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The Categorified Heisenberg Algebra I: A Combinatorial Representation

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Abstract

We show how Khovanov’s categorification of the Heisenberg algebra arises as a linearization of a discrete combinatorial structure in the bicategory of spans of groupoids. We also treat a categorification of $U(\mathfrak{sl}_n)$ in a similar way.

1 Introduction

1.1 Overview

Our objective in this paper is to describe a combinatorial model of Khovanov’s categorification of the Heisenberg algebra [15], using a natural construction based on the groupoidification programme of Baez and Dolan [1, 2]. This gives rise to a narrative which serves to ‘explain’ the structure of the categorified algebra in terms of the combinatorics of finite sets. It is also fundamental, giving rise to known linear representations via a canonical process of linearization. This account is one part of a theory of representations of the categorified Heisenberg algebra in terms of free symmetric algebraic structures, described in a forthcoming companion article [20].

The model of the categorified Heisenberg algebra which we describe is based on a groupoidification of the quantum harmonic oscillator, a simple quantum-mechanical system. This system has an infinite-dimensional Hilbert space of states called *Fock space*, which carries an action of a *Heisenberg algebra*. The simplest of these, describing a quantum harmonic oscillator with a single degree of freedom, is the free complex algebra on generators \mathbf{a}^\dagger and \mathbf{a} , called the *creation* and *annihilation* operators respectively, modulo the commutation relation

$$\mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1. \quad (1)$$

The action on Fock space involves unbounded operators, and equation (1) is only required to hold on a dense domain.

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The Fock space for a single oscillator has a standard basis of states called the Fock basis, which are labeled by non-negative integer *energy levels*. Elements of this basis represent states in which the system contains an integer number of energy quanta, interpreted in a quantum field theory setting as ‘particles’. The creation and annihilation operators act on this basis to increase or decrease the number of particles by one.

This description is already understood to be a ‘shadow’, or *decategorification*, of a richer perspective in which the actual sets of particles are themselves the relevant mathematical objects [1, 2, 22]. Thus, for instance, rather than merely describing a change in particle number, one can instead consider the maps between sets of particles which *witness* that change. This makes room for more structure, so that one can consider the action of the symmetric group which corresponds to physically permuting the particles, an act which has no nontrivial mathematical representation in the case of ordinary Fock space. The usual Fock space picture is recovered in the decategorification (or ‘degroupoidification’) of this structure.

The categorified Heisenberg algebra captures the interesting mathematical statements that can be made in this richer setting. The new contribution here is the interpretation of these statements in terms of the combinatorics of finite sets.

These structures have an abstract mathematical description which makes them more general than any one model. In particular, there are applications in computer science [7, 8, 9] for which a Fock space-like structure represents an unlimited number of copies of some logical resource; the constructions of this paper are relevant to categorifications of this scenario. We will explore this more general perspective in a more technical article [20], where we show how representations of the categorified Heisenberg algebra arise generically on free symmetric pseudomonoid structures internal to monoidal bicategories with sufficiently good properties.

1.2 Khovanov’s categorification

Technically, to *categorify* a complex algebra means to find a monoidal category \mathbf{C} with coproducts, such that the algebra can be recovered as the complexification of the monoid of isomorphism classes of objects of \mathbf{C} , with the vector space structure on the algebra arising from the coproduct structure in \mathbf{C} . This can be seen as the composition of two mathematical processes:

$$[\text{Monoidal Categories}] \xrightarrow{K_0} [\text{Rings}] \xrightarrow{\mathbb{C} \otimes -} [\text{Algebras}] \quad (2)$$

Khovanov has given a categorification of the Heisenberg algebra in this sense [15], part of a broader programme which includes the categorification of quantum groups [16, 31]. In general, this programme involves the construction of monoidal categories whose morphisms are classes of diagrams, sometimes with decorations of various kinds, modulo certain topological identifications.

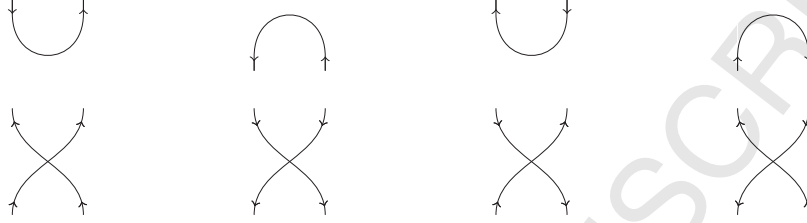
To construct the categorified Heisenberg algebra, Khovanov first constructs the following auxiliary monoidal category. Khovanov requires his monoidal category to be linear over some field k , but we neglect this here.

Definition 1.1. The monoidal category \mathbf{H}' is the free monoidal category on the generators given below, modulo the relations, with a zero object and biproducts. The generating objects are Q_+ and Q_- , which we depict as upwards- and downwards-pointing strands

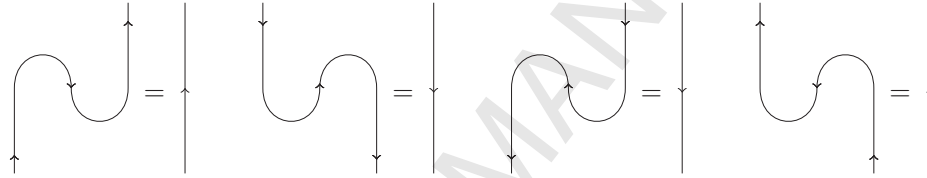
respectively:

$$\begin{array}{c} \downarrow \\ Q_+ \end{array} \quad \begin{array}{c} \downarrow \\ Q_- \end{array}$$

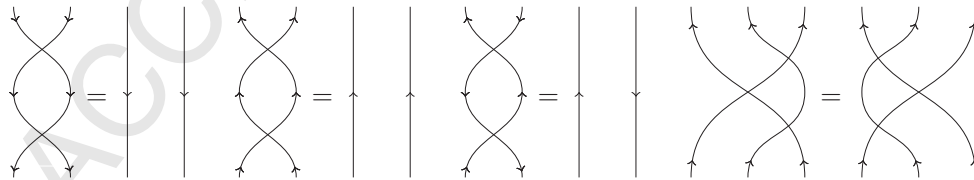
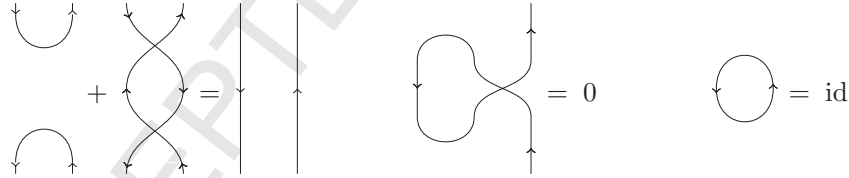
Tensor product is represented by horizontal juxtaposition and composition by vertical concatenation of diagrams. Then the morphisms are classes of string diagrams taken up to planar isotopy. These are generated by diagrams which we represent graphically as follows:



The fact that the morphisms are isotopy classes of string diagrams means in particular that the following relations must hold:



These relations ensure that the upward-pointing and downward pointing arrows satisfy a formal adjunction. It is a common requirement in the categorification of algebras using a graphical calculus. Moreover, we impose the following additional relations on the generators, which are specific to the Heisenberg algebra:



The top-left equation here uses additive structure which is induced canonically by the biproduct structure. These crossings cannot be interpreted as braidings as they are not invertible, as emphasized by the top-left diagram. These equations imply an isomorphism

$$Q_- \otimes Q_+ \simeq (Q_+ \otimes Q_-) \oplus I \quad (3)$$

where I is the monoidal unit object, giving a categorification of the Heisenberg algebra relation (1).

Khovanov goes on to show that this has a representation on a monoidal category whose objects are bimodules describing restriction and induction of representations of symmetric groups. As we will see, this amounts to a representation on *2-vector spaces* in the sense of Kapranov and Voevodsky.

We can make this more precise with the following definition.

Definition 1.2. Let $\Omega_{\mathbf{H}'}$ be the bicategory with one object arising from the monoidal category \mathbf{H}' .

Then one of the results of this paper is to define a 2-functor

$$K : \Omega_{\mathbf{H}'} \rightarrow \mathbf{2Vect} \quad (4)$$

where $\Omega_{\mathbf{H}'}$ is a bicategory with one object, whose 1-morphisms and 2-morphisms come from the objects and morphisms of \mathbf{H}' . This takes the single object of $\Omega_{\mathbf{H}'}$ to a 2-vector space $\coprod_n \text{Rep}(S_n)$. This is the coproduct of the categories of representations of the symmetric groups, or equivalently the representation category of the groupoid \mathbf{S} , which is the coproduct of all the symmetric groups; it plays the role of a categorified Fock space. The same data could alternatively be encoded as a monoidal functor from \mathbf{H}' to the endomorphism category $\text{Hom}(\text{Rep}(\mathbf{S}), \text{Rep}(\mathbf{S}))$.

We will restrict our attention to this monoidal category \mathbf{H}' of diagrams in Section 2, when we give them a combinatorial interpretation. However, in fact, the categorification of the full Heisenberg algebra with multiple generators a_n requires somewhat more. The generators a_n are represented in the categorification by permutation-invariant subobjects of products of the basic objects Q_+ and Q_- . In the diagrammatic categorification, this is done formally by completing \mathbf{H}' to a category called \mathbf{H} , which is k -linear over a field k of characteristic 0, and in which all idempotents split. In Section 3, we will turn to this in more detail; in particular, in section 3.6 we discuss how the particular symmetrizer subobjects which appear in this completion occur naturally in $\mathbf{2Vect}$ as sub-functors, and describe some of them in detail.

1.3 Groupoidification

The concrete model of \mathbf{H}' we find arises from a seemingly different approach to categorifying the Heisenberg algebra, based on the *groupoidification* program of Baez and Dolan [2]. A *groupoid* is a category in which all morphisms are invertible. The central objects of study in the groupoidification programme are *spans of groupoids*, diagrams of the form

$$\begin{array}{ccc} & \mathbf{X} & \\ G \swarrow & & \searrow F \\ \mathbf{B} & & \mathbf{A} \end{array} \quad (5)$$

where \mathbf{A} , \mathbf{B} and \mathbf{X} are groupoids, and F and G are functors. There has been extensive work within category theory on constructions involving spans [6]. The most immediately important fact is that spans of groupoids can be organized into a bicategory $\mathbf{Span}(\mathbf{Gpd})$. The main construction of this paper is a 2-functor

$$C : \Omega_{\mathbf{H}'} \rightarrow \mathbf{Span}(\mathbf{Gpd}), \quad (6)$$

which gives a representation of Khovanov's categorification \mathbf{H}' of the Heisenberg algebra in terms of spans of groupoids.

An important interpretation of groupoidification comes from physics: groupoids represent *physical symmetries*, and spans represent *spaces of histories*. The idea is that configuration spaces of physical systems can be represented as groupoids in a way that usefully encodes their symmetries, and that spans of groupoids encode ways in which these states and their symmetries can be transformed via physical processes. In our example of the harmonic oscillator, these configurations are the non-negative integer-valued energy eigenstates. Then a span represents a space of *histories*, with its source and target maps picking out the starting and ending configurations. Furthermore, these are not just set-maps of the objects (histories and configurations), but functors, which also describe how symmetries of histories act on starting and ending configurations.

The single object of $\Omega_{\mathbf{H}}$ is mapped by C to the groupoid \mathbf{S} of finite sets and bijections. Spans involving this groupoid have an elegant interpretation in term of the combinatorics of finite sets, thanks to work of Joyal [14] and Baez and Dolan [1]. The resulting combinatorial interpretation of our representation gives us a new perspective on the categorified Heisenberg algebra.

A construction called *2-linearization* [23] converts a span of groupoids to a 2-linear map between 2-vector spaces. This resulting 2-linear map can be thought of as being ‘accounted for’ by the underlying span of groupoids, giving rise to a useful combinatorial perspective on the mathematics. The 2-linearization process gives a 2-functor of the following type:

$$\Lambda : \mathbf{Span}(\mathbf{Gpd}) \rightarrow \mathbf{2Vect} \quad (7)$$

Composing this functor with our representation (6) gives us a linear representation of the categorified Heisenberg algebra, which agrees with Khovanov’s representation. This is our main theorem, stated as follows in Section 3.5.

Theorem 3.7. *The following diagram commutes up to pseudonatural equivalence:*

$$\begin{array}{ccc} & \xrightarrow{C} & \mathbf{Span}(\mathbf{Gpd}) \\ & & \downarrow e \\ \Omega_{\mathbf{H}} & \xrightarrow{K} & \mathbf{2Vect} \end{array} \quad (8)$$

In this way, our results give a combinatorial ‘explanation’ of Khovanov’s construction.

In Section 4 we extend these ideas to produce a groupoidification of the universal enveloping algebras $U(\mathfrak{sl}_n)$ of the Lie algebras \mathfrak{sl}_n . Since these algebras are closely related to the Heisenberg algebra it is perhaps not surprising that such a treatment can be given. The treatments of both families of algebras is unified in the accompanying article [20], in which the representations of the categorified Heisenberg algebra and categorified $U(\mathfrak{sl}_n)$ are both seen as arising from free symmetric monoidal groupoids: for the Heisenberg algebra, on the trivial groupoid with one morphism; and for $U(\mathfrak{sl}_n)$, on the discrete groupoid with n morphisms. The construction relating Heisenberg algebras and $U(\mathfrak{sl}_n)$ is related to Kac-Moody algebras and their categorifications [25], so groupoidification may also provide a useful perspective in this case.

In the q -deformed case, categorifications in terms of 2-vector spaces have been described as a part of the Khovanov-Lauda programme [16]. The groupoidification formalism used here, based on the groupoid of finite sets and bijections, cannot be directly applied in this case. Baez, Hoffnung and Walker [2] have suggested that such groupoidifications should not be in terms of bijections of sets, but rather linear bijections of vector spaces over the finite field \mathbb{F}_q with q elements, for q a prime power. These would be used as groupoidifications of the Hecke algebras which appear in place of symmetric group algebras in the

categorifications of quantum groups. If successful, this would yield combinatorial models for these categorified q -deformed algebras, yielding further insight into their structure.

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2 Groupoidification of the Heisenberg algebra

2.1 Spans of groupoids

Our combinatorial representation of the categorified Heisenberg algebra will be given in terms of *spans of groupoids*. A groupoid is a category in which all morphisms are invertible, and a span of groupoids from \mathbf{A} to \mathbf{B} is a diagram of the following form, where \mathbf{A} , \mathbf{B} and \mathbf{X} are groupoids and F and G are functors:

$$\begin{array}{ccc} & \mathbf{X} & \\ G \swarrow & & \searrow F \\ \mathbf{B} & & \mathbf{A} \end{array} \quad (9)$$

A span like this forms a 1-morphism of type $\mathbf{A} \rightarrow \mathbf{B}$ in the monoidal bicategory $\mathbf{Span}(\mathbf{Gpd})$. The symmetry in the definition of a span means that, for any span $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ as above, we can define its *converse* $(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A})^\dagger$ as the span $\mathbf{A} \xleftarrow{F} \mathbf{X} \xrightarrow{G} \mathbf{B}$. Every 1-morphism is adjoint to its converse in $\mathbf{Span}(\mathbf{Gpd})$, in the internal sense of bicategory theory.

If we think of the objects of \mathbf{A} and \mathbf{B} as being the states of some physical system, then any $x \in \text{Ob}(\mathbf{X})$ can be interpreted as a ‘history’ relating the state $F(x) \in \text{Ob}(\mathbf{A})$ to the state $G(x) \in \text{Ob}(\mathbf{B})$. Because we are working with groupoids, our states and histories come equipped with symmetry groups, and the functors F and G show how the symmetries of histories are mapped to symmetries of states.

The combinatorial interpretation that we will explore for these spans arises from the fundamental role played here by the groupoid of finite sets. The Fock space is represented by the groupoid \mathbf{S} and the annihilation and creation operators A and A^\dagger by the following spans of groupoids:

$$\begin{array}{ccc} & \mathbf{S} & \\ \text{id} \swarrow & & \searrow +1 \\ \mathbf{S} & & \mathbf{S} \end{array} \quad \begin{array}{ccc} & \mathbf{S} & \\ +1 \swarrow & & \searrow \text{id} \\ \mathbf{S} & & \mathbf{S} \end{array} \quad (10)$$

Here \mathbf{S} is the groupoid of finite sets and bijections, and $\mathbf{S} \xrightarrow{+1} \mathbf{S}$ is the functor taking the disjoint union with the one-element set.

A different perspective on \mathbf{S} is to see it as the *free symmetric monoidal groupoid* on the trivial groupoid $\mathbf{1}$ with one object and one morphism. The functor $\mathbf{S} \xrightarrow{+1} \mathbf{S}$ then arises by taking the tensor product with the generating object. In fact, this free symmetric monoidal perspective is fundamental, and allows for the construction of a representation of the categorified Heisenberg algebra in completely abstract terms. We develop this perspective in detail in the companion article [20].

