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# Evaluation of Rocking and Coupling Rotational Linear Stiffness Coefficients of Adjacent Foundations

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# 1 Evaluation of Rocking and Coupling Rotational Linear Stiffness Coefficients 2 of Adjacent Foundations

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14

## 15 Abstract

16 This paper presents closed form expressions for rocking spring stiffnesses and coupling-interaction  
17 rotational spring stiffnesses for a set of closely spaced footings. Sub-structuring is employed to derive  
18 analytically the exact reduced order spring models of the system. The stiffness coefficients of this  
19 reduced order model are determined by employing both (i) an extended, novel, application of  
20 Boussinesq's surface displacement of a point loaded half-space and (ii) an empirically derived  
21 formulation that makes use of both Finite Element and experimental results. Further validation  
22 suggests that, within the scope of epistemic uncertainty present in the physical world, the interaction  
23 formulae between two footings is sufficient for more general multi-footing interaction cases.

24 **Keywords** Structure-Soil-Structure Interaction; Discrete Lumped Parameter Model; Rocking Stiffness  
25 Coefficient; Coupling Rotational Stiffness Coefficient.

## 26        **Introduction**

27        Dynamic cross-interaction, also known as Structure-Soil-Structure Interaction (SSSI), among adjacent  
28        structures has received considerable attention in recent decades. Imperative works of Warburton  
29        (Warburton et al. 1971), Lee and Wesley (Lee and Wesley 1973), Luco and Contesse (Luco and  
30        Contesse 1973), Kobori et al (Kobori and Kusakabe 1980; Kobori and Minai 1974; Kobori et al. 1973;  
31        Kobori et al. 1977) and Qian and Beskos (Qian and Beskos 1995) have demonstrated the need to  
32        include cross-interaction effects in the seismic analysis of buildings located in close proximity. In fact,  
33        a Soil-Structure Interaction (SSI) analysis is not considered complete unless it takes into account the  
34        mutual interaction between adjacent structures via the underlying soil medium (Zaman 1982).

35        The analysis of problems involving ground and structure interaction, such as SSI and SSSI, are  
36        conducted predominantly via two approaches, (Stewart et al. 1998) and (Wolf 1985). The first is  
37        referred to as the *direct methodology* where the whole interacting system, i.e. structure and semi-  
38        infinite soil, is analysed in one step using numerical discretisation procedures such as the Finite  
39        Element Method (FEM) or Boundary Element Method (BEM) or a combination of both. One advantage  
40        of using such methods is the possibility to model complex geometries and system nonlinearities,  
41        especially that of the soil continuum. However, because of the large number of degrees of freedom  
42        (*dofs*) involved, these analyses are computationally costly and time consuming, and in addition are  
43        sensitive to changes in soil constitutive model parameters. The second and more popular technique  
44        is the *substructure or impedance method* where each interacting component is dealt with in a separate  
45        step then assembled to form the final solution taking advantage of the superposition principle. The  
46        method starts with the evaluation of the design input motion, *i.e.* kinematic interaction, followed by  
47        determination of the system's impedance function which is a complex valued function that describes  
48        the force/moment-displacement/rotation relationship. Next, dynamic analysis of the structure resting  
49        on the impedances from step two and subjected to the input motion from step one is conducted. The  
50        latter method is a convenient and reliable tool for both time and frequency domains analyses, (Wolf  
51        1994), (Bowles 1996), (Barros and Luco 1990) and (Dutta and Roy 2002). This approach allows a swift

52 calculation of system properties, conducting parametric studies, examining different design schemes  
53 and the appreciation of the essential features of the problem.

54 Although the substructure method has an advantage that is the ability of breaking down the complex  
55 SSI problem into more manageable components that could be easily verified, the analysis using this  
56 method is essentially linear and time invariant which is a simplification. The equivalent linear method,  
57 (Idriss and Seed 1967), is commonly used to approximate the soil nonlinearity during the site response  
58 analysis stage. On the other hand, using the direct method, time domain nonlinear and hysteretic soil  
59 models could be implemented which is in theory more rigorous representation. However, in addition  
60 to the computational expense, thorough understanding and expertise in using such soil models and  
61 parameter selection are required for engineering practice.

62 Results predicted using simplified models have been demonstrated to approximate physical  
63 observations, for example (Kobori et al. 1977) and (Aldaikh et al. 2016; Aldaikh et al. 2015) hence, such  
64 models could serve as a practical civil engineering analysis tool and provide preliminary estimates of  
65 the effects of complex interaction problems until the need for more sophisticated analyses is  
66 determined. Simplified discrete models with limited numbers of degrees of freedom have been well  
67 recognized and applied to the substructure method for the analysis of static and dynamic soil-  
68 structure interaction problems. In these mechanical models, a lumped parameter system treats all  
69 masses, springs and dashpots as if they were lumped into a single mass, single spring and single  
70 damping constant for each mode of vibration. Original works such as (Bycroft 1956) described how to  
71 define the characteristics of discrete models by matching the resulting impedance functions with  
72 those resulting from the use of continuum models, i.e. rigid foundations resting on an elastic half-  
73 space. Many imperative subsequent works on vertically loaded foundations were based on the same  
74 methodology, (Barkan 1962) and (Lysmer and Richart 1966).

75 Some numerical results, for example (Dobry and Gazetas 1986) showed that the impedance function  
76 of the discrete system, i.e. dynamic stiffness and damping characteristics, exhibited a dependency on

77 excitation frequency. This dependency is a result of the influence that frequency has on inertia rather  
78 than on soil properties, particularly (Gazetas 1993). As a result, linear Soil-Structure Interaction  
79 calculations cannot be directly used in time domain analyses and are usually performed in the  
80 frequency domain. By choosing representative frequency independent parameter values, the  
81 frequency dependency of the dynamic properties of the springs and dashpots can be reasonably  
82 approximated. It is suggested that these properties remain nearly constant within the frequency range  
83 of interest for typical building structures subjected to earthquakes, (Jennings and Bielak 1973) and  
84 (ATC 1978). Lumped parameter models have been used by some researchers to model the adjacent  
85 structures problem, i.e. SSSI in a 3D representation as in the work described by (Lee and Wesley 1973).  
86 Of particular mention are the studies presented by Mulliken and Karabalis (Karabalis and Mulliken  
87 1995; Mulliken and Karabalis 1998) where it has been illustrated that this kind of modelling with  
88 frequency independent lumped parameters can be successfully applied in the evaluation of  
89 interaction between rigid massive adjacent two and three identical surface foundations supported by  
90 a homogenous linear elastic half space subjected to various loadings including impulsive force,  
91 moment, sinusoidal and random signals. The coupling effect was incorporated into the solution by  
92 means of empirical stiffness and damping coupling coefficients which were calculated replacing  
93 numerical constants of static coefficients of stiffness and damping evaluated by Wolf (Wolf 1988) with  
94 functions of a dimensionless inter-foundation distance ratio. More recently, (Mykoniou et al. 2016)  
95 have used the same approach and utilised the coupling coefficients in (Mulliken and Karabalis 1998)  
96 to study the interaction of adjacent liquid-storage tanks.

97 Based on the above discussion, the aim for this paper is to introduce the theoretical background and  
98 mathematical formulations of the problem of adjacent surface footings within the linear elastic  
99 domain. The formulation is algebraically solved and simplified in order to obtain closed form solutions  
100 for the frequency-independent rotational foundation and coupling spring coefficients that could be  
101 used in recently developed simplified discrete analyses of SSSI problems (Aldaikh 2013). Only cases of  
102 two and three identical equispaced footings are considered. The paper also will examine if the

103 proposed formulae are comparable to a novel application of Boussinesq’s point loaded half-space  
104 solution and more sophisticated Finite Element analyses. In addition, an analogue experimental  
105 procedure examining the case of two adjacent foundations is described and results are used to  
106 validate the analytical and numerical analyses.

### 107 **Objectives**

108 The objectives of this paper are

- 109 1. To clarify the formulae for rotational coupling spring coefficients for the case of multi-  
110 footing interaction. These formulae are provided as an alternative to a full continuum  
111 model.
- 112 2. To theoretically demonstrate why the rotational coupling springs between adjacent and  
113 alternate footings must have negative values.
- 114 3. To derive a theoretical estimate based on a novel application of Boussinesq’s surface  
115 displacement of a half-space subjected to a point load. The accuracy of this theoretical  
116 estimate is compared with an empirical numerical/experimental fits for rotational  
117 coupling springs between adjacent footings.
- 118 4. To determine the validity of a previous assumption, (Aldaikh et al. 2015), in which the  
119 rotational coupled-interaction springs between alternate footings were ignored in SSSI  
120 analyses.
- 121 5. To determine whether it is sufficiently accurate to make use of the coupled interaction  
122 formulae derived originally in (Alexander et al. 2013) for two adjacent structures for the  
123 case of multiple adjacent footings (i.e. greater than two structures)?

### 124 **Model description**

125 Prior to developing the analytical formulations the following simplifying assumptions are initially  
126 outlined:

- 127 1. The analyses are limited to the linear elastic domain for both rigid foundations and underlying  
 128 half-space. Linear analysis is commonly adopted for the analysis of critical structures such as  
 129 nuclear power plants, also for machine foundation problems, (Wolf 1991).
- 130 2. Only cases of two and three identical equispaced (i.e. equal inter-building spacing) footings  
 131 are considered.
- 132 3. Foundation and coupling stiffnesses are independent of loading frequency, hence static  
 133 analysis is justified.

