

The nonperturbative Hilbert space of quantum gravity with one boundary

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ABSTRACT: We discuss a basis for the nonperturbative Hilbert space of quantum gravity with one asymptotic boundary. We use this basis to show that the Hilbert space for gravity with two disconnected boundaries factorizes into a product of two copies of the single boundary Hilbert space.

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1 Introduction

In this paper we address a puzzle about quantum gravity. The puzzle concerns gravity in universes with two asymptotic boundaries. If the cosmological constant is negative, holographic duality requires that the underlying Hilbert space is the product of two single-boundary gravity Hilbert spaces. However, the gravity path integral with these boundary conditions includes a sum over wormholes that connect the asymptotic regions. These wormholes seem to threaten factorization of the Hilbert space [1, 2]. This is our puzzle: does the two-boundary Hilbert space of quantum gravity factorize into single-boundary Hilbert spaces?

The puzzle can be resolved if we have access to a complete basis of gravity states. Recently, such a basis was constructed by cutting open the Euclidean gravitational path integral for universes with two boundaries [3–7], even though this path integral seems to only be an effective description calculating coarse-grained observables of an underlying fine-grained theory [8–13]. Here, using the toolkit constructed in [7] for deriving equality between quantities in the underlying fine-grained theory with access only to the gravity path integral,

we will show that the shell states discussed in [14] span the entire single-boundary Hilbert space. This explicit basis allows us to show that the complete two-boundary Hilbert space factorizes into a product of two single-boundary Hilbert spaces. Previous work had shown that traces taken in the Hilbert space associated to two-boundary black holes are products of factors, but did not establish what those factors were or what the mechanism of factorization was (see [15] for 2d JT gravity, and [6] for a leading saddlepoint analysis in higher dimensions).

Four sections follow. Section 2 reviews aspects of the gravitational path integral and the toolkit of [7]. Section 3 constructs a basis for the single-boundary Hilbert space. Section 4 uses this basis to demonstrate that the two-boundary Hilbert space is a product of two one-boundary Hilbert spaces. We conclude in section 5 with a summary and discussion.

2 The gravitational path integral

2.1 Review

The Euclidean gravity path integral is a map from boundary conditions — topology, metric and matter sources (collectively Σ_b) — to the complex numbers. We calculate it by summing over bulk topologies, geometries (g), and fields (ϕ), compatible with the boundary conditions:

$$\zeta[\Sigma_b] = \int_{g, \phi \rightarrow \Sigma_b} \mathcal{D}g \mathcal{D}\phi e^{-I_{\text{bulk}}[g, \phi]}$$

$$I_{\text{bulk}} = -\frac{1}{16\pi G_N} \int_{\mathcal{M}} \sqrt{g}(R - 2\Lambda + \mathcal{L}_{\text{matter}}) - \frac{1}{8\pi G_N} \int_{\partial\mathcal{M}=\mathcal{M}_b} \sqrt{h}K. \quad (2.1)$$

Here I_{bulk} is the gravitational action; for example the Einstein-Hilbert action, \mathcal{M} is a bulk manifold filling in the boundary, and a sum over topologies is understood. This is an effective description below a cutoff for a more complete fundamental theory. To construct states we formally cut open the boundary conditions put into the Euclidean path integral along some cut \mathcal{X} as in figure 1. The two parts produce functionals of data on the cut, defining states $|M_1\rangle$ and $\langle M_2|$ such that $\langle M_2|M_1\rangle = \zeta[\Sigma_b]$. Note that defining a state by partitioning boundary data is a standard way of producing states in the *bulk* quantum gravitational Hilbert space. For instance, such methods were used in [16] and [11] to construct states in JT gravity and a topological toy model respectively, and in this work we will apply this method in higher dimensional Einstein-Hilbert gravity. Crucially, we do not assume any kind of dual boundary CFT construction. It is clear our states live in the bulk Hilbert space as we can, in principle, compute the wavefunctionals $\Psi_{\mathcal{M}_1}(h, \phi) \equiv \langle h, \phi|M_1\rangle$ between our states and states defined by some bulk time-slice data $h_{\mu\nu}, \phi$ by performing the path integral subject to these joint boundary conditions. For example, consider the gravity path integral subject to the periodic asymptotic boundary condition $\mathbb{S}^1_\beta \times \mathcal{X}$ where S^1_β denotes a boundary circle of length β . We will denote this gravitational path integral as $Z(\beta)$ and call it the “Z partition function”. By cutting this path integral in two it can be interpreted as computing the norm of a state $|\beta\rangle$ defined as $Z(\beta) = \langle\beta|\beta\rangle$.

We can construct states $|i\rangle$ by performing the path integral after inserting operators \mathcal{O}_i on the boundary ([11, 17, 18]). Inserting \mathcal{O}_i^\dagger instead constructs $\langle i|$, or equivalently $|i^*\rangle$. To calculate the overlap $\langle j|i\rangle$ we sew the boundary conditions defining $\langle j|$ and $|i\rangle$ together along \mathcal{X} and integrate over all geometries satisfying this completed boundary condition. To

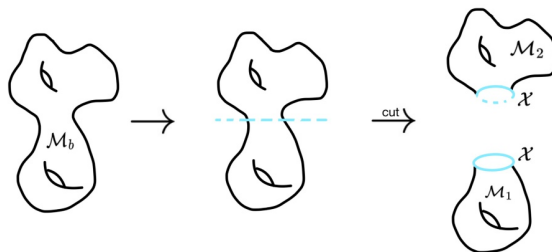


Figure 1. Cartoon of cutting a gravity path integral boundary condition along \mathcal{X} to define a state.

compute the square of this overlap, $\langle i|j\rangle\langle j|i\rangle$, we insert $\mathcal{O}_{i,j}$ on two copies of the boundary manifold and sum over all geometries and topologies that fill in the boundary conditions, including connected topologies (wormholes) interpolating between the two boundaries. These wormholes can lead to an apparent paradox,

$$\langle i|j\rangle = \delta_{ij} \quad \text{while} \quad \langle i|j\rangle\langle j|i\rangle = \delta_{ij}^2 + Z_{\text{WH}}, \tag{2.2}$$

where Z_{WH} is the wormhole contribution. Following [3–6, 8–13, 19–21] we interpret this situation as arising because the gravitational path integral is an effective theory computing coarse-grained averages of quantities in an underlying fine-grained theory. We therefore write an overbar over quantities computed using the gravitational path integral, e.g.,

$$\overline{\langle i_1|j_1\rangle\langle i_2|j_2\rangle\cdots\langle i_n|j_n\rangle}. \tag{2.3}$$

Let $\mathcal{H}_\kappa \equiv \text{span}\{|i\rangle, i = 1, 2, \dots, \kappa\}$ be the span of the states described above. Overlaps between these states are summarized by the Gram matrix $G_{ij} \equiv \langle i|j\rangle$, whose rank equals the dimension of \mathcal{H}_κ . To construct an orthonormal basis from the overcomplete $|i\rangle$ basis we write

$$|v_\gamma\rangle = G_{ji}^{-1/2} U_{i\gamma} |j\rangle, \tag{2.4}$$

where U is a unitary transformation such that $G = UDU^\dagger$ with D diagonal (details in appendix A of [6, 7]). Here and in what follows repeated indices are summed. Following [6, 15], we define the inverse of the Gram matrix via analytic continuation of G_{ij}^n at positive integer n

$$G_{ij}^{-1} \equiv \lim_{n \rightarrow -1} G_{ij}^n. \tag{2.5}$$

This procedure inverts nonzero eigenvalues while preserving zero eigenvalues. The trace over \mathcal{H}_κ is independent of U :

$$\text{Tr}_{\mathcal{H}_\kappa}(O) = G_{ij}^{-1} \langle j|O|i\rangle. \tag{2.6}$$

Thus, even if $\{|i\rangle\}$ is overcomplete we can use these states to compute operator traces. The resolution of the identity on \mathcal{H}_κ becomes

$$\mathbb{1}_{\mathcal{H}_\kappa} = \sum_\gamma |v_\gamma\rangle\langle v_\gamma| = G_{ij}^{-1} |i\rangle\langle j|. \tag{2.7}$$

Now suppose that we want to show that two quantities \mathcal{A} and \mathcal{B} are equal in the fine-grained theory, but only have access to the gravitational path integral. Since the latter

computes coarse grained averages $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ as discussed above, it is not enough to show that $\bar{\mathcal{A}} - \bar{\mathcal{B}} = 0$. We have to additionally show that $\overline{(\mathcal{A} - \mathcal{B})^2} = 0$, as this means that $\mathcal{A} - \mathcal{B} = 0$ for all the fine-grained configurations that are being averaged. Explicit computation of the relevant path integrals is generally intractable, except in the saddlepoint approximation. But, as pointed out in [7], if we only want to show that $\mathcal{A} = \mathcal{B}$ without explicitly calculating either quantity, then it suffices to show that each contribution to \mathcal{A} is matched by an equal contribution to \mathcal{B} and vice versa. Following [7] we can achieve this in three steps: **(1)** Construct an overcomplete basis of states by inserting operators into the Euclidean path integral; **(2)** Take a limit in which the basis is infinitely overcomplete, leading to drastic simplifications of the topologies contributing to amplitudes; **(3)** Demonstrate an action-preserving bijection between geometries contributing to the path integrals of interest after this topological simplification. This task can be simplified by insertions of the identity (2.7). The authors of [7] used the two-sided shell states of [3, 4, 10, 22] (details in appendix A and [23]) to demonstrate this procedure for universes with two boundaries, and showed these states indeed form an (over-) complete basis for this space. We will extend this approach to universes with a single boundary.

2.2 The factorisation puzzle

We can build states for gravity with two boundaries by cutting open the boundary conditions for Euclidean gravity path integral (perhaps with operators inserted on the boundary) along a cut with two disconnected components $\mathcal{X} = \mathcal{B}_L \cup \mathcal{B}_R$. The latter analytically continue to two disconnected components of the Lorentzian asymptotic boundary. We will call the full two-sided Hilbert space \mathcal{H}_{LUR} . States in \mathcal{H}_{LUR} come in two categories. First, we could cut open a *single* connected closed boundary manifold, as in figure 2(a). Let us call the span of these states \mathcal{H}_{LR} . Second, we could cut open two disconnected closed boundary manifolds, as in figure 2(b). Since these disconnected components specify separate path integrals for single-boundary gravity theories, this construction prepares states in a tensor product Hilbert space $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$. For states in \mathcal{H}_{LR} there can exist bulk geometries contributing to the path integral in which the two components of the cut are connected through the bulk. Moreover, as we are also summing over topologies in the path integral, this can also be the case for geometries contributing to overlaps in $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$.

Generically, we might expect that the full Hilbert space would be the direct sum $\mathcal{H}_{LR} \oplus \mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$ or perhaps $\mathcal{H}_{LUR} = \{u+v \mid u \in \mathcal{H}_{LR}, v \in \mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}\}$. But holography with AdS boundary conditions requires that the full Hilbert space with two disconnected Lorentzian boundaries has a tensor product structure because the dual CFT Hilbert space must.

This leads to a Hilbert space factorization puzzle: *show the fine-grained, all-order equality in figure 3:*

$$\mathcal{H}_{LUR} = \mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R} = \mathcal{H}_{LR}. \tag{2.8}$$

The factorization puzzle has sometimes been posed in Lorentzian signature by asking how we can split the effective field theory Hilbert space in, say, the two-boundary AdS black hole background into factors as expected from AdS/CFT for the complete Hilbert space. However, this formulation is misleading: the code subspace around a single entangled state in a large

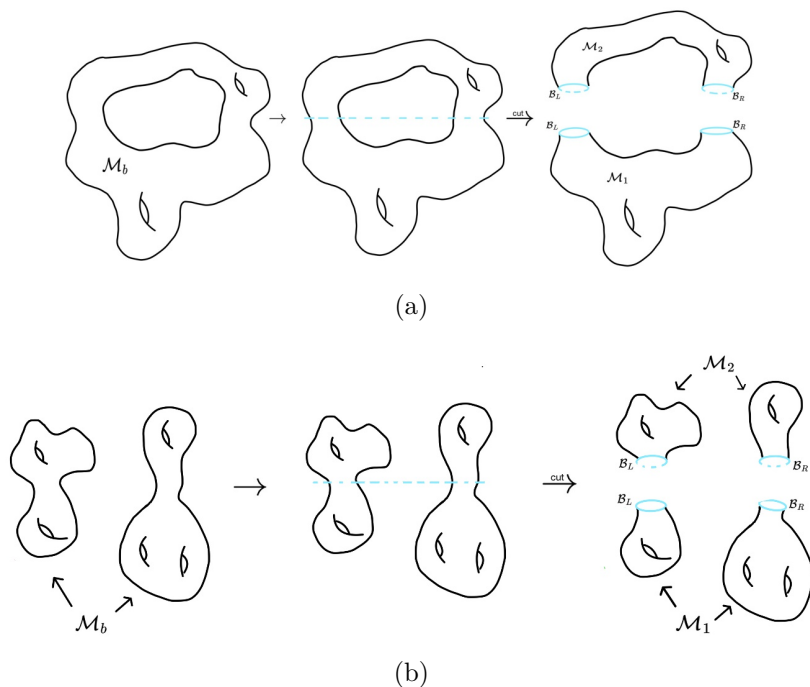


Figure 2. Two distinct ways of preparing states in $\mathcal{H}_{L \cup R}$. (a) Construction of an example state in \mathcal{H}_{LR} . (b) Construction of an example state in $\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$. Note that here we are drawing the *boundary conditions* of the Euclidean geometry. The bulk geometry between the boundary components $B_{L,R}$ may be either be connected (like a spatial section of the eternal black hole) or disconnected (like a product of two thermal AdS geometries).

factorized Hilbert space need not itself factorize. Likewise the effective field theory Hilbert space in the two-sided black hole background need not factorize. Rather, AdS/CFT says that (2.8) should be true in the full, non-perturbative theory, and this is what we seek to show.

There have been some earlier steps towards showing (2.8). In two dimensional JT gravity, [15] showed that traces in the two-sided Hilbert space are a product of factors to all orders in e^{-1/G_N} , and [6] extended the argument to any dimension in the leading saddle approximation. Likewise, [7] argued that the fine-grained thermal partition function of \mathcal{H}_{LR} factorises exactly:

$$\text{Tr}_{\mathcal{H}_{LR}}(e^{-H_L \beta_1} e^{-H_R \beta_2}) = Z(\beta_1) \times Z(\beta_2) \quad (2.9)$$

for asymptotically flat or AdS gravity in any dimension. As H_L, H_R commute, it follows from (2.9) that the two-sided Hilbert space \mathcal{H}_{LR} has a tensor product representation (see section 3 in [6]). More generally, [18] used operator algebras on the Hilbert spaces formed by cutting open a generalized notion of a gravitational path integral, and showed that \mathcal{H}_{LR} must be a product of factors.¹ But to resolve the factorization problem we have to additionally show that the two factors are actually the non-perturbative single-boundary Hilbert spaces, as we will do below. This does not follow simply from algebraic considerations arising from the axioms in [18], and is therefore a non-trivial constraint on the gravity theory.

¹More generally a sum over products of factors $\mathcal{H}_{LR} = \bigoplus_{\mu} \mathcal{H}_{LR}^{\mu,L} \otimes \mathcal{H}_{LR}^{\mu,R}$.

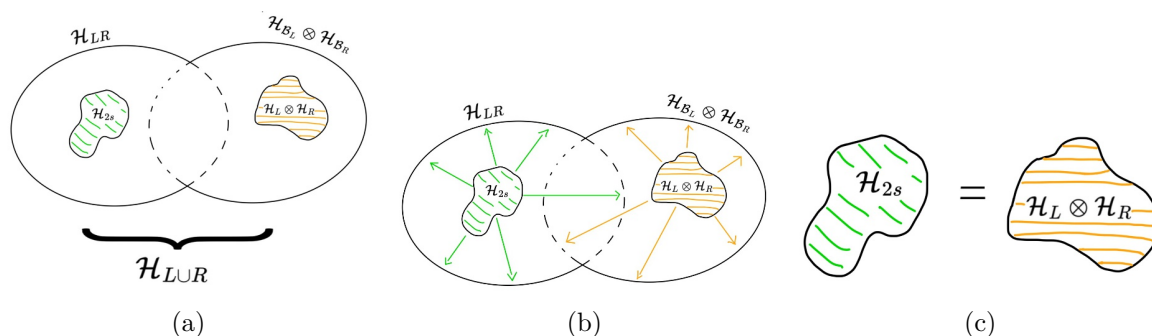


Figure 3. Resolving the factorization problem. (a) The two-sided Hilbert space \mathcal{H}_{LR} , the tensor product Hilbert space $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$, and the total two-sided Hilbert space \mathcal{H}_{LUR} . The shaded regions labeled \mathcal{H}_{2s} and $\mathcal{H}_L \otimes \mathcal{H}_R$ represent the spans of a set of two-sided “shell states” and a set of tensor products of single-sided “shell states” respectively, both of which are constructed from the Euclidean path integral. See [3, 4, 10, 22] for 2-boundary shell states and section 3 for 1-boundary shell states. (b) We will argue that the two- and single-sided shell states span \mathcal{H}_{LR} and $\mathcal{H}_{\mathcal{B}_L, \mathcal{B}_R}$ respectively when we take the number of shell states to infinity. The two-sided shell state span was shown in [7]. (c) We will then show that the two-sided shell states spanning \mathcal{H}_{LR} also span $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$ and vice versa.

There is also another sort of factorization puzzle. Suppose we use the Euclidean gravity path integral to calculate some amplitude or correlation \mathcal{A} . Then the path integral for $\mathcal{A}\mathcal{A}$ may not equal $(\mathcal{A})^2$ because of wormhole contributions. This failure of factorization is usually attributed to coarse-graining of the ultraviolet theory implicit in doing a path integral just over metrics and geometries. Despite this coarse graining there are techniques for using the gravitational path integral to show fine-grained equalities [7, 15]. We will use these techniques to show that it is precisely the wormholes that are responsible for non-factorization of coarse-grained amplitudes that lead reveal the factorization of the fine-grained two-boundary Hilbert space. A similar observation was made in [15].

