Logic and Modality

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Contents

Abstract

In this thesis I consider and argue against various versions of the view that logical truth and necessity are identical. I criticise purely modalist accounts of logical consequence, according to which consequence is just necessary truth-preservation, and I also criticise views on which necessity is identified with some prior understanding of logical truth (as semantic validity, or substitutional validity, or as syntactic derivability). In the process I develop various systems of modal semantics in detail and investigate the logic of logical truth under various alternative hypotheses.

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Part I

Preliminaries

1
Introduction

In this chapter I will be introducing the topic, explaining why it is of interest, and outlining my background assumptions and my use of certain key terms.

1.1 On a Recurrent Intuition

On many occasions in the history of philosophy, modal and logical notions have been taken as related in an especially strict sense, such that accounts of one set of notions have been given in terms of the other. Aristotle defines the syllogism as, roughly, an argument in which the conclusion follows by necessity from the premises,[1] and an essentially similar definition of valid argumentation has been rehearsed in introductory logic texts and classrooms down to today. Conversely, definitions of possibility in terms of logical consistency, and hence definitions of necessity in terms of logical truth, have had a presence in Western philosophy since at least the 17th century.[2] Indicating the importance

[1] Aristotle [1], *Pr. An.* 24b19-21.

[2] In Leibniz: "the connection or consecution is of two kinds, the one is absolutely necessary, whose contrary implies a contradiction [...] the other is necessity only *ex hypothesi*, and, so to speak, only by accident, but it is contingent in itself, since the contrary does not imply a contradiction." ([36], 17). And in the early Spinoza: "I call a thing impossible if its nature implies that it would be a contradiction for it to exist; necessary, if its nature implies that it would be a contradiction for it not to exist; and possible, if, by its very nature, neither its existence nor its nonexistence implies a contradiction" ([64], 14). Though such views are more common starting with the earlier Analytic tradition. In Wittgenstein: "A necessity for one thing to happen because another has happened does not exist. There is only *logical* necessity" ([70], 6.37), a statement which was influential on the positivists. Especially Carnap: "After defining semantical concepts like logical truth and related ones, I proposed to interpret the modalities as those properties of propositions which correspond to certain semantical properties [...] Foe example, a proposition is logically necessary if and only if it a sentence expressing it is logically true." ([11], 62). The view survives into more recent writing too. In

of such a view, for many it has been one of the great advances of contemporary analytic philosophy since Kripke [34] to *reject* equating necessity with logical truth or the *a priori*. Nevertheless, often enough there have been hints of renewed sympathy for the identification of logic and necessity, from several different directions. For some it has simply meant a willingness to give to the broadest kind of necessity (understood as a property of propositions) the title of 'logical necessity', without going so far as to claim that the broadest necessity shares in any of the commonly conceived features of logical truth. Such a willingness can be justified by the fact that tautologies, being paradigms of logical truth, possess this very kind of maximally broad necessity.[3] Such a stance allows one to indulge in calling a proposition 'logically necessary' just as long as it possesses the same necessity as that of propositions expressed by tautologies and other paradigm cases of logically true sentences.[4] For others, it has meant a thoroughgoing modalism about logical truth and consequence, according to which there is nothing more to consequence than necessary truth preservation, and nothing more to logical truth than necessity.[5] For still others, the proposal to identify logical truth and necessity has constituted part of a theory of necessity (or at least of the broadest sort of necessity) which tries to analyse—or else analogise—necessity down to a conceptually prior understanding of logical truth, and necessitation to consequence.[6]

([20], 9, 17-20), Hartry Field expresses strong approval for a theory of possibility in terms of logical consistency. See also later footnotes.

[3]This first attitude is reflected in comments like the following from Roberts [54]:

> According to one popular view, a natural candidate for absolute necessity is logical necessity. However, it is often unclear how logical necessity—thought of as a propositional operator, not a predicate of expressions—is intended to be characterised. Nevertheless, under the assumption of Booleanism, according to which propositions form a Boolean algebra under the usual truth-functional operations, logical necessity can be characterised quite naturally. In the Boolean setting, there is a unique tautologous proposition—the top element of the algebra—which may be thought of as the logical truth. Thus, in that setting, it is highly natural to characterise logical necessity as simply being identical with the tautologous proposition. (705)

[4]Especially if one adopts some coarse-grained view of propositions, on which any proposition that is necessary in the broadest sense is *identical* to the proposition expressed by a tautology, it is tempting to allow the proposition in question to be called a 'logical truth'.