134 ***Reduced order models and mechanical analogue systems***

135 The static analysis of any linearly elastic mechanical system can be defined by the following algebraic  
 136 equations:

137 
$$\begin{bmatrix} \mathbf{f}_s \\ \mathbf{f}_m \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{ss} & \mathbf{K}_{sm} \\ \mathbf{K}_{sm}^T & \mathbf{K}_{mm} \end{bmatrix} \begin{bmatrix} \mathbf{u}_s \\ \mathbf{u}_m \end{bmatrix} \quad (1)$$

138 where  $\mathbf{u}_m$  is the vector of ‘master’ degrees of freedom (in this paper these will be the rotations at  
 139 footings) and  $\mathbf{u}_s$  is the vector of ‘slave’ degrees of freedom (which are all other displacement and  
 140 rotation *dofs*). Similarly  $\mathbf{f}_m$  is the vector actions applied at the ‘master’ *dofs* (in this paper these will be  
 141 applied moments at footings) and  $\mathbf{f}_s$  is vectors actions at all other *dofs*. Block matrices  $\mathbf{K}_{mm}, \mathbf{K}_{sm}, \mathbf{K}_{ss}$   
 142 are classical stiffness matrices. Eq.(1) can be condensed, (by *partitioning* or *sub-structuring* see Guyan  
 143 (Guyan 1965.)) to achieve the following reduced order model which is a reduced rank system:

144 
$$\mathbf{f} = \mathbf{K}\mathbf{u} \quad (2)$$

145 where matrices are defined as follows

146 
$$\mathbf{K} = \mathbf{K}_{mm} - \mathbf{K}_{sm}^T \mathbf{K}_{ss}^{-1} \mathbf{K}_{sm}, \quad \mathbf{f} = \mathbf{f}_m - \mathbf{K}_{sm}^T \mathbf{K}_{ss}^{-1} \mathbf{f}_s, \quad \mathbf{u} = \mathbf{u}_m \quad (3)$$

147 Note that if actions are only applied at ‘master’ degrees of freedom then  $\mathbf{f}_s = 0$  and the action vector  
 148  $\mathbf{f} = \mathbf{f}_m$ . If the displacement/rotations at ‘slave’ *dofs* are required then the following equation, Eq.(4),  
 149 could be employed although this equation would be equivalent to solving Eq.(1) directly.

$$150 \quad \mathbf{u}_s = \mathbf{K}_{ss}^{-1} (\mathbf{f}_s - \mathbf{K}_{sm} \mathbf{u}_m) \quad (4)$$

151 From energy considerations (Zienkiewicz et al. 2013) the global stiffness matrix of the system in Eq.(1)  
 152 is symmetric, hence the block matrices  $\mathbf{K}_{mm}$  and  $\mathbf{K}_{ss}$  must also be symmetric. Matrix  $\mathbf{K}_{sm}$  is not, in  
 153 general, symmetric.

154 A question arises as to whether the reduced order model stiffness matrix  $\mathbf{K}$  is necessarily symmetric.  
 155 It may be assumed from energy considerations that this should be true. Nevertheless, the following  
 156 simple proof demonstrates this. Two matrix theorems are employed (Petersen and Pedersen 2008),  
 157 first  $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$  which states that the inverse of a symmetric matrix is symmetric; hence  $\mathbf{K}_{ss}^{-1}$  is  
 158 symmetric. Second, using  $(\mathbf{ABC})^T = \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T$  it can be concluded that  $\mathbf{K}_{sm}^T \mathbf{K}_{ss}^{-1} \mathbf{K}_{sm}$  is also symmetric.  
 159 Hence it is known, without any loss of generality, that any reduced order model stiffness matrix  $\mathbf{K}$  is  
 160 symmetric.

161 While the reduced order system in Eq.(2) has been obtained from a condensed system in Eq.(1) it can  
 162 also be obtained from an independent system of three *dofs* interconnected with three springs. Fig.1(a)  
 163 displays a system of three static moments applied to a linear elastic half-space. This can be analysed  
 164 using the finite element method; which generally results in a large set of linear algebraic equations. In  
 165 the case at hand here it is desirable to define ‘master’ degrees of freedom as  $\mathbf{u}_m = [\theta_1, \theta_2, \theta_3]^T$ . The  
 166 reduced order model of this system has the form of Eq.(2) and in this particular case is a set of three  
 167 linear algebraic equations in terms of just the rotational degrees of freedom  $\theta_1, \theta_2$  and  $\theta_3$ .

168 It is clear mathematically that the mechanical system in Fig.1(b) is a *completely identical analogue* to  
 169 the condensed version of the system in Fig.1(a). If appropriate stiffness coefficients are assigned to  
 170 the springs in Fig.1(b) then its stiffness matrix (which is a general diagonal matrix) mathematically

171 equals the condensed stiffness matrix  $\mathbf{K}$  of the system in Fig.1(a). This is because both stiffness  
 172 matrices are arbitrary symmetric matrices.

173 However, while the reduced order model and mechanical analogue system have identical stiffness  
 174 matrices it may not be possible to ensure that the stiffness coefficients of all springs in the mechanical  
 175 analogue system are positive. In the case herein it turns out that all the coupled interaction springs  
 176  $k_{12}$ ,  $k_{23}$  &  $k_{13}$  that cross-couple the footings must be negative. By physical reasoning, (i.e. by  
 177 considering applied moments at the surface) it is clear that an anticlockwise rotation of a footing is  
 178 likely to produce a clockwise rotation of an adjacent footing. Therefore a 'spring' connecting these  
 179 two footings must have a negative stiffness. Thus, it is not easy to envisage a physical incarnation of  
 180 the mechanical analogue system Fig.1(b). It exists principally as a mathematical abstraction.

181 The potential energy of the system Fig.1(b) is given in Eq.(5) and its Euler-Lagrange equations are given  
 182 in Eq.(6)

$$183 \quad U = -\sum_{i=1}^3 M_i \theta_i + \frac{1}{2} \sum_{i=1}^3 k_i \theta_i^2 + \frac{1}{2} k_{12} (\theta_2 - \theta_1)^2 + \frac{1}{2} k_{23} (\theta_3 - \theta_2)^2 + \frac{1}{2} k_{13} (\theta_3 - \theta_1)^2 \quad (5)$$

$$184 \quad \begin{aligned} M_1 &= k_1 \theta_1 - k_{12} (\theta_2 - \theta_1) - k_{13} (\theta_3 - \theta_1) \\ M_2 &= k_2 \theta_2 + k_{12} (\theta_2 - \theta_1) - k_{23} (\theta_3 - \theta_2) \\ M_3 &= k_3 \theta_3 + k_{23} (\theta_3 - \theta_2) + k_{13} (\theta_3 - \theta_1) \end{aligned} \quad (6)$$

185 Using Eq.(6) (which are  $\partial U / \partial \theta_i = 0$ ) for any given set of moments (and their associated surface  
 186 rotation field) the stiffness coefficients  $k_i$  and  $k_{ij}$  can be evaluated. Castigliano's theorem states that  
 187 more than one load regime may be required to determine all stiffness coefficients in a general case.  
 188 However, not all combinations of load cases result in a rank sufficient system in terms of the stiffness  
 189 coefficients  $k_i$  and  $k_{ij}$  as variables, so care is required. Here, an analysis of the system in Fig.1(a) is  
 190 used to obtain the associated surface moments  $M_1$ ,  $M_2$  &  $M_3$  and rotations  $\theta_1$ ,  $\theta_2$  &  $\theta_3$ . Thus, the  
 191 spring stiffnesses for the mechanical analogue system can be derived.

192 **Surface displacement field caused by applied surface moments**

193 To determine the stiffness coefficients in Eq.(6) the surface moment-rotation relationship must be  
 194 determined. In this paper, two approaches are presented: (i) an analytic approximation based on a  
 195 combination of the application of the Boussinesq solution (Poulos and Davis 1974) and (M I Gorbunov-  
 196 Possadov et al. 1961) results and (ii) an empirical fit of finite element and experimental results.

197 For small deflections, the surface displacement field  $U(x)$  is defined in Eq.(7) in terms of a decay  
 198 function  $\Delta(x)$  (see Fig.2 ), where  $x$  is an arbitrary horizontal coordinate in the free surface plane

199 
$$U(x) = \frac{b}{2} \phi \Delta(x), \quad \text{for } x \geq \frac{b}{2}, \quad \Delta\left(\frac{b}{2}\right) = 1 \quad (7)$$

200 where  $\phi$  is the rotation of the rigid footing and  $b$  is the actual width of the footing. This equation is  
 201 non-dimensionalised by the introduction of the non-dimensional length  $x = \xi b$  (where  $\xi$  is a non-  
 202 dimensional horizontal coordinate) and non-dimensional surface vertical displacement  $u(x)$  (where  
 203  $U(x) = u(x)b$ ). Hence Eq. (7) becomes.