3 Single-sided shell states

To derive the factorisation (2.8) we require a complete basis for the single-boundary Hilbert space $\mathcal{H}_{\mathcal{B}}$. We start by arguing that a set of $\kappa \rightarrow \infty$ single-boundary shell states (which we will refer to as $\mathcal{H}_L, \mathcal{H}_R$ for L, R copies of this set) span the full single-sided Hilbert space $\mathcal{H}_{L,R} = \mathcal{H}_{\mathcal{B}_L, \mathcal{B}_R}$ and therefore $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R} = \mathcal{H}_L \otimes \mathcal{H}_R$, see figure 3(b). In particular, we will construct a basis for the single-sided Hilbert space consisting of states obtained by a cutting open boundary manifolds with topology $\mathbb{R} \times \mathbb{S}^{d-1}$, so that the cut \mathcal{B} has topology \mathbb{S}^{d-1} . In the remainder of this paper we simply denote this Hilbert space as $\mathcal{H}_{\mathcal{B}}$. If we consider two copies of this space we will label them $\mathcal{H}_{\mathcal{B}_L}$ and $\mathcal{H}_{\mathcal{B}_R}$.

3.1 Construction

Our candidate basis states for $\mathcal{H}_{\mathcal{B}}$ are the single-sided thin matter shell states discussed in [14]. To define these states we fix a half-boundary with topology $\mathbb{R}^{<0} \times \mathbb{S}^{d-1}$ (where $\mathbb{R}^{<0}$ is the half line) and insert an \mathbb{S}^{d-1} symmetric heavy ($m \sim O(1/G_N)$) dust shell operator \mathcal{O}_S separated by a boundary length $\frac{\beta}{2}$ from the cut \mathcal{B} . The cut-open path integral boundary



Figure 4. Cut-open path integral boundary condition defining the single-sided shell states. (a) Euclidean boundary with topology $\mathbb{R}^{<0} \times \mathbb{S}^{d-1}$ for preparation of the shell states. In AdS/CFT we can also perform the path integral in the boundary CFT with insertion of a \mathbb{S}^{d-1} symmetric operator dual to the shell. The shell operator \mathcal{O}_i is pictured in red and $\beta/2$ is the Euclidean “preparation time”. (b) Euclidean boundary with the \mathbb{S}^{d-1} suppressed. We adopt this convention for the rest of the paper, and sometimes depict this boundary with a curve or a kink to clarify diagrams.

condition defining such states thus has a Euclidean boundary with time τ extending from $-\infty$ to $\tau = 0$, followed by insertion of \mathcal{O}_S at $\tau = 0$, and finally another Euclidean time section up to $\tau = \beta/2$. This boundary condition is depicted in figure 4 where the arrows denote a half infinite Euclidean time segment. We can obtain an infinite family of shell states in this way by varying the mass m_i of the shell operators \mathcal{O}_i to produce states $|i\rangle$. We will see below that the Lorentzian continuation of states prepared in this way will either be a black hole with a shell behind the horizon or a thermal geometry along with an additional closed universe. In either case the preparation time parameter β is related to temperature of the noncompact part of the Lorentzian geometry.

Our construction will apply to both asymptotically flat and AdS universes. But for concreteness we will focus on the asymptotically AdS case, where there is a dual CFT defined on the boundary with geometry $\mathbb{S}^{d-1} \times \mathbb{R}$. Although our analysis is formulated entirely in the bulk and remains agnostic about any specific boundary dual, it is instructive to briefly review how these states are constructed on the CFT side. From the CFT perspective the shell states are defined by preparing the vacuum state, acting with an operator \mathcal{O}_S and subsequently evolving with the CFT Hamiltonian H_{CFT} for a Euclidean time $\frac{\beta}{2}$:

$$|\Psi_S\rangle = e^{-\frac{\beta}{2} H_{\text{CFT}}} \mathcal{O}_S |0\rangle. \tag{3.1}$$

The shell states are created by operators \mathcal{O}_S that are holographically dual to heavy spherically-symmetric dust shells inserted at the asymptotic boundary of Euclidean AdS_{d+1} . Such operators are constructed in the CFT by taking products of n scalar operators \mathcal{O}_Δ of dimension $\Delta \sim O(1)$. To make sufficiently heavy states with $m \sim O(1/G_N)$ we will need to insert $n \sim l^{d-1}/G$ operators at spherically symmetric points on the \mathbb{S}^{d-1} , where l is the AdS radius.²

Many path integrals of interest in this paper will be subject to boundary conditions given by the Euclidean half line $[-\infty, 0]$ followed by a series of operator insertions and Euclidean

²See [24, 25] for recent discussion regarding the existence of the large N limit of these states in the CFT dual.

time evolutions up to some time T_E and finally another infinite segment half line $[T_E, \infty]$. Formally this should be thought of the limit of time evolutions $[-\alpha, 0]$ and $[T_E, \alpha + T_E]$ where $\alpha \rightarrow \infty$. We will refer to the time interval T_E as the “length of the strip” and denote such an asymptotic boundary condition as $S(T_E)$. For example, the difference in length of the asymptotic boundaries $S(T_E)$ and $S(0)$ is given by $\lim_{\alpha \rightarrow \infty} 2\alpha + T_E - 2\alpha = T_E$.

We will want to compute the Gram matrix $G_{ij} \equiv \langle i|j \rangle$ of overlaps between shell states. The ket in this overlap is constructed by the path integral in figure 4a, while the bra is defined by a similar path integral on the time interval $0 < \tau < \infty$. The boundary condition defining G_{ij} consists of a boundary time interval $[-\infty, 0]$ followed \mathcal{O}_i insertion, followed by β_R boundary time evolution, followed by an \mathcal{O}_j^\dagger insertion and another infinite segment half line $[\beta, \infty]$ (figure 5). Using the above terminology this boundary condition is therefore a strip of length β separating the \mathcal{O}_j and \mathcal{O}_i^\dagger operator insertions, and we will refer to such boundary conditions as *shell strip* boundaries.

The coarse-grained overlap $\overline{\langle i|j \rangle}$ is evaluated by the path integral with the shell strip boundary condition (figure 6). Following [3], shell states with different mass can be made orthogonal by taking their inertial mass differences to be arbitrarily large. This is because we expect that it will take $|m_i - m_j|$ bulk interactions in Planck units for such shells to annihilate, leading to $\overline{\langle i|j \rangle} = \delta_{ij} Z_1$. Later we will show that wormholes in the path integral that are stabilized by the shell matter can modify the overlap to $|\overline{\langle i|j \rangle}|^2 = \overline{\langle i|j \rangle} \langle j|i \rangle = Z_2 + \delta_{ij} Z_1^2$, where Z_2 is contribution from the wormhole saddlepoint (figure 7(a)). These non-perturbative corrections are of order e^{-1/G_N} . We therefore pick shell labels $\{m_i\}$ with spacings $|m_i - m_j|$ large enough that (i) any perturbative overlap suppression is subleading to e^{-1/G_N} , and (ii) each shell’s local energy density ρ , curvature invariants R , $R_{\mu\nu}R^{\mu\nu}$, and typical quantum wavelengths satisfy $\rho^{1/4}$, $R^{1/2}$, $\lambda^{-1} \ll M_{\text{Pl}}$. With these constraints the semiclassical saddle analysis and the resulting structure of the Gram matrix remain within the validity of the gravitational EFT.

3.2 Norm and wormhole saddles

The path integral for the overlap $\overline{\langle i|j \rangle} = \delta_{ij} Z_1$ can be evaluated in the saddlepoint approximation. Saddles satisfying the shell strip boundary condition are obtained by filling in one side of the shell-worldvolume with a saddle geometry of the Z partition function (defined in section 2.1) and the other with the Euclidean vacuum geometry with non-compact Euclidean time (which we refer to as the “strip”) respectively. This procedure can be done for any Z partition saddle — we refer to any such saddle as a “disk”. These two geometries are glued together along the trajectory of the shell using the Israel junction conditions [26], see figure 6. In particular, we consider a Euclidean AdS strip with a shell operator \mathcal{O}_i on the boundary at $\tau = 0$ and the conjugate operator \mathcal{O}_i^\dagger inserted at time $\tau = \Delta T_S$. Similarly consider a Euclidean AdS disk with boundary insertions \mathcal{O}_i and \mathcal{O}_i^\dagger separated by a time ΔT_D on a disk of boundary length $\beta + \Delta T_D$. The saddle geometry is then constructed by discarding the shell homology regions (purple regions in figure 6) from the strip and disk and gluing together the shell world-volumes across the disk and strip via the junction conditions, which dynamically determine ΔT_S and ΔT_D such that the resulting geometry satisfies the equations of motion.

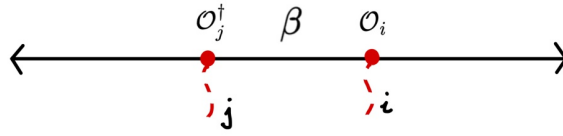


Figure 5. Shell-strip asymptotic boundary condition for the overlap $\langle j|i \rangle$ consisting of the line $\lim_{\alpha \rightarrow \infty} [-\alpha, \beta + \alpha]$ on which \mathcal{O}_i and \mathcal{O}_j^\dagger are inserted at $\tau = 0$ and $\tau = \beta$ respectively.

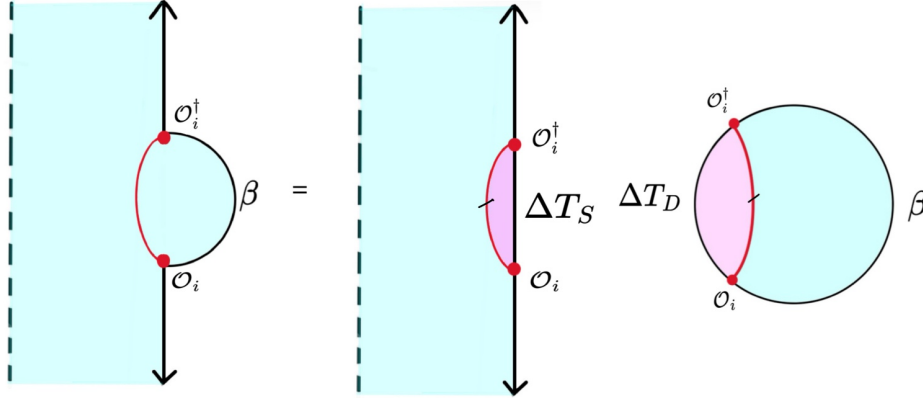


Figure 6. The saddle for the norm $\overline{\langle i|i \rangle}$ is constructed by considering the shell propagating on a disk and strip separately for some propagation times $\Delta T_{S,D}$ and then gluing them together along the i -shell worldvolume by discarding the shell homology region (purple). The junction conditions dynamically determine $\Delta T_{S,D}$ to yield an on shell glued geometry. We have suppressed the angular directions in these diagrams, and represent the radius-time plane with the dashed line representing the origin.

Large shell mass limit. The junction conditions for the single-sided shell states can be obtained from those for the two-sided shell states [3, 7] by taking one of the two-sided preparation temperatures, say β_L , to infinity. The two sided-shell states are reviewed in appendix A and, as discussed there, in the heavy shell mass ($m_i \rightarrow \infty$) limit the propagation times $\Delta T_{1,2}$ on the L and R disks go to zero, and the shell homology regions pinch off so that each shell contributes a factor $Z_{m_i} \sim e^{-2(d-1)\log(G_N m_i)}$ that is a universal function of its mass. In [3, 4, 22] this behavior was shown to be independent of which saddle of the Z -partition function is glued along the shell. Furthermore, as this result is independent of β_L, β_R and the junction conditions are local, by considering $\beta_L \rightarrow \infty$ these results carry over to the single-sided shell state construction.³

Hence, in the $m_i \rightarrow \infty$ limit the bulk turning point of the shell approaches the asymptotic boundary and $\Delta T_S, \Delta T_D \rightarrow 0$. Therefore the shell homology regions pinch off and the remainder grows to cover the entire strip of length⁴ zero and disk of length β , and the shell contribution a universal factor Z_{m_i} . Therefore the overlap is simply given by:

$$\overline{\langle i|j \rangle} = \delta_{ij} Z_{m_i} \times \overline{Z(\beta)} \times \overline{S}(0) \quad (3.2)$$

³The solution to the junction conditions for the single-sided shell states can be found in the appendix of [14], from which this expected behavior is confirmed.

⁴This is “renormalized” strip length as defined in section 3.1.

where $\overline{Z(\beta)}$ the Z partition function and $\overline{S(0)}$ the zero length strip path integral.⁵ As explained in detail in [7] and appendix B, the form of (3.2) is expected to remain valid in the heavy shell mass limit even after perturbative bulk corrections are included. In this regime, corrections to the l.h.s. simply map onto corresponding corrections to the path integrals $\overline{Z(\beta)}$ and $\overline{S(0)}$ appearing on the r.h.s. This expectation was verified explicitly in [27] by embedding the shell construction into a doubly holographic model. The same reasoning applies to all other factorized expressions of the form (3.2) that appear in the remainder of this work.

Wormholes. Generalizing this construction, we construct wormhole saddlepoints contributing to the amplitude $\overline{(G^n)_{ii}} = \overline{\langle i|j_1\rangle\langle j_1|j_2\rangle\cdots\langle j_n|i\rangle}$ (no sum on i, j_k) by connecting n asymptotic boundaries (arising from each factor in $(G^n)_{ii}$) with a single multiboundary wormhole. The k -th shell can only propagate through the bulk of the wormhole from \mathcal{O}_k to \mathcal{O}_k^\dagger on different asymptotic boundaries, and hence the wormhole connectivity reflects the index structure of $\overline{(G^n)_{ii}}$ (figure 7(a)). The wormhole saddle is constructed analogously to the norm by gluing together n copies of the Euclidean strip into a disk along the shell world-volumes (figure 7(b)). In particular, the saddle geometries satisfying this boundary condition can be constructed for a given n by starting with a disk with n shells inserted on the asymptotic boundary. The shell operator \mathcal{O}_{S_i} is separated from $\mathcal{O}_{S_i}^\dagger$ time a boundary time $\Delta T_{D,i}$, and $\mathcal{O}_{S_i}^\dagger$ from $\mathcal{O}_{S_{i+1}}$ by a boundary time β in a circular pattern. For each shell, a separate shell-strip is also constructed by placing a shell \mathcal{O}_{S_i} on the strip boundary separated from $\mathcal{O}_{S_i}^\dagger$ by $\Delta T_{D,i}$ boundary time. The saddle satisfying the boundary consideration is then constructed by cutting off the shell homology regions on the disk and strips and gluing each strip to the disk by identifying corresponding shell worldvolumes using the junction conditions.

In the large shell-mass limit the shell propagation times in the strips and disks limit to zero and hence the saddle is given by the product of the n strips, the disk of length $n\beta$ and the universal shell contributions $\overline{G^n_{ii}} = \overline{Z(n\beta)} \times \left(\overline{S(0)}\right)^n \times \prod_{i=1}^n Z_{m_i}$. After normalising the shell states to have unit norm ($|k\rangle \rightarrow |k\rangle/\sqrt{\overline{Z_1}}$), the latter two factors cancel out altogether and we are left with

$$\overline{G^n_{ii}} = \frac{\overline{Z(n\beta)}}{\overline{Z(\beta)}^n}. \tag{3.3}$$

3.3 Saddle geometry

We can associate a geometry to a shell state by considering the leading gravitational saddlepoint contributing to the norm $\overline{\langle i|i\rangle}$. In the large shell mass limit resulting in (3.2), the leading saddle for the overlap is thus given by the leading saddle for the Z-partition function, $\overline{Z(\beta)}$, which is either thermal Euclidean AdS if $\beta > \beta_{\text{HP}}$ or the large Euclidean AdS Black Hole if $\beta < \beta_{\text{HP}}$, where β_{HP} denotes the inverse temperature for the Hawking-Page transition [28]. The time reflection symmetric bulk slice of these saddles can be analytically continued to Lorentzian signature to associate a Lorentzian geometry to the state. There are thus two classes of single sided shell states:

⁵The regularization of the strip will drop out of all calculations in this paper by normalising the shell states, so we will not specify it.

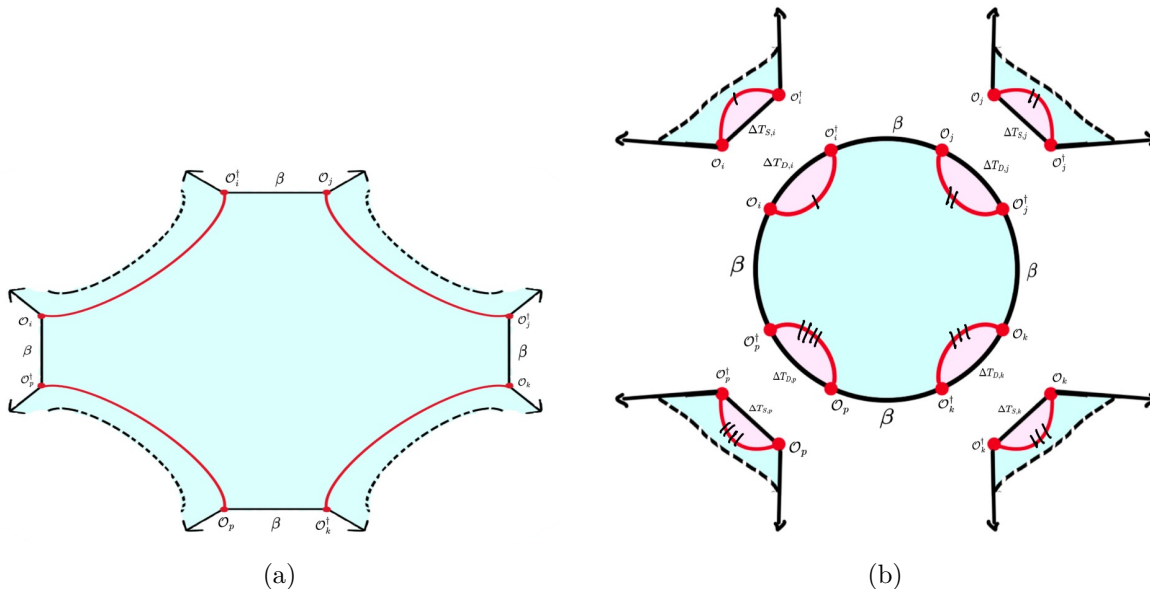


Figure 7. The fully connected single-sided shell wormholes saddles for $\overline{(G^n)_{ii}}$ are constructed by gluing n strips into the disk along the shell worldvolumes, depicted here for $n = 4$.

1. $\beta < \beta_{\text{HP}}$: the leading saddle for the disk portion will be the Euclidean AdS black hole. In this case the Lorentzian geometry corresponds to an AdS-Schwarzschild black hole with the shell propagating in the interior which is capped off by empty AdS behind the shell. We will refer to these states as *type A* single-sided shell states.
2. $\beta > \beta_{\text{HP}}$: the leading saddle for the disk portion will be thermal AdS. In this case the Lorentzian geometry corresponds to thermal AdS at temperature β in addition to a disconnected AdS closed “Big Crunch” universe in which the shell propagates. We will refer to these states as *type B* single-sided shell states.