[5]As I will mention in Chapter 3, this sort of view appears to have been held by Chrysippus (see Gould [25], 80-81), and was well-represented among the medievals (for example, see the conclusions of King [32]). In the more contemporary literature we might point to someone like Stephen Read [52], who rejects a formal component of validity in favour of a purely modal criterion.

[6]Again consider the remarks of Field [20]. This is also a theme of Andrew Bacon's work. In [3] he develops a theory of the broadest necessity as *logical* necessity, which he takes to "stand to the world in the way that logical truth stands to sentences", and attempts to differentiate this from metaphysical necessity. Both here and in [2] he takes the broadest necessity operator in higher-order logic to be defined as $\Box := \lambda p(p = \top)$ where $\top$ is some fixed tautology.

To back up a bit, the subject of our discussion is the intuition just mentioned: that logical truth and necessity are really just the same thing. More generally, our subject is the thesis that necessity is logical truth, that possibility is consistency, and that the relation of necessitation just is logical consequence. Call this the *identity thesis*. My aim is to set out the ways that this thesis can be developed and to critique them. What motivates the investigation is the continued sympathy of some philosophers for the thesis, the historical significance of the thesis, and finally its obvious intrinsic interest. Logical consequence and necessity are two of the central concepts of theoretical philosophy, so the possibility of their being identical is worth serious consideration. In the end, my verdict will be to reject the identity thesis. The significance of this conclusion is that, by reaffirming the distinction between necessity and logic, we also reaffirm the point that metaphysics is autonomous from logic. To explain why, I will say a little more about the relevance of the identity thesis to the relationship between logic and metaphysics in general.

1.1.1 The Independence of Metaphysics

Whether or not the identity thesis is true has broad implications regarding the relationship between logic and metaphysics. For example, taking the identity thesis in one way, suppose that there is nothing more to logical consequence than necessary truth preservation, with 'necessity' here taken in an independent, worldly sense. In this case the discipline of logic as the study of validity would be nothing but an extension of the study of worldly necessity—in particular, a codification of certain general patterns of necessitation, perhaps with an eye to applications in constructing arguments. For example, if this development of the identity thesis is right then the question of whether or not the argument 'someone is in pain ∴ someone's C fibres are firing' is valid is fundamentally a worldly question, equivalent to the question of whether it is possible for someone to be in pain without being in the particular physical state of having firing C fibres. In this case, the logical question of the argument's validity depends on whether a certain strict form of materialism is correct in the philosophy of mind. Logical questions would then be subsumed under the purview of metaphysics, since they would be questions about what is possible and necessity in a worldly sense.

On the other hand, suppose we take the identity thesis in the opposite direction and claim that necessity is nothing but logical truth, with 'logical truth' understood in one of the usual ways—semantic validity, for example. In this case, the study of necessity would be nothing more than sufficiently developed logical theory. In this case, the status of metaphysics itself is left unclear. If necessity is unworldly, then its study is not part of metaphysics proper. In this case the question remains: if this understanding of the identity thesis is true, then what are we left with under the name of 'metaphysics'? The two figures who have had perhaps the greatest impact on how metaphysics is seen and practiced by analytic philosophers since the middle of the previous century are Quine and Kripke.[7] But even though Quine [49] allowed there to be such a thing as metaphysics to house the practice of ontology, Quine's prescribed method made ontological practice into essentially a kind of elaboration or logical reflection upon natural science. Therefore the Quinean picture of metaphysics is not really one of an autonomous discipline, but is more a vision of a legitimate albeit non-exhaustive part of philosophy of science. By contrast, Kripke's [34] identification of metaphysical necessity ensured the possibility of seeing metaphysics as both legitimate and autonomous. The study of metaphysical necessity is one which is not merely part of philosophy of science, or logic for that matter. Therefore metaphysics, if it contains the study of necessity, is autonomous from logic and philosophy of science. But then, if it is on the basis of a concept of worldly necessity that the possibility of autonomous metaphysics currently rests, perhaps to take back the former is to take back the latter too. Perhaps then, accepting the identity thesis in this way, reducing necessity to logical truth, must result in a relegation of metaphysics back to its previous status as legitimate only as a development of logic and the philosophy of science.

