_{G'}$ has non-zero length $d'_e > 0$. The inclusion of this subcategory has a left-adjoint $\mathcal{C} \rightarrow \mathcal{C}_0$ given by sending a graph to the quotient where we collapse all the edges of length 0. After taking classifying spaces an adjunction becomes a homotopy equivalence and so $B\mathcal{C}_0 \simeq B\mathcal{C}$. Now note that \mathcal{C}_0 is in fact a connected groupoid, and so $B\mathcal{C}_0$ is equivalent to $B \operatorname{Aut}_{\mathcal{C}}(G', w', d')$ for any object

(G', w', d') . The automorphism group $\text{Aut}_{\mathcal{C}}(G', w', d')$ is always finite and so its classifying space has trivial rational homology. In summary we have shown that:

$$H_*(r^{-1}([G, w, d]); \mathbb{Q}) \cong H_*(BC; \mathbb{Q}) \cong H_*(B \text{Aut}_{\mathcal{C}}(G', w', d'); \mathbb{Q}) \cong 0.$$

The claim now follows from lemma 9.2.5. □

Now that we know that the map $B\mathcal{T}_g \rightarrow \Delta_g$ is an isomorphism in rational homology we only need to understand the fundamental group on both sides in order to show that it is a rational equivalence. In fact, we can already conclude that the suspended map $SB\mathcal{T}_g \rightarrow S\Delta_g$ is a rational equivalence, which would be sufficient for proving theorem K. However, we still include a comparison of fundamental groups as it might be of independent interest. For the more complicated case of the tropical moduli space this was computed in [ACP19].

Theorem 9.2.7 ([ACP19, Theorem 1.1]). *For any $g \geq 2$ the space Δ_g is simply connected.*

We are now left with showing the analogous result for J_g , which is elementary but tedious.

Proposition 9.2.8. *For any $g \geq 2$ the space BJ_g is simply connected.*

Proof. We will show that the groupoid \mathcal{G}_g obtained from J_g by formally inverting all morphisms is contractible. Let $F : J_g \rightarrow \mathcal{G}_g$ be the canonical quotient map. It is a bijection on objects and has the property that every morphisms in \mathcal{G}_g can be written as the composite of morphisms of the form $F(f)$ and $F(g)^{-1}$.

Claim 1: Any two parallel morphisms $f, g : (G, w) \rightarrow (G', w')$ where $|G'| \neq *$ are identified by F .

We begin by showing that several morphisms in J_g are identified by F . Let $\mathcal{Q} \in J_g$ denote the weighted graph with one vertex v_0 of weight $g - 1$ and two half-edges h_0 and h'_0 . For any $(G, w) \in J_g$ and non-separating half-edge $h \in H_G$ there is a unique morphism $q_{G,h} : (G, w) \rightarrow \mathcal{Q}$ that sends h to h_0 . If $p : (G, w) \rightarrow (G', w')$ is any other morphism in J_g and $h' \in H_{G'}$ is a half-edge with preimage $h \in H_G$, then $q_{G',h'} \circ p = q_{G,h}$.

Next, we let $\mathcal{O} \in J_g$ be the weighted graph that has one vertex v_0 of weight $g - 2$ and four half-edges $\{h_1, h'_1, h_2, h'_2\}$ such that $s(h_i) = h'_i$. This has an automorphism $\sigma_1 : \mathcal{O} \rightarrow \mathcal{O}$ that flips the first loop, i.e. it fixes h_2 and h'_2 and swaps h_1 with h'_1 . We let $\sigma : \mathcal{Q} \rightarrow \mathcal{Q}$ denote the analogous automorphism that flips the loop in \mathcal{Q} . It follows that

$$q_{\mathcal{O},h_2} \circ \sigma_1 = q_{\mathcal{O},h_2} \quad \text{and} \quad q_{\mathcal{O},h_1} \circ \sigma_1 = q_{\mathcal{O},h'_1} = \sigma \circ q_{\mathcal{O},h_1}.$$

After mapping to the groupoid \mathcal{G}_g with F it follows from the first equation that $F(\sigma_1) = \text{id}_{F(\mathcal{O})}$ and from the second equation that $F(\sigma) = \text{id}_{F(\mathcal{Q})}$. For any other $(G, w) \in J_g$ and half-edge $h \in H_G$ we have $\sigma \circ q_{G,h} = q_{G,s(h)}$ so in \mathcal{G}_g this means $F(q_{G,h}) = F(q_{G,h'})$.

