

# Sensitivity Analysis of Boundary Equilibria

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**Abstract** This paper studies the sensitivity of economic equilibria to perturbations when the implicit function theorem cannot be applied on account of the presence of boundaries. It presents results from the mathematical programming literature which provide conditions under which equilibria are robust to perturbation and are locally unique Lipschitz-continuous functions of parameters. Economic applications include search equilibrium, Cournot equilibrium and general equilibrium.

**Keywords** sensitivity analysis · implicit function theorem · equilibria · variational inequalities · boundaries

**JEL classification** C61 · C62

## 1 Introduction

Economists are often interested in how equilibria vary in response to changes in parameters. A desirable characteristic is that equilibria remain locally unique and vary continuously as parameters change. Such results are of importance if models are to make robust predictions. Since parameters are rarely known exactly, if small changes in data can produce large changes in equilibria, such models are unlikely to be useful in practice.

The implicit function theorem is the tool traditionally used to guarantee that equilibria behave well under perturbations. It implies that if the parameters of a model are altered slightly then, if its conditions hold, an equilibrium

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will continue to exist near the old one and moreover will remain locally unique. In addition, the equilibrium will vary smoothly with the parameters.

In the presence of boundaries, for example if there are non-negativity constraints, the implicit function is not generally applicable. An equilibrium on the boundary of the feasible set may cease to exist or cease to be locally unique in response to changes in parameters if the set of binding constraints changes. This paper gives conditions under which this behavior can be ruled out, so that equilibria continue to exist and remain locally unique even if the set of binding constraints may change. Smoothness is too much to hope for in the presence of boundaries but Lipschitz continuity will hold and directional derivatives can be determined.

This problem has been extensively studied in the mathematical programming literature but these results are not so well known in Economics. This paper is partly expository but also shows how these results can be applied to economic examples. Examples studied include search equilibrium, Cournot equilibrium and general equilibrium.

For simplicity the paper studies the case when agents' actions must lie in sets constrained by finitely many linear inequalities, for example non-negativity constraints. The problematic case for the implicit function theorem is when the set of constraints which bind may change if agents' payoff functions are perturbed. The generalization of the implicit function theorem given requires that a certain family of determinants be non-zero and have the same sign regardless of which constraints one treats as binding. More geometrically, the determinant of the Jacobian matrix is required to have the same non-zero sign when restricted to certain subspaces. The examples given show that how these conditions can be checked simply. In particular in the examples of search equilibrium and Cournot they will hold if own effects are strong enough. Equivalent geometrical conditions are also given. These results come from the mathematical programming literature on variational inequalities (see for example Facchinei and Pang (2003)) but the economic applications are new.

In the case of pure optimization in consumer theory some techniques for establishing Lipschitz-continuity of solutions in the presence of boundaries are already known (see for example Mas-Colell (1985)). These are discussed in Section 5 of the paper. In the case of equilibrium problems, however, the application of results introduced here seems to be largely new in economics.<sup>1</sup>

In the case of supermodular games, or their generalizations, monotone comparative statics offer powerful tools if one is interested in the direction of changes of the equilibrium set — see for example Milgrom and Shannon (1994). These techniques are silent, however, on the local behavior of an individual equilibrium and in particular whether it continues to exist and remain locally unique in response to changes in parameters, which is the focus of this paper. The paper also shows how directional derivatives may be calculated, which is of use in comparative statics.

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<sup>1</sup> Beggs (2015) has some applications to Bayesian games.

The paper proceeds as follows. Section 2 provides an example of search equilibrium to illustrate the issues addressed. Section 3 outlines the general framework and gives further examples. Section 4 contains the main results used from the mathematical programming literature. Section 5 discusses applications to the examples. The examples studied include search equilibrium and Cournot equilibrium. It is shown that equilibria behave well at the boundary if own-effects are strong enough – a diagonal-dominance condition. In addition, applications are given to general equilibrium and consumer theory. Section 6 discusses extensions. Section 7 concludes.

## 2 An Example

This section provides a simple example to illustrate the problems considered. Consider the following simple one-dimensional example based on the Cooper and John (1988) model of strategic complementarities as described in Chapter 2 of Cooper (1999). This may be interpreted as a model of search equilibrium. Each of a finite number of agents must choose an effort level  $e \in [0, 1]$ . An agent's payoff function,  $\pi(e, e_{-i}, u)$ , depends on the effort level he chooses  $e$ , the vector of effort levels of others  $e_{-i}$  and a parameter  $u$ .  $\pi$  is assumed concave in own action and twice continuously differentiable.

The game is symmetric, so it is natural to look for symmetric equilibria. Let  $g(e, u)$  denote the derivative of  $\pi$  with respect to  $e_i$  evaluated at the point where  $e_i$  and all elements of  $e_{-i}$  equal  $e$ .  $u$  will be suppressed when convenient.  $e$  is a symmetric equilibrium if

$$\begin{cases} g(e) = 0 & 0 < e < 1 \\ g(e) \leq 0 & e = 0 \\ g(e) \geq 0 & e = 1 \end{cases} \quad (1)$$

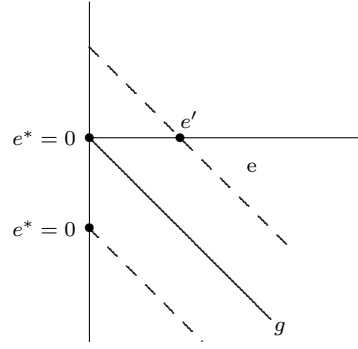
At an interior equilibrium one can apply the implicit function theorem to deduce that the equilibrium will change smoothly as  $u$  changes provided that  $g'(e) \neq 0$ .

Suppose instead there is an equilibrium at  $e = 0$ , that is when activity is at the lowest possible level. If  $g(0) < 0$  then  $e = 0$  remains an equilibrium for small changes in  $u$ . If  $g(0) = 0$ , however, it is straightforward to see graphically that there are two possible cases. If  $g'(0) < 0$  then there will be an equilibrium at or near  $e = 0$  if  $u$  is changed slightly:

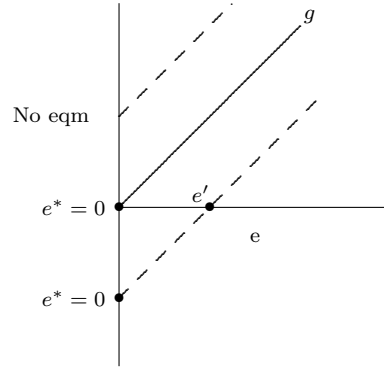
The dashed lines denote perturbations of  $g$ . If  $g$  is perturbed the equilibrium either remains at  $e^* = 0$  or there is a locally unique equilibrium near 0 (at  $e'$  in the figure).

If, though,  $g'(0) > 0$  then if  $u$  changes there may be no equilibria near  $e = 0$  or more than one equilibrium near  $e = 0$ :

In the upper perturbation of  $g$  there are no equilibria near  $e = 0$ . In the lower perturbation there are two:  $e = 0$  and  $e'$ . A similar analysis applies at  $e = 1$ . Note that in both these examples  $g'(0) \neq 0$ , so in the absence of



**Fig. 1** First case



**Fig. 2** Second case

boundaries the usual implicit theorem could be applied and the equilibrium would continue to exist and be locally unique.

$g'(e)$  represents the effect of a simultaneous change in all agents' effort levels. It follows, that given the assumed concavity of  $\pi$ ,  $g'(e) < 0$  if the effect of the change in own action outweighs that of other agents. This will be a general theme.

To rule out the phenomena in Fig. 2 it is enough to assume that  $g'(0) < 0$ . If one were to strengthen this to assume that  $g'(e) < 0$  for all  $e$ , then there would be a unique equilibrium. The same will be true of many of the conditions given. The paper focuses, however, on local properties. The possibility of multiple equilibria is often of interest and in any case information on global properties usually requires strong assumptions.