In what follows we will denote a single-sided shell state of mass m_i and preparation temperature β_R as $|i\rangle_R$, and the Hilbert space spanned by κ_R such states of varying mass as \mathcal{H}_R .

3.4 A complete basis

We now use methods developed in [7] to establish that the single-sided shell states span the full single-sided Hilbert space \mathcal{H}_B by showing that the projector onto \mathcal{H}_R acts as the identity on \mathcal{H}_B at the fine-grained level:

$$\mathbb{1}_{\mathcal{H}_B} = \prod_{\mathcal{H}_R} \equiv G_{ij}^{-1} |i\rangle \langle j| \tag{3.4}$$

for any shell preparation temperature β_R . We derive this equality from the gravitational path integral by showing the two path integral equalities

$$\overline{\langle \Psi | \prod_{\mathcal{H}_R} | \Psi \rangle} = \overline{\langle \Psi | \Psi \rangle} \tag{3.5}$$

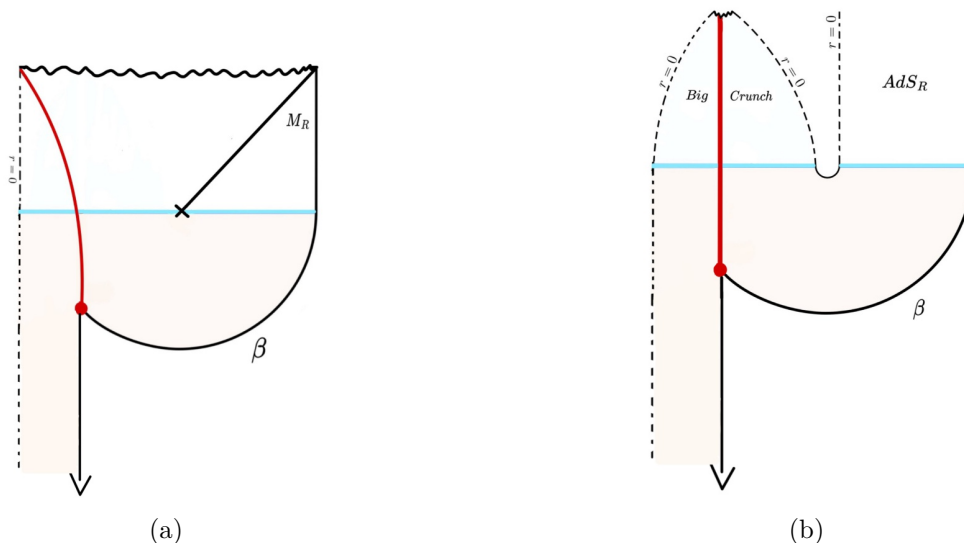


Figure 8. Diagrams showing the analytic continuation of the type A-B single-sided shell states to the Lorentzian signature Penrose diagram. (a) Type A shell state with $\beta < \beta_{\text{HP}}$ corresponding to a single-sided black hole. (b) Type B shell state with $\beta > \beta_{\text{HP}}$ and consisting of thermal AdS with the addition of a disconnected compact Big-Crunch AdS cosmology.

and

$$\overline{(\langle \Psi | \prod_{\mathcal{H}_R} |\Psi\rangle - \langle \Psi | \Psi \rangle)^2} = 0. \tag{3.6}$$

for $\forall |\Psi\rangle \in \mathcal{H}_B$. As we discussed in section 2.1, these equations together imply the fine-grained equality $\langle \Psi | \Psi \rangle = \langle \Psi | \prod_{\mathcal{H}_R} |\Psi\rangle$. As the fine-grained inner product must satisfy the Cauchy-Schwartz identity, it then follows that $\mathcal{H}_R = \mathcal{H}_B$.

By definition \mathcal{H}_B is the space of states made by cutting open the path integral. Therefore all states $|\Psi\rangle \in \mathcal{H}_B$ can be written as superpositions of states constructed by boundary time evolution over the half infinite strip followed by insertion of some (possibly non-local) operator \mathcal{O}_Ψ on the asymptotic boundary followed by $\tilde{\beta}_R$ boundary time evolution. In particular we denote such states as $|\Psi(\tilde{\beta}_R)\rangle = e^{-\frac{\tilde{\beta}_R H_R}{2}} \mathcal{O}_\Psi |0\rangle$ and define

$$\langle \Psi | \Psi \rangle \equiv S_{\mathcal{O}_\Psi}(\tilde{\beta}_R), \tag{3.7}$$

where the vacuum state $|0\rangle$ should be understood as representing the half line boundary condition⁶ and \mathcal{O}_Ψ may include boundary time evolution. This does not restrict the states under consideration but rather sets up our notation.

The path integral (3.5) is $\overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi\rangle} = \lim_{n \rightarrow -1} \overline{G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle}$ where each shell overlap insertion $\langle i | j \rangle$ corresponds to a shell strip boundary condition (figure 5). Similarly, the boundary condition for the $\langle \Psi | i \rangle$ insertion consists of a boundary time interval $[-\infty, 0]$, an \mathcal{O}_i insertion at $\tau = 0$, followed by an \mathcal{O}_Ψ^\dagger insertion at $\tau = \frac{\beta_R + \tilde{\beta}_R}{2}$ and finally another infinite segment half line $[\frac{\beta_R + \tilde{\beta}_R}{2}, \infty]$.⁷ We refer to these as *operator strip* boundaries.

⁶In holographic theories the bulk-boundary dictionary maps this to the path integral for the CFT vacuum.

⁷As noted above this should be understood as the boundary condition $\lim_{\alpha \rightarrow \infty} [\frac{\beta_R + \tilde{\beta}_R}{2}, \frac{\beta_R + \tilde{\beta}_R}{2} + \alpha]$.

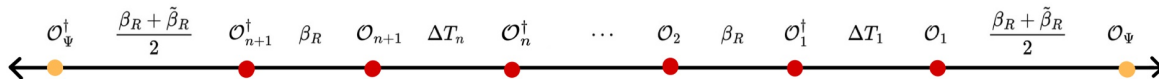


Figure 9. Asymptotic boundary condition for the operator-strip wormhole into which $n + 1$ shell strips are glued, forming the connected wormhole contribution to (3.8).

As the full single-sided Hilbert space \mathcal{H}_B is expected to be infinite dimensional, we need an infinite number of basis vectors. Hence we take \mathcal{H}_R to be the span of $\kappa_R \rightarrow \infty$ number of single-sided shell states obtained by fixing the preparation temperature β_R but varying the mass. As explained in [7], in this limit only geometries that do not break any shell index loops contribute to the sum over topologies.

Coarse-grained completeness. We start by establishing the coarse-grained equality (3.5), which is computed as $\overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle} = \lim_{n \rightarrow -1} \overline{G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle}$, within the saddle point approximation. Following [7], as we take $\kappa_R \rightarrow \infty$ we will use the inverse of the Gram matrix G^{-1} in (3.4) to account for overlaps between the shell states in the resolution of the identity. This procedure also takes care of potential overcompleteness in the basis. The gravity path integral $\overline{G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle}$ contains only a single maximal shell index loop, and hence in the $\kappa_R \rightarrow \infty$ limit we only have to consider the maximally connected topology. These geometries connect the n shell boundaries and the two operator boundaries via a single bulk wormhole geometry (figure 10). We will refer to these wormholes as *strip wormholes*. In principle there can be multiple saddles of this kind depending on the properties of $|\Psi\rangle$, all of which will turn out to be independent of the shell mass index running in the loop. We will show below that such saddles always exist if $\overline{\langle \Psi | \Psi \rangle}$ has a saddle. Denoting the sum over these saddles as $\hat{S}_{n+2}(\Psi)$ we therefore obtain:

$$\overline{G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle} = \kappa^{n+1} \hat{S}_{n+2}(\Psi). \quad (3.8)$$

We proceed as follows. Time evolve the state $|\Psi\rangle$ by an amount $\frac{(n+1)\beta_R}{2}$. Consider saddlepoints contributing to the norm of this time evolved state, which in the notation of (3.7) computes $S_{\mathcal{O}_\Psi}(\tilde{\beta}_R + (n+1)\beta_R)$. From any such a saddle we can obtain a strip wormhole saddle and vice versa, as we will now show by construction. The strip wormhole saddles are constructed by gluing $n + 1$ shell strips into one central strip along the shell worldvolumes (see the right side of figure 10). This is in contrast with the wormholes for $\overline{G_{ii}^n}$ discussed in section 3.2 which involved gluing n shell strips into a central disk. The to-be-glued shell strip boundary condition for the i -th shell consists of the half line $[-\infty, 0]$, followed by $\mathcal{O}_{S_i}, \mathcal{O}_{S_i}^\dagger$ insertions separated by \tilde{T}_i and another half line. The central strip geometry has an asymptotic boundary condition consisting of the $[-\infty, 0]$ half line, a \mathcal{O}_Ψ operator insertion at $\tau = 0$, followed by the first shell insertion \mathcal{O}_{S_1} at $\tau = \frac{\tilde{\beta}_R + \beta_R}{2}$, and insertion of $\mathcal{O}_{S_1}^\dagger$ at time T_1 later. A repeating pattern then follows where every $\mathcal{O}_{S_j}^\dagger$ insertion is followed β_R time evolution, insertion of $\mathcal{O}_{S_{j+1}}$ and T_{j+1} time evolution, followed by $\mathcal{O}_{S_{j+1}}^\dagger$ insertion until $\mathcal{O}_{S_{n+1}}^\dagger$. This is further followed by a $\frac{\tilde{\beta}_R + \beta_R}{2}$ time evolution, \mathcal{O}_Ψ^\dagger and a half line time evolution, completing the strip boundary, see figure 9. We will denote the sum over saddlepoints for the path integral subject to this central strip boundary condition as $S_{\text{Central}}(\tilde{\beta}_R + \sum_i T_i + (n+1)\beta_R)$.

On both the shell- and central-strips the shells propagate into the bulk before being re-absorbed a time \tilde{T}_i, T_i later respectively. The strip wormhole saddle is obtained by cutting off the shell homology regions on all the strips and gluing the shell strips into the central strip along the corresponding shell worldvolumes via the junction conditions, which dynamically determine $\{T_i, \tilde{T}_i\}$ to yield an on-shell geometry (figure 10). In the large shell mass limit the propagation times $\{T_i, \tilde{T}_i\}$ go to zero, the shell homology regions on the operator and shell strips pinch off, and the effect of each shell is only to contribute a universal factor Z_{m_i} . Hence in this limit the remaining part of the saddle action is given by action of the $n + 1$ shell strips resulting in a factor $\overline{S}^{n+1}(0)$ and the action of the central strip. Note that neither of these contain a contribution from the shells, as the shell action is already accounted for. As the central strip now has no shell insertions and $T_i \rightarrow 0$ what remains is simply the action of the $\overline{S}_{\mathcal{O}_\Psi}((n+1)\beta_R + \tilde{\beta}_R)$ saddles⁸ defined above. We therefore obtain $\hat{S}_{n+2}(\Psi) = \prod_{i=1}^{n+1} Z_{m_i} \times \overline{S}^{n+1}(0) \times \overline{S}_{\mathcal{O}_\Psi}((n+1)\beta_R + \tilde{\beta}_R)$. After normalising the shell states to unity we get:

$$\hat{S}_{n+2}(\Psi) = \frac{\overline{S}_{\mathcal{O}_\Psi}((n+1)\beta_R + \tilde{\beta}_R)}{\overline{Z}(\beta_R)^{n+1}}. \tag{3.9}$$

The upshot is that given a saddle for the norm of a time-evolved version of the state $|\Psi\rangle$ given by $\overline{S}_{\mathcal{O}_\Psi}((n+1)\beta_R + \tilde{\beta}_R)$ the saddles for (3.8) can automatically be constructed in the large shell mass limit and are independent of the shell index running in the loop, because of the universal pinching-off behavior of the shell insertions in this limit. Taking $n \rightarrow -1$, we find

$$\overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle} = \lim_{n \rightarrow -1} \kappa^{n+1} \frac{\overline{S}_{\mathcal{O}_\Psi}((n+1)\beta_R + \tilde{\beta}_R)}{\overline{Z}(\beta_R)^{n+1}} = \overline{S}_{\mathcal{O}_\Psi}(\tilde{\beta}_R) \equiv \overline{\langle \Psi | \Psi \rangle}, \tag{3.10}$$

showing (3.5) within the saddlepoint approximation to the path integral.

Fine-grained completeness. Next, we need to also show (3.6) to obtain the fine-grained equality (3.4). If $\overline{\langle \Psi | \Psi \rangle^2} \neq \left(\overline{\langle \Psi | \Psi \rangle}\right)^2$ due to connected wormhole contribution to the l.h.s. supported by \mathcal{O}_Ψ matter insertions there will be multiple saddles of different topology for each term in (3.6). We consider the family of wormhole saddles $\overline{\langle \Psi | \Psi \rangle \langle \tilde{\Psi} | \tilde{\Psi} \rangle}_{\text{conn}}$ where $|\tilde{\Psi}\rangle$ is a state obtained by boundary time evolution of $|\Psi\rangle$ by an amount Δ_R ; we denote this as $\overline{\langle \Psi | \Psi \rangle \langle \tilde{\Psi} | \tilde{\Psi} \rangle}_{\text{conn}} \equiv \overline{S_{\Psi\tilde{\Psi}}(\Delta_R)}$. Similar to the discussion above, in the large shell mass limit any of the \mathcal{O}_Ψ matter supported wormhole saddles relevant for (3.6) can be obtained by gluing shell strips into the $S_{\Psi\tilde{\Psi}}(\Delta_R)$ wormhole for different values of Δ_R .

We are considering the $\kappa_R \rightarrow \infty$ limit in which the shell index loop cannot be broken. Taking, for example, the cross term in (3.6)

$$\overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle \langle \Psi | \Psi \rangle} = \lim_{n \rightarrow -1} \overline{G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle \langle \Psi | \Psi \rangle} \tag{3.11}$$

there can then be two types of saddle: saddles connecting the various \mathcal{O}_Ψ insertions across $\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle$ and $\langle \Psi | \Psi \rangle$, and ones that do not. For the latter we simply have

⁸We stress that again $\overline{S}_{\mathcal{O}_\Psi}((n+1)\beta_R + \tilde{\beta}_R)$ collectively denotes the sum over saddles to $\overline{\langle \Psi | \Psi \rangle}((n+1)\beta_R + \tilde{\beta}_R)$ and for each of these saddles individually the above construction follows through.

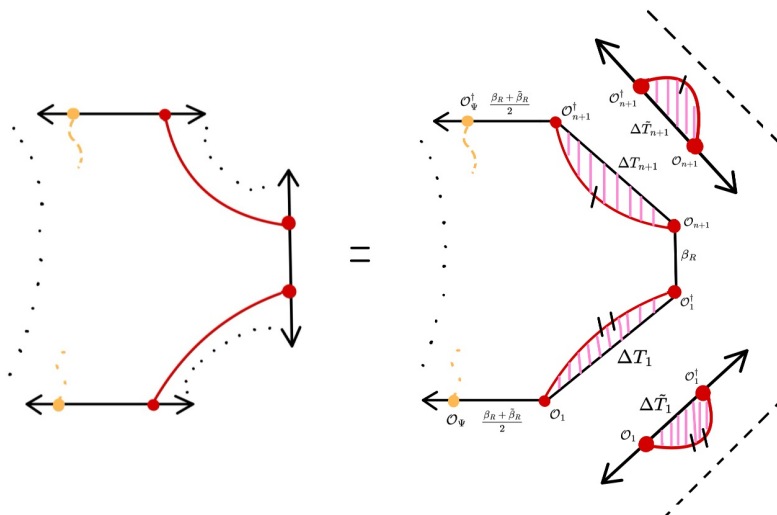


Figure 10. The fully connected wormhole saddle for (3.8) is constructed by gluing $n + 1$ shell strips into the central strip, depicted here for $n = 1$.

$\overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle \langle \Psi | \Psi \rangle}_{\text{disconn}} = \overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle} \times \overline{\langle \Psi | \Psi \rangle}$ which by (3.10) equals $(\overline{\langle \Psi | \Psi \rangle})^2$. The fully connected contributions can be constructed by starting with the wormhole connecting the operator boundary $\langle \Psi | \Psi \rangle$ to boundary of the central strip (figure 9), and again gluing in $n + 1$ shell strips along the shell worldvolumes (figure 11) into the central strip as above. In the large shell mass limit the shell homology regions pinch off and the shells contribute universally to the action. The action is therefore given by that of $n + 1$ strips of length zero in addition to the action of the wormhole without any shell insertions, which is simply the action of the $\overline{S_{\Psi\tilde{\Psi}}((n + 1)\beta_R)}$ saddles defined above. Hence, after normalising the shell states we obtain

$$\lim_{n \rightarrow -1} \overline{G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle \langle \Psi | \Psi \rangle}_{\text{conn}} = \lim_{n \rightarrow -1} \kappa^{n+1} \frac{\overline{S_{\Psi\tilde{\Psi}}((n + 1)\beta_R)}}{\overline{Z}(\beta_R)^{n+1}} = \overline{S_{\Psi\tilde{\Psi}}(0)} \equiv \overline{\langle \Psi | \Psi \rangle^2}_{\text{conn}}. \tag{3.12}$$

Hence the connected and disconnected saddles to (3.11) account for the connected and disconnected saddles to $\overline{\langle \Psi | \Psi \rangle \langle \Psi | \Psi \rangle}$, and we have $\overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle \langle \Psi | \Psi \rangle} = \overline{\langle \Psi | \Psi \rangle \langle \Psi | \Psi \rangle}$. A similar argument applies to the term $(\overline{\langle \Psi | \prod_{\mathcal{H}_R} |\Psi \rangle})^2 = \overline{\langle \Psi | \Psi \rangle \langle \Psi | \Psi \rangle}$, where now the fully-connected contributions are constructed by gluing $n + 1$ shells strips into each of two central strip boundaries connected by the \mathcal{O}_Ψ -supported wormhole. Therefore in the $n \rightarrow -1$ limit each term in (3.6) limits to $\overline{\langle \Psi | \Psi \rangle^2}$, proving (3.6).