Next, consider the graph $\Theta \in J_g$ that has two vertices v_0 and v_1 where v_0 has weight $g - 2$ and v_1 has weight 0. This has three edges $\{h_i, h'_i\}_{i=1}^3$ where we set $r(h_i) = v_0$ and $r(h'_i) = v_1$. The symmetric group on three elements acts on this by permuting the three edges. Write $\rho_{(1,2)} : \Theta \rightarrow \Theta$ for the automorphism that swaps h_1 and h_2 . Then we have:

$$q_{\Theta, h_3} \circ \rho_{(1,2)} = q_{\Theta, h_3} \quad \text{and} \quad q_{\Theta, h_1} \circ \rho_{(1,2)} = q_{\Theta, h_2}$$

The first equation implies $F(\rho_{(1,2)}) = \text{id}_{F(\Theta)}$ and the second equation implies $F(q_{\Theta, h_1}) = F(q_{\Theta, h_2})$. There also is a quotient map $p : \Theta \rightarrow \infty$ that collapses $\{h_3, h'_3\}$ and satisfies $p(h_i) = h_i$ for $i = 1, 2$. We can write $q_{\Theta, h_i} = q_{\infty, h_i} \circ p$ for $i = 1, 2$ and so we can conclude that $F(q_{\infty, h_1}) = F(q_{\infty, h_2})$. In summary we have shown that all morphisms $\infty \rightarrow \mathcal{Q}$ become identified by $F : J_g \rightarrow \mathcal{G}_g$.

Let now $(G, w) \in J_g$ be any object with more than one edge. Then for any two half-edges $h_1, h_2 \in (G, w)$ that belong to different edges and such that $|G| \setminus \{h_1, h_2\}$ is connected, there is a morphism $r_{G, h_1, h_2} : (G, w) \rightarrow \infty$ such that $q_{G, h_i} = q_{\infty, h_i} \circ r_{G, h_1, h_2}$. Together with our previous observation that $F(q_{\infty, h_1}) = F(q_{\infty, h_2})$ this implies that $F(q_{G, h_1}) = F(q_{G, h_2})$. If the graph (G, w) has the property that $|G|$ has at least two loops, then this implies that all morphisms $(G, w) \rightarrow \mathcal{Q}$ are identified by F . To see this, note that for any two non-separating edges h_1 and h_2 either the tuple (h_1, h_2) is non-separating, or we can pick another non-separating edge h_3 such that both tuples (h_1, h_3) and (h_3, h_2) are non-separating and we have $F(q_{G, h_1}) = F(q_{G, h_3}) = F(q_{G, h_2})$.

We have so far shown claim 1 for the case that $b_1(|G|) \geq 2$ and $(G', w') = \mathcal{Q}$. Let us now consider the general case of two morphisms $f, g : (G, w) \rightarrow (G', w')$ as in claim 1. Since $|G'|$ is not contractible we may pick a morphism $q : (G', w') \rightarrow \mathcal{Q}$. Further, we can also find a morphism $p : (G'', w'') \rightarrow (G, w)$ where $b_1(|G''|) \geq 2$. Namely, if $b_1(|G|) \geq 2$ we can take $p = \text{id}_{(G, w)}$, and otherwise there is at least one vertex v_0 of weight ≥ 1 , as we have genus $g \geq 2$. In the latter case we let $p : (G'', w'') \rightarrow (G, w)$ be the morphism where the preimage of v_0 is v with one less weight and a loop $\{h, h'\}$ at v . The graph G'' has one more loop than G , which already has to have one loop. So claim 1 applies to the morphisms $q \circ f \circ p, q \circ g \circ p : (G'', w'') \rightarrow \mathcal{Q}$:

$$F(q) \circ F(f) \circ F(p) = F(q \circ f \circ p) = F(q \circ g \circ p) = F(q) \circ F(g) \circ F(p)$$

Since \mathcal{G}_g is a groupoid we can cancel $F(p)$ and $F(q)$ on both sides to conclude $F(f) = F(g)$ as claimed.