2.2 Combinatorics

Spans of groupoids can be used to reason about the combinatorics of structures on finite sets. The theory of *stuff types* [1] has been developed to describe how this works, generalizing Joyal's theory of *structure types* [14]. Given some particular structure of interest constructed from a finite set, the groupoid \mathbf{G} of *models* of this structure can be built, with morphisms given by symmetries that relate one model to another. This comes equipped with a functor $\mathbf{G} \xrightarrow{F} \mathbf{S}$, called a *stuff type*, where \mathbf{S} is the groupoid of finite sets and bijections. This functor assigns to each model its underlying set, and to symmetries of models the induced bijections on the underlying sets. We can interpret any abstract stuff type as describing some combinatorial structure; in general, a finite set equipped with some structure which satisfies some property. A stuff type with good properties can be *de-groupoidified*, a form of *linearization*, giving rise to a generating function for the structure in the ordinary sense.

Since every groupoid has a unique functor to the 1-element set, stuff types are precisely spans of the form

$$\begin{array}{ccc} & \mathbf{G} & \\ F \swarrow & & \searrow \\ \mathbf{S} & & \mathbf{1}, \end{array} \quad (11)$$

which are morphisms of type $\mathbf{1} \rightarrow \mathbf{S}$ in $\mathbf{Span}(\mathbf{Gpd})$. A *stuff operator* is a span of groupoids of type $\mathbf{S} \rightarrow \mathbf{S}$:

$$\begin{array}{ccc} & \mathbf{H} & \\ K \swarrow & & \searrow J \\ \mathbf{S} & & \mathbf{S} \end{array} \quad (12)$$

A stuff operator thus contains a single groupoid \mathbf{H} of models, but two potentially different descriptions K and J of underlying sets; it tells us how the same structure can be built in different ways. In the physical interpretation we have mentioned, these models — the objects in the middle groupoid — are seen as *histories*, and the two underlying sets are the starting and ending configurations of these histories. Stuff operators can act on stuff types by composition in $\mathbf{Span}(\mathbf{Gpd})$ as described in Section 2.3, producing new stuff types whose structures are composites of those from the original stuff type with the histories from the stuff operator.

For example, consider the stuff type $\mathbf{S} \xrightarrow{+1} \mathbf{S}$ which takes the union of each set with a chosen 1-element set. This represents the structure ‘finite sets with a chosen element’, with symmetries given by bijections that leave the chosen element fixed. Then our annihilation operator A defined in (10) is a stuff operator which treats an n -element set in the middle copy of \mathbf{S} in two different ways: by the right leg, as an $n+1$ -element set with a chosen element; and by the left leg, simply as an n -element set. It can be thought of as a ‘rule’ for how to consider an $n+1$ -element set as an n -element set — or more simply, as a way to *remove an element* from a finite set.

2.3 Composition of Spans and Identities

Given spans of groupoids $P = (\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A})$ and $Q = (\mathbf{C} \xleftarrow{K} \mathbf{Y} \xrightarrow{J} \mathbf{B})$, we can compose them to obtain a span $Q \circ P = (\mathbf{C} \xleftarrow{K \circ P_Y} (J \downarrow G) \xrightarrow{F \circ P_X} \mathbf{A})$. The construction is in terms

of a *pseudo-pullback* (also termed *homotopy pullback*) groupoid $(J \downarrow G)$:

$$\begin{array}{c}
 \text{---} (J \downarrow G) \text{---} \\
 \text{K} \circ P_Y \quad P_Y \quad P_X \quad F \circ P_X \\
 \swarrow \quad \searrow \quad \swarrow \quad \searrow \\
 \mathbf{Y} \xrightarrow{\alpha} \mathbf{X} \\
 \downarrow J \quad \downarrow G \\
 \mathbf{B}
 \end{array}
 \quad (13)$$

This groupoid $(J \downarrow G)$ is equipped with functors $(J \downarrow G) \xrightarrow{P_Y} \mathbf{Y}$ and $(J \downarrow G) \xrightarrow{P_X} \mathbf{X}$, and a natural isomorphism $J \circ P_Y \xrightarrow{\alpha} G \circ P_X$. It is defined to satisfy a universal property: for any groupoid \mathbf{Z} equipped with functors $\mathbf{Z} \xrightarrow{Z_Y} \mathbf{Y}$ and $\mathbf{Z} \xrightarrow{Z_X} \mathbf{X}$ and a natural isomorphism $J \circ Z_X \xrightarrow{\zeta} G \circ Z_Y$, there must exist a functor $\mathbf{Z} \xrightarrow{L} (J \downarrow G)$, unique up to isomorphism, such that $L \circ \alpha = \zeta$.

Based on the universal property described above, a standard construction for $(J \downarrow G)$ is the following groupoid:

- **Objects** are triples $(x \in \text{Ob}(\mathbf{X}), y \in \text{Ob}(\mathbf{Y}), G(x) \xrightarrow{f} J(y))$.
- **Morphisms** $(x_1, y_1, f_1) \rightarrow (x_2, y_2, f_2)$ are pairs of morphisms $x_1 \xrightarrow{a} x_2$ and $y_1 \xrightarrow{b} y_2$ satisfying the following commuting diagram:

$$\begin{array}{ccc}
 G(x_1) & \xrightarrow{f_1} & J(y_1) \\
 \downarrow G(a) & & \downarrow J(b) \\
 G(x_2) & \xrightarrow{f_2} & J(y_2)
 \end{array}
 \quad (14)$$

This construction is essentially a weak form of the fibred product, where instead of taking pairs (x, y) whose images in \mathbf{B} agree, we choose a specific isomorphism between them. We can unpack what this means for stuff types, for which $\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{S}$, in a concise way: an object in the pseudo-pullback is a pair of models in \mathbf{X} and \mathbf{Y} equipped with a bijection of underlying sets; and a morphism is a pair of symmetries of models which induce the same bijections on the underlying sets. This construction is essentially a weak form of the fibred product, where instead of taking pairs (x, y) whose images in \mathbf{B} agree, we choose a specific isomorphism between them.

The fact that composition of spans is by homotopy pullback has the consequence that $\mathbf{Span}(\mathbf{Gpd})$ has only weak units. The identity span on a groupoid \mathbf{A} is the span $(\mathbf{A} \xleftarrow{\text{id}_{\mathbf{A}}} \mathbf{A} \xrightarrow{\text{id}_{\mathbf{A}}} \mathbf{A})$. However, this is only a weak identity. To see this, consider the composite span $P \circ \text{id}_{\mathbf{A}}$, for P as given above: its central object $(F \downarrow \text{id}_{\mathbf{A}})$ has objects given by triples (x, f, a) for $a \in \text{Ob}(\mathbf{A})$, $x \in \text{Ob}(\mathbf{X})$ and $f : F(x) \rightarrow a$. This is not isomorphic in general to the set of objects of \mathbf{X} , so clearly $P \circ \text{id}_{\mathbf{A}} \neq P$. However, there is an *equivalence* of categories $(F \downarrow \text{id}_{\mathbf{A}}) \simeq \mathbf{X}$, which yields an equivalence of spans. In brief, the equivalence of groupoids is given by the following pair of functors between groupoids. First, $e : (F \downarrow \text{id}_{\mathbf{A}}) \rightarrow \mathbf{X}$ such that $e(x, f, a) = x$ and similarly, e acts as projection onto the first morphism in each pair. Secondly, its weak inverse $e' : \mathbf{X} \rightarrow (F \downarrow \text{id}_{\mathbf{A}})$ is such that $e'(x) = (x, \text{id}_{F(x)}, x)$, and similarly doubles each morphism.

2.4 The categorified commutation relation

The categorified form of the commutation relation (1) is an isomorphism of spans of the following form:

$$A \circ A^\dagger \simeq A^\dagger \circ A \oplus \text{id}_{\mathbf{S}} \quad (15)$$

The symbol ' \oplus ' represents the *direct sum* of spans, which is formed from the disjoint union of groupoids of histories. We begin with an intuitive argument for why this isomorphism should exist. (The formal proof, which requires the notion of a morphism between spans to be formalised, appears as Lemma 24.)

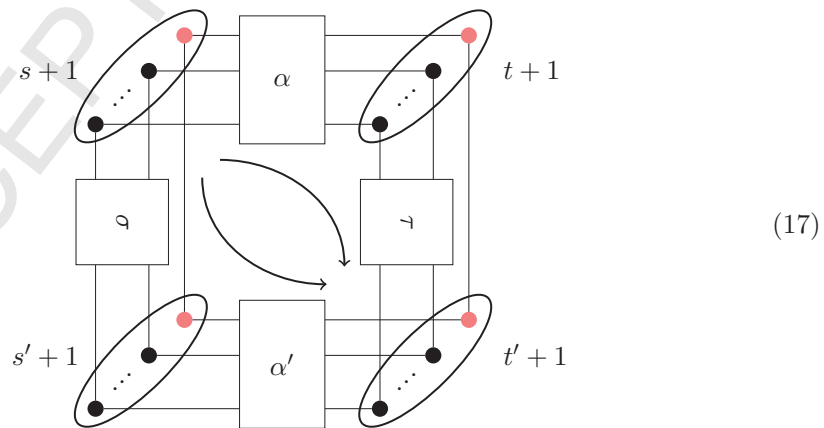
We saw above that the histories of the span A^\dagger represent all the ways to add an element to a finite set, and the histories of A represent all the ways to remove an element. The histories of $A \circ A^\dagger$ therefore represent all the ways to add an element x , and then remove an element y . Such histories can be divided into two distinct classes: those for which $x \neq y$, and those for which $x = y$. Restricting to the first case, one might as well remove y before adding x , and so we have a bijection to the histories for the span $A^\dagger \circ A$; in the second case, the set remains unchanged, corresponding to the identity span. These two cases give the two terms on the right-hand side, and the explicit case-by-case bijection we have constructed gives the isomorphism of spans.

We now verify this intuition with direct calculation. The composite $A \circ A^\dagger$ is the following span:

$$\begin{array}{ccc} & (+1 \downarrow +1) & \\ P \swarrow & & \searrow Q \\ \mathbf{S} & & \mathbf{S} \end{array} \quad (16)$$

The groupoid $(+1 \downarrow +1)$ has objects which are triples (s, t, α) , where s and t are sets and $s + 1 \xrightarrow{\alpha} t + 1$ is an isomorphism. The projection maps P and Q have the actions $P(s, t, \alpha) = s$ and $Q(s, t, \alpha) = t$ on objects.

A morphism $(s, t, \alpha) \rightarrow (s', t', \alpha')$ is a pair of isomorphisms $s \xrightarrow{\sigma} s'$ and $t \xrightarrow{\tau} t'$ such that $\alpha' \circ (\sigma + 1) = (\tau + 1) \circ \alpha$, with the projection maps P and Q taking this to σ and τ respectively. We can visualize this condition in the following way:



The extra element in each set is drawn in red. We show explicitly in this diagram how the permutations α , α' , σ and τ act on the different parts of each set. For the condition to be satisfied, the two composites from the top-left to the bottom-right must be equal.

With this diagram to aid us, we can draw some interesting conclusions. Firstly, for (s, t, α) and (s', t', α') to be isomorphic, then α must fix the extra element if and only if α'

does — and this is sufficient as long as $s \simeq s'$ and $t \simeq t'$. So for each isomorphism class of set S , there are exactly two isomorphism classes of object in its preimage under P or Q in $(+1 \downarrow +1)$. Secondly, if α and α' fix the extra element, then for every σ there exists some τ making the diagram commute, so there is an S_n -worth of morphisms between any two such objects for $n = |S|$. Thirdly, if neither α nor α' fix the extra element, then a given τ can only be part of a commuting diagram if $\tau(\alpha(1)) = \alpha'(1)$; in this case we can define α as the restriction to S of the composite $\alpha'^{-1} \circ (\tau + 1) \circ \alpha$. This means that, in this case, there is a S_{n-1} worth of morphisms between any two such objects for $n = |S|$.

This gives us a complete understanding of the isomorphism classes of object in $(+1 \downarrow +1)$ and their symmetries, which are the union of the entries in the following table:

	0	1	2	3	...	n	...
Extra element not fixed		S_0	S_1	S_2	...	S_{n-1}	...
Extra element fixed	S_0	S_1	S_2	S_3	...	S_n	...

(18)

The column headings indicate the cardinality of the set to which each object is projected, under the projection maps P and Q . As a result, we see that the span (16) can be written in the following way:

$$\begin{array}{ccc}
 & \mathbf{S} \cup \mathbf{S} & \\
 (\text{id}_{\mathbf{S}}, +1) \swarrow & & \searrow (\text{id}_{\mathbf{S}}, +1) \\
 \mathbf{S} & & \mathbf{S}
 \end{array}
 \quad (19)$$

Here $\mathbf{S} \cup \mathbf{S}$ represents the disjoint union of two copies of \mathbf{S} , and $(\text{id}_{\mathbf{S}}, +1)$ represents the functor which acts as the identity on the first copy of \mathbf{S} and as $+1$ on the second. We can consider this to be the union of spans $(\mathbf{S} \xleftarrow{\text{id}_{\mathbf{S}}} \mathbf{S} \xrightarrow{\text{id}_{\mathbf{S}}} \mathbf{S})$ and $(\mathbf{S} \xleftarrow{+1} \mathbf{S} \xrightarrow{+1} \mathbf{S})$, which gives us the isomorphism (15) we are seeking.

2.5 Forming a bicategory

So far our construction essentially follows that given in [22], but we now need to go further and use the bicategorical structure of $\mathbf{Span}(\mathbf{Gpd})$. This requires a notion of morphism between spans. For this, we define a *span of spans* of type $(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}) \rightarrow (\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A})$ is a span $\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}$ equipped with natural transformations $G \circ S \xrightarrow{\mu} J \circ T$ and $F \circ S \xrightarrow{\nu} K \circ T$, as indicated by the following diagram:

$$\begin{array}{ccccc}
 & & \mathbf{X} & & \\
 & G \swarrow & \uparrow S & \searrow F & \\
 \mathbf{B} & & \mathbf{Z} & & \mathbf{A} \\
 & \downarrow \mu & & \downarrow \nu & \\
 & J \swarrow & \downarrow T & \searrow K & \\
 & & \mathbf{Y} & &
 \end{array}
 \quad (20)$$

We say that two such spans of spans $(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)$ and $(\mathbf{X} \xleftarrow{S'} \mathbf{Z}' \xrightarrow{T'} \mathbf{Y}, \mu', \nu')$ are *equivalent* when there is an equivalence of groupoids $\mathbf{Z} \xrightarrow{U} \mathbf{Z}'$, and natural transformations $S \xrightarrow{\sigma} S' \circ U$ and $T \xrightarrow{\tau} T' \circ U$, such that the following pasted composites give μ and ν

respectively:

$$\begin{array}{ccc}
 & \text{X} & \\
 \text{G} \swarrow & & \searrow \text{F} \\
 \text{B} & & \text{A} \\
 \downarrow \mu' & \text{Z} \xleftarrow{U} \text{Z}' & \downarrow \nu' \\
 \text{Y} & & \text{Y} \\
 \text{J} \swarrow & & \searrow \text{K} \\
 & \text{X} &
 \end{array}
 \quad (21)$$

We use this to form a bicategory $\mathbf{Span}(\mathbf{Gpd})$ of spans of groupoids. Since this has been described in some detail elsewhere [23], in the case of finite groupoids, we will avoid much formal discussion of it here. However, we wish to use groupoids which are discrete but not finite, and there is an extra assumption which must be added for this to have a good linear representation theory. This is addressed in work on groupoidification [2] for the 1-category case, where it is noted that groupoids and spans must be *tame*. The following definitions indicate what this means:

Definition 2.1. The *cardinality* of a groupoid is the sum of rational numbers $|\mathrm{Hom}(A, A)|^{-1}$ over representatives A of every isomorphism class of object, if this sum is well-defined. Otherwise, the cardinality is not defined.

Definition 2.2. A groupoid is *essentially small* if it is equivalent to one with a set of objects, rather than a proper class.

Definition 2.3. A groupoid is *tame* if it is essentially small, with all hom-sets being finite, and with finite groupoid cardinality.

Note that we will make almost exclusive use of the groupoid of finite sets and bijections, which is indeed tame: the objects are determined, up to isomorphism, by an integer n ; the automorphism groups are isomorphic to the permutation groups S_n , which are finite; the groupoid cardinality is therefore $\sum_n \frac{1}{n!} = e$, which is finite.

Definition 2.4. If \mathbf{Y} is a tame groupoid, then a groupoid $\mathbf{Z} \xleftarrow{J} \mathbf{A}$ equipped with a map J into \mathbf{A} is *tame over \mathbf{A}* provided that for each $a \in \mathbf{A}$, the preimage $F^{-1}(a)$ is tame.

Notice that we do not require \mathbf{Z} itself to be tame, but only the individual fibres of J . Then we can define a tame span:

Definition 2.5. A span of groupoids $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ is *tame* if, for any $\mathbf{Z} \xleftarrow{J} \mathbf{A}$ which is tame over \mathbf{A} , the pseudo-pullback $(F \downarrow J)$ with the induced map into \mathbf{B} is tame over \mathbf{B} . A span is *cotame* if its converse span is tame.