This inference is defeasible; since Kripke's identification of metaphysical necessity, a number of other phenomena have been proposed to make for additional, proper objects of metaphysical enquiry. For example, metaphysical grounding is seen by some to constitute a topic which belongs properly to metaphysics understood as an autonomous discipline. Similar claims have been made for essence and for various conceptions of fundamentality. If there are such things as metaphysical ground or essence, or fundamentality, that are irreducible to necessity, and if these are properly objects of metaphysical enquiry, then metaphysics may be autonomous even if necessity is reduced to logical truth. This possibility allows some metaphysicians like Sider [62] to adopt a radically reductionist view of necessity while retaining a proper domain for autonomous metaphysical enquiry in the form of questions of fundamentality and 'structure'.[8] However, it is not clear to me that any of the proposed 'post-modal' domains for metaphysical enquiry are yet developed enough to bear the weight of metaphysics' nascent legitimacy. Theorising in modal metaphysics is supported and rigidified by the technical edifice of modal logic; by virtue of this solid underpinning, it has

[7] A third figure of equal if not greater influence on contemporary metaphysics is David Lewis [41], for whom modality takes center stage, but in a reductionist vein—necessity being analysed in terms of quantification over causally isolated island universes and trans-world counterparts. Lewis, with his insistence on an extensionalist approach to modality via counterpart theory (rather than quantified modal logic) places him in something of the position of an ideological intermediary between the Quinean and Kripkean understandings of modality.

[8] Sider's 'Humean' theory of necessity says that a proposition is necessary iff it is derivable from the modal axioms by the modal rules, where the identity of the modal axioms and modal rules are decided by convention ([62], Ch. 12). It is in this sense a version of the kind of 'deducibility logicism' examined in Chapter 8, though framed in terms of propositions rather than sentences.

been able to develop some amount of maturity as a domain of enquiry, more uniform standards of theory-evaluation via the formal construction of models and systems of logic, and a narrowing of the range of live options. By contrast, we can take the theory of grounding for instance, which boasts one of the liveliest literatures in post-modal metaphysics. Thought concerning grounding does not yet have the same level of maturity as modal metaphysics. It is still broadly the case in the grounding literature that every theory is as live an option as every other. There is not yet any significant uniformity in technical methodology, nor, by that same token, in standards of theory-evaluation. No set of principles yet has a privileged position, and no worked-out 'orthodox' picture of ground has emerged to better structure the dialectic. Now this state of affairs may only be temporary, but indicates a need in the present to reaffirm the independence of modality from logical truth, and hence the robustness of modal metaphysics against would-be reductions to logical theory. Chapters 6-8 will be most important in this regard.

One last point before moving on. Rejecting the identity thesis does not ensure by itself that logic is independent from metaphysics. There are accounts of logical consequence besides the 'necessary truth-preservation' view that would imply that logic is symbiotic on metaphysics in a different way.[9] All that I have tried to indicate in this section is that rejecting the identity thesis supports the autonomy of metaphysics from logical theory. Now having introduced our topic, and outlined its interest, in the remainder of this chapter it will be my aim to explain my use of key terms, and to set out some of the background assumptions I am adopting to constrain discussion.

[9]For example, the conception of logical truth as metaphysical universality as introduced in Williamson [68]. Take any formula ϕ of a formal language (say first-order predicate logic), and let $\forall(\phi)$ be the result of replacing all non-logical symbols including predicates in ϕ with fresh variables and prefixing the whole thing with a block of the relevant universal quantifiers. (Note that $\forall(\phi)$ may not be a first-order formula itself since it may contain predicate variables.) The formula ϕ is metaphysically universal on some given interpretation iff $\forall(\phi)$ is true on that interpretation. This property is appropriately termed *metaphysical* universality since it involves objectual (rather than substitutional) quantifications over individuals, which serve as denotations for singular terms, and properties, which serve as denotations for predicates (etc.). Identifying logical truth and metaphysical universality would make logic dependant on metaphysics. Since $\forall(\phi)$ can involve objectual quantification over higher-order entities, then, this account of logical truth makes certain logical matters dependant on metaphysical questions—for example, concerning the existence of properties. More interestingly, the account rules out certain views in the metaphysics of properties. For example: some philosophers, who think that properties are ontologically dependant on their instances, say that every property must be instantiated (suppose we symbolise this in the monadic case as: $\forall X \exists x Xx$). But if they were right, then on this account of logical truth, $\exists x Fx$ would be counted as a logical truth (since $\forall(\exists x Fx) = \forall X \exists x Xx$), which is not plausible. Hence on this view of logical truth, it is implausible that all extant properties are instantiated.

1.2 Assumptions

1.2.1 Classical Logic

I will be assuming a fully classical understanding of logic throughout. By 'classical logic' I mean to include classical propositional logic, quantification theory, and classical laws governing identity including reflexivity and Leibniz's Law. The assumption of classical logic is employed throughout the discussion. The assumption of classical logic is very well justified theoretically. But it is also necessary from a pragmatic point of view. A daunting range of alternatives to classical logic have been proposed and studied in detail over the years; if we had adopted any of these alternatives in the place of classical logic, our methods and conclusions would have looked quite different (Boolean algebras could not be used in the discussion of Chapters 4 and 5, for example). Thus, to have properly considered these alternatives in parallel to the classical case would have required an essay at least twice as long as this one.

