ISSN 1471-0498

DEPARTMENT OF ECONOMICS

DISCUSSION PAPER SERIES

Eliciting the Just-Noticeable Difference

Paweł Dziewulski

Number 798
First Published: October, 2016
Revised: June 2017

Manor Road Building, Oxford OX1 3UQ

Eliciting the just-noticeable difference*

Paweł Dziewulski†

June 2017

Abstract

The evidence from psychophysics suggest that people are unable to discriminate between alternatives unless the options are significantly different. Since this assumption implies non-transitive indifferences, it can not be reconciled with utility maximisation. We provide a method of eliciting consumer preference from observable choices when the agent is incapable of discerning between similar bundles. It is well-known that the issue of noticeable differences can be modelled with *semiorder* maximisation. We introduce a necessary and sufficient condition under which a finite dataset of consumption bundles and corresponding budget sets can be rationalised with such a relation. The result can be thought of as an extension of Afriat's (1967) theorem to semiorders, rather than utility optimisation. Our approach is constructive and allows us to infer the just-noticeable difference that is sufficient for the agent to differentiate between bundles as well as the "true" preferences of the consumer (i.e., as if perfect discrimination were possible). Furthermore, we argue that the former constitutes a natural measure of how well the preference revealed in the data could be approximated by a weak order. We conclude by applying our test to household-level scanner panel data of food expenditures.

Keywords: revealed preference, testable restrictions, semiorder, just-noticeable difference, GARP, Afriat's efficiency index, money-pump index
JEL Classification: C14, C60, C61, D11, D12

*I am grateful to John Quah for his invaluable help during writing this paper. Moreover, I would like to thank Abi Adams, Miguel Ballester, Laurens Cherchye, Ian Crawford, Federico Echenique, John Hey, Ole Jann, Josh Lanier, Paulo Natenzon, Ludovic Renou, Bram de Rock, David Ronayne, Bart Smeulders, and André Veiga for helpful comments and suggestions. Finally, I am indebted to Chris Chambers for the encouragement and early discussions on this project.

†Department of Economics, University of Oxford, Manor Road Building, Manor Road, Oxford OX1 3UQ, United Kingdom. Email: pawel.dziewulski@economics.ox.ac.uk.

1 Introduction

The evidence from psychophysics suggest that people are unable to discriminate between two physical stimuli unless there is a significant difference between their intensities.[1] A similar idea may be applied to the problem of consumer choice. Referring to the classical thought experiment by Luce (1956), suppose an agent is presented with two cups of coffee containing n and $(n+1)$ grains of sugar, respectively. Given that the two commodities are indistinguishable, we would expect the consumer to behave as if she were indifferent between the options, for all n, regardless of the actual preference for sugar. Since utility representation of preferences requires for the indifference relation to be transitive, such an observation can not be explained in the above sense unless we assume that the consumer is indifferent between any two amounts of sugar in her coffee. However, such an approach is neither realistic nor useful. Specifically, the revealed indifference between the cups of coffee need not exhibit the actual preference over the commodities; rather, it might be a result of the consumer's inability to discern between the two objects.

The aim of this paper is to provide a method that allows to infer consumer preference from the observable choices when the agent is unable to discriminate perfectly between similar bundles. Moreover, we are interested in eliciting the just-noticeable difference that is sufficient for the decision maker to distinguish between options.

Following Luce (1956) or Beja and Gilboa (1992), we address the problem of noticeable differences by modelling consumer choice using *semiorders*. The formal definition of this notion is postponed until Section 2.[2] However, a semiorder can be thought of as an asymmetric binary relation P such that aPb if and only if $u(a) > u(b) + 1$, for some utility function u. Thus, option a is strictly preferred to b if and only if it yields a sufficiently higher utility than b. In particular, any semiorder P is inherently related to a weak order induced by the utility function u. Additionally, as its symmetric counterpart I is given by: aIb if and only if $\big|u(a) - u(b)\big| \leq 1$, the latter relation is *not* transitive.

Our main results are presented in Section 3. We begin by determining the testable implications for semiorder maximisation. Suppose we observe an agent making a choice from ℓ goods. Following the tradition of Afriat (1967), an observation consists of a budget

[1] See, e.g., Laming (1997) or a short survey in Algom (2001) .

[2] See also the discussion in Luce (1956), Scott and Suppes (1958), Fishburn (1975), Beja and Gilboa (1992), Gilboa and Lapson (1995), or more recent Argenziano and Gilboa (2015, 2017).

set $B \subseteq \mathbb{R}^\ell_+$ and a bundle $x \in B$ selected from it. We say that a semiorder P rationalises a finite set of observations $\mathcal{O}$, whenever

$$(B, x) \in \mathcal{O} \quad \text{and} \quad y \in B \quad \text{implies not} \quad yPx.$$

Clearly, without any further restrictions, the above notion has no empirical content, as an arbitrary set of observations can be trivially rationalised with the semiorder $P = \emptyset$.[3] In order to obtain any observable implications of the model, we restrict our attention to a class of semiorders P for which there exists a number $\lambda > 1$ such that

$$\lambda' \geq \lambda \quad \text{implies} \quad (\lambda' a)Pa, \tag{1}$$

for all non-zero $a \in \mathbb{R}^\ell_+$. This condition is inspired by the so-called *Weber-Fechner law* which postulates that people can not discriminate between two physical stimuli, unless the ratio between their magnitudes exceeds a particular value — a Weber's constant.[4] In our specification, the parameter λ may be interpreted as the ratio between sizes of two bundles that is sufficient for the agent to correctly discern between the options. One example of such preferences is discussed in Argenziano and Gilboa (2017), where the authors characterise semiorders P that satisfy: $(a_i, c_{-i})P(b_i, c_{-i})$ if and only if $a_i/b_i > \delta_i$, for some predetermined $\delta_i > 1$, for all $i = 1, \ldots, \ell$. That is, the agent prefers one bundle over another if the ratio between volumes of at least one commodity i exceeds the constant δ_i. In particular, Argenziano and Gilboa show that such relations can be represented with a Cobb-Douglas utility function u, in the sense specified previously. Clearly, such preferences satisfy condition (1) for any $\lambda \geq \min\{\delta_i : i = 1, \ldots, \ell\}$.

We find condition (1) to be convenient for our analysis. Apart from allowing us to formalise a notion of noticeable differences, the above class of semiorders is appealing for empirical applications. Specifically, by restricting our attention to such relations, we find it possible to construct computationally efficient methods that determine whether a set of observations is rationalisable by a semiorder P satisfying the restriction for some $\lambda > 1$. Nevertheless, in Section 3 of the Online appendix we extend our analysis to other classes of semiorders that exhibit alternative properties.

[3]This is to say that aIb, for all $a, b \in \mathbb{R}^\ell_+$. An analogous issues arises in the case of utility maximisation. Clearly, any set of observations can be rationalised by a constant utility function u. Unless we impose some additional restrictions on u, e.g., local non-satiation, this model has no testable implications.

[4]Although *Stevens' power law* seems to better explain sensory discrimination than Weber-Fechner's law, the latter is considered to be the best first approximation. See, e.g., Algom (2001).

In Axiom 1 we present a necessary and sufficient condition under which a dataset $\mathcal{O}$ is rationalisable by a semiorder P satisfying property (1) for some $\lambda > 1$. We summarise this observation in the Main Theorem. The restriction has a structure similar to the generalised axiom of revealed preference, or GARP for short, introduced in Afriat (1967) and Varian (1982). GARP requires for the dataset to admit no revealed preference cycles, where bundle x is revealed preferred to y if $(B, x) \in \mathcal{O}$ and $y \in B$. It is well-established that this is both a necessary and sufficient condition for the set of observations to be rationalisable by a locally non-satiated utility function. Analogously, the purpose of our condition is to exclude a particular type of cycles from the dataset. However, the class of sequences admissible by our restriction is broader than in the case of GARP. In fact, we argue that $\mathcal{O}$ satisfies GARP if and only it is rationalisable by a semiorder satisfying property (1) for *any* $\lambda > 1$. Thus, the condition is a special case of Axiom 1.

The proof of our result is constructive and provides a direct method of formulating a semiorder P with the corresponding utility function u that rationalise the dataset in the aforementioned sense. However, since the number of observations is finite, such orderings are not unique. In particular, they need not coincide with the semiorder and the utility that originally generated the data. We address this issue in Section 3.4, where we show how to recover (partially) such objects. Specifically, we introduce two revealed preference relations P^* and $\succeq^*$ that are consistent with any semiorder P and the corresponding utility u that generated the observations, respectively. Once we interpret u as the "true" tastes of the agent, our method allows us to elicit partially the preferences of the consumer as if the perfect discrimination between bundles were possible.

Representing consumer choice with a semiorder can be interpreted as a utility maximisation with a particular type of error. In Section 4 we show that, whenever a set of observations $\mathcal{O}$ is rationalisable by a semiorder P satisfying condition (1) for some $\lambda > 1$, there is a utility function u such that, for any observation (B, x) in $\mathcal{O}$:

$$u(z) \geq u(x) \text{ implies } u(\lambda z) > u(y), \text{ for all } y \in B \text{ and non-zero } z \in \mathbb{R}^{\ell}_{+}.$$

That is, even though the consumer determines the choice using her utility u, she is making a consistent error in evaluating which bundles are superior to x. Although there might be an available option y such that $u(y) > u(x)$, the choice x remains optimal unless there is some z such that $u(y) \geq u(\lambda z)$ and $u(z) \geq u(x)$. That is, the indifference curves intersecting bundles y and x must be sufficiently far apart for y to be chosen over x.

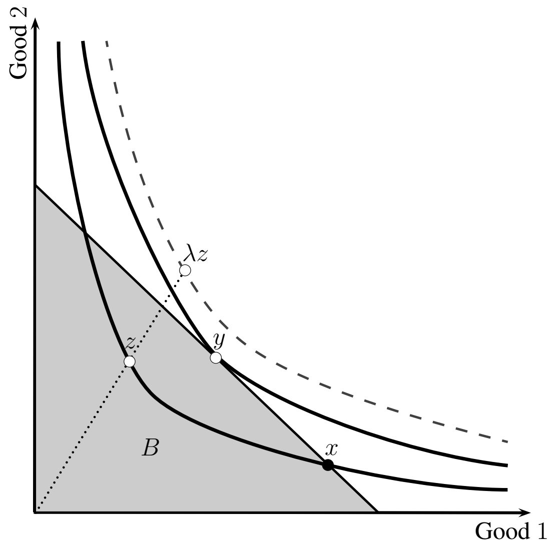

Figure 1: The dashed line depicts set $\{\lambda z : u(z) = u(x)\}$. Although $y \in B$ and $u(y) > u(x)$, we have $u(\lambda z) > u(y)$, for all z with $u(z) \geq u(x)$. Thus, bundle x is optimal.

Consider the example in Figure 1. In fact, the opposite implication is also true. That is, if there is a utility function u rationalising the set of observations in the above sense for some $\lambda > 1$, the data can be supported by a semiorder P' satisfying property (1) for λ. Therefore, the two models of consumer choice are empirically equivalent.

Our results induce a natural index of revealed preference violations. Let λ_* denote the infimum over all numbers $\lambda > 1$ for which a dataset $\mathcal{O}$ satisfies Axiom 1 — henceforth, the *just-noticeable difference*. We show in Section 5 that this parameter, or rather its inverse $1/\lambda_*$, constitutes an informative measure of how severely the set of observations violates GARP. Additionally, it can be evaluated using computationally efficient methods, which makes it convenient for empirical implementations. We argue that this approach combines the desirable properties of Afriat's inefficiency index and the money-pump index, while responding to the main criticism pointed at those two measures. Since the just-noticeable difference is more sensitive to changes in the dataset than the former, it is immune to the critique by Echenique, Lee, and Shum (2011). On the other hand, unlike the latter, it provides an alternative model of consumer choice rationalising the data whenever $\mathcal{O}$ fails GARP. We conclude the section by introducing a procedure based on our measure that allows us to determine if a violation of GARP is significant.

We are not the first to analyse consumer choice under semiorder maximisation. Fishburn (1975) characterises such relations in terms of choice correspondences. The author imposes conditions on correspondence $C(B) := \{x \in B : \text{ if } y \in B \text{ then not } yPx\}$, defined over all subsets B of the domain, that are both necessary and sufficient for the relation P to be a semiorder. The axiomatisation crucially depends on the fact that the researcher observes the value of the correspondence C for *any* subset B of the domain and that $C(B)$ contains *all* choices from the set B. In contrast, we follow Afriat (1967) by assuming that the observer gathers only partial information about C, by monitoring a single choice from each of a finite collection of budget sets.

We conclude the paper with an empirical illustration of our methods. In Section 6 we apply our test to a household-level panel data containing time series of food grocery purchases, collected at checkout scanners in supermarkets. We elicit the value of the just-noticeable difference for 396 households whose choices violated GARP, out of 494 available in the dataset, and find that the median value of λ_* is approximately 1.07. Hence, a 7% increase in the bundle size is sufficient for the consumer to notice the difference.

Proofs of the results not displayed in the main body of the article are moved to the Appendix. The Online appendix provides additional discussion and extensions.

2 Semiorders

A *semiorder* over a set of alternatives X is a binary relation P that satisfies the following three properties: (S1) it is *irreflexive*, i.e., it is not true that aPa; (S2) it satisfies the *interval order* condition, i.e., if aPb and cPd then aPd or cPb; and (S3) it is *semitransitive*, i.e., aPb and bPc imply aPd or dPc; for all a, b, c, and d in X. In particular, the three conditions guarantee that any semiorder is both transitive and asymmetric.[5] Throughout this paper, we interpret P as the *strict preference*. Alternatively, if neither aPb nor bPa, we say that a is *indifferent* to b and denote it by aIb. Clearly, the later relation is both reflexive and symmetric, but not transitive. We denote aRb whenever aPb or aIb.

The notion of semiorders is inherently related to weak orders. Recall that a *weak order* over X is a complete, reflexive, and transitive binary relation $\succeq$, with its symmetric and asymmetric parts denoted by $\sim$ and $\succ$, respectively. Theorem 1 in Luce (1956) guarantees that any semiorder P induces the following weak order: Let $a \succ b$ whenever there is some c such that either (W1) aPc and cIb; or (W2) aIc and cPb. Specifically, if aPb then $a \succ b$. Moreover, we set $a \sim b$ if neither $a \succ b$ nor $b \succ a$. In the remainder of our paper,

[5] Property (S3) implies that aPb and bPc only if aPa or aPc. As P is irreflexive, it must be that aPc. Thus, relation P is transitive. By (S1) it has to be asymmetric.

we say that such a weak order is *induced* by the semiorder P.

Even though the indifference relation I is not transitive, it does convey some information about the semiorder P and the induced weak order $\succsim$. The following property plays a significant role in our main argument presented in Section 3. As the proof of this result is rather extensive, we provide it the Appendix.