In this one-dimensional example it is easy to determine behavior at the boundary graphically. In higher dimensions the analysis is not so obvious. This paper surveys some tools developed in the Operations Research literature to deal with this problem and provides some economic applications.

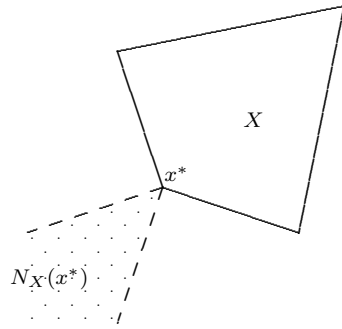
### 3 The Formulation

This section formulates the framework of the paper and shows that it covers a number of economic problems, for example constrained optimization and equilibria.

Let  $X$  be a closed, convex subset of  $\mathbb{R}^n$  and let  $f$  be a function from  $\mathbb{R}^n$  to  $\mathbb{R}$ . The normal cone of  $X$  at  $x^*$ ,  $N_X(x^*)$ , is the empty set if  $x^* \notin X$  and is the set

$$N_X(x^*) = \{y : \langle y, z - x^* \rangle \leq 0, \quad \forall z \in X\} \quad (2)$$

otherwise, where  $\langle u, v \rangle$  denotes the scalar product of  $u$  and  $v$ . Note that if  $x^*$  is in the interior of  $X$  then  $N_X(x^*) = \{0\}$ . The Normal cone is the set of vectors that make an obtuse angle with any direction at  $x^*$  pointing into  $X$  — see Fig. 3.



**Fig. 3** The normal cone

$x^*$  is said to solve the variational inequality  $VI(f, X)$  if

$$-f(x^*) \in N_X(x^*) \quad (3)$$

Attention will focus on the case in which  $X$  is polyhedral, that is it is defined by finitely many linear inequalities. In other words,  $X$  has the form  $X = \{x \in \mathbb{R}^n : Ax \leq b\} = \{x \in \mathbb{R}^n : \langle a_i, x \rangle \leq b_i, i = 1, \dots, m\}$ , where  $A$  is a  $m \times n$  matrix with rows  $a_1, \dots, a_m$ .

Variational inequalities can be formulated as equations, although this fact will not be of direct use. By the projection theorem<sup>2</sup>, an equivalent definition of the Normal cone is that  $v \in N_X(x)$  if and only if

$$\Pi_X(x + v) = x \quad (4)$$

where  $\Pi_X$  is the projection mapping taking in each point in  $\mathbb{R}^n$  to the closest point in  $X$ .

<sup>2</sup> See for example Ruszczynski (2006) Chapter 2, Lemma 2.11.

It follows that  $x^*$  solves (3) if and only if it is a zero of the ‘natural map’

$$f^{nat}(x) = x - \Pi_X(x - f(x)) \quad (5)$$

or less obviously (see Facchinei and Pang (2003) Proposition 1.5.9 for a proof) that  $x^* = \Pi_X(z)$  where  $z$  is a zero of the ‘normal map’

$$f^{norm}(z) = f(\Pi(z)) + z - \Pi(z) \quad (6)$$

The latter map has the advantage that it is defined for all  $z$  even if  $f$  is only defined on  $X$ . Because the projection mapping is in general only piecewise smooth these equations cannot be treated directly by the standard implicit function theorem.

This framework encompasses a number of problems.

*Example 1* Constrained Optimization and Complementarity Problems

If  $X$  is closed convex subset of  $\mathbb{R}^n$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  then  $x^*$  satisfies the first-order conditions for minimizing  $g$  over  $X$  if and only if  $x^*$  solves  $VI(f, X)$  where  $f = Dg$ .<sup>3</sup> For example if  $X = \mathbb{R}_+^n$ , the positive orthant, the first-order conditions are

$$\begin{cases} f_i(x^*) = 0 & x_i^* > 0 \\ f_i(x^*) \geq 0 & x_i^* = 0 \end{cases} \quad (7)$$

This is equivalent to the condition that  $-f \in N_X(x^*)$  since  $N_{\mathbb{R}_+^n}(x^*)$  is the set of vectors  $v$  such that

$$\begin{cases} v_i = 0 & x_i^* > 0 \\ v_i \leq 0 & x_i^* = 0 \end{cases} \quad (8)$$

This follows from Lemma 2 in Section 4 but is obvious geometrically:

(7) is a well-defined problem even when  $f$  is not the gradient of another function and is an example of a complementarity problem, which is an important special class of variational inequality problems.

More generally, a mixed complementarity problem has the form:

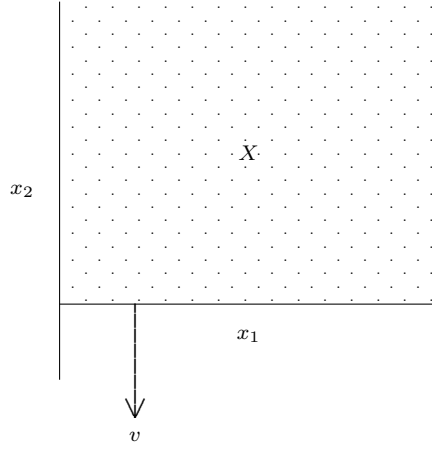
$$\begin{cases} f_i(x^*) = 0 & a_i < x_i < b_i \\ f_i(x^*) \geq 0 & x_i^* = a_i \\ f_i(x^*) \leq 0 & x_i^* = b_i \end{cases} \quad (9)$$

which as above can easily be shown to be equivalent to the condition  $-f \in N_X(x^*)$ , where  $X = \Pi_{i=1}^n[a_i, b_i]$ .

The example in Section 2 fits in this framework. The equilibrium conditions in (1) are an example of a mixed complementarity problem (with  $f = -g$  as agents are maximizing rather than minimizing payoffs).

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<sup>3</sup> See for example Facchinei and Pang (2003) Section 1.3.1.



**Fig. 4** Normal cone for complementarity problem

*Example 2* General Equilibrium

If  $z(p)$ ,  $p \in \mathbb{R}^n$ , represents the excess demand vector for an economy with free disposal then  $p^*$  is a competitive equilibrium price vector if and only if

$$p_i^* \geq 0, z_i(p^*) \leq 0 \quad \forall i, \quad z_i(p) = 0 \quad \text{if } p_i > 0 \quad (10)$$

or equivalently

$$p^* \geq 0, z(p^*) \leq 0 \quad \langle p^*, z(p^*) \rangle = 0 \quad (11)$$

Taking  $K = \mathbb{R}_+^n$ , the positive orthant, then this is equivalent to the condition that

$$z(p^*) \in N_K(p^*) \quad (12)$$

In other words  $p^*$  solves  $VI(-z, K)$ . (11) is again an example of a complementarity problem.

Other formulations are also useful. In Section 5 it is shown that Kehoe (1980)'s model of general equilibrium with a production technology given by linear activities is naturally formulated and analyzed as general (polyhedral) variational inequality problem rather than a complementarity problem.

*Example 3* Kuhn-Tucker

Consider the problem of minimizing  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  subject to the constraints  $g_i(x) \leq 0$ ,  $i = 1, \dots, s$ ,  $g_i(x) = 0$ ,  $i = s + 1, \dots, m$ . Introduce the Lagrangian  $L = f(x) + y'g(x)$ . The Kuhn-Tucker<sup>4</sup> first-order conditions are

$$\begin{aligned} L_{x_j} &= 0 & j &= 1, \dots, n \\ L_{y_j} &\leq 0 & \text{if } y_j = 0 \text{ and } 1 \leq j \leq s \\ L_{y_j} &= 0 & \text{otherwise } j = 1, \dots, m \end{aligned} \quad (13)$$

<sup>4</sup> In the Operations Research literature these are often referred to as the Karush-Kuhn-Tucker conditions.