Conclusion. Since we have shown both (3.5) and (3.6) we obtain the fine grained equality

$$\mathcal{H}_B = \mathcal{H}_R \tag{3.13}$$

in the saddlepoint approximation to the path integral. The relation holds if we take \mathcal{H}_R to be the span of $\kappa_R \rightarrow \infty$ shell states of fixed preparation temperature but varying mass. Interestingly, (3.13) holds for *any* choice of preparation temperature, since the derivations of (3.5) and (3.6) were both independent of the temperature used to prepare the shell states.⁹

⁹Moreover the above argument extends trivially to $\langle \Psi | \Phi \rangle = \langle \Psi | \prod_{\mathcal{H}_R} |\Phi \rangle + \mathcal{O}(e^{G_N^0})$ for any $|\Psi \rangle, |\Phi \rangle \in \mathcal{H}_B$.

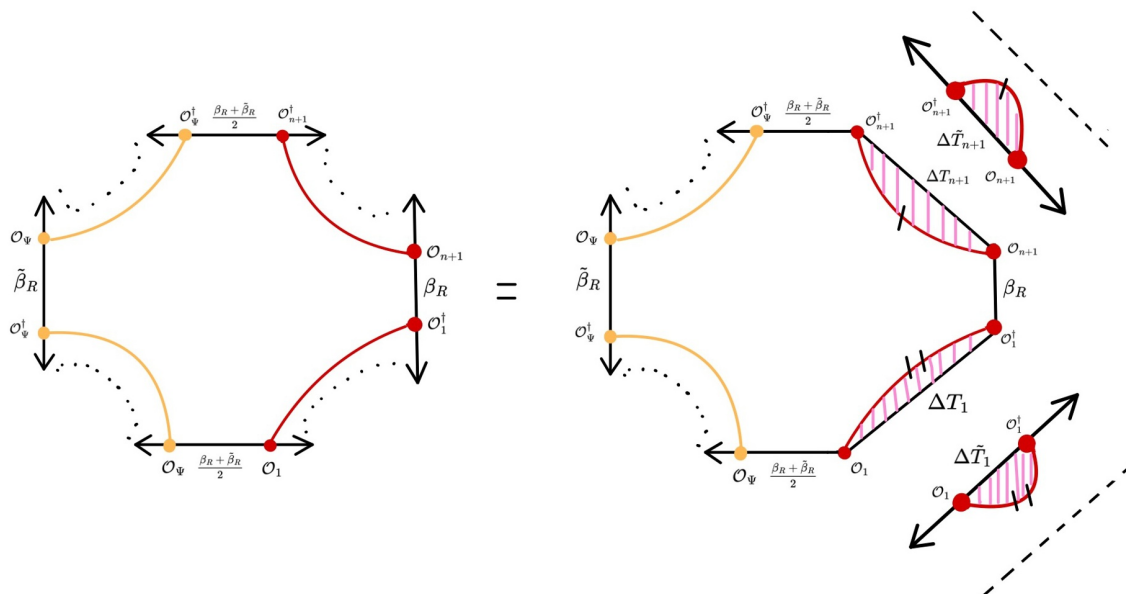


Figure 11. The additional fully connected wormhole contributions to (3.6) can be constructed analogously to those for (3.8) by gluing $n + 1$ shell strips into the central bulk.

We could therefore take this basis to consist of type A shell states, which contain a horizon, or type B shell states which do not but instead have a compact universe. As a $\kappa_R \rightarrow \infty$ set of either of these states spans, a *type A black hole geometry can be written as a superposition of horizonless type B geometries and vice versa*. In [7] we reached the same conclusion for the two-sided shell states and proceeded to argue that the notion of a fixed geometry associated to a state made by cutting the gravity path integral must be emergent in the semiclassical limit. Our results here show that the same observations apply to the single-boundary gravity theory. Finally, we can use the “surgery methods” of the toolkit proposed in [7] to extend the argument that *the shell states span \mathcal{H}_B to all orders in the fine-grained theory (see appendix B.1)*.

4 The two-sided Hilbert space is a product of single-sided Hilbert spaces

We showed in [7] that all states of quantum gravity with two boundaries constructed by the Euclidean gravity path integral with a single connected boundary \mathcal{H}_{LR} (figure 2(a)) are in the span \mathcal{H}_{2s} of the two-sided shell states reviewed in appendix A. Likewise, it follows from section 3 that the states of two-boundary quantum gravity constructed by cutting the Euclidean path integral with a boundary consisting of two closed disconnected components $\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$ (figure 2(b)) are in the span of $\mathcal{H}_L \otimes \mathcal{H}_R$ of products of the single-sided shell states. In this section we will show that $\mathcal{H}_{2s} = \mathcal{H}_L \otimes \mathcal{H}_R$, which in turn shows the factorisation of the full Hilbert spaces $\mathcal{H}_{LR} = \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$.

First define the L, R single-sided shells states with preparation temperatures $\beta_{L,R}$ respectively and the two-sided shell states with temperatures $\tilde{\beta}_{L,R}$. We denote a two-sided state defined by the shell operator insertion \mathcal{O}_i in boldface as $|\mathbf{i}\rangle_{2s}$, and L, R single-sided states defined by that same operator as $|i\rangle_{L,R}$. First we want to determine the normalization of the product state $|i\rangle_L \otimes |j\rangle_R$. There are two saddlepoint contributions — the product

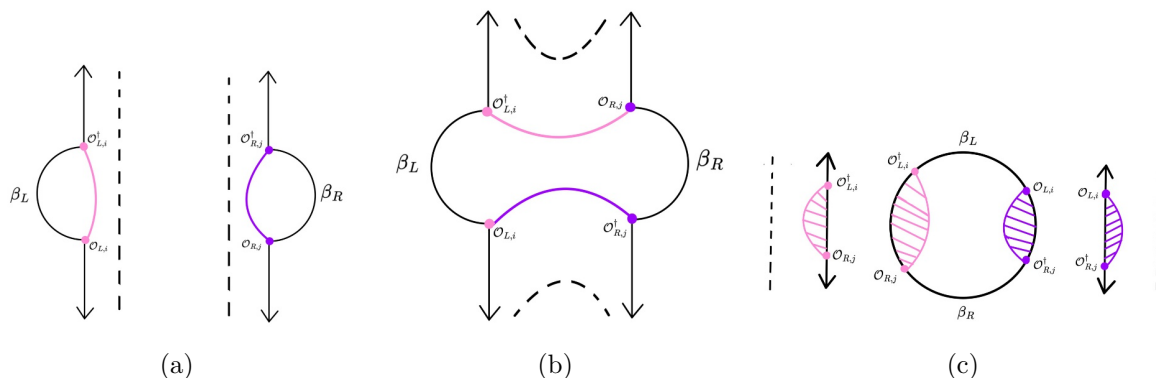


Figure 12. Saddle geometries for $\overline{\langle i|i \rangle_L \langle j|j \rangle_R}$: (a) Disconnected contribution corresponding to two copies the saddle (3.2). (b) Connected wormhole contribution present when $i = j$. (c) This wormhole contribution is constructed by gluing L, R strips into a disk.

$\overline{\langle i|i \rangle_L} \times \overline{\langle j|j \rangle_R}$ of the norms of the L and R states, and the wormhole shown in figure 12. The wormhole only contributes if the shell creating the L state can annihilate with the shell creating the R state, which is possible only if $m_i = m_j$ in view of the large inertial mass gaps between shells (see section 2.1). Thus we have:

$$\overline{\langle i|i \rangle_L \langle j|j \rangle_R} = \bar{Z}(\beta_L) \bar{Z}(\beta_R) \bar{S}(0)^2 + \delta_{ij} \bar{Z}(\beta_L + \beta_R) \bar{S}(0)^2, \quad (4.1)$$

where \bar{Z} denotes the Z -partition function (see section 2.1) and \bar{S} denotes the path integral with the strip boundary condition (see section 3). For simplicity will choose the bases for $\mathcal{H}_{L,R}$ to contain shells of different mass so that the wormhole contribution is absent.

Next we want to calculate the overlap between a product of one-sided states and a two-sided one: $\langle i|_L \langle j|_R | \mathbf{p} \rangle_{2s}$. The boundary condition for the path integral calculating this overlap consists of a half infinite line, followed by an $\mathcal{O}_{L,i}^\dagger$ insertion and time evolution by $\frac{\beta_L}{2}$ from the state $\langle i|_L$, followed by $\frac{\tilde{\beta}_L}{2}$ insertion of $\mathcal{O}_{2s,\mathbf{p}}$, and time evolution by $\frac{\tilde{\beta}_R}{2}$ from $|\mathbf{p} \rangle_{2s}$, and finally $\frac{\beta_R}{2}$ evolution and insertion of $\mathcal{O}_{R,j}^\dagger$ half infinite line coming from $\langle j|_R$ (see figure 13). Strictly speaking, constructing the saddle points for this boundary condition requires a detailed analysis of shell splitting and joining dynamics. However, if we choose the L, R and two-sided shell bases such that all the shell operators in any one of them have large mass differences, then these splitting interactions will be highly suppressed, as it takes $|m_{\mathbf{p}} - (m_i + m_j)|$ bulk interactions in Planck units to match the shells, resulting in very small overlap. Nevertheless, if we instead consider the square of the overlap $|\overline{\langle i|_L \langle j|_R | \mathbf{p} \rangle_{2s}}|^2$ we find a universal wormhole contribution constructed by gluing a L, R disk together along the $\mathcal{O}_{2s,\mathbf{p}}$ insertion and gluing a strip into each of disks along the insertions $\mathcal{O}_{L,j}^\dagger, \mathcal{O}_{R,i}^\dagger$ shells respectively as in figure 14. In the large shell mass limit, the shell homology regions again pinch off, and we are left with two disks and two strips. After normalising the states, this wormhole saddle is therefore given by:

$$|\overline{\langle i|_L \langle j|_R | \mathbf{p} \rangle_{2s}}|^2 = \frac{\bar{Z}(\beta_L + \tilde{\beta}_L) \bar{Z}(\beta_R + \tilde{\beta}_R)}{\bar{Z}(\beta_L) \bar{Z}(\tilde{\beta}_L) \bar{Z}(\beta_R) \bar{Z}(\tilde{\beta}_R)} \quad (4.2)$$

where the strip contributions have canceled out with normalizations.

$$\langle i|_L \langle j|_R | \mathbf{p} \rangle_{2s} = \leftarrow \begin{array}{c} \mathcal{O}_{L,i}^\dagger \quad \frac{\beta_L + \tilde{\beta}_L}{2} \quad \mathcal{O}_{2s,\mathbf{p}} \quad \frac{\beta_R + \tilde{\beta}_R}{2} \quad \mathcal{O}_{R,j}^\dagger \\ \text{---} \end{array} \rightarrow$$

Figure 13. Boundary condition for the overlap between a two-sided shell state and the tensor product of single-sided shell states: $\langle i|_L \langle j|_R | \mathbf{p} \rangle_{2s}$.

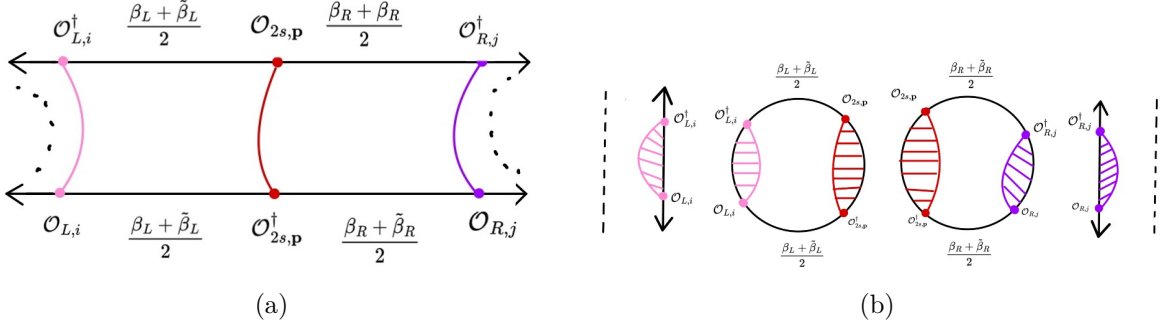


Figure 14. (a) Universal wormhole contribution to $|\overline{\langle i|_L \langle j|_R | \mathbf{p} \rangle_{2s}}|^2$. (b) This wormhole is constructed by gluing a L, R strip into the L, R disk and then gluing the disk together along the shell worldvolumes.

The non-zero value of (4.2) shows that $\mathcal{H}_L \otimes \mathcal{H}_R$ and \mathcal{H}_{2s} are not orthogonal. To show that they are equal we will establish below that $\mathcal{H}_L \otimes \mathcal{H}_R \subseteq \mathcal{H}_{2s}$ and $\mathcal{H}_{2s} \subseteq \mathcal{H}_L \otimes \mathcal{H}_R$ in the fine-grained theory by demonstrating that: (a) For any $|\Psi_\otimes\rangle \in \mathcal{H}_L \otimes \mathcal{H}_R$

$$\langle \Psi_\otimes | \prod_{\mathcal{H}_{2s}} |\Psi_\otimes\rangle = \langle \Psi_\otimes | \Psi_\otimes \rangle, \quad (4.3)$$

and (b) For any $|\Phi_{2s}\rangle \in \mathcal{H}_{2s}$

$$\langle \Phi_{2s} | \prod_{\mathcal{H}_L} \prod_{\mathcal{H}_R} |\Phi_{2s}\rangle = \langle \Phi_{2s} | \Phi_{2s} \rangle. \quad (4.4)$$

4.1 Two-sided shell states the span tensor product of one-sided shell states

We start with the case where $|\Psi_\otimes\rangle$ is a tensor product. To show (4.3) it suffices that

$$\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R = \langle i|i\rangle_L \langle j|j\rangle_R. \quad (4.5)$$

where $\prod_{\mathcal{H}_{2s}}$ is the projector onto the two-sided shell Hilbert space, because the product states $|i\rangle_L |j\rangle_R$ form a basis of $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$. Following [7] and section 2.1, the projector $\prod_{\mathcal{H}_{2s}}$ can be resolved in terms of the shell states as

$$\overline{\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R} = \lim_{n \rightarrow -1} \overline{G_{2s,kl}^n \langle i|_L \langle j|_R | \mathbf{k} \rangle_{2s} \langle 2s \mathbf{l} | i\rangle_L | j\rangle_R}. \quad (4.6)$$

where the Gram matrix G accounts for overcompleteness and non-orthogonality of the shell basis. The boundary condition for the path integral computing G_{2s}^n corresponds to the product of n two-sided shell state overlaps $\langle \mathbf{j} | \mathbf{i} \rangle$. Each overlap involves a periodic boundary with $\mathcal{O}_{2s,i}$

and $\mathcal{O}_{2s,j}^\dagger$ insertions separated by $\tilde{\beta}_L, \tilde{\beta}_R$ boundary time (see figure 23 and appendix A) while the $\langle j|_L \langle l|_R | \mathbf{k} \rangle_{2s}$ insertions corresponds to the mixed-shell boundary condition in figure 13.

To evaluate (4.6) we take a limit where the number of 2-sided shells included in the basis becomes infinite ($\kappa_{2s} \rightarrow \infty$). In this limit only geometries with a maximal number of boundary index loops contribute in path integral computations [7]. For (4.6) this means that the n two-shell boundaries and the two mixed-shell boundaries must be connected through a single bulk (figure 15(a)). Saddle points satisfying this boundary condition can be constructed by starting with L, R disks, each containing $n + 1$ two-sided shell insertions and one L, R single-sided shell insertion respectively. An L, R shell strip is glued into the L, R disk along the single-sided shell insertions, and the two disks are then glued together along the remaining $n + 1$ two-shell worldvolumes (figure 15(b)). In the large single- and two-sided shell mass limit the shell homology regions pinch off, so that the shells contribute universally to the action. The wormhole saddle is then: $\bar{S}_L(0) \times \bar{S}_R(0) \times \bar{Z}(\beta_L + (n + 1)\tilde{\beta}_L) \times \bar{Z}(\beta_R + (n + 1)\tilde{\beta}_R) \times Z_{m_i,L} \times Z_{m_j,R} \times \prod_{k=1}^{n+1} Z_{m_k,2s}$. After normalizing the two-sided shell states, this wormhole saddle is independent of the shell indices running in the loop and we therefore obtain $\overline{G_{2s,\mathbf{k}l}^n \langle i|_L \langle j|_R | \mathbf{k} \rangle_{2s} \langle 2s \mathbf{l} | i \rangle_L | j \rangle_R} =$

$$\kappa^{n+1} \frac{\bar{S}_L(0) \times \bar{S}_R(0) \times \bar{Z}(\beta_L + (n + 1)\tilde{\beta}_L) \times \bar{Z}(\beta_R + (n + 1)\tilde{\beta}_R) \times Z_{m_i,L} \times Z_{m_j,R}}{\bar{Z}(\tilde{\beta}_L)^{n+1} \bar{Z}(\tilde{\beta}_R)^{n+1}} \quad (4.7)$$

Finally, we take $n \rightarrow -1$ and find

$$\overline{\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R} = \bar{S}_L(0) \bar{Z}(\beta_L) Z_{m_i,L} \bar{S}_R(0) \bar{Z}(\beta_R) Z_{m_j,R} = \overline{\langle i|i\rangle_L \langle j|j\rangle_R} \quad (4.8)$$

showing that $\overline{\langle \Psi_\otimes | \prod_{\mathcal{H}_{2s}} | \Psi_\otimes \rangle} = \overline{\langle \Psi_\otimes | \Psi_\otimes \rangle}$ within the saddle approximation.

Fine grained. To extend this coarse-grained equality to a fine-grained one (4.3) we have to also show that

$$\overline{\left(\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R - \langle i|i\rangle_L \langle j|j\rangle_R \right)^2} = 0. \quad (4.9)$$

The projector onto two-sided shell states $\prod_{\mathcal{H}_{2s}}$ is defined via an $n \rightarrow -1$ limit as above, and we will show that each term in (4.9) limits to $(\langle i|i\rangle_L \langle j|j\rangle_R)^2$. Note first that $(\langle i|i\rangle_L \langle j|j\rangle_R)^2 = \overline{\langle i|i\rangle_L \langle j|j\rangle_R \langle i|i\rangle_L \langle j|j\rangle_R} = \overline{\langle i|i\rangle_L \langle i|i\rangle_L} \times \overline{\langle j|j\rangle_R \langle j|j\rangle_R}$ where the final equality follows from the discussion around (4.1): L, R shell states are chosen to all have different mass, hence the wormholes connecting L to R shells are disallowed. Even so, each factor in $\overline{\langle i|i\rangle_L \langle i|i\rangle_L} \times \overline{\langle j|j\rangle_R \langle j|j\rangle_R}$ has connected and disconnected saddles as constructed in section 3.2. For example, for unnormalised shell states, we have

$$\overline{\langle j|j\rangle_R \langle j|j\rangle_R} = Z_{m_j,R}^2 \bar{S}(0)^2 (\bar{Z}(2\beta_R) + \bar{Z}(\beta_R)^2) \quad (4.10)$$

where the first term comes from the connected saddles and the second from the disconnected ones, and similarly for L .