Proposition 1. *For any finite sequence $\{(x_{i+1}, x_i)\}_{i=1}^m$ with $x_{i+1}Rx_i$, for all $i = 1, \ldots, m$, let $\mathcal{P}$ and $\mathcal{I}$ denote sets of pairs (x_{i+1}, x_i) such that $x_{i+1}Px_i$ and $x_{i+1}Ix_i$, respectively. Whenever $|\mathcal{P}| \geq |\mathcal{I}|$ then $x_{m+1} \succ x_1$, while $|\mathcal{P}| > |\mathcal{I}|$ implies $x_{m+1}Px_1$.*

The above proposition implies that, for any sequence x_1, x_2, ..., x_m, x_{m+1} of elements in X such that $x_{i+1}Rx_i$, for all $i = 1, \ldots, m$, whenever the number of strict comparisons P is strictly greater than the number of indifferences I, then $x_{m+1}Px_1$. Moreover, if the number of strict comparisons is equal to the number of indifferences, then $x_1 \succ x_{m+1}$, where $\succsim$ is the weak order induced by P. This property proves to be crucial for determining the testable implications of the model under consideration.

An important characteristic of semiorders concerns their utility representation. Following the result by Scott and Suppes (1958), for any semiorder P over a finite domain X there exists a function $u : X \to \mathbb{R}$ such that

$$aPb \ \text{ if and only if } \ u(a) > u(b) + 1. \tag{2}$$

This implies that aRb is equivalent to $u(a) \geq u(b) - 1$, while the weak order $\succsim$ induced by P satisfies $a \succsim b$ if and only if $u(a) \geq u(b)$, without loss of generality.[6] In addition, under some regularity conditions, the representation result can be extended to semiorders defined over uncountable spaces. See Beja and Gilboa (1992) for details.

3 Revealed noticeable difference

A *set of observations* $\mathcal{O} = \{(B_t, x_t)\}_{t\in T}$ consists of a finite number of pairs (B_t, x_t) of budget sets $B_t \subseteq \mathbb{R}^\ell_+$ and bundles $x_t \in B_t$ selected from the corresponding sets, for all $t \in T$.[7] In this section we provide the testable restrictions for the model of consumer

[6] This is to say that, in general, function u satisfying condition (2) does not represent the induced weak order $\succsim$. However, there always exists a function u for which this is true.

[7] We endow space $\mathbb{R}^\ell_+$ with the natural product order $\geq$. This is to say that $x \geq y$ if and only if $x^i \geq y^i$, for all $i = 1, 2, \ldots, \ell$. Moreover, we denote $x \gg y$ whenever $x^i > y^i$, for all $i = 1, 2, \ldots, \ell$.

choice with a *noticeable difference* λ. That is, we determine a tight condition on set $\mathcal{O}$ under which there is a semiorder P and some $\lambda > 1$ such that, for all $t \in T$:

$$\text{if } \; y \in B_t \; \text{ then it is not true that } \; yPx_t,$$

while $\lambda' \geq \lambda$ implies $(\lambda' a)Pa$, for all non-zero $a \in \mathbb{R}^\ell_+$. Before we proceed with our main result, we introduce the framework in which we perform our analysis.

3.1 Preliminaries

Throughout the remainder of the paper we consider consumer choices over *generalised budget sets*, defined as in Forges and Minelli (2009). Specifically, we require sets B_t to be compact and downward comprehensive.[8] Additionally, we assume that there is some $y \in B_t$ such that $y \gg 0$. With a slight abuse of the notation, let

$$\partial B_t := \big\{ y \in B_t : \text{ if } z \gg y \text{ then } z \notin B_t \big\}$$

denote the *upper bound* of the set B_t. Suppose that, for any vector $y \in \partial B_t$ and scalar $\theta \in [0, 1)$, we have $\theta y \in B_t \setminus \partial B_t$. That is, for any element $y \in \mathbb{R}^\ell_+$, ray $\big\{\theta y : \theta \geq 0\big\}$ intersects the boundary ∂B_t exactly once. Finally, we assume that $x_t \neq 0$, for all $t \in T$.[9] It is straightforward to verify that this framework admits linear budget sets, considered in the original work by Afriat (1967) and Varian (1982).[10]

Even though originally the above authors formulated the *generalised axiom of revealed preference*, or GARP for short, for linear budget sets, it is possible to extend the condition to our general framework. Following Forges and Minelli (2009), a set of observations $\mathcal{O}$ obeys GARP if for any sequence $\mathcal{C} = \big\{(a, b), (b, c), \dots, (z, a)\big\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t, s) \in \mathcal{C}$, we have $x_s \in \partial B_t$, for all $(t, s) \in \mathcal{C}$. That is, bundle x_s belongs to the upper bound of set B_t, for any pair (t, s) in the cycle. Specifically, this concerns all one-element sequences. Hence, we have $x_t \in \partial B_t$, for all $t \in T$.

Proposition 3 in Forges and Minelli (2009) guarantees that a dataset $\mathcal{O}$ satisfies GARP if and only if it is rationalisable by a locally non-satiated weak order. That is, the condition is both necessary and sufficient for the existence of a locally non-satiated weak

[8] Set B_t is *downward comprehensive* if $y \in B_t$ and $z \leq y$ imply $z \in B_t$, for all $z \in \mathbb{R}^\ell_+$.

[9] This assumption is not without loss of generality. However, it substantially simplifies our analysis and seems to be insignificant from the empirical point of view.

[10] The authors consider budget sets $B_t := \big\{ y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t \big\}$, some $p_t \in \mathbb{R}^\ell_{++}$, for all $t \in T$.

order $\succeq$ such that $y \in B_t$ implies $x_t \succeq y$, for all $t \in T$. Moreover, without loss of generality, the relation can be represented by a continuous and increasing utility function u that satisfies $u(\lambda' a) > u(a)$, for all non-zero $a \in \mathbb{R}^\ell_+$ and $\lambda' > 1$. This extends the original result by Afriat (1967) and Varian (1982).

3.2 The axiom

In this subsection we introduce and thoroughly discuss the condition imposed on a set of observations that is essential to our analysis. Consider the following axiom.

Axiom 1. *Dataset $\mathcal{O}$ satisfies the axiom for some $\lambda > 1$ if for an arbitrary sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$, we have*

$$- \sum_{(t,s)\in\mathcal{C}} k_{ts} \;<\; |\mathcal{C}|,$$

where we denote $k_{ts} := \inf\big\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\big\}$, for any t, $s \in T$.

To better understand the above restriction, consider a sequence $\mathcal{C}$ specified as in the thesis of the axiom. Note that, by assumptions imposed on set B_t, integer k_{ts} must be negative, for any index pair (t,s) that belongs to the cycle. In particular, $-k_{ts}$ determines the maximal number of times one could scale set B_t by $1/\lambda$ without excluding x_s from it. Equivalently, number $-k_{ts}$ is the greatest integer k such that bundle $(\lambda^k x_s)$ belongs to B_t. Observe that Axiom 1 requires for the sum of all such integers to be strictly less than $|\mathcal{C}|$, which is the total number of index pairs in the sequence. Equivalently, this is to say that the average value of $-k_{ts}$ must be strictly less than 1.

Alternatively, number $\lambda^{k_{ts}}$ is the least power of λ for which bundle x_s belongs to $\lambda^{k_{ts}} B_t$. Therefore, this value may serve as an approximation of the relative distance between x_s and the boundary of the budget set B_t. Notice that, for any cycle $\mathcal{C}$ specified as in Axiom 1, the previously stated restriction can be reformulated as

$$\sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \lambda^{k_{ts}}} \;>\; \lambda^{-1}. \tag{3}$$

Therefore, the geometric mean of such measures $\lambda^{k_{ts}}$, over any sequence $\mathcal{C}$, has to be strictly greater than the inverse of λ. In other words, our condition imposes a bound on the average approximate distance between bundle x_s and the boundary of set B_t, along any cycle. We discuss this condition further in Section 5.

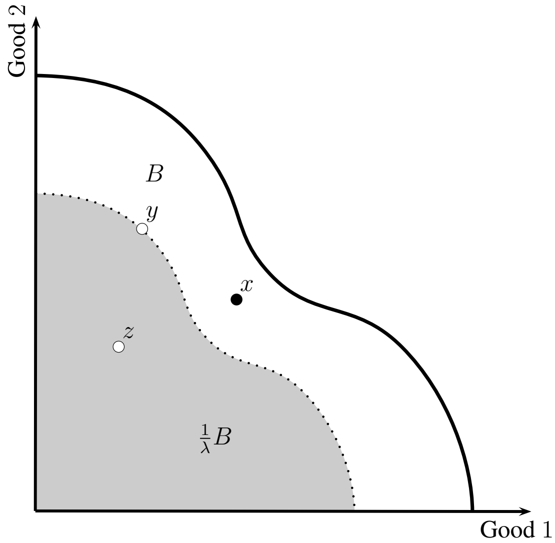

Figure 2: A singleton set $\mathcal{O} = \{(B, x)\}$ obeys Axiom 1 for $\lambda > 1$. However, since x does not belong to ∂B_t, it can not be supported by a locally non-satiated utility function. At the same time, sets $\{(B, y)\}$ and $\{(B, z)\}$ violate Axiom 1 for this value of λ.

Specifically, the property stated in Axiom 1 has to be satisfied by all one-element sequences. This requires for bundle x_t to be outside of the set $\lambda^{-1}B_t$, for all $t \in T$. Therefore, the least integer k for which the vector x_t is contained in $\lambda^k B_t$ must be equal to 0. Nevertheless, unlike GARP, the condition admits observations (B_t, x_t) in which x_t does not belong to the upper boundary of B_t. For instance, see Figure 2.

A closer look at Axiom 1 reveals three important properties of this restriction. First of all, for any set of observations $\mathcal{O}$ there is a sufficiently large number $\lambda > 1$ for which the set obeys the axiom. For example, take any number λ such that $1/\lambda$ is strictly less than $\inf\{\theta > 0 : x_s \in \theta B_t, \text{ for all } t, s \in T\}$. By construction, it must be that $x_s \notin \lambda^{-1}B_t$, for all t, $s \in T$, which implies that number k_{ts}, specified as in the thesis of Axiom 1, must be weakly greater than 0, for all t, $s \in T$. This suffices for $\mathcal{O}$ to satisfy the condition.

Second of all, Axiom 1 is monotone in λ. That is, whenever a dataset satisfies the restriction for some $\lambda > 1$, it obeys the condition for all $\lambda' \geq \lambda$. Take any sequence $\mathcal{C}$ specified in the axiom and construct the corresponding numbers k_{ts}, for all $(t, s) \in \mathcal{C}$ and some $\lambda > 1$. Moreover, define $k'_{ts} := \inf\{k \in \mathbb{Z} : x_s \in (\lambda')^k B_t\}$, for some $\lambda' \geq \lambda$ and all $(t, s) \in \mathcal{C}$. Clearly, since bundle x_s belongs to set B_t, for any pair (t, s) in the cycle, it must be that $k_{ts} \leq k'_{ts}$. Thus, for any such sequence

$$- \sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}| \quad \text{implies} \quad - \sum_{(t,s)\in\mathcal{C}} k'_{ts} < |\mathcal{C}|.$$

In particular, the set of observations satisfies Axiom 1 for λ only if it obeys the condition for λ'. Therefore, the focus of the researcher can be restricted solely to the least value of the parameter λ for which the property stated in the axiom holds.

Finally, our condition is weaker that the generalised axiom of revealed preference stated in preceding subsection. We formalise this claim below.

Proposition 2. *Set $\mathcal{O}$ obeys GARP if and only if it satisfies Axiom 1 for all $\lambda > 1$.*

Proof. Consider any cycle $\mathcal{C} = \{(a,b),(b,c),\ldots,(z,a)\}$ in $T \times T$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$. Whenever the set of observations obeys GARP, it must be that $x_s \in \partial B_t$, for all $(t,s) \in \mathcal{C}$, where ∂B_t denotes the upper boundary of set B_t, defined as in Section 3.1. This implies that the integer k_{ts} is equal to 0, for any pair (t,s) and $\lambda > 1$. Hence, we have $-\sum_{(t,s)\in\mathcal{C}} k_{ts} = 0 < |\mathcal{C}|$, which suffices for $\mathcal{O}$ to obey Axiom 1 for any $\lambda > 1$.

We prove the converse by contradiction. Suppose that $\mathcal{O}$ obeys Axiom 1 for all $\lambda > 1$, but fails to satisfy GARP. Therefore, there is some sequence $\mathcal{C}$, specified as above, and a pair $(t,s) \in \mathcal{C}$ such that $x_s \notin \partial B_t$. Equivalently, we have $\inf\{\theta > 0 : x_s \in \theta B_t\} < 1$. Take any $\lambda > 1$ such that $\lambda^{-|\mathcal{C}|}$ is greater than the above infimum. Since the latter is strictly less than 1, such a λ exists. By construction, we have $x_s \in \lambda^{-|\mathcal{C}|} B_t$, which implies that $k_{ts} \leq -|\mathcal{C}|$. Therefore, it must be that $-\sum_{(i,j)\in\mathcal{C}} k_{ij} \geq |\mathcal{C}|$, which contradicts that the set of observations $\mathcal{O}$ satisfies Axiom 1 for all $\lambda > 1$. □

We find the above result intuitive. The generalised axiom requires that for any cycle $\mathcal{C}$, specified as previously, each bundle x_s belongs to the boundary of set B_t, for all pairs (t,s) in the sequence. Thus, the geometric mean of the corresponding values $\lambda^{k_{ts}}$, as in (3), must be equal to 1, regardless of the value of λ. Hence, whenever the value of the parameter λ is arbitrarily close to 1, GARP is equivalent to Axiom 1.