1.2.2 Monism

The second set of assumptions I will be making concerns the basic arrangement of modal space, meaning the space of necessities. To explain these assumptions, it is necessary to mention the familiar concept of a relative necessity. A relativisation N' of a necessity N is a necessity which holds fixed certain propositions (or sentences; in general, whatever the bearers of necessities actually are) such that to be N'-necessary is to be N-necessitated by the propositions held fixed by N'.[10] A relative necessity is a necessity which is relative to some other necessity. I call any non-relative necessity an *absolute necessity*. Now, the concept of a relative necessity has occasionally been problematised in the literature, but questions raised surrounding the concept until now seem to me to be over points of fine detail and formal implementation, rather than of the sort to undermine its use as such.[11] Our usage of it will not run into these questions, and I take the idea of one necessity being a relativisation of another to be clear enough anyway to provide us with a functioning distinction between those necessities which are relative and those which are absolute.

[10] It might be a contingent matter which particular propositions/sentences a modality holds fixed. For instance, physical necessity, understood as a relative necessity, actually holds fixed the true physical law that massive bodies do not travel faster than light. But, if this had not been a physical law, physical necessity would not have held it fixed, but would instead have held fixed whatever the true physical laws had then been instead.

[11] I am referring to the literature starting with Hale and Leech's [27] and [35], and continued with Roberts' modifications in [54]. The discussions in this literature are illuminating, and there are important points which turn on the technical intricacies—for example, Leech in [35] makes essential use of such intricacies in responding to an argument of Fine's ([22], Ch. 7) for pluralism. However, our discussion does not turn on these same technical details, and so I will omit them here.

Given this, I make two assumptions about the structure of modal space under the relation of relativisation. The first, which I call absolutism, says that every relative necessity is a relativisation of some absolute necessity. The second, which I call monism, says that there is exactly one absolute necessity.[12] Together, these assumptions just say that every necessity besides the sole absolute necessity is a relativisation of the sole absolute necessity. The absolute necessity—or just 'absolute necessity', dropping the article—is the source of all other necessities via relativisation. Monism is to be contrasted with pluralism, which says there are least two absolute necessities. Perhaps the best known defender of pluralism in the literature is Kit Fine, for whom there are three absolute (meaning non-relative) necessities: metaphysical, natural, and moral (Fine [22], Ch. 7). Monism being an assumption of our discussion, rather than an argued-for position, I will not be considering the case for pluralism here. Absolutism on the other hand is to be contrasted with what might be called *relativism*, the view that there are some relative necessities that are not relativisations of any absolute necessity. Assuming, plausibly, that the relation of relativisation is transitive, relativism implies the existence of an infinite regress of relativisation without a lower bound. It is not clear to me that such a picture has any independent plausibility, and conversely, a monist who was also a relativist could not sensibly claim that absolute necessity was really the source of all necessity. Therefore to omit absolutism would result in missing out on of some of the main benefits that are possible given the assumption of monism. I will now explain exactly what these these benefits are.

The assumption of absolutism and monism greatly simplifies and constrains the landscape of alternatives for developing the identity thesis. A natural question posed to the view that logical truth and necessity are identical is: *which* variety of necessity? Everyone agrees that there are many types of necessity, but the absolutist monist seems to have just one sensible answer to this particular question. The absolutist monist who wants to identify logic and necessity will identify logical truth with their unique absolute necessity, since this is the source of all the other necessities via relativisation and thus the only viable candidate referent for talk of necessity *simpliciter*. If one drops either absolutism or monism, and so is either a relativist or a pluralist, then it is no longer the case that there is one 'master' necessity. That means that the relativist and the pluralist must give more considered answers to the question of which necessity logical truth is identical to. Once again, such considerations would greatly lengthen and complicate our discussion.[13] Having adopted monism and absolutism, in what

[12] Fine ([22], Ch. 7) uses the terms 'monism' and 'pluralism' with similar meanings to us.

[13] None of this is to say that there could not be sensible pluralist developments of the identity thesis. Even under the assumption of classical logic, and the exclusion of logical pluralism, there are still multiple varieties of classical logical truth—three of which are examined in Chapters 6-8—so perhaps a pluralist could say that these are precisely the absolute necessities. Although this is a view of much interest, the things that could be said about such a pluralist theory would essentially just be a combination of what we will say anyway about the various monistic forms of 'logicism' developed in Chapters 6-8. So nothing substantial is lost by their omission.

follows I will just use the term 'absolute necessity' to refer to the unique non-relative necessity.