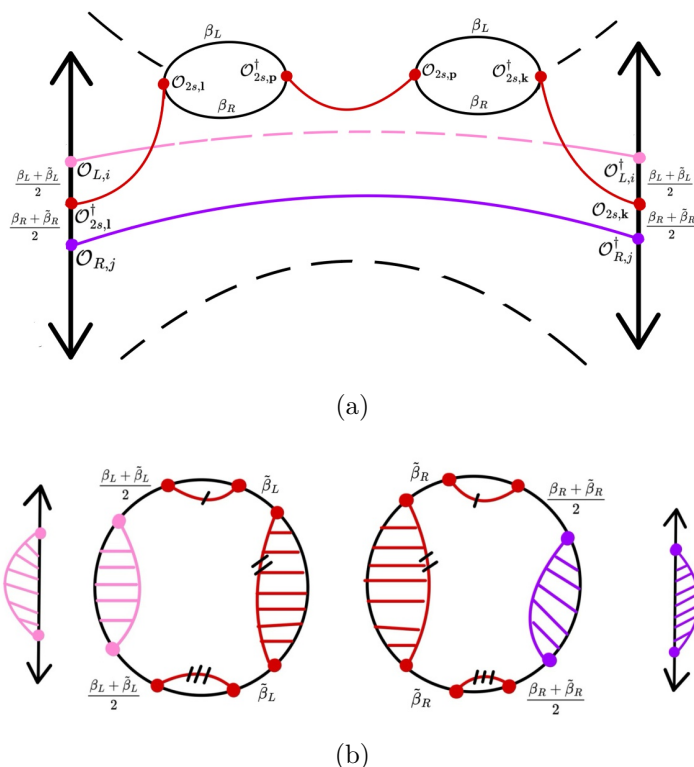


Figure 15. (a) The fully connected contribution to (4.6) depicted for $n = 2$. (b) These saddles are constructed by gluing an L, R disk together along the worldvolume of $n + 1$ two-sided shell wordvolumes and in addition gluing an L, R strip into the respective disks along the single-sided shell insertions.

Now consider, for example, the cross term in (4.9). By using shell states as above to resolve the projector we obtain

$$\overline{\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R \times \langle i|i\rangle_L \langle j|j\rangle_R} = \lim_{n \rightarrow -1} \overline{G_{2s, \mathbf{k}l}^n \langle i|_L \langle j|_R | \mathbf{k} \rangle_{2s} \langle 2s \mathbf{l} | i \rangle_L | j \rangle_R \langle i|i\rangle_L \langle j|j\rangle_R}. \quad (4.11)$$

We must consider the $\kappa_{2s} \rightarrow \infty$ limit in which the two-sided shell states span, and hence the two-sided shell index loop in $G_{2s, \mathbf{k}l}^n | \mathbf{k} \rangle_{2s} \langle 2s \mathbf{l} |$ cannot be broken, as explained in [7]. As (4.11) contains additional $|i\rangle_L, |j\rangle_R$ shell insertions compared to (4.6), there are then two types of saddlepoint for (4.11), depending on whether the single sided shell insertions are connected or not. Firstly there are saddles in which the $\langle i|i\rangle_L \langle j|j\rangle_R$ and $\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R$ boundary insertions are not connected, so that the path integral gives a product of the contributions from each term separately. It was shown in (4.8) that $\overline{\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R} = \overline{\langle i|i\rangle_L \langle j|j\rangle_R}$. Hence these disconnected contributions give contributions of $\overline{\langle i|i\rangle_L \langle j|j\rangle_R} \times \overline{\langle i|i\rangle_L \langle j|j\rangle_R}$, which are exactly the disconnected contributions to $(\overline{\langle i|i\rangle_L \langle j|j\rangle_R})^2$.

Second, there are saddles connecting the different $|i\rangle_L, |j\rangle_R$ shell insertions (figure 16(a)). These wormhole saddles are constructed analogously to those for (4.6) by including an additional i -shell on the L disk separated by boundary time β_L from the original i -shell then gluing in another shell strip along the added i shell, and similarly for the R disk (figure 16(b)). Thus, in the large shell mass limit, after normalizing the two-sided states,

the wormhole saddle contributes:

$$\frac{\overline{S}_L(0)^2 \times Z_{m_i,L}^2 \times \overline{Z}(2\beta_L + (n+1)\tilde{\beta}_L) \times \overline{S}_R(0)^2 \times Z_{m_j,R}^2 \times \overline{Z}(2\beta_R + (n+1)\tilde{\beta}_R)}{\overline{Z}(\tilde{\beta}_L)^{n+1} \overline{Z}(\tilde{\beta}_R)^{n+1}}. \quad (4.12)$$

In the $n \rightarrow -1$ limit this is the same as the wormhole contribution to $\overline{\langle i|i \rangle_L \langle j|j \rangle_R}^2|_{\text{conn}} = \overline{\langle i|i \rangle_L \langle i|i \rangle_L}|_{\text{conn}} \times \overline{\langle j|j \rangle_R \langle j|j \rangle_R}|_{\text{conn}}$ where as above $\overline{\langle j|j \rangle_R \langle j|j \rangle_R}|_{\text{conn}} = Z_{m_j,R}^2 \overline{S}(0)^2 \overline{Z}(2\beta_R)$. Putting everything together we get $\overline{\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R \langle i|_L \langle j|_R \langle i|_L \langle j|_R \langle i|_L \langle j|_R} = (\overline{\langle i|i \rangle_L \langle j|j \rangle_R})^2$ as all saddles are accounted for on both sides of the equality. Similarly, after accounting for the analogous connected and disconnected saddles, we find

$$\overline{\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R \langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i\rangle_L |j\rangle_R} = \overline{\langle i|i \rangle_L \langle j|j \rangle_R}^2. \quad (4.13)$$

Hence (4.9) holds, at least within the sum over saddle points.

The above shows $\overline{\langle \Psi_\otimes | \prod_{\mathcal{H}_{2s}} | \Psi_\otimes \rangle} = \overline{\langle \Psi_\otimes | \Psi_\otimes \rangle}$ and $\overline{(\langle \Psi_\otimes | \prod_{\mathcal{H}_{2s}} | \Psi_\otimes \rangle - \langle \Psi_\otimes | \Psi_\otimes \rangle)^2} = 0$, within the saddlepoint approximation to the path integral. As the single- and two-sided shell states form a basis for $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$ and \mathcal{H}_{LR} respectively, this establishes the fine-grained equality (4.3) and therefore $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R} \subseteq \mathcal{H}_{LR}$. We extend this argument to hold as full path integral equalities by using the surgery tool of the toolkit in [7] in appendix B.3.

4.2 Tensor product shell states span two-sided shell states

Similarly we will establish $\mathcal{H}_{2s} \subseteq \mathcal{H}_L \otimes \mathcal{H}_R$ by showing (4.4) for arbitrary $|\Phi_{2s}\rangle$ two-sided shell states, say $|\mathbf{p}\rangle_{2s}$. This shows (4.4) for any state in \mathcal{H}_{LR} because the 2-sided shell states form a basis (as shown in [7]). Thus we want to use the path integral to show that

$$\overline{\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} |\mathbf{p}\rangle_{2s} - \langle \mathbf{p} | \mathbf{p} \rangle_{2s}} = 0 \quad (4.14)$$

$$\overline{\left(\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} |\mathbf{p}\rangle_{2s} - \langle \mathbf{p} | \mathbf{p} \rangle_{2s} \right)^2} = 0, \quad (4.15)$$

for any $|\mathbf{p}\rangle_{2s}$.

As the Gram matrix of overlaps for $\mathcal{H}_L \otimes \mathcal{H}_R$ is the tensor product $G_{L \otimes R} = G_L \otimes G_R$ the projector onto this space is simply given by $\prod_{\mathcal{H}_L} \prod_{\mathcal{H}_R}$:

$$\overline{\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} |\mathbf{p}\rangle_{2s}} = \lim_{n,m \rightarrow -1} \overline{G_{L,ac}^n G_{R,il}^m \langle \mathbf{p}|_{2s} a \rangle_L |i\rangle_R \langle c|_L \langle l|_R | \mathbf{p} \rangle_{2s}}. \quad (4.16)$$

From section 3, path integrals computing G_L^n require n shell strip boundaries of length β_L (similarly for G_R^m), and $\langle j|_L \langle l|_R | \mathbf{p} \rangle_{2s}$ requires the mixed-shell boundary defined above (figure 13). Also, as we discussed, when we take $\kappa_L, \kappa_R \rightarrow \infty$, the single-sided shell states span $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$, and only geometries with a maximal number of shell index loops contribute to the path integral. These geometries must connect the $n+m$ asymptotic strip boundaries with the two mixed-shell boundary in a single bulk (figure 17(a)). Such wormholes are constructed by gluing an L, R disk together along the $\mathcal{O}_{2s,p}$ shell and gluing n, m strips into

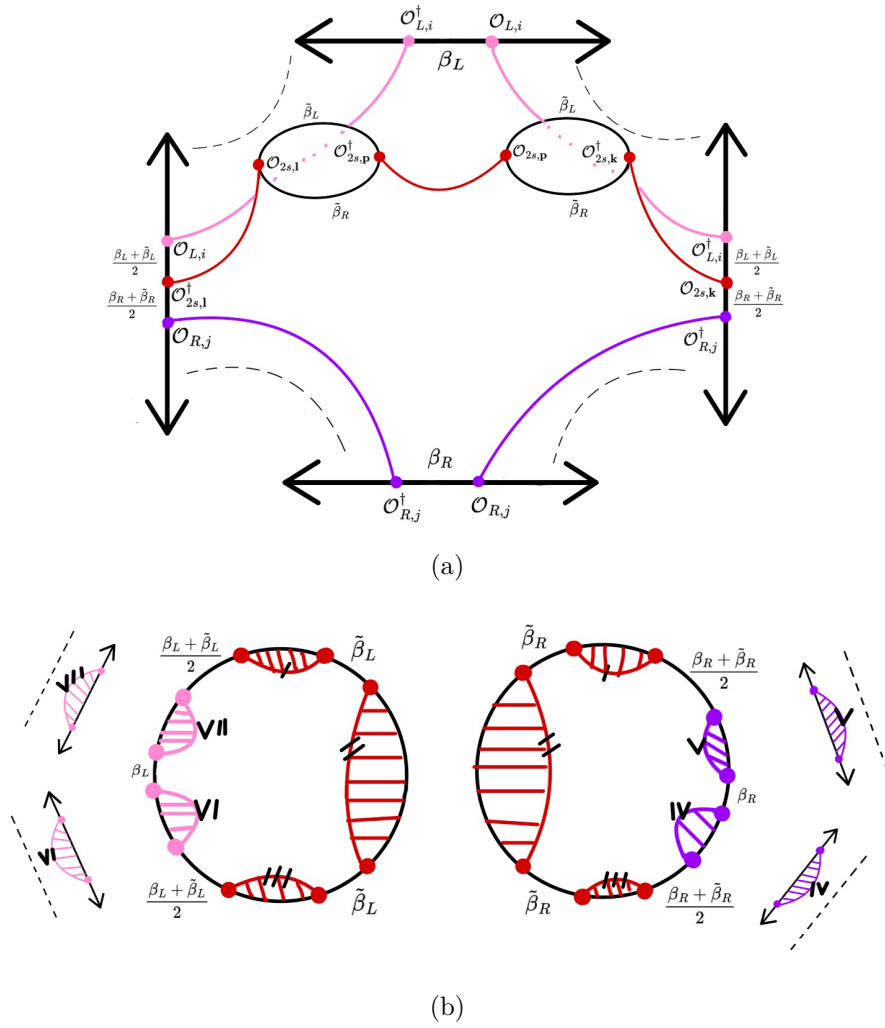


Figure 16. (a) The fully connected contribution to $\overline{G_{2s,kl}^n \langle i|_L \langle j|_R | \mathbf{k} \rangle_{2s} \langle 2s \mathbf{1} | i \rangle_L | j \rangle_R \langle i | i \rangle_L \langle j | j \rangle_R}$ depicted for $n = 2$. (b) These saddles are constructed by gluing an L, R disk together along the worldvolume of $n + 1$ two-sided shell worldvolumes and in addition gluing two L, R strip separates by $\beta_{L,R}$ into the respective L, R disks along the single-sided shell insertions.

the L, R disk long the L, R single-sided shell insertions respectively (figure 17(b)). Thus, in the large shell mass limit the wormhole saddle contribution to (4.16) is:

$$\overline{Z}(\tilde{\beta}_L + (n + 1)\beta_L) \overline{Z}(\tilde{\beta}_R + (m + 1)\beta_R) \overline{S}_L(0)^{n+1} \overline{S}_R(0)^{m+1} Z_{m_{\mathbf{p}}, 2s} \prod_{k=1}^{n+1} Z_{m_k, L} \prod_{k=1}^{m+1} Z_{m_k, R}. \quad (4.17)$$

After normalizing the L, R single-sided shell states this becomes independent of the shell indices running in the loop and we have:

$$\overline{G_{L,ac}^n G_{R,il}^m \langle \mathbf{p} | 2s a \rangle_L \langle i \rangle_R \langle c |_L \langle l |_R | \mathbf{p} \rangle_{2s}} = \kappa^{n+1} \frac{\overline{Z}(\tilde{\beta}_L + (n + 1)\beta_L) \overline{Z}(\tilde{\beta}_R + (m + 1)\beta_R) Z_{m_{\mathbf{p}}, 2s}}{\overline{Z}(\beta_L)^{n+1} \overline{Z}(\beta_R)^{m+1}}. \quad (4.18)$$

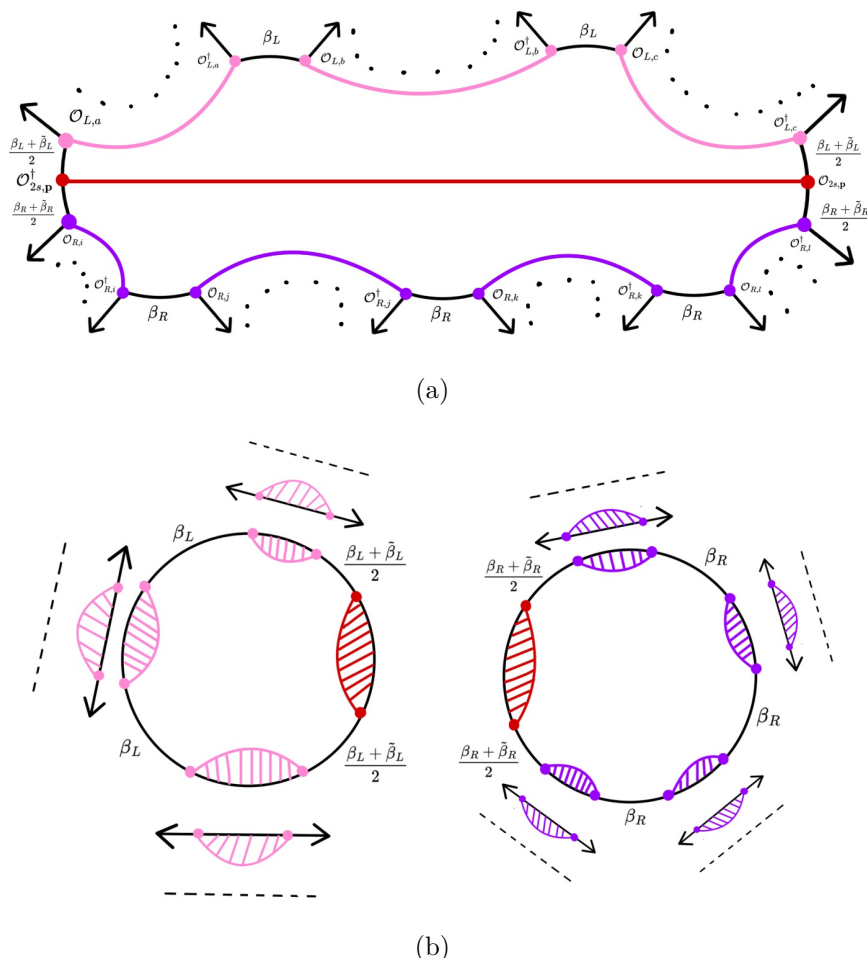


Figure 17. (a) The fully connected contribution to (4.16) depicted for $n = 2, m = 3$. (b) These saddles are constructed by gluing an L, R disk together along the worldvolume of two-sided \mathbf{p} -shell worldvolume and in addition gluing n, m strips into the respective L, R disks along the single-sided shell insertions.

Taking $m, n \rightarrow -1$ we recover the norm (A.4) of the two-sided shell state $|\mathbf{p}\rangle_{2s}$:

$$\overline{Z}(\tilde{\beta}_L)\overline{Z}(\tilde{\beta}_R)Z_{m\mathbf{p},2s} = \overline{\langle \mathbf{p} | \mathbf{p} \rangle}_{2s}, \tag{4.19}$$

thereby deriving (4.14).

Fine grained. To arrive at the fine-grained statement (4.4) we have to also show (4.15). The logic here closely resembles that of the fine-grained result in section 4.1 above. Once again we show that each term in (4.15) limits to $\overline{\langle \mathbf{p} | \mathbf{p} \rangle}_{2s}^2 = Z_{m\mathbf{p}}^2 \left(\overline{Z}(\tilde{\beta}_L)^2 \overline{Z}(\tilde{\beta}_R)^2 + \overline{Z}(2\tilde{\beta}_L)\overline{Z}(2\tilde{\beta}_R) \right)$ where the first term corresponds to the disconnected contribution $\overline{\langle \mathbf{p} | \mathbf{p} \rangle}_{2s} \times \overline{\langle \mathbf{p} | \mathbf{p} \rangle}_{2s}$ and the latter the wormhole contribution, see (A.3) and appendix A.

