

On Cleavability



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Abstract

This thesis concerns *cleavability*. A space X is said to be cleavable over a space Y along a set $A \subseteq X$ if there exists a continuous function $f : X \rightarrow Y$ such that $f(A) \cap f(X \setminus A) = \emptyset$. A space X is cleavable over a space Y if it is cleavable over Y along all subsets A of X .

In this thesis we prove three results regarding cleavability.

First we discover the conditions under which cleavability of an infinite compactum X over a first-countable scattered linearly ordered topological space (LOTS) Y implies embeddability of X into Y . In particular, we provide a class of counter-examples in which cleavability does not imply embeddability, and show that if X is an infinite compactum cleavable over ω_1 , the first uncountable ordinal, then X is embeddable into ω_1 .

We secondly show that if X is an infinite compactum cleavable over any ordinal, then X must be homeomorphic to an ordinal. X must also therefore be a LOTS. This answers two fundamental questions in the area of cleavability. We also leave it as an open question whether cleavability of an infinite compactum X over an uncountable ordinal λ implies X is embeddable into λ .

Lastly, we show that if X is an infinite compactum cleavable over a separable LOTS Y such that for some continuous function f from X to Y , the set of points on which f is not injective is scattered, then X is a LOTS.

In addition to providing these three results, we introduce a new area of research developed from questions within cleavability. This area of research is called *almost-injectivity*.

Given a compact T_2 space X and a LOTS Y , we say a continuous function $f : X \rightarrow Y$ is almost-injective if the set of points on which f is not injective has countable cardinality. In this thesis, we state some questions concerning almost-injectivity, and show that if λ is an ordinal, X is a T_2 compactum, and $f : X \rightarrow \lambda$ is almost-injective, then X must be a LOTS.

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Notation and Terminology

As is standard, we use \mathbb{R} to represent the real numbers. When we say a space X is compact, we do not also assume the space X is Hausdorff. (However, it will be seen that while we are not assuming that these compacta are Hausdorff, they will still have this property.) By compactum we mean a compact space. We say a space X is perfect if it is dense in itself and closed. If X and Y are topological spaces, and $f : X \rightarrow Y$ is a function, we use M_f to represent the set $\{x \in X : |f^{-1}(f(x))| > 1\}$; that is, M_f is the set of points on which f is not injective. We use \mathbb{D} to denote the Double Arrow Space, also known as Alexandroff's Double Arrow Space; \mathbb{D} is constructed as $([0, 1) \times \{0\}) \cup ((0, 1] \times \{1\})$, with the lexicographical order, and given the linear order topology. As is also convention, we denote the set of continuous functions from a space X to a space Y as $\mathcal{C}(X, Y)$. We use LOTS as an abbreviation for “linearly ordered topological space”, and “LOTSs” as the plural of LOTS. Lastly, if X is a topological space, then we use the notation X^β to mean the β -th Cantor-Bendixson derivative of X , defined in full in Definition 3.2.3.

All other terms used will be defined within the thesis.

Outline

This thesis is composed of seven chapters.

The first two chapters provide an introduction to cleavability. The first chapter is non-technical, and provides a rich context for the studies contained within this thesis. The context includes motivation for studying cleavability, previous results, and a history of the topic. It also includes outstanding questions related to cleavability, and the questions about which this thesis is concerned.

The second chapter is intended to provide an introduction on how one uses cleavability as a mathematical tool, and offers a list of basic and important results in cleavability. In the material of this thesis, we frequently refer to results contained within this chapter.

The most important results of this thesis are obtained in Chapters 3, 4, and 5. In these chapters, we look at cleavability over various types of LOTSs, and provide partial answers to some of the outstanding questions related to cleavability. Each of these chapters is broken into several sections; the last section of each chapter contains conclusions and open questions. These conclusions and open questions frequently motivate research discussed in the following chapters. Chapters 3 and 5 have been published, and may be found in [21] and [20] respectively; Chapter 4 has been submitted for publication. Each of these chapters is the original research of the author.

The sixth chapter is dedicated to introducing an area of research inspired by methods and results of cleavability: “almost-injectivity”. We first introduce this topic, then provide results related to the area of study, and offer conclusions and open questions.

Lastly, Chapter 7 provides the conclusion to this thesis. It contains a summary of results obtained, a description of why these results are significant, and related open questions.

It is not necessary to read this thesis linearly. Any results used from previous chapters will be cited, and often restated for the benefit of the reader.

Cleavability

Chapter 1

Introduction to Cleavability

1.1 Motivation

The following is a well-known theorem:

Theorem 1.1.1. *If X is compact, Y is Hausdorff, and $f : X \rightarrow Y$ is a continuous injection, then f embeds X into Y .*

This theorem, while useful, is not constructive. Were we to know a continuous injection existed, then we would know as much about X as we do about Y . However, it does not prescribe the conditions under which such a function must exist. Thus unless one is able to describe a specific function, very little about X may be known.

This is one of our focal reasons for the study of cleavability. Cleavability, among other uses, is a mechanism by which we may say for certain when an injective function must exist from a compact space X into a Hausdorff space Y .

Definition 1.1.2. *A space X is cleavable over a space Y along $A \subseteq X$ if there exists a continuous function $f : X \rightarrow Y$ such that $f(A) \cap f(X \setminus A) = \emptyset$.*

Definition 1.1.3. *A space X is said to be **cleavable over** a space Y if X is cleavable over Y along every subset A of X .*

From these definitions, we have the following observation.

Observation 1.1.4. *Let X and Y be topological spaces. If there exists a continuous injection from X to Y , then X is cleavable over Y along every subset A of X .*

Furthermore, we do not assume that if f cleaves X along A over Y , then f cleaves X along $B \subseteq X$ as well.

The following two questions, stated in [6] and [11] respectively, are what have motivated the research in this thesis:

Question 1. *When does cleavability of a compact space X over a Hausdorff space Y imply the existence of a homeomorphism from X to a subspace of Y ?*

Question 2. *Let a compact space X be cleavable over a LOTS. Is X a LOTS?*

To answer either of these questions, it is often useful to consider the following:

Question 3. *If X is cleavable over a space Y with property P , must X have property P ?*

Answering any of the above questions positively has many non-trivial and powerful consequences for the areas of analytic topology and functional analysis, a few of which we will now discuss.

One of the primary functions of cleavability is the distribution of information. Answering the above questions allows us to understand a complex topological space better by associating it with a simpler, well-understood topological space. If cleavability over a space with property P does imply X must have this property, then there is a great deal of information we may import to X based on our knowledge of Y . Were we to show a continuous injection, and therefore an embedding, did exist, then we would have almost as much information about X as we do about Y .

Cleavability is particularly useful when proving the existence (or non-existence) of spaces with various properties. A common question in topology is of the form: “Does there exist a – space which is / is not –”; for example, taken from [9], “Does there exist a CCC pseudocompact space which is not 2-starcompact?”. (For definitions of CCC, pseudocompact, and 2-starcompact, see page 9 of [9].) When seeking to show if such a space exists, it is useful to speculate what other properties this space, if it did exist, would necessarily have. If we may then show that this space is cleavable over a fairly well-known topological space, we may be able to import a wealth of information from the easily understood space to comprehend this difficult space better.

In addition, a research area of popular interest is the characterization of when a space is a LOTS, and many papers have been devoted to this study. (For examples, see [27], [28], and [31].) If we are able to show that cleavability of a space X over a LOTS implies X is itself a LOTS, then we will have added a new characterization of those spaces which are linearly orderable. This is particularly useful, as there is a large amount of information known about linearly ordered topological spaces,

and showing which spaces are in fact linearly orderable gives us a great deal more information about the space. For examples of papers and books detailing this information, see [22], and [13].

Lastly, as we mentioned, answering Question 1 would prescribe the conditions under which an injective function between two spaces must exist. We would be able to show the exact conditions under which we may weaken the property of injectivity in Theorem 1.1.1, and still have the existence of an embedding.

1.2 History

Cleavability was originated by A. V. Arhangel'skiĭ in [2], though it was originally called splittability. Initially, cleavability was a tool in functional analysis to approximate real-valued functions using continuous real-valued functions; we discuss this in Section 1.5. Soon after, in articles such as [4] and [3], Arhangel'skiĭ began to explore cleavability as an independent area of research.

Since the origination of cleavability in [2], several authors in addition to Arhangel'skiĭ have contributed to the field of study. Authors include, but are not limited to, D. J. Marron and T. B. M. McMaster (see: [23], [24], [25]), H. Bennett, R. Byerly, and D. Lutzer (see: [15]), and R. Z. Buzyakova (see: [10], [11]). For a few of several papers authored by Arhangel'skiĭ on the subject, see [4], [3], and [1]. For a basic survey on cleavability, see [6].

1.3 Previous Positive Results

We relied on papers containing the following positive results for an introduction to the topic of cleavability. However, since the topic was introduced, a large amount of research has been conducted, and therefore the list presented is by no means exhaustive.

One of the first major results was published in Arhangel'skiĭ's paper, "On cleavability of topological spaces over \mathbb{R} , \mathbb{R}^n , and \mathbb{R}^ω (see: [5]). In this paper, he proved that every compact space X that is cleavable over \mathbb{R} must be embeddable into \mathbb{R} . He also proved that cleavability of a compactum X over \mathbb{R}^ω must be embeddable into \mathbb{R}^ω .

Several years later, R. Z. Buzyakova in [10] went on to show that if \mathcal{L} is a continuum cleavable over a LOTS Y , then \mathcal{L} must be homeomorphic to a subspace of Y . She also proved in [11], that any compact space C cleavable over the Double Arrow Space \mathbb{D} must be embeddable into \mathbb{D} . In the latter paper, the result was obtained by fixing a function $f : C \rightarrow \mathbb{D}$ such that M_f was countable, and a G_δ set in C . It was this method, and this paper, that inspired the results contained in Chapter

5.

Other positive results can be found in [23], [24], and [25], all of which are about cleavability over semi-groups and partial orders.

1.4 Previous Negative Results

Ideally, we would like it to be true in all cases that cleavability of an infinite compactum over a T_2 space implies embeddability. Unfortunately, there are a few counter-examples that show cleavability does not always imply embeddability.

We provide two types of examples in this section. The first few examples are designed to show why we assume various properties about X and Y . After we explain the necessity of certain assumptions, we then give examples of infinite compacta that are cleavable over various LOTs, but not embeddable.

We begin with a demonstration of why we assume X and Y are infinite spaces throughout this thesis.

Example 1.4.1. *Let $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, both with the discrete topologies. Then X is cleavable over Y , but not homeomorphic to a subspace of Y .*

Proof. Let $A \subseteq X$, and let $f : X \rightarrow Y$ be defined as: $f(x) = y_1$ if $x \in A$, and $f(x) = y_2$ if $x \notin A$. This function is clearly continuous, as X and Y are discrete. Thus X is cleavable over Y , but obviously not embeddable into Y . \square

The same results are reached for any discrete X and finite Y such that $|X| > |Y| > 1$.

Let us now take another example:

Example 1.4.2. *Let X be any finite space, let Y be an infinite T_2 space, and let X be cleavable over Y . Then X is embeddable into Y .*

Proof. As a consequence of Proposition 2.1.1, which we will prove in Chapter 2, X must be discrete. Let $|X| = m < \omega$. Enumerate the elements of X as x_1, \dots, x_m . Let y_1, \dots, y_m be distinct elements of Y . Then $f : X \rightarrow Y$ such that $f(x_j) = y_j$ for every $j \leq m$ is obviously continuous. It is also closed and injective, making it an embedding. \square

In section 1.1, we stated that the fundamental uses of cleavability include gaining information about complex topologies by associating them with well-known spaces, and showing when an injection (respectively: embedding) from a space X into a space Y exists. Therefore, in order for

cleavability to provide any meaningful results, Examples 1.4.1 and 1.4.2 show that we should assume X and Y to be infinite.

If we wish to avoid another trivial counter-example, then we must also assume Y is at least T_2 . Otherwise, consider the following:

Example 1.4.3. *Let $O = \omega \cup \{\omega\} \cup \{\hat{\omega}\}$, where the topology on O is generated by singleton subsets of ω , and final segments of ω unioned with either $\{\omega\}$ or $\{\hat{\omega}\}$. Then there exists a continuous injection from $\omega \cdot 2 + 1$ onto O , but $\omega \cdot 2 + 1$ is not embeddable into O .*

We know from Observation 1.1.4 that since there is a continuous injective function from $\omega \cdot 2 + 1$ to O , $\omega \cdot 2 + 1$ is cleavable over O . However, it is definitely not embeddable into O . Therefore, to avoid the class of spaces that may be condensed into a space without being embedded, we assume Y is at least T_2 .

However, even with these assumptions, there are a few negative results of infinite compacta that are cleavable over various LOTSs, but not embeddable.

The following is from [6]:

Example 1.4.4. *The complete graph with five vertices is cleavable over \mathbb{R}^2 , but is not embeddable into \mathbb{R}^2 .*

This counter-example is actually part of a larger class of counter-examples: namely the class of one-dimensional compact polyhedra that are cleavable over \mathbb{R}^2 , but not embeddable into \mathbb{R}^2 . This class is a subclass of all one-dimensional compact polyhedra that are cleavable over \mathbb{R}^2 .

Another negative result in cleavability is contained in [11], which we restate here:

Example 1.4.5. *Let $X = (\omega_1 \times_l [0, 1)) \cup \{\omega_1\}$ be a long segment (that is, a one-point compactification of the lexicographical product of the space of all ordinals less than ω_1 and the interval $[0, 1)$). Let $C = X \times \{1, 2, 3\}$ be the Cartesian product of X and a discrete three-element space. Consider a partition p on C whose only non-trivial elements are $\{(\omega_1, 1), (\omega_1, 2)\}$ and $\{(\omega_1, 3), ((0, 0), 1)\}$. Denote $p(C)$ with the quotient topology by L . The space L is a connected LOTS with two points of uncountable character. The point $p(\{(\omega_1, 1), (\omega_1, 2)\})$ can be reached in L by two disjoint ω_1 -type sequences while the other point of uncountable character cannot be approached by disjoint ω_1 -type sequences. The compactum C is cleavable over L . However, C is not embeddable in L as the latter suffers from the deficit of points of uncountable character.*

In Chapter 3 we add another class of counter-examples to this list: if $\beta \geq \omega$ is a countable limit ordinal, and $m, n \in \omega \setminus \{0, 1\}$ such that $m > n$, then $\beta \cdot m + 1$ is cleavable over $\beta \cdot n + 1$, but not

embeddable into it. The specific example we give, in Theorem 3.2.1, is that $\omega \cdot 3 + 1$ is cleavable over $\omega \cdot 2 + 1$, but is not embeddable into it.

While these counter-examples show that cleavability does not imply embeddability in all circumstances, the results do seem to indicate that a space X that is cleavable over a space Y looks nearly identical to Y , but may have more points of complexity than Y . By points of complexity we mean, for instance, the points of uncountable character from Example 1.4.5, or the points $\beta \cdot m$ and $\beta \cdot n$ from Theorem 3.2.1. It may be the case, then, that cleavability ensures local embeddability of X into Y . That is, for every $x \in X$, we may find an open neighborhood that is embeddable into Y . We expand upon this thought, and formally state it as an open question within Chapter 5.

1.5 Cleavability and Functional Analysis

As mentioned, cleavability was first used as a tool in functional analysis. Arhangel'skiĭ defined cleavability in an attempt to answer the following question, stated in [7]:

Question 4. *Given a real-valued function f defined on a topological space X , when can one find a countable family $\{f_n : n \in \omega\}$ of continuous real-valued functions on X that approximates f on finite subsets of X ? That is, for any finite set $F \subset X$ and every real number $\epsilon > 0$ one can choose $n \in \omega$ such that $|f(x) - f_n(x)| < \epsilon$ for every $x \in F$.*

In [7], it was shown that such a family of continuous real-valued functions exists if and only if X is cleavable over \mathbb{R} .

Cleavability still plays a role in current research in functional analysis as well, specifically in the area of C_p -theory, which we now discuss.

We remind the reader here that cleavable and splittable are two names for the same definition.

In [17], D. Jardón defined when a space X is weakly cleavable over a space Y :

Definition 1.5.1. *Call a space X **weakly splittable** (also known as **weakly Eberlein compact**) if, for each $f \in \mathbb{R}^X$ there exists a σ -compact $F \subset C_p(X)$ such that $f \in \overline{F}$ (the bar denotes the closure in \mathbb{R}^X).*

In [17], Jardón went on to prove that every Eberlein compactum (a space X such that $\mathcal{C}(X, \mathbb{R})$ has certain properties) is weakly Eberlein compact, and therefore that weak splittability is a weakening of both splittability and Eberlein compactness. He then proved the following corollary:

Corollary 1.5.2. *A space X is weakly Eberlein compact if and only if it is splittable over the class of Eberlein compact spaces.*

When a space X is cleavable over a class of spaces \mathcal{Y} , we mean that for every $A \subseteq X$, there exists a $Y \in \mathcal{Y}$ and $f \in \mathcal{C}(X, Y)$ such that $f(A) \cap f(X \setminus A) = \emptyset$. Obviously, if X is cleavable over a space Y , then X is also cleavable over any class of spaces containing Y .

The results of Corollary 1.5.2 are strengthened if one is also able to answer the following question positively (taken from [17]):

Question 5. *Does weak Eberlein compactness imply Eberlein compactness?*

Similarly, since Corson and Gul'ko compactness are also non-trivial properties for some space of continuous functions, we have the following open questions:

Question 6. *Let a compact space X be splittable over the class of Corson (respectively: Gul'ko) compact spaces. Must X be a Corson (respectively: Gul'ko) compact space?*

For definitions of these properties, see [14].

Answering any of these open questions in the affirmative would allow mathematicians to use the property of cleavability (or splittability) to gain detailed information about $\mathcal{C}(X, \mathbb{R})$ just by associating X with a class of well-understood spaces.

Chapter 2

General Theorems, Properties, and Methods

A core part of research related to cleavability is answering the following question:

Question 3. *If X is cleavable over a space Y with property P , must X have property P ?*

With respect to this thesis, if our aim is to provide partial answers to Questions 1 and 2, then we would better be able to find an embedding into Y , or a linear order on X , if we knew precisely the properties a space X must have when it is cleavable over a given space Y .

In this chapter, we give the reader a small tutorial on the strategies used to employ cleavability in order to answer this question. We begin by explaining three methods used within this thesis, then employ these methods to provide a list of theorems useful to cleavability.

2.1 Methods

In order to demonstrate how we use cleavability to import information about a well-known topological space Y to a less well-known space X , we take, for example, the following proposition from [5]:

Proposition 2.1.1. *If X is cleavable over a T_2 space, then X is T_2 .*

We will go through this proof slowly for clarity.

Proof. **Step 1:** We first identify two points, $x_1 \neq x_2$ in X .

We would like to use cleavability in order to find disjoint open subsets $U_1 \ni x_1$ and $U_2 \ni x_2$. To do this, we must find the “right” subset of X to cleave along. To cleave along anything in

$X \setminus \{x_1, x_2\}$ is pointless, as we gain no new information about the T_2 -ness of X ; similarly, if we cleave along $\{x_1, x_2\}$, there is the possibility that the function that cleaves along this set, call it f , may be such that $f(x_1) = f(x_2)$. However, if we cleave along one of the points, say x_1 , we might be able to use the fact that Y is T_2 to find disjoint open subsets of X .

Step 2: Let $f : X \rightarrow Y$ cleave along $\{x_1\}$. Then by definition of a cleaving function, $f^{-1}(f(x_1)) = \{x_1\}$. That is, $f(x_1) \neq f(x_2)$.

Step 3: We can now use the fact that Y is T_2 . Let V_1 and V_2 be disjoint open sets in Y containing $f(x_1)$ and $f(x_2)$ respectively.

Step 4: Since f is a function, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint in X . Furthermore, by continuity of f , $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are open. Lastly, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ contain x_1 and x_2 respectively. Therefore, since the choice of x_1 and x_2 was arbitrary, X must be T_2 . \square

It is important to note the general structure of this proof, as it is the structure used for most other cleavability proofs. We began in X , wisely chose a specific set to cleave along, used the properties of Y to gain information about the set we cleaved along, then used the pre-image of the cleaving function to import the information we gained in Y back to X .

An important question to ask, however, is: how does one know which subset of X to cleave along in order to get the desired results?

There are three distinct methods in this thesis used to address that question, and we examine them in the three following sections.

Focusing on a Singleton

There are many topological properties P such that if a space X has property P , then in order to demonstrate P , one must initially consider a single point of X . For example, first-countability is exhibited by taking a single point, and showing it has a countable local base; local connectedness is exhibited by taking a single point and checking to see if it has a local base of connected open sets. When considering cleavability of a space X over a topological space Y with such a property P , it is often useful to consider a point $x \in X$, find a continuous function f that cleaves along $\{x\}$, and consider $f(x)$. Since Y has property P , $f(x)$ must demonstrate this property; we then use f^{-1} to try to show that X must have property P as well.

In order to understand this strategy more clearly, let us go back to the proof of Proposition 2.1.1.

In this proof, we began by taking single points, x_1 and x_2 . We then used a function that cleaved along the singleton $\{x_1\}$. This allowed us to identify $f(x_1) \in Y$, which we knew could not share the image of any other point in X under f . We were therefore able to use the properties of $f(x_1) \in Y$, and import them to X by taking the pre-image of $V_1 \ni f(x_1)$.

Let us now take another example. The following theorem is stated as a proposition in [6].

Theorem 2.1.2. *If X is a compactum cleavable over a first-countable, T_2 space Y , then X is first-countable.*

We again go through the proof slowly for clarity.

Proof. Step 1: Since we want to show that X is first-countable, we must first identify a single point $x \in X$.

Step 2: We wish to use the fact that Y is first-countable to show that $x \in X$ has a countable local base, therefore we take a function $f : X \rightarrow Y$ that cleaves along $\{x\}$. This allows us to examine $f(x)$ as a point in Y without having any other points of X map onto $f(x)$.

Step 3: Now we use the first-countability of Y . Let $\mathcal{B} = \{B_n : n \in \omega\}$ be a local base for $f(x)$ in Y .

Step 4: As with Proposition 2.1.1, now that we have used the property of Y to examine $f(x) \in Y$, we take pre-images of the sets concerned. That is, we take the collection $\{f^{-1}(B_n) : n \in \omega\}$. What we would like is for this collection to be a countable local base of x . It is certainly countable, and by continuity of f all the elements are open, but we must check whether it is a local base.

Step 5: To check this collection is a local base for x , let $V \subseteq X$ be an open set containing x . Notice that $X \setminus V$ is closed. Since X is compact and Y is T_2 , $f(X \setminus V)$ must be closed in Y . Notice that $Y \setminus f(X \setminus V)$ is an open set containing $f(x)$. Therefore for some $m \in \omega$, $B_m \subseteq Y \setminus f(X \setminus V)$. As f is a function, it must therefore be the case that $f^{-1}(B_m) \subseteq V$.