Consider the example depicted in Figure 3. Clearly, it violates GARP, as $x_1 \in B_2$ and $x_2 \in B_1$, but none of the bundles belongs to the upper bound of the respective set. Nevertheless, we claim that those observations obey Axiom 1 for all $\lambda > \sqrt{5/4}$. Take any such λ. Since $x_1 \in \partial B_1$ and $x_2 \in \partial B_2$, it suffices to verify that the condition specified in the axiom holds for cycle $\mathcal{C} = \{(1,2),(2,1)\}$. Given that $x_2 \in (9/10)\partial B_1$ and $(9/10) > \lambda^{-1}$, the integer k_{12} is equal to 0. Similarly, as $x_1 \in (4/5)\partial B_2$ and $(4/5) > \lambda^{-2}$, we obtain $k_{21} > -2$. Therefore, we have $-k_{12} - k_{21} < 2$, which is sufficient for the set of observations to satisfy Axiom 1. At the same time value $\sqrt{5/4}$ constitutes the lower

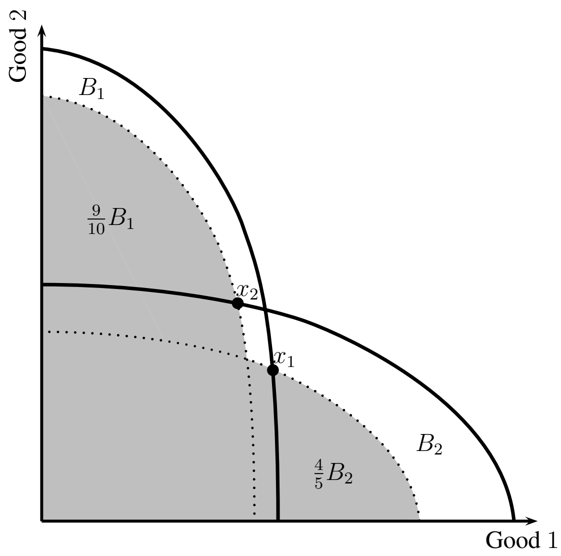

Figure 3: Set $\mathcal{O} = \{(x_1, B_1), (x_2, B_2)\}$ violates GARP. In fact, it fails to satisfy the weak axiom of revealed preference. However, it obeys Axiom 1 for any $\lambda > \sqrt{5/4}$.

bound for all such parameters λ. In fact, the restriction is violated for $\lambda = \sqrt{5/4}$.

Verifying whether a set of observations satisfies Axiom 1 is computationally efficient. Specifically, the test can be reduced to the minimum cost-to-time ratio cycle problem.[11] Take any $\lambda > 1$. Notice that set $\mathcal{O}$ obeys the condition if and only if the minimum of $\sum_{(t,s)\in\mathcal{C}} k_{ts}/|\mathcal{C}|$ over all cycles $\mathcal{C}$ specified in the axiom is strictly greater than -1, where integers k_{ts} are defined as previously. Construct a graph with each vertex $t \in T$ corresponding to a single observation in $\mathcal{O}$. Additionally, let the graph contain a directed arc (t, s) with the corresponding cost $c_{ts} = k_{ts}$ and time $\tau_{ts} = 1$ if and only if $x_s \in B_t$, i.e., if $k_{ts} \leq 0$. Clearly, solving the minimum cost-to-time ratio cycle problem for the above graph is equivalent to determining the least value of $\sum_{(t,s)\in\mathcal{C}} k_{ts}/|\mathcal{C}|$ over all cycles $\mathcal{C}$ induced by $\mathcal{O}$. Since the above class of problems is solvable in a polynomial time, determining whether a set of observations obeys Axiom 1 is efficient.[12]

3.3 The main result

Consider the main theorem of this paper.

Main Theorem. *Set $\mathcal{O}$ obeys Axiom 1 for $\lambda > 1$ if and only if it is rationalisable by a semiorder P that satisfies $(\lambda' a)Pa$, for all $\lambda' \geq \lambda$ and non-zero a.*

[11] See Chapter 5.7 in Ahuja, Magnanti, and Orlin (1993) for a handbook introduction to this method.

[12] Note that, in order to reject the axiom, it suffices to find at least one cycle that violates the condition. Hence, it is not necessary to determine the minimal price-to-time ratio to conclude that the set of observations does not meet our restriction. Thus, the algorithm can be rendered even more efficient.

We devote this subsection to the proof of this result. Given that the argument highlights both the properties of Axiom 1 and its applicability, we find it convenient to present it in several steps. First, we show that the restriction is necessary for a set of observations to be rationalisable in the sense specified at the beginning of this section.

Consider a cycle $\mathcal{C} = \{(a,b),(b,c),\dots,(z,a)\}$ in $T \times T$ with $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$. Take any such pair (t,s) and evaluate the corresponding integer k_{ts}, which is negative. To make our notation compact, denote $k = -k_{ts}$ and construct a sequence

$$\mathcal{S}_{ts} = \{(x_t, \lambda^k x_s),\ (\lambda^k x_s, \lambda^{k-1} x_s),\ \dots\ ,\ (\lambda^2 x_s, \lambda x_s),\ (\lambda x_s, x_s)\}$$

consisting of $(1 - k_{ts})$ elements. Since relation P rationalises set $\mathcal{O}$ in the aforementioned sense and $(\lambda^k x_s) \in B_t$, it must be that $x_t R(\lambda^k x_s)$, while aPb for any other pair (a,b) in $\mathcal{S}_{ts}$. Thus, the number of elements (a,b) in $\mathcal{S}_{ts}$ for which aPb is greater than $-k_{ts}$.

Since the above observation holds true for any $(t,s) \in \mathcal{C}$, combining all such sequences $\mathcal{S}_{ts}$ allows us to construct another sequence $\{(z_{i+1}, z_i)\}_{i=1}^{m}$, where $m = |\mathcal{C}| - \sum_{(t,s)\in\mathcal{C}} k_{ts}$, with $z_{i+1} R z_i$, for all $i = 1,\dots,m$, and $z_1 = z_{m+1}$. Moreover, the total number of pairs (z_{i+1}, z_i) for which $z_{i+1} P z_i$ is greater than $-\sum_{(t,s)\in\mathcal{C}} k_{ts}$. By Proposition 1, this requires that $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$. Otherwise, it would have to be that $z_1 \succ z_{m+1}$, contradicting that the induced weak order relation $\succ$ is irreflexive.

A careful read of the above argument provides a deeper understanding of the revealed preference relations that are induced by the set of observations. Note that, for any $\lambda > 1$ and sequence $\mathcal{S} = \{(a,b),(b,c),\dots,(y,z)\}$ such that $x_s \in B_t$, for all $(t,s) \in \mathcal{S}$:

$$-\sum_{(t,s)\in\mathcal{S}} k_{ts} \geq |\mathcal{S}| \ \text{ implies } \ x_a \succ x_z \ \text{ and } \ -\sum_{(t,s)\in\mathcal{S}} k_{ts} > |\mathcal{S}| \ \text{ implies } \ x_a P x_z,$$

where integers k_{ts} are defined as previously. Therefore, any sequence satisfying one of the above conditions reveals that the bundle x_a is strictly superior to x_z — either with respect to the induced weak order $\succ$ or the original semiorder P. Clearly, as both relations are irreflexive, this excludes the possibility that $x_a = x_z$. In fact, the exact role of Axiom 1 is to guarantee that the set of observations $\mathcal{O}$ admits no such cycles. We exploit this observation in Section 3.4, where we present a method of recovering and extrapolating consumer preference from the observable data.

Establishing the sufficiency part of the Main Theorem is more demanding. For this reason, we conduct the remainder of the proof in three stages. In the lemma that follows, we argue that whenever a set of observations satisfies Axiom 1 for some $\lambda > 1$, there exists a solution to a particular system of linear inequalities.

Lemma 1. *If set $\mathcal{O}$ obeys Axiom 1 for $\lambda > 1$, there are numbers $\{\phi_t\}_{t\in T}$ and $\mu > 1$ such that $k_{ts} \leq 0$ implies $\phi_s - 1 \leq \phi_t + \mu k_{ts}$, where $k_{ts} := \inf\left\{k \in \mathbb{Z} : x_s \in \lambda^k B_t\right\}$, for $t, s \in T$.*

Proof. Notice that, a set of observations $\mathcal{O}$ satisfies Axiom 1 if and only if, for any cycle $\mathcal{C} = \left\{(a,b),(b,c),\ldots,(z,a)\right\}$ in $T \times T$ such that $k_{ts} \leq 0$, for all $(t,s) \in \mathcal{C}$, we have $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$. The rest follows from Proposition A.1 in the Appendix. $\square$

As discussed in Section 3.2, there is an efficient algorithm that allows us to verify whether a set of observations satisfies Axiom 1. Nevertheless, the above lemma provides an alternative method that pertains to linear programming.[13] According to the result, determining the existence of a solution to the above system of inequalities is necessary for the dataset $\mathcal{O}$ to obey Axiom 1. In fact, as we argue in the remainder of this section, it is also a sufficient condition. Given that linear programs are solvable in a polynomial time, the alternative method is also computationally efficient.

Apart from establishing an alternative method of verifying Axiom 1, Lemma 1 provides us with a foundation for constructing a semiorder that rationalises the set of observations. In the next result, we argue that determining a solution to the linear system specified previously is sufficient for the existence of a particular utility function.

Lemma 2. *Whenever there is a solution to the system of inequalities specified in Lemma 1, there exists a utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ such that $\lambda' \geq \lambda$ implies $u(\lambda' y) > u(y) + 1$, for all non-zero $y \in \mathbb{R}^\ell_+$, and $y \in B_t$ implies $u(x_t) \geq u(y) - 1$, for all $t \in T$.*

Proof. Define function $g_t : \mathbb{R}^\ell_+ \to \mathbb{Z}$ by $g_t(y) := \inf\left\{k \in \mathbb{Z} : y \in \lambda^k B_t\right\}$, for all $t \in T$. Note that, we have $g_t(y) \leq 0$ if and only if $y \in B_t$. Take any non-zero $y \in \mathbb{R}^\ell_+$ and $\lambda' \geq \lambda$. Whenever $(\lambda' y) \in \lambda^{k+1} B_t$ then $(\lambda'/\lambda) y \in \lambda^k B_t$, for any $\lambda' \geq \lambda$, $k \in \mathbb{Z}$, and non-zero y. As B_t is comprehensive, this implies that $y \in \lambda^k B_t$. Thus, we obtain $g_t(\lambda' y) \geq g_t(y) + 1$. Finally, observe that $g_t(x_s) = k_{ts}$, where k_{ts} is defined as previously, for all t, $s \in T$.

Take any numbers $\{\phi_t\}_{t\in T}$ and μ that satisfy the inequalities in Lemma 1. Moreover, let $\nu > 1$ be such that $\phi_s - 1 \leq \phi_t + \nu k_{ts}$, for all t, s in T satisfying $k_{ts} > 0$. Clearly, it

[13] Note that, this linear system is closely related to *Afriat's inequalities*, as in Forges and Minelli (2009).

exists. Construct function $v_t : \mathbb{R}^\ell_+ \to \mathbb{R}$ as follows:

$$v_t(y) := \begin{cases} \phi_t + \mu g_t(y) & \text{if } g_t(y) \leq 0, \\ \phi_t + \nu g_t(y) & \text{otherwise.} \end{cases}$$

First, we claim that $v_t(\lambda' y) > v_t(y) + 1$, for any non-zero $y \in \mathbb{R}^\ell_+$ and $\lambda' \geq \lambda$. In fact, whenever we have $g_t(y) \leq 0$ and $g_t(\lambda' y) \leq 0$, then it must be

$$v_t(\lambda' y) - v_t(y) \;=\; \mu\big[g_t(\lambda' y) - g_t(y)\big] \;\geq\; \mu \;>\; 1.$$

As $\nu > 1$, the claim holds analogously if $g_t(y) > 0$ and $g_t(\lambda' y) > 0$. Therefore, it suffices to prove the property for $g_t(y) \leq 0$ and $g_t(\lambda' y) > 0$. In such a case, we have

$$v_t(\lambda' y) - v_t(y) \;=\; \nu g(\lambda' y) - \mu g_t(y) \;>\; g_t(\lambda' y) - g_t(y) \;\geq\; 1,$$

since μ, $\nu > 1$. Therefore, our initial claim holds. Moreover, as $g_s(x_t) = k_{st}$, by construction of numbers $\{\phi_t\}_{t \in T}$, μ, and ν, we obtain $\phi_t - 1 \leq \min_{s \in T} v_s(x_t)$, for all $t \in T$.

Define function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ by $u(y) := \min_{s \in T} v_s(y)$. Given the properties of functions v_s, for $s \in T$, it is clear that $\lambda' \geq \lambda$ implies $u(\lambda' y) > u(y) + 1$, for any non-zero $y \in \mathbb{R}^\ell_+$.[14] To complete the proof, take any $t \in T$ and $y \in B_t$. Since $g_t(y) \leq 0$, we obtain

$$u(y) - 1 \;\leq\; \phi_t + \mu g_t(y) - 1 \;\leq\; \phi_t - 1 \;\leq\; \min_{s \in T} v_s(x_t) \;=\; u(x_t),$$

which concludes our argument. $\square$

In order to complete the proof of the Main Theorem, take any function u specified as in Lemma 2 and define a binary relation P by

$$aPb \;\text{ if and only if }\; u(a) > u(b) + 1.$$

It is straightforward to show that P is a semiorder. Moreover, if $\lambda' \geq \lambda$ then $(\lambda' a)Pa$, for all non-zero a. Finally, since $u(x_t) \geq u(y) - 1$, for all $y \in B_t$ and $t \in T$, it can never be that yPx_t. Thus, the semiorder P rationalises the set $\mathcal{O}$, which completes our proof.

The argument presented above establishes a fourfold equivalence. First of all, it implies that Axiom 1 is necessary and sufficient for a set of observations $\mathcal{O}$ to be rationalisable by a semiorder P with some noticeable difference $\lambda > 1$. Moreover, our restriction is

[14] By definition, there is some t, $s \in T$ such that $u(\lambda' y) = v_t(\lambda' y)$ and $u(y) = v_s(y) \leq v_t(y)$. This guarantees that $u(\lambda' y) - u(y) = v_t(\lambda' y) - v_s(y) \geq v_t(\lambda' y) - v_t(y) > 1$.

equivalent to the existence of a solution to the system of linear inequalities specified in Lemma 1. Third, solving the linear problem is both necessary and sufficient for the set of observations to admit a utility function u that satisfies the two properties listed in Lemma 2. This establishes the fourth equivalence, according to which the semiorder P induced by any such function u rationalises $\mathcal{O}$ with the noticeable difference λ.

The proof supporting Lemma 2 proposes only one possible utility function u that satisfies the conditions listed in its thesis. In the Online appendix we provide an alternative proof of the Main Theorem. Specifically, we show that without loss of generality the utility function specified in the lemma is continuous, increasing, and satisfies $u(\lambda' a) > u(a)$, for all non-zero $a \in \mathbb{R}^\ell_+$ and $\lambda' > 1$.[15] More importantly, we argue that the weak order $\succeq$ induced by the corresponding semiorder P is represented by u. That is, we have $a \succeq b$ if and only if $u(a) \geq u(b)$. This observation will be significant in Section 4.

3.4 Recoverability and extrapolation

In this subsection we apply the tools developed in the preceding part of the paper to the problem of recovering preferences from the observable data and forecasting choices from hypothetical budget sets. As it was pointed out previously, our framework crucially depends on two types of preferences revealed in the data: the semiorder P that the agent is using whilst making a choice, and the induced weak order $\succeq$. Below we show how to recover both relations from a set of observations.