Claim 2: Any morphism $F(\mathcal{Q}) \rightarrow F(\mathcal{Q})$ in \mathcal{G}_g can be written as $F(g_n) \circ F(f_n)^{-1} \circ \dots \circ F(g_1) \circ$

$F(f_1)^{-1}$ where the f_i and g_i are morphisms:

$$\begin{array}{ccccccc}
 & & (\Gamma_1, u_1) & & \dots & & (\Gamma_n, u_n) \\
 & \swarrow f_1 & & \searrow g_1 & & \swarrow f_n & \searrow g_n \\
 \mathbb{Q} = (G_0, w_0) & & (G_1, w_1) & & \dots & & (G_n, w_n) = \mathbb{Q}
 \end{array}$$

Moreover, these graphs can be chosen such that each G_i has exactly one edge and each Γ_i has at most two edges.

It follows from the definition of \mathcal{G}_g that every morphism in it can be written in the form $F(g_n) \circ F(f_n)^{-1} \circ \dots \circ F(g_1) \circ F(f_1)^{-1}$ for some morphisms f_i and g_i . We need to show that this presentation can be simplified so that the G_i and Γ_j are of the required form.

For each G_i that has more than one edge we can pick some edge $e \in G_i$ and collapse all other edges. This defines a quotient map $q : (G_i, w_i) \rightarrow (G', w')$ to a graph G' with only one edge. This graph G' now has to either be \mathbb{Q} where the vertex is labelled by $g - 1$ or it is $a \leftrightarrow_{g-a}$ for some $1 \leq a \leq g - 1$. We may now replace (G_i, w_i) by (G', w') in the zig-zag and set $g'_i := q \circ g_i$ and $f'_{i+1} := q \circ f_{i+1}$. This still represents the same morphism since

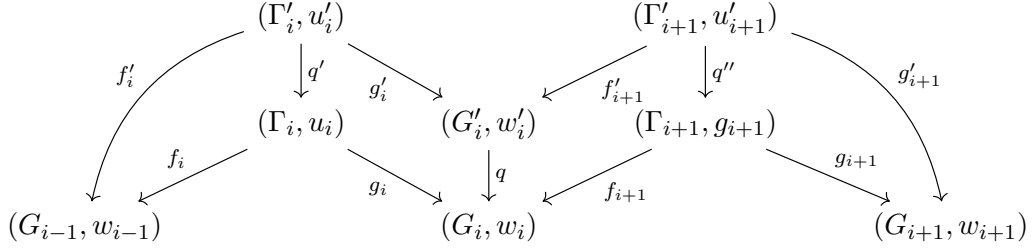
$$F(f'_{i+1})^{-1} \circ F(g'_i) = F(f_{i+1})^{-1} \circ F(q)^{-1} \circ F(q) \circ F(g_i) = F(f_{i+1})^{-1} \circ F(g_i).$$

A similar strategy can be used to bring the Γ_i in the desired form. Since we may now assume that both (G_{i-1}, w_{i-1}) and (G_i, w_i) only have one edge, there are now at most two edges in Γ_i that are not collapsed by both f_i and g_i . Let $q : (\Gamma_i, u_i) \rightarrow (\Gamma', g')$ be the quotient that collapses all edges that are collapsed by both f_i and g_i . We have factorisation $f_i = f'_i \circ q$ and $g_i = g'_i \circ q$, which we may use to replace the zig-zag by one where the i th graph in the upper line is (Γ', g') , which has at most two edges. This proves claim 2.

Claim 3: In the presentation from claim 2 we may assume that each of the (G_i, w_i) is \mathbb{Q} .