Step 6: Since the choice of V was arbitrary, we have therefore shown that $\{f^{-1}(B_n) : n \in \omega\}$ is a local base of x . As the choice of x was arbitrary as well, we must have that X is first-countable. \square

In addition to demonstrating how we import information from Y to X , this proof also demonstrates the importance of assuming compactness of X . With compactness, since we are usually cleaving over T_2 spaces, our continuous functions will be closed. We thus have greater knowledge of how a given cleaving function works on closed sets, and often on open ones as well.

Collapsing vs. Expanding

One of the many strengths of cleavability as a tool is the ability to isolate subsets of X , and see how their images behave in Y . One of the few weaknesses, however, is that if we are cleaving along $A \subseteq X$ with f , then $f(A)$ might be a singleton; this tells us little about A , and potentially nothing about A in relation to $X \setminus A$. Furthermore, when studying a space X cleavable over a space Y , if we wish to show that X is embeddable into Y , it is useful to find a function f such that M_f is sufficiently small and “well-behaved”; we may then find a function g , based on f such that $g : X \rightarrow Y$ is an embedding. This strategy is used in both [10] and this thesis, specifically in Chapter 5.

How, then, do we prevent the collapse of X into Y ?

Depending on the properties of X and Y , one strategy is by taking a dense, and co-dense (if possible) $A \subset X$; then if f cleaved along A , the collapse of X would force $f(A)$ and $f(X \setminus A)$ to have non-empty intersection, a contradiction. This is the strategy we use in Theorem 2.2.2. As another example, take the following lemma:

Lemma 2.1.3. *If $\omega + 1$ is cleavable over a T_2 space Y , then $\omega + 1$ is embeddable into Y .*

Proof. Let $f : \omega + 1 \rightarrow Y$ cleave along $\{\omega\}$. While the statement of this lemma requires us to show an embedding exists, we will first make a few observations about how cleaving over such a subset of $\omega + 1$ requires f to behave.

1. $f^{-1}(f(\omega)) = \{\omega\}$.

This is obvious, since f cleaves along $\{\omega\}$.

2. $|f^{-1}(f(n))|$ is finite for every $n < \omega$.

To see why is this true, first note that since $\omega + 1$ is compact, and $f^{-1}(f(n))$ is a closed subset, it must be compact as well. Were $f^{-1}(f(n))$ infinite, it would not be closed, as $\omega \notin f^{-1}(f(n))$.

3. $|f(\omega + 1 \setminus \{\omega\})| = \aleph_0$.

This is a consequence of the second property.

Due to the second and third property, we may therefore assume without loss of generality that $f^{-1}(f(n)) = \{n\}$ for every $n \in \omega$. We may do so, because if this were not the case, we would be able to create a continuous function $g : \omega + 1 \rightarrow Y$, based on f , such that $g^{-1}(g(n)) = \{n\}$.

4. f is a closed function.

This follows from the fact that $\omega + 1$ is compact, and Y is T_2 .

Since f is a closed, continuous injection from $\omega + 1$ to Y , we therefore know that f is an embedding. \square

The main observation to make about this proof, is that by cleaving along $\{\omega\}$, we were able to prevent any infinite fibers, and therefore any major collapse of X into Y . We rely on this method heavily in our exploration of cleavability over ordinals.

Using the Space of all Continuous Functions

In the previous two examples, the proofs used were both straightforward: if p , then q . However, many of the proofs used in this thesis rely on arriving at a contradiction, and the methods used in those types of proof require a bit of explanation as well. Specifically, if we want to show that any X cleavable over a space Y must have property P , then it is sometimes useful to assume X has property $\neg P$, and show that X cannot be cleavable over Y . We do so by showing that any space X with property $\neg P$ must contain a subset A along which no continuous function may cleave over Y . We of course provide a lemma below that illustrates this strategy.

Before we state and prove the following lemma, though, it would be useful to give further details about the method described. Let $\mathcal{C}(X, Y)$ denote the collection of all continuous functions from a space X to a space Y ; also, let $M_f = \{x \in X : |f^{-1}(f(x))| > 1\}$. If for every function in $\mathcal{C}(X, Y)$, M_f is non-empty, then what we would like to do is somehow systematically choose $x_1 \neq x_2 \in X$ such that $f(x_1) = f(x_2)$, then put x_1 into A , and x_2 into $X \setminus A$. In such a way, at the end of our construction, we will arrive at the result that no f may cleave A from its complement; this will be obvious, since if such an f did exist, we would have chosen x_1 and x_2 to contradict this.

We demonstrate this method in the following lemma; the statement may be found in [5], but we provide a proof for the reader here. Note that in the following, τ is always an infinite cardinal number, and is also considered an initial ordinal number of this cardinality.

Lemma 2.1.4. *[Vanja's Lemma] Let $\{f_\alpha : \alpha < \tau\}$ be an infinite family of continuous mappings of a space X that cleaves X along all subsets of X , that is, for every $A \subset X$ there exists $\alpha < \tau$ such*

that $f_\alpha(A) \cap f_\alpha(X \setminus A) = \emptyset$. Then $|M_{f_\alpha}| < \tau$ for some $\alpha < \tau$.

Proof. Assume for a contradiction that for every $\alpha < \tau$ we have $|M_{f_\alpha}| \geq \tau$. To prove this lemma we construct a set $A \subset X$ along which no function may cleave. We do so by using transfinite induction on $\alpha < \tau$. Specifically, for each $\alpha < \tau$, we consider M_{f_α} ; from this set, we hope to identify two elements $x_{\lambda,1} \neq x_{\lambda,2}$ such that A will contain $x_{\alpha,1}$, but not $x_{\alpha,2}$. If we are able to identify these elements for every α , then we will arrive at the conclusion that no continuous f may cleave along A .

Base Case: Consider f_0 . Since $|M_{f_0}| \geq \tau$, there exists $x_{0,1} \neq x_{0,2}$ such that $f_0(x_{0,1}) = f_0(x_{0,2})$. Let $A_0 = \{x_{0,1}\}$, and $B_0 = \{x_{0,2}\}$.

Successor and Limit Case, λ : Assume we have created A_α and B_α , where $\alpha < \lambda < \tau$, and consider f_λ . Let $\hat{A}_\lambda = \bigcup_{\alpha < \lambda} A_\alpha$, and $\hat{B}_\lambda = \bigcup_{\alpha < \lambda} B_\alpha$. (Note that if λ is a successor ordinal equal to $\beta + 1$, then $\hat{A}_\lambda = A_\beta$.) By construction, since $\lambda < \tau$, and $|M_{f_\lambda}| \geq \tau$, it must be that $|\hat{A}_\lambda \cup \hat{B}_\lambda| < |M_{f_\lambda}|$. Thus $M_{f_\lambda} \setminus (\hat{A}_\lambda \cup \hat{B}_\lambda)$ must contain $x_{\lambda,1} \neq x_{\lambda,2}$ such that $f_\lambda(x_{\lambda,1}) = f_\lambda(x_{\lambda,2})$. Let $A_\lambda = \hat{A}_\lambda \cup \{x_{\lambda,1}\}$, and let $B_\lambda = \hat{B}_\lambda \cup \{x_{\lambda,2}\}$.

Let $A = \bigcup_{\alpha < \tau} A_\alpha$. I claim no function f_α can cleave X over Y along A . If there were such a function, this would be one of the f_β , for $\beta < \tau$, and by construction there exist $x_{\beta,1} \in A$ and $x_{\beta,2} \in X \setminus A$ such that $f_\beta(x_{\beta,1}) = f_\beta(x_{\beta,2})$; therefore $f_\beta(A) \cap f_\beta(X \setminus A) \neq \emptyset$. Thus no function may cleave X over Y along A .

As we have arrived at a contradiction, it must be the case that for some $\alpha < \tau$, $|M_{f_\alpha}| < \tau$. \square

As mentioned, this method has been used by Arhangel'skiĭ in [5], and Buzyakova in [11].

2.2 General Theorems

Now that we have a general idea of the methods used in exploring cleavability, we give a small list of those theorems that are useful to this thesis. This list is by no means comprehensive or exhaustive. One survey on cleavability that does provide many strong and non-trivial results is [6], in which all of the following theorems and lemmas may be found.

We begin by restating two theorems already proved in the section entitled, "Methods".

Theorem 2.2.1. *If X is cleavable over a T_2 (respectively T_1 ; respectively T_0) space Y , then X is T_2 , (respectively T_1 ; respectively T_0).*

Theorem 2.1.2. *If X is a compactum cleavable over a first-countable, T_2 space Y , then X is first-countable.*

The following demonstrates the method described in “Collapsing vs. Expanding”.

Theorem 2.2.2. *If X is an infinite compactum cleavable over a scattered space Y , then X is scattered.*

Proof. Assume for a contradiction that X contains a dense in itself subset D , and consider \overline{D} . We know \overline{D} is compact T_2 , and perfect, therefore by [16] it is resolvable; that is, there exist non-empty $A, B \subset \overline{D}$ such that $\overline{D} = A \cup B$, $\overline{A} = \overline{B} = \overline{D}$, and $A \cap B = \emptyset$. Then no continuous function can cleave apart A and B . Therefore X cannot contain a dense in itself subset. \square

Proposition 2.2.3. *If X is cleavable over Y , and Y is cleavable over Z , then X is cleavable over Z .*

Proof. Let $A \subseteq X$, and let $f : X \rightarrow Y$ cleave along A . Consider $f(A)$; let $g : Y \rightarrow Z$ cleave along $f(A)$. Then $g \circ f : X \rightarrow Z$ is continuous since f and g are continuous, and $g(f(A)) \cap g(f(X \setminus A)) = \emptyset$. Therefore X cleaves over Z . \square

Lemma 2.2.4. *If X is cleavable over Y , and $A \subseteq X$, then A is cleavable over Y .*

Proof. If $B \subset A$, then let $f : X \rightarrow Y$ cleave along B . Then $f|_A : A \rightarrow Y$ also cleaves along B , as the restriction of a continuous function is continuous (see [13]). \square

Theorem 2.2.5. *If X is cleavable over a totally disconnected space Y , then X is totally disconnected.*

Proof. Let $A \subseteq X$ be a connected component. The continuous image of a connected set must be connected; therefore for every $f : X \rightarrow Y$, $f(A)$ must be a singleton. It is thus impossible to cleave X along any proper subset of A . \square

The following is from [11]; one may find a proof of this in [11] as well.

Proposition 2.2.6. *Let X be an infinite compactum cleavable over a separable LOTS L . Then all of the following are true:*

1. X is separable.
2. X is first-countable.
3. There exists $f \in \mathcal{C}(X, L)$ such that $|M_f| < \mathfrak{c}$.

Now that we have a firm background on which properties P a (compact) X must have when cleavable over a space with property P , we are ready to try to answer Questions 1 and 2 for certain types of LOTSs. We first consider cleavability over a scattered first-countable LOTS.

Chapter 3

Cleavability over Scattered First-countable LOTSs

In this chapter, we answer Questions 1 and 2 for cleavability of an infinite compactum X over a first-countable scattered LOTS Y . That is, we first characterize those infinite compacta that are cleavable over first-countable scattered LOTSs, then detail the conditions under which an embedding exists. Our motivating question throughout the section is whether cleavability over ω_1 , the first uncountable ordinal, implies embeddability into ω_1 .

We break this chapter into two sections. In the first section, we derive basic properties an infinite compactum X must have when it is cleavable over a scattered first-countable LOTS Y . We also prove that cleavability over ω_1 implies embeddability into ω_1 . The most important and interesting results, however, are contained in the second section, which begins with a counter-example (Theorem 3.2.1). Once our counter-example is stated, we then go on to describe the necessary conditions on X for when cleavability over a first-countable LOTS implies the existence of a homeomorphism.

3.1 The General Picture

We commence our exploration by discussing those properties an infinite compactum X must have if it is cleavable over a scattered first-countable LOTS. We then use these properties to prove the main theorems of this section, Theorems 3.1.7 and 3.1.8: that any infinite compactum cleavable over ω_1 must embed into ω_1 , and must also therefore be a LOTS.

To begin, we state two theorems, found in [1] and [13] respectively.

Theorem 3.1.1. *Every first-countable compact T_2 scattered space is metrizable.*

Theorem 3.1.2. *Every uncountable compact metric space includes a closed dense-in-itself subspace.*

Theorem 3.1.3. *Every first-countable compact T_2 scattered space is countable.*

Proof. This follows from Theorems 3.1.1 and 3.1.2. □

Theorem 3.1.4. *Let X be an infinite compactum cleavable over a scattered first-countable LOTS. Then X is countable.*

Proof. This follows from Theorems 2.1.2, 2.2.2, and 3.1.3. □

The following is a result from [26].

Theorem 3.1.5. *If X is a compact countable metric space then X is homeomorphic to a countable ordinal.*

We can now definitively answer Question 2 for the case when Y is a scattered first-countable LOTS, as well as answer our motivating question regarding cleavability over ω_1 .

Theorem 3.1.6. *If X is an infinite compactum cleavable over a first-countable scattered LOTS, then X is homeomorphic to a countable ordinal.*

Proof. This follows from Theorems 2.2.2, 2.1.2, 3.1.3, and 3.1.5. □

The following two theorems therefore follow from Theorem 3.1.6.

Theorem 3.1.7. *If X is an infinite compactum cleavable over a first-countable scattered LOTS, then X must be a LOTS.*

Theorem 3.1.8. *If X is an infinite compactum cleavable over ω_1 , then X must be homeomorphic to a subspace of ω_1 .*

In the next section, we will be able to establish the conditions under which cleavability over a general first-countable scattered LOTS Y implies embeddability.

3.2 Necessary Conditions for Injectivity

We now have an answer to our motivating question: if X is cleavable over ω_1 , it must be homeomorphic to a subspace of ω_1 . Note that ω_1 is just one given scattered first-countable LOTS, and we will soon find that we cannot prove the same for all scattered first-countable X and Y .

As a counter-example, let us examine two countable ordinals, $\omega \cdot 3 + 1$ and $\omega \cdot 2 + 1$.

Theorem 3.2.1. *The infinite compactum $\omega \cdot 3 + 1$ is cleavable over $\omega \cdot 2 + 1$, but is not homeomorphic to a subspace of $\omega \cdot 2 + 1$.*

Proof. To show that $\omega \cdot 3 + 1$ is cleavable over $\omega \cdot 2 + 1$, we must show that for each $A \subset \omega \cdot 3 + 1$ there exists a continuous $f_A : X \rightarrow Y$ that cleaves along A . There are several cases to consider for this set.

If A is finite and contains no limit points, let $m \in A$, and let

$$f_A(\alpha) = \begin{cases} m & \alpha \in A \\ \alpha & \alpha \in \omega \cdot 2 + 1 \setminus A \\ \omega \cdot 2 & \alpha = \omega \cdot 3 \\ \omega + k & \alpha = \omega \cdot 2 + k \in (\omega \cdot 2, \omega \cdot 3) \setminus A \text{ for } k > 0. \end{cases}$$

This function clearly cleaves along A . If A is finite and contains one limit point, λ , one can modify f_A so that f_A still cleaves along A . For example, if $\lambda = \omega \cdot 2$, then f_A could be defined as:

$$f_A(\alpha) = \begin{cases} m & \alpha \in A \\ \alpha & \alpha \in \omega + 1 \setminus A \\ \omega + k & \alpha = \omega \cdot 2 + k \setminus A \text{ for } k \geq 0 \\ \alpha & \alpha \in (\omega, \omega \cdot 2) \setminus A. \end{cases}$$

The same can be done when A contains two limit points. In fact, the previous example of a function serves to cleave along A when A contains ω and $\omega \cdot 3$. Modifications of the above functions can also serve to cleave along A if A is equal to a single limit point, if A is equal to two limit points, or if A is equal to all three limit points.

Now we consider an infinite A . One function, and simple modifications of this function, serve to cleave along all cases of infinite A . Enumerate the elements of A as α_n in the following way: $\alpha_{3(k+1)}$ is the least ordinal in $A \cap \omega$ greater than α_{3k} (where α_0 is the least element of $A \cap \omega$), $\alpha_{3(k+1)+1}$ is the least ordinal in $A \cap (\omega, \omega \cdot 2)$ greater than α_{3k+1} , and $\alpha_{3(k+1)+2}$ is the least ordinal in $A \cap (\omega \cdot 2, \omega \cdot 3)$ greater than α_{3k+2} . Enumerate all of the $\beta_n \in [0, \omega \cdot 3) \setminus A$ similarly. It appears as if we are assuming that $A \cap (\omega \cdot (j-1), \omega \cdot j)$ for $j = 1, 2, 3$ is both infinite and co-infinite, however, this enumeration works for both a co-finite and co-infinite A . For example, assume $\omega \cdot 3 + 1 \setminus A$ is finite; just enumerate the β_n as needed. The same can be done if any of the $A \cap (\omega \cdot (j-1), \omega \cdot j)$

for $j = 1, 2, 3$ is finite or empty. If A also contains one limit point, such as ω , then let

$$f_A(\alpha) = \begin{cases} \alpha & \alpha \in [0, \omega] \\ \omega + 2k + 1 & \alpha = \beta_{3k+1} \text{ or } \beta_{3k+2} \\ \omega + 2(k+1) & \alpha = \alpha_{3k+1} \text{ or } \alpha_{3k+2} \\ \omega \cdot 2 & \alpha = \omega \cdot 2 \text{ or } \omega \cdot 3. \end{cases}$$

In the case of the above function, it also cleaves along A when A contains two limit points, namely $\omega \cdot 2$ and $\omega \cdot 3$. If A contained two different limit points, such as ω and $\omega \cdot 2$, the function could be modified so that $f_A(\omega) = f_A(\omega \cdot 2) = \omega$, etc. If A contained no limit points, either of the functions described above would cleave along A .

Since for every $A \subset \omega \cdot 3 + 1$ there exists a continuous function that cleaves along A , $\omega \cdot 3 + 1$ is cleavable over $\omega \cdot 2 + 1$. However, any continuous injective function must map limit ordinals of $\omega \cdot 3 + 1$ to limit ordinals of $\omega \cdot 2 + 1$. As there are three limit ordinals to consider in $\omega \cdot 3 + 1$ and only two in $\omega \cdot 2 + 1$, no continuous injective function exists from $\omega \cdot 3 + 1$ to $\omega \cdot 2 + 1$. \square

What, then, are the conditions under which cleavability of an infinite compactum X over a scattered first-countable LOTS Y implies the existence of a homeomorphism from X to a subspace of Y ? In order to answer this question, we must first provide a few more details about the properties of X and Y . We begin by introducing some further definitions.

Definition 3.2.2. Let X be a topological space, and let A be a subset of X . The *derived set* of A , written as A' , is the set of all limit points of A .

Definition 3.2.3. For ordinal numbers α , the α -th *Cantor-Bendixson derivative* of a topological space X is defined by transfinite induction as follows:

- $X^0 = X$
- $X^{\alpha+1} = (X^\alpha)'$
- $X^\lambda = \bigcap_{\alpha < \lambda} X^\alpha$ for limit ordinals λ .

The smallest ordinal α such that $X^{\alpha+1} = X^\alpha$ is called the *Cantor-Bendixson rank* of X , written as $\text{CB}(X)$.

Definition 3.2.4. Let X be a scattered topological space. We use the notation $I_\alpha(X)$ to represent $X^\alpha \setminus X^{\alpha+1}$.

We will be depending on derived sets and the Cantor-Bendixson rank of a space for the remainder of the proofs, and so it would be useful to state a few introductory observations. They may also be found in [29].

Observation 3.2.5. *For a scattered space X , the Cantor-Bendixson rank is the least ordinal μ such that X^μ is empty.*

Observation 3.2.6. *If δ is the Cantor-Bendixson rank of X , then X^μ is closed for all $\mu \leq \delta$.*

Observation 3.2.7. *If X is a compact scattered topological space, then the Cantor-Bendixson rank of X must be a successor ordinal. In addition, the rank of a single element x must be a successor ordinal.*

In particular, from Observation 3.2.7 we have:

Definition 3.2.8. *Let X be a scattered topological space, and $x \in X$. We use the notation $\text{rank}(x)$ to mean the least ordinal α such that $x \notin X^\alpha$.*

Definition 3.2.9. *Let X be a scattered topological space. For $x \in X$, we use $\text{CB}^*(x)$ to be the greatest ordinal β such that $x \in X^\beta$.*

From these observations it is easy to verify that the Cantor-Bendixson rank of any countable ordinal is countable, and the rank of ω_1 is ω_1 .

Lastly, we may also make the following observation:

Observation 3.2.10. *If X is a topological space, $x \in X$, and $U \subseteq X$ is an open set containing x , then the rank of x relative to X is equal to the rank of x relative to U .*

Proof. This follows from the fact if $\text{rank}(x) = \beta + 1$, then x is a limit point of X^γ for every $\gamma \leq \beta$. \square

The proof of Theorem 3.1.5 used in [26] relies on the idea of derived sets, and in fact shows the following:

Theorem 3.2.11. *If X is a compact countable metric space such that X^β is the last non-empty derived set of X , and such that X^β contains m -many elements, then X is homeomorphic to the ordinal $(\omega^\beta \cdot m) + 1$.*

Lemma 3.2.12, Definition 3.2.13, and Theorem 3.2.14 are due to Richard Lupton, and significantly improve an analogous result proved for compact X in an earlier draft of this thesis.

Lemma 3.2.12. *Suppose X is scattered and $A \subseteq X$ with $x \in \overline{A} \setminus A$. Then $\text{CB}^*(x)$ must be greater than $\inf_{a \in A} (\text{CB}^*(a))$.*

Proof. Let $\alpha = \inf_{a \in A} \text{CB}^*(a)$. Then $A \subseteq X^\alpha$, and, since X^α is closed, $\overline{A} \subseteq X^\alpha$. In particular, $x \in X^\alpha$. We know x is not isolated in X^α , since x is not an element of A , but every open set about x must have non-empty intersection with A . So $x \in X^{\alpha+1}$, and hence $\text{CB}^*(x) \geq \alpha + 1$. \square

Definition 3.2.13. We say that an ordinal α is an **even ordinal** if α is of the form $\lambda + (2 \cdot n)$, where λ is a (necessarily unique) limit ordinal and n is a natural number. Let us denote the class of even ordinals by \mathbb{EON} .

Theorem 3.2.14. Suppose X and Y are scattered and X cleaves over Y . Then $\text{CB}(X) \leq \text{CB}(Y)$.

Proof. Let $f : X \rightarrow Y$ cleave along A , where $A = \bigcup_{\alpha \in \mathbb{EON}} (X^\alpha \setminus X^{\alpha+1})$. We show by transfinite induction on $\text{CB}^*(x)$, that for each $x \in X$, $\text{CB}^*(x) \leq \text{CB}^*(f(x))$. It follows that $\text{CB}(X) \leq \text{CB}(Y)$.