Let  $X = \mathbb{R}^n$  and  $Y = \mathbb{R}_+^s \times \mathbb{R}^{m-s}$  and  $h = (L_x, -L_y)$  then (13) can be written equivalently as

$$-h \in N_{X \times Y} \quad (14)$$

Note that  $N_{X \times Y} = N_X \times N_Y$  for any convex sets  $X$  and  $Y$ .

The Kuhn-Tucker framework covers for example a consumer maximizing utility subject to standard budget and non-negativity constraints but also cases with more complex constraints such as time constraints or non-linear budget constraints.

#### Example 4 Nash Equilibrium

Let  $S_i$  be a closed convex subset of  $\mathbb{R}^{m_i}$  for  $i = 1, \dots, l$ . Let  $S = \prod_{i=1}^n S_i$  and let  $R_i : S \rightarrow \mathbb{R}$  be differentiable<sup>5</sup> in  $s_i$  for each  $i$ .  $s = (s_1^*, \dots, s_n^*)$  is a Nash equilibrium if for each  $i$ ,  $R_i(s_i^*, s_{-i}^*) \geq R_i(s_i, s_{-i}^*)$  for all  $s_i \in S_i$ . The first-order conditions for a Nash equilibrium can be written as

$$-f(s^*) \in N_S(s^*) \quad (15)$$

where  $f = (-D_{s_1} R_1, \dots, -D_{s_l} R_l)$ . These are sufficient for  $s^*$  to be a Nash equilibrium if, for example, each  $R_i$  is concave in  $s_i$ .

In all these examples one may be interested in how solutions depend on certain parameters. Let  $f$  depend on some parameters  $u \in U$ , where  $U$  is an open subset of some Banach space. In most applications  $U$  will be a subset of some finite-dimensional Euclidean space but this formulation allows for an infinite-dimensional parameterization. Let  $f : \Omega \times U \rightarrow \mathbb{R}^n$  be continuously differentiable, where  $\Omega$  is an open subset of  $\mathbb{R}^n$  containing  $X$ . In Example 4, for example,  $u$  may represent the payoff functions of different players.

In equilibrium problems, such as Example 2 and Example 4, there will typically be multiple equilibria. Attention will focus on conditions under which an isolated equilibrium continues to exist and remains locally unique, varying a Lipschitz-continuous fashion, as  $u$  is varied. As noted in the introduction, differentiability in general cannot be guaranteed. More generally, if  $x = x^*$  is a solution to

$$-f(x, u) \in N_X(x) \quad (16)$$

when  $u = u^*$  then attention will focus on the behavior of solutions in the neighborhood of  $x^*$  as  $u$  varies.

## 4 Analysis

In this section general conditions are given under which solutions to variational inequalities remain locally unique and are Lipschitz-continuous as parameters are varied. Essentially a generalization of the implicit function is introduced.

<sup>5</sup> Recall that a function is differentiable on  $S$  if it can be extended to a differentiable function defined on an open set containing  $S$ .



Readers interested in applications may wish to peruse this section briefly initially and then proceed to the next section to see how these can be applied in examples. An approach based on Lagrange multipliers is used as these are familiar to economists. No originality is claimed for these results or approach and the section draws heavily on Facchinei and Pang (2003) and Robinson (2003).

By the definition of the normal cone the condition

$$-f(x^*) \in N_X(x^*) \quad (17)$$

is equivalent to the condition that

$$\langle -f(x^*), x - x^* \rangle \leq 0 \quad \forall x \in X$$

or equivalently

$$x^* \min x' f(x^*) \quad \text{subject to} \quad x \in X \quad (18)$$

The Lagrangian for this problem is

$$\mathcal{L} = x' f(x^*) + \lambda'(Ax - b)$$

Since the constraints are affine and the objective function linear the Kuhn-Tucker first-order conditions are necessary and sufficient. Hence

**Lemma 1**  *$x^*$  solves the variational inequality (17) if and only if there exists a vector  $\lambda$  such that*

$$f(x^*) + A'\lambda = 0 \quad (19)$$

$$\lambda'(b - Ax^*) = 0 \quad (20)$$

$$\lambda \geq 0, \quad Ax^* \leq b \quad (21)$$

This system, (19)–(21), contains the usual complementary slackness conditions:

$$\lambda_i > 0 \implies (Ax^*)_i = b_i \quad i = 1, \dots, m \quad (22)$$

Suppose that the reverse condition is true:

$$\lambda_i = 0 \implies (Ax^*)_i < b_i \quad i = 1, \dots, m \quad (23)$$

In this case it is clear that, provided the solution is continuous, constraints which do not bind will remain non-binding for small perturbations of  $f$ , with their corresponding multipliers zero. The system can then be reduced to

$$\begin{aligned} f(x^*, u) + B'\mu &= 0 \\ c - Bx &= 0 \end{aligned} \quad (24)$$

where  $B$  and  $c$  consist of the rows of  $A$  and  $b$  respectively corresponding to the binding constraints and  $\mu$  consists of the non-zero elements of  $\lambda$ .

According to the implicit function theorem  $x$  and  $\mu$  will be single valued functions of  $u$  if the Jacobian determinant of the system with respect to  $x$  and  $\mu$  is non-zero, that is if

$$\begin{vmatrix} Df_x & B' \\ -B & 0 \end{vmatrix} \quad (25)$$

is non-zero, where  $Df_x$  denotes the derivative of  $f$  with respect to  $x$ .

If (23) does not hold, then constraints with  $\lambda_i = 0$  and  $(Ax^*)_i = b_i$  may become non-binding for small perturbations. It will be required that the condition above holds whether or not they are treated as binding at  $x^*$ . Let  $\alpha$  denote the set of indices for which  $\lambda_i > 0$  and let  $\beta$  be the subset of indices for which  $\lambda_j = 0$  but  $(Ax)_j = b_j$ . Let  $\Gamma$  be the collection of subsets of indices  $\gamma$  such that  $\gamma \supseteq \alpha$  and  $\gamma \subseteq \alpha \cup \beta$ . For each  $\gamma \in \Gamma$  let  $B_\gamma$  be a matrix composed of a maximal linearly independent<sup>6</sup> subset of the rows  $\{a_i\}_{i \in \gamma}$ .

**Theorem 1** *Let  $f$  be continuously differentiable in  $(x, u)$  and let  $X$  be polyhedral. Suppose that  $x^*$  solves the variational inequality*

$$-f(x, u) \in N_X(x) \quad (26)$$

for  $u = u^*$ . If the determinant

$$\begin{vmatrix} Df_x & B'_\gamma \\ -B_\gamma & 0 \end{vmatrix} \quad (27)$$

has the same non-zero sign for each  $\gamma \in \Gamma$ , then there are neighborhoods  $V$  of  $u^*$  and  $\Xi$  of  $x^*$  and a Lipschitz-continuous function  $x : V \rightarrow \Xi$  such that  $x(u)$  is the unique solution in  $\Xi$  of (26). Moreover  $x(u)$  is directionally differentiable at  $u = u^*$ .

References to proofs can be found at the end of the section. Directional derivatives in this paper are one-sided ones.<sup>7</sup> For brevity, the conclusion of Theorem 1 that  $x(u)$  is locally unique and Lipschitz continuous will be abbreviated by saying that (3) is **strongly regular** at  $x = x^*$  and  $u = u^*$ .

The multipliers, and so index sets, may not be unique but if the condition in Theorem 1 holds for one set of multipliers it holds for any other, so the choice is unimportant. This follows from the geometric discussion below.<sup>8</sup>

If  $h$  is a linear map and  $S$  a subspace then the restriction of  $h$  to  $S$ , mapping  $S$  to  $S$ , is defined to be  $\pi_S \circ h$ , where  $\pi_S$  is the orthogonal projection onto  $S$  and  $\circ$  composition of mappings. (27) is (positively) proportional to the determinant of the restriction of the linear map corresponding to the matrix  $Df_x(x^*, u^*)$  to the linear subspace  $S_\gamma = \{x : B_\gamma x = 0\}$ .<sup>9</sup> The requirement

<sup>6</sup> This requirement covers in particular the case when an equality constraint is represented as a pair of inequality constraints.