1.3 Absolute and Metaphysical Necessity

Having explained my assumptions, I will now move to explain my usage of certain important terms. Importantly, I will be working with a definition of metaphysical necessity as the broadest objective necessity. So I will need to explain what is meant by breadth and by objectivity. While explaining breadth, I will also have the opportunity to discuss the relationship between absolute necessity and the broadest necessity.

1.3.1 Breadth and Absolute Necessity

Much has been made of the idea that certain necessities are 'broader' than others, and that there might be such a thing as a broadest necessity. The definition of breadth as such is somewhat problematic, but here we can say a little about the general shape of the concept. In the first place, there is a very basic relation between necessities that we can call extensional breadth. A necessity N is extensionally as broad as another necessity N' iff whatever is N-necessary is also N'-necessary.[14] Extensional breadth can be seen as a first pass at the intuitive idea of one necessity being 'stricter' than another, or there being 'higher and lower degrees' of necessity, such that necessities can be compared with respect to strength. Here, extensionally broader necessities are in a clear sense stronger than extensionally less broad ones. Extensional breadth itself is not a problematic concept, but it is a very weak one, and probably inadequate to fully encode the aforementioned intuitive idea of relative strength between necessities. True to its name, it only refers to the extensions of necessities. This means that, in general, whether one necessity is as extensionally broad as another can be a highly contingent matter, and so may not reflect the kinds of differences between necessities that the intuitive idea of relative strength seems to aim at. If it just so happens that N is extensionally broader than N', but that it could easily have been otherwise, then it seems unnatural to say that N is stronger than N' in the way that physical necessity is stronger than (say) biological necessity.[15]

[14] Additionally, we say that N is extensionally *broader* than N' iff N is extensionally as broad as N' and not vice versa. If N and N' are each as extensionally broad as the other, we say that N and N' coincide.

[15] For example, suppose that there are neighboring countries A and B, both with very rudimentary codified legal systems consisting entirely of simple commands such as 'do not steal', 'do not wear hats on Fridays', and the like. Due to a common history, many of these commands are shared between A and B; as a result, their codified legal systems are almost identical. There is only one difference. As it happens, many years ago a man living in country B gained a monopoly by cornering the market on olive presses just before a particularly good yield. Ever since, the command 'do not have a monopoly in olive oil' has been a law in country B and not in country A. If we say that p is *legally possible* in country A (B) iff it is not in conflict with their laws, and legally necessary if $\neg p$ is legally impossible, then legal necessity according to B is extensionally broader than legal necessity according to country A. Even though,

Here is where the concept of breadth comes in. Breadth is intended as a strengthening of extensional breadth which allows for non-contingent comparisons between necessities. We can see this much in Bacon's [2] proposed definition of breadth. Roughly, this definition says that N is as broad as N' iff for all necessities N'', it is N''-necessary that N is as extensionally as broad as N'.[16] Other definitions of breadth follow similar lines. That is, if claims of extensional breadth are understood as a generalised material conditional, of the form "for all x, x is $N \rightarrow x$ is N'", claims of breadth switch out the material conditional for something stronger. We can make this more precise later on, but this gives us a clear enough idea of breadth to proceed for now.

Many writers have postulated the existence of a *broadest* necessity—one which is at least as broad as every other necessity. Among such writers, some have additionally used the term 'absolute necessity' simply as shorthand to refer to the broadest necessity (cf. Roberts [54], 704). So it is important that we clarify the differences between 'absolute necessity' understood in this way and 'absolute necessity' as we are using it. On our usage, the term 'absolute necessity' is used to designate the unique non-relative necessity; it does not follow from this definition alone that absolute necessity is as broad as every other necessity. Therefore the claim that absolute necessity is as broad as every other necessity is, for us, a substantive thesis rather than a definitional point. Whether it can be established depends on what further assumptions we make, and of the various theories discussed in the following chapters, there are some for which it is true and others for which it is false.[17] However, it is possible to argue that absolute necessity is as *extensionally* broad as every other necessity based on the assumptions made so far plus:[18]

(Closure) If it is absolutely necessary that ϕ and absolutely necessary that $\phi \rightarrow \psi$, then it is absolutely necessary that ψ.

clearly, there is no sense in which legal necessity according to B is an intrinsically stronger kind of necessity than A's. After all, the olive oil magnate might very easily have decided to live in country A, in which case A's variety of legal necessity might have instead been extensionally broader than B's.