The base case is clear since $0 \leq \text{CB}^*(f(x))$ for all $x \in X$. Let us now suppose, as an inductive hypothesis, that for x with $\text{CB}^*(x) \leq \alpha$ we have $\text{CB}^*(x) \leq \text{CB}^*(f(x))$. Suppose x satisfies $\text{CB}^*(x) = \alpha + 1$. Clearly $(\alpha + 1 \in \mathbb{EON} \leftrightarrow \alpha \notin \mathbb{EON})$, so $f(x) \notin f(X^\alpha \setminus X^{\alpha+1})$. However, $x \in \overline{X^\alpha \setminus X^{\alpha+1}}$, so by continuity of f , $f(x) \in \overline{f(X^\alpha \setminus X^{\alpha+1})}$. Therefore, by Lemma 3.2.12:

$$\text{CB}^*(f(x)) > \inf_{y \in (X^\alpha \setminus X^{\alpha+1})} \text{CB}^*(f(y)) \geq \alpha$$

where the last inequality is from our inductive hypothesis. Hence $\text{CB}^*(f(x)) \geq \alpha + 1 = \text{CB}^*(x)$.

Finally, suppose λ is a limit ordinal and, as an inductive hypothesis, for any $x \in X$ with $\text{CB}^*(x) < \lambda$ we have $\text{CB}^*(x) \leq \text{CB}^*(f(x))$. Observe that $\lambda \in \mathbb{EON}$. Suppose $x \in X^\lambda \setminus X^{\lambda+1}$, so $\text{CB}^*(x) = \lambda$. Suppose $\alpha < \lambda$. As λ is a limit ordinal, there is an ordinal β with $\beta \notin \mathbb{EON}$, and $\alpha \leq \beta < \lambda$ (one of α or $\alpha + 1$ will work). In particular, $f(x) \notin f(X^\beta \setminus X^{\beta+1})$. Nonetheless, $x \in \overline{X^\beta \setminus X^{\beta+1}}$, hence by continuity of f , $f(x) \in \overline{f(X^\beta \setminus X^{\beta+1})}$. By Lemma 3.2.12 and inductive hypothesis,

$$\text{CB}^*(f(x)) > \inf_{y \in (X^\beta \setminus X^{\beta+1})} \text{CB}^*(f(y)) \geq \beta \geq \alpha.$$

Since $\alpha < \lambda$ was arbitrary, $\text{CB}^*(f(x)) \geq \lambda$, concluding the induction. \square

Theorem 3.2.15. If Y is a scattered first-countable LOTS with Cantor-Bendixson rank γ , and $y \in Y$ is such that the rank of y is a countable ordinal $\mu \leq \gamma$, then there exists a compact subspace of Y containing y such that the rank of y in this compact subspace is μ .

Proof. We will prove this by transfinite induction on the rank of $y \in Y$. As the base case is vacuous, and by Observation 3.2.7 the limit case is impossible, we will only consider the successor case, where $\mu = \alpha + 1$.

Let the rank of y be $\alpha + 1$. Let $\langle y_n \rangle$ be a sequence that converges to y such that $\langle \text{rank}(y_n) \rangle$ converges to α . Denote $\text{rank}(y_n)$ as λ_n . This sequence exists since Y is a LOTS, $\text{rank}(y) = \alpha + 1$, and y is a limit point of Y^β for every $\beta < \alpha$. Without loss of generality, assume this sequence is monotonically increasing. As y is the only limit point of this sequence, we can find disjoint clopen intervals U_n around each y_n that contains no other y_m for $m \neq n$; U_n is a scattered first-countable LOTS, in which the rank of y_n is still λ_n (by Observation 3.2.10). We may therefore apply our inductive hypothesis, and find a compact subspace of U_n , call it D_n , in which the rank of y_n is λ_n . We claim that $D = \bigcup_{n \in \omega} D_n \cup \{y\}$ satisfies our theorem. It is obvious by our construction that the rank of y in C is $\alpha + 1$. Compactness follows since every open set containing y contains all but finitely many of the D_n , and each D_n is compact. \square

We are now ready to define the conditions under which cleavability of an infinite compactum X over a scattered first-countable LOTS Y implies a homeomorphism exists from X to a subspace of Y .

Theorem 3.2.16. *If X is an infinite compactum cleavable over a compact, scattered, first-countable LOTS Y such that Y has only one element, y , in its last non-empty derived set, then X is homeomorphic to a subspace of Y .*

Proof. First observe that by Theorems 2.2.2, 3.1.3, and 2.1.2, both X and Y are countable, and therefore by Theorem 3.1.5, must be homeomorphic to countable ordinals. By Theorem 3.2.14, if Y is homeomorphic to δ , then X must be homeomorphic to an ordinal less than $\delta \cdot \omega$. If X is homeomorphic to an ordinal less than or equal to δ , then the proof is complete. Therefore assume X is homeomorphic to $\delta \cdot m$, where $m \in \omega \setminus \{0, 1\}$. For ease of notation, we will refer to X as if it were $\delta \cdot m$; that is, if we say $x \neq z \in X$ are such that $x < z$, we mean if $h : X \rightarrow \delta \cdot m$ is a homeomorphism, then $h(x) < h(z)$. We will show X cannot be cleavable over Y .

Let x_1, x_2 be elements within the last non-empty derived set of X , and assume without loss of generality that $x_1 < x_2$. Let $A = [0, x_1]$, and let $B = (x_1, x_2]$. As in the proof of Theorem 3.2.14, let C be a subset of A equal to $(\bigcup_{\alpha \in \mathbb{EON}} (A^\alpha \setminus A^{\alpha+1})) \setminus \{x_1\}$; similarly, let D be a subset of B equal to $(\bigcup_{\alpha \in \mathbb{EON}} (B^\alpha \setminus B^{\alpha+1})) \setminus \{x_2\}$. I claim no continuous function can cleave along $E = C \cup D \cup \{x_2\}$.

We know from the proof of Theorem 3.2.14 that if such a cleaving function f existed, then $\text{rank}(x_1) \leq \text{rank}(f(x_1))$. As by assumption there does not exist any element of Y with rank greater than $\text{rank}(x_1)$, we must have that $\text{rank}(x_1) = \text{rank}(f(x_1))$. Therefore x_1 must be mapped to y , the only element of the last non-empty derived set of Y . By the same argument, however, f must map

x_2 to y as well. However, x_1 is an element of E , and x_2 is an element of $X \setminus E$, implying f does not cleave along E , a contradiction. \square

We therefore know that cleavability implies embeddability when Y is ω_1 , and when it is a compact, scattered first-countable LOTS with only one element in its last non-empty derived set.

Theorem 3.2.17. *If X is an infinite compactum cleavable over a scattered first-countable LOTS Y , then X is homeomorphic to a subspace of Y if and only if one of the following two conditions holds:*

1. *the Cantor-Bendixson rank of Y is strictly greater than that of X .*
2. *the Cantor-Bendixson ranks of X and Y are equal, and the number of elements in the last non-empty derived set of X is less than or equal to the number of elements in the last non-empty derived set of Y .*

Proof. As this is an “if and only if” statement, we must prove the theorem true in two directions. The “if” direction, (if X is homeomorphic to a subspace of Y , then one of the two properties holds) is obvious. For the “only if” direction, let us first assume that the first property holds.

Let the Cantor-Bendixson rank of Y be δ , and the Cantor-Bendixson rank of X be $\mu + 1$. From Theorem 3.1.6, we know $\mu + 1$ must be countable, and thus by Theorem 3.2.15 we know there exists a compact subspace of Y with rank $\mu + 2$. Therefore by 3.2.11, X is homeomorphic to $\omega^\mu \cdot n + 1$ for some $n \in \omega$, and Y contains a subspace homeomorphic to $\omega^{\mu+1} + 1$. As $\omega^\mu \cdot n + 1 \subset \omega^{\mu+1} + 1$, X is homeomorphic to a subspace of Y .

Now assume the Cantor-Bendixson rank of X and Y are equal to $\beta + 1$. Since X must be homeomorphic to a countable ordinal, we know $\beta + 1$ must be countable as well. If $|Y^\beta| = m$, then using the process described in the proof of Theorem 3.2.15, we may construct a subset of Y homeomorphic to $(\omega^\beta \cdot m) + 1$. The fact that X is homeomorphic to a subspace of Y follows easily from Theorem 3.2.11. \square

We have therefore described those conditions under which cleavability of an infinite compactum X over an infinite scattered first-countable LOTS Y implies embeddability of X into Y .

3.3 Conclusions and Open Questions

In this chapter, we have shown that if X is an infinite compactum cleavable over an infinite first-countable scattered LOTS, then X must be homeomorphic to a countable ordinal. We have therefore

given affirmative partial answers to Questions 1 and 2.

As mentioned, however, the motivating question to this chapter was cleavability over ω_1 , the first uncountable ordinal. Many of the theorems and lemmas used in this chapter were dependent upon the first-countability of Y , a property of ω_1 and no larger ordinal. The following questions therefore remain:

Question 7. *If X is an infinite compactum cleavable over an uncountable ordinal $\lambda > \omega_1$, must X be embeddable into λ ?*

Question 8. *If X is an infinite compactum cleavable over an uncountable ordinal $\lambda > \omega_1$, must X be a LOTS?*

We consider Question 8 in the next chapter, entitled “Cleavability over ordinals”, and show that if such an X is cleavable over λ , it must be homeomorphic to an ordinal.

Many of the results in this chapter relied on the fact that an infinite, scattered, first-countable compactum must be countable; how crucial was countability of X or Y in determining these results? That is, if we weaken our assumptions on X or Y , may we still arrive at the same, or similar, results? The following questions are therefore also still open:

Question 9. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must X be a LOTS?*

Question 10. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must X be embeddable into Y ?*

Another natural question to look at is how vital first-countability was in obtaining our results. We also therefore have the following questions.

Question 11. *Let X be an infinite compactum cleavable over a first-countable LOTS Y . Must X be a LOTS?*

Question 12. *Under what conditions does cleavability of an infinite compactum X over a first-countable LOTS Y imply that X must be embeddable into Y ?*

We attempt to answer Questions 9 and 10 for a general separable LOTS in the chapter entitled “Cleavability and Scattered Sets of Non-trivial Fibers”. In that chapter, we will be able to provide partial answers, though the general questions are still open.

Chapter 4

Cleavability over Ordinals

In this chapter, we answer the following questions:

Question 8. *If X is an infinite compactum cleavable over an uncountable ordinal $\lambda > \omega_1$, must X be a LOTS?*

Problem 13. *Characterize those infinite compacta that are cleavable over an ordinal.*

In particular, we will be able to show that if X is an infinite compactum cleavable over an ordinal, then X must be homeomorphic to an ordinal.

As mentioned in Section 1.1, there are many benefits to using cleavability in one's research. In addition to those mentioned, such as describing sufficient conditions for the linear orderability of a space X , the results of this chapter provide us with another benefit.

A popular area of research is the characterization of those spaces that are homeomorphic to an ordinal (see [26], [8], and [27] for examples.) The results of this chapter add another characterization of those spaces homeomorphic to an ordinal. The novelty of this characterization is that, whereas the results of [26], [8], and [27] rely on topological properties of a space X , our new characterization relies on finding an appropriate ordinal λ , and an appropriate subset of $\mathcal{C}(X, \lambda)$. This shifts the focus from a topological exercise, to a functional one.

This chapter is written in four sections. In the first section, we provide introductory definitions, observations and lemmas. The most important of these is Theorem 4.1.11; in this theorem, we show that any compact X cleavable over an ordinal such that X “hereditarily has a spine” must be homeomorphic to an ordinal. (A definition for this property is provided in Definition 4.1.5.) In the second and third sections, we show that every compact X cleavable over an ordinal must hereditarily have a spine. In the fourth section, we provide answers to Question 8 and Problem 13. The second

and third sections of this chapter are heavily technical, and the reader should take special notice of statements of lemmas, and notation used. The main result of this chapter, that such a space X is homeomorphic to an ordinal, is stated and proven in Theorem 4.4.4.

4.1 Introductory Proofs

In this section we provide introductory definitions, observations, and lemmas. The most important lemma of this section is Theorem 4.1.11, in which we show that every infinite compactum that “hereditarily has a spine” (see Definition 4.1.5), and is cleavable over an ordinal, must be homeomorphic to an ordinal. This provides the foundation for the rest of the chapter, in which we prove that every infinite compactum cleavable over an ordinal must hereditarily have a spine.

We begin by restating several well-known definitions, observations, and lemmas. It is suggested the reader acquaint himself with Theorem 2.2.2, Definition 3.2.3, and Observations 3.2.5 and 3.2.6. We also remind the reader that every space cleavable over a T_2 space must be T_2 .

Definition 4.1.1. *Let X be a compact scattered space. If $\text{CB}(X) = \beta + 1$, we say X is **simple** if $|X^\beta| = 1$. We say X is **simple with** \hat{x} if \hat{x} is the only element of X^β .*

Definition 4.1.2. *Let X be a compact scattered space. If X^β is finite and contains exactly n -many points, the pair (β, n) is called the **characteristic** of X .*

The following lemma is also easily provable, but may be found in [8]:

Lemma 4.1.3. *Any closed subset of an ordinal is homeomorphic to an ordinal.*

We now state an important definition.

Definition 4.1.4. *Let X and Y be such that X is infinite, compact, scattered, and simple with \hat{x} , and Y is an infinite ordinal. We say X has a **spine** (S, k) , where S is a subset of $X \setminus \{\hat{x}\}$, and $k : X \rightarrow Y$ is a continuous function, if the following properties are satisfied:*

1. k cleaves along $S \cup \{\hat{x}\}$,
2. $k|_{S \cup \{\hat{x}\}}$ is an embedding into Y , and
3. $k(S)$ is club in $[0, k(\hat{x}))$.

For example, if k were injective on X , then $(k^{-1}([0, k(\hat{x}))), k)$ would satisfy this definition.

At this point, it may help for us to aid the reader’s intuition. If X and Y are as described in Definition 4.1.4, what we are essentially trying to do is find a function k from X to Y such that for

some club $C \subseteq [0, k(\hat{x}))$, $|k^{-1}(c)| = 1$ for every $c \in C$. That is, only one element of X maps onto one element of this club.

It may also aid the reader to think of (S, k) as creating what looks like a spine in $k(X)$.

Definition 4.1.5. Let X and Y be infinite spaces such that X is compact, scattered, and simple with \hat{x} , and Y is an infinite ordinal. We say X **hereditarily has a spine** if X has a spine, and every closed, infinite, simple $A \subseteq X \setminus \{\hat{x}\}$ also has a spine.

Definition 4.1.6. Let X and Y be such that X is infinite, scattered, compact, and simple with \hat{x} , and Y is an infinite ordinal. We say (R, j) is a **semi-spine** of X if (R, j) satisfies properties (2) and (3) of Definition 4.1.4.

The following definition and theorem may be found in [8].

Definition 4.1.7. We shall say that a point $x \in X$ **satisfies (D)** in X if x has a neighborhood base consisting of a decreasing (possibly transfinite) sequence $\{U_\alpha\}_{\alpha < \tau}$ of clopen sets with the additional property that $(\bigcap_{\alpha < \beta} U_\alpha) \setminus U_\beta$ contains at most one point for each limit ordinal β with $\beta < \tau$. We say a space X **has property (D)** if every $x \in X$ satisfies (D).

Theorem 4.1.8. Let X be a compact scattered space with characteristic (λ, n) . If X has property (D), it is homeomorphic to $(\omega^\lambda \cdot n) + 1$.

We now prove two lemmas to aid us in the proof of Theorem 4.1.11.

Lemma 4.1.9. Let X be a compact T_2 space, and let Y be an ordinal. For every $x \in X$ and $f \in \mathcal{C}(X, Y)$, we may find an ordinal Z and a function $g \in \mathcal{C}(X, Z)$ such that $g(x)$ is the greatest element of $g(X)$, and such that $M_g = M_f$.

Proof. Let $x \in X$, and let $f \in \mathcal{C}(X, Y)$. If $f(x)$ is not the greatest element of $f(X)$, then let y be the greatest. (A greatest element must exist since $f(X)$ is compact.) It is obvious that $[0, f(x)]$ is equal to some ordinal γ . Since $(f(x), y]$ is closed, by Lemma 4.1.3 it must be homeomorphic to an ordinal as well, say β ; let b be such a homeomorphism. Consider the space $\beta + \gamma$. Then the function g such that $g|_{f^{-1}((f(x), y])} = b$ and such that $g|_{f^{-1}([0, f(x)])} = f|_{[0, f(x)]}$ is clearly continuous, $M_g = M_f$, and by construction $g(x)$ is the greatest element of $\beta + \gamma = Z$. \square

Lemma 4.1.10. Let X be a T_2 compactum, Y an ordinal, and $f \in \mathcal{C}(X, Y)$. Furthermore, let $\{U_\lambda : \lambda \in \Lambda\}$ be a family of disjoint clopen intervals of Y such that $U_\lambda \cap f(X) \neq \emptyset$ for every $\lambda \in \Lambda$. If each $V_\lambda = f^{-1}(U_\lambda)$ is linearly orderable, and Z is the LOTS resulting from replacing each U_λ with V_λ , then there exists a $g \in \mathcal{C}(X, Z)$ such that $M_g \subseteq M_f$, and $M_g \cap V_\lambda = \emptyset$ for every $\lambda \in \Lambda$.

Proof. Let 0_Z be the least element of Z , and let β be the greatest element. We will prove this lemma by defining such a function g . Let $i_\lambda : V_\lambda \subset X \rightarrow V_\lambda \subset Z$ be similar (in the obvious way) to the identity, and let $\hat{f} : X \setminus \bigcup_{\lambda \in \Lambda} V_\lambda \rightarrow Z$ be similar (in the obvious way) to $f|_{X \setminus \bigcup_{\lambda \in \Lambda} V_\lambda}$. Then we define $g : X \rightarrow Z$ to be the following:

$$g(x) = \begin{cases} i_\lambda(x) & x \in V_\lambda, \lambda \in \Lambda \\ \hat{f}(x) & x \in X \setminus \bigcup_{\lambda \in \Lambda} V_\lambda \end{cases}$$

To show g is continuous, it is sufficient to show $g^{-1}([0_Z, a))$ and $g^{-1}((b, \beta])$ are open in X .

First consider $g^{-1}([0_Z, a))$. If $a \in V_\lambda$ for some $\lambda \in \Lambda$, let μ be the least element of this particular V_λ . (This exists since V_λ is a clopen subset of a compact space X .) Then $g^{-1}([0_Z, a)) = f^{-1}([0, \mu)) \cup [\mu, a)$. The left side of the union is open by continuity of f , and the right side is open since V_λ is clopen and linearly orderable; the finite union of open sets is again open. If, however, a is not an element of V_λ for any $\lambda \in \Lambda$, then $g^{-1}([0_Z, a)) = f^{-1}([0, a))$, which is open by the continuity of f . Therefore $g^{-1}([0_Z, a))$ must be open in all cases.

The proof that $g^{-1}((b, \beta])$ is open is nearly identical.

By construction, it is obvious that $M_g \cap V_\lambda = \emptyset$ for every $\lambda \in \Lambda$. It is also obvious by construction that $M_g \subseteq M_f$. □

We are now ready to state and prove the main result of this section.

Theorem 4.1.11. *Let X be a compact space cleavable over an ordinal (and thus scattered by Theorem 2.2.2). If X hereditarily has a spine, then X is homeomorphic to an ordinal.*

Proof. As X is compact, the last non-empty derived set of X must have finitely many elements. We may therefore assume without loss of generality that X is simple with \hat{x} . (Otherwise we may partition X into finitely many clopen subsets, each of which isolates an element of the last non-empty derived set of X , and proceed with our proof on the clopen subsets.) We will complete this proof using transfinite induction on $\text{CB}(X)$. We need only consider the successor case, as by Observation 3.2.6, $\text{CB}(X)$ must be a successor ordinal.

Let $\text{CB}(X) = \alpha + 1$, and assume we have shown that if X hereditarily has a spine, and $\text{CB}(X) < \alpha + 1$, then X is homeomorphic to an ordinal. Let (S, k) be a spine of X . By Lemma 4.1.9 we may assume that $k(\hat{x})$ is the greatest element of $k(X)$. As a consequence of being a spine of X , $S \cup \{\hat{x}\}$ must be closed in X ; since k is continuous, we thus know from Lemma 4.1.3 that $k(S \cup \{\hat{x}\})$ is homeomorphic to an ordinal. Let $\lambda + 1$ be the ordinal to which it is homeomorphic. Let g be a

homeomorphism such that if we enumerated the elements of $k(S \cup \{\hat{x}\})$ as y_β , where $g(y_\beta) = \beta \in \lambda$, then $g(y_\beta) < g(y_\gamma)$ if and only if $\beta < \gamma$ in $k(X)$.

For each $\alpha \in \lambda$, consider the interval $(y_\alpha, y_{\alpha+1}]$. In the case when $\alpha = 0$, we consider the interval $[0, y_1]$. Note that it is possible these intervals contain only one element. Each interval is clopen, therefore $k^{-1}((y_\alpha, y_{\alpha+1}])$ is clopen. Since this set is compact, scattered, by assumption has Cantor-Bendixson rank less than X , and hereditarily has a spine, this subset is homeomorphic to an ordinal $\lambda_{\alpha+1}$ by the inductive hypothesis. Replace $(y_\alpha, y_{\alpha+1}]$ with a copy of $\lambda_{\alpha+1}$. Repeat this process for every clopen interval considered, and call the resulting space Y . That is, $Y = \bigcup_{\alpha \in \lambda} \{\alpha + 1\} \times \lambda_{\alpha+1} \cup \bigcup_{\alpha \in (\lambda+1)^1} \{\alpha\} \times \{y_\alpha\}$. Let the order on Y be lexicographical, and impose the order topology on Y .

By Lemma 4.1.10, we know there exists $g \in \mathcal{C}(X, Y)$ such that $M_g = \emptyset$. (This is true from the definition of a spine.) By Lemma 4.1.3 and Theorem 4.1.8, as Y is T_2 , X is compact, and g is continuous, $g(X)$ is homeomorphic to an ordinal. Therefore since g is closed, continuous, and injective, X must be homeomorphic to an ordinal as well. \square

We now know that every infinite compactum X cleavable over an ordinal such that X hereditarily has a spine must be homeomorphic to an ordinal. We will use this to show that every infinite compactum cleavable over an ordinal is homeomorphic to an ordinal by proving that every infinite compactum cleavable over an ordinal must hereditarily have a spine.