204 
$$u(\xi) = \frac{1}{2} \phi \Delta(\xi), \quad |\xi| \geq \frac{1}{2}, \quad \Delta\left(\frac{1}{2}\right) = 1 \quad (8)$$

205 By differentiating Eq.(8), an expression for the surface rotation field is obtained.

206 
$$\theta(\xi) = u'(\xi) = \frac{1}{2} \phi \Delta'(\xi), \quad |\xi| \geq \frac{1}{2}, \quad (9)$$

207 The prime notation in this equation is defined as  $\Delta' = d\Delta/d\xi$ .

208 **Boussinesq approximation for surface rotation field**

209 Boussinesq (Poulos and Davis 1974) suggested that the vertical surface displacement field  $\rho_z$  due to a  
 210 vertical point load  $P$  applied to a linear elastic half-space is given by the following equation

211 
$$\rho_z(x) = \frac{P(1-\nu^2)}{\pi E} \frac{1}{|x|}: \quad |x| \gg 0 \quad (10)$$

212 where  $\nu$  is the Poisson's ratio and  $E$  is the elastic modulus of the half space. The ordinate  $x$  is any radial  
 213 distance from the point load in the surface plane. If this formula, Eq.(10), is applied to the case of a  
 214 couple of equal and opposite forces one located at  $x = -b/2$  and the other at  $x = +b/2$  an estimate of  
 215 the surface vertical displacement function  $U(x)$  due to an applied moment  $m = Pb$  can be obtained by  
 216 superposition, as follows:

$$\begin{aligned}
 217 \quad U(x) &= \rho_z(x - \frac{b}{2}) - \rho_z(x + \frac{b}{2}) = \frac{P(1-\nu^2)}{\pi E} \left( \frac{1}{|x - \frac{b}{2}|} - \frac{1}{|x + \frac{b}{2}|} \right) \\
 &= \frac{m(1-\nu^2)}{\pi Eb^2} \frac{\text{sgn}(x)}{\left(\left(\frac{x}{b}\right)^2 - \frac{1}{4}\right)} : |x| \gg \frac{b}{2}
 \end{aligned} \tag{11}$$

218 The simplification above is easily obtained by assuming two cases one where  $x < -b/2$  and the other  
 219 when  $x > b/2$ . These two cases can be combined into equation (11) by employing the Signum  
 220 function. The Signum function used above is defined as  $\text{sgn}(x) = x/|x|$ . Note that this expression is  
 221 not valid in the range  $b/2 > x > -b/2$  where we assume that the footing imposes a linear  
 222 displacement field. Introducing the non-dimensional coordinate  $x = \xi b$  and displacements  
 223  $U(x) = u(x)b$

$$224 \quad u(\xi) = \frac{m}{k_s} \frac{1}{\pi} \frac{\text{sgn}(\xi)}{\xi^2 - \frac{1}{4}}, \quad k_s = \frac{Eb^3}{1-\nu^2} : |\xi| \gg \frac{1}{2}, \tag{12}$$

225 The rotation of the surface is given by differentiation for the case of small deflection theory, (Boas  
 226 2006)

$$227 \quad u'(\xi) = \theta(\xi) = -\frac{m}{k_s} \frac{1}{\pi} \frac{2\xi \text{sgn}(\xi)}{\left(\xi^2 - \frac{1}{4}\right)^2} \tag{13}$$

228 Note that the derivative of  $\text{sgn}(\xi)$  is a Dirac delta  $\delta(\xi)$  hence we would expect to see this in  
 229 equation (13). However, since the range of analysis here is limited to  $|\xi| \gg \frac{1}{2}$  the derivate terms  
 230 involving  $\delta(\xi)$  can be safely neglected as it is zero for  $\forall \xi \neq 0$ . This result is reasonably accurate away

231 from the application of the point loads but is, unfortunately, singular (infinite) at the edge of the  
 232 foundation, i.e.  $\xi = \frac{1}{2}$ , due to the limitation of Boussinesq's conjecture. So the formula suggested by  
 233 (M I Gorbunov-Possadov et al. 1961) is used instead for the rotation  $\phi$  at the footing itself.

$$234 \quad \phi = \frac{m}{k_s} = m \frac{1-\nu}{0.5Gb^3} \quad (14)$$

235 The term  $k_s$  in Eq.(12) is assumed to be the rotational stiffness of the footing;  $G$  is the elastic shear  
 236 modulus of the half-space. Therefore expressing  $m/k_s$  in terms of  $\phi$  and recalling the form of Eq.(8),  
 237 an estimate of the surface displacement at any point a non-dimensional distance  $\xi$  away from a  
 238 footing subject to a rotation  $\phi$  is obtained:

$$239 \quad u(\xi) = \frac{1}{2}\phi \Delta(\xi) \quad (15)$$

$$\Delta(\xi) = \frac{1}{2\pi} \frac{\text{sgn}(\xi)}{\xi^2 - \frac{1}{4}}: \quad |\xi| \gg \frac{1}{2}$$

240 It should be noted that this formula (15) gives  $\Delta(\frac{1}{2}) = \infty$  rather than 1. This is a consequence of the  
 241 singularity embedded in Boussinesq's result. By differentiation an estimate of the surface rotation  
 242 function  $\Delta'(\xi)$  is obtained:

$$243 \quad \Delta'(\xi) = -\frac{1}{\pi} \frac{\xi \text{sgn}(\xi)}{(\xi^2 - \frac{1}{4})^2}: \quad |\xi| \gg \frac{1}{2} \quad (16)$$

#### 244 ***Empirical fit surface decay function using finite element analysis (FEA)***

245 The weakness of Eq.(16) is that its accuracy is likely to reduce as  $\xi$  reduces i.e. as the footings get  
 246 closer together, and this is when it needs to be most accurate. Additionally, it does not include the  
 247 constraining effects of the footing itself, that is a footing applies a moment but also constrains  
 248 displacements locally. Finally, Eq.(16) is only applicable for a very simple case of a linearly elastic,  
 249 homogeneous isotropic half-space. For more complex cases finite element analysis is required.

250 From a finite element (PLAXIS2D, (PLAXIS-BV 2012)) solution of the problem of this single moment  
 251 applied to an isotropic linear elastic half-space, (Aldaikh 2013), a good least squares match ( $R^2=0.99$ )  
 252 to the decay function  $\Delta(\xi)$  is obtained by using the following inverse square relationship

$$253 \quad \Delta(\xi) = \frac{\text{sgn}(\xi)}{(2.83|\xi| - 0.415)^2} \quad : \quad |\xi| \geq \frac{1}{2} \quad (17)$$

254 The FE model is a two-dimensional (2-D) plane strain model (i.e. results are represented per unit length  
 255 in the out of plane direction) with linear elastic underlying material conditions which have the elastic  
 256 properties of the Polyurethane foam hereinafter described, Fig.3. Adjacent footings were modelled  
 257 using 2-D plate elements of 1m unit width, composed of beam elements with three degrees of  
 258 freedom: two translational *dofs* and one rotational *dof* in the *x-y* plane. The beam elements are  
 259 perfectly rigid and based on Mindlin's beam theory (See PLAXIS2D reference manual). The soil was  
 260 modelled using an unstructured mesh of 15 node triangular elements with finer mesh coarseness in  
 261 regions close to the foundation plates. It has been recommended that finite element mesh for shallow  
 262 foundations of width *r* on isotropic homogeneous soil usually includes an area extending to about  $5r$   
 263 laterally and  $8r$  vertically, an area within most of the stresses variation are expected to occur, (Azizi  
 264 2000).

265 Thus, by differentiating Eq.(17) we obtain an estimate of the surface rotation function  $\Delta'(\xi)$

$$266 \quad \Delta'(\xi) = -\frac{5.66}{(2.83|\xi| - 0.415)^3} \quad : \quad |\xi| \geq \frac{1}{2} \quad (18)$$

267 This empirical curve-fit in Eq.(17) is an inverse quadratic and as such is of the same order as Eq.(15). It  
 268 should be noted, however, that this equation, Eq.(17), is also constrained to give  $\Delta(\frac{1}{2})=1$  which is  
 269 the correct value and so it differs at small  $\xi$  from the Boussinesq derived Eq. (15) which is singular.  
 270 Fig.4(a) and Fig.4(b) respectively display comparisons between the surface decay function and surface

271 rotation function using the Boussinesq results, Eq.(15) and Eq.(16), with the FEA fitted functions,  
 272 Eq.(17) and Eq.(18). It can be seen that the form of both functions in Eq.(15) and Eq.(17) is very similar.

### 273 **Example applications**

274 In this section, two example cases are considered: (i) two identical footings, and (ii) three equispaced  
 275 footings. These are considered here to conjecture whether simple formulae for rotational spring  
 276 stiffnesses can be determined, that are sufficiently accurate (for practising engineering  
 277 analyses/design) for a range of different system geometries, (i.e. for a different number of footings  
 278 and non-identical ones).

#### 279 ***Analysis case 1: two identical rigid footings with interaction***

280 Consider Fig.1(b), where  $k_3 = k_{23} = k_{13} = 0$  and  $k_1 = k_2$ . For a load case it is assumed that a single  
 281 moment  $M_1 = m$  is applied to rigid footing 1, and  $M_2 = M_3 = 0$ . According to Eq.(9) the rotations of  
 282 the footings are, for this load case,  $\theta_1 = \phi$ ,  $\theta_2 = \frac{1}{2}\phi\Delta'(\xi)$  and  $\theta_3 = \frac{1}{2}\phi\Delta'(2\xi)$ . Hence, Eq.(6) can be  
 283 solved to determine the unknown stiffness coefficients  $k_1$  and  $k_{12}$

$$k_1 = k_2 = k_s \frac{2}{\Delta'(\xi) + 2}$$

$$284 \quad k_{12} = -k_1 \frac{\Delta'(\xi)}{\Delta'(\xi) - 2} \quad (19)$$

$$k_s = \frac{m}{\phi}$$

285 It should be noted that  $k_s$  would be the rotational spring stiffness of a single, completely isolated,  
 286 rigid footing; that is to say, the  $k_s$  value could be obtained directly from Eq.(14) (M I Gorbunov-  
 287 Possadov et al. 1961). The rotational spring stiffness  $k_1 \neq k_s$  as it includes the additional stiffening effect  
 288 of the adjacent footing.