[16] "Say that a necessity operator, $\Box_1$, is *as broad as* another, $\Box_2$, if and only if [...] $\Box_3(\Box_1 P \rightarrow \Box_2 P)$ whenever $\Box_3$ is a necessity operator, and P is a proposition." Bacon, ([2], 733).

[17] For example, I will argue that the intensionalist view considered in Chapter 4 does imply that absolute necessity is as broad as every other necessity. Conversely in the course of Chapters 6-8, we will see versions of logicism on which absolute necessity is *not* as broad as every necessity. These are the versions of logicism on which modal operators are themselves not treated as logical constants. Absolute necessity fails to be the broadest necessity on such views because, to use the notation introduced below, $\Box_\alpha(\Box_\alpha\psi \rightarrow \Box_N\psi)$ will always be false since the argument $(\Box_\alpha\psi \rightarrow \Box_N\psi)$ is not a non-modal logical truth.

[18] Throughout, I reserve '$\rightarrow$' for the material conditional. In formal terms, this assumption amounts to the assertion that the propositional logic of absolute necessity contains the distribution axiom $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$. This is in addition to said logic containing $\Box\phi$ wherever ϕ is a tautology, in keeping with the assumptions regarding the necessity of classical logic made in the previous section.

This assumption is validated on every one of the theories considered in this essay. Now to sketch the argument itself. Recall that on monism, for every necessity N besides absolute necessity, something is N-necessary iff it is absolutely necessitated by the propositions (or sentences) that are held fixed by N. Roughly, this means that where some set of formulas Φ express the propositions (or are the sentences which are) held fixed by N, it is N-necessary that ϕ iff it is absolutely necessary that $\bigwedge\Phi \to \phi$, where $\bigwedge\Phi$ expresses the conjunction of the propositions held fixed (or is the conjunction of the sentences held fixed). To express this more succinctly via the use of symbols, letting '$\Box_N$' abbreviate 'it is N-necessary that' and $\Box_\alpha$ abbreviate 'it is absolutely necessary that', we get

$$(*)\ \Box_N\phi \leftrightarrow \Box_\alpha(\bigwedge\Phi \to \phi)$$

Given this, extensional breadth comes as follows. Making use of our notation:[19]

(1)	$\Box_\alpha(\psi \to (\bigwedge\Phi \to \psi))$	Necessity of PC.
(2)	$\Box_\alpha(\psi \to (\bigwedge\Phi \to \psi)) \to (\Box_\alpha\psi \to \Box_\alpha(\bigwedge\Phi \to \psi))$	Closure.
(3)	$\Box_\alpha\psi \to \Box_\alpha(\bigwedge\Phi \to \psi)$	From (1, 2).
(4)	$\Box_\alpha\psi \to \Box_N\psi$	From (3, $*$).

To sum up this and the previous section: an absolute necessity is a necessity which is not a relativisation of some other necessity. Monism, which we assume throughout our discussion, is the view that there is only one such necessity, which can be referred to accordingly as 'absolute necessity' in the singular. Unlike other uses of the term 'absolute necessity', this definition does not imply that absolute necessity is as broad as (or even extensionally as broad as) every other necessity. We will see in later chapters that some views imply that absolute necessity is as broad as every other necessity, while others imply that it is not. However, the thesis that absolute necessity is as extensionally broad as every other necessity can be retrieved on relatively minimal background assumptions, which are shared by all the theories considered in the main discussion.

1.3.2 Objectivity and Metaphysical Necessity

The term 'metaphysical necessity' was introduced into the analytical philosophical vocabulary in the wake of Kripke's *Naming and Necessity* in order to refer to what he speaks of as 'necessity in the

[19] On (1): since all the usual theorems of classical propositional logic are admitted as logical truths, every relevant development of the identity thesis will imply that every theorem of the classical propositional calculus is absolutely necessary.

highest degree—whatever that means'.[20] On this basis one might think that metaphysical necessity is to be defined simply as the broadest necessity. But such a definition is problematic, at least if given prior to the settling of certain questions about what counts as a necessity. In particular, are varieties of so-called 'epistemic necessity' to be counted? If so, then whatever is metaphysically necessary is epistemically necessary and, conversely, whatever is epistemically possible is metaphysically possible. Therefore, perhaps, since it is epistemically possible (at time of writing) that Goldbach's conjecture (GC) is true and also epistemically possible that ¬GC is true, then GC is *metaphysically* contingent, which is false. One could avoid this particular problem by restricting their understanding of 'epistemic necessity' to some sufficiently idealised notion of *a priori* knowability;[21] in this sense, GC is not epistemically contingent since either GC or ¬GC is in principle discoverable *a priori*. However even on such a restricted conception of the epistemic necessities, there are still problematic consequences if we say that metaphysical necessity is broader than epistemic necessities. To say why I must mention an important property of metaphysical necessity.