4.2 Finding the Semi-spine of X

In this section, we show that every infinite compactum X cleavable over an ordinal must have a semi-spine. We do so by first finding a set $A \subseteq X$, indexing A using ordinals, then using this index to find $T \subseteq A$; this set T , along with a function f that we will define, will be our semi-spine. The main results of this section are contained in Theorem 4.2.18. We will then show in the next section that T contains a subset that when paired with f is a spine.

We begin with several definitions, observations, and a well-known lemma. We then use these to construct the set we will encode.

Definition 4.2.1. The *cofinality* of a partially ordered set A , $\text{cf}(A)$, is defined as the least of the cardinalities of the cofinal subsets of A .

Observation 4.2.2. If a compact space X is cleavable over an ordinal λ , then for every $x \in X$, if $f \neq g$ both cleave along $\{x\}$ over λ , then $\text{cf}([0, f(x)) \cap f(X)) = \text{cf}([0, g(x)) \cap g(X))$.

Proof. If $[0, f(x)) \cap f(X)$ is bounded, then $\text{cf}([0, f(x)) \cap f(X)) = 1$. This implies $f(x)$ is isolated in $f(X)$; since f cleaves along $\{x\}$, x must be isolated in X . By continuity of g and compactness of X , $g(x)$ also must be isolated in $g(X)$. Thus $\text{cf}([0, f(x)) \cap f(X)) = \text{cf}([0, g(x)) \cap g(X))$.

If, alternatively, $[0, f(x)) \cap f(X)$ is unbounded, then assume for a contradiction that $\text{cf}([0, f(x)) \cap f(X)) < \text{cf}([0, g(x)) \cap g(X))$. Let $\Delta \subset [0, f(x)) \cap f(X)$ be a cofinal subset of cardinality $\text{cf}([0, f(x)) \cap f(X))$. For every $\delta \in \Delta$, let $x_\delta \in f^{-1}(\delta)$. Let $D = \{x_\delta : \delta \in \Delta\}$. (Since Δ must be non-empty, D must be non-empty.) Then $|D| = \text{cf}([0, f(x)) \cap f(X))$, and as f cleaves along $\{x\}$, $x \in \overline{D}$. Since g cleaves along $\{x\}$, $g(D)$ must be cofinal in $[0, g(x)) \cap g(X)$, implying $\text{cf}([0, g(x)) \cap g(X)) \leq \text{cf}([0, f(x)) \cap f(X))$, a contradiction. \square

Definition 4.2.3. Let a compact space X cleave over an ordinal λ , let $x \in X$, and let f cleave along $\{x\}$ over λ . We say the **pseudo-cofinality** of x , $\text{pcf}(x)$, is the least of the cardinalities of the cofinal subsets of $[0, f(x)) \cap f(X)$. That is, $\text{pcf}(x) = \text{cf}([0, f(x)) \cap f(X))$.

Observation 4.2.2 implies this definition is well defined.

Lemma 4.2.4. Let X be a compactum cleavable over an ordinal. Then for every $x \in X'$, $\text{pcf}(x) = \inf \{|A| : A \subseteq X, x \in \overline{A} \setminus A\}$.

Proof. Let $x \in X'$, and let f cleave along $\{x\}$. Let $A \subseteq X$ be such that $x \in \overline{A} \setminus A$. Consider $f(A)$. By continuity of f , $f(A)$ must be cofinal in $[0, f(x))$. Let $B \subseteq f(A)$ be cofinal in $[0, f(x))$, and such that $|B| = \text{pcf}(x)$. For each element β of B , let $x_\beta \in A$ be such that $f(x_\beta) = \beta$, and let $C = \{x_\beta : \beta \in B\}$. Then $|C| = \text{pcf}(x)$, and $x \in \overline{C} \setminus C$ since f cleaves along $\{x\}$. This implies $\text{pcf}(x) \leq |A|$ for every $A \subseteq X$ such that $x \in \overline{A} \setminus A$. By the definition of $\text{pcf}(x)$, it follows that $\text{pcf}(x)$ is not just a lower bound of $\{|A| : A \subseteq X, x \in \overline{A} \setminus A\}$, but is in fact the infimum. \square

Definition 4.2.5. If β is any ordinal, then we use β^* to refer to β with reversed order, and the order topology. For example, the least element of $(\omega+1)^*$ is ω , and the greatest element is 0 . Furthermore, ω^* has no least element.

Lemma 4.2.6. There do not exist ordinals $\alpha, \beta, \gamma \geq \omega$, where $\text{cf}(\alpha)$ or $\text{cf}(\beta)$ is uncountable, such that $X = \alpha + 1 + \beta^*$ is cleavable over γ .

Proof. Let $z = \{\alpha\} = \{\beta^*\}$; that is, z is the point that joins α and β^* . If f cleaves along $\{z\}$, then we can immediately see by Lemma 4.2.4 that $\text{cf}(\alpha)$ must equal $\text{cf}(\beta)$.

Now let $A = \alpha' \cup I_0(1 + \beta^*)$. We claim no function can cleave along A .

To see this, assume for a contradiction that some continuous function g does cleave along A . Note that $g((\alpha+1)')$ and $g((1+\beta^*)')$ must be closed in $g(X)$, since X is compact, γ is Hausdorff, and g is

continuous. Also note that from the way we have chosen A , $g(\alpha') \cap [0, g(z))$ and $g((\beta^*)') \cap [0, g(z))$ must be unbounded in $[0, g(z))$: $g(\alpha')$ must be unbounded since $z \notin A$, and $g((\beta^*)')$ must be unbounded since g is continuous, and $\text{cf}(\beta)$ is uncountable. As $g(\alpha')$ and $g((\beta^*)')$ must be club in $[0, g(z))$, $g(\alpha') \cap g((\beta^*)')$ must be non-empty, a contradiction since g is assumed to cleave them apart. \square

Corollary 4.2.7. *If X is a compactum cleavable over an ordinal, then X cannot contain a subset equal to $\alpha + 1 + \beta^*$, where $\alpha, \beta \geq \omega$, and $\text{cf}(\alpha)$ or $\text{cf}(\beta)$ is uncountable.*

Lemma 4.2.8. *If X is a compactum cleavable over an ordinal λ , then X cannot contain a subspace that is homeomorphic to the one-point compactification of an uncountable discrete space.*

Proof. Let $A \subseteq X$ be homeomorphic to the one-point compactification of an uncountable discrete space, and let $x \in A$ be the one point that compactifies it. Then every open set containing x must contain all but finitely many elements of x . Let $f \in \mathcal{C}(X, \lambda)$ cleave along $\{x\}$. By continuity, $f(A \setminus \{x\})$ must be unbounded in $[0, f(x))$. Since λ is an ordinal, we may find an open set U containing $f(x)$ and not containing countably infinitely many elements of $f(A \setminus \{x\})$. Then by continuity of f , $f^{-1}(U)$ must be open in X , containing x and not countably infinitely many elements of A , a contradiction. \square

Corollary 4.2.9. *If X is an infinite compactum cleavable over an ordinal λ such that X is simple with \hat{x} , and $f : X \rightarrow \lambda$ cleaves along $I_0(X) \cup \{\hat{x}\}$, then for every $x \in I_0(X) \cup \{\hat{x}\}$ we may assume without loss of generality that $f^{-1}(f(x)) = \{x\}$.*

Proof. If $\hat{x} \notin \overline{f^{-1}(f(x)) \setminus \{\hat{x}\}}$, then we may easily modify f so that all elements of $f^{-1}(f(\hat{x})) \setminus \{\hat{x}\}$ are now mapped to 0. If, however, $\hat{x} \in \overline{f^{-1}(f(x)) \setminus \{\hat{x}\}}$, then by continuity of f and Lemma 4.2.4, we know $\text{pcf}(\hat{x}) = \omega$, and $\text{cf}(f(\hat{x})) = \omega$. Then we may just as easily modify f so that all elements of $f^{-1}(f(\hat{x})) \setminus \{\hat{x}\}$ are mapped to a countable, cofinal sequence in $[0, f(\hat{x}))$. Therefore we may assume it is the case that $f^{-1}(f(\hat{x})) = \{\hat{x}\}$.

By continuity of f , $|f^{-1}(x)|$ must be finite for every $x \in I_0(X)$. For every $\delta \in L = \lambda' \cup \{0\}$, consider $U_\delta = (\delta, \delta + \omega]$. (In the case that $\delta = 0$, let $U_0 = [0, \omega]$.) As f is continuous, and cleaves over $I_0(X) \cup \{\hat{x}\}$, $V_\delta = f^{-1}(U_\delta)$ must be homeomorphic to $\omega + 1$ for every $\delta \in L$. Then by Lemma 4.1.10, there exists a LOTS Z and $g \in \mathcal{C}(X, Z)$ such that $M_g \cap (I_0(X) \cup \{\hat{x}\})$ is empty. That is, $g^{-1}(g(x)) = \{x\}$ for every $x \in I_0(X) \cup \{\hat{x}\}$. In fact, since for every $\delta \in L$, $U_\delta = (\delta, \delta \cdot \omega]$, it must be the case that Z is equal to λ . Obviously, g still cleaves along $I_0 \cup \{\hat{x}\}$. \square

Theorem 4.2.10. *Let X be an infinite compactum cleavable over an ordinal such that X is simple with \hat{x} , and such that $\text{pcf}(\hat{x}) \leq \omega$. Then X has a spine.*

Proof. If $\text{pcf}(\hat{x}) < \omega$, then X must be finite, and is thus homeomorphic to a finite ordinal. If $\text{pcf}(\hat{x}) = \omega$, then let f cleave along $I_0(X)$. By a similar proof to Corollary 4.2.9, we may assume without loss of generality that for every element $x \in I_0(X)$, $f^{-1}(f(x)) = \{x\}$. As $\text{pcf}(\hat{x}) = \omega$, we may find a countable sequence $\langle a_n \rangle$ of isolated points that converges to \hat{x} . The pair $(\{a_n : n \in \omega\}, f)$ obviously satisfies all properties of Definition 4.1.4, and is therefore a spine of X . \square

Lemma 4.2.11. *Every singular ordinal β such that $\text{cf}(\beta) = \gamma$ contains a cofinal club homeomorphic to $\text{cf}(\beta)$.*

Proof. By definition there is a strictly increasing cofinal map f of γ into β (see Lemma 10.31 in [18]). Define $g : \gamma \rightarrow \beta$ to be: $g(0) = f(0)$, $g(\alpha + 1) = f(\alpha + 1)$, and for limit ordinals $\lambda < \gamma$, $g(\lambda) = \bigcup_{\alpha < \lambda} f(\alpha)$. One may easily verify that g is continuous, injective, and thus that $g(\gamma)$ is homeomorphic to $\gamma = \text{cf}(\beta)$. \square

We now construct $A \subseteq X$, which we will use to find the spine of X . To give a short explanation of what we will be doing, our first step is to generate a set $A \subseteq X$ that we will index using ordinals; that is, each element will be of the form $x_\lambda : \lambda \in \Lambda$. (We will soon be more specific about what λ and Λ are.) We will then find a function f that cleaves X along A . It will be shown that for some subset of A , when a given element $x_\lambda \in A$ is mapped using f into the ordinal over which X cleaves, and $f(x_\lambda)$ “looks ahead”, it will see that only elements with indices greater than λ are mapped to elements greater than or equal to $f(x_\lambda)$; it will then “look back” and see that only those elements with indices less than λ are mapped to elements less than $f(x_\lambda)$. This will be enough to show that this subset, with f , is a semi-spine of X . From the way we construct A , f , and our given subset of A , it will be easy to verify in Section 4.3 that X in fact has a spine.

We now begin this process by finding $A \subseteq X$. Note that due to Theorem 4.2.10, when creating A we need only consider those infinite compacta such that $\text{pcf}(\hat{x}) > \omega$.

Construction of $A \subseteq X$:

Let X be an infinite compactum cleavable over an ordinal λ , such that X is simple with \hat{x} , and such that $\text{pcf}(\hat{x}) > \omega$. Let f cleave along $I_0(X) \cup \{\hat{x}\}$. By Corollary 4.2.9, we may assume without loss of generality that $f^{-1}(f(y)) = \{y\}$ for every $y \in I_0(X) \cup \{\hat{x}\}$; by Lemma 4.1.9, we may assume that $f(\hat{x})$ is the greatest element of $f(X)$.

As $f(X)$ is a closed subset of an ordinal, by Lemma 4.1.3 we know it is homeomorphic to an ordinal. Therefore, without loss of generality, let $f(X) = \mu + 1 \subseteq \lambda$. If μ is singular, let $M \subset \mu$ be a cofinal club homeomorphic to $\text{cf}(\mu)$; if μ is regular, let $M = \mu$. Let $g : M \rightarrow \text{cf}(\mu)$ be an order preserving homeomorphism. (Obviously if μ is regular, g is the identity.) Note $\text{cf}(\mu) > \omega$, as $\text{pcf}(\hat{x}) > \omega$.

Let A be chosen in the following way: for each $\alpha \in \text{cf}(\mu)$, let $x_\alpha \in f^{-1}(M)$ be such that $g(f(x_\alpha)) = \alpha$. Let $A = \{x_\alpha : \alpha \in \text{cf}(\mu)\}$.

We will use the subscripts of the elements of A , and the continuity of various functions from X to λ , to show there exists a set $T \subseteq A$ such that $T \cup \{\hat{x}\}$ is closed in X , and such that $\{\gamma : x_\gamma \in T\}$ is unbounded in $\text{cf}(\mu)$. By construction, $f|_A$ is injective, therefore $f|_T$ will be injective; this will be all we need to give us that (T, f) is a semi-spine of X . To see how we will use the subscripts of A , however, we must state an observation and a well-known lemma, also known as Fodor's Lemma.

Observation 4.2.12. *For every unbounded $B \subseteq \text{cf}(\mu)$, $\hat{x} \in \overline{\{x_\alpha : \alpha \in B\}}$.*

Proof. For ease of notation, assume μ is regular. (Otherwise we may replace μ with $\text{cf}(\mu)$, etc.) Consider $\overline{\{x_\alpha : \alpha \in B\}}$. Since f is continuous, $f(\overline{\{x_\alpha : \alpha \in B\}})$ is closed in $\mu + 1$. If $\hat{x} = x_\mu$ were not an element of $\overline{\{x_\alpha : \alpha \in B\}}$, this would cause an immediate contradiction. \square

Lemma 4.2.13 (Pressing Down Lemma). *Let $\kappa > \omega$ be regular, S a stationary subset of κ , and $f : S \rightarrow \kappa$ such that $\forall \gamma \in S, f(\gamma) < \gamma$; then for some $\alpha < \kappa$, $f^{-1}(\alpha)$ is stationary.*

Lemma 4.2.14. *Let X, f, μ , and A be as described in the **Construction of A** . If $j : X \rightarrow \lambda$ cleaves along A , then either $j = f$ and X has a spine, or $j \neq f$ and there exists a club subset of $\text{cf}(\mu)$ such that for every element β in this club, and for every $\eta \in \text{cf}(\mu)$, $j(x_\eta) \geq j(x_\beta)$ implies $\eta \geq \beta$.*

Proof. If $j = f$, the function used in our construction of A , then we immediately know that (A, f) is a spine of X . Therefore, assume $j \neq f$. We may assume by Lemma 4.1.9 that $j(\hat{x})$ is the greatest element of $j(X)$. We define a function based on the subscripts of the elements of A to show a club subset of $\text{cf}(\mu)$ exists as described above. As we will soon see, it does not affect the proof whether we assume μ is singular or regular. (If μ is singular, we simply change all occurrences of μ to $\text{cf}(\mu)$.) Therefore, for ease of notation, assume μ is regular.

Let $g : \mu \rightarrow \mu$ be defined to be

$$g(\alpha) = \min \{\eta \in \mu : j(x_\eta) \geq j(x_\alpha)\}.$$

That is, g is the identity for those elements δ such that if $j(x_\delta) \leq j(x_\gamma)$, then $\delta \leq \gamma$. Let $\hat{B} = \{\delta \in \mu : g(\delta) = \delta\}$. For every $\eta \in \mu \setminus \hat{B}$, it is obvious that $g(\eta) < \eta$. If $\mu \setminus \hat{B}$ were stationary in μ , then by the Pressing Down Lemma, for an unbounded number of elements $\eta \in \mu$, x_η would be mapped to a point less than some x_γ ; by Observation 4.2.12, this contradicts either the continuity of j , or the fact that j cleaves along A , or that $j(\hat{x})$ is the greatest element of $j(X)$. Therefore $\mu \setminus \hat{B}$ cannot be stationary in μ , and \hat{B} must contain a club subset of μ . \square

We now use this lemma to construct $T \subseteq A$ such that (T, f) is a semi-spine of X .

Construction of $T \subseteq A$:

Let X, f, A, M , and $\mu + 1$ be as described in **Construction of A** , and let j cleave along A . If $f = j$, then X has a spine. Otherwise, assume $f \neq j$. Let C be the club subset of μ described in Lemma 4.2.14. Note that by construction, C is either homeomorphic to $\text{cf}(\mu)$, or by Lemma 4.2.11, contains a subset homeomorphic to $\text{cf}(\mu)$; therefore assume without loss of generality that C is homeomorphic to $\text{cf}(\mu)$. Furthermore, remember that the elements of A are written as x_α , where $\alpha \in \text{cf}(\mu)$.

Let $T = \{x_{\beta_\gamma} : \beta_\gamma \in \text{cf}(\mu)\}$, where β_γ is defined as:

Base Step: Let β_0 be the least element of C such that $\beta_0 > \eta$ for every $x_\eta \in j^{-1}([0, j(x_0)])$.

Successor Step: Let $\beta_{\alpha+1}$ be the least element of C such that $\beta_{\alpha+1} > \eta$ for every $x_\eta \in j^{-1}([0, j(x_{\beta_\alpha})])$.

Limit Step: Consider step δ , where δ is a limit ordinal. Since the ordinals we have chosen are increasing at every step, and C is club in μ , by construction there is only one element in $\overline{\{\beta_\alpha : \alpha < \delta\}} \setminus \{\beta_\alpha : \alpha < \delta\}$; let β_δ be this element.

We now explore the properties of T .

Lemma 4.2.15. *The set $\{\beta_\alpha : x_{\beta_\alpha} \in T\}$ is club in μ .*

Proof. This is obvious by construction. \square

Lemma 4.2.16. *For every $\gamma \in M$, $\delta \in M'$, and $\beta_\delta \in C$, $\gamma \geq \beta_\delta$ if and only if $j(x_\gamma) \geq j(x_{\beta_\delta})$.*

Proof. This follows from the description of C in Lemma 4.2.14, and the construction of T . \square

We now rely on Lemma 4.2.16 to show that $T \cup \{\hat{x}\}$ is closed. It will then be an obvious consequence that (T, f) is a semi-spine of X .

Lemma 4.2.17. *If X , f , \hat{x} , and T are all as described in **Construction of T** , then $T \cup \{\hat{x}\}$ is closed in X .*

Proof. Let $z \in \overline{T}$; we know if $z = \hat{x}$, then z is already an element of our set. Therefore assume $z \neq \hat{x}$. Take $\{\beta_\alpha : j(x_{\beta_\alpha}) < j(z)\}$. We know this set has to be unbounded in $j(z)$ by continuity of j . Let δ be the least ordinal greater than every α such that β_α is in this set, and consider x_{β_δ} .

By continuity of f and j , and by construction of T , we know $\{\beta_\alpha : j(x_{\beta_\alpha}) < j(z)\}$ must also be unbounded in $j(x_{\beta_\delta})$; therefore $j(z) = j(x_{\beta_\delta})$. (This is a result of how we chose x_{β_δ} in our construction of A , and our choice of β_δ in our construction of T .) Thus $z \in A$, and may be written as x_η . However, by Lemma 4.2.16, η must be greater than or equal to β_δ . Since $z = x_\eta$ is in the closure of $\{\beta_\alpha : j(x_{\beta_\alpha}) < j(z)\}$, η cannot be greater than β_δ ; thus η must equal β_δ , and hence $z \in T$. Therefore T contains all of its limit points. \square

Using these lemmas, we may now show that (T, f) is a semi-spine of X .

Lemma 4.2.18. *(T, f) is a semi-spine of X .*

Proof. By Lemma 4.2.17, $T \cup \{\hat{x}\}$ is closed in X ; it is also obvious that $f|_{T \cup \{\hat{x}\}}$ is injective. Therefore $f|_T$ is an embedding. It is also obviously unbounded in $f(X) \setminus \{f(\hat{x})\}$. \square

The problem we are now faced with is whether or not (T, f) satisfies property (1) of Definition 4.1.4. We consider this in the next section.

4.3 Finding the Spine of X

We now know that (T, f) is a semi-spine of X . To show that X has a spine, we prove that since X is cleavable over λ , then for some club subset β of $f(X \setminus \{\hat{x}\})$, it must be that for every $\alpha \in \beta$, $|f^{-1}(\alpha)| = 1$. This result is found in Lemma 4.3.5. For a less technical and more intuitive explanation of how we will show this, it is suggested the reader reacquaint himself with the paragraph given after the proof of Lemma 4.2.11.

We begin, however, with a few observations.

Lemma 4.3.1. *If X is an infinite compactum cleavable over an ordinal such that X is simple with \hat{x} , then X contains a subset homeomorphic to $\text{pcf}(\hat{x}) + 1$.*

Proof. This follows from Lemma 4.2.18. \square

Lemma 4.3.2. *If X is an infinite compactum cleavable over an ordinal such that X is simple with \hat{x} , and $\text{pcf}(\hat{x}) > \omega$, then X cannot contain two subsets homeomorphic to $\text{pcf}(\hat{x}) + 1$ that intersect only at \hat{x} .*

Proof. This follows from Lemmas 4.3.1 and 4.2.6. □

We rely on this lemma to prove X must have a spine. We do so by trying to construct sets similar to A and T , disjoint from A and T , and showing that it cannot be done; we must therefore have that some subset of T , along with f , must act as a spine of X .

We begin by constructing those sets similar to A and T .

Construction of \hat{A} and $S \subseteq \hat{A}$:

Let X , f , and M be as described in **Construction of A** . Let \hat{A} be chosen in the following way: for each $\alpha \in \text{cf}(\mu)$, let $z_\alpha = x_\alpha$ if $f^{-1}(f(x_\alpha)) \setminus \{x_\alpha\} = \emptyset$, or an element of $f^{-1}(f(x_\alpha)) \setminus \{x_\alpha\}$ if $f^{-1}(f(x_\alpha)) \setminus \{x_\alpha\} \neq \emptyset$.

Let $\hat{A} = \{z_\alpha : \alpha \in \text{cf}(\mu)\}$.

Let $S \subseteq \hat{A}$ be constructed in the same way $T \subseteq A$ was constructed.

We will show $(S \cap T, f)$ is a spine of X , and we begin to do so with a few consequences of Lemma 4.3.2.