Take any $\lambda > 1$ and let $\mathcal{X} := \{x_t\}_{t \in T}$ be the set of observable demands. For any $x_a \in \mathcal{X}$ and $x_0 \in \mathbb{R}^\ell_+$, bundle x_a is revealed preferred to x_0 with respect to the revealed *strict* relation $\succ^*$, or simply $x_a \succ^* x_0$, if and only if there is a sequence $\mathcal{S} = \big\{(a,b),(b,c),\dots,(y,z)\big\}$ in $T \times T$ such that $k_{ts} \leq 0$, for $(t,s) \in \mathcal{S}$, $k_{z0} \leq 0$, and

$$-k_{z0} - \sum_{(t,s)\in\mathcal{S}} k_{ts} \;\geq\; 1 + |\mathcal{S}|,$$

where numbers k_{ts} are defined as previously, for all pairs $(t,s) \in \mathcal{S}$ and $(z,0)$.[16] Additionally, if the above inequality is strict, then x_a is revealed strictly preferred to x_0 with respect to the revealed relation P^*, or equivalently $x_a P^* x_0$. Be aware that the two binary comparisons are defined conditionally on the number λ. Moreover, it is straightforward

[15] In addition, if the complement of set B_t is convex, for all $t \in T$, the function is also quasiconcave.
[16] We allow for set $\mathcal{S}$ to be empty.

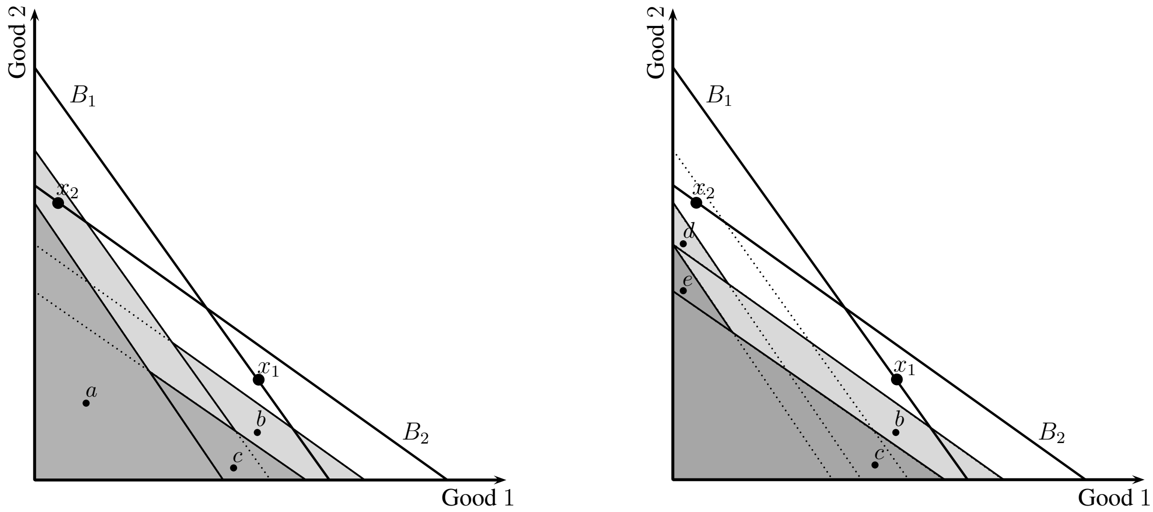

Figure 4: Recovering preferences from observations (B_1, x_1) and (B_2, x_2). The dotted lines represent upper bounds of sets B_1 and B_2 scaled by λ^{-1}, λ^{-2}, and λ^{-3}. The shaded areas depict bundles that are revealed inferior to x_1 (left) and x_2 (right) with respect to $\succ^*$ and P^*.

to verify that both relations are transitive, while $x_a P^* x_0$ implies $x_a \succ^* x_0$.

In order to better understand the two types of revealed preferences, consider the example depicted in Figure 4. Consider the graph on the left. Clearly, as bundle x_2 belongs to $\lambda^{-1}B_1$, we have $k_{12} \leq -1$, which implies that $x_1 \succ^* x_2$. Analogously, since option a is in $\lambda^{-2}B_1$, it must be that $x_1 P^* a$. By applying the same argument, we can show that both $x_2 \succ^* b$ and $x_2 P^* c$ must hold, as depicted in the graph on the right.

Regarding the remaining revealed comparisons, transitivity of $\succ^*$ guarantees that $x_1 \succ^* x_2$ and $x_2 \succ^* b$ implies $x_1 \succ^* b$, as shown on the left. In order to prove $x_1 P^* c$, denote $x_0 = c$. As $k_{12} = -1$ and $k_{20} \leq -2$, there is a one-element sequence $\mathcal{S} = \big\{(1,2)\big\}$ such that $-k_{12} - k_{20} \geq 3 > |\mathcal{S}| + 1$. Therefore, according to the definition of P^*, we obtain $x_1 P^* c$. Similarly, let $x_0 = d$ and observe that $k_{21} = 0$, while $k_{10} = -2$, which implies $x_2 \succ d$. Using an analogous approach, we can show that $x_2 P^* e$.

Proposition 3. *Suppose that set $\mathcal{O}$ can be rationalised by a semiorder P with a noticeable difference $\lambda > 1$ and the induced weak order $\succeq$. Then the revealed relation P^* is consistent with P and the revealed strict relation $\succ^*$ is consistent with $\succ$.*

The above result is a direct application of Proposition 1 and the argument presented in Section 3.3. In fact, whenever $x_a \succ^* x_0$, it is possible to construct a sequence $\big\{(z_{i+1}, z_i)\big\}_{i=1}^{m}$ such that $z_{i+1} R z_i$, for all i, while $z_1 = x_0$ and $z_{m+1} = x_a$. Moreover, in any such sequence, the number of pairs (z_{i+1}, z_i) for which $z_{i+1}Pz_i$ is greater than the number of couples with $z_{i+1}Iz_i$. Therefore, by Proposition 1, it must be that $x_a \succ^* x_0$ implies $x_a \succ x_0$. Similarly, whenever $x_aP^*x_0$ then x_aPx_0. The following corollary proposes an alternative characterisation of Axiom 1.

Corollary 1. *Set $\mathcal{O}$ obeys Axiom 1 for $\lambda > 1$ if and only if $x_t \succ^* x_s$ implies not $x_s \succ^* x_t$, for all t, $s \in T$, where the revealed relation $\succ^*$ is defined conditionally on λ.*

Proof. Suppose that set $\mathcal{O}$ obeys the axiom for some $\lambda > 1$. Whenever $x_t \succ^* x_s$ and $x_s \succ^* x_t$, there is a cycle $\mathcal{C} = \big\{(t, a), \ldots, (z, s), (s, a'), \ldots, (z, t)\big\}$ such that $k_{ij} \leq 0$, for all $(i, j) \in \mathcal{C}$, and $-\sum_{(i,j)\in\mathcal{C}} k_{ij} \geq |\mathcal{C}|$. However, this contradicts Axiom 1. Conversely, set $\mathcal{O}$ violates the axiom only if $x_t \succ^* x_t$, for some $t \in T$. $\square$

We now turn to the problem of extrapolation. Suppose that a dataset $\mathcal{O}$ satisfies Axiom 1 for some $\lambda > 1$, or equivalently, it is rationalisable by a semiorder P. The revealed relations $\succ^*$ and P^* summarize all the information on consumer preference that is contained in the data. In particular, following Proposition 3, this implies that any semiorder P and the induced weak order $\succeq$ supporting the observations must contain the respective revealed relations. In the remainder of this subsection we are interested in out-of-sample predictions of consumer choices. That is, given a budget set B_0 that has not been previously observed, which bundles $x_0 \in B_0$ are consistent with the past behaviour? In other words, we would like to determine which vectors belong to set

$$S(B_0) := \Big\{x_0 \in B_0 : \text{ set } \mathcal{O} \cup \big\{(B_0, x_0)\big\} \text{ satisfies Axiom 1 for } \lambda\Big\}.$$

Given that set $\mathcal{O}$ satisfies Axiom 1 for $\lambda > 1$, it is straightforward to show that bundle x_0 belongs to $S(B_0)$ if and only if for any cycle $\mathcal{C} = \big\{(0, a), (a, b), \ldots, (z, 0)\big\}$ of pairs of indices t, s in $T \cup \{0\}$ such that $k_{ts} \leq 0$, for all $(t, s) \in \mathcal{C}$, we have $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|$, where integers k_{ts} are defined as previously. Verifying the above condition directly might be cumbersome. Therefore, we find it convenient to provide some necessary conditions that have to be satisfied by each element of $S(B_0)$. In particular, those properties make an extensive use of the revealed preference relations defined previously.

Suppose that x_0 is an element of $S(B_0)$. The following conditions have to be satisfied. Clearly, we require that $x_0 \notin \lambda^{-1}B_0$. Moreover, it must be that (i) $x_tP^*x_0$ implies $x_t \notin B_0$. In fact, we have $x_tP^*x_0$ only if there is some sequence $\mathcal{S} = \big\{(t, a), (a, b), \ldots, (z, 0)\big\}$ such that $k_{ij} \leq 0$, for $(i,j) \in \mathcal{S}$, and $-\sum_{(i,j)\in\mathcal{S}} k_{ij} > |\mathcal{S}|$. As numbers $-k_{ij}$ are integers, the above condition requires that their sum is weakly greater than $|\mathcal{S}|+1$. Moreover, whenever $x_t \in B_0$ then $k_{0t} \leq 0$. In particular, this allows to construct a cycle $\mathcal{C} = \mathcal{S} \cup \{(0,t)\}$ with $k_{ij} \leq 0$, for all $(i,j) \in \mathcal{C}$, where the sum of integers $-k_{ij}$ is weakly greater that $|\mathcal{C}| = |\mathcal{S}|+1$. However, this contradicts that $x_0 \in S(B_0)$. Analogously, we may show that (ii) whenever $x_t \succ^* x_0$ then $x_t \notin \lambda^{-1}B_0$; and (iii) if $x_0 \in B_t$ then $x_t \notin \lambda^{-2}B_0$.

The conditions (i)–(iii) are necessary but *not* sufficient for a bundle x_0 to be an element of $S(B_0)$. Consider a dataset $\mathcal{O}$ consisting of vectors $x_1 = (5,2)$ and $x_2 = (4,5)$ selected respectively from budget sets $B_t = \{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t\}$, for $t = 1,2$, where $p_1 = (3,1)$ and $p_2 = (2,1)$. Note that, this two-element dataset satisfies Axiom 1 for $\lambda = 5/4$. Given a previously unobserved linear budget set B_0, defined as above for $x_0 = (5,3)$ and $p_0 = (1,5)$, we would like to determine the corresponding set $S(B_0)$. Observe that, vector $x_0 \in B_0$ satisfies conditions (i)–(iii). However, it does not belong to $S(B_0)$. In particular, neither x_1 nor x_2 is revealed preferred to x_0 with respect to P^* or $\succ^*$. Thus, properties (i) and (ii) hold trivially. Moreover, even though x_0 is an element of B_2, it is outside of $\lambda^{-2}B_2$. Therefore, condition (iii) holds as well. At the same time, observe that it is possible to construct a cycle $\mathcal{C} = \{(0,1),(1,2),(2,0)\}$ such that $k_{01} = -3$, $k_{12} = 0$, $k_{20} = 0$. Since $-\sum_{(t,s)\in\mathcal{C}} k_{ts} = |\mathcal{C}|$, this contradicts that the extended dataset $\mathcal{O} \cup \{(B_0, x_0)\}$ satisfies the axiom.

Whenever we restrict our attention to linear budget sets, the corresponding set of consistent choices admits a property that proves to be useful in applications. Formally, a budget set is *linear* if $B_t := \{y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq m_t\}$, for some $p_t \in \mathbb{R}^\ell_{++}$ and $m_t > 0$.

Proposition 4. *Consider a dataset $\mathcal{O}$ in which B_t is linear, for all $t \in T$. If $\mathcal{O}$ obeys Axiom 1 for some $\lambda > 1$, then the corresponding set $S(B_0)$ is convex, for any linear B_0.*

Proof. Whenever B_t is linear, then $k_{ts} := \inf\{k \in \mathbb{Z} : (p_t \cdot x_s)/m_t \leq \lambda^k\}$, for any x_s. Let x_0 be a convex combination of any x_0', x_0'' in $S(B_0)$. Clearly, we have $x_0 \in B_0$. Since $p_t \cdot x_0 \geq \min\{p_t \cdot x_0', p_t \cdot x_0''\}$, for all $t \in T$, the above set violates the axiom only if either $\mathcal{O} \cup \{(B_0, x_0')\}$ or $\mathcal{O} \cup \{(B_0, x_0'')\}$ fails to satisfy the the restriction. □

In fact, linear budget sets imply that set $S(B_0)$ is a polyhedron. However, since any of its elements must be outside of $\lambda^{-1}B_0$, in general, the set is not closed.

4 Interpreting just-noticeable differences

In the preceding section we established the necessary and sufficient conditions under which a set of observations is rationalisable by a semiorder P with some noticeable difference λ. Additionally, we provided a method of eliciting the utility function u that represented the semiorder in the sense specified in Section 2. In this part of the paper we claim that the semiorder rationalisation may be interpreted as maximisation of the utility u with a particular type of error summarised by the parameter λ.

Take any $\lambda > 1$ and assume that preferences of a consumer can be represented by a continuous utility function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ that satisfies $u(\lambda' a) > u(a)$, for all $\lambda' > 1$ and non-zero a. Consider a model of choice such that, for all $(B_t, x_t) \in \mathcal{O}$:

$$u(z) \geq u(x_t) \text{ implies } u(\lambda z) > u(y), \text{ for all } y \in B_t \text{ and non-zero } z \in \mathbb{R}^\ell_+. \tag{4}$$

Even though the consumer determines the choice using her actual utility u, she is making a consistent error in evaluating which bundles are superior to x_t. In particular, although there might exist an available option y such that $u(y) > u(x_t)$, the choice x_t remains optimal in the above sense unless there is some z such that $u(y) \geq u(\lambda z)$ and $u(z) \geq u(x_t)$. That is, the indifference curves intersecting bundles y and x_t must be sufficiently far apart for y to be chosen over x_t.[17] Roughly speaking, the consumer optimises as if her indifference curves were thick, where the thickness is determined by λ. For a graphical interpretation of this condition, recall Figure 1 in the Introduction. Moreover, as λ tends to 1, condition (4) coincides with maximisation of u.[18] Finally, a fundamental feature of the above model is that it is specified in ordinal terms. Therefore, cardinality of the utility u plays no role in determining optimal choices.

It is straightforward to show that if set $\mathcal{O}$ is rationalisable by a semiorder P with a noticeable difference λ, then it can be supported by the above model of consumer choice. Let $\succeq$ be the weak order induced by P. Following our discussion in Section 1 of the Online appendix, we may assume that $\succeq$ can be represented by a continuous function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ satisfying $u(\lambda' a) > u(a)$, for all $\lambda' > 1$ and non-zero a. Since P rationalises the data, it must be that $y \in B_t$ implies $u(x_t) \geq u(y) - 1$. In particular,

[17] Suppose that u and u' denote indifference curves that contain x_t and y, respectively. Bundle x_t is optimal only if the value $\sup\{\theta > 0 : z \in u \text{ and } (\theta z) \in u'\}$ is strictly lower than λ.