<sup>7</sup> That is  $f$  has a directional derivative at  $x$  in the direction  $v$  if  $\lim_{h \downarrow 0} (f(x+hv) - f(x))/h$  exists. It is directionally differentiable if it has directional derivatives in all directions.

<sup>8</sup> See also Facchinei and Pang (2003) Theorem 5.3.24.

<sup>9</sup> See for example Facchinei and Pang (2003) Proposition 4.2.7 or Mas-Colell (1985) 1.B.5.2.

is then that this determinant has the same non-zero sign regardless of which subspace one looks at.

The result can be applied without introducing a Lagrangian directly. Note that Lemma 1 implies that  $-f(x^*)$  lies in the Normal cone of  $X$  at  $x^*$  if and only if it can be expressed as a non-negative combination of the rows of  $A$  (strictly speaking transposed, so they are column vectors) corresponding to the binding constraints at  $x^*$ . Since  $-f(x^*)$  can be chosen arbitrarily this allows<sup>10</sup> one to identify the Normal cone:

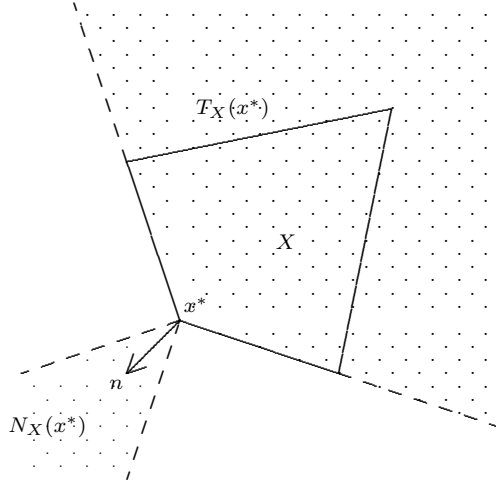
**Lemma 2**  $N_X(x^*) = \{y : y = \sum_{i \in I} \lambda_i a'_i, \lambda_i \geq 0\}$ , where  $a'_i$  are the rows of  $A$  transposed and  $I$  denotes the set of binding constraints at  $x^*$ ,  $(Ax^*)_i = b_i$ .

The indices  $\beta$  above are therefore those of the rows of  $A$  which correspond to binding constraints at  $x^*$  but have zero weight in the representation of  $-f(x^*)$ .

To gain further intuition note that the set of directions in which it is feasible to move from  $x^*$  is the **tangent cone**<sup>11</sup>

$$T_X(x^*) = \{v : (Av)_i \leq 0 \quad i \in I\} \quad (28)$$

where  $I$  denotes the set of binding constraints at  $x^*$ . The tangent cone is the polar of the Normal cone, that is  $v \in T_X(x^*)$  if and only if  $\langle v, n \rangle \leq 0$  for all  $n \in N_X(x^*)$ :



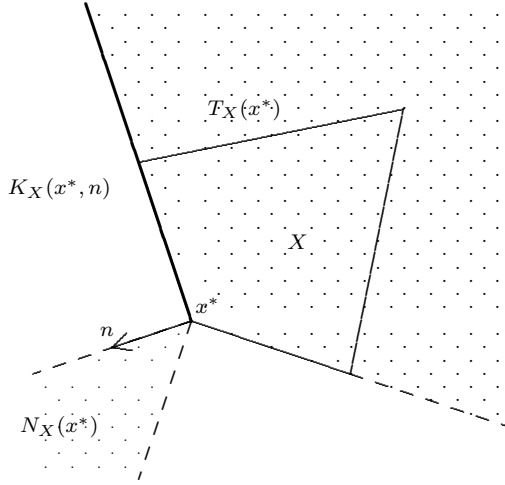
**Fig. 5** Normal and tangent cones

$n$  in Fig. 5 equals  $-f(x^*)$ .<sup>12</sup> If  $f$  is slightly perturbed then it is clear that  $x^*$  will remain a solution to the variational inequality and both constraints

<sup>10</sup> As pointed out by the referee, Lemma 2 also follows directly from Farkas' lemma.

<sup>11</sup> For a general closed convex set the tangent cone is the closure of the set of feasible directions.

<sup>12</sup> For convenience the tangent and normal cones are drawn based at  $x^*$  rather than 0.



**Fig. 6** Normal, tangent and critical cones

(corresponding to the intersection of the faces at  $x^*$ ) will continue to bind. On the other hand if the picture looked like:

then if  $f$  were slightly perturbed the solution might either remain at  $x^*$  or move in the direction denoted by  $K_X(x^*, n)$ . The latter is the **critical cone** or the set of feasible directions at  $x^*$  orthogonal to  $n$  and gives the potential directions the solution can move if  $f$  is perturbed slightly. Algebraically, with  $n = -f(x^*)$  and  $\alpha$  and  $\beta$  the indices defined above

$$K_X(x^*, n) = \{v \in T_X(x^*) : \langle v, n \rangle = 0\} = \{v : (Av)_i \leq 0 \ i \in \beta, (Av)_i = 0 \ i \in \alpha\} \quad (29)$$

That is the condition in the Theorem requires that the determinants of the restrictions of  $Df(x^*)$  to the linear directions of the faces of the critical cone have the same sign. The Lagrange multipliers and so index sets need not be unique but the critical cone is, hence the unimportance of the choice of multipliers in Theorem 1.

To gain some more geometric intuition consider taking a first-order linear approximation to  $f$  around  $(x^*, u^*)$ :

$$-y - C(x - x^*) - v \in N_X(x) \quad (30)$$

where  $y = f(x^*, u^*)$ ,  $C = Df_x(x^*, u^*)$  and  $v = Df_u(x^*, u^*)(u - u^*)$ . In the standard case where there are no constraints,  $x$  can be solved for uniquely in terms of  $u$  provided  $C$  is invertible, which is the usual implicit function theorem condition. It is natural to ask when (30) can be solved for  $x$  in terms of  $u$ .

The analysis can be simplified here by noting that  $z = x - x^*$  must belong to the critical cone  $K = K_X(x^*, -f(x^*))$  introduced above so it is enough to

be able to solve the linearized system

$$-v - Cz \in N_K(z) \quad (31)$$

uniquely for  $z$ . That is only potentially binding constraints need be taken into account.

Solving this variational inequality is, as noted in (6) in Section 3, equivalent to finding points  $w$  and  $z$  with  $z = \Pi_K(w)$  and  $L(w) = -v$  where  $L$  is the piecewise-linear map

$$L(w) = w - \Pi_K(w) + C(\Pi_K(w)) \quad (32)$$

Euclidean space can be divided up into cells which project onto a particular face of  $K$ .  $L$  is linear on each cell.

(31) is strongly regular if and only if  $L$  is a homeomorphism (is 1-1, continuous and onto). This in turn is true if and only if the determinants of the linear restrictions to each cell have the same sign. This **coherent orientation** condition turns out to be equivalent to the condition derived above.