Lemma 4.3.3. *The set $S \cap T$ is non-empty.*

Proof. If $S \cap T$ were empty, then following from Lemma 4.3.1, $S \cup \{\hat{x}\}$ and $T \cup \{\hat{x}\}$ would be homeomorphic to $\text{pcf}(\hat{x}) + 1$. Since $S \cup \{\hat{x}\} \cup T$ is cleavable over λ , this would contradict Lemma 4.3.2. □

We can do even better than showing $S \cap T$ is non-empty:

Lemma 4.3.4. *$f(S \cap T)$ is club in $f(X) \setminus \{f(\hat{x})\}$.*

Proof. Unboundedness of $f(S \cap T)$ is proved in the exact same way as in Lemma 4.3.3. The fact that $f(S \cap T)$ is closed in $f(X) \setminus \{f(\hat{x})\}$ follows from Lemma 4.2.17, as $(S \cup \{\hat{x}\}) \cap (T \cup \{\hat{x}\})$ must be closed, and f is continuous. □

Lemma 4.3.5. *For every $\beta \in f(S \cap T)$, $|f^{-1}(f(\beta))| = 1$.*

Proof. This follows from how we have constructed S and T . □

Lemma 4.3.6. *$(S \cap T, f)$ is a spine of X .*

Proof. By construction, $f|_{S \cap T}$ is injective. By Lemmas 4.2.17 and 4.3.5, $(S \cap T, f)$ satisfies all properties of Definition 4.1.4. \square

We are now ready to prove the main theorems of this chapter, which we do in the next section.

4.4 Results

In this section, we answer Question 8 and Problem 13.

Theorem 4.4.1. *Every infinite compactum X cleavable over an ordinal such that X is simple has a spine.*

Proof. This follows from Theorem 4.2.10 and Lemma 4.3.6. \square

Theorem 4.4.2. *Every infinite compactum X cleavable over an ordinal such that X is simple hereditarily has a spine.*

Proof. Every closed, infinite, simple $A \subseteq X$ is also cleavable over an ordinal. \square

Theorem 4.4.3. *If X is an infinite compactum cleavable over an ordinal such that X is simple, then X is homeomorphic to an ordinal.*

Proof. By Lemma 4.4.2, X must hereditarily have a spine. By Lemma 4.1.11, X must therefore be homeomorphic to an ordinal. \square

We may now give a definitive answer to Question 8, as well as Question 2 for cleavability over ordinals:

Theorem 4.4.4. *If X is an infinite compactum cleavable over an ordinal, then X is homeomorphic to an ordinal, and is therefore a LOTS.*

Proof. The last non-empty derived set of X , call it X^β , must have finitely many elements. We may partition X into finitely many clopen sets, such that each clopen set contains one element of X^β . Each clopen set, by Theorem 4.4.3, must be homeomorphic to an ordinal; in fact, following from Theorem 2 in [8], they must all be homeomorphic to the same ordinal. Therefore if λ is the ordinal to which they are all homeomorphic, X will be homeomorphic to $\lambda \cdot n$, where $n = |X^\beta|$. \square

4.5 Conclusions and Open Questions

We have now shown that if X is an infinite compactum cleavable over an ordinal λ , then X is homeomorphic to an ordinal and is therefore a LOTS. If X is countable, then from Chapter 3, we know the specific conditions under which X is embeddable into λ . The following, however, is still an open question:

Question 7. *If X is an infinite compactum cleavable over an uncountable ordinal $\lambda > \omega_1$, must X be embeddable into λ ?*

While we do not have a definitive answer for this question, we do know the following:

Theorem 3.2.14. *Suppose X and Y are scattered and X cleaves over Y . Then $\text{CB}(X) \leq \text{CB}(Y)$.*

Corollary 4.5.1. *If X is an infinite compactum cleavable over an ordinal λ , then X is homeomorphic to an ordinal δ , and δ must be less than $\lambda \cdot \omega + 1$.*

Lemma 4.5.2. *If β is an uncountable regular ordinal, with $\text{cf}(\beta) > \omega$, then $\beta \cdot j + 1$, where $j \in \{2, 3, 4\}$ is not cleavable over $\beta \cdot k + 1$, where $k < j$.*

Proof. We will prove the lemma true in the case where $j = 4$ and $k = 3$. The proof may then be modified in the case where j is either 2 or 3.

Let C be club in $\beta \setminus \{0\}$; similarly, let C_{m+1} be a copy of C in $(\beta \cdot m, \beta \cdot (m + 1))$, where $m + 1 \in \{1, 2, 3, 4\}$. Let $A = C_1 \cup C_3 \cup ((\beta, \beta \cdot 2) \setminus C_2) \cup ((\beta \cdot 3, \beta \cdot 4) \setminus C_4) \cup \{\beta, \beta \cdot 2\}$. That is, A contains the clubs C_1 and C_3 , the complements of C_2 and C_4 in their respective intervals, and the endpoints β and $\beta \cdot 2$. We claim no function can cleave X along A over $\beta \cdot 3 + 1$.

Assume for a contradiction that there does exist such an f . Then from the way we have chosen A , $f(C_{m+1})$ must be unbounded in $f(\beta \cdot (m + 1))$ for every $m + 1 \in \{1, 2, 3, 4\}$. As β is regular, this implies $f(\beta \cdot (m + 1))$ must be mapped to one of the points $\beta \cdot m$, and as A and $X \setminus A$ both contain two of the $\beta \cdot (m + 1)$, we know $f(\beta) = f(\beta \cdot 2)$ or $f(\beta \cdot 3) = f(\beta \cdot 4)$. Without loss of generality, assume it is the first. Furthermore, as f is continuous, X is compact, each C_{m+1} is club in its respective interval, and $\beta \cdot 3 + 1$ is T_2 , $f(C_1)$ and $f(C_2)$ must be club in $f(\beta)$. Therefore, $f(C_1) \cap f(C_2)$ must be non-empty. However, A contains C_1 and $((\beta, \beta \cdot 2) \setminus C_2)$, contradicting the fact that f cleaves along A . Thus no f can cleave along A . \square

The results of this lemma lie in contrast with that of Theorem 3.2.1; in Theorem 3.2.1, we proved that $\omega \cdot 3 + 1$ is cleavable over $\omega \cdot 2 + 1$; clearly, the uncountability of β versus the countability of ω plays an important role when showing whether or not cleavability implies embeddability.

We have developed the instinct, however, that the following is true:

Question 14. *If β is an uncountable ordinal, must it be the case that $\beta \cdot n + 1$, where $n \geq 5$, is cleavable over $\beta \cdot m + 1$, where $m \geq 4$, but is not embeddable unless $n \leq m$?*

Chapter 5

Cleavability and Scattered Sets of Non-trivial Fibers

5.1 Introduction

In this chapter, we explore cleavability over separable LOTSs. We show that if X is an infinite compactum cleavable over a separable LOTS Y such that for some continuous $f : X \rightarrow Y$, M_f is scattered, then X is a LOTS.

There are two motivations for this exploration.

The first motivation is the work previously described regarding cleavability over scattered first-countable LOTSs. Many of the proofs presented in Chapter 3 depended on either the countability of the infinite compactum X , or the fact that such a LOTS Y must contain subsets that are homeomorphic to countable ordinals. If we were to weaken our conditions on our LOTS so that we might approximate countability using separability, would we arrive at similar results? That is, would we obtain positive results to the following questions:

Question 9. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must X be a LOTS?*

Question 10. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must X be embeddable into Y ?*

The other motivation for this chapter is the library of previous results on cleavability. Two of the best-known results on this area of research come from the papers [5] and [11]. In these papers,

Arhangel'skiĭ and Buzyakova respectively proved that if X is an infinite compactum cleavable over \mathbb{R} (resp: \mathbb{D}), then X must be embeddable into \mathbb{R} (resp: \mathbb{D}). We therefore hope to generalize the results obtained in [5] and [11] to those compacta that are cleavable over a general separable space.

This chapter is written in two sections. In “Totally Disconnected X ”, we first consider the case where X is totally disconnected. We then use those results to prove in “True for all X ” that for any compactum X cleavable over a separable LOTS Y such that for some continuous $f : X \rightarrow Y$, M_f is scattered, X is a LOTS.

5.2 Totally Disconnected X

In this section we show that if X is a totally disconnected compactum cleavable over a separable LOTS Y such that for some continuous $f : X \rightarrow Y$, M_f is scattered, then X is a LOTS. We do so by finding a LOTS \hat{Y} into which X is embeddable, and then invoking Lemma 5.2.2. The main results of this section are given by Theorems 5.2.7 and 5.2.10, with the rest of the section containing tools needed for the proofs of these theorems. The most important of these tools are Lemmas 5.2.3 and 5.2.4, and we explain their importance before the statements of the lemmas.

We begin by providing a definition and a lemma from [19].

Definition 5.2.1. *Let X be a compact LOTS, and let $A \subset X$ be closed. We say a non-empty open interval $(a, b) \subset X \setminus A$ is **maximal** if either a and b are both elements of A , or one is an element of A and the other is an endpoint of X .*

Lemma 5.2.2. *If X is a LOTS and H is a compact subset of X , then the relative topology and the order topology agree on H .*

We want to answer Question 9 in the affirmative. If we had an injective and continuous f from X to Y , we would have an immediate answer, as f would be an embedding, and since $f(X)$ would be a compact subspace of a LOTS, X would be a LOTS as well. In a very informal sense, there are two reasons why we may not be able to find an injective map from X to Y . Either the topology on X is too complicated for the elements of X to be linearly ordered, or there isn't “enough room” in Y to continuously and injectively map all of the points of X . What Lemmas 5.2.3 and 5.2.4 ensure is that for any single $y \in f(M_f)$, we may find a LOTS \hat{Y} and a continuous $f : X \rightarrow \hat{Y}$ with enough room to accommodate the points of $f^{-1}(y)$. Since we will be assuming M_f is scattered for some f , we will be able to systematically repeat the method contained in Lemma 5.2.4 to find a \hat{Y} that accommodates all points of M_f .

What Lemma 5.2.3 actually does is strategically partition X so that, when we do find a \hat{Y} with “more room” than Y (Lemma 5.2.4 finds this \hat{Y}), our function from X to \hat{Y} is continuous.

Lemma 5.2.3. *Let X be a totally disconnected, compact T_2 space, and A a countable, closed subset of X such that A is homeomorphic to some countable ordinal λ , and such that for every $x \in A$, x has a countable local base. Let $h : A \rightarrow \lambda$ be a homeomorphism, and let $A = \{x_\beta : \beta < \lambda\}$, where $h(x_\beta) = \beta$. For each $\alpha \in \lambda \setminus \lambda'$, we may then find a clopen set $U_\alpha \ni x_\alpha$ such that the following are satisfied:*

1. For $\alpha_1 \neq \alpha_2$, where $\alpha_1, \alpha_2 \in \lambda \setminus \lambda'$, $U_{\alpha_1} \cap U_{\alpha_2} = \emptyset$.
2. Let $E \subseteq \lambda$ be clopen. Then $\bigcup_{\alpha \in (\lambda \setminus \lambda') \cap E} U_\alpha \cup \{x_\beta : \beta \in \lambda' \cap E\}$ is clopen.
3. $\bigcup_{\alpha \in \lambda \setminus \lambda'} U_\alpha \cup \{x_\beta : \beta \in \lambda'\} = X$.

Proof. We will prove this by transfinite induction on $\text{CB}(A)$. Since the base case is trivial, and the limit case is impossible by Observation 3.2.7, we need only consider the successor case.

Successor Case: Let $\text{CB}(A) = \alpha + 1$. Since A is a closed subset of a compact space X , A must also be compact, and therefore $|A^\alpha|$ must be finite. Without loss of generality, assume A^α contains only one element, and call it x_α . Since A is first-countable, and X is zero-dimensional, we know x_α must have a countable local base of clopen sets, $\{D_n : n \in \omega\}$. We may require that $D_0 = X$, $\bigcap_{n \in \omega} D_n = \{x_\alpha\}$, and since A is homeomorphic to λ under the given homeomorphism h , then for every $n \in \omega$, $h((X \setminus D_n) \cap A)$ is equal to some initial segment of λ . For each $n \in \omega$, let $C_n = (X \setminus D_{n+1}) \setminus (X \setminus D_n)$. Each C_n is clopen, and $\bigcup_{n \in \omega} C_n \cup \{x_\alpha\} = X$.

By assumption, and Observation 3.2.10, we know $C_n \cap A$ must be a closed subset of X homeomorphic to some countable ordinal μ , such that $\text{CB}(\mu) < \text{CB}(\lambda)$. Thus by the inductive hypothesis, we may partition each C_n in such a way that satisfies all of the listed requirements. But does the collective partitioning, the one in which we consider all clopen sets created from partitioning each C_n , satisfy the theorem’s three requirements? It is obvious that this partition satisfies requirements 1 and 3. We must now check property (2) is satisfied.

Since A is compact, let λ , the ordinal to which A is homeomorphic, be equal to $\beta + 1$. Let us also use the notation F_E for the set $\bigcup_{\alpha \in (\lambda \setminus \lambda') \cap E} U_\alpha \cup \{x_\beta : \beta \in \lambda' \cap E\}$, described in the statement of property 2.

Firstly notice that a single clopen interval of an ordinal $\beta + 1$, which is of the form $[a + 1, b]$, $a, b \in X$, is the complement of $[0, a] \cup [b + 1, \beta]$, both of which are clopen as well. (Note that a may

equal 0, and if $b = \beta$, then $[b + 1, \beta]$ will be empty.) Further, if $[c + 1, \beta]$ is a clopen interval, then $[0, c]$ is a clopen interval. Lastly, each clopen subset of $\beta + 1$ is the union of at most finitely many clopen intervals. For these three reasons, to show requirement 2 is satisfied for all clopen $E \subseteq \beta + 1$, it is sufficient to show requirement 2 of the theorem is satisfied whenever we take $E = [0, a]$, for some $a \in \beta + 1$.

Thus, let $E \subseteq \beta + 1$ be equal to $[0, a]$. If $a \in \{0, \beta\}$, then F_E is trivially clopen. Therefore let $a \in C_m \cap A$ for some $m \in \omega$. Then $F_E = \bigcup_{j < m} C_j \cup F_{E \cap A}$. The left part of this union is clopen since each C_j is clopen, and it is a finite union of clopen sets; the right part of the union is clopen by the inductive hypothesis. Therefore requirement 2 is satisfied, and the successor case is proven.

This completes the proof. \square

Lemma 5.2.4. *Let X be a totally disconnected, first-countable, separable, compact T_2 space, let Y be a separable LOTS, and let there exist a continuous function $f : X \rightarrow Y$ such that M_f is countable. Then for every $x \in M_f$, there exists a separable LOTS Y_1 and a continuous function $f_1 : X \rightarrow Y_1$ such that $x \notin M_{f_1}$, and such that $M_{f_1} \subset M_f$.*

Proof. We will be relying on the notation used in Lemma 5.2.3. Let $x \in M_f$, let $y = f(x)$, and let $A = f^{-1}(y)$. By Theorem 3.1.5 A is homeomorphic to an ordinal. Let λ be the ordinal to which A is homeomorphic, where $h : A \rightarrow \lambda$ is a homeomorphism, and enumerate the points of A to be x_α such that $h(x_\alpha) = \alpha$. (It does not matter for the sake of the proof which x_α is equal to our original x .) Partition X as described in Lemma 5.2.3, with $U_\alpha \ni x_\alpha$. Let Y_1 be λ with all of the isolated ordinals α replaced with $f(U_\alpha)$. That is, $Y_1 = \bigcup_{\alpha \in I_0(\lambda)} \{\alpha\} \times U_\alpha \cup \bigcup_{\alpha \in \lambda^1} \{\alpha\} \times \{x_\alpha\}$. Let the order on Y_1 be lexicographical, and impose the order topology on Y . Note that this order preserves the order between the ordinals, and preserves the order already on each $f(U_\alpha) \subset Y$. Let $g_\alpha : U_\alpha \rightarrow f(U_\alpha) \subset Y_1$ be identical to $f|_{U_\alpha}$, and let $h' : A' \rightarrow \lambda' \subset Y_1$ be such that $h'(x_\alpha) = \alpha$. The functions g_α and h' are clearly continuous. We first claim that there exists a continuous function $f_1 : X \rightarrow Y_1$.

Let f_1 be defined as:

$$f_1(x) = \begin{cases} g_\alpha(x) & x \in U_\alpha \\ h'(x) & x \in A'. \end{cases}$$

First note that each $f_1(U_\alpha)$ is clopen in Y_1 . To show f_1 is continuous, let V be an open set in Y_1 , let $z \in V \cap f_1(U_\alpha)$ for some α , and let $x \in f_1^{-1}(z)$. Then we know there exists an open set in $f_1^{-1}(V)$ containing x , namely $f_1^{-1}(V \cap f(U_\alpha))$. (This is true by continuity of f_1 on U_α .) If, however, for some

$y \in V$, $y = h'(x_\delta)$ for some $\delta \in \lambda'$, then by construction we may find an ordinal $\gamma \in \lambda$ such that if $B = \{\alpha \in I_0([\gamma, \delta]) : f(U_\alpha) \subset V\}$, and $C = \{\beta \in [\gamma, \delta]' : f_1(x_\beta) \in V\}$, then $B \cup C \cup \{\delta\}$ is a clopen set of ordinals. This implies, by property 2 of Lemma 5.2.3, that $\bigcup_{\alpha \in B} U_\alpha \cup \{x_\beta : \beta \in C\} \cup \{x_\delta\}$ is a clopen set in X containing x_δ , contained in $f_1^{-1}(V)$. Therefore f_1 is continuous, $M_{f_1} \subset M_f$, and $x \notin M_{f_1}$.

Since f_1 is a continuous function from a separable space X onto a LOTS Y_1 , we know Y_1 must be separable. \square

We now have two powerful lemmas we may use to prove X is a LOTS. The following two lemmas ensure that if M_f is scattered in X for some continuous f , then the points of $f(M_f)$ behave in Y well enough for us to systematically implement the methods contained in Lemmas 5.2.3 and 5.2.4.

Lemma 5.2.5. *Let X be an infinite compact T_2 space, and Y a totally disconnected LOTS. If there exists a continuous $f : X \rightarrow Y$ such that M_f is scattered, then $f(M_f)$ is scattered in Y .*

Proof. Let $f : X \rightarrow Y$ be the continuous function such that M_f is scattered, and consider $f(M_f)$. Assume for a contradiction, and without loss of generality, that $f(M_f)$ is dense in itself. Let $y \in f(M_f)$, and consider $f^{-1}(y)$. This set contains an element x such that for every other $x' \in f^{-1}(y)$, $\text{rank}(x') \leq \text{rank}(x)$; the rank we are referring to here and for the rest of this proof is its rank with respect to M_f . Since $f(X)$ is totally disconnected and zero-dimensional, we know there exists a clopen set V_1 containing y such that $f^{-1}(V_1)$ does not contain any other elements of M_f whose rank is greater than or equal to the rank of x . (Otherwise by continuity there would exist a point $\hat{x} \in f^{-1}(y)$ such that $\text{rank}(\hat{x}) > \text{rank}(x)$.) Let $y_1 \in f(M_f) \cap (V_1 \setminus \{y\})$, and consider $f^{-1}(y_1)$. We again know that there exists an element $x_1 \in f^{-1}(y_1)$ such that for every other $x' \in f^{-1}(y_1)$, $\text{rank}(x') \leq \text{rank}(x_1)$. Since $f(X)$ is totally disconnected and zero-dimensional, we know there exists a clopen set $V_2 \subset V_1$ containing y_1 such that $f^{-1}(V_2)$ does not contain any other elements of M_f whose rank is greater than or equal to the rank of x_1 . Note that from this process we are creating a strictly decreasing sequence of ordinals (namely, the rank of each x_n). Since this sequence must be finite, we know for some step in this process, we will get to the point where $y_j \in M_f \cap (V_j \setminus \{y, y_1, \dots, y_{j-1}\})$ is such that the greatest rank of the elements in $f^{-1}(y_j)$ is 1. That is, the points of $f^{-1}(y_j)$ must be isolated in M_f . By continuity of f , and the fact that X is both limit point and sequentially compact, y_j must be isolated in $f(M_f)$, a contradiction. Therefore $f(M_f)$ must be scattered. \square

Lemma 5.2.6. *Let X be a totally disconnected, first-countable, separable, compact T_2 space, and let Y be a totally disconnected separable LOTS. If there exists a continuous $f : X \rightarrow Y$ such that*

M_f is scattered, then the rank of $f(M_f)$ must be less than ω_1 .

Proof. Assume for a contradiction that $\text{rank}(f(M_f)) = \omega_1$. (If $\text{rank}(f(M_f)) > \omega_1$, just take an appropriate subspace.) Let A be the set of elements $y \in f(X)$ such that $(f(M_f) \cup \{y\})^{\omega_1} = \{y\}$. We know A is non-empty since $f(X)$ is compact; furthermore, A is sequentially closed, and since $f(X)$ is first-countable, A is a closed subset of $f(X)$. Note that $A \subset f(X) \setminus f(M_f)$.

Take $f(X) \setminus A$. This is open, and therefore made up of the union of open intervals. Since $f(X)$ is a compact LOTS, we can assume these open intervals are maximal. Consider one (a, b) such that $(a, b) \cap f(M_f) \neq \emptyset$, and such that a and b are both in A . (If one of $\{a, b\}$ is an endpoint of $f(X)$, we may modify the proof accordingly.) Since this is a non-empty open interval, and $f(X)$ is compact and totally disconnected, there exist gap points c_L and c_R such that $(a, b) = (a, c_L] \cup [c_R, b)$. By construction, $([c_R, b) \cup \{b\})^{\omega_1} = \{b\}$. As $[c_R, b]$ is a totally disconnected compact separable LOTS, we may partition $[c_R, b)$ into countably many disjoint clopen intervals V_m , $m \in \omega$, such that $\bigcup_{m \in \omega} V_m = [c_R, b)$.

Now since $b \in A$, we know that $[c_R, b) \cap f(M_f)$ is uncountable. Thus for at least one $m \in \omega$, $V_m \cap f(M_f)$ is uncountable as well. But by construction, $\text{rank}(V_m \cap f(M_f)) < \omega_1$. Therefore, for some $\beta < \text{rank}(V_m \cap f(M_f))$, $(V_m \cap f(M_f))^\beta \setminus (V_m \cap f(M_f))^{\beta+1}$ is uncountable as well. Around each point $x \in (V_m \cap f(M_f))^\beta \setminus (V_m \cap f(M_f))^{\beta+1}$ we may take an open interval containing no other point of $(V_m \cap f(M_f))^\beta \setminus (V_m \cap f(M_f))^{\beta+1}$, thereby creating uncountably many disjoint open intervals, which contradicts the fact that $f(X)$ is separable.