Consider for example, the cross term:

$$\overline{\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} |\mathbf{p}\rangle_{2s} \langle \mathbf{p} | \mathbf{p} \rangle_{2s}} = \lim_{n,m \rightarrow -1} \overline{G_{L,ac}^n G_{R,il}^m \langle \mathbf{p} |_{2sa} \rangle_L |i\rangle_R \langle c|_L \langle l|_R | \mathbf{p} \rangle_{2s} \langle \mathbf{p} | \mathbf{p} \rangle_{2s}}. \tag{4.20}$$

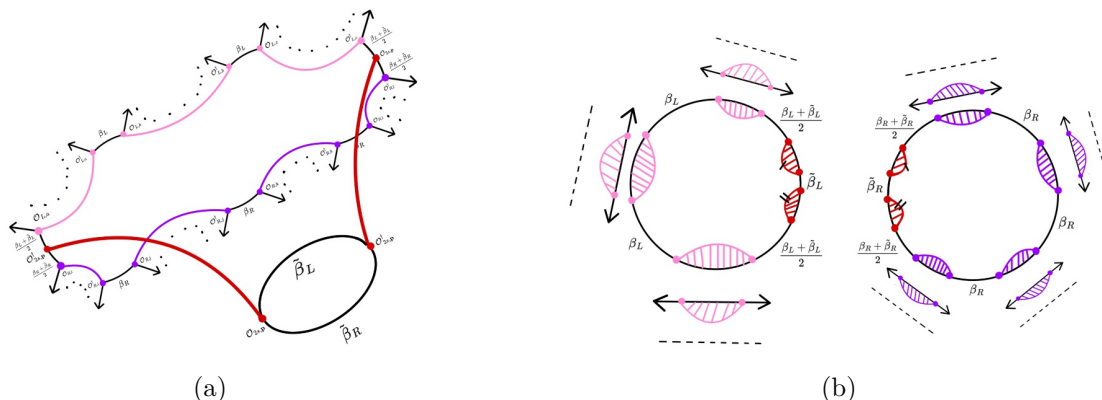


Figure 18. (a) Fully connected saddle contributing to (4.21). (b) These saddles are constructed by gluing an L, R disk together along the worldvolumes of two copies of the two-sided \mathbf{p} -shell worldvolume and in addition gluing n, m strips into the respective L, R disks along the single-sided shell insertions.

Recall that we require the $\kappa_{L,R} \rightarrow \infty$ limit in order for the single-sided shell states to span. Hence the single-sided shell index loops cannot be broken [7]. Thus there are again two classes of saddle; ones in which the multiple \mathbf{p} insertions are connected and ones in which they are not. The disconnected saddles correspond to $\overline{\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} | \mathbf{p} \rangle_{2s}} \times \overline{\langle \mathbf{p} | \mathbf{p} \rangle_{2s}}$, which we have shown around (4.18) to equal $\overline{\langle \mathbf{p} | \mathbf{p} \rangle_{2s}} \times \overline{\langle \mathbf{p} | \mathbf{p} \rangle_{2s}}$. The fully-connected saddles for (4.20) (see figure 18(a)) account for the connected contribution to $\overline{\langle \mathbf{p} | \mathbf{p} \rangle_{2s}^2}$. The construction of these fully connected wormhole saddles is analogous to that of the saddles for (4.16) apart from the addition of an extra identified \mathbf{p} -shell worldvolume on the L, R disks separated by $\tilde{\beta}_{L,R}$ from the other \mathbf{p} -shell insertion (see figure 18(b)). Upon gluing the L, R disks together, this extra identification results in the additional $\overline{\langle \mathbf{p} | \mathbf{p} \rangle_{2s}}$ asymptotic boundary.

Hence in the large shell mass limit, after normalising the shell states, these connected saddles contribute

$$\overline{G_{L,ac}^n G_{R,il}^m \langle \mathbf{p} |_{2s} a \rangle_L | i \rangle_R \langle c |_L \langle l |_R | \mathbf{p} \rangle_{2s} \langle \mathbf{p} | \mathbf{p} \rangle_{2s}} |_{\text{conn}} = \frac{\overline{Z}(2\tilde{\beta}_L + (n+1)\beta_L) \overline{Z}(2\tilde{\beta}_R + (m+1)\beta_R) Z_{m_{\mathbf{p}}, 2s}^2}{\overline{Z}(\beta_L)^{n+1} \overline{Z}(\beta_R)^{m+1}}. \quad (4.21)$$

In the $m, n \rightarrow -1$ limit this recovers the connected contribution to the squared norm of the un-normalised $|\mathbf{p}\rangle$ state $\overline{Z}(2\tilde{\beta}_L) \overline{Z}(2\tilde{\beta}_R) Z_{m_{\mathbf{p}}, 2s}^2 = \overline{(\langle \mathbf{p} | \mathbf{p} \rangle)_{2s}^2} |_{\text{conn}}$ (see appendix A). Again, $\overline{\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} | \mathbf{p} \rangle_{2s} \langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} | \mathbf{p} \rangle_{2s}} = \overline{(\langle \mathbf{p} | \mathbf{p} \rangle)_{2s}^2}$ follows by similar logic. Putting everything together, we have derived the fine grained result (4.4), and therefore $\mathcal{H}_{LR} \subseteq \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$, within the saddlepoint approximation. Using the surgery approach of the toolkit in [7] the argument can again be extended to hold for the full path integral (see appendix B.2).

Conclusion. Above we have shown that $\mathcal{H}_L \otimes \mathcal{H}_R = \mathcal{H}_{2s}$. These Hilbert spaces coincide with $\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$ and \mathcal{H}_{LR} respectively, so $\mathcal{H}_{LR} = \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$. This resolves the factorization problem (2.8) because the total two-sided Hilbert space is $\mathcal{H}_{L \cup R} = \mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R} \cup \mathcal{H}_{LR}$ and these two sets in the union are the same.

Interestingly, wormhole geometries connecting mixed single- and two- sided shell state boundaries were crucial in deriving this result. The perturbative gravity Hilbert space

obtained in the $G_N \rightarrow 0$ limit does not factorise because such wormholes are not included. This is consistent with the observation that the L, R von Neumann algebras of observables, $\mathcal{A}_{L,R}$, describing low-energy back reaction around the two-sided black hole saddle are of type III [29, 30]. Such algebras do not have well-defined traces, states or density matrices. Interestingly, after imposing the gravitational constraints the resulting crossed product L, R algebras are of type II [31–33]. While type II algebras contain density matrices and traces up to constants, they do not contain any pure states and do not have a representation on a tensor product Hilbert space $\mathcal{H}_{\mathcal{B}_L} \otimes \mathcal{H}_{\mathcal{B}_R}$. The tensor product factorisation (2.8) of the nonperturbative, fine-grained two-sided Hilbert space implies that the full nonperturbative L, R algebras must be of type I, for which there exist a unique trace, pure states and tensor product representation. Furthermore, as the two-sided Hilbert space factorises into copies of the single-sided theory, these nonperturbative L, R algebras must be unitarily equivalent to the nonperturbative single-sided operator algebra. This conclusion is exactly what is expected if the single-sided theory is described by a holographic dual theory on the asymptotic boundary, as these statements then follow trivially from the dual quantum mechanics. Note however that all the steps leading to the factorisation (2.8), are applicable in Euclidean path integral formulation of asymptotically flat quantum gravity too.

4.3 Disentangling the shell states

The discussion above resolves the generalized version of the Hilbert space factorisation problem introduced in section 2.2. However, initial discussions of the factorisation problem were formulated in terms of an Einstein-Rosen bridge connecting two asymptotic boundaries of a two-sided black hole [1, 2, 6, 15]. Recall that the geometry associated to a state created by cutting open the gravity path integral is determined by the leading saddle computing the norm. So we can only pose the version of the factorization problem based on the presence of an Einstein-Rosen bridge in a perturbative semiclassical limit where we keep the leading saddle and ignore the others. By contrast, above we have shown the factorisation 3 requires inclusion of non-perturbative saddle contributions that are omitted in this approximation. Indeed, as explained in section 2.2, the fact that some geometries computing the norm of a gravity state may analytically continue to Lorentzian wormholes is not in tension with factorization. Still, one might worry about the fact the geometries of some states in the intrinsically two-sided Hilbert spaces \mathcal{H}_{LR} do not “look” like they arise from an underlying tensor product.

Consider for example the two-sided shell basis for \mathcal{H}_{LR} used above and reviewed in appendix A. As discussed in [7], the semiclassical geometry associated to these states come in three classes, determined by whether the left and right preparation temperatures $\tilde{\beta}_L, \tilde{\beta}_R$ are above/below the Hawking-Page transition scale β_{HP} . The resulting geometry is either a two-sided black hole with a long wormhole interior, a single-sided black hole with disconnected copy of thermal AdS or two disconnected copies of thermal AdS entangled with a compact universe. The authors of [7] showed that, non-perturbatively, each of these three sets provides a basis. The latter two bases consist of disconnected geometries, and may initially appear factorized because they are not geometrically connected, but are not actually separable tunable. For example, none of these states is a product of one-sided black holes.

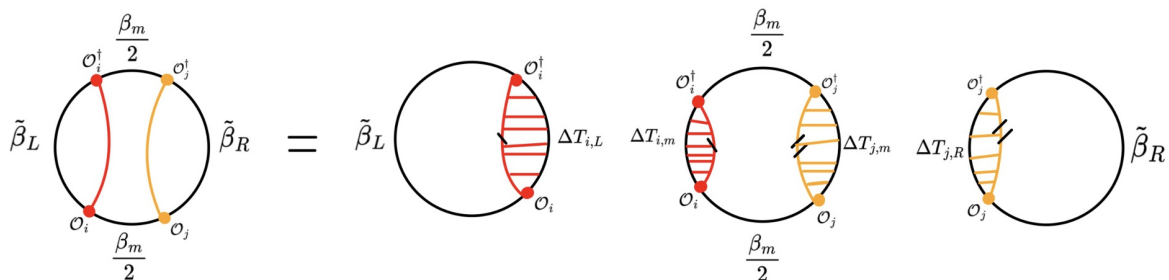


Figure 19. Construction of the saddle for $\overline{\langle \mathbf{i}, \mathbf{j} | \mathbf{i}, \mathbf{j} \rangle}_{2s}$: the L and R disks are glued together into the central disk by identifying the \mathbf{i} and \mathbf{j} shell worldvolumes.

In holographic settings the two-sided single-shell states map to partially entangled states (PETS) in the dual CFT [34]. In general, it is hard to deduce the underlying factorizing structure of a tensor product Hilbert space with access only to such a basis of highly entangled states. Since we have derived that the two-sided Hilbert space must factorize, reducing the entanglement between the L, R Hilbert space factors should produce a basis corresponding to explicitly factorized Lorentzian geometries. We will show that this is indeed the case. To do so we consider an alternative basis for the two-sided Hilbert space consisting of states with an additional shell operator inserted. The two-shell state $|\mathbf{i}, \mathbf{j}\rangle_{2s}$ consists of an \mathcal{O}_i and \mathcal{O}_j shell insertions separated by a Euclidean time β_m :

$$|\mathbf{i}, \mathbf{j}\rangle_{2s} = |e^{-\frac{\tilde{\beta}_L H_L}{2}} \mathcal{O}_i e^{-\frac{\beta_m H_R}{2}} \mathcal{O}_j e^{-\frac{\tilde{\beta}_R H_R}{2}}\rangle. \tag{4.22}$$

We have written a time evolution by H_R , but could equally have used H_L . If we have AdS boundary conditions, from the dual CFT perspective, varying β_m amounts to tuning the entanglement between the L, R Hilbert space factors. A simple adaptation of the argument that the single-shell states span \mathcal{H}_{LR} given in [7] shows that a $\kappa \rightarrow \infty$ set of the two shell states also form a basis of the full two-sided Hilbert space \mathcal{H}_{LR} . In particular, whenever a geometry is glued together along the shell world-volume in the single-shell case the analogous operation for the two shell case is to glue together two distinct shell worldvolumes via an additional disk containing each shell separated by a boundary time $\beta_m/2$ on either side. Compare for example figure 19 with figure 24 in appendix A. In the large shell mass limit the only difference is the presence of additional factors of $\overline{Z}(\beta_m)$. In the completeness argument this will induce a factor $\lim_{n \rightarrow -1} \overline{Z}(\beta_m)^{n+1}$, leaving the conclusion unchanged. Hence the two-shell states span \mathcal{H}_{LR} regardless of the value of β_m .

Similarly, the saddle points computing the norms of the two-shell states are constructed analogously to the single-shell two-sided case. The single-shell saddles are constructed by gluing together portions of L, R disks whereas the two shell saddles are constructed by gluing the both the L, R disks into central disk via the Israel junction conditions (figure 19). In the large shell mass limit the shell homology regions pinch off ($\Delta T_{i,L}, \Delta T_{i,m}, \Delta T_{j,m}, \Delta T_{j,R} \rightarrow 0$) and the shells contribute universally to the action, as they did in earlier sections. Hence $\overline{\langle \mathbf{i}, \mathbf{j} | \mathbf{i}, \mathbf{j} \rangle}_{2s} = \overline{Z}(\beta_L) \overline{Z}(\beta_R) \overline{Z}(\beta_m) Z_{m_i} Z_{m_j}$. The Lorentzian geometry for the two-shell states is determined by the leading saddles for each of the three Z-partition functions, which depends on whether the preparation temperatures $\beta_L, \beta_m, \beta_R$ are above/below the Hawking-Page

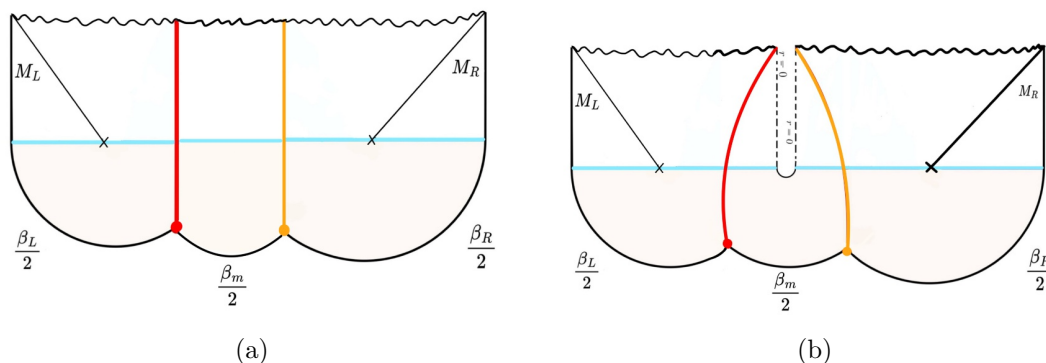


Figure 20. (a) Lorentzian geometry for $\beta_L, \beta_R < \beta_{\text{HP}}$ and $\beta_m < \beta_{\text{HP}}$ corresponding to a two-sided AdS black hole with a long interior supported by the shell states. (b) Lorentzian geometry for $\beta_L, \beta_R < \beta_{\text{HP}}$ and $\beta_m > \beta_{\text{HP}}$ corresponding to two disconnected AdS black holes.

transition. This produces $2 \times 2 \times 2 = 8$ different classes of two-shell states. These Lorentzian geometries can be thought of as taking any of the 4 classes of single-shell two-sided states discussed in [7], cutting the geometry along the shell, and gluing in between the two sides either a portion of the AdS black hole ($\beta_m < \beta_{\text{HP}}$) or thermal AdS ($\beta_m > \beta_{\text{HP}}$), again giving 8 possibilities in all. For example, choosing $\tilde{\beta}_L, \tilde{\beta}_R < \beta_{\text{HP}}$ and $\beta_m > \beta_{\text{HP}}$ results in a Lorentzian geometry consisting of two disconnected AdS black holes (figure 20(b)), while $\beta_m < \beta_{\text{HP}}$ corresponds to a two-sided AdS Black hole with a very long interior (figure 20(a)).

The $\beta_m > \beta_{\text{HP}}$ regime is particularly interesting as the Lorentzian geometry associated to $|\mathbf{i}, \mathbf{j}\rangle_{2s}$ consists of disconnected L and R geometries identical to the geometry of the tensor product state $|i\rangle_L \otimes |j\rangle_R$ if we choose $\beta_L = \tilde{\beta}_L$ and $\beta_R = \tilde{\beta}_R$. Hence each state in this basis for \mathcal{H}_{LR} has a geometry that is in one-to-one correspondence with the geometries in the basis for $\mathcal{H}_{B_L} \otimes \mathcal{H}_{B_R}$ discussed above. Any other state in \mathcal{H}_{LR} can therefore be expanded into a linear superposition of states corresponding to factorised Lorentzian geometries.

We have recovered a factorised basis by tuning down the entanglement between the two factors. However, for any finite β_m there should still be *some* entanglement, unlike in a tensor product state. The fact that $|\mathbf{i}, \mathbf{j}\rangle_{2s}$ and $|i\rangle_L \otimes |j\rangle_R$ cannot be the same state but still their semiclassical geometry is identical might seem puzzling. The resolution is that the semiclassical approximation of considering only the leading saddle does not capture the finer correlations that distinguish between these two states. To see this we consider the saturation of the Cauchy-Schwarz inequality for the inner product ${}_{2s}\langle \mathbf{i}, \mathbf{j} | i\rangle_L | j\rangle_R$ when $\beta_L = \tilde{\beta}_L$ and $\beta_R = \tilde{\beta}_R$. The saddles computing ${}_{2s}\langle \mathbf{i}, \mathbf{j} | i\rangle_L | j\rangle_R$ are constructed by gluing two disks into a strip of length $\beta_m/2$ along the two shells (figure 21). In the large shell mass limit we obtain:

$$\frac{\overline{{}_{2s}\langle \mathbf{i}, \mathbf{j} | i\rangle_L | j\rangle_R}}{\sqrt{\langle \mathbf{i}, \mathbf{j} | \mathbf{i}, \mathbf{j} \rangle_{2s} \langle i | i \rangle_L \langle j | j \rangle_R}} = \frac{\overline{S(0)} \overline{Z(\beta)_L} \overline{Z(\beta)_R} \overline{S}(\beta_m/2)}{\sqrt{S(0) \overline{Z(\beta_L)} S(0) \overline{Z(\beta_R)} \sqrt{\overline{Z(\beta_L)} \overline{Z(\beta_R)} \overline{Z(\beta_m)}}}} = \frac{\overline{S}(\beta_m/2)}{\sqrt{\overline{Z(\beta_m)}}}. \quad (4.23)$$

In the regime $\beta_m > \beta_{\text{HP}}$ where these states have identical semiclassical geometries, the dominant saddle for $\overline{Z(\beta_m)}$ is thermal AdS, for which $\sqrt{\overline{Z(\beta_m)}} = \overline{Z(\beta_m/2)}$ and $\overline{S}(\beta_m/2) =$

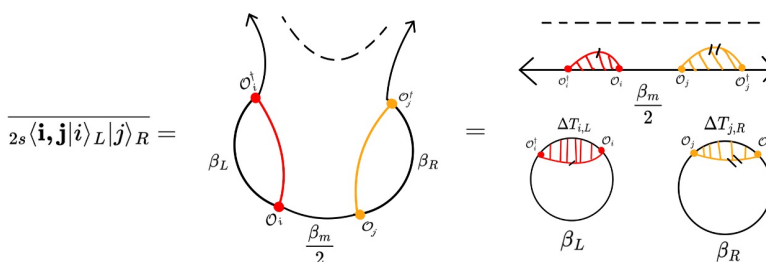


Figure 21. Saddle for the direct overlap (4.23) is constructed by gluing two disks into a strip of length $\beta_m/2$.