289 **Analysis case 2: three identical, equispaced, rigid footings with interaction**

290 In this symmetrical case with identical rigid footings,  $k_1 = k_3$  and  $k_{12} = k_{23}$ . For this problem, there are  
 291 four unknown stiffness coefficients  $k_1, k_2, k_{12}$  &  $k_{13}$ . Hence two load cases are required. First, a  
 292 moment  $M_1 = m$  is applied to rigid footing 1 and  $M_2 = M_3 = 0$ . According to Eq. (9) the rotations of  
 293 the footings are, for this load case,  $\theta_2 = \phi$ ,  $\theta_2 = \frac{1}{2}\phi\Delta'(\xi)$  and  $\theta_3 = \frac{1}{2}\phi\Delta'(2\xi)$ . In the second load case,  
 294 a moment  $M_2 = m$  is applied to rigid footing 2 and  $M_1 = M_3 = 0$ . According to Eq. (9) the rotations  
 295 of the footings are, for this load case,  $\theta_2 = \phi$ ,  $\theta_1 = \theta_3 = \frac{1}{2}\phi\Delta'(\xi)$ . Hence Eq. (6) can be solved to  
 296 determine the unknown stiffness coefficients  $k_1, k_2, k_{12}$  &  $k_{13}$ .

$$k_1 = k_3 = k_s \frac{\Delta'(\xi) - 2}{\Delta'(\xi)^2 - \Delta'(2\xi) - 2} \quad (20)$$

$$k_2 = k_1 \frac{2\Delta'(\xi) - \Delta'(2\xi) - 2}{\Delta'(\xi)^2 - 2}$$

$$k_{12} = k_{23} = -k_1 \frac{\Delta'(\xi)}{\Delta'(\xi) - 2} \quad (21)$$

$$k_{13} = -k_1 \frac{\Delta'(\xi)^2 - \Delta'(2\xi)}{(\Delta'(\xi) - 2)(\Delta'(2\xi) - 2)},$$

299 **Experimental Evaluation of Spring Coefficients**

300 To physically validate the theoretical expressions proposed for the rotational coupling and foundation  
 301 springs, a simple experiment was performed for the case of two identical adjacent rigid foundations  
 302 as described in the following paragraphs. The aim here is to produce physical similitude of the  
 303 analytical method used to evaluate the rotational springs stiffnesses, i.e.  $k_i$  and  $k_{ij}$ .

## 304 **Setup and Procedure**

305 The two foundations were modelled with square Perspex plates (width  $B= 80$  mm and  $t=5$  mm thick)  
306 and were firmly glued using an epoxy adhesive to the surface of a Polyurethane foam block  
307 (dimensions:  $1000 \times 1000 \times 750$  mm, Young's modulus  $120$  kN/m<sup>2</sup>; Poisson's ratio  $0.11$  and density  $50$   
308 kg/m<sup>3</sup>). The foam block proved suitable as a representation of the linear elastic half-space, (Aldaikh et  
309 al. 2015), (Aldaikh et al. 2016) and (Soubestre et al. 2012). The experiment setup is depicted in Fig.5.  
310 A moment was applied at the centre of one plate (active plate) and the resulting rotations of the active  
311 plate itself and at the second plate (passive plate) were measured. This procedure was followed for  
312 different spacing intervals  $z$ , as shown in Table A.1, between the two plates to eventually derive a  
313 function between rotational springs stiffnesses and spacing. It was not, however, experimentally  
314 straightforward to apply a moment at the centre of the active plate, hence, an aluminium rod of  
315 negligible weight was fixed at the middle of the active plate which was pulled by a wire running  
316 through a pulley. The wire carried weights which would generate a tension force pulling the aluminium  
317 bar and creating a moment at the centre of the first plate.

318 The moment was equivalent to the tension force  $T$  multiplied by the lever arm  $l$ . Vertical displacements  
319 at the edges of each plate were recorded using Linear Variable Differential Transformer (LVDT)  
320 transducers, two per plate as shown in Fig. 5. Values of rotations  $\phi_1$  and  $\phi_2$  (Appendix A) at the centre  
321 of each plate were calculated as follows:

$$\phi_1 = \frac{y_2 - y_1}{B} \quad (22)$$

$$\phi_2 = \frac{y_4 - y_3}{B}$$

323 where  $y_1$  and  $y_2$  are the vertical displacements at the edges of the active plate (ends 1 and 2) where  
324 the moments were applied while  $y_3$  and  $y_4$  are the vertical displacements at the edges of the second

325 plate (ends 3 and 4). By rearranging Eq.(6) the formulae for  $k_1$  and  $k_{12}$  as functions of  $\phi_2$  and  $\phi_1$  are as  
326 follows:

$$\begin{aligned} k_1 &= k_s \frac{\phi_1}{\phi_1 + \phi_2} \\ k_{12} &= k_1 \frac{\phi_2}{\phi_1 - \phi_2} \\ k_s &= \frac{m_1}{\phi_1} \end{aligned} \tag{23}$$

328 where  $k_s$  is the experimentally determined foundation stiffness of an isolated footing (with no  
329 neighbouring footing).

## 330 **Results**

### 331 ***Analysis case 1: results***

332 Fig.6(a) and Fig.6(b) respectively present the variation of foundation rotational stiffness  
333 (normalised by  $k_s$ ) and interaction (coupling) rotational stiffness (normalised by  $k_1$ ) with the non-  
334 dimensional inter-footing spacing for the case of two identical adjacent footings. It can be seen from  
335 Fig.6(a) and Fig.6(b) that the increase in the rotational stiffness of a single foundation (i.e. separation  
336 distance independent) could reach up to 25% when there is a negligible distance between the edges  
337 of the adjacent foundations. Similarly, it can be seen that as the inter-foundation spacing increases  
338 the interaction effect diminishes. At a spacing of approximately 2.5 times the foundation's width, the  
339 rotational coupling stiffness is negligible. It can also be observed that results from the proposed  
340 formulation for both individual foundation and coupling interaction stiffness coefficients agree very  
341 well with both FEA and experimental data. Moreover, the current results for the coupling coefficients,  
342 Fig6.(b), are compared to those resulted from the logarithmic curve fitting formula proposed by  
343 Mulliken and Karabalis (Mulliken and Karabalis 1998). However, using the Boussinesq approximate  
344 Eq.(15) resulted in a slightly stiffer estimate of stiffness coefficients. It should be noted that the

345 experimental stiffness ratios shown in Fig.6(a) and Fig.6(b) are the average values resulted from all  
346 applied bending moment levels.

### 347 ***Analysis case 2: results***

348 In this section, the following questions are considered: (i) in the case where there are more than two  
349 adjacent foundations, would adjacent footing coupling springs  $k_{12}$  and  $k_{23}$  be sufficient to model the  
350 mutual interaction i.e. is the additional alternate footing coupling spring  $k_{13}$  necessary? (ii) are the  
351 resultant numerical values for  $k_{12}$  significantly different in the two and three footings case? (iii) are  
352 stiffness coefficients  $k_1$  and  $k_2$  significantly different in the two and three footings case?

353 These questions are examined in Fig.7(a) and Fig.7(b) where they respectively present the variation of  
354 foundation rotational stiffness and interaction (coupling) rotational stiffness with the inter-footing  
355 centre-to-centre spacing for the case of two adjacent footings in comparison to that where a third  
356 foundation is present.

357 The value of the alternate footing coupling spring coefficient  $k_{13}$  decreases as the footing spacing  
358 increases and it approximately equals one-quarter of that of the adjacent footing coupling  $k_{12}$  at  
359 spacing where footings touch, i.e. at  $\xi = 1$ , (see Fig.7(b)). Given other epistemic uncertainty present  
360 in the application of this theory to physical problems (e.g. due to the site characterisation of soil) it  
361 appears that the alternate footing coupling spring coefficient  $k_{13}$  may be neglected without significant  
362 error, as was done in (Aldaikh et al. 2015).

363 The values of the adjacent coupling spring coefficient  $k_{12}$  are almost identical for the case of two and  
364 three footings; i.e. formulae Eq.(19) and Eq.(21) for  $k_{12}$  produce almost identical results regardless of  
365 centre-to-centre footing spacing  $\xi$ . This suggests that Eq.(19) for adjacent coupling spring coefficients  
366 is a reasonable and simple approximation for a more general case.

367 Finally, the values of spring coefficients  $k_1$  and  $k_2$  for the two and three footing cases show very  
 368 similar qualitative forms. However, these coefficients in the three footing case are slightly stiffer than  
 369 the two footing case. Fig.8 displays these relative stiffening effects graphically when moving from two  
 370 to three closely spaced footings. The central spring  $k_2$  is generally greater than the outer spring  $k_1$  in  
 371 this case. It should be noted here that these small relative stiffening effects were neglected in (Aldaikh  
 372 et al. 2015).

373 ***Relative errors in employing the two footing formulation more generally***

374 Eq.(19) along with the surface slope decay function, Eq.(18) are simple and easy to adopt for a more  
 375 general case of multiple footings (greater than 2). The results in Analysis case 2 section suggest that  
 376 the Eq.(19) estimate of adjacent footing coupling rotational springs  $k_{i,i+1}$  are almost exactly the same  
 377 as the more complex and accurate Eq.(21). Additionally, these results suggest that there is an  
 378 argument to completely neglect alternate footing coupling rotational springs  $k_{i,i+2}$ . However, the  
 379 same results also suggest that if the estimate of foundation springs  $k_i$  from Eq.(19) is employed for a  
 380 more general case of multiple footings (greater than 2) then it tends to underestimate the stiffnesses  
 381 (see Fig.8 ).

382 Therefore the question remains if formulation in Eq.(19) is used for three footings (with  $k_1 = k_2 = k_3$   
 383 and  $k_{13} = 0$ ) rather than Eq.(20) and Eq.(21), what errors would be introduced?

384 For any given rotations of footings,  $\theta_i$  the resulting norm of moments  $\|M_i\|$  can be evaluated using  
 385 Eq.(6). This analysis is performed for both cases (a) stiffness from Eq.(19) with  $k_1 = k_2 = k_3$  and  $k_{13} = 0$   
 386 and (b) stiffness from Eq.(20) and Eq.(21). Therefore the relative percentage error  $\varepsilon$  of using  
 387 formulation Eq.(19) is expressed as follows.