[18] Clearly, utility maximisation implies (4), for any $\lambda > 1$. Conversely, suppose that condition (4) holds for all $\lambda > 1$, but there is some $y \in B_t$ such that $u(y) > u(x_t)$. By continuity of u, we can always find a $\lambda' > 1$ that satisfies $u(y) \geq u(\lambda' x_t) > u(x_t)$. However, this contradicts (4).

whenever condition (4) is violated, there is some $y \in B_t$ and a non-zero $z \in \mathbb{R}^\ell_+$ such that $u(y) \geq u(\lambda z) > u(z) + 1 \geq u(x_t) + 1$, which contradicts that $\mathcal{O}$ is rationalisable by P.

Interestingly, the converse implication also holds. Therefore, given a set of observations $\mathcal{O}$ specified as in Section 3.1, it is not possible to distinguish between the two forms of rationalisation. We state this observation formally in the following proposition.

Proposition 5. *Suppose that set $\mathcal{O}$ is rationalisable according to (4) for $\lambda > 1$ and a continuous function $u : \mathbb{R}^\ell_+ \to \mathbb{R}$ satisfying $u(\lambda' a) > u(a)$, for all $\lambda' > 1$ and non-zero a. Then it is rationalisable by a semiorder P with a noticeable difference λ.*

Proof. Define relation P by: aPb if and only if $u(a) \geq u(\lambda' z)$ and $u(z) \geq u(b)$, for some $\lambda' \geq \lambda$ and a non-zero $z \in \mathbb{R}^\ell_+$. Clearly, it admits the noticeable difference λ.

We argue that P is a semiorder. Note that aPa only if there is a non-zero z and $\lambda' \geq \lambda$ such that $u(a) \geq u(\lambda' z)$ and $u(z) \geq u(a)$. As $u(\lambda' z) > u(z)$, this implies that $u(a) > u(a)$, yielding a contradiction. Thus, it is irreflexive. To show that is satisfies ($S2$), let aPb and cPd. By construction, there is some non-zero z, z' and λ', $\lambda'' \geq \lambda$ such that $u(a) \geq u(\lambda' z)$ and $u(z) \geq u(b)$, while $u(c) \geq u(\lambda'' z')$ and $u(z') \geq u(d)$. If $u(b) \geq u(d)$, then $u(a) \geq u(\lambda' z)$ and $u(z) \geq u(d)$, thus, aPd. Otherwise, it must be that cPb. Using an analogous argument, we show that property (S3) is also satisfied.

To prove that P rationalises $\mathcal{O}$, take any observation (B_t, x_t) and $y \in B_t$. Whenever yPx_t, there must be some non-zero z such that $u(y) \geq u(\lambda z)$ and $u(z) \geq u(x_t)$. However, this contradicts that u rationalises the data according to (4). □

Proposition 5 provides an alternative interpretation of the Main Theorem. Suppose that a set of observations $\mathcal{O}$ violates GARP, but satisfies Axiom 1 for some $\lambda > 1$. Clearly, there is no locally non-satiated utility function u that rationalises the data. However, the above result suggests that λ may be interpreted as the consistent error that the consumer is making whilst maximising her preference. Since the testable implications of this model are equivalent to those of semiorder maximisation presented in the preceding section, the two interpretations describe the same observable behaviour.

5 Measuring revealed preference violations

There is an extensive literature on various methods of evaluating the severity of revealed preference violations. For a detailed discussion see Apesteguia and Ballester (2015) or

Chapter 5.1 in Echenique and Chambers (2016). In this section we explore the properties of the just-noticeable difference as an alternative measure of GARP violations.

Formally, let $\lambda_* \geq 1$ denote the infimum over all numbers $\lambda > 1$ for which a set of observations $\mathcal{O}$ satisfies Axiom 1. Henceforth, we refer to λ_* as the *just-noticeable difference*. In the remainder of this section, we find it convenient to focus on the inverse of λ_* as a possible measure of revealed preference violations. Nevertheless, any property established for $1/\lambda_*$ can be easily reformulated for λ_*.

As $\lambda_* \geq 1$, the value of its inverse ranges from 0 to 1. In particular, Proposition 2 implies that the index is equal to 1 if and only if the set of observations satisfies GARP. To better understand the meaning of this measure, let $\theta_{ts} := \inf\big\{\theta > 0 : x_s \in \theta B_t\big\}$, for all t, $s \in T$. Therefore, the number θ_{ts} is the least value by which one can scale set B_t without excluding x_s. This may be interpreted as the relative distance of the bundle x_s from the upper bound of B_t. Notice that, for any $\lambda > 0$ it must be that $\lambda^{k_{ts}} \geq \theta_{ts} > \lambda^{k_{ts}-1}$, where integer k_{ts} is defined as previously. Given this and (3), it is easy to show that

$$\lambda_*^{-2} \;\leq\; \min\left\{ \sqrt[|\mathcal{C}|]{\prod\nolimits_{(t,s)\in\mathcal{C}} \theta_{ts}} \;:\; \mathcal{C} \text{ is a cycle in } T\times T \right\} \;\leq\; \lambda_*^{-1}.$$

Hence, the just-noticeable difference serves as an approximation of the least geometric mean of numbers θ_{ts} over all cycles in the dataset. In other words, our index measures a violation of GARP by evaluating the minimal average distance of bundle x_s from the upper bound of the corresponding set B_t, along all observable cycles. To provide a better context, we discuss the remaining properties of our index by comparing it to two well-known measures of revealed preference violations. Specifically, we argue that our approach combines the desirable features of the Afriat's efficiency index and the money-pump index, while responding to the main criticism pointed at those two measures.[19]

5.1 Afriat's efficiency index and just-noticeable differences

First, we compare the inverse of the just-noticeable difference to a generalised version of *Afriat's efficiency index*, proposed originally in Afriat (1967). A dataset $\mathcal{O}$ obeys GARP for an efficiency index $e \in [0,1]$, if for any sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T\times T$ with $x_s \in B_t$, for all $(t,s) \in \mathcal{C}$, we have $x_s \in \partial(eB_t)$, for all $(t,s) \in \mathcal{C}$.[20] Afriat's efficiency

[19]We select measures that are both commonly used in the literature and can be evaluated using efficient algorithms. For other notable indices see, e.g., Houtman and Maks (1985) and Dean and Martin (2016).

[20]That is, bundle x_s belongs to the upper bound of set eB_t, defined as in Section 3.1, for all $(t,s) \in \mathcal{C}$.

index e_* is the supremum over all efficiency indices for which $\mathcal{O}$ obeys GARP.

Similarly to Axiom 1, Afriat's efficiency index imposes a weaker restriction on a set of observations than GARP. In particular, a dataset $\mathcal{O}$ obeys the latter condition only if $e_* = 1$; however, the converse need not be true (see Example 1 below). Before discussing further distinctions between this measure and the inverse of the just-noticeable difference, we point out an important relation satisfied by the two indices.

Proposition 6. *Whenever dataset $\mathcal{O}$ obeys Axiom 1 for some $\lambda > 1$, then it satisfies GARP for any efficiency index $e \leq \lambda^{-1}$.*

Proof. Suppose that set $\mathcal{O}$ obeys Axiom 1 for some $\lambda > 1$. Whenever the dataset admits a sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ with $x_s \in \lambda^{-1}B_t$, for all $(t,s) \in \mathcal{C}$, the corresponding integers k_{ts} must be less than -1, for all (t,s). This implies that

$$-\sum_{(t,s)\in\mathcal{C}} k_{ts} \geq |\mathcal{C}|,$$

which yields a contradiction. Thus, no such cycle is admissible. This suffices for the dataset to satisfy GARP for any efficiency index $e \leq \lambda^{-1}$. $\square$

Proposition 6 implies that the inverse of the just-noticeable difference is always lower than Afriat's efficiency index. However, the opposite implication need not hold. That is, there are observation sets that obey GARP for some efficiency index e, but fail to satisfy Axiom 1 for $\lambda = e^{-1}$. For instance, recall the dataset from Figure 3. Notice that, the corresponding Afriat's efficiency index is equal to $e_* = 9/10$. At the same time, the inverse of the just-noticeable difference is $\lambda_*^{-1} = \sqrt{4/5} < \sqrt{81/100} = 9/10$.

The above property is implied by one crucial feature of Afriat's efficiency index. As noted in Echenique, Lee, and Shum (2011), this measure seeks to "break" a violation of GARP at its weakest link. Consider the example in Figure 5. Clearly, both sets $\mathcal{O}$ and $\mathcal{O}'$ violate GARP. Moreover, the inverse of the just-noticeable difference corresponding to the latter set is lower than the one evaluated for the former. This would suggest that the violation in $\mathcal{O}'$ is more severe than in $\mathcal{O}$, as it would require a larger value of the just-noticeable difference λ_* to rationalise $\mathcal{O}'$ in the sense discussed in Section 3. At the same time, Afriat's efficiency index does not discriminate between these two cases.

More generally, suppose that set $\mathcal{O}$ admits a sequence $\mathcal{C} = \big\{(a,b),(b,c),\ldots,(z,a)\big\}$ in $T \times T$ with $x_s \in B_t$, for all (t,s) in $\mathcal{C}$. Clearly, we have $\theta_{ts} \leq 1$, for any pair of indices

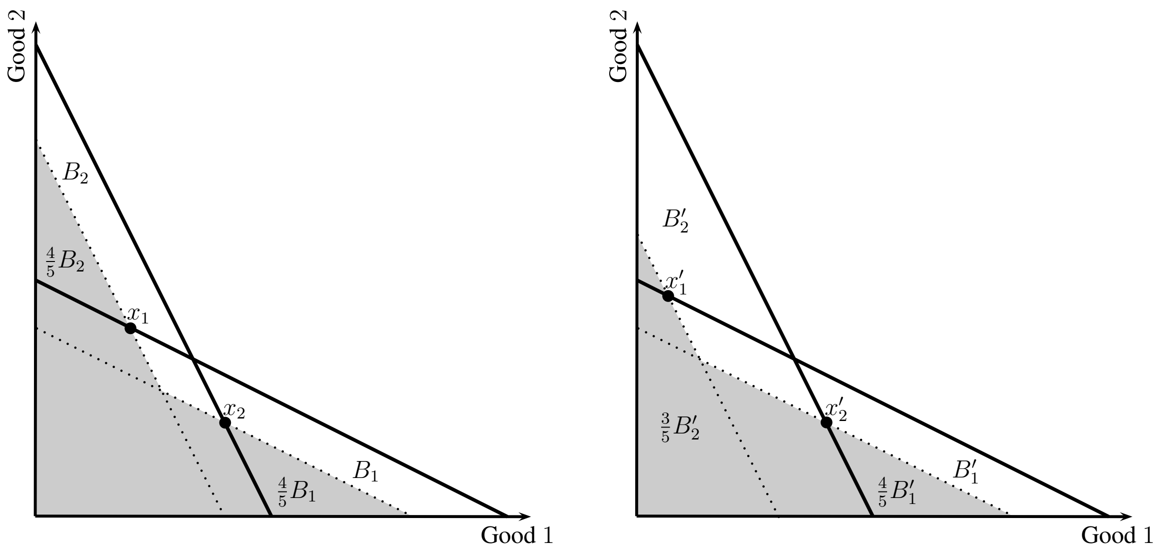

Figure 5: Sets $\mathcal{O} = \{(B_1, x_1), (B_2, x_2)\}$ and $\mathcal{O}' = \{(B_1', x_1'), (B_2', x_2')\}$ violate GARP. In either case, the corresponding Afriat's efficiency index is 4/5. However, the inverse of the just-noticeable difference is equal to 4/5 for $\mathcal{O}$ and $\sqrt{3/5}$ for $\mathcal{O}'$.

(t, s) in the cycle, where numbers θ_{ts} are specified as previously. Note that, in order to "break" the above cycle, it suffices to scale each budged set by a number strictly less than $\bar{\theta}_{\mathcal{C}} := \max\{\theta_{ts} : (t, s) \in \mathcal{C}\}$. In fact, Afriat's efficiency index is the minimal $\bar{\theta}_{\mathcal{C}}$ over all such sequences $\mathcal{C}$. Hence, it is the least value by which one has to scale each budget set B_t in order to break every cycle in the set of observations.

Unlike Afriat's efficiency index, the purpose of the inverse of the just-noticeable difference is not to "break" cycles. Rather, by approximating the geometric mean of numbers θ_{ts} along such a sequence, our measure quantifies the severity of a GARP violation in each cycle revealed in the set of observations. We revisit an example from Echenique, Lee, and Shum (2011) to highlight the extent to which the same violation of revealed preference can be evaluated differently by these two measures.

Example 1. Take any $\epsilon \in (0, 1)$ and consider a set of observations $\mathcal{O} = \{(B_t, x_t)\}_{t=1}^{3}$, where $x_1 = (\epsilon^2, 0)$, $x_2 = \big(\epsilon^2/(1+\epsilon^2), \epsilon^2/(1+\epsilon^2)\big)$, and $x_3 = (0, 1)$, while

$$B_t = \{y \in \mathbb{R}^2_+ : p_t \cdot y \leq p_t \cdot x_t\},$$

for all $t = 1, 2, 3$, with $p_1 = (1/\epsilon, \epsilon)$, $p_2 = (\epsilon, 1/\epsilon)$, and $p_3 = (1, 1)$. Note that, by construction of this example, we obtain $\theta_{12} = 1$, $\theta_{23} = 1$, as well as $\theta_{31} = \epsilon^2$.

Take any number $\lambda > 1$ and the sequence $\mathcal{C} = \{(1, 2), (2, 3), (3, 1)\}$. Evaluate the corresponding integers k_{ts} as in Axiom 1. In particular, recall that $\lambda^{k_{ts}} \geq \theta_{ts} \geq \lambda^{k_{ts}-1}$.

Therefore, we have $k_{12} = k_{23} = 0$, while set $\mathcal{O}$ obeys the axiom only if $k_{31} > -3$, or

$$\sqrt[3]{\lambda^{k_{12}}\ \lambda^{k_{23}}\ \lambda^{k_{31}}} \ = \ \sqrt[3]{\lambda^{k_{31}}} \ > \ \lambda^{-1},$$

where $\lambda^{k_{31}} \geq \epsilon^2 > \lambda^{k_{31}-1}$. Specifically, this implies that $\lambda^{-1} < \sqrt{\epsilon}$. As ϵ tends to 0, the infimum over numbers λ under which the set $\mathcal{O}$ satisfies Axiom 1 tends to infinity, while its inverse approaches 0. At the same time, it is easy to verify that Afriat's efficiency index is equal to 1, irrespectively of the value of the parameter ϵ.