To summarize:

**Theorem 2** *Let  $f$  be continuously differentiable in  $(x, u)$  and let  $X$  be polyhedral. Suppose that  $x^*$  solves the variational inequality*

$$-f(x, u) \in N_X(x) \quad (33)$$

*for  $u = u^*$ . This problem is strongly regular at  $x = x^*, u = u^*$  if and only if one of the following equivalent conditions holds*

$$(a) \quad -f(x^*, u^*) - Df_x(x^*, u^*)(x - x^*) - v \in N_X(x)$$

*is strongly regular at  $x = x^*, v = 0$ .*

(b) *The map*

$$L(x) = Df_x(x^*, u^*)(\Pi_K(x)) + x - \Pi_K(x) \quad (34)$$

*is a homeomorphism.*

(c) *The map  $L$  is coherently oriented.*

(d) *The determinant condition in Theorem 1 holds.*

(e) *The determinant of the linear map corresponding to  $Df_x(x^*, u^*)$  has the same non-zero sign when restricted to each subspace  $\{v : B_\gamma v = 0\}$ .*

The directional derivative in the direction  $w = u - u^*$  can be computed by solving (31) (see for example Robinson (2003) or Facchinei and Pang (2003) Theorem 5.4.12 specialized to this context).<sup>13</sup>

**Theorem 3** *Under the conditions of Theorem 2, the derivative of  $x(u)$  at  $u = u^*$  in the direction  $w$  is the solution to (31) for  $z$  when  $v = Df_u(x^*, u^*)w$  or equivalently using the normal map formulation  $\Pi_K L^{-1}(-Df_u(x^*, u^*)w)$ .*

<sup>13</sup> Robinson (2003) uses the term B-differentiable, which for Lipschitz functions in finite-dimensional space reduces to directional differentiability.

Some further equivalent conditions in addition to those in Theorem 2 can be found in Robinson (2003) and Dontchev and Rockafellar (1996). (d) is not stated in these sources but follows from Facchinei and Pang (2003) Proposition 4.2.7, as does (e) (which does appear in Robinson (2003)). Robinson (2003) does not contain proofs but these can be found in Robinson (1995). A book source is Facchinei and Pang (2003) — see Theorem 5.3.17 for Theorems 1 and 2 (noting that only the polyhedral case is required). If the set of perturbations is rich enough then the conditions are also necessary — see for example Dontchev and Rockafellar (2009) Proposition 2.E.5. The latter book contains a broad perspective on such results and extensions to cases where the solution mapping may not be single-valued.

## 5 Examples

This section provides economic examples of the approach in the previous section. The first subsection develops the results of the previous section further in the special case of complementarity problems and the following subsections apply it to the Cooper-John model of Section 2, more general search equilibrium and Cournot equilibrium. Section 5.5 applies the results to general equilibrium. Section 5.6 applies the results to optimization and consumer demand and compares with other approaches in that application.

### 5.1 Complementarity Problems

Consider the complementarity problem in Example 1:

$$\begin{cases} f_i(x^*) = 0 & x_i^* > 0 \\ f_i(x^*) \geq 0 & x_i^* = 0 \end{cases} \quad (35)$$

Let  $I_1$  be the set of indices with  $x_i^* > 0$ ,  $I_2$  those with  $x_i^* = 0$  and  $f_i(x^*) > 0$  and  $I_3$  the remainder. Since the constraint set can be written in the form  $X = \{x : (-e_i)'x \leq 0, i = 1, \dots, n\}$ , where  $e_i$  is a vector with 1 in the  $i$ th position and zero elsewhere, the indices in  $I_2$  correspond to the indices  $\alpha$  in the previous subsection: those which have a strictly positive weight in the expression of  $-f(x^*)$  as a linear combination of the  $e_i$ . These constraints will continue to bind if the system is slightly perturbed. Those in  $I_3$  may not.

For a set of indices  $J$ , let  $E_J$  denote a matrix whose rows are the standard unit vectors in the  $j$ th direction,  $e_j$ , for  $j \in J$ . The determinant condition (27) therefore requires that

$$\begin{vmatrix} Df & E_J' \\ -E_J & 0 \end{vmatrix} \quad (36)$$

have the same non-zero sign for each subset of indices  $J$  with  $I_2 \subseteq J \subseteq I_2 \cup I_3$ .

It is easy to see by elementary row operations that (36) equals

$$|(Df)_{J^c}| \quad (37)$$

where  $(Df)_{J^c}$  denotes the matrix obtained from  $Df$  by dropping the rows and columns in  $J$  or equivalently by treating these variables as fixed at 0 in the differentiation. If  $I_3$  is empty, this is simply the usual determinant condition. More generally the requirement is that this have the same sign whichever of the constraints in  $I_3$  one treats as binding. If  $J^c$  is empty (all constraints bind) the sign of (37) is treated as positive.

A **P-matrix** is a square matrix all of whose principal minors are strictly positive. If  $Df$  is a P-matrix then (37) is guaranteed to hold. A sufficient condition for an  $n$ -dimensional matrix  $A$  to be a P-matrix is that it have all diagonal entries strictly positive and be (row) **diagonally dominant**: that is there exists a set of strictly positive  $\lambda_1, \lambda_2, \dots, \lambda_n$  such that  $\lambda_i |a_{ii}| > \sum_{j \neq i} \lambda_j |a_{ij}|$ .<sup>14</sup> That is, own-effects outweigh cross-effects.

The analysis generalizes easily to mixed complementarity problems. It is simple to check that (37) and the P-matrix condition guarantee strong regularity of solutions if some or all variables are also upper bounded. If the Jacobian matrix is a P-matrix everywhere then the solution to the mixed complementarity problem is unique (see for example Facchinei and Pang (2003) Section 5.3.2). The focus here, however, is on local conditions, and global information need not be available.

## 5.2 Cooper-John model

In the case of the Cooper-John model in Section 2 a symmetric equilibrium  $e \in [0, 1]$  must satisfy:

$$\begin{cases} g(e) = 0 & 0 < e < 1 \\ g(e) \leq 0 & e = 0 \\ g(e) \geq 0 & e = 1 \end{cases} \quad (38)$$

where  $g(e)$  is the derivative of payoffs with respect to own action,  $\partial \pi(e, e_{-i}, u) / \partial e_i$ , when all actions equal  $e$ . An equilibrium at  $e = 0$  falls into the framework of the previous subsection with  $f = -g$ .

Consider an equilibrium at  $e = 0$  and  $g(e) = 0$ . According (37) the determinant of  $-Dg$  needs to be a P-matrix, that is have all principal minors positive. In this case the Jacobian is simply  $-g'(0)$ . The condition of the previous subsection is therefore met if and only if  $g'(0) < 0$ .

It follows that if  $g'(0) < 0$  an equilibrium will continue to exist near  $e = 0$  if the system is perturbed. Furthermore it will be a locally unique Lipschitz-continuous function of the parameters. This confirms the graphical analysis of Section 2. A similar analysis is easily performed for equilibria at  $e = 1$ .

To illustrate the use of Theorem 3 to calculate directional derivatives, suppose that  $u$  is a scalar parameter affecting  $g$ ,  $g(e, u)$ , and when  $u = 0$  there is an equilibrium with  $e = 0$  and  $g = 0$ . Let subscripts denote partial derivatives. According to Theorem 3 if  $g_1(0, 0) < 0$  the directional derivative in the

<sup>14</sup> See for example Nikaido (1968).

direction  $w$  is found by solving for  $z$  the following linearized version of the problem, dropping all strictly non-binding constraints

$$g_1(0,0)z + g_2(0,0)w \leq 0, \quad = 0 \text{ when } z > 0 \quad (39)$$

Hence, setting  $w = \pm 1$ , if  $g_2(0,0) > 0$  then the right-hand derivative at  $e = 0$ ,  $u = 0$  is  $(de/du)_+ = -g_2(0,0)/g_1(0,0)$  and the left-hand one  $(de/du)_- = 0$ .

### 5.3 Two-dimensional search equilibrium

A more elaborate example can be obtained by looking at a version of the model in the previous section where payoffs need not be symmetric. Let there be two players for simplicity and let player  $i$  have payoff function

$$\pi_i(e_i, e_j) = \alpha e_i + \beta_i e_i e_j - \frac{1}{2} e_i^2 \quad (40)$$

where  $e_i$  is the effort of agent  $i$  and  $e_j$  of the other agent, both belonging to  $[0, 1]$ .  $\alpha$  and  $\beta_i > 0$  are parameters.