388 
$$\varepsilon = 100 \frac{\|M_i\|_{\text{case (a)}} - \|M_i\|_{\text{case (b)}}}{\|M_i\|_{\text{case (b)}}} \quad (24)$$

389 The relative percentage error  $\varepsilon$  must be evaluated for a random set of footing rotations  $\phi_i \in [-1,1]$  ,  
390 i.e. a range of different load cases. Fig.9 displays the results of such an analysis, plotting the relative  
391 error, Eq.(24), as a function of the centre to centre footing spacing  $\xi$  .The mean error  $\mu(\varepsilon)$  at a touching  
392 distance ( $\xi = 1$ ) is approximately -7%. Given other epistemic uncertainties present (e.g. in site soil  
393 classification) in the application of this theory to physical problems, this is a small error.

## 394 **Conclusions**

395 The current study presented a simplified analytical formulation for the evaluation of frequency-  
396 independent stiffness coefficients for the problem of adjacent identical footings resting on a linear  
397 elastic half-space. A derivation of the formulae was presented for the case of two and three adjacent  
398 foundations. Boussinesq's solution for the surface displacement field caused by a point load is  
399 extended to the case of a moment and combined with the Gorbunov-Possadov moment-rotation  
400 relationship for an isolated footing.

401 The extended Boussinesq's solution, along with a rigorous finite element model and analogue physical  
402 model, showed excellent agreement with the proposed formulae for both foundation rotational and  
403 coupling spring stiffness coefficients. Contrary to the common assumption in past literature, the  
404 dependency of rocking stiffness of individual foundations on the inter-foundation spacing has been  
405 demonstrated which indicates that reliance on such spacing-independent rocking stiffness could lead  
406 to over-conservative analyses. Results have also shown that there exists only a small difference in the  
407 value of adjacent footing rotational stiffnesses when more than two foundations are considered in the  
408 analysis. Hence, omitting springs connecting alternate footings is permissible given the other  
409 epistemic uncertainties in a physical setting. Bearing in mind this limiting assumption, the formulae  
410 proposed in Eq. (18) and Eq.(19) are simple and straightforward to adopt for a more general case of  
411 multiple footings (greater than two). These can be directly used in the straight-forward  
412 implementation of discrete lumped parameter modelling of adjacent structure interaction problems  
413 which could save considerable computational effort in preliminary design.

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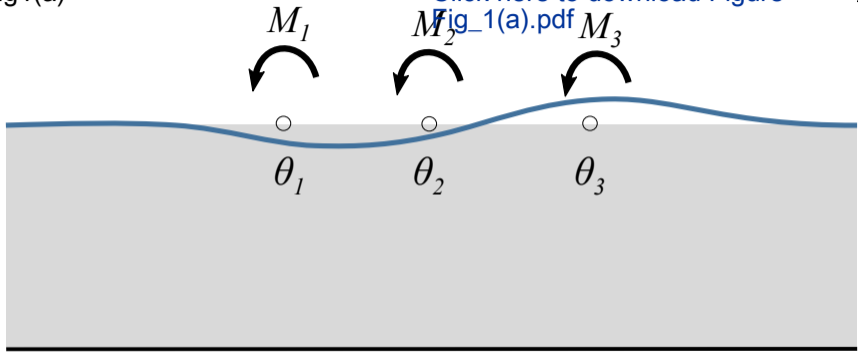
## Appendix A

**Table A.1** Foundations rotations in radians

<i>Spacing z [mm]</i>		8	16	24	32	40	48	56	64	72	80
<i>Moment m<sub>1</sub> [N.mm]</i>											
68	$\phi_1$	0.00355	0.0053	0.0057	0.0058	0.006113	0.00465	0.00485	0.0049	0.00497	0.005
	$\phi_2$	-0.0006	-0.00042	-0.00042	-0.0005	-0.00017	-0.00013	-0.00013	-0.00013	-0.00017	-0.00013
98	$\phi_1$	0.0071	0.0079	0.0083	0.0083	0.0091	0.0071	0.00751	0.0069	0.00764	0.0072
	$\phi_2$	-0.0009	-0.00066	-0.00063	-0.00063	-0.0003	-0.0002	-0.0003	-0.00013	-0.00023	-0.00017
173	$\phi_1$	0.0126	0.0144	0.0149	0.0153	0.0185	0.0128	0.0132	0.012	0.0128	0.0130
	$\phi_2$	-0.0016	-0.00126	-0.00099	-0.00099	-0.00053	-0.0004	-0.0005	-0.00037	-0.00027	-0.00023
248	$\phi_1$	0.0197	0.0215	0.0239	0.0261	0.0287	0.0191	0.0185	0.0186	0.0186	0.0191
	$\phi_2$	-0.0026	-0.00126	-0.00143	-0.00133	-0.0008	-0.00059	-0.00063	-0.0006	-0.00043	-0.00036
323	$\phi_1$	0.027	0.031	0.0356	0.0389	0.0399	0.0257	0.0254	0.0255	0.0255	0.0257
	$\phi_2$	-0.0034	-0.00253	-0.00193	-0.00169	-0.00106	-0.0008	-0.00086	-0.00076	-0.00063	-0.00049
473	$\phi_1$	0.0413	0.046	0.0539	0.0596	0.0597	0.0385	0.0392	0.0385	0.0385	0.0406
	$\phi_2$	-0.0049	-0.0038	-0.0026	-0.00243	-0.00159	-0.00133	-0.00123	-0.0011	-0.001	-0.00083
773	$\phi_1$	0.0777	0.088	0.0931	0.0912	0.0906	0.0744	0.0769	0.0750	0.0760	0.077
	$\phi_2$	-0.008	-0.00624	-0.0042	-0.0035	-0.00238	-0.00229	-0.00199	-0.00175	-0.00149	-0.0015

Fig1(a)

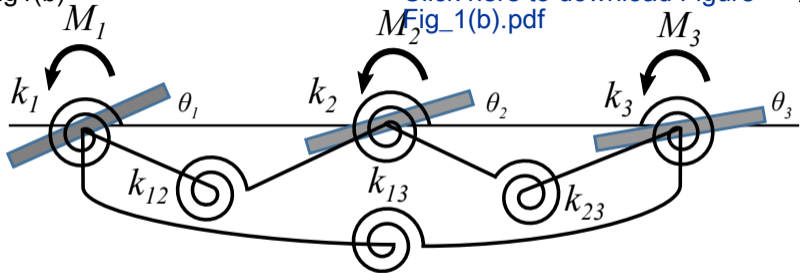
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(a)

Fig1(b)

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(a)

Fig2

[Click here to download Figure Fig\\_2.pdf](#)

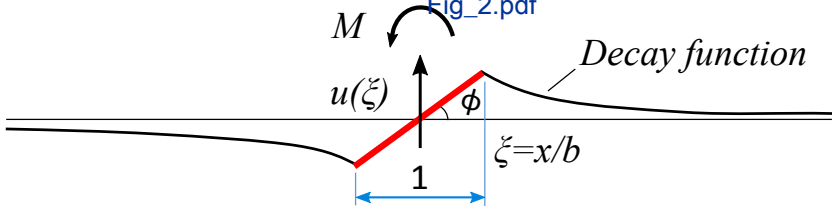


Fig4(a)

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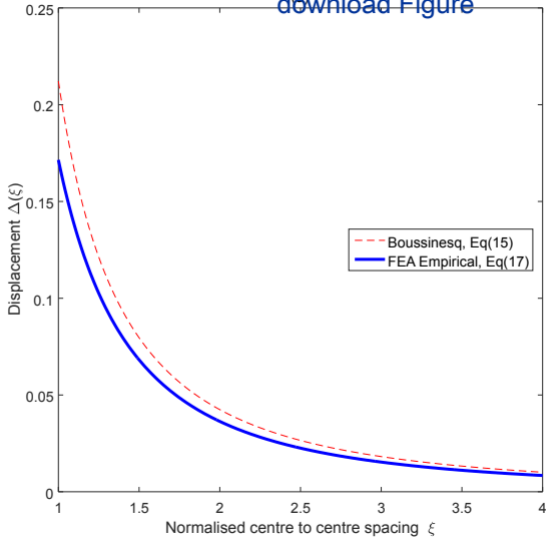
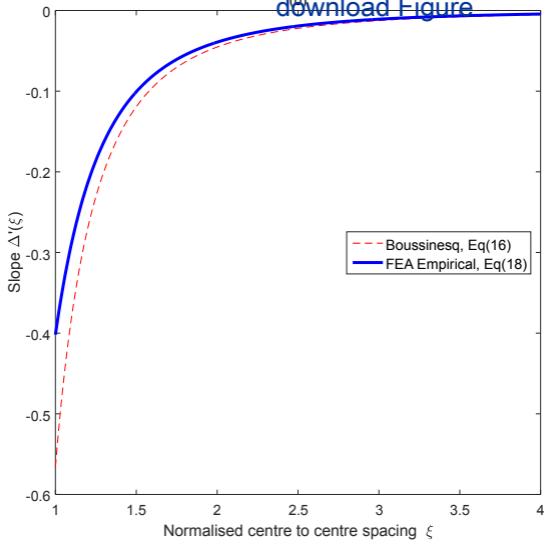


Fig4(b)