Varian (1990) modified the Afriat's measure by considering a vector $(e_t)_{t\in T}$ of efficiency indices, rather than a scalar. The objective is to choose numbers e_t in $[0, 1]$, as close to 1 as possible (according to some metric), such that the corresponding dataset $\{(e_tB_t, x_t)\}_{t\in T}$ satisfies GARP. Varian's idea is similar to the original efficiency index, as numbers e_t are selected in order to eliminate all cycles in the dataset.[21] In fact, it is easy to show that the index is bounded from below by the Afriat's measure. Thus, given Proposition 6, it always dominates the inverse of the just-noticeable difference.

The interpretation of efficiency indices differs from the just-noticeable difference. The former can be perceived as a margin of error the agent is allowed to make in her choices in terms of the tolerance for wasted expenditure. Analogously, the measure determines the size of consideration sets — the greater is the deviation from GARP, the smaller is the set of options considered by the decision maker. Alternatively, our approach pertains to the consumer's inability to discern between bundles that are similar. That is, we increase the number of binary comparisons in which the agent is indifferent.

An important feature of Afriat's (and Varian's) efficiency index is that it allows to infer consumer preference from the data. Specifically, by assuming that the observed bundle x_t is superior to every option in the perturbed set e_*B_t, one is able to construct a revealed preference relation that is consistent with any utility function rationalising the observations over the restricted consideration sets.[22] Similarly, our approach provides not only a model of consumer behaviour that supports choices which fail to satisfy GARP, but also allows us to recover partially preferences from the observable data — both the semiorder P generating the observations and the corresponding utility function u.

[21]However, the measure possesses some additional properties. See Apesteguia and Ballester (2015).

[22]See also Halevy, Persitz, and Zrill (2017) for an extensive discussion on how efficiency indices can be applied to recover preferences from observable data using parametric methods.

5.2 Money-pump index and just-noticeable differences

Echenique, Lee, and Shum (2011) introduce an alternative measure of revealed preference violations, called the *money-pump index*. As this notion refers to choices over linear budget sets, in this subsection we restrict our attention to datasets $\mathcal{O} = \{(B_t, x_t)\}_{t \in T}$ such that, for all $t \in T$ and some price $p_t \in \mathbb{R}^{\ell}_{++}$, we obtain

$$B_t = \{y \in \mathbb{R}^{\ell}_{+} : p_t \cdot y \leq p_t \cdot x_t\}.$$

In such a case, the dataset satisfies GARP if for any sequence $\mathcal{C} = \{(a,b),(b,c),\dots,(z,a)\}$ in $T \times T$ such that $p_t \cdot x_s \leq p_t \cdot x_t$, for all $(t,s) \in \mathcal{C}$, all the inequalities are binding.

Suppose that the dataset $\mathcal{O}$ admits a cycle $\mathcal{C}$ specified as above. The money-pump index $m_{\mathcal{C}}$ corresponding to sequence $\mathcal{C}$ is given by

$$m_{\mathcal{C}} \;=\; \frac{\sum_{(t,s)\in\mathcal{C}} p_t \cdot x_s}{\sum_{(t,s)\in\mathcal{C}} p_t \cdot x_t}.$$[23]

To obtain a single value of the measure, Echenique, Lee, and Shum propose to take the mean or the median of money-pump indices, over all such cycles $\mathcal{C}$. Alternatively, Smeulders, Cherchye, De Rock, and Spieksma (2013) suggest to use the least or the greatest value of the index, as it reduces the computational burden of evaluating the measure. In order to guarantee that the value of the index is equal to 1 if and only if $\mathcal{O}$ satisfies GARP, we find it appropriate to use its mean or minimum.[24]

The money-pump index has an intuitive interpretation. Consider a sequence $\mathcal{C}$ specified as previously. Suppose there is a devious "arbitrageur" who is aware of the choices made by the consumer. In particular, by buying bundle x_s from the agent at prices p_t and reselling it at p_s, for any pair (t,s) in $\mathcal{C}$, the arbitrageur would make a positive profit equal to $p_a \cdot (x_a - x_b) + p_b \cdot (x_b - x_c) + \ldots + p_z \cdot (x_z - x_a)$. This value is the *money-pump cost* associated with the sequence. Hence, measure $(1 - m_{\mathcal{C}})$ is the money-pump cost normalised by the total expenditure in the cycle $\mathcal{C}$.

Similarly to the inverse of the just-noticeable difference, the purpose of the money-pump index is to evaluate the extent to which a cycle observed in the dataset is violating GARP. To show the distinction between our approach and the above measure, take any

[23] The original definition of the money-pump index for sequence $\mathcal{C}$ is equivalent to $(1-m_{\mathcal{C}})$. We modified the notion to make it comparable with the other measures discussed in this paper.

[24] Notice that, it is possible for a set of observations to violate GARP, even though the median or the maximal value of the corresponding money-pump index is equal to 1.

cycle $\mathcal{C} = \{(a,b),(b,c),\ldots,(z,a)\}$ specified as before. Since we focus on linear budget sets, we obtain $\theta_{ts} = p_t \cdot x_s / p_t \cdot x_t$, for all $t,\, s \in T$. Once we let $r_t := p_t \cdot x_t / \sum_{(i,j)\in\mathcal{C}} p_i \cdot x_i$ to be the share of budget $p_t \cdot x_t$ in the total expenditure along the cycle $\mathcal{C}$, this allows us to reformulate the money-pump index associated with cycle $\mathcal{C}$ as

$$m_{\mathcal{C}} \;=\; \sum_{(t,s)\in\mathcal{C}} r_t \theta_{ts}.$$

Therefore, the value of $m_{\mathcal{C}}$ is equivalent the to the algebraic mean of scalars θ_{ts}, for all $(t,s) \in \mathcal{C}$, weighted by the shares r_t. Just like the inverse of the just-noticeable difference, rather than focusing on "breaking" cycles, the money-pump index measures GARP violations by evaluating the average deviation from rationality along each such sequence. Hence, apart from the obvious differences, the idea of the two approaches is similar. Specifically, as both measures are sufficiently sensitive to changes in the dataset, they are not vulnerable to the critique of efficiency indices presented in Echenique, Lee, and Shum, that was also discussed in Figure 5 and Example 1.

Despite the similarities, there are several significant differences between the two approaches. First of all, since $m_{\mathcal{C}}$ depends on weights r_t, scaling prices $\{p_t\}_{t\in T}$ by different constants affects the value of the index. This is because absolute values of expenditures are relevant for establishing the money-pump cost associated with each cycle. In contrast, the just-noticeable difference (as well as efficiency indices) focus only on the actual budget sets. Thus, they are independent of such transformations. Please note that it is arguable which property is more desirable.

Second of all, unlike efficiency indices or the just-noticeable difference, the money pump-index does not provide a method of eliciting preference from the observable data. Even though it measures the extent to which an agent violates GARP, it does not propose an alternative model that would explain her behaviour.

Finally, suppose that we normalise each price in the dataset, so that $p_t \cdot x_t = 1$, for all $t \in T$. Clearly, such a transformation is irrelevant to whether the set of observations satisfies GARP. However, the corresponding money-pump index changes its interpretation. Namely, since $r_t = 1/|\mathcal{C}|$ for each $t \in T$, value $(1 - m_{\mathcal{C}})$ determines the average share of each budget set along the cycle that the consumer is losing to the arbitrageur. Thus, it is a money-pump in terms of fractions of expenditure losses. In fact, this interpretation is almost analogous to the just-noticeable difference, which approximates the geometric mean of shares $\theta_{ts} = p_t \cdot x_s / p_t \cdot x_t$ along each cycle $\mathcal{C}$.

5.3 A significant just-noticeable difference

When measuring revealed preference violations, it is fundamental to determine the value of the index at which it is considered significant. In this subsection, we propose one simple method of establishing such a critical value by employing the notion of the just-noticeable difference. Our approach is not a formal statistical test; rather, it is an objective way of establishing which values of our measure can be regarded as "large".

Assume that consumer preference can be represented by a continuous utility function u that satisfies $u(\lambda' a) > u(a)$, for all $\lambda' > 1$ and non-zero $a \in \mathbb{R}^{\ell}_{+}$. Given our discussion in Section 3.1, any set of observations generated via maximisation of the above relation would satisfy GARP. However, whilst evaluating the available options, the agent is prone to a particular error. To be precise, for each $t \in T$, there is a number $\lambda_t > 1$ drawn from a cumulative probability distribution $F : (1, \infty) \to [0, 1]$ such that, for all $t \in T$:

$$u(z) \geq u(x_t) \text{ implies } u(\lambda_t z) > u(y), \text{ for all } y \in B_t \text{ and non-zero } z \in \mathbb{R}^{\ell}_{+}. \tag{5}$$

Thus, bundle x_t need not maximise u over B_t, as there might be some available option y such that $u(y) > u(x_t)$. Nevertheless, it remains optimal in the above sense as long as the indifference curves intersecting x_t and y are sufficiently close together. This model of consumer choice resembles the one introduced in Section 4. However, in this case, we do not assume that the agent is making a consistent error; rather, we require for λ_t to be drawn independently in each observation and vary between choices.

In the remainder of this section we assume that $\lambda_t = \exp |\epsilon_t|$, where ϵ_t is drawn independently from a normal distribution with mean 0 and a standard deviation σ, for each $t \in T$. Equivalently, this is to say that $F(z) = \Phi_\mu(\log z)$, where Φ_μ denotes the half-normal cumulative distribution with mean $\mu = \sigma\sqrt{2/\pi}$.

Suppose we observe the numbers $\{\lambda_t\}_{t \in T}$ and denote $\lambda := \max\{\lambda_t : t \in T\}$. In the null hypothesis of our test we assume that the realisations are drawn independently from the F. Our decision to reject the hypothesis is based on the statistic λ. Given the construction of the variable, its cumulative distribution is given by $G(\lambda) := \big[F(\lambda)\big]^{|T|}$.[25] We reject the null hypothesis if λ is strictly greater than the critical value $c_\alpha := G^{-1}(1-\alpha)$,

[25] This follows from the fact that $\text{Prob}\big(\lambda \leq z\big) = \text{Prob}\big(\lambda_t \leq z, \text{ for all } t \in T\big)$

where $\alpha \in [0, 1]$ is a predetermined confidence level. That is, we consider it unlikely for the set of observations to be generated by the above model if the probability of drawing a number greater than λ from G is lower than α.

The intuition of our test is the following. Suppose an agent is a utility maximiser. However, in each observation t the choice is affected by a noticeable difference λ_t drawn independently from distribution F. Unless the maximal value of the observed realisations exceeds the $(1-\alpha)$'th quantile of the the corresponding distribution G, we have no basis for to reject the hypothesis that the set of observations was generated by the above model. Otherwise, we consider it unlikely for the data to be generated by utility maximisation with the particular form of random error.

Unfortunately the actual realisation of the statistic λ is unobservable. Nevertheless, we claim that it is always bounded below by the just-noticeable difference λ_*. In fact, it is easy to verify that, whenever a set of observations $\mathcal{O}$ is rationalisable in the above sense, then $u(z) \geq u(x_t)$ implies $u(\lambda z) > u(y)$, for all $y \in B_t$ and non-zero $z \in \mathbb{R}^\ell_+$, $t \in T$. Therefore, any such dataset can be supported by the model discussed in Section 4. Moreover, Proposition 5 and the argument in Section 4 guarantees that the set of observations satisfies Axiom 1 for λ. Hence, it must be that $\lambda_* \leq \lambda$.

Given the above observation, we employ λ_* as an alternative statistic.[26] Thus, we reject the null hypothesis if $\lambda_* > c_\alpha$. Introducing λ_* to our test has a significant drawback. Namely, whenever λ_* is strictly greater than c_α, the null hypothesis may be rejected with at least the imposed confidence level α. However, substituting the true value of λ with the just-noticeable difference leads to under-rejection of the null hypothesis, affecting the actual power of the test. This is because $\lambda_* \leq c_\alpha$ does not exclude the possibility that the actual realisation of λ is strictly greater that the critical value.

The result of our test depends crucially on the choice of the parameter σ that determines the cumulative probability distribution F.[27] In the following section we perform the test for several values of σ to provide ourselves with a better intuition of what could be regard as a significantly "large" value of the just-noticeable difference.

[26] Constructing a statistical test by bounding the true statistic from below has been performed in, e.g., Varian (1985) and Echenique, Lee, and Shum (2011).

[27] A similar issue was encountered in, e.g., Varian (1985), Echenique, Lee, and Shum (2011).

	Just-noticeable difference	Inverse of the just-noticeable difference	Afriat's efficiency index	Minimal money-pump index
Mean	**1.0898**	**.9210**	**.9440**	**.9064**
	[.0690]	[.0529]	[.0502]	[.0616]
Minimum	1.0042	.6868	.6868	.5990
First quartile	1.0436	.8946	.9260	.8726
Median	1.0707	.9340	.9563	.9203
Third quartile	1.1178	.9583	.9760	.9511
Maximum	1.4559	.9958	1.0000	.9951

Table 1: Descriptive statistics for the just-noticeable difference, its inverse, Afriat's efficiency index, and the minimal money-pump index evaluated for the available sample of households. The values within brackets indicate standard deviations of the variables. Recall that Afriat's efficiency index may take the value of 1 even if the set of observations violates GARP.

6 Implementation

We implement our test using household-level scanner panel data from the so-called Stanford Basket Dataset. The set of observations was collected by Information Resources Inc. and has been thoroughly analysed, among others, in Echenique, Lee, and Shum (2011) or Smeulders, Cherchye, De Rock, and Spieksma (2013). The panel contains grocery expenditure data for 494 households, aggregated monthly, with 26 purchase observations per household, and the total number of 375 available commodities. For further details, see Section IV of Echenique, Lee, and Shum (2011).

In our analysis we assume that, for each household, an observation set $\mathcal{O}$ consists of elements (B_t, x_t), for $t = 1, \ldots, 26$, where $x_t \in \mathbb{R}^\ell_+$ denotes the bundle of $\ell = 375$ consumption goods selected at observation t from a linear budget set

$$B_t = \left\{ y \in \mathbb{R}^\ell_+ : p_t \cdot y \leq p_t \cdot x_t \right\},$$

given the observed shelf prices $p_t \in \mathbb{R}^\ell_{++}$ of all the available commodities. In the preliminary analysis we established that, out of all 494 households, 396 violated GARP. The remainder of our analysis pertains to the latter subset of households.