$\overline{Z}(\beta_m/2)$,¹⁰ and thus $\frac{\overline{2s\langle \mathbf{i}, \mathbf{j} | i \rangle_L | j \rangle_R}}{\sqrt{\langle \mathbf{i}, \mathbf{j} | \mathbf{i}, \mathbf{j} \rangle_{2s\langle i \rangle_L \langle j \rangle_R}}} = 1$. As the Cauchy-Schwarz inequality is saturated, $|\mathbf{i}, \mathbf{j}\rangle$ and $|i\rangle_L \otimes |j\rangle_R$ must be linearly dependent when $\beta_m > \beta_{\text{HP}}$. However, if you keep the sub-dominant black hole saddle for $\overline{Z}(\beta_m)$ this is not longer the case; the states are distinguishable. Also note that in the $\beta_m < \beta_{\text{HP}}$ regime the leading disk saddle for $\overline{Z}(\beta_m)$ is the Euclidean AdS black hole, while the strip saddle is still given by Euclidean AdS, and therefore (4.23) will be smaller than unity, as the semiclassical geometry of the two-shell and tensor product state are not equal when $\beta_m < \beta_{\text{HP}}$.

5 Discussion

In this paper we have shown that the Hilbert space of quantum gravity with two asymptotic boundaries factorises into two copies of the single-boundary Hilbert space. To show this we used a toolkit introduced in [7] that drastically simplifies non-perturbative gravity path integral calculations, and allows extraction of fine-grained equalities from the coarse-grained path integral. In [7] these tools were used to show that the two-sided shell states described in [3, 4, 10, 22, 23] and appendix A span the entire intrinsically two-boundary, fine-grained Hilbert space. Similarly, in this paper we used these methods to show that a single-sided version of the shell states discussed in [14] span the entire single-sided, fine-grained Hilbert space. The semiclassical geometry of the single-sided shell states is either thermal AdS entangled with a compact baby universe, or a single-sided black hole, depending on the preparation temperature of the state. As the single-sided shell states form a basis for any preparation temperature, our results imply that a black hole can be written as a superposition of horizonless geometries, and vice versa. The authors of [7] reached the same conclusion for the two-sided case, and used it to argue against the existence of a “geometry” observable in the quantum theory. The same logic applies to the single boundary gravity theory.

As an application of these results, we showed that the two-boundary, fine-grained gravity Hilbert space factorises into two copies of the single-boundary, fine-grained gravity Hilbert space. This result followed from novel non-perturbative wormhole contributions to the path integral that connect the asymptotic boundaries preparing single- and two-sided states. The toolkit from [7] rendered the sum over topologies tractable, and allowed us to extend coarse-grained path integral calculations to a fine-grained result. Since we have shown that

¹⁰Thermal AdS is just a section of the strip of length β with the endpoints identified.

the two-boundary gravity Hilbert space factorises, and have also explicitly constructed bases with semiclassical interpretations as connected or disconnected geometries, we conclude that connected geometries can be written as superpositions of disconnected ones and vice versa. Therefore, there cannot be a linear operator that measures “connectedness” of a state in quantum gravity.

Interestingly, after normalizing the shell states every wormhole geometry encountered in this work, whether connecting two-sided states (as in [7]), single-sided states, or a mix, evaluates to products of the Z -partition function (see section 2.1) which is thought to compute the thermal partition function in theories of gravity [35]. This echoes recent proposals [12, 13] that the gravity path integral captures the maximal ignorance matrix model constrained by a set of input parameters, one of which is the expectation value for thermal partition function as computed by the gravity path integral. Wormhole contributions to the gravity path integral should be captured by moments of the matrix model. It would be helpful to explicitly show from the proposed matrix model why the quantities receiving the wormhole contributions considered in this work evaluate to products of the thermal partition function.

Recently [18] showed that the operator algebra on the Hilbert space obtained by cutting open the gravity path integral is of Type I, if the theory satisfies a certain general set of axioms. Type I algebras admit a trace, which [18] used to give a new understanding of the HRT entropy formula [36–38] from a purely gravitational point of view. However, their results were not sufficient to establish that the two-boundary Hilbert space is a product of single boundary factors. It would be interesting if our results could help to establish which axioms should be added to those in [18] to imply factorization.

We have shown factorization of the two-sided Hilbert space using just the gravitational path integral, and our results apply equally to asymptotically flat or AdS gravity, without any assumption about holographic duality. Interestingly, the factorization we found is *required* if there is a dual quantum mechanics living on the asymptotic boundary of spacetime, as there is for AdS gravity. We do not know if such duals exist for asymptotically flat gravity but the factorisation we showed is consistent with their existence.

Acknowledgments

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A Two-sided shell states

We use the shell state basis for the two-sided Hilbert space developed in [3, 4, 7, 10, 22, 23].¹¹ These states are defined by the Euclidean gravity path integral with boundary $\mathbb{I} \times \mathbb{S}^{d-1}$. The cut surface of this boundary has two \mathbb{S}^{d-1} components which we call $\mathcal{B}_{L,R}$ (figure 22). To create states we insert \mathbb{S}^{d-1} symmetric dust shell operators \mathcal{O}_S of mass $m_i \sim O(1/G_N)$ on the boundary, at distances $\frac{\beta_L}{2}$ and $\frac{\beta_R}{2}$ from the $\mathcal{B}_{L,R}$ cuts. By varying m_i we obtain an infinite family of shell states $|i\rangle$. To calculate entries of the Gram matrix $G_{ij} \equiv \langle i|j\rangle$ of

¹¹In the AdS/CFT context, these are related to Partially Entangled Thermal States (PETS) [34].

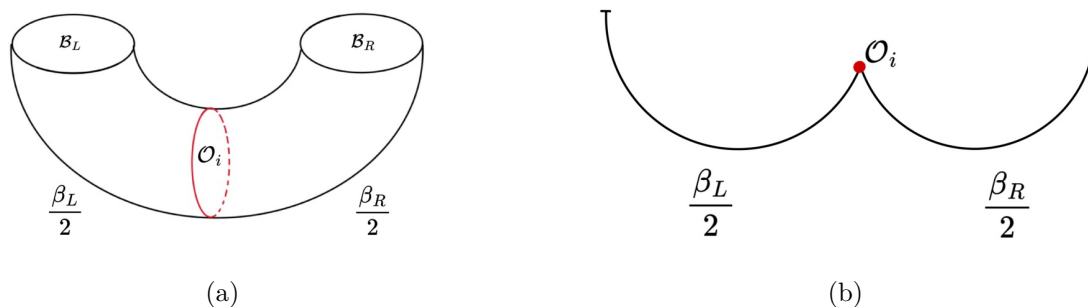


Figure 22. Asymptotic boundary condition for the gravity path integral defining the shell state. (a) Cut-open Euclidean boundary with topology $\mathbb{I}_{\frac{\beta_L+\beta_R}{2}} \times \mathbb{S}^{d-1}$ for preparation of the shell states. The shell operator \mathcal{O}_i is pictured in red. In AdS/CFT we can also perform the path integral in the boundary CFT with insertion of a \mathbb{S}^{d-1} symmetric operator dual to the shell. (b) Euclidean boundary with the \mathbb{S}^{d-1} suppressed. We adopt this convention for the rest of the paper. Here $\beta_{L,R}/2$ are Euclidean “preparation times”. Figure adapted from [7].

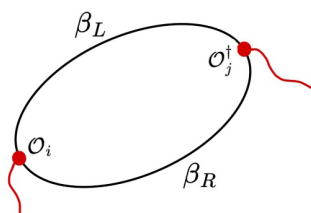


Figure 23. Shell asymptotic boundary condition for $\overline{\langle j|i \rangle}$, consisting of the operator insertions \mathcal{O}_i and \mathcal{O}_j^\dagger separated by asymptotic time extent β_L and β_R respectively. The red lines represent the shells propagating into the bulk. Figure adapted from [7].

overlaps between the shell states, we must sew together the $|i\rangle$ and $|j\rangle$ state preparation path integrals at their cuts. We will say that the overlap path integral has a *shell boundary* because it includes boundary operator insertions \mathcal{O}_j and \mathcal{O}_i^\dagger separated by asymptotic times β_L and β_R on the closed boundary manifold (figures 23, 24). To leading order

$$\overline{\langle i|j \rangle} = \delta_{ij} Z_1, \tag{A.1}$$

if the shell inertial masses $m_{i,j}$ are sufficiently large and different, because it takes $|m_i - m_j|$ bulk interactions in Planck units to match such shells [3]. However, higher topology contributions (figure 25) stabilized by the shell matter modify the overlap to

$$|\overline{\langle i|j \rangle}|^2 = \overline{\langle i|j \rangle \langle j|i \rangle} = Z_2 + \delta_{ij} Z_1^2, \tag{A.2}$$

where Z_2 is a wormhole saddlepoint contribution [3, 7]. The wormhole is constructed by identifying shell worldvolumes across two disks subject to the Israel junction conditions (figure 25). The saddlepoint geometry on either side of the shell can be anything that locally solves the equations of motion; hence by symmetry these are portions of any of the saddlepoints contributing to the Z -partition function in section 2.1.

Similarly, we can construct saddlepoints contributing to $\overline{(G^n)_{ii}} = \overline{\langle i|j_1 \rangle \langle j_1|j_2 \rangle \cdots \langle j_n|i \rangle}$ (no sum on i, j_k) by connecting n asymptotic boundaries (arising from each factor in $(G^n)_{ii}$)

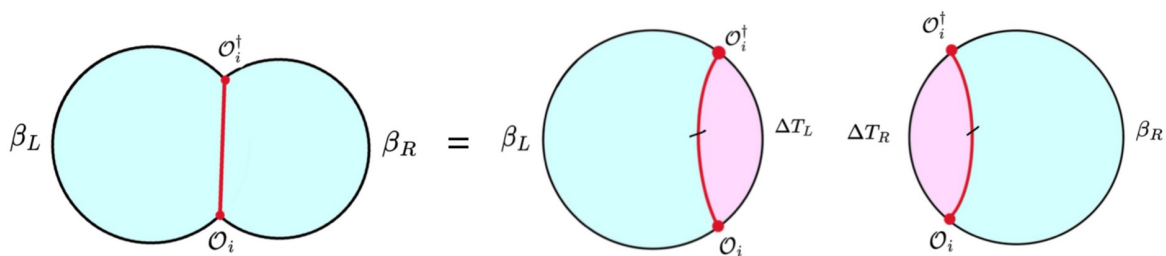


Figure 24. The saddlepoints for the shell norm path integral $\overline{\langle i|i \rangle}$ are constructed by gluing together two disks. In particular the shell homology regions (purple) on each disk are discarded and the resulting geometries glued together along the shell worldvolume. Figure adapted from [7].

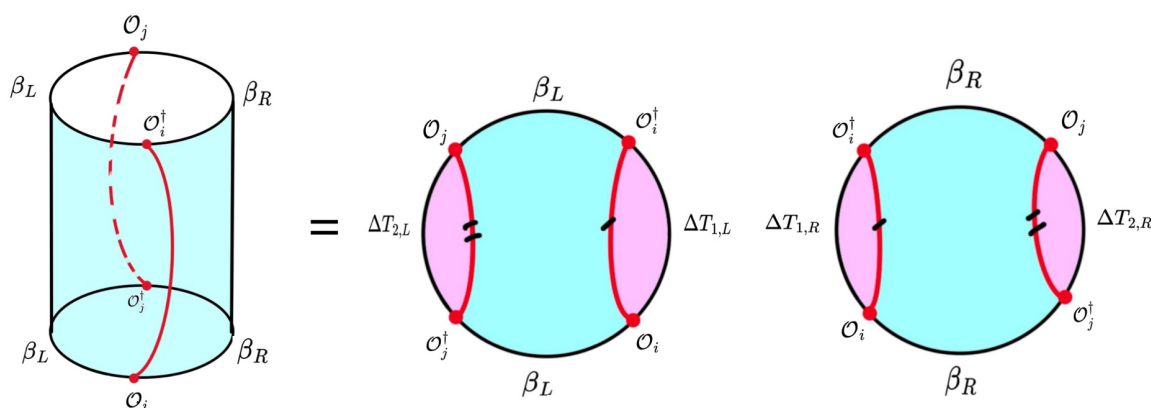


Figure 25. Construction of shell wormhole saddles Z_2 . We glue two disks together by discarding the regions indicated in purple and attaching the remainder along the shell worldvolumes (red, black dashes indicate the glued shells) while obeying the Israel junction conditions. Saddlepoints, i.e., solutions to the equation of motion, exist with this topology [3]. Figure adapted from [7].

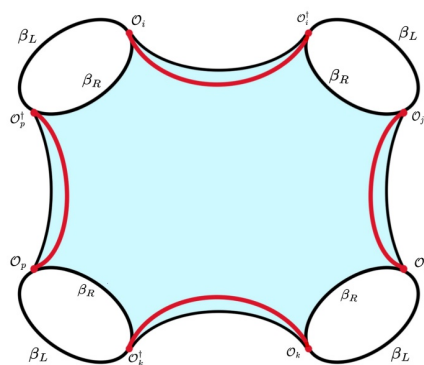


Figure 26. Fully connected wormhole geometry for $\overline{\langle G^n \rangle_{ii}}$ consisting of n shell boundaries connected by a single bulk, pictured here for $n = 4$. Figure adapted from [7].

with a single multiboundary wormhole. The k -th shell can only propagate through the wormhole from \mathcal{O}_k to \mathcal{O}_k^\dagger on different asymptotic boundaries, and so the index structure of $\overline{(G^n)_{ii}}$ is reflected in the geometry of the wormhole (figure 26). We will call this wormhole geometry the *pinwheel*. There are also disconnected saddlepoints connecting different subsets of the boundaries together which look like collections of smaller pinwheels.

To construct a pinwheel saddlepoint we slice figure 26 open along the shell worldvolumes producing L - and R - *sheet diagrams* in figure 27(a). We can also produce these sheets by removing the pink shell homology regions from the L and R disks in figure 27(b). The L (R) disk boundary contains n shell operator insertions $\mathcal{O}_{1,2,\dots,n}, \mathcal{O}_{1,2,\dots,n}^\dagger$ with \mathcal{O}_i and \mathcal{O}_i^\dagger separated by a time $\Delta T_{L,i}$ ($\Delta T_{R,i}$) and $\mathcal{O}_i^\dagger, \mathcal{O}_{i+1}$ separated by a time β_L (β_R). So to construct the wormhole we can start with the disks in figure 27(b) with shell homology regions removed, and then glue along the indicated shell worldvolumes by using the Israel junction conditions [26]. The shell propagation times ΔT_i are then determined dynamically by the shell equations of motion. We can apply this procedure started with any saddle geometry to the Z -partition function in section 2.1.

As shown in [3, 4, 22] the turning points of the shell trajectories approach the asymptotic boundary of the disk as the shells masses increase, so that the shell homology regions pinch off. In this limit, $\Delta T_i \rightarrow 0$, and each shell contributes simply contributes a factor $Z_{m_i} \sim e^{-2(d-1)\log(GNm_i)}$. This behavior is independent of the Z -partition saddle geometries that are glued along the shells [3, 4, 22], so that the fully connected saddlepoint contribution to the n -boundary path integral decomposes into two Z -partition functions and a shell contribution:

$$\overline{(G^n)_{ii}}|_{\text{connected}} = \overline{Z}(n\beta_L)\overline{Z}(n\beta_R) \prod_{i=1}^n Z_{m_i}. \tag{A.3}$$

As $m_i \rightarrow \infty$ the diagram in figure 24 computing $\langle i|i \rangle$ also simplifies — the propagation time of the shell vanishes, so that the diagram decomposes into two disks of circumference $\beta_{L,R}$ attached via a pointlike shell trajectory. Thus (A.1) becomes $\langle i|j \rangle = \delta_{ij} Z_1 = \delta_{ij} \overline{Z}(\beta_L)\overline{Z}(\beta_R) Z_{m_i}$ in the sum over saddlepoints approximation. We can also see this by setting $n = 1$ in the above expression for $\overline{(G^n)_{ii}}|_{\text{connected}}$. Normalizing the states as $|k\rangle / \sqrt{Z_1}$, we get

$$\overline{(G^n)_{ii}}|_{\text{connected}} = \frac{\overline{Z}(n\beta_L)\overline{Z}(n\beta_R)}{\overline{Z}(\beta_L)^n \overline{Z}(\beta_R)^n}, \tag{A.4}$$

where again the overbars indicate a coarse-grained averages path integral computation. Each Z partition sum on the right side is a sum over all saddlepoints, accounting for all the saddle contributions to $\overline{(G^n)_{ii}}|_{\text{connected}}$.

B All order extensions

The main text showed various results in the sum over saddlepoints approximation to the gravity path integral. Here we extend those results to the full path integral by using the “surgery” method of [7]. Our arguments are adapted from section 5 of [7], to which we refer the reader for details.

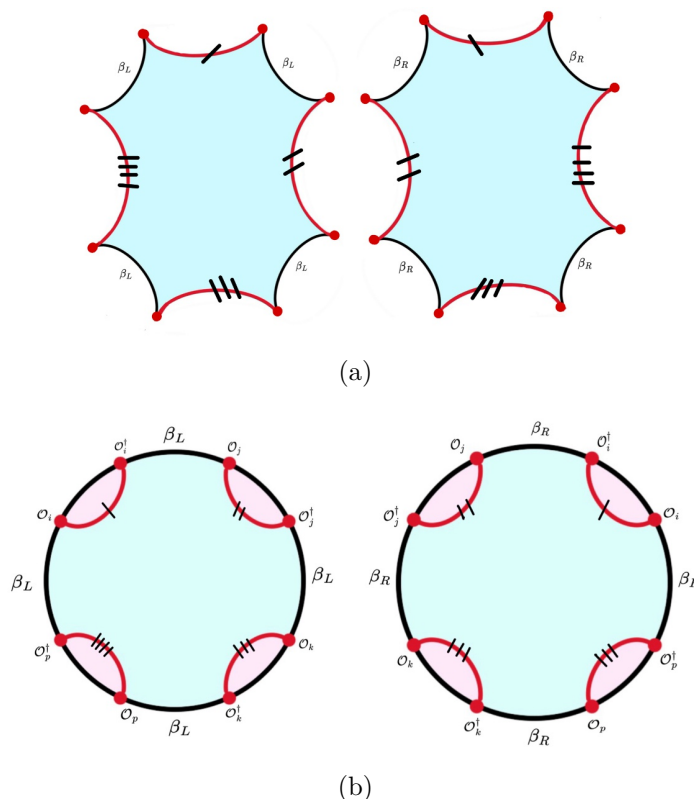


Figure 27. Schematic of the construction of the pinwheel wormhole. (a) Sheet diagrams for the pinwheel obtained by cutting figure 26 along the shell worldvolumes. (b) Each of the pinwheel sheets can be regarded a portion of a disk. The pinwheel saddle constructed by gluing to disks together along the corresponding shell worldvolumes by the junction conditions. Figure adapted from [7].