These definitions play a crucial role in groupoidification, because they are the conditions which ensure the well-definition of a functor from groupoids and spans into vector spaces and linear maps. Under this functor, a tame groupoid \mathbf{A} determines a vector space, while groupoids which are tame over \mathbf{A} determine vectors in that vector space. The convergence of cardinalities ensured by tameness amount to the requirement that the components of this vector be well-defined. Similarly, tameness of a span ensures that it determines a well-defined linear map from the vector space associated to \mathbf{A} and that associated to \mathbf{B} . For more detail see [2], where the following condition is shown to imply tameness:

Proposition 2.6 (Baez, Hoffnung, Walker). *A span of groupoids $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ is tame when, for any pair of objects $(b, a) \in \mathbf{B} \times \mathbf{A}$, the joint preimage $G^{-1}(b) \cap F^{-1}(a)$ is tame, and for any object b in the target groupoid \mathbf{B} , for only finitely many a in the source \mathbf{A} is this joint preimage nonempty.*

Note that tameness of a span is an asymmetrical condition, so the converse notion of cotame span is also interesting. These conditions of tameness and cotameness can also be applied to spans-of-spans as in our bicategorical situation, and is similarly required of the 2-morphisms in $\mathbf{Span}(\mathbf{Gpd})$ to ensure that Λ is well-defined. Thus:

Definition 2.7. A span $\mathbf{B} \xleftarrow{G'} \mathbf{Z} \xrightarrow{F'} \mathbf{A}$ equipped with a span map (S, μ, ν) into the span $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ is *tame over* $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ provided that for each $x \in \mathbf{X}$, the preimage $S^{-1}(x)$, together with the restrictions to it of G' and F' , form a tame span.

As before, we can define a tame span-of-spans, exactly analogous to Definition 2.5:

Definition 2.8. A span of spans is *tame* provided that the pseudo-pullback with any span which is tame over its source gives a tame span.

We claim that there is a characterization of such 2-morphisms analogous to Proposition 2.6. Indeed, in the special case where \mathbf{A} and \mathbf{B} are both the terminal groupoid, the two propositions are equivalent.

Proposition 2.9. *A span of spans*

$$(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}) \xrightarrow{(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)} (\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A})$$

is tame if, for each pair of objects $(b, a) \in \mathbf{B} \times \mathbf{A}$, the joint preimage

$$(G^{-1}(b) \cap F^{-1}(a)) \xleftarrow{G} \mathbf{X} \xrightarrow{F} (J^{-1}(b) \cap K^{-1}(a))$$

is a tame span, and for any $b \in \mathbf{B}$, for only finitely many $a \in \mathbf{A}$ is it nonempty.

With the following natural definition we can also define *cotame* for spans of spans exactly as above:

Definition 2.10. For any 2-morphism

$$(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}) \xrightarrow{(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)} (\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A}),$$

we define its *converse* $(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)^\dagger$ as the following span of spans:

$$(\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A}) \xrightarrow{(\mathbf{Y} \xleftarrow{T} \mathbf{Z} \xrightarrow{S} \mathbf{X}, \mu^{-1}, \nu^{-1})} (\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}).$$

Then a span of spans is *cotame* if its converse is tame.

Given these definitions, we build the following bicategory.

Definition 2.11. The bicategory $\mathbf{Span}(\mathbf{Gpd})$ has:

- **Objects:** tame groupoids.
- **Morphisms:** spans of groupoids which are tame and cotame, with composition defined by pseudo-pullback.
- **2-Morphisms:** equivalence classes of spans of spans which are tame and cotame.

Composition is by pseudo-pullback as before.

This bicategory will be the formal setting for our results. The symmetric monoidal bicategory structure of a similar category has been described by Hoffnung and Stay [13, 28], although in the simpler situation where the 2-morphisms are span maps rather than spans of spans. We will not prove formally that our bicategory is well-defined, since a full proof would be a substantial paper in its own right, so we state for the record here our expectation.

Conjecture 2.12. *The bicategory $\mathbf{Span}(\mathbf{Gpd})$ can be equipped with the structure of a symmetric monoidal bicategory whose monoidal product is disjoint union of groupoids.*

We remark here that there are various possible bicategories one might use to generalize \mathbf{Gpd} . In particular, one common way to generalize functors between categories, which in this case happen to be groupoids, is to use *profunctors* (or distributors, to use the terminology of Bénabou [3]). This gives a bicategory which is somewhat different. Why does groupoidification use, instead, the bicategory $\mathbf{Span}(\mathbf{Gpd})$?

Our answer is essentially a positive remark about $\mathbf{Span}(\mathbf{Gpd})$ and its useful properties. In Section 2.9, we see that $\mathbf{Span}(\mathbf{Gpd})$ has the properties that it contains \mathbf{Gpd} as a sub-bicategory, and that every 1-morphism has a two-sided adjoint. Moreover, there are reasons to believe that $\mathbf{Span}(\mathbf{Gpd})$ is the universal bicategory with respect to these properties, in the sense that it is initial among all such bicategories [24] (though see also [5]). The property of having such adjoints is critical here, because the morphisms in Khovanov's monoidal category \mathbf{H}' have ambidextrous adjoints. $\mathbf{Span}(\mathbf{Gpd})$ is therefore a good choice for our setting.

2.6 The commutation relation

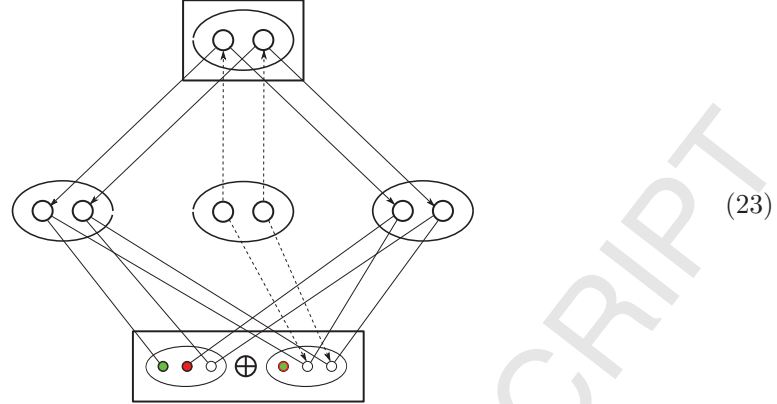
Based on the calculations in the previous section, we can define the following spans of spans, called i_{id_S} and $i_{A \circ A^\dagger}$:

$$\begin{array}{ccc}
 & \mathbf{S} & \\
 \text{id}_S \swarrow & & \searrow \text{id}_S \\
 \mathbf{S} & \downarrow \text{id} & \mathbf{S} \\
 \downarrow \text{id} & & \downarrow \text{id} \\
 (\text{id}_S, +1) & \mathbf{S} \cup \mathbf{S} & (\text{id}_S, +1) \\
 & \downarrow I_1 & \\
 \text{id}_S & \xrightarrow{i_{\text{id}_S}} & A \circ A^\dagger
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathbf{S} & \\
 +1 \swarrow & & \searrow +1 \\
 \mathbf{S} & \downarrow \text{id} & \mathbf{S} \\
 \downarrow \text{id} & & \downarrow \text{id} \\
 (\text{id}_S, +1) & \mathbf{S} \cup \mathbf{S} & (\text{id}_S, +1) \\
 & \downarrow I_2 & \\
 A^\dagger \circ A & \xrightarrow{i_{A^\dagger \circ A}} & A \circ A^\dagger
 \end{array}
 \tag{22}$$

Here, I_1 and I_2 are functors embedding the first and second copy of \mathbf{S} respectively.

We look more closely at the definition of i_{id_S} , looking ‘inside’ the groupoids for the

case of histories acting on the 2-element set:



Across the top of this picture, we see one object of the middle \mathbf{S} , namely the two-element set. The identity span maps this object down to the two-element set on either side. On the bottom, though, we see the two objects in $(+1 \downarrow +1)$ which map down to two-element sets. Each is a 3-element set, but as in the table (18), we see that the marked elements—the element added and the element removed—are either the same or different. The symmetry groups noted in (18) are those which permute the unmarked elements. The span of spans shows how objects in the identity relate to particular objects in $(+1 \downarrow +1)$. This determines an inclusion.

We use these ideas to demonstrate that the categorified commutation relation holds.

Definition 2.13. A *dagger-category* is a category \mathbf{C} equipped with an contravariant involutive endofunctor $\dagger : \mathbf{C} \rightarrow \mathbf{C}$, which is the identity on objects.

The involution on a dagger-category takes a morphism to its ‘converse’, in some sense. A basic model is the operation of taking adjoints of linear maps between finite-dimensional Hilbert spaces. In our setting, this structure is given by the converse span construction given in subsection 2.1.

Definition 2.14. In a dagger-category, a *dagger-biproduct* of objects A and B is an object $A \oplus B$ equipped with morphisms

$$\begin{aligned} i_A &: A \rightarrow A \oplus B \\ i_B &: B \rightarrow A \oplus B \end{aligned}$$

satisfying the following equations:

$$\begin{aligned} \text{id}_A &= (i_A)^\dagger \circ i_A \\ 0_{B,A} &= (i_A)^\dagger \circ i_B \\ 0_{A,B} &= (i_B)^\dagger \circ i_A \\ \text{id}_B &= (i_B)^\dagger \circ i_B \\ \text{id}_{A \oplus B} &= i_A \circ (i_A)^\dagger + i_B \circ (i_B)^\dagger \end{aligned}$$

Lemma 2.15. *There is an isomorphism of spans of groupoids*

$$A \circ A^\dagger \simeq (A^\dagger \circ A) \oplus \text{id}_{\mathbf{S}}, \quad (24)$$

where \oplus represents the dagger-biproduct of spans of groupoids.

Proof. The \dagger -biproduct is witnessed by the following equations involving the injection 2-morphisms i_{id_S} , $i_{A^\dagger \circ A}$ and their converses:

$$\text{id}_{\text{id}_S} = (i_{\text{id}_S})^\dagger \circ i_{\text{id}_S} \quad (25)$$

$$0_{A^\dagger \circ A, \text{id}_S} = (i_{\text{id}_S})^\dagger \circ i_{A^\dagger \circ A} \quad (26)$$

$$0_{\text{id}_S, A^\dagger \circ A} = (i_{A^\dagger \circ A})^\dagger \circ i_{\text{id}_S} \quad (27)$$

$$\text{id}_{A^\dagger \circ A} = (i_{A^\dagger \circ A})^\dagger \circ i_{A^\dagger \circ A} \quad (28)$$

$$\text{id}_{A \circ A^\dagger} = i_{\text{id}_S} \circ (i_{\text{id}_S})^\dagger + i_{A^\dagger \circ A} \circ (i_{A^\dagger \circ A})^\dagger \quad (29)$$

We will see that these injection and projection maps, which characterize the \dagger -biproduct, are related to the adjointness of the spans A and A^\dagger . Correctness of these equations follows from the combinatorial interpretations of these spans of spans, which we develop below. \square

2.7 Graphical notation

We now introduce a graphical notation for certain spans of type $S \leftarrow X \rightarrow S$, and for the 2-morphisms going between them in $\mathbf{Span}(\mathbf{Gpd})$. This is the notation of Khovanov [15], and is an application of the standard graphical calculus for morphisms in a monoidal category [27]. Creation operators A^\dagger and annihilation operators A are represented as vertical lines with upwards and downwards orientation, respectively:

$$\begin{array}{c} \uparrow \\ A^\dagger \end{array} \quad \begin{array}{c} \downarrow \\ A \end{array} \quad (30)$$

Composition of operators is represented by horizontal juxtaposition. The identity span $S \xleftarrow{\text{id}_S} S \xrightarrow{\text{id}_S} S$ is represented by the empty diagram.

We denote 2-morphisms with string diagrams. In particular, our 2-morphisms i_{id_S} , $(i_{\text{id}_S})^\dagger$, $i_{A^\dagger \circ A}$ and $(i_{A^\dagger \circ A})^\dagger$ have the following representations:

$$\begin{array}{cccc} \begin{array}{c} \downarrow \\ \text{cap} \\ \uparrow \end{array} & \begin{array}{c} \downarrow \\ \text{cup} \\ \uparrow \end{array} & \begin{array}{c} \downarrow \quad \uparrow \\ \text{cross} \\ \uparrow \quad \downarrow \end{array} & \begin{array}{c} \downarrow \quad \uparrow \\ \text{cross} \\ \uparrow \quad \downarrow \end{array} \\ i_{\text{id}_S} & (i_{\text{id}_S})^\dagger & i_{A^\dagger \circ A} & (i_{A^\dagger \circ A})^\dagger \end{array} \quad (31)$$

Using this notation, the biproduct equations (25–29) have the following representation:

$$\begin{array}{ccc} \begin{array}{c} \downarrow \\ \text{cap} \\ \uparrow \end{array} = \text{id}_{\text{id}_S} & \begin{array}{c} \downarrow \quad \uparrow \\ \text{cross} \\ \uparrow \quad \downarrow \end{array} = 0_{A^\dagger \circ A, \text{id}_S} & \begin{array}{c} \downarrow \quad \uparrow \\ \text{cross} \\ \uparrow \quad \downarrow \end{array} = 0_{\text{id}_S, A^\dagger \circ A} \\ (25) & (26) & (27) \end{array}$$

(28)
(29)

Khovanov's graphical axioms for the categorified Heisenberg algebra include equations (25), (28) and (29). Equations (26) and (27) are not explicitly part of his algebra, but they can be straightforwardly derived from the other three in a setting where hom-set addition distributes over composition and addition is cancellable, properties which hold in both $\mathbf{Span}(\mathbf{Gpd})$ and also in Khovanov's bimodule category setting.

Although the graphical representations of $i_{A^\dagger \circ A}$ and its converse look like the braiding of a braided monoidal category, we do not have such a structure here, as witnessed explicitly by equation (29): the 'braidings' are not invertible. It even fails to be a lax braiding, as naturality does not hold. We will see in Section 2.11 how to deduce this from a theorem of Yetter [34].

2.8 Combinatorial interpretation

We can extend our combinatorial interpretation to the morphisms $i_{\mathbf{id}_S}$, $(i_{\mathbf{id}_S})^\dagger$, $i_{A^\dagger \circ A}$ and $(i_{A^\dagger \circ A})^\dagger$ as depicted in the diagrams (31). This will allow us to see intuitively why equations (25–29) should hold.

A span of spans gives a way to relate one history to another, in such a way that related histories have isomorphic sources and isomorphic targets. We can therefore understand our spans of spans by listing the histories which they relate. This does not completely define a span of spans, as the actions on the symmetry groups of objects must also be taken into account, but it is nevertheless a useful way to develop intuition.

- $\mathbf{id}_S \xrightarrow{i_{\mathbf{id}_S}} A \circ A^\dagger$. This relates a history where no change is made to the underlying set, to a history in which some element is added and then removed.
- $A \circ A^\dagger \xrightarrow{(i_{\mathbf{id}_S})^\dagger} \mathbf{id}_S$. Histories in $A \circ A^\dagger$ representing adding and removing the same element are related to the trivial history representing no action. Histories representing adding one element and removing a different element are related to nothing.
- $A^\dagger \circ A \xrightarrow{i_{A^\dagger \circ A}} A \circ A^\dagger$. Histories where x is removed and then y is added are related to histories where y is added and then x is removed.
- $A \circ A^\dagger \xrightarrow{(i_{A^\dagger \circ A})^\dagger} A^\dagger \circ A$. Given a history where y is added and x is removed, then if $x \neq y$ this is related to the history where x is removed and then y is added; otherwise it is related to nothing.

This intuition allows us to understand why equations (25–29) should hold, filling out the proof of Lemma 2.15.

- (25) If we begin with the trivial action on a set, and then pass to a history where we add and remove the same element, and then verify that indeed the same element has been added as has been removed, then this leaves our initial history unchanged.

- (26) Beginning with a history where we remove x and add y , we then ensure that $x \neq y$ and pass to a history where we add y and then remove x . We then ensure that x and y are the same, which is clearly impossible, so our original history is related to nothing.
- (27) We begin by choosing a history where we add and remove the same element. We then reverse the order of these operations, which is clearly impossible, so our original history is related to nothing.
- (28) We begin with a history where we remove x and add y . This is related to a history where we add y and then remove x , which in turn is related to a history where we remove x and add y . This clearly gives the identity on histories.
- (29) The left-hand side of this equation is comprised of two separate relations on histories. Suppose we begin with a history for which we add x and then remove y . The first summand selects the case that $x = y$, and relates it to itself. The second ensures that $x \neq y$ and relates the initial history to the case that we remove y and then add x , which in turn is related to the history where we add x and then remove y ; so this relates every history to itself, except for the case that $x = y$. Overall the sum of these relations give the identity on histories, and so the equation is satisfied.

2.9 Adjunction of A and A^\dagger

The bicategory $\mathbf{Span}(\mathbf{Gpd})$ is useful because it contains every object and 1-cell of \mathbf{Gpd} , with the additional property that every 1-cell has a two-sided adjoint [6]. In particular, for every span, its converse is both a left and a right adjoint. This situation is called an *ambidextrous adjunction*, or just *ambiadjunction* for short.