Kripke also provided us with one of the key logical properties of metaphysical necessity. Namely, that contexts generated by the phrase 'it is (metaphysically) necessary that…' admit substitution of co-referring, rigidly designating terms. This category of terms includes proper names like 'Hesperus' and 'Phosphorous'. From this substitution property one can deduce the metaphysical necessity of identity, meaning that one can deduce '(metaphysically) necessarily, $a = b$' whenever $a = b$ and where 'a' and 'b' are rigid designators. This is simply because it is a logical truth that $a = a$, as from this we get that it is metaphysically necessary that $a = a$, so that by substitution of rigid designators we have that it is metaphysically necessary that $a = b$. So, for instance, it is metaphysically necessary that Hesperus = Phosphorus. Returning to the question of epistemic modals: if epistemic necesities were necessities properly so-called, and metaphysical necessity were simply the broadest necessity, it would need to be *epistemically necessary* that Hesperus is Phospherous. But this is false if by epistemic necessity we mean *a priori* knowability, since "Hesperus = Phosphorus" is *a posteriori*. This highlights the basic problem with simply defining metaphysical necessity as the broadest necessity. For such a definition to be made acceptable, in light of the above substitution property, we must first rule out the view that epistemic necessities are necessities

[20] See Kripke, ([34], 99). In fact Kripke only uses the phrase '(metaphysically) possible' once in the book ([34], 137), and never uses the phrase 'metaphysical necessity'. Though there is little dispute that *Naming and Necessity* is the origin of contemporary discourse surrounding metaphysical necessity. This introduction of what is basically a technical term places constraints on admissible interpretations and applications of the concept. The sense of the concept depends far more on the general idea of maximal breadth than it does on the various and often controversial examples of metaphysically necessary truths that have been proposed in the literature (like Kripke's own example of the necessity of origin). Here I follow the basic line of thought expressed in Dorr ([18], 69).

[21] Perhaps along the lines of Chalmers [12].

properly so-called. Indeed, we must do the same for any kind of 'necessity' which does not apply to the statement that Hesperus = Phosphorus, among other informative identities. This would also include logical necessity, at least on most understandings, since on standard characterisations, 'Hesperus = Phosphorus' is not a logical truth. Therefore, especially for our purposes, we cannot assume that metaphysical necessity is the broadest necessity *tout court*, if we are to be even minimally charitable to the views under consideration.[22]

To avoid these problems we will posit, alongside other writers in the literature, a class of necessities called the *objective* necessities. We will then define metaphysical necessity as the broadest objective necessity. There is not a simple accepted definition of the objective necessities that we can recite here, but there are several characterisations which pin-down the idea nicely.[23] First of all, true to the name, objective necessities are *worldly* in a way that other proposed classes of necessities are not. This is why substitution works. When a property is ascribed to a certain object by metaphysical necessity, or any other objective necessity, the truth of the ascription must be independent of the guise under which the object is presented. Therefore if two names rigidly refer to the same object, merely presenting the same object under distinct guises, then they must be inter-substitutable as claimed. But epistemic necessities, at least as they are typically theorised, can be highly sensitive to guise. Properties like *a priori* knowability are not preserved even when rigidly co-referring terms are switched out. Again, it is *a priori* that Hesperus = Hesperus, but not *a priori* that Hesperus = Phosphorus. In this way, objective necessities are often characterised in negative terms. For instance, what we have said so far accords with Williamson, who writes that the objective necessities are "non-epistemic, non-psychological, non-intentional [...] they are not sensitive to the guises under which the objects, properties, relations, and states of affairs at issue are presented." (Williamson, [69], 454). To accompany such characterisations, appeals are also made to linguistics, where there is often a basic distinction made between epistemic and non-epistemic modals. Objective necessities are thus described as corresponding to the categories of 'circumstantial'/'root' modals and 'dynamic modals' as described in the work of Kratzer and Portner respectively.[24] Though importantly, the latter are classes of modal *words*, rather than necessities themselves. Therefore the idea here is that an objective necessity is the kind of necessity which comes to be expressed by a circumstantial/root/dynamic

[22] This does not mean that metaphysical necessity *cannot* be simply the broadest necessity on any of the views we consider. For instance, on the view considered in Chapter 4, this claim turns out to be true.