B.1 Single-sided shell completeness

The proof in the main text that the single-sided shell states form a basis was restricted to sum over saddle points approximation to the gravity path integral. In this section we argue that the full path integrals on the l.h.s. and r.h.s. of (3.5) and (3.6) are identical, and hence (3.13) is exact in the fine-grained theory. Throughout this section we adopt the notation of section 3.4. All our considerations will be in the strict $\kappa \rightarrow \infty$ limit in which only wormhole geometries that do not break an index loop contribute. The argument made in section 3.4 is easily seen to include perturbative corrections around the $\hat{Z}_{n+2}(\Psi)$ saddle points by considering the limit in which the shells are parametrically heavy. In this limit the shells propagate in the asymptotic region, which is protected from quantum corrections, and contribute universally to the action. The shells therefore only serve to glue the parts of the different strips together. In this limit corrections to the $\hat{Z}_{n+2}(\Psi)$ saddle points correspond to the corrections to the time evolved overlap $\langle \Psi | \Psi \rangle_{(\frac{\beta_R + (n+1)\beta_R}{2})}$. Hence the corrections contribute equally to both sides of (3.5) and similarly for (3.6).

We might even extend the equality (3.13) to topologies that do not support a saddlepoint using the surgery tool from [7]. This argument is adapted from that of section 5.1 in [7] and we invite the reader to look at that section for details. Mainly, we argue that the

structure of the full gravity path integral

$$\overline{G_{ij}^n \langle \Psi|i \rangle \langle j|\Psi \rangle} = \int_{g, \phi \rightarrow G_{ij}^n \langle \Psi|i \rangle \langle j|\Psi \rangle} \mathcal{D}g \mathcal{D}\phi e^{-I_{\text{tot}}[g]} \tag{B.1}$$

at a fixed positive integer n is such that in the $n \rightarrow -1$ limit this path integral reduces to the full path integral for $\overline{\langle \Psi|\Psi \rangle}(\tilde{\beta}_R)$, where $g, \phi \rightarrow G_{ij}^n \langle \Psi|i \rangle \langle j|\Psi \rangle$ denotes that we path integrate over metrics and quantum fields satisfying the asymptotic boundary conditions. As we are in the $\kappa \rightarrow \infty$ limit we only have to consider the fully connected contributions. First, we organize the full sum over connected geometries by topology and consider a geometry g . If the shells are parametrically heavy enough the shell matter path integral $\int \mathcal{D}\phi$ subject to the asymptotic shell insertion boundary condition localizes to geodesics on the background g . Recall that the saddle points for (B.1) were constructed by gluing $n + 1$ shell strips into a so-called central strip. We will use the same terminology here to denote the analogous regions on g . We imagine dividing up g by cutting out a small region around the i -th shell worldvolume of size ϵ_{s_i} (of which there are $n + 1$ in total), see figure 28(a). The complement of these regions is denoted by g_{bulk} and looks like a cut-out of a central strip and cut-out shells strips. The cut-outs ϵ_{s_i} are chosen such that all handles of g are contained within in g_{bulk} .

As the Einstein-Hilbert action consists of local and boundary terms we may write the total gravitational action of g as the sum of the action of these individual patches:

$$I_{\text{tot}}[g] = \sum_{i=1}^{n+1} I_{\epsilon_{s_i}} + I_{g_{\text{bulk}}} . \tag{B.2}$$

By shrinking the cut-out regions ϵ_{s_i} to include just the shell world-volumes the contribution to the action from the shells is given by the co-dimension 1 integral $\int_{\mathcal{W}_i[g]} \sigma_i$ of the shell matter density (σ_i) over the world-volume ($\mathcal{W}_i[g]$)

$$\sum_{i=1}^{(n+1)} I_{\epsilon_{s_i}} = \sum_{i=1}^{(n+1)} \int_{\mathcal{W}_i} \sigma_i \tag{B.3}$$

and hence we write

$$\zeta[G_{ij}^n \langle \Psi|i \rangle \langle j|\Psi \rangle] = \int_{g \rightarrow G_{ij}^n \langle \Psi|i \rangle \langle j|\Psi \rangle} \mathcal{D}g e^{-I_{\text{bulk}}} \prod_{i=1}^{n+1} e^{-\int_{\mathcal{W}_i[g]} \sigma_i} . \tag{B.4}$$

Now consider the g_{bulk} region, which consists cut-outs of both the central and shell strip regions (and possibly handles connecting them) with incomplete asymptotic boundaries, see figure 28(a). To make contact with the desired result, we imagine completing the boundary of each of these sections into a full strip boundary. Take the central strip cut-out for example, for every shell that was cut out we add in ‘‘homology’’ regions $g_{\mathcal{C}_i}$ between an added-in piece of asymptotic boundary of length T_i and the shell cut-out contour ϵ_{s_i} . There $n + 1$ such pieces needed to complete the central strip boundary resulting an asymptotic boundary of length $(n + 1)\beta_R + \tilde{\beta}_R + \sum_{i=1}^{n+1} T_i$, see figure 28(b). We can sum over all possible completions of the central strip cut-out $\subset g_{\text{bulk}}$ into a completed strip by summing over all possible homology region geometries $g_{\mathcal{C}_i}$ ending on the worldvolume \mathcal{W}_i : $\int \mathcal{D}g_{\mathcal{C}_i} e^{-I_{g_{\mathcal{C}_i}}} \equiv Z_{\mathcal{C}_i}[g]$. Similarly the i -th cut-out shell strip can be completed into a full strip by adding in a homology region

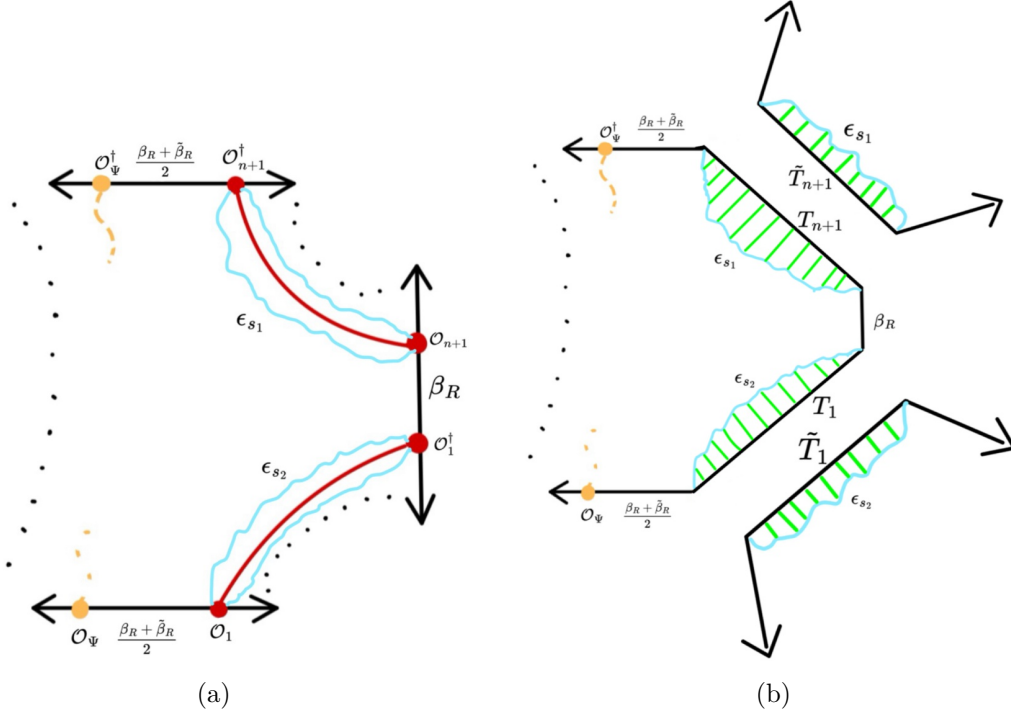


Figure 28. (a) Cartoon of the ϵ_{s_i} cut-outs around the shell worldvolumes for some contribution to (B.1), depicted for $n = 2$. (b) The homology regions are filled in (green) to complete the bulk regions into two vacuum strips and one central strip.

g_{S_i} corresponding to a boundary section of length \tilde{T}_i and we define $\int \mathcal{D}g_{S_i} e^{-I_{g_{S_i}}} \equiv Z_{S_i}[g]$. While these terms were not present in the original path integral we can rewrite (B.4) in a helpful way by inserting $1 = \frac{\prod_{i=1}^{n+1} Z_{C_i}[g]}{\prod_{i=1}^{n+1} Z_{C_i}[g]}$ and $1 = \frac{\prod_{i=1}^{n+1} Z_{S_i}[g]}{\prod_{i=1}^{n+1} Z_{S_i}[g]}$:

$$\begin{aligned}
 & \int_{g \rightarrow G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle} \mathcal{D}g e^{-I_{\text{bulk}}} \prod_{i=1}^{n+1} e^{-\int \mathcal{W}_i[g] \sigma_i} \\
 &= \int_{g \rightarrow G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle} \mathcal{D}g \frac{\prod_{i=1}^{n+1} Z_{C_i}[g]}{\prod_{i=1}^{n+1} Z_{C_i}[g]} \frac{\prod_{i=1}^{n+1} Z_{S_i}[g]}{\prod_{i=1}^{n+1} Z_{S_i}[g]} e^{-I_{\text{bulk}}} \prod_{i=1}^{n+1} e^{-\int \mathcal{W}_i[g] \sigma_i} \\
 &= \int_{g \rightarrow G_{ij}^n \langle \Psi | i \rangle \langle j | \Psi \rangle} \mathcal{D}g e^{-I_{\text{bulk}}} \prod_{i=1}^{n+1} \int \mathcal{D}g_{Y,i} \mathcal{D}g_{C,i} e^{-I_{g_{C,i}}} e^{-I_{g_{S,i}}} \left(\frac{e^{-\int \mathcal{W}_i[g] \sigma_i}}{Z_{C_i}[g] Z_{S_i}[g]} \right)
 \end{aligned} \tag{B.5}$$

The path integral $\int \mathcal{D}g \mathcal{D}g_{C,i} \mathcal{D}g_{S,i}$ therefore sums over all geometries satisfying the boundary condition of a completed central strip of length $(n+1)\beta_R + \tilde{\beta}_R + \sum_{i=1}^{n+1} T_i$ and $n+1$ “vacuum” strips with no operator insertion where the i -th strip has length \tilde{T}_i . Note that due to the cut-out these boundary conditions do not contain any shell insertions, and that the sum over geometries includes those with handles connecting, say, the central strip bulk to the k -th vacuum strip. In terms of the notation used through this work, this is nothing more than the boundary condition $\langle \Psi | \Psi \rangle (n+1)\beta_R + \tilde{\beta}_R + \sum_{i=1}^{n+1} T_i \prod_{i=1}^{n+1} S(\tilde{T}_i)$, where $\langle \Psi | \Psi \rangle (\dots)$ was defined to denote the norm of a time evolved version of the state $|\Psi\rangle$. Although there is the possibility of handles in the g_{bulk} region connecting the central and

shells strips, the cut-outs ϵ_{s_i} where shrunken to just include the shell worldvolumes. The shell worldvolume contributions to the action are therefore insensitive to these handles. The above path integral can therefore alternatively be written as an integral over metrics \hat{g} satisfying the $\hat{g} \rightarrow \langle \Psi | \Psi \rangle (n+1)\beta_R + \tilde{\beta}_R + \sum_{i=1}^{n+1} T_i \prod_{i=1}^{n+1} S(\tilde{T}_i)$ asymptotic boundary condition:

$$\int_{\hat{g} \rightarrow \langle \Psi | \Psi \rangle (n+1)\beta_R + \tilde{\beta}_R + \sum_{i=1}^{n+1} T_i \prod_{i=1}^{n+1} S(\tilde{T}_i)} \mathcal{D}\hat{g} e^{-I_{\hat{g}}} \prod_{i=1}^{n+1} \left(\frac{e^{-\int w_i[\hat{g}] \sigma_i}}{Z_{C_i}[\hat{g}] Z_{S_i}[\hat{g}]} \right), \quad (\text{B.6})$$

and we can now see that the $n \rightarrow -1$ limit recovers

$$\int_{\hat{g} \rightarrow \langle \Psi | \Psi \rangle (\tilde{\beta}_R)} \mathcal{D}\hat{g} e^{-I_{\hat{g}}} = \overline{\langle \Psi | \Psi \rangle}. \quad (\text{B.7})$$

This shows full equality between the path integrals $\overline{\langle \Psi | \prod_{\mathcal{H}_R} | \Psi \rangle} = \overline{\langle \Psi | \Psi \rangle}$. The argument can be extended to show $(\overline{\langle \Psi | \prod_{\mathcal{H}_R} | \Psi \rangle} - \overline{\langle \Psi | \Psi \rangle})^2 = 0$ by including all handles connecting $\langle \Psi | \prod_R | \Psi \rangle$ with $\langle \Psi | \prod_R | \Psi \rangle$ (or $\langle \Psi | \Psi \rangle$ in the cross term) into the I_{bulk} region and repeating the above argument. This concludes our argument that the $\kappa \rightarrow \infty$ set of shell states at any preparation temperature spans the fine-grained Hilbert space \mathcal{H}_{LR} in the full gravitational path integral.

B.2 All order argument for $\mathcal{H}_{2s} \subseteq \mathcal{H}_L \otimes \mathcal{H}_R$

In this section we adopt the notation of section 4.2. In the limit where the shells are parametrically heavy the extension to include perturbations around the saddlepoints follows immediately by the same logic as in section B.1. We can again extend further to topologies that do not allow saddlepoints. The argument in section B.1 was made by cutting up the connected contributions to (B.1) along the shell world volumes, resulting in a central strip and $n+1$ vacuum strips. Consider now the connected contributions to

$$\overline{G_{L,ac}^n G_{R,il}^m \langle \mathbf{p} |_{2s} a \rangle_L | i \rangle_R \langle c |_L \langle l |_R | \mathbf{p} \rangle_{2s}} \quad (\text{B.8})$$

as in figure 17(a). If we cut a given geometry along both the L and R shell worldvolumes we end up with a set of cut-out L and R strips and a central region that looks like a cut-out circle containing the two-sided shell \mathbf{p} insertions. Whereas before in the $n \rightarrow -1$ limit the strips disappeared and the cut-out of the central strip was glued together into the operator strip boundary condition defining $\langle \Psi | \Psi \rangle$, here we find that the cut-out circle gets glued into the boundary circle defining $\langle \mathbf{p} | \mathbf{p} \rangle_{2s}$ as $n, m \rightarrow -1$. In particular, the asymptotic boundary of this circle and of the L, R strips can again be completed by adding in homology regions of time extent $T_k, \tilde{T}_{i,L}, \tilde{T}_{j,R}$ respectively in a completely analogous manner as above. For the cut-out circle this results in an asymptotic boundary condition that is simply the norm $\langle \mathbf{p} | \mathbf{p} \rangle_{2s}$ of a time-evolved version of the $|\mathbf{p}\rangle_{2s}$ state. Hence we end up rewriting the path integral (4.16) as one that sums over geometries and topologies with a boundary consisting of several disconnected components: (a) a circular asymptotic boundary condition consisting of the $\mathcal{O}_{\mathbf{p}}$ and $\mathcal{O}_{\mathbf{p}}^\dagger$ operators separated by an L, R time of $\sum_{k=1}^{n+1} T_k + \tilde{\beta}_L, \sum_{l=1}^{m+1} T_l + \tilde{\beta}_R$ respectively, (b) n copies of a vacuum L strip, and (c) m copies of a R vacuum strip.

In the limit that $n, m \rightarrow -1$ the shell dependent terms vanish and we are left with the path integral computing $\overline{\langle \mathbf{p} | \mathbf{p} \rangle_{2s}}$. We therefore have $\overline{\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} | \mathbf{p} \rangle_{2s}} = \overline{\langle \mathbf{p} | \mathbf{p} \rangle_{2s}}$ to

as a full gravity path integral equality. The argument can again be extended to show $\left(\langle \mathbf{p} | \prod_{\mathcal{H}_L \otimes \mathcal{H}_R} | \mathbf{p} \rangle_{2s} - \langle \mathbf{p} | \mathbf{p} \rangle_{2s}\right)^2$ by including any handles connecting the various boundary insertions into the g_{bulk} region.

B.3 All order argument for $\mathcal{H}_L \otimes \mathcal{H}_R \subseteq \mathcal{H}_{2s}$

In this section we adopt the notation of section 4.1. The argument extending (4.8) and (4.9) now proceeds analogously to sections B.1 and B.2. Again, in the limit where the shells are parametrically heavy the extension to include perturbations around the saddlepoints follows from the same logic as in section B.1. As the relevant connected contributions to (4.8) and (4.9) are wormholes of mixed single- and two-sided boundary conditions, the extension to include topologies that do not allow saddlepoints is done by combining elements of the argument in section B.1 above and that of section 5.1 of [7]. In particular, for any geometry that contributes we can produce a so-called “sheet diagram” by cutting the geometry along the two-sided shell worldvolumes, resulting in one L and R sheet, potentially connected by handles. We now again imagine cut-outs around each two-sided shell, and complete the two sheets into two strips by filling in homology regions. The rest of the argument now carries over straightforwardly from above: in the $n \rightarrow -1$ limit we again find that the shell contributions vanish and the boundary condition reduces to the strip boundary conditions defining $\overline{\langle i|i \rangle_L \langle j|j \rangle_R}$. The argument for $(\langle i|_L \langle j|_R \prod_{\mathcal{H}_{2s}} |i \rangle_L |j \rangle_R - \langle i|i \rangle_L \langle j|j \rangle_R)^2 = 0$ is again done by including any handles connecting the various boundary insertions into the g_{bulk} region.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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