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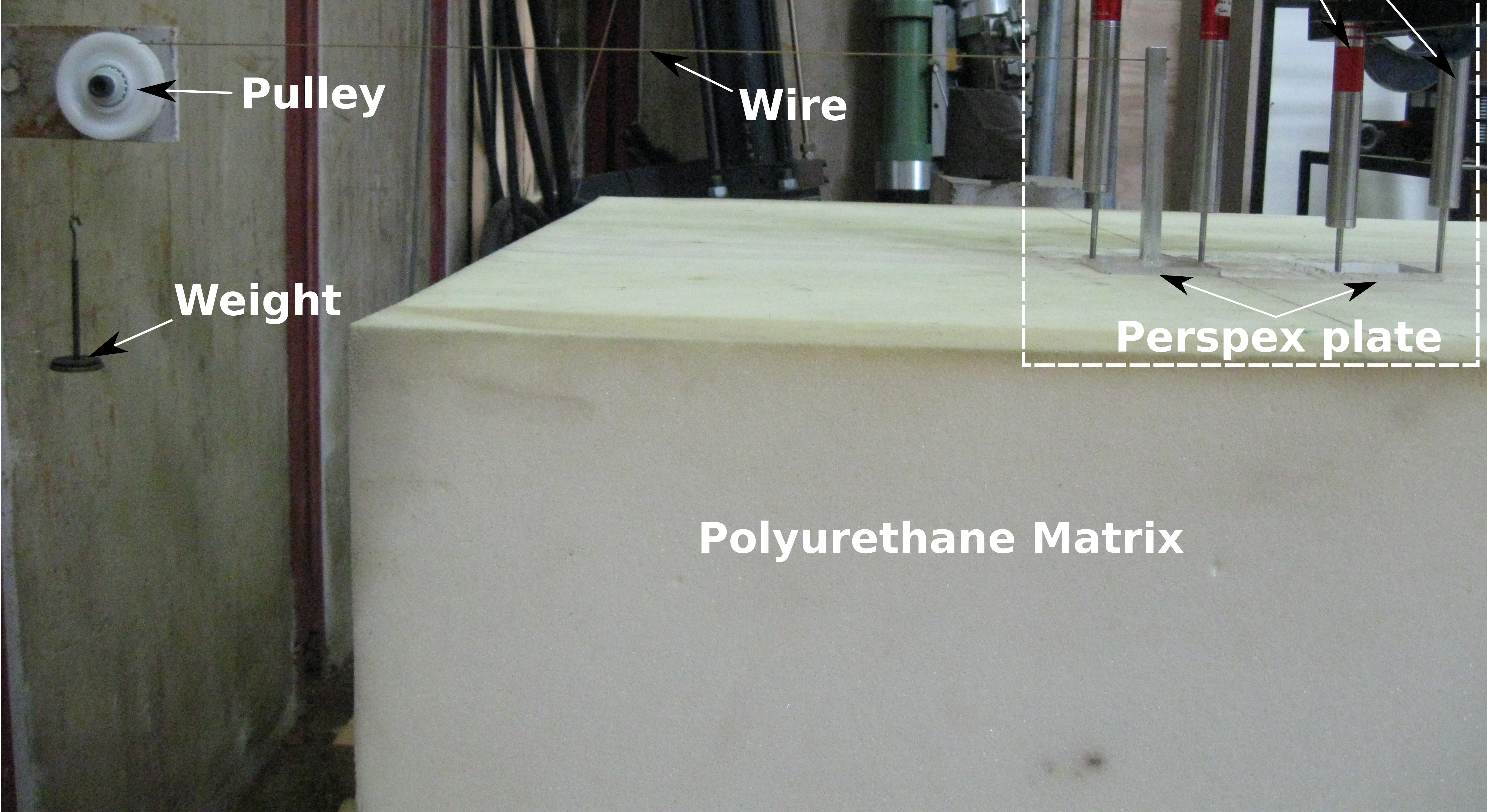
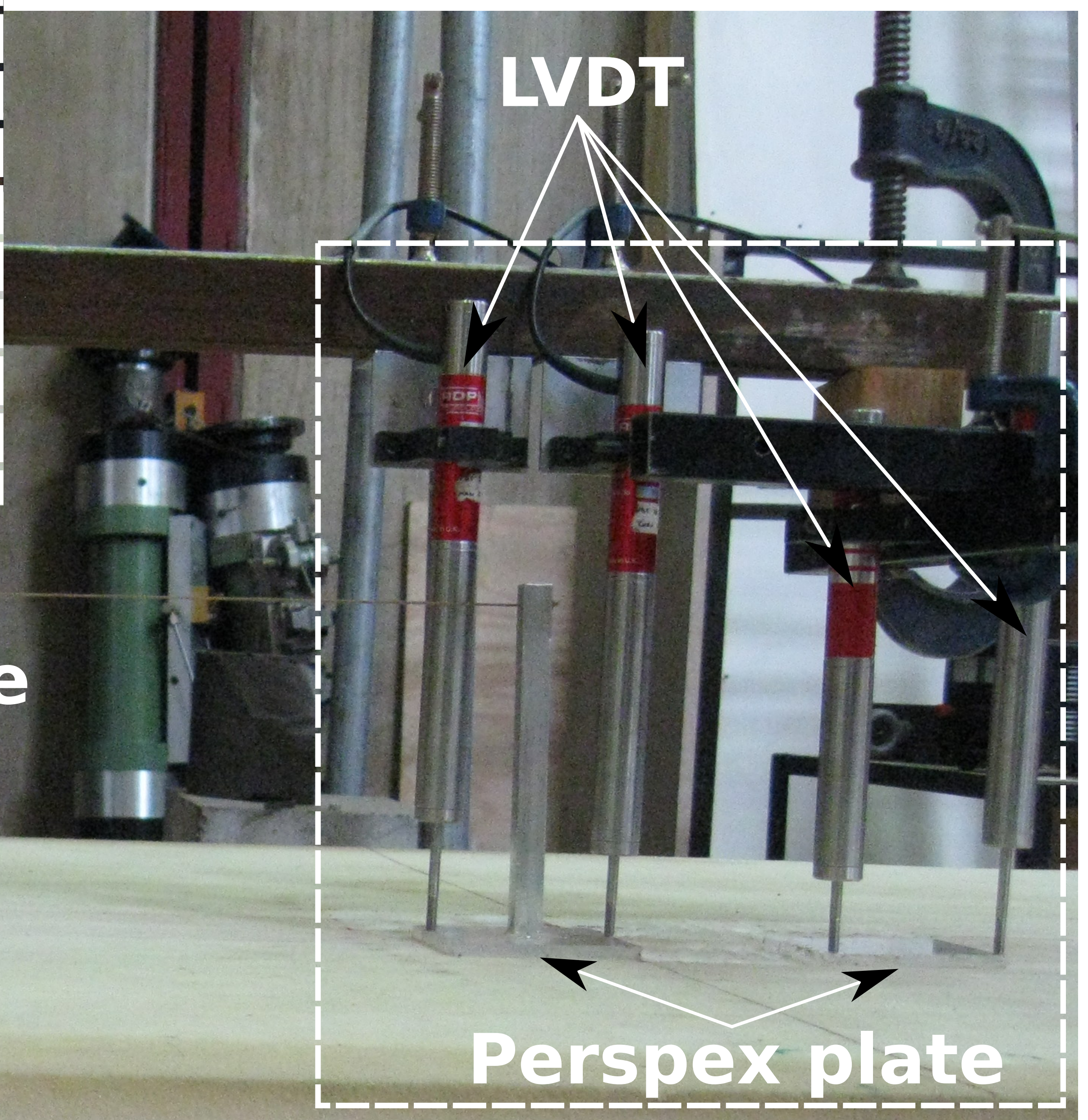
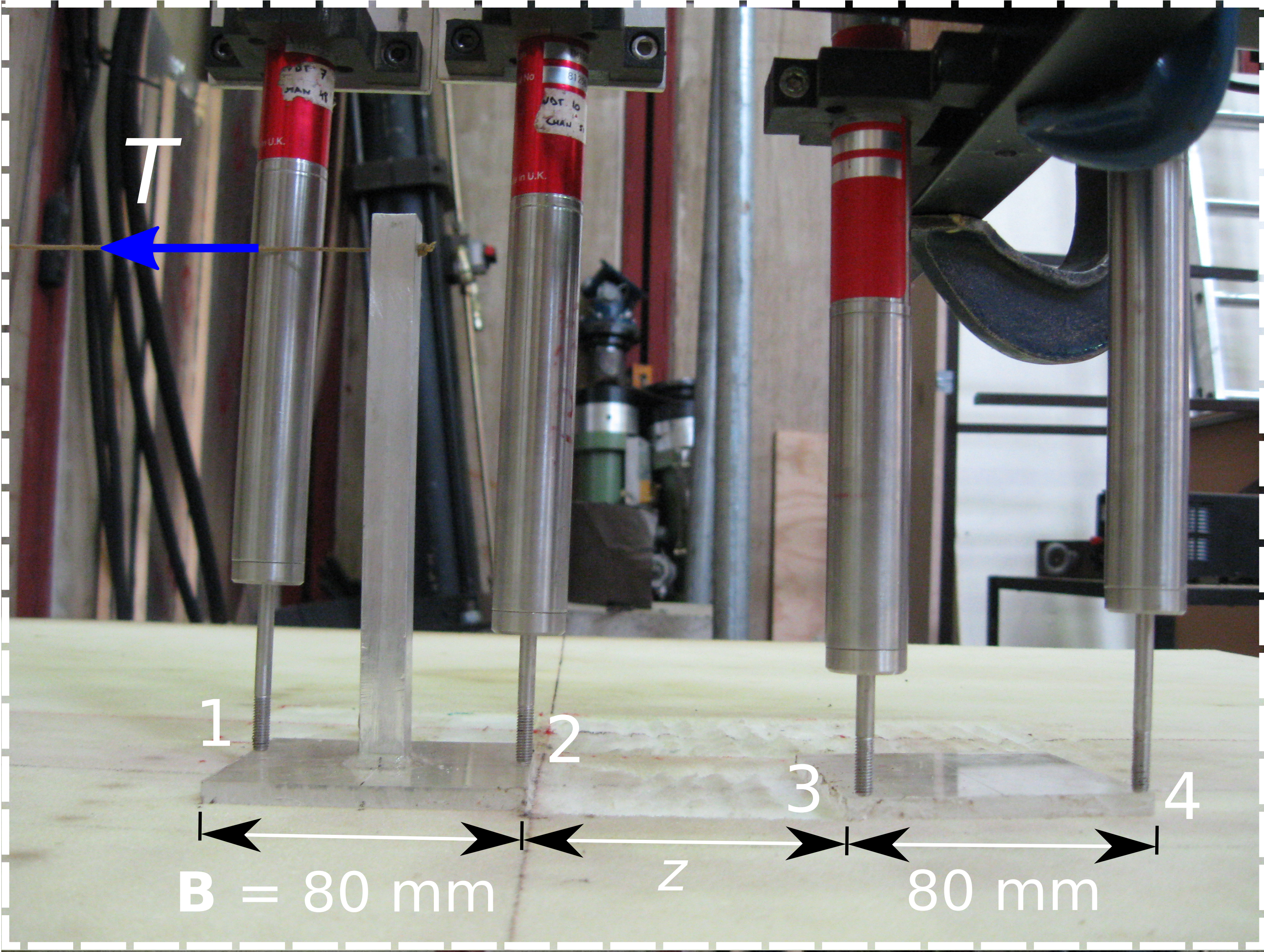