The derivative of  $\pi_i$  with respect to own effort is

$$g_i(e_i, e_j) = \alpha + \beta_i e_j - e_i \quad (41)$$

Again this is an example of a (mixed) complementarity problem:  $e = (e_1, e_2)$  is an equilibrium if for all  $i$ :

$$\begin{cases} g_i(e) = 0 & 0 < e_i < 1 \\ g_i(e) \leq 0 & e_i = 0 \\ g_i(e) \geq 0 & e_i = 1 \end{cases} \quad (42)$$

If  $\alpha = 0$  then there is always an equilibrium with  $e_1 = e_2 = 0$  and  $g_1 = g_2 = 0$ . One can apply the criteria of the first subsection with  $f = (f_1, f_2) = (-g_1, -g_2)$  to explore the effect of varying  $\alpha$ . The Jacobian of  $f$  is

$$Df = \begin{pmatrix} 1 & -\beta_1 \\ -\beta_2 & 1 \end{pmatrix} \quad (43)$$

Since the diagonal elements are positive, this will have all principal minors positive if its determinant is positive, that is if

$$1 - \beta_1 \beta_2 > 0 \quad (44)$$

If this condition is met then  $Df$  is a P-matrix, so if  $\alpha$  is varied slightly equilibria will continue to exist near  $(0, 0)$  and be locally unique and equilibrium efforts will be Lipschitz-continuous functions of it. Note that (44) holds in particular when  $1 > \beta_1$  and  $1 > \beta_2$ , that is  $Df$  is a dominant-diagonal matrix with a positive diagonal, illustrating the sufficiency of this condition noted in the first subsection.



In the symmetric case when  $\beta_1 = \beta_2 = \beta$ , the necessity of (44) is easily seen. When  $\beta < 1$ , the unique equilibrium is  $e = (0, 0)$  when  $\alpha \leq 0$  and  $(\frac{\alpha}{1-\beta}, \frac{\alpha}{1-\beta})$  if  $\alpha > 0$  for  $\alpha$  small enough. When  $\beta > 1$ , there are equilibria at  $(0, 0)$ ,  $(1, 1)$  and  $(\frac{\alpha}{1-\beta}, \frac{\alpha}{1-\beta})$  when  $\alpha \leq 0$  (and small enough in absolute value) but if  $\alpha > 0$  the unique equilibrium is  $(1, 1)$  and in particular no equilibria exist close to  $(0, 0)$ . The equilibrium set therefore changes radically when  $\alpha$  crosses zero if  $\beta > 1$  and (43) fails to be a P-matrix.

The same approach can be applied when  $g$  does not have the particular linear form. An equilibrium at  $(0, 0)$ , or at  $(1, 1)$ , will be strongly regular if  $Df$  is a P-matrix, for which the dominant-diagonal condition is sufficient.

As in the previous example one can compute directional derivatives. Suppose that  $g$  may be non-linear and depends on a scalar parameter  $u$ . At a strongly regular equilibrium at  $(0, 0)$  the directional derivatives of  $e_1$  and  $e_2$  in the direction  $w$  can be found by solving the linearized version of the problem, with all non-binding constraints omitted, for  $z_1$  and  $z_2$  respectively:

$$g_{1,e_1}z_1 + g_{1,e_2}z_2 + g_{1,u}w \leq 0, = 0 \text{ when } z_1 > 0 \quad (45)$$

$$g_{2,e_1}z_1 + g_{2,e_2}z_2 + g_{2,u}w \leq 0, = 0 \text{ when } z_2 > 0 \quad (46)$$

Subscripts  $e_1$  and so on denote partial derivatives. If one applies this to the symmetric version of the linear example (41) above with  $u = \alpha$ , it is easy to check, solving for  $z_1$ , that  $(\partial e_1 / \partial u)_+ = 1/(1 - \beta)$ ,  $(\partial e_1 / \partial u)_- = 0$ , as one obtains directly from the formulae for the equilibrium solutions.

#### 5.4 Cournot Equilibrium

Consider a Cournot oligopoly with  $n$  firms with cost function  $C_i(q_i)$ ,  $i = 1, \dots, n$ , with  $q_i$  the output of firm  $i$ . If  $Q$  is aggregate output and  $P(Q)$  the inverse demand function then firm  $i$ 's profit function is  $\Pi_i(Q) = P(Q)q_i - C_i(q_i)$ . The inverse demand functions and cost functions may depend on some parameters  $u$ , so can be written as  $P(Q, u)$  and  $C_i(q_i, u)$ , and one may be interested in how equilibria vary with  $u$ . For convenience  $u$  will be suppressed from the notation.

The only constraints are that  $q_i \geq 0$ , so assuming the first-order conditions are sufficient (for example if  $\Pi_i$  is pseudo-concave), the equilibrium conditions are:

$$-\frac{\partial \Pi_i}{\partial q_i} \geq 0, \quad q_i \geq 0, \quad q_i \frac{\partial \Pi_i}{\partial q_i} = 0 \quad (47)$$

In other words this is a complementarity problem. Let  $I_1$ ,  $I_2$  and  $I_3$  have the same meaning as in the first subsection.

As seen in the first sub-section, an equilibrium  $q^* = (q_1^*, \dots, q_n^*)$  will be strongly regular if all matrices of the form  $(Df)_{J^c}$ , with  $f = (-\frac{\partial \Pi_1}{\partial q_1}, \dots, -\frac{\partial \Pi_n}{\partial q_n})$ ,

have the same sign. For convenience, abbreviate these as  $D_{J^c}$ . The elements of  $D_{J^c}$  have the form:

$$\begin{aligned} D_{ii} &= -\frac{\partial^2 \Pi_i}{\partial q_i^2} = -P''(Q)q_i - 2P'(Q) + C''(q_i) \\ D_{ij} &= -\frac{\partial^2 \Pi_i}{\partial q_i \partial q_j} = -P''(Q)q_i - P'(Q) \quad i \neq j \end{aligned} \quad (48)$$

It is straightforward to check (see for example Kolstad and Mathiesen (1987) Theorem 3) that

$$\det(D_{J^c}) = \left\{ 1 - \sum_{j \in J^c} \left[ \frac{P'(Q^*) + q_j^* P''(Q^*)}{C_j''(q_j^*) - P'(Q^*)} \right] \right\} \sum_{j \in J^c} [C_j''(q_j^*) - P'(Q^*)] \quad (49)$$

If one assumes that  $C_j''(q_j^*) > P'(Q^*)$  for all  $j$ , which is certainly true if costs are convex and demand downward-sloping, then

$$\text{sign } \det(D_{J^c}) = \text{sign} \left\{ 1 - \sum_{j \in J^c} \left[ \frac{P'(Q^*) + q_j^* P''(Q^*)}{C_j''(q_j^*) - P'(Q^*)} \right] \right\} \quad (50)$$

If  $q_j^* = 0$ , the term corresponding to  $j$  in the inner sum is simply  $-P'(Q^*)/(C_j''(q_j^*) - P'(Q^*))$ . Hence if demand is downward sloping, this term is positive. It follows that if

$$\left\{ 1 - \sum_{j \in I_1} \left[ \frac{P'(Q^*) + q_j^* P''(Q^*)}{C_j''(q_j^*) - P'(Q^*)} \right] \right\} > 0 \quad (51)$$

then the equilibrium is strongly regular. In other words one need only check the determinant condition with respect to the  $q_j$  which are strictly positive at equilibrium.

Kolstad and Mathiesen (1987) prove uniqueness of equilibrium if (51) holds everywhere under the assumption that  $I_3$  is empty in all equilibria, that is  $q_j^* = 0$  implies  $\partial \Pi_j / \partial q_j < 0$ . They refer to this as non-degeneracy. Gaudet and Salant (1991) prove uniqueness under this assumption without assuming non-degeneracy. The result here shows that even if it does not hold everywhere, any equilibria at which (51) holds are strongly regular, that is they are and remain locally unique and vary in a Lipschitz manner with respect to the perturbations,  $u$ . In particular, behavior does not change dramatically when shifts in cost or demand cause some firms' outputs to fall to zero.