In particular, suppose we have a 1-morphism $\mathbf{A} \xrightarrow{F} \mathbf{B}$ in $\mathbf{Span}(\mathbf{Gpd})$, which is written as the following span:

$$\begin{array}{ccc} & \mathbf{X} & \\ G \swarrow & & \searrow F \\ \mathbf{B} & & \mathbf{A} \end{array} \quad (32)$$

Then its ambiadjoint morphism $\mathbf{B} \xrightarrow{F^\dagger} \mathbf{A}$ is given by the converse span:

$$\begin{array}{ccc} & \mathbf{X} & \\ F \swarrow & & \searrow G \\ \mathbf{A} & & \mathbf{B} \end{array} \quad (33)$$

To fully specify the ambiadjunction we need four 2-morphisms

$$\eta_L : \text{id}_A \Rightarrow F \circ F^\dagger \quad (34)$$

$$\eta_R : \text{id}_B \Rightarrow F^\dagger \circ F \quad (35)$$

$$\epsilon_L : F^\dagger \circ F \Rightarrow \text{id}_B \quad (36)$$

$$\epsilon_R : F \circ F^\dagger \Rightarrow \text{id}_A \quad (37)$$

satisfying the appropriate adjunction equations:

$$(\epsilon_R \circ \text{id}_F) \cdot (\text{id}_F \circ \eta_R) = \text{id}_F \quad (38)$$

$$(\text{id}_{F^\dagger} \circ \epsilon_R) \cdot (\eta_R \circ \text{id}_{F^\dagger}) = \text{id}_{F^\dagger} \quad (39)$$

$$(\epsilon_L \circ \text{id}_{F^\dagger}) \cdot (\text{id}_{F^\dagger} \circ \eta_L) = \text{id}_{F^\dagger} \quad (40)$$

$$(\text{id}_F \circ \epsilon_L) \cdot (\eta_L \circ \text{id}_F) = \text{id}_F \quad (41)$$

We construct these 2-morphisms in the following way. First, η_L is formed as follows:

$$\eta_L := \begin{array}{ccccc} & & \mathbf{A} & & \\ & \nearrow F & \uparrow F & \nwarrow F & \\ \mathbf{A} & \xleftarrow{F} & \mathbf{X} & \xrightarrow{F} & \mathbf{A} \\ & \nwarrow F \circ \pi_1 & \downarrow \Delta_F & \nearrow F \circ \pi_2 & \\ & & (F \downarrow F) & & \end{array} \quad (42)$$

Here $(G \downarrow G)$ is the iso-comma category given by the weak pullback of G along itself. Its objects are given by triples (x_1, α, x_2) such that $\alpha : G(x_1) \rightarrow G(x_2)$, and the morphisms are compatible pairs of morphisms. Then the diagonal map Δ_F is the map into this groupoid from X which on objects maps $x \mapsto (x, \text{id}_{F(x)}, x)$ and on morphisms maps $g \mapsto (g, g)$. We form η_R in a similar way:

$$\eta_R := \begin{array}{ccccc} & & \mathbf{B} & & \\ & \nearrow G & \uparrow G & \nwarrow G & \\ \mathbf{B} & \xleftarrow{G} & \mathbf{X} & \xrightarrow{G} & \mathbf{B} \\ & \nwarrow G \circ \pi_1 & \downarrow \Delta_G & \nearrow G \circ \pi_2 & \\ & & (G \downarrow G) & & \end{array} \quad (43)$$

The counit 2-cells are then formed as the converses of these:

$$\epsilon_L := \eta_R^* \quad (44)$$

$$\epsilon_R := \eta_L^* \quad (45)$$

Lemma 2.16. *The 2-cells (42)-(45) are the units and counits of an ambidextrous adjunction.*

Proof. It is obvious that these 2-morphisms have the correct source and target. To see that they satisfy (38), there are four properties to check (two for each adjunction), but the proofs are all essentially the same. Consider the identity

$$(\text{id} \circ \eta_R) \cdot (\epsilon_R \circ \text{id}) = \text{id} \quad (46)$$

The left hand side is given by the following diagram, representing the composite of

2-morphisms:

$$\begin{array}{c}
 \begin{array}{ccccccc}
 B & \xleftarrow{G} & X & \xrightarrow{F} & A & \xlongequal{\quad} & A & \xlongequal{\quad} & A \\
 \parallel & & \parallel & & \parallel & & \uparrow F & & \parallel \\
 B & \xleftarrow{G} & X & \xrightarrow{F} & A & \xleftarrow{F \circ \pi_1} & (G \downarrow G) & \xrightarrow{G \circ \pi_2} & B \\
 \parallel & & \parallel & & \parallel & \nearrow \pi_1 & \xleftarrow{\alpha} & \nwarrow \pi_2 & \parallel \\
 B & \xleftarrow{G} & X & \xrightarrow{F} & A & \xleftarrow{F} & X & \xrightarrow{G} & B & \xleftarrow{G} & X & \xrightarrow{F} & A \\
 \parallel & & \parallel & & \parallel & \nwarrow \pi_1 & \xrightarrow{\beta} & \nearrow \pi_2 & \parallel & & \parallel & & \parallel \\
 B & \xleftarrow{G \circ \pi_1} & (F \downarrow F) & \xrightarrow{F \circ \pi_2} & B & \xleftarrow{G} & X & \xrightarrow{F} & A \\
 \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
 B & \xlongequal{\quad} & B & \xlongequal{\quad} & B & \xleftarrow{G} & X & \xrightarrow{F} & A
 \end{array}
 \end{array} \quad (47)$$

Note that η_R appears in the top right, and ϵ_R in the bottom left of this diagram. The middle rows simply relate the composite $F \circ F^\dagger \circ F$ to the composites in the unit and counit by exhibiting the pseudo-pullback which gives the composite.

To see that the composite of these 2-cells is just the identity, we simply use the fact that the diagonal map has a special role relative to the pseudo-pullback. Namely, in the following diagram we have that $\pi_1 \circ \Delta_G = \pi_2 \circ \Delta_G$:

$$\begin{array}{ccc}
 & & X \\
 & \nearrow \pi_1 & \downarrow G \\
 X & \xrightarrow{\Delta_G} & (G \downarrow G) \Downarrow \alpha & B \\
 & \searrow \pi_2 & \uparrow G \\
 & & X
 \end{array} \quad (48)$$

Furthermore, by construction of Δ_G , the composite $\alpha \circ \Delta_G$ gives the identity natural transformation. So this 2-morphism is just the same as the identity:

$$\begin{array}{ccc}
 & & X \\
 & \nearrow & \downarrow G \\
 X & \xrightarrow{\quad} & B \\
 & \searrow & \uparrow G \\
 & & X
 \end{array} \quad (49)$$

Applying this at both the unit and counit in (47) one readily verifies that the whole composite is just the identity. The other three identities for an ambiadjunction are proved

in precisely the same way, and hence our given 2-morphisms are indeed unit and counit cells for an ambiadjunction. \square

In particular, we have the immediate special case:

Corollary 2.17. *There is an ambiadjunction between the spans A and A^\dagger .*

Thus we have, for example, the unit 2-morphism:

$$\begin{array}{ccccc}
 & & \mathbf{S} & & \\
 \text{id}_{\mathbf{S}} \swarrow & & \uparrow +1 & \searrow \text{id}_{\mathbf{S}} & \\
 \mathbf{S} & \xrightarrow{\text{id}} & \mathbf{S} & \xrightarrow{\text{id}} & \mathbf{S} \\
 \swarrow +1 & & \downarrow \text{id}_{\mathbf{S}} & & \searrow +1 \\
 & & \mathbf{S} & & \\
 \text{id}_{\mathbf{S}} \xrightarrow{\eta} A^\dagger \circ A & & & &
 \end{array} \tag{50}$$

The graphical representation is extended to depict η and its converse in the following way:

$$\begin{array}{ccc}
 \begin{array}{c} \uparrow \\ \downarrow \end{array} & & \begin{array}{c} \uparrow \quad \downarrow \\ \downarrow \quad \uparrow \end{array} \\
 \eta & & \eta^\dagger
 \end{array} \tag{51}$$

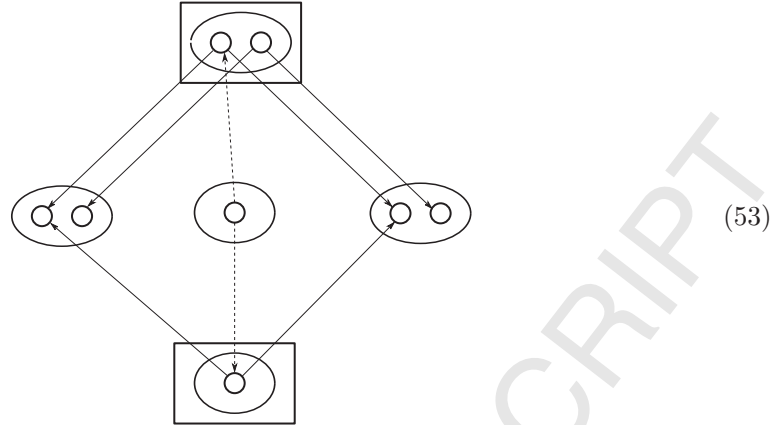
The equations for the ambiadjunction then have the following representation in our graphical notation:

$$\begin{array}{ccc}
 \begin{array}{c} \uparrow \\ \downarrow \end{array} & = & \begin{array}{c} \uparrow \\ \downarrow \end{array} \\
 \begin{array}{c} \uparrow \\ \downarrow \end{array} & = & \begin{array}{c} \uparrow \\ \downarrow \end{array} \\
 \begin{array}{c} \uparrow \\ \downarrow \end{array} & = & \begin{array}{c} \uparrow \\ \downarrow \end{array} \\
 \begin{array}{c} \uparrow \\ \downarrow \end{array} & = & \begin{array}{c} \uparrow \\ \downarrow \end{array}
 \end{array} \tag{52}$$

One can readily see that these diagrams show the ‘path’ of a chain of id_X arrows in the composite (47).

We can understand how η acts with the following diagram which depicts the action of

η at the 2-element set; a similar picture could be drawn for each nonempty set in \mathbf{S} .



This can be compared to (23), which similarly illustrates $i_{\text{id}_{\mathbf{S}}}$. The converse spans of spans are just the same, with the top-to-bottom orientation reversed.

This picture is a helpful aid to extending our combinatorial interpretation for other structures, given in Section 2.8, to include η and the snake equations (52). The span of spans η , and its converse, are interpreted in the following way.

- $\text{id}_{\mathbf{S}} \xrightarrow{\eta} A^\dagger \circ A$. This relates the identity history to the history where an element is removed, and then replaced. There are n possible ways to do this for an n -element set. In particular it is impossible on the zero-element set, and so in that case η relates the identity history to nothing. Otherwise, each possible history is covered by this relation.
- $A^\dagger \circ A \xrightarrow{\eta^\dagger} \text{id}_{\mathbf{S}}$. This relates any history to the identity history.

The equations (52) can be established combinatorially in a similar way to equations (25–29). We examine the first in detail. On the left-hand side, we begin with the operator A^\dagger , a history of which corresponds to adding some element x to a set. We then apply $i_{\text{id}_{\mathbf{S}}}$, passing to a history for which we add some element y , remove it, and then add our element x . Finally we apply our interpretation of η^\dagger , giving a history where we simply add the element y . This is not the same as our original history, for which we added the element x , but it is *equivalent* to it, which is our definition of equality for 2-morphisms in $\mathbf{Span}(\mathbf{Gpd})$ as given by equation (21). The other snake equations can be interpreted in a similar way.

2.10 Symmetric group actions

The construction of the full categorified Heisenberg algebra makes use of certain symmetrizer objects. These are the largest subobjects invariant under the action of the symmetric groups S_n on spans of the form A^n or $(A^\dagger)^n$, the n -fold composites of the spans A and A^\dagger . Roughly, the effect of this action is to permute the order in which elements of a set are added or removed. More explicitly:

Definition 2.18. There exist actions of S_n on the n -fold composites A^n and $(A^\dagger)^n$ given as follows. Up to isomorphism, spans of the form A^n and $(A^\dagger)^n$ have the following form:

$$\begin{array}{ccc}
 & \mathbf{S} & \\
 \text{id}_{\mathbf{S}} \swarrow & & \searrow +n \\
 \mathbf{S} & & \mathbf{S}
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathbf{S} & \\
 +n \swarrow & & \searrow \text{id}_{\mathbf{S}} \\
 \mathbf{S} & & \mathbf{S}
 \end{array}
 \quad (54)$$

Here we define $(+n) := (+1)^n$, the functor which takes the disjoint union with an n -element set. It has natural endo-transformations $\hat{\mu}$ for each $\mu \in S_n$, defined for each set T by the set maps

$$\hat{\mu}_T : T \sqcup \mathbf{n} \rightarrow T \sqcup \mathbf{n}, \quad (55)$$

which act by $\hat{\mu}_T(t) = t$ for all $t \in T$ and $\hat{\mu}_T(j) = \mu(j)$ for all $j \in \mathbf{n}$. These give rise to the following spans of spans:

$$(56)$$

Intuitively these permute the extra n elements, and leave the remaining elements unchanged.

For the action of the non-identity element of S_2 on A^2 and $(A^\dagger)^2$, we use the following graphical representation:

$$(57)$$

Actions of S_n on A^n and $(A^\dagger)^n$ can be built up from these basic permutations of adjacent operations. Because they are actions of the symmetric group, they satisfy the following equations:

$$(58)$$

They also satisfy the following equations, making them generators of S_n for any n :

$$(59)$$

As relations on histories, the symmetry on $A \circ A$ relates the history ‘remove x , and then remove y ’ to the history ‘remove y , and then remove x ’. The symmetry on $A^\dagger \circ A^\dagger$ can be described in a similar way. The equations satisfied by these symmetries can then be accounted for using these combinatorial interpretations.

Together with the morphism $i_{A^\dagger \circ A}$ and its converse, these morphisms allow us to interpret arbitrary string diagrams involving strands labeled A or A^\dagger . These are not

invertible in general, since $i_{A^\dagger \circ A}$ is not an isomorphism but an inclusion, and so we do not get a representation of the symmetric group in these cases where both A and A^\dagger appear.

This is important for connections to quantum field theory, for which powers of the field operator $\Phi = A + A^\dagger$ play a major role. The inner product $\langle \psi, p(\phi)\psi' \rangle$ (for a polynomial p) is the groupoid cardinality of the stuff type inner product

$$\langle \Psi, p(\Phi)\Psi' \rangle \quad (60)$$

where Ψ, Ψ' are any groupoidifications of the vectors ψ, ψ' . As described in [22], this can be interpreted as a *sum over histories*, which are represented by Feynman diagrams. These are given weights which are exactly as determined by the groupoid cardinality.

2.11 Failure of naturality

The crossings in our graphical notation are self-inverse when all the strands are oriented in the same direction. This is the basis of the symmetric group action on powers such as A^n and $(A^\dagger)^n$. However, this is not the case for ‘mixed’ crossings in which the strands are oriented in opposite directions, a fact implied by (29). Indeed, this is essential to the categorification of the commutation relations.

By a theorem of Yetter [34], a lax braiding for an object with a right dual is automatically invertible. Since the crossing vertices in our string diagrams are not invertible, one of the axioms of a lax braiding must fail, and the culprit is naturality. If we *assume* naturality, we can use Yetter’s argument to show that the braiding is invertible, in contradiction with equation (29):

Combinatorially, the following naturality property fails to hold:

Lemma 2.19. *The crossings (31) violate naturality in the combinatorial representation.*

Proof. The following composites are not equal:

$$\text{Figure-eight span} \neq \text{U-loop span} \quad (62)$$

The reason this equality fails is that the left-hand side factors through the composite $A^\dagger \circ A \circ A$, and hence annihilates the history that removes the unique element from a 1-element set. The right-hand does not annihilate this history, and hence the two spans of spans cannot be equal. \square

2.12 The Twist

Lemma 2.20. *Khovanov's 'left twist equals zero' axiom holds in the combinatorial representation:*

$$\text{Left twist} = 0_{A^\dagger, A^\dagger} \quad (63)$$

Proof. We verify this combinatorially as follows. The initial span is A^\dagger , and we begin with a history in this span which adds an element x to a set. This is related to a history in which we add x , then add and remove some new element y , according to our interpretation of i_{id_S} as described above. We then apply the symmetry map, obtaining a history in which we add y , add x , and then remove y . Finally we apply our interpretation of $(i_{\text{id}_S})^\dagger$ to the final add-remove pair, which only relates histories in which the element we remove is the same as the element which we add. However, in this case this is impossible, as x and y are different elements, and so our history is related to nothing. Combinatorially, this just says that we are *distinguishing* x and y . \square

2.13 Main theorem

To this point, we have described a combinatorial interpretation of the graphical notation for the monoidal category \mathbf{H}' . This interpretation may be understood as a representation as living in the bicategory $\mathbf{Span}(\mathbf{Gpd})$, as a functor from \mathbf{H}' to hom-category of spans on the object \mathbf{S} . This interpretation is summarized in Table 1. We formalize this in terms of a functor defined on the bicategory $\Omega_{\mathbf{H}'}$ with one object, whose morphisms and 2-morphisms are the objects and morphisms of \mathbf{H}' . This is formalized by the following theorem.

Theorem 2.21. *The combinatorial interpretation we have sketched yields a 2-functor*

$$\Omega_{\mathbf{H}'} \xrightarrow{C} \mathbf{Span}(\mathbf{Gpd}).$$

which sends the unique object to the groupoid \mathbf{S} of finite sets and bijections.