Thus the rank of $f(M_f)$ must be less than ω_1 . □

We are now ready to prove one of the main theorem of this section. Note that while we are assuming Y is totally disconnected for now, we will be able to drop this assumption and prove the theorem holds for any separable Y .

Theorem 5.2.7. *Let X be a totally disconnected, first-countable, separable, compact T_2 space, and let Y be a totally disconnected separable LOTS. If there exists a continuous $f : X \rightarrow Y$ such that M_f is scattered, then X is a LOTS.*

Proof. We will prove this by transfinite induction on the Cantor-Bendixson rank of $f(M_f)$.

Base Case: If the rank of $f(M_f)$ is 0, then it is empty, and the theorem is true vacuously. It would be useful, however, to exhibit the proof for the case where $\text{rank}(f(M_f)) = 1$. Therefore let $f(M_f)^0$ be the last non-empty derived set of $f(M_f)$. Enumerate the elements of $f(M_f)$ as y_j ,

where $j \in v \subseteq \omega$. Note that since $\text{rank}(f(M_f)) = 1$, v could be finite. Let U_j be a clopen interval containing y_j and no other y_k for $k \neq j$. By Lemmas 5.2.3 and 5.2.4 we know we may find a LOTS \hat{U}_j and a continuous function $f_j : f^{-1}(U_j) \rightarrow \hat{U}_j$ such that $M_{f_j} = \emptyset$. Let $\hat{Y} = Y$ with each U_j replaced with \hat{U}_j , let $\hat{f}_j : f^{-1}(U_j) \rightarrow \hat{U}_j \subseteq \hat{Y}$ be identical to the function f_j , let $\hat{f} : X \setminus \bigcup_{j \in v} f^{-1}(U_j) \rightarrow \hat{Y}$ be identical to f , (note that $X \setminus \bigcup_{j \in v} f^{-1}(U_j)$ may be empty), and let $g : X \rightarrow \hat{Y}$ be defined as the following:

$$g(x) = \begin{cases} \hat{f}(x) & x \in X \setminus \bigcup_{j \in v} f^{-1}(U_j) \\ \hat{f}_j(x) & x \in f^{-1}(U_j). \end{cases}$$

Then g is continuous, M_g is empty, and since g is an injective continuous function from X to \hat{Y} , X is a LOTS.

Successor Case: Let $\text{rank}(f(M_f)) = \alpha + 1$, and assume it is true that if the rank of $f(M_f)$ is less than or equal to α , then X is a LOTS. Enumerate the elements of $f(M_f)^\alpha$ as y_j , where $j \in v \subseteq \omega$. Since $f(X)$ is a totally disconnected LOTS, we may find clopen intervals U_j containing y_j and no other y_k for $k \neq j$ such that $f(M_f) \subseteq \bigcup_{j \in v} U_j$. Now consider a single U_j . This clopen set has only one element of $f(M_f)^\alpha$; label it x_j . Since Y is separable, and U_j is clopen, U_j must be separable as well. Therefore we know there exists a sequence of distinct elements $\langle z_{j,n} \rangle_{n \in \omega}$ in U_j that converges to x_j . Since U_j is also a totally disconnected compact LOTS, we may find clopen intervals $V_{j,m} \subset U_j$ containing $z_{j,m}$ and no other $z_{j,k}$ for $k \neq m$, and such that $\bigcup_{m \in \omega} V_{j,m} = U_j$. The rank of each $V_{j,m} \cap f(M_f)$ is less than $\alpha + 1$, therefore by the inductive hypothesis, we may find a compact LOTS $\hat{V}_{j,m}$ and a continuous function $f_{j,m} : f^{-1}(V_{j,m}) \rightarrow \hat{V}_{j,m}$ such that $M_{f_{j,m}} = \emptyset$. Let Y_1 be Y with each $V_{j,m}$ replaced with $\hat{V}_{j,m}$. By Lemma 4.1.10 we may then find a continuous function g_1 from X to Y_1 such that $M_{g_1} \subseteq f^{-1}(f(M_f)^\alpha)$. We are now left with a situation where $\text{rank}(g_1(M_{g_1})) = 1$. Therefore we know we may find a LOTS Y_2 and a continuous function $g_2 : X \rightarrow Y_2$ such that $M_{g_2} = \emptyset$, making X a LOTS.

Limit Case: Let the Cantor-Bendixson rank of $f(M_f)$ be equal to γ , where γ is a limit ordinal, and assume we have shown that if $\text{rank}(g(M_g)) < \gamma$, then X is a LOTS.

Let $A = \overline{(f(M_f))^\gamma}$. Notice that A is nowhere dense in $f(X)$. Also notice that $A \subseteq f(X) \setminus f(M_f)$, since $\text{rank}(f(M_f)) = \gamma$.

Take $f(X) \setminus A$. This is open, and therefore made up of the union of open intervals. Since $f(X)$ is a compact LOTS, we know we can take these open intervals to be maximal. The set $f(M_f)$ must

be contained within $X \setminus A$, and since $f(X)$ is separable, we know there may only be countably many of these maximal open intervals. Enumerate them as (a_n, b_n) , where $n \in v \subseteq \omega$.

Take a single maximal open interval, (a_m, b_m) . Assume both a_m and b_m are elements of A , and $\overline{(a_m, b_m)}^\gamma = \{a_m, b_m\}$. Since this is a non-empty open interval, and $f(X)$ is compact and totally disconnected, there exist gap points c_L and c_R such that $(a_m, b_m) = (a_m, c_L] \cup [c_R, b_m)$.

Now take $[c_R, b_m)$, and let $\langle y_{m,n} \rangle$ be a sequence contained in $[c_R, b_m) \cap f(M_f)$ that converges to b_m . As $[c_R, b_m)$ is a totally disconnected, compact LOTS, we may partition $[c_R, b_m)$ into clopen intervals $V_{m,n}$, $n \in \omega$, such that $V_{m,n}$ contains $y_{m,n}$, and such that $\bigcup_{n \in \omega} V_{m,n} = [c_R, b_m)$. By construction, each $V_{m,n} \cap f(M_f)$ has Cantor-Bendixson rank less than γ . Therefore by the inductive hypothesis, we know there exists a LOTS $\hat{V}_{m,n}$ and a continuous function $f_{m,n} : f^{-1}(V_{m,n}) \rightarrow \hat{V}_{m,n}$ such that $M_{f_{m,n}} = \emptyset$. Let Y_1 be Y with each $V_{m,n}$ replaced by $\hat{V}_{m,n}$, for every $m \in v$, $n \in \omega$. Then there exists a continuous function $g_1 : X \rightarrow Y_1$, composed piecewise of functions identical to $f_{m,n}$, for all $m \in v$, $n \in \omega$, such that $M_{g_1} = \emptyset$, proving X is a LOTS. \square

We have now proved that X is a LOTS when Y is totally disconnected, but what if Y is not totally disconnected? The following lemma allows us to find a totally disconnected, separable LOTS \hat{Y} that we may use instead of Y in order to complete the proof of the theorem. Note that this lemma also looks very similar to Lemma 5.2.4. While the statements differ only slightly, they are actually exhibiting very different properties. What Lemma 5.2.4 showed is that we may enlarge Y so that f is injective on more points of X ; Lemma 5.2.9 shows that we may find a totally disconnected separable \hat{Y} and a continuous $\hat{f} : X \rightarrow \hat{Y}$ such that we may implement Theorem 5.2.7 to show X is a LOTS. To begin, we must state a new definition, and restate two theorems from Chapter 3.

Definition 5.2.8. *Let X be a topological space, $Y = [a, b]$ a LOTS, and let f be a continuous function from X to Y . We say a point $y \in Y$ can be **separated** if there exist $x_1, x_2 \in X$ such that $f(x_1) = f(x_2) = y$, a space $\hat{Y} = [a, y_L] \cup [y_R, b]$, where $\hat{Y} \setminus \{y_L, y_R\}$ is a subspace of Y , and a continuous function $g : X \rightarrow \hat{Y}$ such that if h embeds $[a, y) \cup (y, b] \subset Y$ into \hat{Y} in the obvious way, then $g(x) = h(f(x))$ when $f(x) \neq y$, $g(x) = y_L$ if $x \in \overline{f^{-1}([a, y))} \setminus f^{-1}([a, y))$, and $g(x) = y_R$ if $x \in \overline{f^{-1}((y, b])} \setminus f^{-1}((y, b])$. For example, if the Double Arrow Space were mapped onto the unit interval $[0, 1]$ in the obvious way, then every point in $(0, 1)$ can be separated.*

Theorem 3.1.3. *Every first-countable compact T_2 scattered space is countable.*

Theorem 3.1.5. *If X is a compact countable metric space then X is homeomorphic to a countable ordinal.*

Lemma 5.2.9. *Let X be a totally disconnected infinite compactum cleavable over a separable LOTS Y . If $f : X \rightarrow Y$ is a continuous function such that M_f is countable, then there exist a totally disconnected, separable LOTS \hat{Y} and a continuous $\hat{f} : X \rightarrow \hat{Y}$ such that $M_{\hat{f}} \subseteq M_f$.*

Proof. Let $X = A \cup B$, where A is perfect, B is scattered, and $A \cap B = \emptyset$. If A is empty, then by Theorems 3.1.3 and 3.1.5, we must have that X is homeomorphic to a countable ordinal. Then our totally disconnected separable LOTS \hat{Y} would be this countable ordinal, and our \hat{f} would be a homeomorphism.

If A is non-empty, then let $D \subset f(X)$ be a connected component; we must have that $f(A) \cap D = D$. To see this, assume without loss of generality that $f(X)$ is connected. We will show that $f(A) = f(X)$. Assume for a contradiction that $f(A) \neq f(X)$. Since A is perfect and f is continuous, $f(A)$ must be closed, and $f(X) \setminus f(A)$ must be non-empty and open. Let (a, b) be a non-empty open interval contained in $f(X) \setminus f(A)$, and let $C \subset (a, b)$ be a closed interval. We must have that C is uncountable since it is perfect, and therefore $f^{-1}(C) \subset B$ is a compact, first-countable, scattered set which is uncountable. This contradicts Theorem 3.1.3.

Therefore, to prove this lemma, it is sufficient to assume X is perfect.

There are three cases to consider: either $f(X)$ is totally disconnected already (and then we have completed the proof), $f(X)$ is connected, or $f(X)$ contains a non-trivial connected component. If it is either of the latter two cases, we will prove this by showing that if D is the set of points of $f(X)$ that can be separated, then D is dense in each of the connected components of $f(X)$.

Without loss of generality, we will assume $f(X) = [c, d]$ is connected. (If it were not connected, we could modify the proof to consider an arbitrary and non-trivial connected component of $f(X)$.) Now take $x_1, x_2 \in M_f$ such that $f(x_1) = f(x_2)$. Let U_1 and U_2 be clopen sets containing x_1 and x_2 respectively such that $U_2 = X \setminus U_1$. Then both $f(U_1)$ and $f(U_2)$ are closed in $f(X)$, and $f(X) \setminus f(U_1)$ is open. Therefore there exists a maximal open interval (a_2, b_2) contained in $f(X) \setminus f(U_1)$.

Assume without loss of generality that $b_2 \in f(U_1) \cap f(U_2)$. (Note that b_2 may equal $f(x_1)$.) Since M_f is countable, and this intersection is closed in $f(X)$, then by Theorem 3.1.5, $f(U_1) \cap f(U_2)$ must be homeomorphic to a countable ordinal. For the sake of this example, we may assume b_2 is isolated in $f(U_1) \cap f(U_2)$, otherwise we may take another element that is isolated in this intersection. As b_2 is isolated in $f(U_1) \cap f(U_2)$, $f(X)$ is connected, and X is dense-in-itself, this implies that there exists a maximal open subset of $f(X) \setminus f(U_2)$, namely (b_2, c) . That is, $(a_2, b_2) \subseteq f(X) \setminus f(U_1)$ is maximal, $b_2 \in f(U_1) \cap f(U_2)$, and $(b_2, c) \subseteq f(X) \setminus f(U_2)$. Therefore b_2 can be separated.

Note that $f(M_f)$ must be dense in $f(X)$, as otherwise we would have a non-trivial connected

subset of X , a contradiction as X is totally disconnected. Now to see that D is dense in $f(X)$, let (a, b) be an open interval in $f(X)$, and let $[a', b'] \subset (a, b)$ be closed. If we take $\hat{X} = f^{-1}([a', b'])$, the topology on \hat{X} to be the subset topology, and $f|_{\hat{X}} : \hat{X} \rightarrow Y$ as our fixed continuous function, we may repeat the previous argument and find a $z \in f|_{\hat{X}}(M_{f|_{\hat{X}}}) \subset (a, b)$ that can be separated. This implies D is dense in $f(X)$.

Since $D \subseteq f(M_f)$, and $f(M_f)$ is countable, we may enumerate the points of D as y_n , where $n \in v \subseteq \omega$. Let Y_1 be the LOTS created after we have separated y_1 into $y_{1,L}$ and $y_{1,R}$, with $f_1 : X \rightarrow Y_1$; let Y_{j+1} be the LOTS created from Y_j after we have separated y_{j+1} , in which $y_{j+1,L} < y_{j+1,R}$, and f_j is the continuous function mapping X into Y_j . Let \hat{Y} be the space created after all points y_j have been separated into $y_{j,L}$ and $y_{j,R}$ for all $j \in v$, and let the order on \hat{Y} be such that $y'_1 \leq y'_2$ if and only if for some $m \in v$, $y'_1 \leq y'_2$ in Y_k for every $k \geq m$. Note that this relation preserves the order between those points of $f(X)$ that were not separated, and orders the points added during separation. This is obviously a linear order. Let the topology on \hat{Y} be the linear order topology. Note that the space is totally disconnected since the points that can be separated are dense in $f(X)$.

We will now show that there exists a continuous function from X to \hat{Y} . Informally, what this function will be doing is mapping fibers of points that have been separated to the appropriate places in \hat{Y} , and mapping the fibers of points that have not been separated to where they would “normally” go.

Now to define \hat{f} formally, let S be those elements $x \in X$ such that $f(x)$ maps onto a point that can be separated, and let the set of points of \hat{Y} that were not created by separating points in Y be written as T . Notice that T can be embedded into Y . Let h be such an embedding. Let $g : X \setminus S \rightarrow \hat{Y}$ be such that $g(x) = h^{-1}(f(x))$. Let $\hat{f} : X \rightarrow \hat{Y}$ be defined as

$$\hat{f}(x) = \begin{cases} g(x) & x \in X \setminus S \\ f_j(x) & x \in f^{-1}(y_j). \end{cases}$$

Let us show this is continuous.

Let $\hat{Y} = [d, e]$, and consider $[d, c) \subset \hat{Y}$. If c was not separated in the construction of \hat{Y} , then $\hat{f}^{-1}([d, c)) = f^{-1}([d, c))$, which we know is open in X by continuity of f . If, on the other hand, c was separated, then let $c = c_L$. Then $\hat{f}^{-1}([d, c_L)) = f^{-1}([d, c))$, which is again open in X . If $c = c_R$, then $\hat{f}^{-1}([d, c)) = f_j^{-1}([d, c_R))$, which we know is open in X by continuity of f_j . By a nearly identical argument, the sets $\hat{f}^{-1}((c, e])$ are open in X as well. Therefore \hat{f} is a continuous

map from X to a totally disconnected LOTS Y . It is obvious that $M_{\hat{f}} \subseteq M_f$. □

Theorem 5.2.10. *Let X be a totally disconnected, infinite compactum cleavable over a separable LOTS Y . If there exists a continuous $f : X \rightarrow Y$ such that M_f is scattered, then X is a LOTS.*

Proof. This follows directly from Theorems 5.2.7 and 5.2.9. □

5.3 True for all X

We have now shown that if X is a totally disconnected infinite compactum cleavable over a separable LOTS Y such that for some continuous $f : X \rightarrow Y$ we have M_f is scattered, then X is a LOTS. In order to explain how we will use this result to prove that X is a LOTS even when it is not totally disconnected, we must first state a theorem from [10]:

Theorem 5.3.1. *If C is a continuum cleavable over a LOTS Y , then C is homeomorphic to a subspace of Y , and is also therefore a LOTS.*

By continuum, we mean a connected, compact topological space.

In this section, we will combine the results from Theorems 5.2.10 and 5.3.1 to show X is a LOTS. That is, let X be an infinite compactum cleavable over a separable LOTS Y . X is formed by a combination of trivial connected components (single points), and non-trivial connected components. We will first show that if we remove the interior of each of the non-trivial connected components, the result is a closed, totally disconnected subspace of X , which by Theorem 5.2.10 is a LOTS under the subspace topology. We will then use this linear order, combined with the linear order on each connected component (given to us by Theorem 5.3.1), to show that the topology derived from the combined linear order on X is equivalent to the original topology on X .

There are two obstacles in our way, however. While each of the connected components of X may be a LOTS under the subspace topology, we must first ensure that the interiors of these connected components do not interact with the rest of the space (see Lemma 5.3.5), and second ensure that each family of connected components behaves as if X were a LOTS (see Lemma 5.3.6). After we have proved these properties true, we show X is a LOTS in Theorem 5.3.7, which is the main result of this chapter.

We begin by citing another result from [10], and one result from [5]. The latter is equivalent to Lemma 2.1.4, but stated differently.

Theorem 5.3.2. *Let $X = [a, b]$ be a linearly ordered continuum. Let f be a continuous mapping of X onto a LOTS such that $f(a) = f(b)$. Let c, d be elements in X whose images are the two endpoints of $f(X)$. Then for any $x \in X \setminus \{c, d\}$ there exists $y \in X \setminus \{x\}$ such that $f(x) = f(y)$.*

Lemma 5.3.3. *Let A and B be disjoint subsets of a set X , and let $\{f_\alpha : \alpha < \tau\}$ be a family of mappings of the set X into sets Y_α , where τ is an infinite cardinal number, and let us also assume that for every $\alpha < \tau$ the cardinality of the set $M_\alpha = \{x \in X \setminus (A \cup B) : f_\alpha(x) \in f_\alpha(X \setminus \{x\})\}$ is not less than τ . Then there exist disjoint subsets U and V of X such that $A \subset U$, $B \subset V$, and $f_\alpha(U) \cap f_\alpha(V) \neq \emptyset$ for every $\alpha < \tau$.*

Lemma 5.3.4. *The space $X = ([0, 1] \times \{\omega\}) \cup (\{\frac{1}{2}\} \times \omega)$, with the subspace topology inherited from the product $[0, 1] \times (\omega + 1)$, is not cleavable over \mathbb{R} .*

Proof. Arhangel'skiĭ proved this to be true in [5], but it would be useful for us to give an illustration as to why. Let \mathbb{S} be the irrationals, and let $A = ((\mathbb{S} \cap [0, \frac{1}{4}]) \cup (\frac{1}{4}, \frac{3}{4}) \cup (\mathbb{Q} \cap (\frac{3}{4}, 1])) \times \{\omega\}$. Then no continuous $f : X \rightarrow \mathbb{R}$ can cleave along A .

The problem of finding a function that cleaves along A , is finding a place in $[0, 1]$ to which $\{\frac{1}{2}\} \times \omega$ may be continuously mapped. If such a function f did exist, then since $\langle \frac{1}{2}, \omega \rangle \in A$ and $\{\frac{1}{2}\} \times \omega \subset X \setminus A$, f would have to map cofinitely many of $\{\frac{1}{2}\} \times \omega$ to points greater than $\langle \frac{1}{2}, \omega \rangle$, or cofinitely many to points less than $\langle \frac{1}{2}, \omega \rangle$. Without loss of generality, assume they are mapped to points greater than $\langle \frac{1}{2}, \omega \rangle$. As f cleaves along A , and is continuous, f would be forced to map $(\frac{1}{2}, \frac{3}{4}) \times \{\omega\}$ onto $(\frac{1}{4}, \frac{1}{2}) \times \{\omega\}$. This would then force $f(\mathbb{S} \cap [0, \frac{1}{4}]) \times \{\omega\}$ and $f((\mathbb{Q} \cap (\frac{3}{4}, 1]) \times \{\omega\})$ to have non-empty intersection, contradicting the fact that f cleaves along A . \square

Lemma 5.3.5. *Let X be an infinite compactum cleavable over a separable LOTS Y . By Theorem 5.3.1, each non-trivial connected component of X is linearly orderable: let $\{C_n = [a_n, b_n] : n \in \omega\}$ be a family of non-trivial connected components of X such that the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ both converge to x for some $x \in X$. If $\langle w_n \rangle$ is a sequence made up of elements such that $w_n \in (a_n, b_n)$ for every $n \in \omega$, then we must have that $\langle w_n \rangle$ converges to x as well.*

Proof. What this lemma is trying to show is that families of connected components of X behave in the same way as if X were a LOTS.

Let $\{C_n = [a_n, b_n] : n \in \omega\}$ be a family of non-trivial connected components of X . Assume for a contradiction that both $\langle a_n \rangle$ and $\langle b_n \rangle$ converge to a single point x , but some sequence $\langle w_n \rangle$, where each w_n belongs to a different connected component C_n , converges to a point $\hat{x} \neq x$. (If $\langle w_n \rangle$ does

not converge, it must at least contain a convergent subsequence, and we may therefore modify the proof accordingly.) We will show that such an X cannot be cleavable over a separable LOTS Y .

Each continuous function from X to Y either cleaves apart x and \hat{x} , or it does not. Of those that do cleave apart x and \hat{x} , if none of them was injective on each C_n , then by Theorem 5.3.2 and Lemma 5.3.3 we could construct a set A along which no continuous $f : X \rightarrow Y$ could cleave in the following way.

If there did exist $f \in \mathcal{C}(X, Y)$ such that f cleaved x apart from \hat{x} and was injective on each C_m , we would be done. Therefore assume there is no such f . Let $A_0 = \{x\}$ and $B_0 = \{\hat{x}\}$. Since X is a compactum cleavable over a separable Y , we know that $|\mathcal{C}(X, Y)| = \aleph_1$. Therefore enumerate the elements of $\mathcal{C}(X, Y)$ as f_α , where $\alpha \in \omega_1 \setminus \{0\}$. Consider f_1 . If $f_1(x) = f_1(\hat{x})$, let $A_1 = A_0 \cup \emptyset$. Otherwise, by assumption f_1 is not injective on some C_m ; by Theorem 5.3.1 there exists uncountably many elements in M_{f_1} . Let x_1 and y_1 be such that $f_1(x_1) = f_1(y_1)$; let $A_1 = A_0 \cup \{x_1\}$ and $B_1 = B_0 \cup \{y_1\}$. We continue to find A_α and B_α in the same way to that shown in Lemma 5.3.3. Let $A = \bigcup_{\alpha \in \omega_1} A_\alpha$ and $B = \bigcup_{\alpha \in \omega_1} B_\alpha$ then by construction $A \cap B = \emptyset$, and no $f \in \mathcal{C}(X, Y)$ would be able to cleave A apart from B .