With a rich enough set of perturbations non-degeneracy is generic in this one-shot game. In broader contexts this need not be so. For example Gaudet and Salant (1991) give an example of a two-stage game with Cournot competition at the second stage where a degenerate equilibrium is always played at the second stage in every equilibrium of the overall game. Degenerate situations are therefore of interest.

### 5.5 General Equilibrium

As noted in Example 2 the price equilibrium conditions for a pure exchange economy are an example of a complementarity problem:

$$p_i^* \geq 0, z_i(p^*) \leq 0 \quad \forall i, \quad z_i(p) = 0 \quad \text{if } p_i > 0$$

It follows that if the Jacobian matrix  $J = \left(-\frac{\partial z_i}{\partial p_j}\right)$  is a P-matrix at an equilibrium  $p^*$  then  $p^*$  is strongly regular even if some prices are zero. If the Jacobian matrix is a P-matrix everywhere then equilibrium is unique — see for example McKenzie (2002) — but this is not required here. The interest of this results is perhaps limited as it is usually assumed that equilibria cannot occur on the boundary and excess demand may be ill-defined if some prices are zero.

Of more interest is the application to the model of general equilibrium with production of Kehoe (1980). Kehoe (1980) considers general equilibrium models where  $\xi(p)$  represents consumer excess demand and there is  $n \times l$  matrix  $A$  of linear production activities.<sup>15</sup> These activities include the free disposal activities, that is minus the  $n$ -dimensional identity matrix is a sub-matrix of  $A$ .

An equilibrium for such an economy consists of a price vector  $p^*$  such that

- (a)  $p^{*'} A \leq 0$ ,
- (b) there exists a vector  $y^* \geq 0$  such that  $\xi(p^*) = Ay^*$ ,
- (c)  $\sum_i p_i^* = 1$

(a) requires that each activity make non-negative profit, (b) requires that supply equal demand and (c) is the standard normalization condition, innocuous since demand is homogeneous of degree zero.

Let  $S_A = \{p : \sum_i p_i = 1, p' A \leq 0\}$ . Under a boundedness assumption (no outputs without inputs),  $S_A$  is non-empty. Kehoe (1980) shows that the equilibria of this economy are fixed points of the map  $g : S_A \rightarrow S_A$  given by  $g(p) = \pi_{S_A}(p + \xi(p))$ . From (5), this implies that  $p^*$  is an equilibrium price vector if and only if  $\xi(p^*) \in N_{S_A}(p^*)$ , so this is again an example of a variational inequality problem.<sup>16</sup>

Kehoe (1980) assumes that

1. the columns of  $A$  are linearly independent,
2. at any equilibrium any activities which earn zero profits are used at strictly positive levels.

<sup>15</sup>  $A$  is used as a symbol for the activity matrix as is conventional. No confusion should result with that in the definition of  $X$ .

<sup>16</sup> This also follows from Lemma 2

Under these assumptions if  $B(p^*)$  denotes the sub-matrix of  $A$  corresponding to the active processes,  $e$  an  $n$ -vector of 1's, and  $C = [e : B(p^*)]$ , then Kehoe (1980) defines an equilibrium to be regular<sup>17</sup> if the determinant:

$$\begin{vmatrix} -D\xi(p^*) & C \\ -C' & 0 \end{vmatrix} \quad (52)$$

is non-zero.

To relate this to the previous section, note that at an equilibrium

$$\xi(p^*) = \sum y_j a_j$$

where  $y_j$  are the activity levels of the various processes or equivalently Lagrange multipliers for the Kuhn-Tucker formulation of the variational inequality  $\xi(p^*) \in N_{S_A}(p^*)$  — see Section 4.  $a_j$  are the columns of  $A$ .<sup>18</sup> Let  $I$  denote the set of activities which make zero profits. Let  $\alpha$  denote the set of activities which make zero profits and  $y_i > 0$  and let  $\beta$  denote the remaining activities in  $I$ . Condition (2) above requires that  $\beta$  be empty. The analysis of the previous section allows one to extend the analysis if this condition does not hold.

Provided

$$\begin{vmatrix} -D\xi(p^*) & C_\gamma \\ -C'_\gamma & 0 \end{vmatrix} \quad (53)$$

has the same non-zero sign for all  $\alpha \subseteq \gamma \subseteq I$ , where  $C_\gamma$  denotes the matrix composed of the corresponding activities, augmented by the vector  $e$  as the constraint  $e'p = 1$  always binds, then  $p^*$  is strongly regular.

Kehoe (1980) shows that with sufficient perturbations assumption (b) is generic but these results allow the analysis to be applied without invoking genericity arguments.

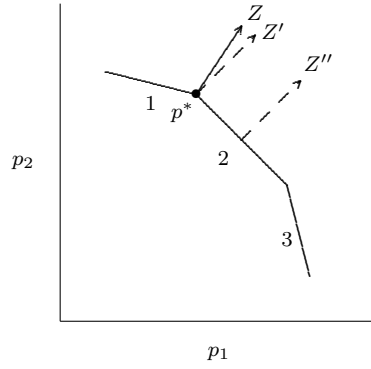
As an elementary example, suppose that there are two consumption goods and labor with joint production, so that each of the finite number of activities  $a_1, \dots, a_l$  produces the two consumption goods in fixed proportions using labor. A typical activity vector has the form  $a_j = (a_{1j}, a_{2j}, -1)'$ , where the final component denotes labor. For convenience normalize the wage to 1 rather than use the normalization above. The break-even constraints are then

$$p_1 a_{1j} + p_2 a_{2j} \leq 1 \quad , = 1 \text{ if } j \text{ is used} \quad (54)$$

This is illustrated in Fig. 7. There are three activities 1, 2 and 3. On each face only one activity, corresponding to the label, breaks even. At the vertices more than one activity is used. A typical equilibrium demand vector,  $Z$ , for goods is shown. It is a linear combination of the currently active activities and is produced using activities 1 and 2. Equivalently the projection of  $p^* + Z(p^*)$  on the set of feasible price vectors,  $S_A$ , equals  $p^*$ .

<sup>17</sup> He gives a different definition but the calculation on p. 1225 of Kehoe (1980) shows the condition below is equivalent. Mas-Colell (1985) Chapter 6 calls such an equilibrium properly regular.

<sup>18</sup>  $e$  does not appear in the expression on account of Walras' law.



**Fig. 7** Activity analysis example

As demands change eventually one activity will cease to be used but just break even. This is illustrated by the demand vector  $Z'$  which uses only activity 2 but at the prices shown activity 1 just breaks even. If (53) is satisfied then equilibrium prices will change in a Lipschitz-continuous manner as demand is perturbed: if  $Z'$  is perturbed further then only activity 2 will be used and prices will adjust so equilibrium will occur for example at  $Z''$ . Eventually if demand is perturbed enough the next vertex will be reached and activity 3 will become active.

To interpret the condition on (53), note that it will hold at  $p^*$  if the law of demand holds on each of the affine subspaces  $p^* + \{v : C'_\gamma v = 0\}$ . That is if prices are adjusted so that the activities in  $\gamma$  continue to break even then demand moves in the opposite direction to prices on average — see for example Mas-Colell (1985) chapter 2.9. More precisely if  $v'D\xi(p^*)v < 0$  for all non-zero  $v$  with  $C'_\gamma v = 0$ , which is a differential form of the law of demand, then (53) has a positive determinant (see Mas-Colell (1985) 1.B.5.2 and section 1.B.6). In other words own effects should again outweigh cross-effects.