Fig6(a)

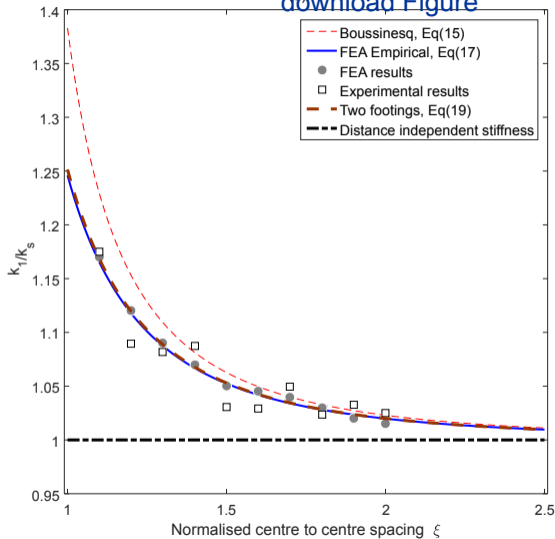
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Fig6(b)

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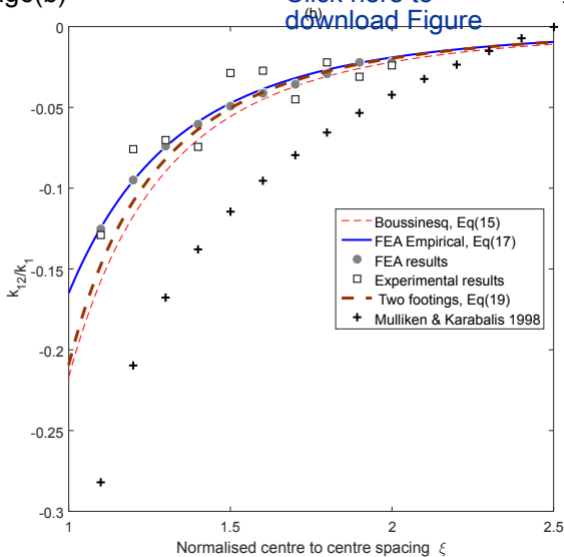


Fig7(a)

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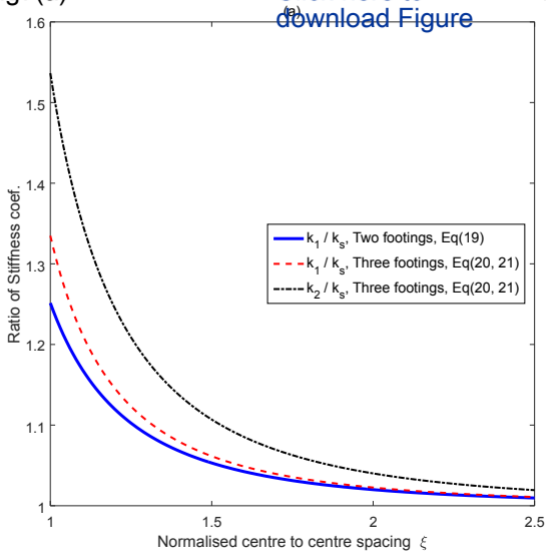


Fig7(b)

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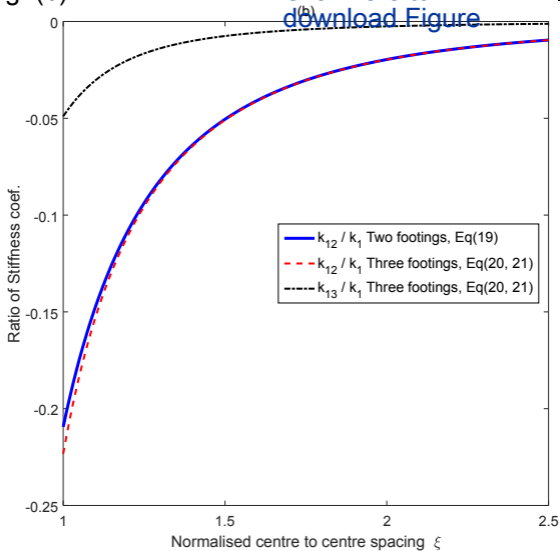


Fig8

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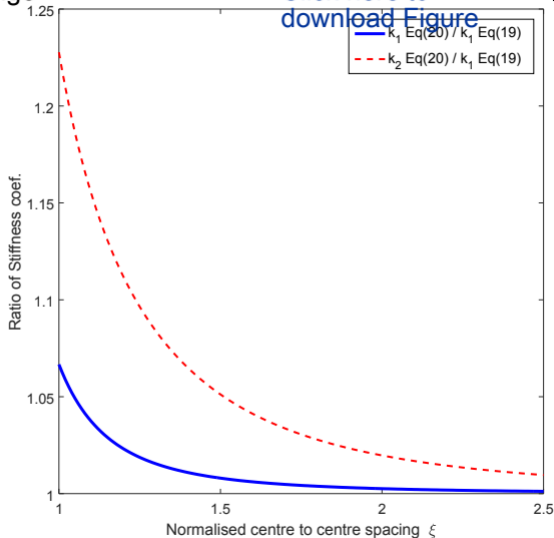
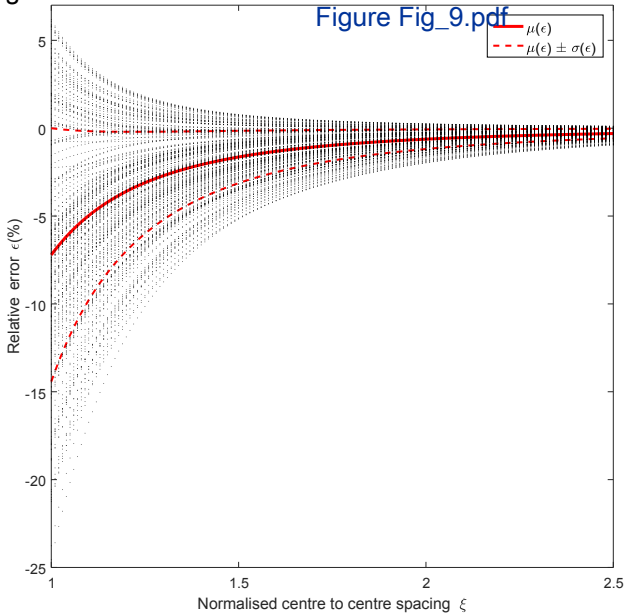
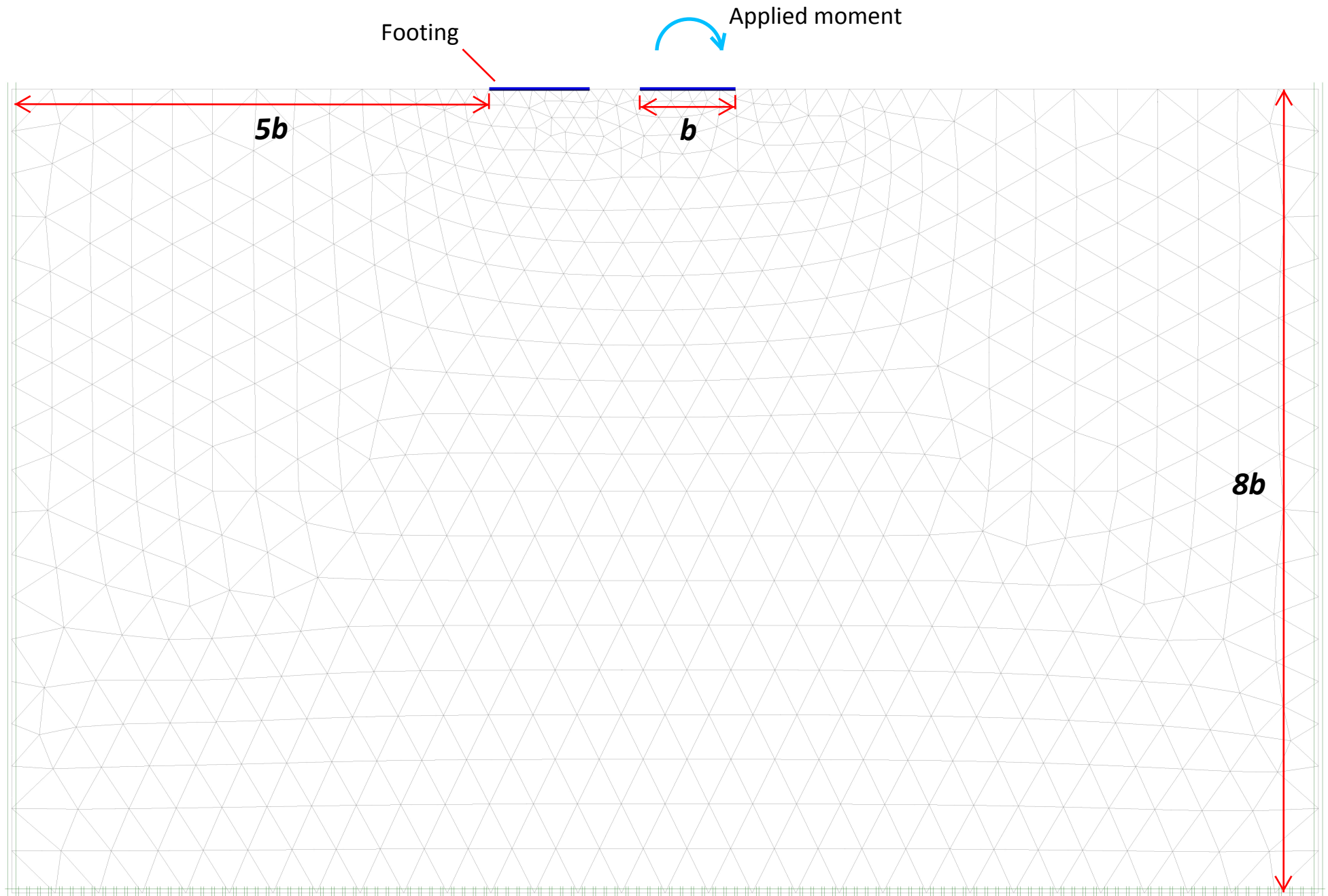