Therefore there must exist a continuous f that cleaves apart x and \hat{x} , and such that every $f|_{C_m}$ is injective.

Consider $f(x)$. Since f cleaves apart x and \hat{x} , we know $f(x) \neq f(\hat{x})$, and since Y is Hausdorff, let V_1 and V_2 be disjoint open intervals containing $f(x)$ and $f(\hat{x})$ respectively. By continuity of f , and by assumption, V_1 must also contain all but finitely many of the $f(a_n)$ and $f(b_n)$. Let j be the least element of ω such that $f(a_m)$ and $f(b_m)$ are contained in V_1 for every $m > j$. As f is injective on each C_n , V_1 is an open interval, and Y is a LOTS, this implies that if $f(a_m)$ and $f(b_m)$ are elements of V_1 , then $f(C_m) \subset V_1$ as well. Therefore $f(C_m) \subset V_1$ for every $m > j$. But V_2 must contain all but finitely many of the $f(w_m)$. Thus for every open $V_1 \ni f(x)$ and open $V_2 \ni f(\hat{x})$, $V_1 \cap V_2 \neq \emptyset$. This contradicts the fact that Y is Hausdorff. Therefore the sequence $\langle w_n \rangle$ must converge to x . \square

Lemma 5.3.6. *Let X be an infinite compactum cleavable over a separable LOTS Y . If $C \subseteq X$ is a non-trivial connected component, and $x \in C$ is such that $x \in \overline{X \setminus C}$, then x must be an endpoint of C . By endpoint, we mean that since C is a LOTS (Theorem 5.3.1), $C = [a, b]$, and $x \in \{a, b\}$.*

Proof. Were x not an endpoint of C , this situation would be very similar to the one described in Lemma 5.3.4, and we know such a C and x would imply that X is actually not cleavable over Y . Though we were assuming Y to be \mathbb{R} in Lemma 5.3.4 and we are not assuming that in this theorem, we may find a subset of Y along which no continuous function can cleave for the same reason as to

why the set A in Lemma 5.3.4 could not be cleaved from its complement. Therefore x must be an endpoint of C . \square

Up to this point, we have shown that X behaves like a LOTS on certain subsets. All that is left to do is to use these lemmas to prove that the topology on X is equivalent to a LOTS. The following theorem is the main result of this chapter.

Theorem 5.3.7. *Let X be an infinite compactum cleavable over a separable LOTS Y such that for some $f : X \rightarrow Y$, M_f is scattered. Then X is a LOTS.*

Proof. If X is either totally disconnected, or connected, then we know by Theorems 5.3.1 and 5.2.10 that X is a LOTS. Therefore assume X is neither totally disconnected, nor connected. Define $D \subseteq X$ to be the set of those elements of X that are in the interior of non-trivial connected components, and let $C = X \setminus D$. We first claim that C is a closed, totally disconnected subset of X .

By Lemma 5.3.6, we know that elements of connected components are either in the interior of the component, or an endpoint of the component. Thus if C did contain any connected components, it would contradict how C was defined. Therefore C must be totally disconnected.

To prove it is closed, it is sufficient to show that C is sequentially closed, as X is first-countable and therefore sequential. Let $\langle x_n \rangle$ be a sequence contained in C . If it does not converge, nor contain a convergent subsequence, then $\{x_n : n \in \omega\}$ is obviously sequentially closed in C . Now assume it does converge, without loss of generality, to a point x . If x belongs to a non-trivial connected component, then by Lemma 5.3.6 x must be an endpoint of the connected component, and therefore a member of C . If x does not belong to a non-trivial connected component, then it belongs to C by assumption. Therefore C is sequentially closed, and closed in X . To prove it is closed, it is sufficient to show that C is sequentially closed, as X is first-countable and therefore sequential. Let $\langle x_n \rangle$ be a sequence contained in C . If it does not converge, nor contain a convergent subsequence, then $\{x_n : n \in \omega\}$ is obviously sequentially closed in C . Now assume it does converge, without loss of generality, to a point x . If x belongs to a non-trivial connected component, then by Lemma 5.3.6 x must be an endpoint of the connected component, and therefore a member of C . If x does not belong to a non-trivial connected component, then it belongs to C by assumption. Therefore C is sequentially closed, and closed in X .

By Theorem 5.2.10, C is a LOTS under the subspace topology of X . We also know each connected component of X has a topology equivalent to the linear order topology. Thus there is a linear order on X that matches the linear orders on both C and D . We claim the topology derived from this linear order on X , τ_O , is equal to the original topology of X , τ_X . To do this, we will show that

the identity i from (X, τ_X) onto (X, τ_O) is continuous; since X is compact and T_2 , i will be a homeomorphism, implying that $\tau_X = \tau_O$.

To show i is continuous, it is sufficient to show that $[-\infty, b)$ and $(a, \infty]$ are both open in τ_X , where b , without loss of generality, is the left endpoint of a non-trivial connected component. (If it is not an endpoint of some connected component, we may modify the proof accordingly.) We will show that $[b, \infty]$ is closed in τ_X , and thus $X \setminus [b, \infty] = [-\infty, b)$ is open. To show $[b, \infty]$ is closed in X , since X is first-countable, it is enough to show it is sequentially closed.

Let $\langle x_n \rangle$ be a sequence contained in $[b, \infty]$ that converges to a point x . (X is first-countable and compact thus sequentially compact.) If $x \in D$, then all but finitely many points of the sequence must be contained within the same connected component to which x belongs. Thus if $\{x_n : n \in \omega\} \subseteq [b, \infty]$, then x must be contained within $[b, \infty]$ as well. If, however, $x \in C$, then assume without loss of generality it is the left endpoint of a non-trivial connected component. (If it is not, we may easily modify the proof.)

If $\langle x_n \rangle$ is a monotonically decreasing sequence, we would have the same case as when $x \in D$, so x must be contained within $[b, \infty]$ as well. Therefore without loss of generality assume the sequence is monotonically increasing. If all but finitely many of these elements are contained within C , then by Theorem 5.2.10 we know if $\{x_n : n \in \omega\} \subseteq [b, \infty]$, then x must be contained within $[b, \infty]$. Assume, however, that infinitely many of these elements are members of D ; that is, they belong to non-trivial connected components. We know only finitely many terms of this sequence may belong to the same connected component (otherwise there exists a subsequence that converges to a point other than x), thus without loss of generality, assume each member of D in this sequence belongs to a different connected component. From the way we have defined the order on X , we know the right endpoint y_n of the connected component to which x_n belongs will be contained within $[b, \infty]$. We know by Theorem 5.3.5 that this new sequence, $\langle y_n \rangle$, converges to x as well. We are now left with the same situation we have just considered, in which there exists a monotonically increasing sequence of elements of C converging to x ; therefore we know if $\{x_n : n \in \omega\} \subseteq [b, \infty]$, then x must be contained within $[b, \infty]$. Since we have considered all types of sequences contained in $[b, \infty]$, and have proven this set is sequentially closed, it must therefore be closed in X . This implies $[-\infty, b)$ is open in X . By a similar argument, we may show $(a, \infty]$ is open in X as well. Thus i is continuous, $\tau_O = \tau_X$, and therefore X is a LOTS. \square

5.4 Conclusion and Open Questions

We have now provided a partial answer to Question 9, showing that if X is a compact space cleavable over a separable LOTS Y such that there exists a continuous $f : X \rightarrow Y$ with M_f scattered, then X is a LOTS. As mentioned, proving that X is a LOTS under such conditions has several positive consequences. It firstly gives us an alternative method by which to say a space is linearly orderable - a popular topic of concern. Secondly, as a great amount of research has been done on first-countable LOTSs, proving that X is a LOTS under various conditions provides a vast amount of information about our topological space.

However, the following question is still unanswered:

Question 15. *If X is an infinite compactum cleavable over a separable LOTS Y such that for some $f \in \mathcal{C}(X, Y)$, M_f is scattered, must X be embeddable into Y ?*

As briefly mentioned in Section 1.4, results on cleavability seem to indicate that even if cleavability of an infinite compactum X over a LOTS Y does not guarantee embeddability into Y , it may at least guarantee local embeddability into Y . That is, for every $x \in X$, we may find an open set $U \ni x$ such that U is embeddable into Y . We therefore have the following open question:

Question 16. *If X is an infinite compactum cleavable over a LOTS Y , must X be locally embeddable into Y ?*

One place to begin answering Questions 15 and 16 is by considering the following spaces, X and Y . Before we define these spaces though, we must state a theorem from [12]:

Theorem 5.4.1. *There exists a subset E of \mathbb{R} such that $|E| = \mathfrak{c}$, E is dense in \mathbb{R} , and there does not exist a continuous order-preserving injection from E to a proper subset of itself.*

We now define X and Y .

Assume without loss of generality that $\frac{1}{2}$ is not contained in E . Let $X = (E \cup \{\frac{1}{2}\}) \times \{0, 1\} \cup \mathbb{R} \setminus (E \cup \{\frac{1}{2}\}) \times \{0\}$, with the lexicographical order. Let $Y = E \times \{0, 1\} \cup \mathbb{R} \setminus E \times \{0\}$. Notice that Y and X are identical, with the exception of how the point $\frac{1}{2}$ is considered.

Now consider the function $f : X \rightarrow Y$ such that $f(\langle x_1, x_2 \rangle) = \langle x_1, x_2 \rangle \in Y$ when $x_1 \in \mathbb{R} \setminus \{\frac{1}{2}\}$, and such that $f(\langle \frac{1}{2}, 0 \rangle) = f(\langle \frac{1}{2}, 1 \rangle) = \langle \frac{1}{2}, 0 \rangle$. This is obviously continuous. It is also the case that $M_f = \{\langle \frac{1}{2}, 0 \rangle, \langle \frac{1}{2}, 1 \rangle\}$. Thus M_f is obviously finite. However, it is unknown whether the existence of this f implies X is cleavable over Y . If X were cleavable over Y , then we would have a negative answer to Question 15, as X is clearly not embeddable into Y due to the rigidity and density of E in \mathbb{R} .

In addition to the fact that Questions 15 and 16 are still open, our strategy within this chapter relied on the hypothesis that there exists a scattered M_f . Therefore the following question also remains open:

Question 17. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must there exist a continuous $f : X \rightarrow Y$ such that M_f is countable and scattered in X ?*

It is this author's intuition that the space X described above is not actually cleavable over Y , and that there is a positive answer to Question 16. We also feel that the answer to Question 17 must be yes; it may be that if $\mathcal{C}(X, Y)$ contained only functions with dense-in-itself M_f , then one would be able to construct a set $A \subset X$ from the various M_f along which no function can cleave.

Chapter 6

Beyond Cleavability

In [30], Tkachuk states, “Another sign of the importance of a topological notion is the number of equivalent definitions for it. One can easily give ten or more equivalences for compactness but if a concept is of little use then it is difficult to find two.” Similarly, we believe that a sign of importance for a topological notion is if there are many open questions that result from exploration of the topic.

This chapter is therefore devoted to giving a brief introduction to an area of research inspired by cleavability, almost-injectivity.

6.1 Almost-Injectivity

6.1.1 Introduction

In “Cleavability and scattered sets of non-trivial fibers”, the method we used to show that our compactum X is a LOTS relied on the existence of a particular $f \in \mathcal{C}(X, Y)$; in particular, a function f such that M_f is scattered. As one can see from the following corollary, M_f must be countable as well:

Corollary 6.1.1. *Let X be an infinite compactum cleavable over a separable LOTS Y . If there exists a continuous $f : X \rightarrow Y$ such that M_f is scattered, then $|M_f| \leq \aleph_0$.*

Proof. By Proposition 2.2.6 and Theorem 5.3.7, we know X must be a separable LOTS. Now assume for a contradiction that $|M_f| > \aleph_0$. By separability of X , each $|M_f^\beta| = \aleph_0$, for every $\beta < \text{rank}(M_f)$, where $\beta \geq 1$. This is because if $|M_f^\beta|$ were greater than \aleph_0 , we would be able to isolate each element of M_f^β with disjoint open intervals, contradicting separability.

We may also show that $\text{rank}(M_f) < \omega_1$. To see this, assume for a contradiction and without loss of generality that it is equal to ω_1 . (If it is greater, we may modify the proof by considering an appropriate subspace.) Following the proof of Lemma 5.2.6, let $A = \{x \in X : (M_f \cup \{x\})^{\omega_1} = \{x\}\}$. A is non-empty and closed in X , and $X \setminus A$ is made up of the union of maximal open intervals. Consider one interval, (a, b) such that $(a, b) \cap M_f \neq \emptyset$. Let $c \in (a, b)$, and consider $[c, b)$. Let $\{V_n : n \in \omega\}$ be a family of closed intervals such that $\bigcup_{n \in \omega} V_n = [c, b)$, and such that for any $j \neq k$, V_j and V_k either have empty intersection, or only intersect at an endpoint. That is, if $V_j = [a_j, b_j]$ and $V_k = [a_k, b_k]$, then $V_j \cap V_k \neq \emptyset$ if and only if $a_j = b_k$ or $a_k = b_j$.

Since $b \in A$, it must be that $[[c, b) \cap M_f]$ is uncountable. Thus there exists $m \in \omega$ such that $|V_m \cap M_f|$ is uncountable. But by construction $\text{rank}(V_m \cap M_f) < \omega_1$. Thus for some $\beta < \text{rank}(V_m \cap M_f)$, $(V_m \cap M_f)^\beta \setminus (V_m \cap M_f)^{\beta+1}$ is uncountable. Since V_m is a closed interval, we may find disjoint open intervals isolating each element of $(V_m \cap M_f)^\beta \setminus (V_m \cap M_f)^{\beta+1}$, contradicting separability of X . Therefore $\text{rank}(M_f) < \omega_1$.

Thus, since the countable union of countable sets is countable, $|M_f| \leq \aleph_0$. □

Not only did we rely on the existence of a function f such that M_f is countable in order to help prove our main theorem of Chapter 5, but Buzyakova relied on such functions to prove her main theorem in [11]. As a consequence, we believe the following is a valuable question to explore:

Question 18. *If X is an infinite T_2 compactum, Y a LOTS, and there exists $f \in \mathcal{C}(X, Y)$ such that $|M_f| \leq \aleph_0$, then is X a LOTS?*

Additionally, we give this functional property a name.

Definition 6.1.2. *Let X and Y be spaces, and let $f : X \rightarrow Y$ be continuous. We say f is **almost-injective** if $|M_f| \leq \aleph_0$. We also say f is an **almost-injection**.*

Many of the characterizations for linear orderability of a topological space rely on descriptions of subbasic open sets (see [31], [28], and [8]). The novelty of almost-injectivity lies in the ability to characterize linear orderability of a space based on the existence of a LOTS Y , and a single function from X to Y .

In this chapter, we provide an answer to Question 18 for the case when Y is an ordinal. Many of the lemmas and theorems are in the style of those related to cleavability. Namely, we use the following question to begin our exploration:

Question 19. *If X is an infinite T_2 compactum, Y a LOTS with property P , and there exists $f \in \mathcal{C}(X, Y)$ such that $|M_f| \leq \aleph_0$, must X have property P ?*

We divide this chapter into two further sections. In the first, we discuss how we may use almost-injectivity to explore Question 19, and the results obtained. In the second, we show that if X is an infinite compactum, Y is an ordinal, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is a LOTS.

6.1.2 General Results

In this section, we provide a number of lemmas that answer Question 19 for various properties P . Before we begin, however, it should be noted that in all of these lemmas and theorems, we assume X is T_2 . This is because of the following example:

Example 6.1.3. *Let $X = \{x_1, x_2\}$ with the trivial topology, and let $Y = \{y_1, y_2\}$ with the discrete topology. Then there exists an almost-injective function from X to Y (in particular, the constant function onto y_1) but even though Y is T_2 , X is not.*

Furthermore, if we do not assume that X is T_2 , then there are infinitely many counter-examples to Question 18: any countable non- T_2 space, and any LOTS Y . We wish to avoid these trivialities, and therefore assume X is T_2 .

Notice that we also consider any constant function from a countable space X to a LOTS Y to be an almost-injective function. This, however, still provides a positive answer to Question 18:

Theorem 6.1.4. *If X is countable, compact, and T_2 , Y is a topological space, and $f : X \rightarrow Y$ is a constant map, then f is almost-injective, X is homeomorphic to a countable ordinal, and therefore X is a LOTS.*

Proof. Almost-injectivity follows by definition. To show X is a LOTS, we know if there exists a constant function from X to Y , then X must be countable as $X = M_f$. Then following from Theorems 3.1.3 and 3.2.11, X must be homeomorphic to a countable ordinal, and is therefore a LOTS. □

Now that we have stated our assumptions, we provide partial answers to Question 19.

Theorem 6.1.5. *If X is compact T_2 , Y is a first-countable T_2 space, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is first-countable.*

Proof. As X is compact, f is continuous, and Y is T_2 , we know $f(X)$ is compact and first-countable in Y .

Let $x \in X$, and consider $f(x)$. As Y is first-countable and T_2 , X is compact, and f is closed, we know there exists a countable nested local base of $f(x) \in Y$, $\mathcal{B} = \{V_m : m \in \omega\}$, such that $V_{m+1} \subseteq \overline{V_m} \subseteq V_m$, and such that $\bigcap_{n \in \omega} V_n = \{f(x)\}$. This nested local base may be finite, but for ease of notation, we assume it is infinite. Let $U_n = f^{-1}(V_n)$.

If $f^{-1}(f(x)) = \{x\}$, then $\{U_n : n \in \omega\}$ is a countable local base for x . To see this, let $W \ni x$ be open. Then $X \setminus W$ is closed, and $f(X \setminus W)$ is closed in Y . This follows since f must be a closed map. Notice that $Y \setminus f(X \setminus W)$ is an open set that must contain $f(x)$. Therefore for some $m \in \omega$, $V_m \subseteq Y \setminus f(X \setminus W)$, and since f is a continuous function, $f^{-1}(V_m) = U_m \subseteq W$.

Now consider when $\{x\} \subsetneq f^{-1}(f(x))$. If there exists an open set W containing x and no other element of $f^{-1}(f(x))$, then consider $\{U_n \cap W : n \in \omega\}$. We may show this is a local base for x by the same argument as before.

If, however, there does not exist such a W , then enumerate the points of $f^{-1}(f(x))$ as x_m , where $m \in \omega$. We know we may do this since M_f is countable. Since X is compact T_2 , we may choose a nested family of open sets W_m such that $x_j \notin W_m$ for $j \leq m$, and such that $W_{m+1} \subseteq \overline{W_m} \subseteq W_m$. Then consider $\{W_m \cap U_m : m \in \omega\}$. We assert this is a countable local base for x .

It is clearly countable. To show it is a local base, assume for a contradiction it is not. Let V be any open set containing x such that no $W_m \cap U_m \subseteq V$. Note that $\bigcap_{m \in \omega} (\overline{U_m} \cap \overline{W_m}) = \bigcap_{m \in \omega} U_m \cap W_m = \{x\}$. To see this, assume there exists some $z \neq x \in \bigcap_{m \in \omega} U_m \cap W_m = \{x\}$. From the way we have chosen W_m , z cannot be an element of $f^{-1}(f(x))$. Therefore $f(x) \neq f(z)$, and since Y is T_2 , there exists some open A containing $f(x)$ and not $f(z)$. Therefore for some $n \in \omega$, $U_n \subseteq A$. Therefore z cannot be an element of $U_{n+1} \cap W_{n+1}$. This implies there does not exist $z \neq x \in \bigcap_{m \in \omega} U_m \cap W_m = \{x\}$.

Remember that we are assuming for some open set $V \ni x$ and any $m \in \omega$, we do not have $W_m \cap U_m \subseteq V$. As $x \in V$, we also have that $X \setminus V \subset X \setminus \{x\} = X \setminus \bigcap_{m \in \omega} U_m \cap W_m = X \setminus \bigcap_{m \in \omega} \overline{U_m} \cap \overline{W_m} \subseteq \bigcup_{m \in \omega} X \setminus \overline{U_m} \cap \overline{W_m}$. We may see from this that $\{X \setminus \overline{U_m} \cap \overline{W_m} : m \in \omega\}$ is an open cover of closed $X \setminus V$. Since X is compact, $X \setminus V$ is compact, and therefore this open cover has a finite subcover. This implies, however, that $\bigcap U_n \cap W_n \subseteq V$ for some finite number of U_n , a contradiction. Therefore $\{W_m \cap U_m : m \in \omega\}$ is a countable local base for x , and X is first-countable. \square

In a nearly identical way, we may show the following:

Theorem 6.1.6. *If X is compact T_2 , Y is a second-countable T_2 space, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is second-countable.*

Proof. Since X is compact, f is continuous, and Y is T_2 , we know $f(X)$ is compact and second-countable in Y . As second-countability is strictly stronger than first-countability, we know X must be first-countable by Theorem 6.1.5. Enumerate the elements of M_f as x_j , $j \in \omega$, and for each x_j let $\{W_{j,m} : m \in \omega\}$ be a countable local base. Let \mathcal{B} be a countable base of Y . Remove any element of \mathcal{B} that has empty intersection with $f(X)$. Enumerate the remaining elements of \mathcal{B} as $\{V_n : n \in \omega\}$. Let $U_n = f^{-1}(V_n)$ for every $n \in \omega$. We claim $B = \{W_{j,m} : j, m \in \omega\} \cup \{U_n : n \in \omega\}$ is a countable base of X .

B is obviously countable. To show it is a base, let A be open in X . We will show that for every $x \in A$, we may find an element of B containing x and contained in A .

If $x \in A \setminus M_f$, then it must be the case that $f(x) \in f(X) \setminus f(X \setminus A)$. To see why, if $f(x) \notin f(X) \setminus f(X \setminus A)$, then $f(x) \in f(X \setminus A)$, implying there exists $x' \in X \setminus A$ such that $f(x) = f(x')$. Thus $x \in M_f$, a contradiction.

By assumption, since $f(X) \setminus f(X \setminus A)$ is non-empty and open in $f(X)$, there exists a $k \in \omega$ such that $f(x) \in V_k \subseteq f(X) \setminus f(X \setminus A)$. Also notice that $f(X) \setminus f(X \setminus A) \subseteq f(A)$. Thus $f^{-1}(V_k) = U_k \subseteq f(A)$, and x is an element of U_k .

Now let $x \in A \cap M_f$. Since $x \in M_f$, it must be equal to x_j for some $j \in \omega$. Therefore for some $m \in \omega$, $W_{j,m} \subseteq A$.