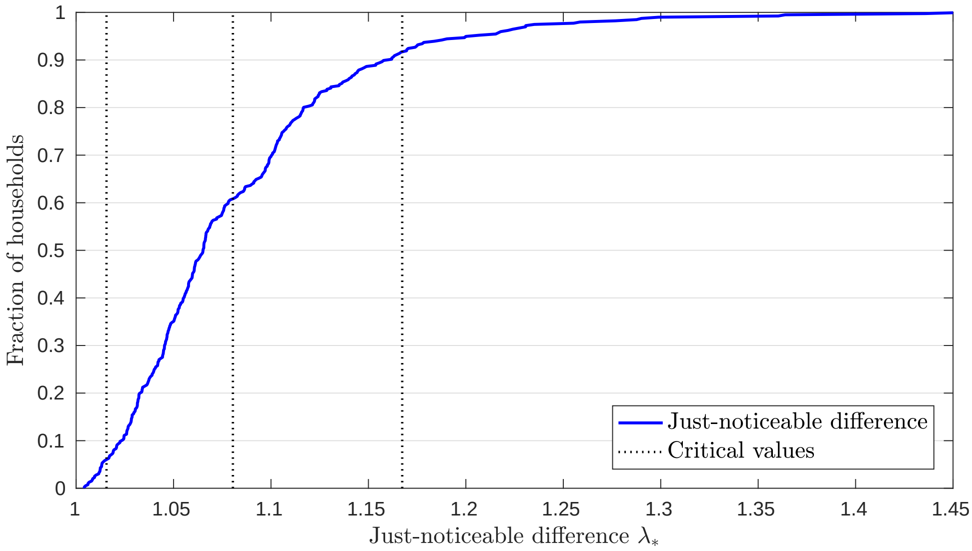

Figure 6: Empirical cumulative distribution of the just-noticeable difference λ_*. The vertical dotted lines mark the critical values of the variable evaluated for the parameter σ equal to .005, .025, and .05, respectively. See Table 2 for details.

6.1 Eliciting the just-noticeable difference

For each household, we determined the infimum over all numbers $\lambda > 1$ under which the corresponding dataset obeyed Axiom 1.[28] Thus, the just-noticeable difference λ_*. In the first column of Table 1 we present descriptive statistics of the empirical distribution of the parameter, evaluated for the analysed sample of households. The median of λ_* was approximately 1.07. Roughly speaking, this is to say that in order to rationalise the median set of observations by a semiorder P with a noticeable difference, we had to assume that a 7% increase in the size of a bundle was sufficient to notice the difference. For 65% of datasets, the just-noticeable difference did not exceed 1.10.

The existing literature on sensory discrimination provides a variety of estimates for the so-called *Weber's constant*, for different types of stimuli. Recall that, the above value is defined as the ratio between the magnitudes of two physical stimuli that is necessary for a subject to discriminate between the sensations. See Laming (1997) for a detailed discussion on this and alternative measures of sensory discrimination. Depending on the type of the stimuli, the estimates of this ratio range from 1.017 for the perception of cold and temperature, to 1.25 when it concerns discriminating between two bitter substances. See Tables 8.1-8.3 in Laming (1997) for a summary of estimates obtained in the literature.

[28]Recall that, for a given value of $\lambda > 1$, it is possible to verify Axiom 1 in a polynomial time. We approximate the infimal value of the parameter using the *binary search* algorithm.

	Value of σ		
	.005	.025	.05
Critical value	1.0156	1.0804	1.1673
% of households	5.8 (24.5)	60.6 (68.4)	91.6 (93.3)

Table 2: Percentage of households for which the value of the just-noticeable difference was *not* significantly large, for the confidence level of .05. Values within the brackets indicate the total percentage of datasets, including the 98 that initially satisfied GARP.

Our evaluation of the median/mean just-noticeable difference λ_* is included in this range. Nevertheless, one should be cautious with such comparisons. First of all, the estimates of Weber's constant were obtained for one-dimensional stimuli, rather than a combination of sensations. Second of all, these fractions were estimated in probabilistic terms. That is, they determined a threshold for which an agent correctly indicated the more intense stimuli with a predetermined frequency. At the same time, parameter λ_* represents the minimal value by which one should scale a bundle x in order to guarantee that the agent prefers $(\lambda_* x)$ to x, for *all* non-zero x. Finally, it is arguable whether preference over consumption bundles and physical stimuli are comparable notions.

In order to provide ourselves with an intuition whether the obtained values of the just-noticeable difference were significantly high, we employed the test introduced in Section 5.3. We performed our analysis for three values of the parameter σ: .005, .025, and .05. To put those numbers in to perspective, this implied that the theoretical mean value of the random variable λ_t was approximately 1.004, 1.020, and 1.041, respectively. Table 2 presents the evaluated critical values for the confidence level $\alpha = .05$ as well as the percentage of households for which we were unable to reject the null hypothesis. That is, for the indicated share of households we did not dismiss the possibility that the data was generated by maximisation of a locally non-satiated utility function with the particular random error. Equivalently, we concluded that choices of those households were not significantly violating GARP, conditional on the value of σ.

For the middle value of the parameter $\sigma = .025$, the critical value of our test was 1.0804. This is to say that, if one was willing to accept that the mean of the random error affecting consumer preferences was 1.020, i.e., on average the agent was able to notice a 2% increase in the size of a bundle, then the probability of observing a just-noticeable

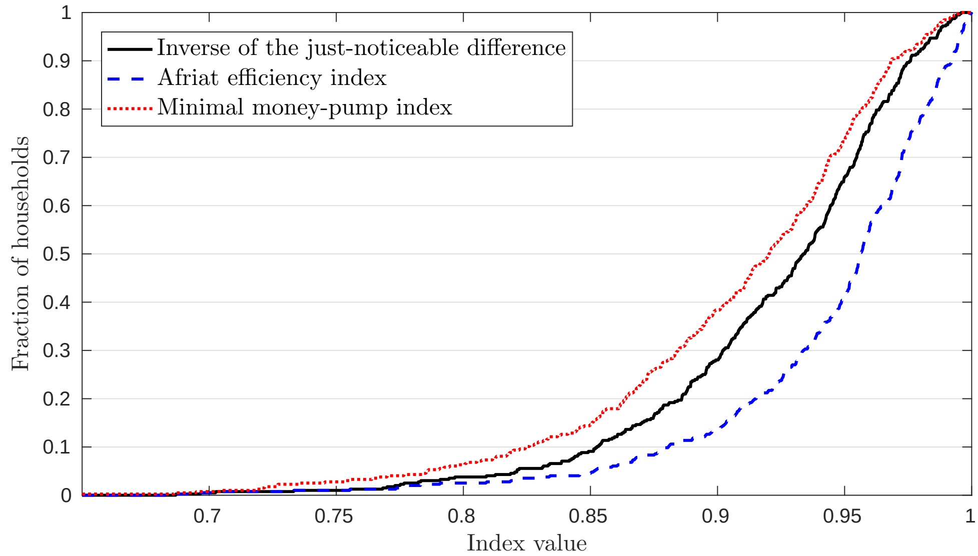

Figure 7: Empirical cumulative distributions of the inverse of the just-noticeable difference, Afriat's efficiency index, and the minimal money-pump index.

difference λ_* greater than 1.0804 was lower than $\alpha = .05$. Under this assumption, the null hypothesis had to be rejected for almost 40% of datasets. However, since the test generally under-rejected the the null hypothesis, the actual number of such subjects was possibly higher. Finally, in order to conclude that choices of all households were not significantly violating GARP, it was necessary to assume that σ was at least .1213. Thus, the theoretical mean of the variable λ_t had to be greater than 1.10.

6.2 Comparing measures of revealed preference violations

In the second part of our analysis we performed a comparison between different measures of revealed preference violations. More precisely, we focused on three indices that were discussed in Section 5. Namely, the inverse of the just-noticeable difference, Afriat's efficiency index, and the minimal money-pump index. Concerning the latter measure, we decided to restrict our attention to the minimum over money-pump indices for two reasons. First of all, unlike the median or the maximum, the minimum attains the value of 1 if and only if the corresponding set of observations satisfies GARP. Second of all, unlike the mean, determining the exact value of the minimum over all money-pump indices is computationally efficient and can be performed in a polynomial time. See Smeulders, Cherchye, De Rock, and Spieksma (2013) for details.

The remaining three columns of Table 1 provide descriptive statistics of the evaluated

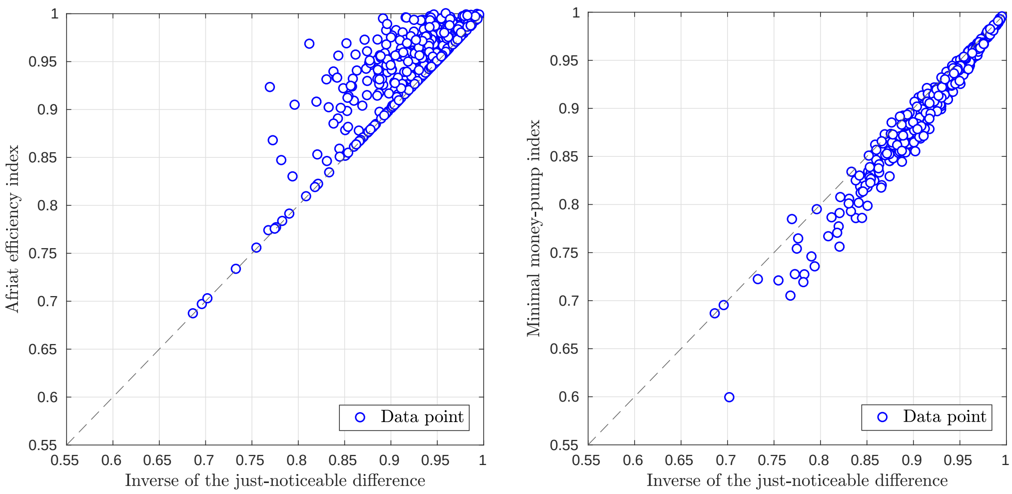

Figure 8: Inverse of the just-noticeable difference relatively to Afriat's efficiency index (on the left) and the minimal money-pump index (on the right).

distributions of the indices. See also Figure 7. First of all, the distributions of all measures were statistically distinct.[29] In addition, for each household, the corresponding inverse of the just-noticeable difference and the minimum of the money-pump index were lower than Afriat's efficiency index. By Proposition 6, such a relation was expected from our measure, as it is always dominated by Afriat's efficiency index. However, given the discussion in Section 5, it is easy to show that the same property generally holds for the minimum of the money-pump index.[30] In addition, the distribution of the minimal money-pump index was stochastically dominated by the distribution of the inverse of the just-noticeable difference, even though there was no such relation between the two measures in the individual-level data.[31]

We found it instructive to determine correlations between the three measures of revealed preference violations. The correlation between the inverse of the just-noticeable difference and the minimal money-pump index was .9822. Thus, the values of the two measures seemed to be very closely related in the analysed dataset. In fact, this is even more visible in Figure 8 (on the right). Simultaneously, the two indices were less correlated with Afriat's efficiency index, although values of the corresponding coefficients remained relatively high — respectively .8514 and .8614 for the inverse of the just-noticeable dif-

[29]In each pairwise comparison, we rejected the null hypothesis of the Anderson-Darling test that the random variables were drawn from the same distribution, for all conventional confidence levels.

[30]By construction of the efficiency index, there is a cycle $\mathcal{C}$ such that $e_* = \max\{\theta_{ts} : (t,s) \in \mathcal{C}\}$. Hence, we have $m_{\mathcal{C}} = \sum_{(t,s)\in\mathcal{C}} r_t\theta_{ts} \leq e_*$. In particular, this holds for the minimal money-pump index.

[31]To be precise, for 378 out of 396 households the minimal money-pump index was strictly lower that the inverse of the just-noticeable difference. Values of the two measures were never equal.

ference and the minimal money-pump index. Nevertheless, following the discussion in Section 5, this observations should not be surprising. Recall that, by construction, the inverse of the just-noticeable difference and the money-pump index are more sensitive to the distribution of observations in a dataset than Afriat's efficiency index. Therefore, they should respond similarly to particular violations of GARP.

Finally, we analysed differences in how each measure ordered individual datasets using the Spearman's rank correlation. In case of the just-noticeable difference and the money-pump index the correlation was .9731. Hence, the two approaches ordered datasets in a similar manner. At the same time, these rankings were less consistent with the one induced by Afriat's efficiency index. The corresponding correlations were .7953 and .8227 for the just-noticeable difference and the money pump index, respectively.

7 Conclusion

We provided the testable implications for a model of consumer choice in which an agent was unable to discriminate between similar options. Analogously to GARP, which is a special case of this condition, the purpose of our axiom was to exclude a specific type of cycles from the dataset. Furthermore, the method allowed us to elicit the just-noticeable difference which was sufficient for the agent to discern between bundles as well as the underlying preference that generated the observations. Finally, we showed that the evaluated noticeable difference provided an informative measure of deviations from rationality.

In the empirical analysis performed on a scanner panel data of food purchases we evaluated the just-noticeable difference for individual households. The median value of the parameter was approximately 1.07. That is, a 7% increase in the bundle size was sufficient for the consumer to notice the difference.

A Appendix

We begin this section by presenting one auxiliary result to which we refer in the main body of the paper. In the second part of the appendix we prove Proposition 1.

A.1 Auxiliary result

Take any numbers $\delta > 0$ and k_{ts}, for all indices t, s that belong to some finite set T. Consider the following proposition.

Proposition A.1. *Suppose that for any sequence $\mathcal{C} = \{(a,b),(b,c),\ldots,(z,a)\}$ in $T \times T$ such that $k_{ts} \leq 0$, for all $(t,s) \in \mathcal{C}$, we have $-\sum_{(t,s)\in\mathcal{C}} k_{ts} < |\mathcal{C}|\delta$. There are real numbers $\{\phi_t\}_{t\in T}$ and $\mu > 1$ such that $k_{ts} \leq 0$ implies $\phi_s - \delta \leq \phi_t + \mu k_{ts}$, for all t, s in T.*

Proof. We construct those numbers recursively. First, choose any $\mu > 1$ such that, for any cycle $\mathcal{C} = \{(a,b),(b,c),\ldots,(z,a)\}$ in $T \times T$ with $k_{ts} \leq 0$, for all $(t,s) \in \mathcal{C}$, we have $-\mu\sum_{(t,s)\in\mathcal{C}} k_{ts} \leq |\mathcal{C}|\delta$. Clearly, this is always possible.