Khovanov's \mathbf{H}'	Image in $\mathbf{Span}(\mathbf{Gpd})$
(none)	\mathbf{S}
Q_-, Q_+	A, A^\dagger
\downarrow, \uparrow	$\text{id}_A, \text{id}_{A^\dagger}$
\otimes	\circ
\cap, \cup	η, ϵ
crossings	$\hat{\mu}, i_{A^\dagger \circ A}, (i_{A^\dagger \circ A})^*$

Table 1: The correspondence between categorifications

Proof. Most of the necessary facts have already been proved. The functor takes the down- and up-arrows in the graphical notation for \mathbf{H}' to the spans A and A^\dagger respectively, which determines all products. The cups, caps, and crossings are taken to those spans of spans which we have previously described. These also determine all monoidal products of these 2-morphisms.

Since the functor is defined by its action on a generating set of objects and morphisms in \mathbf{H}' , by construction it will automatically respect composition, identities and the monoidal structure, provided it is well-defined. In particular, one must verify that the spans of spans we have defined satisfy the relations imposed on the graphical notation for \mathbf{H}' , since otherwise the definition might give different spans of spans for equivalent 2-morphisms in \mathbf{H}' .

The adjointness relations on i_{id_S} and $(i_{\text{id}_S})^\dagger$ were verified in Lemma 2.17. The relations on crossings for like-oriented strands are established by the existence of the symmetric group action shown in Lemma 2.18. The crossing relations for strands with unlike orientations are exactly the biproduct equations (28) and (29).

The remaining relations are the ‘left twist equals zero’ relation and the ‘loop cancellation’ relation. The first of these is the only nontrivial one, which was verified in Lemma 2.20. \square

Some other questions remain about this functor. In particular, several readers have asked us whether this functor is faithful. Clearly, the functor is not full, since there are many spans which cannot be produced from our generators. Faithfulness, however, is less obvious. This amounts to asking whether the collection of relations in the graphical calculus presentation of \mathbf{H}' is necessary and sufficient to characterize the combinatorics of our operations. If the functor is not faithful, then the combinatorial representation imposes relations not already present in the graphical calculus which defines \mathbf{H}' .

We do not address this question in the present paper, but we note its resemblance to the question of whether Khovanov’s homomorphism γ from the integral Heisenberg algebra into the Grothendieck ring of \mathbf{H} is surjective. Both questions revolve around the issue of whether additional relations are needed in the presentation of \mathbf{H}' to properly characterize, respectively, the combinatorics of sets, and the relations of the Heisenberg algebra. We do not yet have an answer to either question.

3 Functorial representations

3.1 Introduction

Until now we have been describing a representation of the categorified Heisenberg algebra \mathbf{H}' on the groupoid \mathbf{S} of finite sets as an object in $\mathbf{Span}(\mathbf{Gpd})$. It is more usual to look

for representations of categorified algebras in \mathbf{Cat} , the bicategory of categories, functors, and natural transformations [19]. Often, one asks further that the categories be equipped with extra structure, as with abelian, additive, or triangulated categories, and functors are required preserve this structure.

In this section, we show two ways to get representations by functors. First, we take a detour through a somewhat different representation by functors on a category, which relates to earlier work on groupoidification.

The key point is that the combinatorial representation in $\mathbf{Span}(\mathbf{Gpd})$ acts as the ‘combinatorial core’ of these functorial representation. This is precisely in the spirit of the groupoidification program which inspired the structures in $\mathbf{Span}(\mathbf{Gpd})$. The bulk of this Section will explain these, and in particular will demonstrate how to recover one which is equivalent to the representation described by Khovanov in terms of bimodules [15], which also includes certain ‘symmetrizer’ 2-morphisms.

3.2 Stuff types

It was remarked in the introduction that the combinatorial representation of the categorified Heisenberg algebra extends the Baez-Dolan groupoidification of the Fock space representation. In that situation, the spans A and A^\dagger act by pseudo-pullback on *stuff types*. A stuff type consists of a groupoid \mathbf{X} ‘over \mathbf{S} ’; that is, equipped with a functor $\mathbf{X} \xrightarrow{\Psi} \mathbf{S}$. A stuff type can be interpreted as a generalized class of structures on finite sets. They form the objects of the slice category \mathbf{Gpd}/\mathbf{S} , whose morphisms from (\mathbf{X}, Ψ) to (\mathbf{X}', Ψ') are maps $f : \mathbf{X} \rightarrow \mathbf{X}'$ forming commuting triangles, so that $\Psi' \circ f = \Psi$.

As shown in [1, 22], the degroupoidification of a stuff type gives a choice of vector in Fock space (more precisely, a map from \mathbb{C} , so that the image of $1 \in \mathbb{C}$ gives the vector). This is sufficient to represent the 1-category $\mathbf{Span}_1(\mathbf{Gpd})$, but not quite enough to represent the 2-morphisms. However, a closely related class of representations follows immediately from the one we have described.

Corollary 3.1. *The combinatorial interpretation C of Theorem 2.21 induces a representation of \mathbf{H}' on the categories $\mathrm{Hom}_{\mathbf{Span}(\mathbf{Gpd})}(\mathbf{G}, \mathbf{S})$ by functors and natural transformations.*

Proof. In the bicategory $\mathbf{Span}(\mathbf{Gpd})$, there are composition maps

$$\circ : \mathrm{Hom}(\mathbf{G}, \mathbf{S}) \times \mathrm{Hom}(\mathbf{S}, \mathbf{S}) \rightarrow \mathrm{Hom}(\mathbf{G}, \mathbf{S}) \quad (64)$$

This amounts to a functor:

$$\circ(-) : \mathrm{Hom}(\mathbf{S}, \mathbf{S}) \rightarrow \mathrm{Hom}(\mathrm{Hom}(\mathbf{G}, \mathbf{S}), \mathrm{Hom}(\mathbf{G}, \mathbf{S})) \quad (65)$$

Note that the outer Hom on the right hand side is in \mathbf{Cat} , and the inner Hom operations are in $\mathbf{Span}(\mathbf{Gpd})$. which takes any 1-morphism from \mathbf{S} to itself to the operation which acts by composition with that 1-morphism. Similarly, $\circ(-)$ takes a 2-morphism in $\mathrm{Hom}(\mathbf{S}, \mathbf{S})$ to a natural transformation between these functors. Composing this with the representation of Theorem 2.21 gives the result. \square

The first part of this argument would work in any bicategory, and does not depend on any details about $\mathbf{Span}(\mathbf{Gpd})$. As a special case, we have an action on a category whose objects correspond directly to stuff types.

Corollary 3.2. *There is a representation of \mathbf{H}' by functors and natural transformation on the category*

$$\mathrm{Hom}_{\mathbf{Span}(\mathbf{Gpd})}(\mathbf{1}, \mathbf{S}) \cong \mathbf{Span}(\mathbf{Gpd}/\mathbf{S}) \quad (66)$$

Proof. The existence of such a representation on $\text{Hom}_{\mathbf{Span}(\mathbf{Gpd})}(\mathbf{1}, \mathbf{S})$ is just a special case of Corollary 3.1.

The objects of this category are in 1-1 correspondence with stuff types, since any groupoid X is equipped with a unique functor into $\mathbf{1}$, so a span from $\mathbf{1}$ to \mathbf{S} determined uniquely by a stuff type. However, the morphisms between these spans are not just the maps of stuff types, but rather isomorphism classes of spans of such maps. That is, the morphisms of this category correspond exactly to those of $\mathbf{Span}(\mathbf{Gpd}/\mathbf{S})$, in a way which agrees with units and composition. \square

3.3 Linearization of spans

We will now consider a different way to get representations of \mathbf{K} on a category by functors and natural transformations, which is also related to the Baez-Dolan groupoidification program, but in a different way.

Linearization is the process of turning geometrical structures into linear ones. An important form of linearization for our purposes is *degroupoidification*, first described in [1] and surveyed in [2], which is a monoidal functor

$$\mathbf{Span}_1(\mathbf{Gpd}) \xrightarrow{D} \mathbf{Vect},$$

where $\mathbf{Span}_1(\mathbf{Gpd})$ is the monoidal 1-category obtained by identifying isomorphic 1-cells in the monoidal bicategory $\mathbf{Span}(\mathbf{Gpd})$. This functor D takes groupoids to the free vector space on their set of equivalence classes of objects, and spans of groupoids to linear maps between these. Tameness properties of the spans are crucial for this functor D to be well-defined.

We will use a higher-categorical generalization called *2-linearization*, which is a monoidal 2-functor

$$\mathbf{Span}(\mathbf{Gpd}) \xrightarrow{\Lambda} \mathbf{2Vect}, \quad (67)$$

where $\mathbf{2Vect}$ is the monoidal bicategory of *Kapranov-Voevodsky 2-vector spaces*. This has objects given by \mathbb{C} -linear semisimple additive categories, 1-morphisms given by linear functors, and 2-morphisms given by natural transformations. Again, tameness properties of $\mathbf{Span}(\mathbf{Gpd})$ as described in Section 2.5 are necessary for this construction to be valid.

The 2-functor Λ is described in more detail in [23], but the essential starting point is that Λ takes a groupoid \mathbf{A} to its category $\text{Rep}(\mathbf{A})$ of finite-dimensional representations. A span of groupoids $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ is mapped to the composite functor

$$\text{Rep}(\mathbf{A}) \xrightarrow{F_*} \text{Rep}(\mathbf{X}) \xrightarrow{G^*} \text{Rep}(\mathbf{B}), \quad (68)$$

where F_* is the pullback functor (also known as the restriction functor) for F , and G^* is the adjoint of the pullback functor (also known as the induction functor) for G .

For a general span-of-spans of the form

$$(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}) \xrightarrow{(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)} (\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A}),$$

we construct its image under Λ as the following composite natural transformation:

$$\begin{array}{ccccc}
 & & \text{Rep}(\mathbf{X}) & & \\
 & \swarrow G^* & \downarrow S^* \quad S_* & \nwarrow F_* & \\
 \text{Rep}(\mathbf{B}) & & \text{Rep}(\mathbf{Z}) & & \text{Rep}(\mathbf{A}) \\
 & \searrow J^* & \downarrow T^* \quad T_* & \swarrow K_* & \\
 & & \text{Rep}(\mathbf{Y}) & &
 \end{array} \quad (69)$$

$(\mu^*)^{-1}$ (vertical arrow from $\text{Rep}(\mathbf{B})$ to $\text{Rep}(\mathbf{Z})$)
 ν_* (vertical arrow from $\text{Rep}(\mathbf{Z})$ to $\text{Rep}(\mathbf{Y})$)

The central 2-cells arise from adjunctions $S_* \dashv S^*$ and $T^* \dashv T_*$. Notice that this makes essential use of the 2-sidedness of the adjunctions between the restriction and induction functors denoted with upper and lower stars respectively.

In Section 1.2 we described Khovanov's construction of the categorified Heisenberg algebra as a monoidal category \mathbf{H}' . In Section 2, we described how this acts in a natural way by spans and spans-of-spans over the groupoid \mathbf{S} of finite sets and bijections, yielding a combinatorial interpretation that comprises the definition of a 2-functor:

$$\Omega_{\mathbf{H}'} \xrightarrow{C} \mathbf{Span}(\mathbf{Gpd}) \quad (70)$$

We can compose C and Λ to obtain a new representation of $\Omega_{\mathbf{H}'}$ in $\mathbf{2Vect}$. In fact, this composite representation $\Lambda \circ C$ is exactly Khovanov's original representation as described in [15], up to completeness issues which we address below.

Corollary 3.3. *The composite 2-functor*

$$\Omega_{\mathbf{H}'} \xrightarrow{C} \mathbf{Span}(\mathbf{Gpd}) \xrightarrow{\Lambda} \mathbf{2Vect} \quad (71)$$

determines a representation of \mathbf{H}' on $\Lambda(\mathbf{S})$ by functors and natural transformations.

This Corollary 3.3 would be immediate, except that Λ was defined in [21] only for essentially finite groupoids, whereas \mathbf{S} is only a locally finite groupoid, and has infinitely many isomorphism classes of objects. Thus Corollary 3.3 requires that Λ be extended to a larger class of groupoids for the statement to be well-defined.

We will not discuss this issue in full generality, since our focus is on the combinatorial representation, but a few remarks are in order. First, there is no difficulty defining Λ on \mathbf{S} , since it has countably many isomorphism classes of object, and is locally finite. That is, the automorphism group of every object is a finite group. As a union, \mathbf{S} is a colimit of a diagram of essentially finite groupoids (\mathbf{FS}_n) , whose n^{th} member is the groupoid of all finite sets with at most n elements. The maps in this diagram are the natural inclusions. There is then a unique extension of Λ to such groupoids by assuming Λ is cocontinuous.

Thus, applying Λ to \mathbf{S} , we get a colimit of finite dimensional 2-vector spaces (Ind-objects for $\mathbf{2Vect}$). This is exactly analogous to the case of an infinite direct sum of vector spaces. So it consists of the category of representations of \mathbf{S} (that is, direct sums of any of the irreducible representations of any of the \mathbf{S}_n). Any summand of an object in this category will eventually appear in the image $\Lambda(\mathbf{FS}_n)$ for sufficiently large n , just as any object of \mathbf{S} eventually appears in sufficiently large \mathbf{FS}_n . For the same reason, there is no serious issue in defining the effect of an extended Λ on 1-morphisms: linear functors (also known as 2-linear maps) between finite-dimensional 2-vector spaces are, up to isomorphism, determined by a matrix of vector spaces, which are again locally finite.

The only real issue relates to the effect of Λ on 2-cells. In particular, the components of the natural transformations that Λ assigns to spans of spans are given as sums. In the 1-category groupoidification program, this is precisely where the definitions of tameness encountered in Section 2.5 become important [2] for guaranteeing convergence. An extension of the tameness concept will be needed to guarantee convergence in general in the categorified case.

In our current example, it is a matter of direct calculation to confirm that the spans-of-spans that we make use of in our combinatorial representation have convergent linearizations, and indeed they do, so we will not consider the general case here.

3.4 2-Linearized ladder operators

We now investigate the images of the spans A and A^\dagger . This will make it clear why the composite (71) is isomorphic to Khovanov's representation.

To begin with, the 2-vector space analog of 'Fock space' is:

$$\Lambda(\mathbf{S}) = [\mathbf{S}, \mathbf{Vect}] \simeq \coprod_{n \in \mathbb{N}} \text{Rep}(S_n) \quad (72)$$

This is a category generated by the irreducible representations of all the symmetric groups S_n . The categorified Heisenberg algebra acts on $\Lambda(\mathbf{S})$ by endofunctors and natural transformations, and we would like to understand this action concretely.

For each isomorphism class of object $n \in \mathbf{S}$, the irreducible representations of the group $\text{Aut}(n) = S_n$ are indexed by Young diagrams with n boxes, so there is a countable basis for $\Lambda(\mathbf{S})$ consisting of all Young diagrams. See [11] for the mechanics of Young diagrams in representation theory the symmetric groups, and [29] for a good discussion in a physical context.

The 1-cell $\Lambda(A^\dagger)$ is given by the functor

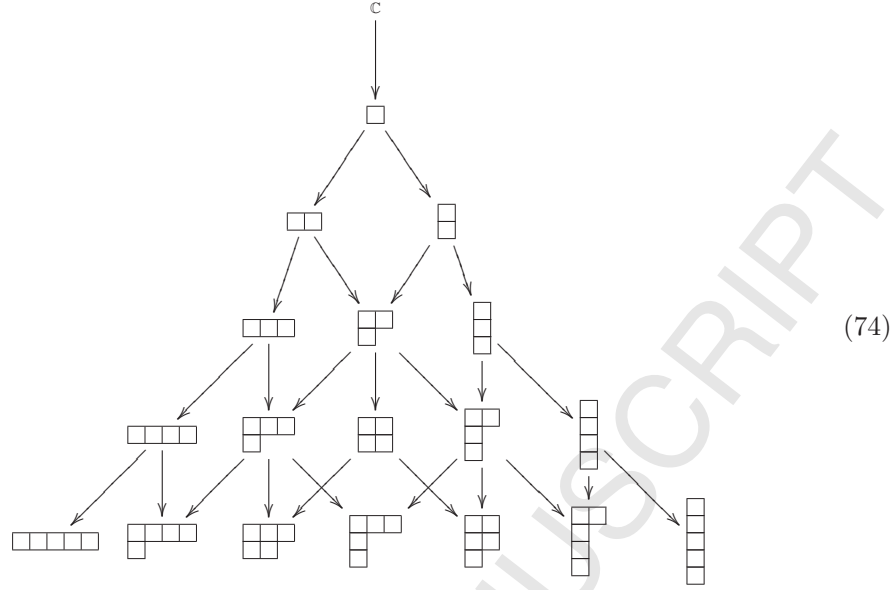
$$\Lambda(\mathbf{S}) \xrightarrow{\Lambda(+1)} \Lambda(\mathbf{S}), \quad (73)$$

and $\Lambda(A)$ by its adjoint. For any n -element set, the functor $(+1)$ maps the automorphism group S_n into S_{n+1} , by the natural inclusion where S_n is the subgroup of S_{n+1} fixing the newly-added element.

To understand $\Lambda(A^\dagger)$ we must therefore see how a representation of \mathbf{S} is pulled back by the functor $(+1)$, and to understand $\Lambda(A)$ we must see how the adjoint of this pullback operation acts. This is described by the theory of restricted and induced representations, a standard part of the representation theory of the symmetric groups [26]. Given an irreducible representation of S_{n+1} represented by a Young diagram D , the corresponding restricted representation of S_n will be a direct sum of irreducibles, which correspond to all the valid Young diagrams obtained by deleting a single box from some row of D . Each representation corresponding to such a diagram appears in the restricted representation with multiplicity one. Likewise, given an irreducible representation of S_n represented by Young diagram D' , the induced representation of S_{n+1} is a representation in which every Young diagram arising by adding a box to any row of D' (possibly of zero length) appears with multiplicity one.