Fig9

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**ASCE Authorship, Originality, and Copyright Transfer Agreement**Publication Title: ASCE International Journal of GeomechanicsManuscript Title: Evaluation of Rocking and Coupling Rotational Linear Stiffness Coefficients of Adjacent Surface Foundations**Author(s) – Names, postal addresses, and e-mail addresses of all authors**Hesham Aldaikh, Ph.D. School of Science and Engineering, University of Dundee. Dundee DD1 4HN, UK. h.aldaikh@dundee.ac.uk OR cexha@my.bristol.ac.ukNicholas A. Alexander, Ph.D., C. Math., C. Sci. Department of Civil Engineering, University of Bristol. Bristol, BS8 1TR, UK. nick.alexander@bristol.ac.ukErdir Ibrahim, Ph.D. Department of Civil Engineering, University of Bristol. Bristol, BS8 1TR, UK. erdirin.ibraim@bristol.ac.ukJonathan A. Knappett, Ph.D. School of Science and Engineering, University of Dundee. Dundee DD1 4HN, UK. j.a.knappett@dundee.ac.uk**I. Authorship Responsibility**

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“Evaluation of Rocking and Coupling Rotational Linear Stiffness Coefficients of Adjacent Surface Foundations”

Authors: Hesham Aldaikh, PhD; Nicholas A Alexander, PhD, C. Math., C. S; Erdin Ibrahim, PhD; Jonathan Knappett, PhD.

*Dear Editor/Editorial Team, ASCE's International Journal of Geomechanics*

The authors would like to thank you and the referees for the time and effort spent on reviewing the manuscript. The reviewers' comments are valuable and very constructive, the authors highly appreciate the contributions of the reviewers which helped improving the presentation, readability and overall quality of the paper. The authors have thoroughly addressed and revised all issues indicated by the referees' and believe that the revised version can meet the publication requirements. Please find below the authors' responses (highlighted normal font) below each of the reviewers' comments (italic font). All suggested changes have been highlighted in a revised annotated version of the manuscript (Response to Reviewers\_RevisedManuscript).

Yours sincerely,

Dr Hesham Aldaikh  
Dr Nick A. Alexander

Date: 26 April 2017

## Response to Reviewers

**Reviewer #1:** *The paper deals with the evaluation of rocking and coupling rotational stiffness of adjacent surface foundations. The paper is really interesting and well written. The following minor comments might benefit the authors in the revised version of the paper.*

1) *In the introduction would be meritorious also to mention pros and cons of direct and impedance methods in the case of nonlinear cases.*

A paragraph has been added to the introduction briefly mentioning some of the pros and cons of direct and impedance methods in nonlinear analysis, (page 2, starting line 54).

2) *Model description. Is also necessary to mention that the footings are identical? It seems it is not a condition but both numerical and experimental analyses are dealing with identical footings.*

We agree with the reviewer. The word "identical" has been added to the text to indicate that the analysis is limited to the case of identical foundations. However, the formulation presented could in fact be extended to the case of two or more surface foundation of dissimilar widths.

3) *Figure 2 and Equation 7: "b" is not defined*

Thank you. A clarification has been added to the text, (page 8, line 198).

4) *Equation 7: does x needs to be in absolute value?*

Yes, please see Boussinesq's formula.

5) *Equation 11: the idea to determine the displacement function from the principle of superposition is interesting, but in the reviewer opinion comments are necessary regarding the difference between this solution unconstrained between  $-b/2$  and  $b/2$  and the case of the footing that will impose constraints in the displacement field.*

Thanks for the suggestion. The suggested solution (eqn 11) is only valid for the cases  $|x| \gg b/2$ . So it cannot and does not provide a solution for the constrained displacements under the footing itself. This displacement field in range  $b/2 > x > -b/2$  for a rigid footing is linear so is easily obtained.

6) Equation 11: *few algebra might also help the readers to derived it*

Textual comments and a little further algebra have been added to explain this simplification more clearly, (highlighted in page 9).

7) Equation 13: *it might be pedantic but the derivative of the sign(x) should be also added. If it is defined through the Heaviside (Unit step function) it might will lead to a Dirac's delta function. Clearly it depends how the unit step function is defined.*

Some textual comments have been added to clarify the math, (highlighted in page 9).

8) *Again in the Conclusions it needs to be clearly state if the analytical formulation is valid for identical footings or if it can be extended to different ones.*

A note has been added to the conclusion.

**Reviewer #2:** *This paper presents closed form expressions for rocking spring stiffnesses and coupling-interaction rotational spring stiffnesses for closely spaced two or three surface footings. In general, this paper is expected to be beneficial to the geotechnical and structural engineering communities. In addition, the topic fits within the scope of this Journal. However, some concerns need to be addressed before acceptance for publication.*

1. *Line 242: As the reference Aldaikh (2013) [PhD dissertation] cannot be easily obtained by readers, it is suggested to add details of the finite element (FE) model in PLAXIS2D, e.g. structural geometry and material, ground stiffness (elastic modulus of the half-space), out-of-plane thickness of the plain-strain element, mesh properties, boundary conditions, etc.*

We thank the reviewer for pointing this out. More information on the FE model has been added, (page 11 lines 252 to 262).

2. *Line 243: Again a concern on the FE model. In the paper, the authors compare the FE analysis (FEA) results with the experimental results. So I guess, in the FE model, you adopted the same parameters as the experiment, isn't it? Also, it is suggested to address the influence of the*

*structural and ground stiffnesses (or their ratio) on the empirical Equations (17) and (18)? Please clarify.*

Yes, the same parameters as the experiment were used in the FE model. This has been clarified as per the preceding response to comment 1. Although not thoroughly, both issues (1) different ground (soil) stiffnesses and (2) structure to ground stiffness (represented through sets of frequency ratios) have been previously considered in other publications by the authors, (Alexander 2013 and Aldaikh 2015). It was found that SSSI is most pronounced on smaller structures and on weaker soils (i.e. loose sand). These issues will be further considered in future research by the authors.

*3. Line 254: At this stage, the FEA empirical equation has not been validated by experimental results. Therefore, it is not appropriate to say "accurate" when discussing the result at smaller inter-footing spacing. Please revise.*

We agree with the reviewer. The sentence has been omitted.

*4. In Figure 6: How to obtain the Experimental Results (square marks) based on the test data listed in Table 1? By which bending moment level? or in average? Please clarify.*

We thank the reviewer again for pointing this out. Yes, the experimental results of stiffness ratios ( $k_1/k_s$ ) and ( $k_{12}/k_1$ ) presented in Figure 6 are the average values of all stiffnesses ratios resulted from different moment levels. A clarification has been added, (page 15, line 343).

*5. Line 347: For the result of  $k_{13}/k_{11}$  in Fig. 7(b), the authors mentioned that it does not exceed "one quarter" of that of the adjacent footing coupling at spacing where footings touch. However, from Fig. 7(b), it seems to be 0.05 rather than "one quarter".*

What is intended to be said here is that the value of ( $k_{13}/k_1$ ) to that of ( $k_{12}/k_1$ ) at zero separation distance approximately equals  $\approx 0.05/0.2 = 0.25$ . This has been clarified in the text, (page 16, line 356).

*6. In Section 5.3: The authors should detail the considered parameters of the analyzed cases, i.e., what is the considered range of the given rotations of footings; I believe it will impact the mean and standard deviation shown in Figure 9.*

As the problem in question is a linear elastic one, the magnitude of the rotations is not deemed to be important. However, in the authors' opinion, exploring a range of positive and negative rotations is needed. A note has been added, (page 15, line 388).

7. Line 380: Minor mistake. The authors mentioned "... for both case (a) stiffness Eqs(19) with  $k_1=k_2=k_3$  and  $k_{13}=0$  and ...". Here, in case (a), it should be  $k_3=0$  and  $k_{23}=0$  as well.

Figure 9 compares the case of 3 adjacent foundations when the outer (connecting alternate foundations) spring,  $k_{13}$ , is taken into account with that when only internal (connecting adjacent foundation) springs,  $k_{12}$  and  $k_{23}$ , are considered (i.e. only  $k_{13}=0$ ).

## Response to Editorial Comments

The authors thank the editorial team for their comments. All comments 1 to 5 by the Editorial Coordinator have been addressed in the revised version of the paper.



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