Since our choice of A was arbitrary, we have shown that B is a countable base of X . □

As a consequence of Theorem 6.1.6, we have:

Theorem 6.1.7. *If X is compact T_2 , Y is a metric space, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is metrizable.*

Proof. As Y is T_2 , X is compact, and f is continuous, we know $f(X)$ must be second-countable. By Theorem 6.1.6, X must also be second-countable, and therefore by Urysohn's Metrization Theorem, X is metrizable. □

If we are going to prove X is a LOTS when we assume Y is an ordinal, then the following two theorems will be useful:

Theorem 6.1.8. *If X is compact T_2 , Y is scattered T_2 , and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is scattered.*

Proof. Let $A \subseteq X$ be closed and dense-in-itself. Consider $f(A)$. Since Y is scattered, $f(A)$ contains an isolated point, y . We know since M_f is countable, that $f^{-1}(y)$ must be countable, and as f is

continuous, must be closed and compact as well. Therefore by Theorems 3.2.11 and 3.1.3, $f^{-1}(y)$ is homeomorphic to a countable ordinal. Thus for some $x \in f^{-1}(y)$, x is isolated. Notice that $f^{-1}(y)$ is open since y is open, and therefore x must be isolated in the entire space X . However $x \in A$, contradicting the assumption that A is dense-in-itself. Therefore X must be scattered. \square

Theorem 6.1.9. *If X is a space, Y is infinite, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then $|X| \leq |Y|$.*

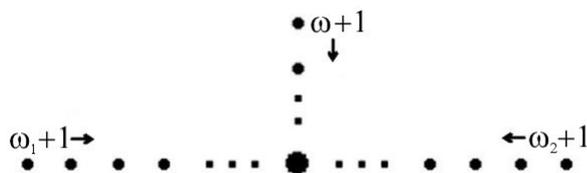
Proof. This follows from the fact that we are assuming M_f is countable. \square

As we can see, there are many properties P such that if Y is T_2 and has property P , and X is compact T_2 , then the existence of an almost-injective continuous function from X to Y implies X must also have property P . This is particularly useful when we show that if Y is an ordinal, then X must be a LOTS.

6.1.3 A Simple Counter-Example

The previous section contains many positive answers to Question 19.

We now provide an answer to Question 18 by stating a counter-example. Before giving the formal definition of the counter-example space X , we illustrate the space here to aid the reader.



Example 6.1.10. *Let X be the one-point compactification of the disjoint union of ω_1 , ω_2 , and ω , and Y be the one-point compactification of the disjoint union of ω_1 and ω_2 . Then there exists an almost-injection from X to Y , but X is not a LOTS.*

If δ is the point added during the compactification, then the continuous almost-injection, f , is the identity on $\omega_1 \cup \delta \cup \omega_2^*$, and for every $n \in \omega$, $f(n) = \delta$. Since ω is countable, M_f is countable, but clearly, due to varying cofinalities, X cannot be linearly ordered.

Though we have found a counter-example to Question 18, we feel as a consequence that the following is a valuable question:

Question 20. *If X is an infinite compact T_2 space, Y is a LOTS, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then must it be the case that X can be decomposed into a finite union of LOTS?*

That is, even if X is not a LOTS, perhaps it looks quite like a LOTS.

6.1.4 Almost-Injectivity and the Ordinals

Though we have found a counter-example, we feel Question 18 may still be answered positively for many general spaces Y . In this section, we prove that if X is a compact T_2 space, Y is an ordinal, and $f \in \mathcal{C}(X, Y)$ is almost-injective, then X is a LOTS (Theorem 6.1.12). We do so by implementing the same method used in Chapter 5: namely, we find a LOTS Z , show there exists a continuous injection g from X to Z , and then apply Lemma 5.2.2.

To make the proof of Theorem 6.1.12 as straightforward as possible, we first restate Lemmas 4.1.9 and 4.1.10, and prove a new lemma, Lemma 6.1.11.

Lemma 4.1.9. *Let X be a compact T_2 space, and let Y be an ordinal. For every $x \in X$ and $f \in \mathcal{C}(X, Y)$, we may find an ordinal Z and a function $g \in \mathcal{C}(X, Z)$ such that $g(x)$ is the greatest element of $g(X)$, and such that $M_g = M_f$.*

Lemma 4.1.10. *Let X be a T_2 compactum, Y an ordinal, and $f \in \mathcal{C}(X, Y)$. Furthermore, let $\{U_\lambda : \lambda \in \Lambda\}$ be a family of disjoint clopen intervals of Y such that $U_\lambda \cap f(X) \neq \emptyset$ for every $\lambda \in \Lambda$. If each $V_\lambda = f^{-1}(U_\lambda)$ is linearly orderable, and Z is the LOTS resulting from replacing each U_λ with V_λ , then there exists a $g \in \mathcal{C}(X, Z)$ such that $M_g \subseteq M_f$, and $M_g \cap V_\lambda = \emptyset$ for every $\lambda \in \Lambda$.*

Lemma 6.1.11. *If X is a compact T_2 space, Y an ordinal, and there exists $f \in \mathcal{C}(X, Y)$ such that f is almost-injective and $|f(M_f)| = 1$, then X is a LOTS.*

Proof. Without loss of generality, assume $f(M_f) = y$ is the greatest element of $f(X)$. (Otherwise, we may modify the proof accordingly: see Lemma 4.1.9.) We also assume that f maps X onto Y ; if it did not, then since $f(X)$ is a compact subset of an ordinal, and is therefore homeomorphic to an ordinal β by Lemma 4.1.3, we would merely take Y to be equal to β and continue with the proof. Also note that since Y is an ordinal, and therefore scattered, by Theorem 6.1.8, X must be scattered; since X is compact, it must therefore be zero-dimensional.

In order to prove this lemma, there are three cases for y we must consider: when $\text{cf}(y) > \omega$, when $\text{cf}(y) = \omega$, and when $\text{cf}(y) < \omega$.

Case 1: $\text{cf}(y) > \omega$

Let $\text{cf}(y) > \omega$. By the fact that y has uncountable cofinality, we know $|f^{-1}(y) \cap \overline{f^{-1}([0, y])}| = 1$. (If it had cardinality greater than 1, for example 2, then let x_1, x_2 be in this intersection, and let C_1, C_2 be disjoint clopen subsets of X containing x_1 and x_2 respectively. As f is continuous and closed, $f(C_1)$ and $f(C_2)$ must both be closed and unbounded in $[0, y)$. It must be the case then that $f(C_1) \cap f(C_2)$ is closed and unbounded in $[0, y)$ as well, implying that $|f(M_f)| > \aleph_0$.) Let $z \in X$ be such that $\{z\} = f^{-1}(y) \cap \overline{f^{-1}([0, y])}$. Then $f^{-1}(y) \setminus \{z\}$ is open, since $X \setminus (f^{-1}(y) \setminus \{z\}) = f^{-1}([0, y)) \cup \{z\}$ is closed.

If z is isolated in $f^{-1}(y)$, then $f^{-1}(y) \setminus \{z\}$ is clopen. Therefore as M_f is countable, we know by Theorem 6.1.4 that $f^{-1}(y) \setminus \{z\}$ must be homeomorphic to a countable ordinal λ . Also note that $f|_{X \setminus (f^{-1}(y) \setminus \{z\})}$ is injective into an ordinal, and therefore is an embedding. Let $X \setminus (f^{-1}(y) \setminus \{z\})$ be homeomorphic to δ . It must be the case, then, that X is homeomorphic to $\delta + \lambda$. (This is actually just homeomorphic to δ , since $\delta > \lambda$.)

If, alternatively, z is not isolated in $f^{-1}(y)$, then let the rank of z in $f^{-1}(y)$ be $\alpha + 1$, and let $V \subset X$ be a clopen set containing z and no other elements of $f^{-1}(y)$ of greater or equal rank. By Theorem 3.1.5, $V \cap f^{-1}(y)$ must be homeomorphic to some ordinal, $\lambda + 1$. (It must be a compact ordinal since $V \cap f^{-1}(y) \subset X$ is closed and therefore compact.) Then if $f(X) = \delta + 1$, since $\text{cf}(z) > \omega$, and $\overline{f^{-1}([0, y])} \cap \overline{f^{-1}(y) \setminus \{z\}} = \{z\}$, X must be homeomorphic to $\delta + 1 + \lambda^*$.

Case 2: $\text{cf}(y) = \omega$.

Let $\text{cf}(y) = \omega$. Partition X as described in Lemma 5.2.3, with $A = f^{-1}(y)$. Let λ be the ordinal to which $f^{-1}(y)$ is homeomorphic, and let each clopen subset created be denoted as U_α , where if $h : A \rightarrow \lambda$ is a homeomorphism, then U_α isolates $h^{-1}(\alpha)$. Notice that for each isolated $\alpha \in \lambda$, $f|_{U_\alpha}$ is injective, and since U_α is also closed, $f|_{U_\alpha}(U_\alpha)$ must be homeomorphic to an ordinal. (This follows from Lemma 4.1.3.) This implies U_α is homeomorphic to an ordinal, δ_α , by some homeomorphism f_α .

For each isolated $\alpha \in \lambda$, replace α with δ_α . Call the resulting space \hat{Y} . That is, $\hat{Y} = \bigcup_{\alpha \in I_0(\lambda)} \{\alpha\} \times \lambda_\alpha \cup \bigcup_{\alpha \in (\lambda+1)^1} \{\alpha\} \times \{f|_{U_\alpha}(U_\alpha)\}$. Give \hat{Y} the lexicographical order, and impose on it the order topology. Then due to a similar proof as given in Theorem 4.1.11, \hat{Y} is an ordinal, and there exists a continuous, injective function from X to \hat{Y} ; this shows X is homeomorphic to an ordinal. This implies X is a LOTS as well.

Case 3: $\text{cf}(y) < \omega$.

If $\text{cf}(y) < \omega$, then y is isolated in $f(X) \subseteq Y$. Then $f^{-1}(y)$ is a clopen subset of X , and homeomorphic to an ordinal by Theorem 6.1.4. $Y \setminus \{y\}$ is compact, and homeomorphic to an ordinal by Lemma 4.1.3; call it β . Then if $f^{-1}(y)$ is homeomorphic to γ , X is homeomorphic to

$\gamma + \beta$, following the proof given in Lemma 4.1.9. \square

We are now able to state and prove the main theorem of this section. Lemmas 6.1.11 and 4.1.10 are used repeatedly in the following proof.

Theorem 6.1.12. *Let X be a compact T_2 space, let Y be an ordinal, and let there exist an almost-injective $f \in \mathcal{C}(X, Y)$. Then X is a LOTS.*

Proof. We prove this using transfinite induction on the Cantor-Bendixson rank of $f(M_f)$. Note that since M_f is countable, $f(M_f)$ must be countable, and therefore $\text{CB}(f(M_f))$ must be countable. Also, we define $v \subseteq \omega$ to be either a finite initial segment of ω , or ω itself. Since the base case is trivial, we will begin with the successor case.

Successor Case: Let $\text{CB}(f(M_f)) = \alpha + 1$. Enumerate the elements of $f(M_f)^\alpha$ as y_m , where $m \in v$.

Let $\alpha = 0$. For each $m \in v$, let U_m be a clopen interval of Y that isolates y_m . Then by Lemma 6.1.11, we know $V_m = f^{-1}(U_m)$ is linearly orderable. By Lemma 4.1.10, we know there exists a LOTS Z , resulting from replacing each U_m with V_m , and a continuous function $g : X \rightarrow Z$ such that $M_g \cap V_m = \emptyset$ for every $m \in v$. This implies that M_g is empty, and that g is injective. As X is compact and Z is a LOTS, g must be closed, and is therefore an embedding. By Lemma 5.2.2, this makes X a LOTS.

Now let $\alpha \neq 0$. Let U_m be a clopen interval isolating y_m such that $f(M_f) \subseteq \bigcup_{m \in v} U_m$, and such that y_m is the greatest element of U_m . We may choose U_m this way since Y is an ordinal. Then since Y is an ordinal, and $\text{rank}(y_m) = \alpha + 1$, it must be the case that $\text{cf}([0, y_m))$ is countable. Therefore there exists a countable strictly increasing sequence of elements in $U_m \subseteq Y$ converging to y_m . Let $\langle z_{m,n} \rangle_{n \in \omega \setminus \{0\}}$ be such a sequence, such that for no $n \in \omega \setminus \{0\}$ is $z_{m,n+1}$ equal to $z_{m,n} + 1$. Let $z_{m,0}$ be the least element of U_m , and consider the intervals $[z_{m,0}, z_{m,1}]$ and $[z_{m,n} + 1, z_{m,n+1}]$. Since each $\text{CB}(f(M_f)) \cap [z_{m,n} + 1, z_{m,n+1}]$ and $\text{CB}(f(M_f)) \cap [z_{m,0}, z_{m,1}]$ must be less than $\alpha + 1$, then by the inductive hypothesis, each $f^{-1}([z_{m,n} + 1, z_{m,n+1}])$ and $f^{-1}([z_{m,0}, z_{m,1}])$ is linearly orderable.

Let \hat{U}_m be the space created by taking U_m and replacing each $[z_{m,n} + 1, z_{m,n+1}]$ and $[z_{m,0}, z_{m,1}]$ with $f^{-1}([z_{m,n} + 1, z_{m,n+1}])$ and $f^{-1}([z_{m,0}, z_{m,1}])$ respectively. By Lemma 4.1.10, there exists a continuous function $g_m : V_m \rightarrow \hat{U}_m$ such that $M_{g_m} \subseteq M_{f|_{V_m}}$, and such that $M_{g_m} \cap f^{-1}([z_{m,n} + 1, z_{m,n+1}])$ is empty. (Similarly for $M_{g_m} \cap f^{-1}([z_{m,0}, z_{m,1}])$.) This implies, however, that $\text{CB}(g_m(M_{g_m})) = 1$, as the only element not contained in the considered intervals was y_m . Thus by Lemma 6.1.11, V_m is linearly orderable. Using Lemma 4.1.10 again, we know there exists a function $g : X \rightarrow Z$ such that

$M_g \cap V_m = \emptyset$ for every $m \in v$. This implies, however, that g is injective, and thus X is a LOTS.

Limit Case: Let $\text{CB}(f(M_f)) = \gamma$, where γ is a limit ordinal, and assume we have shown that if $\text{CB}(f(M_f)) < \gamma$, then X is a LOTS.

Let $A = \overline{(f(M_f))}^\gamma$. Notice that $A \cap f(M_f) = \emptyset$, since $\text{CB}(f(M_f)) = \gamma$.

Take $f(X) \setminus A$. This is open, and therefore made up of the union of open intervals; since $f(X)$ is compact, we may require these open intervals to be of the form (a, b) , where either $a, b \in A$, or one of a or b is an endpoint of $f(X)$ and the other is an element of A . We may also require that $(a, b) \cap A = \emptyset$.

We now only consider those intervals such that $(a, b) \cap f(M_f)$ is non-empty. As $f(M_f)$ must be countable, there are only countably many open intervals to consider. Enumerate the endpoints of these intervals as (a_m, b_m) , where $m \in v$. Since Y is an ordinal, it must be the case that $(a_m, b_m]$ is a clopen subset of Y . Furthermore, $f(M_f) \cup \{b_m : m \in v\}$ is countable, and has Cantor-Bendixson rank $\gamma + 1$. (If $a_m = 0$, then consider instead the interval $[0, b_m]$: this is still clopen, and the Cantor-Bendixson rank is still equal to $\gamma + 1$.) We may therefore follow the proof detailed in the successor case, where $(a_m, b_m]$ are the clopen intervals U_m isolating each b_m , to show that X is a LOTS. \square

6.1.5 Conclusion

We have now shown that if X is compact T_2 , Y is an ordinal, and $f \in \mathcal{C}(X, Y)$ is almost-injective, then X is homeomorphic to an ordinal, and is therefore a LOTS as well. These initial results on almost-injectivity, including the counter-example in Section 6.1.3, seem promising, indicating that when considering a compact, T_2 X , and a LOTS Y , the existence of an almost-injective function either guarantees that X is a LOTS, or that X may be decomposed into a finite union of LOTSs.

While it has proved impossible thus far to test that almost-injectivity into any LOTS Y guarantees linear orderability, it would be interesting if one were able to answer the following questions:

Question 21. *Does there exist an example under ZFC such that X is a compact T_2 space, Y is a LOTS, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, but X cannot be decomposed into a finite union of LOTSs?*

Question 22. *For which LOTSs Y must the existence of an almost-injective continuous function from a compact T_2 space X to Y imply that X is a LOTS?*

Chapter 7

Conclusion

7.1 Results and Open Questions

In this thesis, we have provided a background of cleavability, discussed the various uses, proved several theorems, and given an introduction to a topic inspired by this area of research. We now highlight the applications of our results, showcase the major theorems of this thesis, and restate the open questions.

There are several reasons why cleavability is a useful and non-trivial tool for mathematicians. Firstly, it resolves a difficulty in using the following theorem:

Theorem 1.1.1. *If X is compact, Y is Hausdorff, and $f : X \rightarrow Y$ is a continuous injection, then f embeds X into Y .*

Theorem 1.1.1 tells us that any continuous injection from a compact space to a T_2 space must be an embedding. The difficulty in using this theorem is that we may not be able to say such an injective function exists. Cleavability resolves the difficulty by providing sufficient conditions for injectivity. This tells us we do not need to find a specific injective function, which may be difficult, but instead find a family of continuous functions, which might be much easier.

Additionally, cleavability tells us when a space is linearly orderable. Similarly, we have added another characterization for when a topology on a compactum X is equivalent to a topology induced by a well-ordering of the space.

The characterization of linear orderability within this thesis, however, is uniquely distinct from those given in [31], [28], and [27]; these papers provide topological tools to use on a given space. In this thesis, we provide a more functional tool, requiring one to find instead a suitable LOTS Y , and

a family of continuous functions from X to Y .

While the examples given in Section 1.4 show that cleavability of an infinite compactum X over a LOTS Y may not guarantee that X is embeddable into Y , results do seem to indicate that cleavability may ensure local embeddability of X into Y . That is, for every $x \in X$, we may find an open $U \ni x$ such that U is embeddable into Y .

Lastly, we have provided answers to the following question:

Question 3. *If X is cleavable over a space Y with property P , must X have property P ?*

This can easily be seen as one of the most important tools of cleavability, as it allows us to gain information about a potentially complex space X by comparing it with a well-known topological space Y .

There is therefore quite a bit we may learn about an infinite compactum by cleaving it over various types of LOTSs. We showcase here the major theorems of this thesis:

Theorem 3.1.7. *If X is an infinite compactum cleavable over a first-countable scattered LOTS, then X must be a LOTS.*

Theorem 3.1.8. *If X is an infinite compactum cleavable over ω_1 , then X must be homeomorphic to a subspace of ω_1 .*

Theorem 3.2.17. *If X is an infinite compactum cleavable over a scattered first-countable LOTS Y , then X is homeomorphic to a subspace of Y if and only if one of the following two conditions holds:*

1. *the Cantor-Bendixson rank of Y is strictly greater than that of X .*
2. *the Cantor-Bendixson ranks of X and Y are equal, and the number of elements in the last non-empty derived set of X is less than or equal to the number of elements in the last non-empty derived set of Y .*

Theorem 4.4.4. *If X is an infinite compactum cleavable over an ordinal, then X is homeomorphic to an ordinal, and is therefore a LOTS.*

Theorem 5.3.7. *Let X be an infinite compactum cleavable over a separable LOTS Y such that for some $f : X \rightarrow Y$, M_f is scattered. Then X is a LOTS.*

While we have provided many answers in the areas of cleavability, there are still many open questions. We restate several important questions here. The numbering of these questions coincides with their numbering within the main body of this thesis.

Question 1. *When does cleavability of a compact space X over a Hausdorff space Y imply the existence of a homeomorphism from X to a subspace of Y ?*

Question 2. *Let a compact space X be cleavable over a LOTS. Is X a LOTS?*

Question 7. *If X is an infinite compactum cleavable over an uncountable ordinal $\lambda > \omega_1$, must X be embeddable into λ ?*

Question 9. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must X be a LOTS?*

Question 10. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must X be embeddable into Y ?*

Question 11. *Let X be an infinite compactum cleavable over a first-countable LOTS Y . Must X be a LOTS?*

Question 12. *Under what conditions does cleavability of an infinite compactum X over a first-countable LOTS Y imply that X must be embeddable into Y ?*

Question 14. *If β is an uncountable ordinal, must it be the case that $\beta \cdot n + 1$, where $n \geq 5$, is cleavable over $\beta \cdot m + 1$, where $m \geq 4$, but is not embeddable unless $n \leq m$?*

Question 15. *If X is an infinite compactum cleavable over a separable LOTS Y such that for some $f \in \mathcal{C}(X, Y)$, M_f is scattered, must X be embeddable into Y ?*

Question 16. *If X is an infinite compactum cleavable over a LOTS Y , must X be locally embeddable into Y ?*

Question 17. *Let X be an infinite compactum cleavable over a separable LOTS Y . Must there exist a continuous $f : X \rightarrow Y$ such that M_f is countable and scattered in X ?*

Lastly, we have shown that results within cleavability have opened new areas of research. One such area is almost-injectivity.

Almost-injectivity is a tool mathematicians may use to prove the linear orderability of a compact T_2 space. As mentioned, many papers have been dedicated to characterizing the linear orderability of a topological space. Many of these characterizations rely on the ability to construct various subbases for the topological space in question; even cleavability relies on the ability to find a family of continuous functions from X to a LOTS Y . The greatest use of answering Question 22 is that

linear orderability would be dependent on constructing a single continuous function to an appropriate LOTS.

We therefore have the following open questions:

Question 19. *If X is an infinite T_2 compactum, Y a LOTS with property P , and there exists $f \in \mathcal{C}(X, Y)$ such that $|M_f| \leq \aleph_0$, must X have property P ?*

Question 20. *If X is an infinite compact T_2 space, Y is a LOTS, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then must it be the case that X can be decomposed into a finite union of LOTSs?*

Question 21. *Does there exist an example under ZFC such that X is a compact T_2 space, Y is a LOTS, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, but X cannot be decomposed into a finite union of LOTSs?*

Question 22. *For which LOTSs Y must the existence of an almost-injective continuous function from a compact T_2 space X to Y imply that X is a LOTS?*

In an effort to answer these questions, we have been able to show the following:

Theorem 6.1.5. *If X is compact T_2 , Y is a first-countable T_2 space, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is first-countable.*

Theorem 6.1.6. *If X is compact T_2 , Y is a second-countable T_2 space, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is second-countable.*

Theorem 6.1.7. *If X is compact T_2 , Y is a metric space, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is metrizable.*

Theorem 6.1.8. *If X is compact T_2 , Y is scattered T_2 , and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then X is scattered.*

Theorem 6.1.9. *If X is a space, Y is infinite, and there exists an almost-injective $f \in \mathcal{C}(X, Y)$, then $|X| \leq |Y|$.*

Theorem 6.1.12. *Let X be a compact T_2 space, let Y be an ordinal, and let there exist an almost-injective $f \in \mathcal{C}(X, Y)$. Then X is a LOTS.*

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