## 5.6 Kuhn-Tucker and Consumer Demand

The Kuhn-Tucker first-order conditions for minimizing  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  subject to  $g_i(x) \leq 0$ ,  $i = 1, \dots, s$ ,  $g_i(x) = 0$ ,  $i = s + 1, \dots, m$  are with  $L = f(x) + y'g(x)$

$$\begin{aligned} L_{x_j} &= 0 & j &= 1, \dots, n \\ L_{y_j} &\leq 0 & \text{if } y_j = 0 \text{ and } 1 \leq j \leq s \\ L_{y_j} &= 0 & \text{otherwise } j &= 1, \dots, m \end{aligned} \quad (55)$$

Let  $x^*$  and  $y^*$  be a point satisfying these conditions. Assume for simplicity the constraint qualification that the gradients  $Dg_i$  with respect to  $x$  of the binding constraints at  $x^*$  are linearly independent holds, so that the conditions are necessary. Assume also that  $f$  and  $g_i$ ,  $i = 1, \dots, n$  are  $C^2$  functions of  $x$  and parameters  $u$ .

If one assumes strict complementary slackness:  $g_i(x^*) = 0 \Rightarrow y_i > 0$  then one can use the standard implicit function theorem to perform comparative statics. The framework above allows one to relax this condition.

(55) is a complementarity problem for the function  $h = (L_x, -L_y)$  on  $\mathbb{R}^n \times \mathbb{R}_+^s \times \mathbb{R}^{m-s}$ , so from the discussion earlier in the section  $x^*$  and  $y^*$  are locally unique Lipschitz continuous functions of  $u$  provided the determinant

$$\begin{vmatrix} D^2L_{xx} & (Dg_x^I)' \\ -(Dg_x^I) & 0 \end{vmatrix} \quad (56)$$

has the same non-zero sign for all subsets  $I \supset J$  of the binding constraints, where  $J \supseteq \{s+1, m\}$  is the set of equality constraints and the set of binding inequality constraints with  $y_i > 0$  and  $g^I$  denotes the constraints corresponding to the indices in  $I$ .

Such solutions are stationary points of the Lagrangian. A sufficient condition for  $x^*$  to be a local minimum is that<sup>19</sup>

$$w'D^2L_{xx}w > 0 \quad \text{for all} \quad w \in E = \{w : Dg_x^J w = 0\} \quad (57)$$

That is  $L_{xx}$  is strictly positive definite on  $E$ .

(57) implies that (56) is positive for all  $I$  since if  $DL_{xx}$  is strictly positive definite on  $E$  it is strictly positive definite on every subspace, which implies the result by the usual bordered Hessian test. Moreover if (57) holds at  $x = x^*$  and  $u = u^*$  it holds for values of  $x$  and  $u$  near by. Hence to summarize<sup>20</sup>:

**Lemma 3** *If (57) and the constraint qualification that the binding constraints are independent at  $x^*$  hold, then there is a Lipschitz-continuous mapping  $x(u)$  in a neighborhood of  $u^*$  with  $x(u^*) = x^*$  such that  $x(u)$  is a local minimum.*

(57) is a little stronger the standard second-order condition, which would require that  $L_{xx}$  be strictly positive-definite on the smaller space  $E' = \{w : Dg^J w = 0, D^{J'} w \leq 0\}$ , where  $J'$  denotes the set of binding constraints with zero multiplier, rather than on  $E$ .<sup>21</sup> As shown by Robinson (1980), this condition is not strong enough if one is interested in perturbations to the system.

(57) holds automatically if  $L_{xx}$  is differentiable strictly convex. When the constraints are linear  $D^2L$  reduces to  $D^2f$  and the condition holds automatically if  $f$  is differentiable strictly convex or indeed merely differentiable strictly quasi-convex.<sup>22</sup>

The results above can be applied to consumer demand. Let  $u$  be a  $C^2$  utility function defined over goods  $x = (x_1, \dots, x_n)$  and let the consumer face prices  $p$  and have budget  $b$ . Let  $g_1(x, p)$  be the budget constraint,  $p'x \leq b$ , and the other  $g_i$  be additional linear constraints  $c'_i x \leq d_i$ . The latter may represent for example

<sup>19</sup> See for example Ruszczynski (2006) p. 149 after the proof of Theorem 3.47.

<sup>20</sup> See Dontchev and Rockafellar (2009) Theorem 2.G.3 or Robinson (1980) Theorem 4.1 for proofs

<sup>21</sup> See for example Theorem 4 in Chapter 2 of Fiacco and McCormick (1990).

<sup>22</sup>  $f$  is differentiable strictly quasi-convex if the Hessian of  $f$  is positive definite on the null space of  $Df(x)$  for all  $x$ . The Kuhn-Tucker conditions imply that the null space of  $Dg^J$  is contained in that of  $f$ .

- Non-negativity constraints:  $x_l \geq 0$ ,
- Rationing:  $x_l \leq K_l$ , for some  $K_l$ .
- Time constraints (cf. Becker (1965)),  $\sum_l t_l x_l \leq T$ , where  $t_l$  is the time taken to consume a unit of the  $l$ th commodity and  $T$  total time available.

Assume the gradients of the constraints are linearly independent at  $x^*$  and  $p^*$ , as is the case if  $p^*$  is strictly positive and the additional constraints are non-negativity constraints. If  $u$  is differentiable strictly concave in the neighbourhood of  $x^*$ , then the results above imply that  $x^*$  is locally Lipschitz in prices. The method can be generalized to apply to the case of non-linear constraints.

Mas-Colell (1985), in the case of non-negativity constraints, and Bonnisseau and Rivera-Cayupi (2006), for general linear constraints, derive this result by different means assuming that  $u$  is globally strictly quasi-concave.<sup>23</sup> The following well-known result appears, for example, in Mas-Colell (1985) Chapter 1:

*Let  $k_j$ ,  $j \in J$ , be a finite collection of real-valued  $C^1$  functions defined on an open set  $\Omega$ . Let  $x$  be a continuous function which at each  $\omega \in \Omega$  takes the value of some  $k_j(\omega)$ . Then  $x$  is Lipschitz-continuous.*

Mas-Colell (1985) uses this result to prove that demand is Lipschitz-continuous in prices if preferences are  $C^2$ , strictly monotone and strictly quasi-concave. Under the latter assumption demand,  $x$ , is unique and so by Berge's maximum theorem continuous. Applying the ordinary implicit function theorem and  $C^2$  quasi-concavity to each possible subspace where some consumptions are fixed at zero yields the desired family  $k_j$ . A similar approach can be applied in other optimization problems.

The current approach is not restricted to optimization problems. In the context of optimization problems it is also useful when one is interested in stationary points, which may not be local or global optima. In the case of equilibrium problems to apply the result above one would need to show that equilibria are locally continuous and unique functions of parameters, which is exactly what Theorem 1 guarantees.

## 6 Discussion

The paper has considered only the case where equilibrium is constrained to lie in a fixed set  $X$  described by linear inequalities. This seems sufficient for most economic applications. Note that nonlinearly constrained optimization problems fall within this framework (see Section 3 and Section 5).

Facchinei and Pang (2003) present some results extending the analysis to the case where  $X$  is described by non-linear inequalities using a Lagrangian approach. Under some additional constraint qualifications results similar to those presented here can be developed.

<sup>23</sup> Bonnisseau and Rivera-Cayupi (2006) give some conditions under which the gradients of the constraints are independent.

Lu and Robinson (2008) consider the case when  $X$  is described by linear inequalities  $X = \{x : Ax \leq b\}$  but the right hand-side vectors  $b$  are subject to perturbation. This problem is more complicated as the nature of the polyhedron may change as  $b$  changes — in particular faces may appear and disappear. They develop similar results to those here taking this into account.

## 7 Conclusion

This paper has developed some simple results for the persistence of equilibria in the presence of boundaries using results from the mathematical programming literature. As shown these have applications in a number of areas, for example search equilibrium, Cournot oligopoly and general equilibrium. More general results can be developed but these seem sufficient for many applications.

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