We can see the effect of $\Lambda(A)$ and $\Lambda(A^\dagger)$ on small representations of small symmetric groups using the following directed graph of Young diagrams, which is known as Young's

lattice:



Given any diagram in this partial order, the effect of $\Lambda(A)$ is to give the direct sum of every diagram immediately above it, and the effect of $\Lambda(A^\dagger)$ is to give the direct sum of every diagram immediately below it. As a result, the effect of multiple applications of $\Lambda(A)$ and $\Lambda(A^\dagger)$ can be calculated by counting paths.

We can represent this graph in the following way:

$$\Lambda(A) = \begin{pmatrix} 0 & M_{0,1} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & M_{1,2} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & M_{2,3} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & M_{3,4} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & M_{4,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (75)$$

Entries in this matrix are themselves the incidence matrices for the graph (74), so that $M_{i,i+1}$ gives the incidence relations for the Young diagrams in the i^{th} and $(i+1)^{st}$ rows. The analogous matrices for $\Lambda(A^\dagger)$ are then the transposes $M_{i,i+1}^T$.

The commutation relation

$$\Lambda(A)\Lambda(A^\dagger) \oplus \mathbf{1} \cong \Lambda(A^\dagger)\Lambda(A) \quad (76)$$

must be true in $\mathbf{2Vect}$, since Λ preserves all the relevant structures, and since we have already demonstrated the corresponding combinatorial fact (24) about the composition of spans. This fact can be directly verified for actions on small symmetric groups using the matrices given above. In terms of diagrams, the extra factor of $\mathbf{1}$ on the left-hand side corresponds to the operation of first adding a box after the last row of a diagram D , then removing it.

This commutation relation recalls that the 2-linear maps given by the $M_{i,i+1}$ and their adjoints are only the generators of the categorified Heisenberg algebra. In physical applications, derived operators are also very important. In particular there is the ‘number operator’ $\mathbf{n} = \mathbf{a}^\dagger \mathbf{a}$. In the physical application of the Heisenberg algebra to the quantum harmonic oscillator, this is the Hamiltonian of the system. It is a self-adjoint operator

whose eigenvalues correspond to the observable values of ‘energy’, which is just the number of particles in a particular state of the system.

With this physical motivation in mind, it is interesting to consider the analogous span $N = A^\dagger A$, and its 2-linearization:

$$\Lambda(N) = \Lambda(A^\dagger)\Lambda(A) = \begin{pmatrix} N_0 & 0 & 0 & 0 & 0 & \dots \\ 0 & N_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & N_2 & 0 & 0 & \dots \\ 0 & 0 & 0 & N_3 & 0 & \dots \\ 0 & 0 & 0 & 0 & N_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \end{pmatrix} \quad (77)$$

In place of the eigenvalues n of the operator \mathbf{n} , we have functors given by blocks such as the following:

$$N_4 = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C} & \mathbb{C}^2 & \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C}^2 & \mathbb{C} \\ 0 & 0 & 0 & \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (78)$$

We observe here that the 2-linear maps represented by these blocks are endofunctors of particular subcategories of $\Lambda(\mathbf{S})$. The dimension of each entry counts paths of a certain shape (lower-then-raise) in the Young lattice, but they are otherwise rather opaque.

However, the following observation shows that the N_n are quite natural, and indeed analogous to eigenvalues. The categorified number operator, for example, can be understood directly in terms of the combinatorial interpretation. The span $A^\dagger A$ represents the process of removing an element from a set and then adding an element to the resulting set. This amounts to marking a chosen element of the original set, which can be done in exactly n ways for an n -element set. These are all independent, and the symmetry group S_n acts on them exactly as the permutation representation. The bimodule which effects this functor is just the linear span of these elements, namely:

$$N_n \cong (-) \otimes \mathbb{C}[n] \quad (79)$$

We close this section with a remark about how the above picture captures Khovanov’s model of the diagram calculus in the bimodule category \mathcal{S}' . The blocks $M_{n,n+1}$ each act only on representations supported on single objects n . They represent induction functors such as

$$\text{Ind}_n^{n+1} = (+1)|_{\mathbf{n}} : \text{Rep}(S_n) \rightarrow \text{Rep}(S_{n+1}), \quad (80)$$

which induces a representation from S_n to S_{n+1} along the inclusion $S_n \subset S_{n+1}$. The adjoint blocks describe the restriction functors, showing how an S_{n+1} representation restricts to an S_n representation.

3.5 Equivalence with bimodule representation

We now make precise the sense in which the construction we describe captures Khovanov’s representation of \mathbf{H}' in terms of bimodules. This was defined by a map from the category freely defined by the graphical calculus given earlier, into the following:

Definition 3.4 ([15], Section 3.3). *Let \mathcal{S}' be the monoidal category whose objects are composites of induction and restriction functors for representations of symmetric groups along the standard inclusions $S_n \subset S_{n+1}$, and whose morphisms are natural transformations. Then $\mathcal{S} := \text{Kar}(\mathcal{S}')$ is the Karoubi envelope of \mathcal{S}' .*

This is described in terms of groups, rather than the groupoid \mathbf{S} , so it is constructed as the direct sum of a collection of such categories \mathcal{S}_n , labeled by the starting value of n . However, there is a faithful embedding

$$\mathcal{S}' \rightarrow \text{End}_{\mathbf{2Vect}}(\Lambda(\mathbf{S})) \quad (81)$$

which takes a bimodule to the functor which acts by taking the tensor product (over the appropriate group algebra) with that bimodule. For simplicity, we will call the image of this inclusion \mathcal{S}' as well.

A general object in \mathcal{S}' is, in general, a subobject (given by symmetrizers) of one which represents a functor whose matrix representation is composed of the $M_{n,n+1}$ blocks as described above (or their adjoints). Taking the direct sum gives exactly the bimodules corresponding to the functors in $\text{Hom}(\Lambda(\mathbf{S}), \Lambda(\mathbf{S}))$ which are generated in this way. Thus, the image in $\mathbf{2Vect}$ of our groupoidified Heisenberg algebra corresponds exactly to this representation \mathcal{S}' of \mathbf{H}' .

Khovanov's definition uses the identification of induction and restriction functors with the bimodules which represent them. There is therefore some ambiguity regarding which bicategory should be seen as its target: of algebras and bimodules, or of representation categories and functors. Since most of the discussion in [15] is expressed in terms of the former, we take the target to be the bicategory **Bimod**. Its objects are algebras, morphisms are bimodules, and 2-morphisms are bimodule maps.

We follow the notational convention used in [15], so that (n) denotes the group ring $\mathbb{C}[S_n]$, and subscripts show how these rings are seen as bimodules. Without a subscript, (n) is understood to act on itself by the adjoint action of the ring on itself by left or right multiplication. Since the group rings are direct sums over the elements of the S_n , the inclusion and projection maps $i : (n) \rightarrow (n+1)$ and $\pi : (n+1) \rightarrow (n)$ compose with these adjoint actions to give module structures ${}_n(n+1)$ and ${}_{n+1}(n)$ (and similarly on the right). Bimodules with internal subscripts, such as $(n)_{n-1}(n)$ denote relative tensor products of bimodules.

We can thus re-state Khovanov's construction in the form of a definition and a statement.

Definition 3.5. *Define a 2-functor*

$$\hat{K} : \Omega_{\mathbf{H}'} \rightarrow \mathbf{Bimod} \quad (82)$$

as follows:

- \hat{K} takes the unique object of $\Omega_{\mathbf{H}'}$ to the groupoid algebra $\mathbb{C}[\mathbf{S}] \simeq \bigoplus_n \mathbb{C}[S_n]$
- The categorified raising and lowering operators are taken to the bimodules which represent the induction and restriction functors for the inclusion $S_n \subset S_{n+1}$, which are:

$$Q_+ \mapsto \bigoplus_n ({}_n(n+1)_{n+1}) \quad (83)$$

$$Q_- \mapsto \bigoplus_n ({}_{n+1}(n+1)_n) \quad (84)$$

- The image of the generating 2-morphisms of \mathbf{H}' are bimodule maps. The 'cup' and 'cap' diagrams appear as (32-35) in [15, Section 3.3] and correspond to the adjunction of the induction and restriction functors:

- The cup and cap giving ${}_n(n+1)_n \rightarrow (n)$ and $(n) \rightarrow {}_n(n+1)_n$ are the inclusion and projection maps.
- The map $(n+1)_n(n+1) \rightarrow (n+1)$ is given on the standard basis by multiplication, taking $g \otimes h$ to gh , while the map $(n+1) \rightarrow (n+1)_n(n+1)$ its linear adjoint, which takes 1 to:

$$\sum_{i=1}^{n+1} s_i s_{i+1} \dots s_n \otimes s_n \dots s_{i+1} s_i$$

- Crossings are described immediately afterward in [15] Sec. 3.3. The images of the pure upward and downward double crossings are, respectively, automorphisms of $(n+2)_n$ and ${}_n(n+2)$ given by right (left) multiplication by s_{n+1} (the S_{n+2} generator swapping elements $n+1$ and $n+2$).
- The images of the right-crossing and left-crossing generators are maps

$$({}_n)_{n-1}(n) \xrightarrow{n} (n+1)_n \quad (85)$$

and

$${}_n(n+1)_n \rightarrow ({}_n)_{n-1}(n) \quad (86)$$

respectively. The right-crossing acts on the standard basis by taking $g \otimes h$ to $gs_n h$. The left crossing projects $g \in S_{n+1}$ to 0 if $g \in S_n \subset S_{n+1}$, and elements of the form $gs_n h$ to $g \otimes h$ (which is well-defined because the target is a relative tensor product).

In this bicategorical language, Khovanov's representation of \mathbf{H}' on \mathcal{S} can be presented as follows.

Proposition 3.6. *The monoidal category \mathcal{S}' is the image of $\text{Hom}(\star, \star)$ under the 2-functor:*

$$\widehat{K} : \Omega_{\mathbf{H}'} \rightarrow \mathbf{Bimod} \quad (87)$$

Proof. This is simply a rewriting of the original construction in terms of bicategories. \square

The reason for this rewriting is so that we can compare the 2-linearization of our combinatorial representation of $\Omega_{\mathbf{H}'}$ to Khovanov's bimodule representation of \mathbf{H}' on \mathcal{S} . We will see that there is a symmetric monoidal equivalence between the resulting functors.

This symmetric monoidal equivalence assumes a definition of Khovanov's category \mathcal{S} as a structure within $\mathbf{2Vect}$, so that the representation \widehat{K} becomes a 2-functor K into $\mathbf{2Vect}$. The correspondence between restriction and induction functors, and the bimodules which represent them, is implicit in Khovanov's description. Explicitly, the identification is by the 2-functor

$$\text{Rep} : \mathbf{Bimod} \rightarrow \mathbf{2Vect} \quad (88)$$

which takes an algebra A to $\text{Rep}(A)$, its category of representations. An AB -bimodule ${}_A M_B$ is taken to the functor

$${}_A M_B \otimes_B (-) : \text{Rep}(B) \rightarrow \text{Rep}(A) \quad (89)$$

and a bimodule homomorphism $f : {}_A M_B \rightarrow {}_A N_B$ is taken to $f \otimes \text{id}_{(-)}$.

We use this to define the following composite:

$$K = \text{Rep} \circ \widehat{K} : \Omega_{\mathbf{H}'} \rightarrow \mathbf{2Vect} \quad (90)$$

This makes it possible to directly compare the two representations, which gives our main result for this section.

Theorem 3.7. *The following diagram commutes up to pseudonatural equivalence:*

$$\begin{array}{ccc}
 & C \nearrow & \mathbf{Span}(\mathbf{Gpd}) \\
 \Omega_{\mathbf{H}'} & \xrightarrow{\quad K \quad} & \mathbf{2Vect} \\
 & \searrow e & \Lambda
 \end{array} \quad (91)$$

Proof. We define the pseudonatural transformation of 2-functors

$$e : \Lambda \circ C \Rightarrow K \quad (92)$$

and show that it is an equivalence.

To the unique object $\star \in \Omega_{\mathbf{H}'}$, this assigns

$$e_\star : \Lambda(\mathbf{S}) \rightarrow K(\star) = \text{Rep} \left(\bigoplus_n \mathbb{C}[S_n] \right) \quad (93)$$

which is just the natural identification between the representation categories of the groupoid \mathbf{S} and its groupoid algebra, so e_\star is the identity on the underlying vector spaces of a representation.

To each morphism $f \in \Omega_{\mathbf{H}'}$, the pseudonatural transformation e assigns a weakly commuting square. These are determined by the values on the generating 1-morphisms Q_\pm . Thus for Q_+ , we have the following:

$$\begin{array}{ccc}
 \Lambda(\mathbf{S}) & \xrightarrow{e_\star} & K(\star) \\
 \Lambda(A^\dagger) \downarrow & \swarrow e_{Q_+} & \downarrow K(Q_+) \\
 \Lambda(\mathbf{S}) & \xrightarrow{e_\star} & K(\star)
 \end{array} \quad (94)$$

Recall from Khovanov [15] that $K(Q_+)$ is the functor determined by the following action on objects:

$$K(Q_+)(V) = \bigoplus_n ({}_{n+1}(n+1)_n) \otimes_{\mathbb{C}[S_n]} V \quad (95)$$

Therefore, e_{Q_+} is a natural transformation of the following type:

$$e_{Q_+} : \bigoplus_n ({}_{n+1}(n+1)_n) \otimes_{\mathbb{C}[S_n]} e_\star(-) \Rightarrow e_\star(\Lambda(A^\dagger)) \quad (96)$$

It identifies the representations of $K(\star) = \text{Rep}(\bigoplus_n \mathbb{C}[S_n])$ given by the induction either before, or after, identifying groupoid representations with algebra representations. The precise isomorphism depends on the construction used for the induction functor $\Lambda(A^\dagger) = (+1)_\star$. The standard construction gives the induced representation as a direct sum over n , and for each n over the cosets of S_n in S_{n+1} . This gives a canonical basis for the bimodule tensor factor, so the isomorphism is also canonical, by the decomposition of ${}_{n+1}(n+1)_n$ as a direct sum of subspaces spanned by these cosets.

Similarly, e_{Q_-} is a natural transformation of the following type:

$$e_{Q_-} : ({}_n(n+1)_{n+1}) \otimes e_\star(-) \Rightarrow e_\star(\Lambda(A)) \quad (97)$$

This is even simpler, since both functors act as an isomorphism functor on the underlying vector space of a representation. We choose e_{Q_-} to be the natural isomorphism which

assigns, to each representation $\rho \in \Lambda(\mathbf{S})$, the canonical isomorphism between this relative tensor product and the identity.

On morphisms which are composites of the generators Q_{\pm} , the value of e is the corresponding composite of these squares. We need to verify that e as defined above is pseudonatural, and that it is an equivalence. Pseudonaturality means that for any 2-cell $(\omega : f \Rightarrow g) \in \Omega_{\mathbf{H}'}$, the following condition holds:

$$e_f \cdot (K(\omega) \circ e_{\star}) = (e_{\star} \circ (\Lambda \circ C)(\omega)) \cdot e_g \quad (98)$$

We can express this diagrammatically as the following ‘pillow’ condition:

$$\begin{array}{c} \Lambda(\mathbf{S}) \xrightarrow{e_{\star}} K(\star) \\ \Lambda C(g) \xleftarrow{\Lambda C(\omega)} \Lambda C(f) \xrightarrow{e_f} K(f) \\ \Lambda(\mathbf{S}) \xrightarrow{e_{\star}} K(\star) \end{array} = \begin{array}{c} \Lambda(\mathbf{S}) \xrightarrow{e_{\star}} K(\star) \\ \Lambda C(g) \xleftarrow{e_g} K(g) \xrightarrow{K(\omega)} K(f) \\ \Lambda(\mathbf{S}) \xrightarrow{e_{\star}} K(\star) \end{array} \quad (99)$$

We only need to show that this holds for all the generating 2-morphisms (cups, caps, and crossings), since for any horizontal or vertical composite of generators, we have a corresponding composite of commuting ‘pillows’, which will again commute.

First, consider the proof of Lemma 2.15, in which we see that four generators are the injection and projection maps for the categorified commutation relation. These are the image under C of the cap, cup, and two crossing generators shown in (31). These four generators will satisfy (98); the following is a representative argument for the case $\omega = i_{\text{id}_{\mathbf{S}}} : \text{id}_{\mathbf{S}} \Rightarrow Q_- \circ Q_+$ is the cup. (Note that this is η_L for the ambiadjunction of A and A^{\dagger} , in the notation of (34).) Thus we have $C(\omega) = i_{\text{id}_{\mathbf{S}}}$, as in (22). Then $\Lambda \circ C(\omega)$ is again the injection of the identity functor into $\Lambda(A \circ A^{\dagger})$, since functoriality means that Λ preserves biproduct relations.