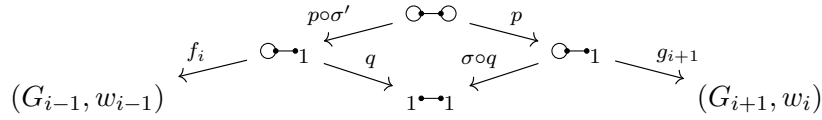
Consider some (G_i, w_i) that is not \mathbb{Q} . Since it has only one edge it has to be $a \leftrightarrow_{g-a}$ for some $1 \leq a \leq g - 1$. Let us assume for now that $a \geq 2$. If $g \geq 3$ then is always possible by swapping a and $g - a$, and we will deal with the case $g = 2$ separately. If we let $v \in a \leftrightarrow_{g-a}$ denote the first vertex, then there is a vertex $v' \in \Gamma_i$ with $g_i(v') = v$ and $u_i(v') \geq 1$. This is true because the preimage $g_i^{-1}(v)$ contains at most one edge and has total genus $a \geq 2$: so it is either \mathbb{Q}_{a-1} , $b \leftrightarrow_c$ or \bullet_d , all of which have a vertex of non-zero weight. Similarly we can find $v'' \in \Gamma_{i+1}$ with $f_{i+1}(v'') = v$ and $u_{i+1}(v'') \geq 1$. Let us now define $q' : (\Gamma'_i, u'_i) \rightarrow (\Gamma_i, u_i)$ as a blow-up at v' , i.e. as the unique weighted graph such that q' is an isomorphism away from v' and $(q')^{-1}(v') = \mathbb{Q}_{u_i(v')-1}$. So Γ'_i is obtained by adding a single loop at v' and reducing $u_i(v')$ by one. Similarly we define $q'' : (\Gamma'_{i+1}, u'_{i+1}) \rightarrow (\Gamma_{i+1}, u_{i+1})$ by adding a loop at v'' , and $q : (G'_i, w'_i) = a-1 \circ \leftrightarrow_{g-a} \rightarrow (G_i, w_i)$

by adding a loop at v . All of these graphs fit into a diagram:



We may therefore replace the original zig-zag with the primed one. In some sense we have made things worse: $G'_i = \circ \rightarrow \bullet$ has two edges and Γ'_i and Γ'_{i+1} have up to three edges. However, G'_i admits a morphism $\circ \rightarrow \bullet \rightarrow \circ$ and following the proof of claim 2 we may first replace (G'_i, w'_i) by \circ and then replace (Γ'_i, u'_i) and $(\Gamma'_{i+1}, u'_{i+1})$ by graphs with at most two edges. In doing so we have managed to replace (G_i, w_i) by \circ without changing any of the other (G_j, w_j) . Applying this iteratively to all of the graphs in the bottom row proves claim 3.

We still have to deal with the case $g = 2$ and $(G_i, w_i) = 1 \leftrightarrow 1$, which we ignored earlier. For $g = 2$ the only graph with two edges that admits a morphism to $1 \leftrightarrow 1$ is $\circ \rightarrow 1$. The only possible morphisms are the projection $q : \circ \rightarrow 1 \rightarrow 1 \leftrightarrow 1$ or $\sigma \circ q$ for $\sigma : 1 \leftrightarrow 1 \rightarrow 1 \leftrightarrow 1$ the flip. If $g_i = f_{i+1}$ we can simply cancel them, so the only case we have to consider is when $g_i = q$ and $f_{i+1} = \sigma \circ q$, or the other way around. However, in this case we can make the following simplification:



Here $p : \circ \rightarrow \bullet \rightarrow \circ \rightarrow 1$ is the morphism that collapses the second loop and $\sigma' : \circ \rightarrow \bullet \rightarrow \circ \rightarrow \bullet \rightarrow \circ$ is the automorphism that swaps the two loops.

Conclusion: By claim 3 the endomorphism $\text{End}_{\mathcal{G}_g}(F(\circ))$ are generated by morphisms of the form $F(g) \circ F(f)^{-1}$ for morphisms $f, g : (G, w) \rightarrow \circ$ in J_g . But claim 1 shows that $F(g) = F(f)$, so we can conclude that the only endomorphism of $F(\circ) \in \mathcal{G}_g$ is the identity. From what we have shown it also follows that \mathcal{G}_g is a connected groupoid and therefore it is equivalent to the trivial category. \square

Proof of theorem 9.2.1. By lemma 9.2.6 the map $B\mathcal{T} \rightarrow \Delta_g$ is an isomorphism on rational homology, by [ACP19, Theorem 1.1] the right-hand-side is simply connected, and by proposition 9.2.8 the left-hand-side is simply connected. \square

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