With a slight abuse of the notation, we shall assume that $T = \{1,2,\ldots,T\}$. Moreover, let $\mathcal{S}_{az} = \{(a,b),(b,c),\ldots,(y,z)\}$ denote a sequence in $T \times T$ satisfying $k_{ts} \leq 0$, for all $(t,s) \in \mathcal{S}_{az}$. Take any number ϕ_1. For an arbitrary $t \geq 2$, let ϕ_t satisfy

$$\phi_s + \mu \sum_{(i,j)\in\mathcal{S}_{st}} k_{ij} + |\mathcal{S}_{st}|\delta \ \geq\ \phi_t \ \geq\ \phi_r - \mu \sum_{(i,j)\in\mathcal{S}_{tr}} k_{ij} - |\mathcal{S}_{tr}|\delta,$$

for all $\mathcal{S}_{st}$, $\mathcal{S}_{tr}$ and s, $r < t$. First, we show that such ϕ_t always exists. If not, then

$$\phi_s + \mu \sum_{(i,j)\in\mathcal{S}_{st}} k_{ij} + |\mathcal{S}_{st}|\delta \ <\ \phi_r - \mu \sum_{(i,j)\in\mathcal{S}_{tr}} k_{ij} - |\mathcal{S}_{tr}|\delta,$$

for some $\mathcal{S}_{st}$, $\mathcal{S}_{tr}$ and s, $r < t$. Consider three cases. Whenever (i) $s < r$, then

$$\phi_s + \mu\left(\sum\nolimits_{(i,j)\in\mathcal{S}_{st}} k_{ij} + \sum\nolimits_{(i,j)\in\mathcal{S}_{tr}} k_{ij}\right) + \big(|\mathcal{S}_{st}| + |\mathcal{S}_{tr}|\big)\delta \ <\ \phi_r.$$

However, as $\mathcal{S}_{st}$ and $\mathcal{S}_{tr}$ form a sequence $\{(s,a),\ldots,(z,t),(t,a'),\ldots,(z',r)\}$, this would be inconsistent with the construction of ϕ_r. Analogously, if (ii) $s > r$, then

$$\phi_s \ <\ \phi_r - \mu\left(\sum\nolimits_{(i,j)\in\mathcal{S}_{st}} k_{ij} + \sum\nolimits_{(i,j)\in\mathcal{S}_{tr}} k_{ij}\right) - \big(|\mathcal{S}_{st}| + |\mathcal{S}_{tr}|\big)\delta,$$

which is inconsistent with the construction of ϕ_s. Finally, whenever (iii) $s = r$, then $-\mu\big(\sum_{(i,j)\in\mathcal{S}_{st}} k_{ij} + \sum_{(i,j)\in\mathcal{S}_{tr}} k_{ij}\big) > \big(|\mathcal{S}_{st}| + |\mathcal{S}_{tr}|\big)\delta$. Given that $\mathcal{S}_{st}$ and $\mathcal{S}_{tr}$ form a cycle $\{(s,a),\ldots,(z,t),(t,a'),\ldots,(z',r)\}$, this is inconsistent with the construction of μ.

Take any such numbers $\{\phi_t\}_{t\in T}$ and μ. We argue that $k_{ts} \leq 0$ implies $\phi_s - \delta \leq \phi_t + \mu k_{ts}$, for all t, s in T. Take any t, s in T such that $k_{ts} \leq 0$. In particular, there is a one-element sequence $\mathcal{S}_{ts} = \{(t,s)\}$. Suppose that $t < s$. By construction of ϕ_t, ϕ_s, it must be that

$$\phi_t + \mu k_{ts} + \delta \ =\ \phi_t + \mu \sum_{(i,j)\in\mathcal{S}_{ts}} k_{ij} + |\mathcal{S}_{ts}|\delta \ \geq\ \phi_s.$$

Analogously, we can show that the inequality holds for $t > s$. Finally, whenever $t = s$, then $\phi_s - \delta \leq \phi_s + \mu k_{ts} = \phi_t + \mu k_{ts}$, since $-\mu k_{ts} \leq \delta$. $\square$

A.2 Proof of Proposition 1

In the remainder of the appendix we present the postponed proof of Proposition 1. The notation employed below corresponds directly to the one introduced in the article. Before we proceed, we find it convenient to state two preliminary results.

Lemma A.1. *If either (i) aPb, bPc, and cId; (ii) aPb, bIc, and cPd; or (iii) aIb, bPc, and cPd; then aPd, for any a, b, c, d in X.*

Proof. To show that (i) implies aPd, notice that, by condition (S3), if aPb and bPc then either aPd or dPc. As cId, it must be that aPd. Next, suppose that (ii) is true. Given aPb and cPd, condition (S2) requires that either aPd or cPb. Since bIc, we have aPd. Finally, we show that (iii) implies aPd. Assuming that bPc and cPd, condition (S3) implies either bPa or aPd. Given aIb, we conclude that aPd. $\square$

The following result provides a useful property of semiorderings.

Lemma A.2. *For any sequence $\mathcal{S} = \big\{(x_{i+1}, x_i)\big\}_{i=1}^{m}$ with $x_{i+1}Rx_i$, for all $i = 1, \ldots, m$, let $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{I}_{\mathcal{S}}$ denote sets containing pairs (x_{i+1}, x_i) such that $x_{i+1}Px_i$ and $x_{i+1}Ix_i$, respectively. Whenever (i) $|\mathcal{P}_{\mathcal{S}}| > |\mathcal{I}_{\mathcal{S}}| \geq 1$; or (ii) $|\mathcal{P}_{\mathcal{S}}| = |\mathcal{I}_{\mathcal{S}}| > 2$ is satisfied, there is a sequence $\mathcal{S}' = \big\{(y_{j+1}, y_j)\big\}_{j=1}^{n}$ with $y_{j+1}Ry_j$, for all $j = 1, \ldots, n$, such that $y_1 = x_1$ and $y_{n+1} = x_{m+1}$, while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_{\mathcal{S}}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_{\mathcal{S}}| - 1$.*

Proof. Take any such sequence $\mathcal{S} = \big\{(x_{i+1}, x_i)\big\}_{i=1}^{m}$. First, we claim that if (i) or (ii) holds, there is some $i = 1, \ldots, (m-2)$ such that either (a) $x_{i+3}Ix_{i+2}$, $x_{i+2}Px_{i+1}$, and $x_{i+1}Px_i$; (b) $x_{i+3}Px_{i+2}$, $x_{i+2}Ix_{i+1}$, and $x_{i+1}Px_i$; or (c) $x_{i+3}Px_{i+2}$, $x_{i+2}Px_{i+1}$, and $x_{i+1}Ix_i$.

We prove the claim by contradiction. Suppose that $|\mathcal{P}_{\mathcal{S}}| = k$. By assumption, we have $k \geq 2$. If there is no $i = 1, \ldots, (m-2)$ that obeys (a), (b), or (c), then it must be $|\mathcal{I}_{\mathcal{S}}| \geq 2(k-1)$.[32] Therefore, if $|\mathcal{P}_{\mathcal{S}}| > |\mathcal{I}_{\mathcal{S}}|$ then $k < 2$. On the other hand, $|\mathcal{P}_{\mathcal{S}}| = |\mathcal{I}_{\mathcal{S}}|$ implies $k \leq 2$. In either case, we reach a contradiction.

[32] In such a case, any two pairs (a, b), (c, d) such that aPb and cPd must be separated by at least two other elements (u, v), (w, y) for which uIv and wIy.

Given the above observation, take any $i = 1, \ldots, (m-2)$ that satisfies (a), (b), or (c). By Lemma A.1, this implies that $x_{i+3}Px_i$. Consider the following sequence:

$$\mathcal{S}' := \big\{(x_{m+1}, x_m),\ \ldots\ ,\ (x_{i+4}, x_{i+3}),\ (x_{i+3}, x_i),\ (x_i, x_{i-1}),\ \ldots\ ,\ (x_2, x_1)\big\}.$$

Clearly, for any $(a, b) \in \mathcal{S}'$, we have aRb, while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_{\mathcal{S}}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_{\mathcal{S}}| - 1$. $\square$

We proceed with the argument supporting Proposition 1.

Proof of Proposition 1. First, we show that $|\mathcal{P}_{\mathcal{S}}| = |\mathcal{I}_{\mathcal{S}}| = k$ implies $x_{m+1} \succ x_1$. Our argument is inductive on the cardinality of sets $\mathcal{P}_{\mathcal{S}}$ and $\mathcal{I}_{\mathcal{S}}$. By conditions (W1) and (W2), the result holds for $k = 1$. Suppose that $k = 2$, hence, $m = 4$. If there is some $i = 1, 2$ such that (a) $x_{i+3}Ix_{i+2}$, $x_{i+2}Px_{i+1}$, and $x_{i+1}Px_i$; (b) $x_{i+3}Px_{i+2}$, $x_{i+2}Ix_{i+1}$, and $x_{i+1}Px_i$; or (c) $x_{i+3}Px_{i+2}$, $x_{i+2}Px_{i+1}$, and $x_{i+1}Ix_i$; then $x_{i+3}Px_i$, by Lemma A.1. In particular, there exists a sequence $\mathcal{S}' = \big\{(y_3, y_2), (y_2, y_1)\big\}$ such that $y_1 = x_1$ and $y_3 = x_5$, while y_3Py_2 and y_2Iy_1, or y_3Iy_2 and y_2Py_1. In either case, (W1) and (W2) imply that $x_5 \succ x_1$. Alternatively, if there is no $j = 1, 2$ that obeys (a), (b), or (c), then

$$x_5Px_4,\ x_4Ix_3,\ x_3Ix_2,\ \text{and}\ \ x_2Px_1.$$

By definition of $\succeq$, we have $x_5 \succ x_3$ and $x_3 \succ x_1$, which implies $x_5 \succ x_1$.

Suppose our initial claim is true for some $k \geq 2$. Take any sequence $\mathcal{S}$, specified as in the thesis of the proposition, such that $|\mathcal{P}_{\mathcal{S}}| = |\mathcal{I}_{\mathcal{S}}| = k + 1$. By Lemma A.2, there is a sequence $\mathcal{S}' = \big\{(y_{j+1}, y_j)\big\}_{j=1}^{n}$ with $y_{j+1}Ry_j$, for $j = 1, \ldots, n$, such that $y_1 = x_1$ and $y_{n+1} = x_{m+1}$, while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_{\mathcal{S}}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_{\mathcal{S}}| - 1$. Given that $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{I}_{\mathcal{S}'}| = k$, our initial assumption guarantees that $x_{m+1} = y_{n+1} \succ y_1 = x_1$.

Next, we show that $|\mathcal{P}_{\mathcal{S}}| > |\mathcal{I}_{\mathcal{S}}| = k$ implies $x_{m+1}Px_1$. Our argument is inductive on the cardinality of the set $\mathcal{I}_{\mathcal{S}}$. By transitivity of P, the result holds for $k = 0$. Suppose the claim is true some $k \geq 0$. Take any sequence $\mathcal{S}$, defined as in Proposition 1, such that $|\mathcal{P}_{\mathcal{S}}| > |\mathcal{I}_{\mathcal{S}}| = k + 1$. By Lemma A.2, there is a sequence $\mathcal{S}' = \big\{(y_{j+1}, y_j)\big\}_{j=1}^{n}$ with $y_{j+1}Ry_1$, for $j = 1, \ldots, n$, such that $y_1 = x_1$ and $y_{n+1} = x_{m+1}$, while $|\mathcal{P}_{\mathcal{S}'}| = |\mathcal{P}_{\mathcal{S}}| - 1$ and $|\mathcal{I}_{\mathcal{S}'}| = |\mathcal{I}_{\mathcal{S}}| - 1$. Given that $|\mathcal{P}_{\mathcal{S}'}| > |\mathcal{I}_{\mathcal{S}'}| = k$, our initial assumption guarantees that $x_{m+1} = y_{n+1}Py_1 = x_1$. Since $x_{m+1}Px_1$ implies $x_1 \succ x_{m+1}$, this concludes the proof. $\square$

References

AFRIAT, S. N. (1967): “The construction of a utility function from expenditure data,” *International Economic Review*, 8(1), 67–77.

AHUJA, R. K., T. L. MAGNANTI, AND J. B. ORLIN (1993): *Network flows: Theory, algorithms, and applications*. Prentice-Hall.

ALGOM, D. (2001): “Psychophysics,” in *Encyclopedia of Cognitive Science*, pp. 800–805. Nature Publishing Group (Macmillan).

APESTEGUIA, J., AND M. A. BALLESTER (2015): “A measure of rationality and welfare,” *Journal of Political Economy*, 123(6), 1278–1310.

ARGENZIANO, R., AND I. GILBOA (2015): “Weighted Utilitarianism, Edgeworth, and the Market,” Discussion paper.

——— (2017): “Psychophysical foundations of the Cobb-Douglas utility function,” *Economic Letters*, 157, 21–23.

BEJA, A., AND I. GILBOA (1992): “Numerical representations of imperfectly ordered preferences (A unified geometric exposition),” *Journal of Mathematical Psychology*, 36, 426–449.

DEAN, M., AND D. MARTIN (2016): “Measuring rationality with the minimum cost of revealed preference violations,” *The Review of Economics and Statistics*, 98(3), 524–534.

DZIEWULSKI, P. (2017): “Eliciting the just-noticeable difference: Online appendix,” Discussion paper, Link: http://www.pawel-dziewulski.com/uploads/7/8/2/0/78207028/dziewulski_jnd_june2017_online_appendix.pdf.

ECHENIQUE, F., AND C. P. CHAMBERS (2016): *Revealed Preference Theory*, Econometric Society Monograph. Cambridge University Press.

ECHENIQUE, F., S. LEE, AND M. SHUM (2011): “The money pump as a measure of revealed preference violations,” *Journal of Political Economy*, 119(6), 1201–1223.

FISHBURN, P. C. (1975): “Semiorders and choice functions,” *Econometrica*, 43(5–6), 975–977.

FORGES, F., AND E. MINELLI (2009): “Afriat’s theorem for general budget sets,” *Journal of Economic Theory*, 144(1), 135–145.

GILBOA, I., AND R. LAPSON (1995): “Aggregation of semiorders: Intransitive indifference makes a difference,” *Economic Theory*, 5, 109–126.

HALEVY, Y., D. PERSITZ, AND L. ZRILL (2017): “Parametric recoverability of preferences,” *Journal of Political Economy*, forthcoming.

Houtman, M., and J. A. H. Maks (1985): "Determining all maximal data subsets consistent with revealed preference," *Kwantitative Methoden*, 19, 89–104.

Laming, D. (1997): *The Measurement of Sensation*, Oxford Psychology Series. Oxford University Press.

Luce, R. D. (1956): "Semiorders and a theory of utility discrimination," *Econometrica*, 24(2), 178–191.

Scott, D., and P. Suppes (1958): "Foundation aspects of theories of measurement," *Journal of Symbolic Logic*, 23(2), 113–128.

Smeulders, B., L. Cherchye, B. De Rock, and F. C. R. Spieksma (2013): "The money pump as a measure of revealed preference violations: A comment," *Journal of Political Economy*, 121(6), 1248–1258.

Varian, H. R. (1982): "The nonparametric approach to demand analysis," *Econometrica*, 50(4), 945–974.

——— (1985): "Non-parametric analysis of optimizing behavior with measurement error," *Journal of Econometrics*, 30, 445–458.

——— (1990): "Goodness-of-fit in optimizing models," *Journal of Econometrics*, 46(1–2), 125–140.