But in Definition 3.5, our restatement of Khovanov’s representation, $K(\omega)$ is just the bimodule inclusion $(n) \rightarrow_n (n+1)_n$ corresponding to the fact that $(n) = \mathbb{C}[S_n]$ is a direct summand of $(n+1) = \mathbb{C}[S_{n+1}]$. The two inclusions of 1-morphisms in the bicategories $\Lambda \circ C(\star) = \Lambda(\mathbf{S})$ and $K(\star) = \bigoplus_n \text{Rep}(\mathbb{C}[S_n])$ are intertwined by e_{\star} which identifies the representations.

Now, $e_{\text{id}} \cdot K(\omega)$ as a natural transformation in $K(\star)$, is just this $K(\omega)$ whiskered on the left by the identification of the source functor with source bimodule. As a natural transformation, it assigns this injection of bimodules to $\Lambda \circ C(\star)$. But $\Lambda \circ C(\omega) \cdot e_{Q_- \circ Q_+}$, which is the injection of functors whiskered on the right by the identification of each target functor with the corresponding bimodule. These are equal since the identification is by the composite of the identifications (96) and (97). These canonical isomorphisms commute. A similar argument holds for the other three 2-cells mentioned in Lemma 2.15.

Now, denote by ω_{\pm} the remaining crossings among the generators of \mathbf{H}' , which interchange two copies of Q_{\pm} respectively. Under C these give the spans of spans of the form (56), with $n = 2$, so μ is the nontrivial permutation of the 2-element set of new elements under the inclusion $T \subset T \sqcup \mathbf{2}$. Applying Λ , this gives the natural transformation between the composite functors $A^{\dagger} \circ A^{\dagger}$ or $A \circ A$ respectively which by permutation of the basis elements of the induced representations of each S_{n+2} . The standard basis elements in the S_n representations are described by Young tableaux as remarked above in Example 3.11.

The permutation of these elements corresponds to changing the order in which two boxes added (or removed) in the Young diagram for each summand of the new representation. This amounts to the permutation of the last two elements in an $n+2$ -element set, for Q_+ . This amounts to conjugation of elements of S_{n+2} by s_{n+1} .

Now, the $K(\omega_\pm)$, following Definition 3.5, correspond to, respectively, left and right module automorphisms of $(n+2)$ over $\mathbb{C}[S_n]$, acting by the permutation s_{n+1} on the standard basis. So $e_{Q_\pm \circ Q_\pm} \cdot K(\omega_\pm)$ is the natural transformation in $K(\star)$ obtained by identifying this bimodule with the corresponding functor, while $\Lambda \circ C(\omega_\pm) \cdot e_{Q_\pm \circ Q_\pm}$ is the natural transformation above, under the identification of groupoid and algebra representations. It is straightforward to confirm that these natural transformations are equal.

Finally, consider the remaining cup and cap generators. We previously considered the injection map, which we remarked above was η_L for the ambiadjunction. The remaining cup is $\eta_R : \text{id} \Rightarrow Q_+ \circ Q_-$. The image of the cap is its converse, η^\dagger . We have already denoted $C(\eta_R)$ by η , the span of spans (50). To apply Λ , we must make use of an explicit formula, for which we refer readers to [21, equations (16), (19)]. Using these formulae, and the fact that one leg of the span-of-spans is just the identity, we get

$$\Lambda(\eta) = \epsilon_{L,+1} \circ N \quad (100)$$

where N is the *Nakayama isomorphism* which, for each $n \in \mathbf{S}$, gives the following:

$$N(\phi) = \frac{1}{|S_n|} \sum_{g \in S_{n+1}} g^{-1} \otimes (\phi(g)) \quad (101)$$

The map $\epsilon_{L,+1}$ is the counit for the left adjunction between the restriction and induction functors along $(+1)$, taking $A^\dagger \circ A$ to the identity. So $\Lambda \circ C(\eta_R)$ is as required.

On the other hand, as noted in [15], the formula for $K(\eta) : \text{id} \Rightarrow \bigoplus_n (n+1)_n (n+1)$ which we restated in Definition 3.5 is a sum over cosets of S_n in S_{n+1} . In particular, the permutations used in the sum are representatives of these cosets. Since there are exactly $|S_n| = n!$ elements in each such coset, the sum is then just the same as in N . By essentially the same argument as before, (98) holds.

The remaining cup was ϵ_R for the ambiadjunction of Q_+ and Q_- in \mathbf{H}' . By Definition 3.5, we have that $K(\epsilon_R) : \bigoplus_n (n+1)_n (n+1) \rightarrow (n+1)$ acts by multiplication. We also have $C(\epsilon_R) = \eta^\dagger$, the converse of η . Thus, we get a natural transformation

$$\Lambda(\eta^\dagger) : \Lambda(A^\dagger) \circ \Lambda(A) \Rightarrow \text{id}_{\Lambda(\mathbf{S})} \quad (102)$$

It is straightforward but somewhat lengthy to check that this makes (98) hold again, so we omit the details here.

To confirm that the pseudonatural transformation e is an equivalence, there must be a weak inverse $e' : K \Rightarrow \Lambda \circ C$ and modifications from $e'e$ and ee' to the identities on $\Lambda \circ C$ and K respectively. But in fact, we have more than this, since the components e_{Q_+} and e_{Q_-} , and therefore all composites built from them, were defined above to be (strict) isomorphisms. \square

Since we have left the structure of $\mathbf{Span}(\mathbf{Gpd})$ as a symmetric monoidal bicategory as a conjecture, we cannot give a proof of the following, but we expect it to hold as well.

Conjecture 3.8. *The 2-functors C , Λ , and K can be equipped with the coherence data of symmetric monoidal 2-functors, and e is a symmetric monoidal pseudonatural equivalence.*

Now, we can apply Theorem 3.7 to comment on a sense in which our categorified Fock space $\Lambda(\mathbf{S})$ can indeed be understood as a higher-categorical analog of the ordinary Fock space. The theorem above notes the correspondence between the 2-linear map $\Lambda(A^\dagger)$, described above in terms of the blocks $M_{n-1,n}$, and $(-) \otimes_{\mathbb{C}[S_n]} n(n+1)_{n+1}$. This amounts, because of the quotient in the relative tensor product, to tensoring with the permutation representation of S_{n+1} on \mathbb{C}^{n+1} . The standard basis corresponds to the $n+1$ cosets of S_n in S_{n+1} .

Similarly, there is a natural equivalence

$$N_n \cong (-) \otimes \mathbb{C}[n] \quad (103)$$

where $\mathbb{C}[n]$ is as above, but seen as $S_n - S_n$ bimodule. So the “number operator”, which in the conventional Fock-space representation acts as “multiplication by the energy level”. This simply refers to the eigenvalue decomposition of $\mathbf{a}^\dagger \mathbf{a}$, where the eigenvalue n is termed the “energy level” of a state vector in Fock space. The energy level n is thus a decategorification of \mathbb{C}^n , which we therefore think of as the “categorified eigenvalue”. This decategorification loses information about the structure of \mathbb{C}^n as a representation of S_n . In the categorified Fock space, each block N_n in $\Lambda(A^\dagger A)$ now has an interpretation as a sort of categorified multiplication by this “eigenvalue”: it is the tensor product with \mathbb{C}^n as a bimodule.

It is precisely the structure lost in this decategorification which supports the extra information appearing in the categorified \mathbf{H}' : the category $\Lambda(\mathbf{S})$ is not merely any categorification of the Fock space representation of the Heisenberg algebra. Rather, it respects the morphism structure of \mathbf{H}' , giving a representation by functors *and natural transformations* of \mathbf{H}' . We will next comment on some of what this extra structure entails.

3.6 Symmetrizers and antisymmetrizers

The algebra categorified by Khovanov is larger than the single-variable Heisenberg algebra described in the introduction. It is a more complicated but closely related algebra also known as the Heisenberg vertex algebra, isomorphic as an ordinary algebra to the infinitely-generated Heisenberg algebra. It has a countably infinite family of generators a_i (which commute amongst themselves) and their adjoints (which also commute amongst themselves), which obey the following relations:

$$\mathbf{a}_i \mathbf{a}_j^\dagger - \mathbf{a}_j^\dagger \mathbf{a}_i = \mathbf{a}_{j-1}^\dagger \mathbf{a}_{i-1} \quad (104)$$

This includes the original relation when $i = j = 1$, if we take $\mathbf{a}_0 = \mathbf{a}_0^\dagger = 1$. This larger algebra has a physical interpretation in terms of conformal field theory, in which operators are replaced by holomorphic operator-valued fields on a Riemann surface. This will not concern us here, though see lecture notes by Thomas [30] for a starting point.

The categorification of this larger algebra uses a larger monoidal category $\mathbf{H} = \text{Kar}(\mathbf{H}')$. This makes use of the *Karoubi envelope* of a category \mathbf{C} (also known in some contexts as the *idempotent completion*, or *pseudoabelian hull*). The Karoubi envelope $\text{Kar}(\mathbf{C})$ is a universal category containing \mathbf{C} such that every idempotent morphism splits. Concretely, up to isomorphism, $\text{Kar}(\mathbf{C})$ can be constructed as the category with objects

$$\{(c, p) | c \in \mathbf{C}, p : c \rightarrow c, p \text{ idempotent}\}, \quad (105)$$

and with morphisms (c_1, p_1) to (c_2, p_2) given by $f : c_1 \rightarrow c_2$ such that $p_2 \circ f = f = f \circ p_1$. This is interpreted as adding new objects so that each idempotent has a kernel and cokernel

which are subobjects of its source and target respectively. The idempotent p can then be interpreted as projection onto the cokernel.

In particular, this introduces symmetrizer and antisymmetrizer idempotents associated to the action of S_n on products such as A^n by the maps $\hat{\mu}$. The new objects introduced can be described as symmetric or antisymmetric tensor products of the object Q_\pm (or A and A^\dagger in our setting). These are called S_\pm^n and Λ_\pm^n respectively, and defined as $S_\pm^n = S^n(Q_\pm)$ and $\Lambda_\pm^n = \Lambda^n(Q_\pm)$. It is a consequence of their construction as symmetrizers and antisymmetrizers that they satisfy relations which are the analogs of the basic commutation relations for the single-variable Heisenberg algebra:

$$S_s^n \otimes \Lambda_+^m \cong (\Lambda_+^m \otimes S_-^n) \oplus (\Lambda_+^{m-1} \otimes S_-^{n-1}) \quad (106)$$

See Khovanov's paper [15] for more details; there is nothing substantially different in our setting.

These summands do not exist as spans of groupoids, where the direct sum is just the disjoint union of spans, and as a result this aspect cannot be groupoidified in our combinatorial model. They do appear, however, after we apply the 2-functor Λ , since $\mathbf{2Vect}$ already contains all such subobjects. In particular, we are using the following fact:

Proposition 3.9. *Given any $\mathbf{V} \in \mathbf{2Vect}$, the category of endofunctors $\text{End}_{\mathbf{2Vect}}(\mathbf{V})$ has the property that all idempotents split.*

Proof. This uses the fact that $\mathbf{2Vect}$ is compact closed (a version of this fact is proved in Yetter [33], Theorem 28.) So the functor category $\text{End}_{\mathbf{2Vect}}(\mathbf{V}) \simeq \mathbf{V}^{op} \otimes \mathbf{V}$ is also a 2-vector space. Since, 2-vector spaces are abelian categories, every morphism has a cokernel, so in particular idempotents split. \square

In particular, we have the following restatement:

Lemma 3.10. *Given a monoidal category \mathbf{C} , any inclusion i of \mathbf{C} into the category of endofunctors of an object $\mathbf{V} \in \mathbf{2Vect}$, extends to an inclusion i' of $\text{Kar}(\mathbf{C})$:*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{i} & \text{End}_{\mathbf{2Vect}}(X) \\ \downarrow & \nearrow i' & \\ \text{Kar}(\mathbf{C}) & & \end{array} \quad (107)$$

Proof. Define i' on objects by

$$i'(c, p) = \text{coker}(i(p)). \quad (108)$$

On morphisms, i' is given by defining

$$i'(f : (c_1, p_1) \rightarrow (c_2, p_2)) \quad (109)$$

to be the corresponding map between the subspaces $\text{coker}(p_1)$ and $\text{coker}(p_2)$. This i' is well-defined since by hypothesis f commutes with the projections to the cokernels. It is a functor because it respects identities and composition. \square

Then since symmetrizers are idempotents, this applies to the case where $\mathbf{C} = \mathbf{H}'$. Treating monoidal categories as one-object bicategories, we can express this in the following diagram:

$$\begin{array}{ccc} \Omega_{\mathbf{H}'} & \xrightarrow{C} & \mathbf{Span}(\mathbf{Gpd}) \\ \downarrow & \searrow F' & \downarrow \Lambda \\ \Omega_{\mathbf{H}} & \xrightarrow{F} & \mathbf{2Vect} \end{array} \quad (110)$$

This says that the representation of \mathbf{H}' into $\mathbf{2Vect}$, which factors through $\mathbf{Span}(\mathbf{Gpd})$, extends in this compatible way to a representation of \mathbf{H} .

3.7 Symmetrizers in $\mathbf{2Vect}$

We will consider here how the symmetrizers appear in concrete terms within $\mathbf{2Vect}$ after 2-linearizing our combinatorial interpretation.

The natural isomorphism

$$\Lambda(A) \circ \Lambda(A) \xrightarrow{\Lambda(\hat{\sigma})} \Lambda(A) \circ \Lambda(A) \quad (111)$$

that permutes the order of the operators $\Lambda(A)$ can be represented in matrix form as a collection of linear maps, each acting on a component of the matrix representation of $\Lambda(A) \circ \Lambda(A)$. That is, each of the vector spaces that appear in this matrix carries a particular action of the symmetric group S_2 . The symmetrizer operation selects the subspace which is fixed by this action, and the antisymmetrizer operation selects the complement of this subspace.

More generally, each component in A^n is a representation of S_n , which decomposes into irreducibles, and the symmetrizer takes the sub-representation consisting of exactly the copies of the trivial representation which appear in this decomposition. Similarly, the antisymmetrizer takes only the copies of the sign representation. For any irreducible representation of S_n there will be a correspondingly ‘symmetrized’ sub-functor of A^n . These operations are the ‘Young symmetrizers’ associated to the Young diagrams which index such representations.

In particular, the subobjects which are needed to produce all the generators of the Heisenberg algebra are those arising from the symmetrizer idempotent for $\Lambda(A^n)$ or $\Lambda((A^\dagger)^n)$ which come from the symmetric group action and the antisymmetrizer, respectively:

$$v \mapsto \sum_{\mu \in S_n} \hat{\mu}(v) \quad v \mapsto \sum_{\mu \in S_n} \text{sign}(\mu) \hat{\mu}(v). \quad (112)$$

We can illustrate this by considering the product $\Lambda(A \circ A) \cong \Lambda(A) \circ \Lambda(A)$. The nontrivial permutation σ of the 2-element set gives the 2-cell

$$A \circ A \xrightarrow{\hat{\sigma}} A \circ A, \quad (113)$$

and thus, combined with the natural isomorphism above, we obtain the natural isomorphism (111).

The composite $\Lambda(A) \circ \Lambda(A)$ is zero except for blocks of the form

$$M_{i,i+2}^2 = M_{i,i+1} \otimes M_{i+1,i+2}. \quad (114)$$

From here on, we will use the notation $M_{i,j}$ to denote the matrix of vector space which is the (i, j) th block entry of the matrix $\Lambda(A)^{j-i}$. Up to a natural isomorphism, these can be expressed in matrix form:

$$M_{0,2} = \begin{pmatrix} \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (115)$$

$$M_{1,3} = \begin{pmatrix} \mathbb{C} & \mathbb{C}^2 & \mathbb{C} \end{pmatrix} \quad (116)$$

$$M_{2,4} = \begin{pmatrix} \mathbb{C} & \mathbb{C}^2 & \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C}^2 & \mathbb{C} \end{pmatrix} \quad (117)$$

As with the basic blocks $M_{n,n+1}$, a general block $M_{i,j}$ can be interpreted as giving a decomposition in matrix form of the $\mathbb{C}[S_i]-\mathbb{C}[S_j]$ -bimodule which effects the induction functor

$$\text{Rep}(S_i) \xrightarrow{\text{Ind}_{S_i}^{S_j}} \text{Rep}(S_j) \quad (118)$$

associated to the natural inclusion of S_i into S_j . That is, given a representation $\rho \in \text{Rep}(S_i)$, the induced representation of S_j is $\rho \otimes_{\mathbb{C}[S_i]} M_{i,j}$.

Example 3.11. The block $M_{2,4}$ has component \mathbb{C}^2 in the position indexed by $(\square\square, \square\square\square)$. As we have remarked, this reflects the fact that there are two paths in the directed graph (74) from $\square\square$ to $\square\square\square$, passing through $\square\square\square$ or $\square\square$.

A Young tableau is a way of filling the boxes of the diagram with the numbers $1, \dots, n$ such that they denote a valid order in which to add the boxes while following a path through (74). It is a standard fact that the tableaux which can fill a given Young diagram label basis elements of the irreducible representation labeled by that diagram. Thus, we see that the component \mathbb{C}^2 acts to exchange the two basis elements. This is a permutation representation on the basis of \mathbb{C}^2 , which decomposes as:

$$\square\square \oplus \begin{array}{c} \square \\ \square \end{array} \quad (119)$$

The symmetrizer and antisymmetrizer each select just one of these summands. The two other components are both trivial S_2 representations. So the symmetrized functor has a matrix form with a $(2, 4)$ block:

$$\begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (120)$$

The antisymmetrized functor has the following block:

$$\begin{pmatrix} 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} & 0 \end{pmatrix} \quad (121)$$

The situation becomes more complex for A^n with larger values of n , but the method is essentially the same.

We have seen here that the functors $\Lambda(A)$ and $\Lambda(A^\dagger)$ have, as sub-functors, precisely the symmetric and antisymmetric powers S_\pm^n and \bigwedge_\pm^n which will give the actual generators of the multivariable Heisenberg algebra when passing to the Grothendieck ring. So, after taking the image under Λ of our groupoidification, passing to the completion of the image recovers Khovanov's full categorification. This is closely related to a construction used by Khovanov in the proof that there is a map $H_{\mathbb{Z}}$ from the integral form of the Heisenberg algebra into the Grothendieck group of \mathbf{H} .

4 Combinatorial models of categorified $U(\mathfrak{sl}_n)$

4.1 Introduction

We now investigate how the techniques described so far can be applied to find combinatorial models of categorifications of $U(\mathfrak{sl}_n)$, the universal enveloping algebras of the Lie algebras \mathfrak{sl}_n .

These categorifications are in the style of the Khovanov-Lauda programme [16, 17, 18]. The main objects of interest in that programme are categorifications of the q -deformed enveloping algebras, the quantum groups $U_q(\mathfrak{sl}_n)$. However, the combinatorial interpretation we describe here only applies to the case $q = 1$, in which case the Grothendieck ring is a module over the integers \mathbb{Z} . In the q -deformed situation, one might expect to obtain a module over the ring $\mathbb{Z}[q, q^{-1}]$ of formal power series in q .

We expect that a similar combinatorial model should exist of the categorification of the full q -deformed algebra, following the pattern of the groupoidification of Hecke algebras [2]. In this approach, the groupoid of finite sets is replaced with the groupoid of finite-dimensional vector spaces over a finite fields \mathbb{F}_q . The special linear groups for such vector spaces take the role of symmetric groups S_n , and their size is related to q -factorials.

4.2 The algebras $U(\mathfrak{sl}_n)$

We write \mathfrak{sl}_n for the Lie algebra associated to the special linear group $SL(n)$, which is given as the group of $n \times n$ matrices with unit determinant. Its Lie algebra would then consist of the traceless $n \times n$ matrices. It is presented as an abstract algebra in terms of generators h_i , e_i and f_i for $i \in \{1, \dots, n-1\}$, satisfying the following relations:

$$[e_i, f_j] = \delta_{ij} h_i \quad (122)$$

$$[e_i, h_j] = 2\delta_{ij} e_i \quad (123)$$

$$[f_i, h_j] = 2\delta_{ij} f_i \quad (124)$$

This is a special case of the standard presentation, in the Chevalley basis, of the Lie algebra generated by Cartan data associated to a particular Dynkin diagram [10, 12]. In the representation of \mathfrak{sl}_2 as the traceless matrices acting on \mathbb{C}^2 , these elements have the following concrete realization:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (125)$$

Just as the Heisenberg algebra has a natural representation on the polynomial ring $\mathbb{C}[[z]]$, the universal enveloping algebras $U(\mathfrak{sl}_n)$ have natural representations on polynomial rings $\mathbb{C}[[z_1, \dots, z_n]]$, defined in the following way:

$$\mathbb{C}[[z_1, \dots, z_n]] = \bigoplus_j (\mathbb{C}^n)^{\otimes_{s,j}} \quad (126)$$

In this polynomial representation, the generators e_i and f_i of $U(\mathfrak{sl}_n)$ become

$$e_i = z_{i+1} \partial_i \quad (127)$$

$$f_i = z_i \partial_{i+1}, \quad (128)$$

and their commutator is

$$h_i = z_{i+1} \partial_{i+1} - z_i \partial_i. \quad (129)$$

This extension gives back the defining representation of \mathfrak{sl}_n when \mathbb{C}^n is identified with the space of degree-1 polynomials.

This representation of $U(\mathfrak{sl}_n)$ suggests defining

$$n_i := z_i \partial_i, \quad (130)$$

so that we have

$$h_i = n_{i+1} - n_i. \quad (131)$$

Rewriting our Lie algebra relations (127–129) in terms of these new variables, and expanding out commutators explicitly and reorganizing to eliminate negatives, we obtain the following:

$$e_i f_j + \delta_{ij} n_i = f_j e_i + \delta_{ij} n_{i+1} \quad (132)$$

$$e_i n_{j+1} + n_j e_i = e_i n_j + n_{j+1} e_i + 2\delta_{ij} e_i \quad (133)$$

$$f_i n_{j+1} + n_j f_i = f_i n_j + n_{j+1} f_i + 2\delta_{ij} f_i \quad (134)$$

In this form, the relations are now suitable for groupoidification. We note that these relations are a special case of the general pattern for the construction of a Kac-Moody algebra from a Cartan matrix. In the case of these Lie algebras, the Cartan matrix is determined by the Dynkin diagram associated to the algebra. So the pattern for groupoidifying these algebras should apply in these more general cases as well.

4.3 Groupoidification

The groupoidification of \mathfrak{sl}_n is based on the groupoid of n -coloured finite sets and colour-preserving bijections, written \mathbf{S}^n . This is equivalent to the category where objects are n -tuples of finite sets and morphisms are n -tuples of bijections, which is the Cartesian product of n copies of \mathbf{S} . The sets of colour i form the i th component of the Cartesian product. A third way to describe \mathbf{S}^n up to equivalence is as the free symmetric monoidal groupoid on the discrete groupoid with n objects.

We write $\mathbf{S}^n \xrightarrow{+1_i} \mathbf{S}^n$ for the functor which acts as $+1$ on the sets of colour i , and the identity otherwise. For any $i \in \{1, \dots, n\}$, we use this to define the spans A_i and A_i^\dagger as follows:

$$\begin{array}{ccc} & \mathbf{S}^n & \\ \text{id} \swarrow & & \searrow +1_i \\ \mathbf{S}^n & & \mathbf{S}^n \end{array} \quad \begin{array}{ccc} & \mathbf{S}^n & \\ +1_i \swarrow & & \searrow \text{id} \\ \mathbf{S}^n & & \mathbf{S}^n \end{array} \quad (135)$$

These are ‘multicoloured’ variants of the ordinary annihilation and creation spans defined in equation (10). They satisfy an adjusted version of the categorified commutation relation:

$$A_i \circ A_j^\dagger \simeq A_j^\dagger \circ A_i \oplus \delta_{ij} \text{id}_{\mathbf{S}^n} \quad (136)$$

Here δ_{ij} represents the identity span if $i = j$, and the zero span otherwise.

To obtain our model of categorified $U(\mathfrak{sl}_n)$, we arbitrarily assign a total ordering to our set n of colours. We then make the following definitions for all $i \in \{1, \dots, n-1\}$:

$$E_i := A_{i+1}^\dagger \circ A_i \quad (137)$$

$$F_i := A_i^\dagger \circ A_{i+1} \quad (138)$$

These are mutually-converse spans for each i . We also define the number operator N_i for each colour i as follows:

$$N_i := A_i^\dagger \circ A_i \quad (139)$$

This span is its own converse. The operators E_i , F_i and N_i satisfy a categorified form of the reorganized relations (132–134) for $U(\mathfrak{sl}_n)$:

$$E_i F_j \oplus \delta_{ij} N_i \simeq F_j E_i \oplus \delta_{ij} N_{i+1} \quad (140)$$

$$E_i N_{j+1} \oplus N_j E_i \simeq E_i N_j \oplus N_{j+1} E_i \oplus 2\delta_{ij} E_i \quad (141)$$

$$F_i N_{j+1} \oplus N_j F_i \simeq F_i N_j \oplus N_{j+1} F_i \oplus 2\delta_{ij} F_i \quad (142)$$

The reason for this is combinatorial, and can be understood by explicitly tracking histories. For instance, in the first relation, consider the span $E_i F_j$. This is equal to $A_j^\dagger \circ A_i \circ A_i^\dagger \circ A_j$, a sequence of adding and removing elements of colours i and j . In the case where $i \neq j$ the operators A_i and A_j commute, so this is isomorphic to $A_i \circ A_i^\dagger \circ A_j^\dagger \circ A_j$. Since the spans A_i and A_j each satisfy the usual categorified commutation relations for the Heisenberg algebra, this is isomorphic to $(N_i \oplus \text{id}) \circ N_j \simeq N_i N_j \oplus N_j$. A similar computation holds on the other side, and so the isomorphism (140) can be established. Applied to a coloured set with n_i elements of colour i , and n_j elements of colour j , both sides compute $n_i n_j + n_i + n_j$. If we add the identity on each side, this relation for E_i and F_j witnesses the isomorphism of two different ways to count $(n_i + 1)(n_j + 1)$, the number of ways to distinguish either zero or one elements of each colour in any given set.

Applying the degroupoidification functor (67) to E_i , F_i and N_i recovers the standard action (122–124) of $U(\mathfrak{sl}_n)$ on spaces of polynomials. The coefficients obtained by the multiplication and differentiation are exactly the numbers we have just calculated in the example. Just as with the Heisenberg algebra, the algebraic facts about the relations they satisfy now appear as consequences of combinatorial facts about coloured sets. In this way, we have a groupoidification of the concrete representation on $\mathbb{C}[[z_1, \dots, z_n]]$ of the algebra $U(\mathfrak{sl}_n)$. In the same way, the abstract categorifications of Khovanov-Lauda type are realized concretely by endomorphisms of an object in $\mathbf{Span}(\mathbf{Gpd})$.

As described in Section 2.2, the groupoidification of vectors in Fock space are stuff types. Such a state $\mathbf{X} \xrightarrow{\Phi} \mathbf{S}$ may be thought of as an ensemble of structures whose underlying finite sets are specified by Φ , which preserves symmetry relations. When Φ is faithful, one obtains a combinatorial species [1]. Considering a stuff type as a span $\mathbf{X} \xleftarrow{\Phi} \mathbf{S} \rightarrow \mathbf{1}$ and degroupoidifying it, we obtain a linear map $\mathbb{C} \rightarrow \mathbb{C}[[z]]$, which specifies a power series in z by considering the image of $1 \in \mathbb{C}$. This is the generating function in the ordinary sense for the combinatorial structure [32], in the case of a combinatorial species.

In the same way, a state in the Fock space for our groupoidified $U(\mathfrak{sl}_n)$ is a functor $\mathbf{X} \xrightarrow{\Psi} \mathbf{S}^n$, thought of as a span out of $\mathbf{1}$. This is a generalization of a *multisort species* [4] which describe structures on coloured sets, in the same way that stuff types generalize species. Thus, we can describe such a state as a *multisort stuff type*. Again, degroupoidification of such a span determines an element of $\mathbb{C}[[z_1, \dots, z_n]]$, so a multisort stuff type determines a vector in the multiple-variable Fock space. The groupoidified $U(\mathfrak{sl}_n)$ acts on multisort stuff types just as the groupoidified Heisenberg algebra acts on stuff types.

References

- [1] John Baez and James Dolan. From finite sets to Feynman diagrams. In Björn Engquist and Wilfried Schmid, editors, *Mathematics Unlimited — 2001 And Beyond*. Springer Verlag, 2001. [arXiv:math/0004133](https://arxiv.org/abs/math/0004133).

- [2] John C. Baez, Alexander E. Hoffnung, and Christopher D. Walker. Higher dimensional algebra VII: Groupoidification. *Theory and Applications of Categories*, 24:489–553, 2010. [arXiv:0908.4305](#), [TAC:24.18](#).
- [3] Jean Bénabou. Distributors at work. <http://www.mathematik.tu-darmstadt.de/streicher/FIBR/DiWo.pdf>.
- [4] Francois Bergeron, Gilbert Labelle, Pierre Leroux, and Margaret Readdy. *Combinatorial Species and Tree-Like Structures*. Encyclopedia of Mathematics and its Applications 67. Cambridge University Press, 1998. [doi:10.1017/cbo9781107325913](#).
- [5] Robert Dawson, Robert Paré, and Dorette A. Pronk. Universal properties of span. *Theory and Applications of Categories*, 13(4):61–85, 2004. [TAC:13.4](#).
- [6] Robert Dawson, Robert Paré, and Dorette A. Pronk. The span construction. *Theory and Applications of Categories*, 24(13):302–377, 2012. [TAC:24.13](#).
- [7] Thomas Ehrhard and Laurent Regnier. The differential lambda-calculus. *Theoretical Computer Science*, 309:1–41, 2003. [doi:10.1016/s0304-3975\(03\)00392-x](#).
- [8] Marcelo Fiore. An axiomatics and a combinatorial model of creation/annihilation operators. Unpublished notes.
- [9] Marcelo Fiore, Nicolo Gambino, Martin Hyland, and Glynn Winskel. The cartesian closed bicategory of generalised species of structures. *Journal of the London Mathematical Society*, 77:203–220, 2008. [doi:10.1112/jlms/jdm096](#).
- [10] Jurgen Fuchs and Christoph Schweigert. *Symmetries, Lie Algebras, and Representations*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1997.
- [11] William Fulton. *Young Tableaux*. London Mathematical Society Student Texts 35. Cambridge University Press, 1997. [doi:10.1017/cbo9780511626241](#).
- [12] Robert Gilmore. *Lie Groups, Lie Algebras, and Some of Their Applications*. Dover, 1974, 2002. [doi:10.1063/1.3128987](#).
- [13] Alexander E. Hoffnung. Spans in 2-categories: a monoidal tricategory. Unpublished preprint, [arXiv:1112.0560](#).
- [14] André Joyal. Foncteurs analytiques et espèces des structures. *Lecture Notes in Mathematics*, 1234:126–159, 1986. [doi:10.1007/bfb0072514](#).
- [15] Mikhail Khovanov. Heisenberg algebra and a graphical calculus. *Fundamenta Mathematicae*, 225:169–210. [arXiv:1009.3295](#). [doi:10.4064/fm225-1-8](#).
- [16] Mikhail Khovanov and Aaron D. Lauda. A diagrammatic approach to categorification of quantum groups III. *Quantum Topology*, 1(1):1–92, 2010. [arXiv:0807.3250](#). [doi:10.4171/QT/1](#).
- [17] Aaron D. Lauda. A categorification of quantum \mathfrak{sl}_2 . *Advances in Mathematics*, 225(6):3327–3424, 2010. [arXiv:0803.3652](#). [doi:10.1016/j.aim.2010.06.003](#).
- [18] Aaron D. Lauda. An introduction to diagrammatic algebra and categorified quantum \mathfrak{sl}_2 . *Bulletin of the Institute of Mathematics, Academia Sinica (New Series)*, 7(2):165–270, 2012. [arXiv:1106.2128](#).

- [19] Volodymyr Mazorchuk. *Lectures on Algebraic Categorification*. The QGM Master Class Series. European Mathematical Society, 2012. doi:10.4171/108.
- [20] Jeffrey Morton and Jamie Vicary. The categorified Heisenberg algebra II: Models on free structures. In preparation.
- [21] Jeffrey C. Morton. Cohomological twisting of 2-linearization and extended tqft. *J. Homotopy Relat. Struct.*, 10:127–187. arXiv:1003.5603. doi:10.1007/s40062-013-0047-2.
- [22] Jeffrey C. Morton. Categorified algebra and quantum mechanics. *Theory and Applications of Categories*, 16(29):785–854, 2006. arXiv:0601458, TAC:16.29.
- [23] Jeffrey C. Morton. Groupoids and 2-vector spaces. *Applied Categorical Structures*, 19(4):659–707, 2010. arXiv:0810.2361. doi:10.1007/s10485-010-9225-0.
- [24] Dorette Pronk. (personal communication).
- [25] Raphael Rouquier. 2-Kac-Moody algebras. Unpublished. arXiv:0812.5023.
- [26] Bruce Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics 203. Springer Verlag, 1991.
- [27] Peter Selinger. A survey of graphical languages for monoidal categories. In Bob Coecke, editor, *New Structures for Physics*, Lecture Notes in Physics 813, pages 289–355. Springer, 2011. arXiv:0908.3347. doi:10.1007/978-3-642-12821-9.
- [28] Michael Stay. Compact closed bicategories. *Theory and Applications of Categories*, 31(26):755–798, 2016. arXiv:1301.1053.
- [29] Shlomo Sternberg. *Group theory and physics*. Cambridge University Press, 1994.
- [30] Justin D. Thomas. The Heisenberg vertex algebra. Unpublished notes.
- [31] Ben Webster. Knot invariants and higher representation theory I: Diagrammatic and geometric categorification of tensor products. arXiv:1001.2020.
- [32] Herbert S. Wilf. *Generatingfunctionology*, 2nd ed. Academic Press, 1993. <http://www.math.upenn.edu/~wilf/gfology2.pdf>. doi:10.1201/b10576.
- [33] David Yetter. Categorical linear algebra: a setting for questions from physics and low dimensional topology. Kansas State University Preprint, <http://math.ucr.edu/home/baez/yetter.ps>.
- [34] David Yetter. Quantum groups and representations of monoidal categories. *Mathematical Proceedings of the Cambridge Philosophical Society*, 108:261–290, 1990. doi:10.1017/s0305004100069139.