[23] These characterisations are due originally to Williamson [69]. One definition of an 'objective necessity' used by Yli-Vakkuri and Strohminger ([65], §1) is that objective necessities are restrictions of metaphysical necessity. This must be an adequate definition in some sense, if we assume that metaphysical necessity is the broadest objective necessity, but it is obviously not so helpful for our purposes since we are trying to get a better handle on metaphysical necessity *via* the category of objective necessities.

[24] Cf. Williamson ([69], 455, 488). The appeal to Krazer and Portner is made in this here; [33] and [47] are cited.

modal. Finally, the distinction between objective and non-objective necessities can be understood as analogous to the distinction between objective and non-objective (or subjective) probabilities, insofar as probabilities themselves can be thought of as gradable modalities.

These explanations of metaphysical necessity are enough for our purposes. We do not need to settle any further questions here about the logic of metaphysical necessity, or consider any contentious cases of supposed metaphysically necessary truths. To recap: metaphysical necessity is defined by us to be the broadest objective necessity. The objective necessities are posited as an important class of necessities which are non-epistemic, non-psychological (etc.) and insensitive to guise. Hence objective necessities admit substitution of co-referring rigid designators. Finally, objective necessities correspond to some independently understood classes of modal words in natural language, and so we can take ourselves to have a decent grasp of the objective necessities even in the absence of a simple definition.

In this chapter I have introduced the topic of discussion, outlined its interest, and covered my own understanding of certain key terms employed throughout. I have also outlined the main general assumptions I will be making in order to constrain our discussion. In the next chapter, to complete the necessary preliminaries, I will be distinguishing between ways of developing the identity thesis in order to provide a basic structure for the main discussion.

2
Modes of Identification

In this chapter I classify the different ways in which one can develop the identity thesis. Recall that the identity thesis claims necessity, possibility, and necessitation to be identical with logical truth, logical consistency, and consequence respectively. In compressed form, the identity thesis says that logic and necessity are identical. I begin by making a general distinction between 'modalist' and 'logicist' ways of identifying logic and necessity. Modalists provide theories of logic in terms of a prior understanding of necessity; logicists provide theories of necessity in terms of a prior understanding of logic. Modalists can be usefully subdivided into those who are intensionalists about propositional individuation and those who are hyperintensionalists about propositional individuation. For our purposes, logicists can be usefully categorised based on which variety of logical truth they try to identify with absolute necessity. We will be dealing with three forms of logicism in this essay, corresponding to three different conceptions of logical truth: the semantic, the substitutional, and the syntactic.[1] In §2.1. I spell all of this out in greater detail. In §2.2. I point out an additional feature of interest shared by all of these developments of the identity thesis, which is that they require one to revise the otherwise standard view of logical properties as properties of formulas and modalities as properties of propositions. Then in §2.3., I provide a detailed overview of the discussion to come.

[1] This is not intended to be exhaustive. For instance we will not be dealing with the quantification account of logical truth advocated by Williamson [68] and mentioned in Bacon and Fine [4].

2.1 Modalisms and Logicisms

In this essay I will be looking at the prospects for using the identity thesis in one of two mutually exclusive ways. In other words, I will be trying to answer two basic questions. First, can the identity thesis be used to provide a satisfactory account of logic in terms of necessity? Second, can the identity thesis be used to provide a satisfactory theory of necessity in terms of logic? In the end, my answer to both of these questions will be in the negative. In this essay, I will use the term 'modalism' to refer to any theory which attempts to identify logic with a prior understanding of necessity, and I will use 'logicism' to refer to any theory which attempts to identify necessity with a prior understanding of logic. Modalist views and logicist views both incorporate the identity thesis. I will be developing multiple modalist theories and multiple logicist theories in order to show some of the general problems with attempts to reduce either necessity to logic or vice versa. In this section I will describe some of the main divisions between types of logicist and modalist theories that I take to be especially salient and helpful in structuring discussion.

To take the modalist camp first. All modalists attempt to understand logical truth and consequence as identical with necessity and necessitation. Working with a prior understanding of necessity then, the modalist will consider the bearers of necessity to be propositions rather than formulas or sentences.[2] Therefore questions concerning propositions are central to considerations of modalism, and in particular, questions of propositional individuation. In this essay I make a distinction between modalists who are intensionalists and modalists who are hyperintensionalists about propositional individuation. Intensionalists believe that propositions are individuated by metaphysically necessary equivalence. In other words, for the intensionalist, it is metaphysically necessary that if propositions x and y metaphysically necessitate each other, then $x = y$. Hyperintensionalists are those who deny intensionalism, believing that it is metaphysically possible for there to be propositions which are metaphysically necessarily equivalent and yet distinct. Though there is only one way to be an intensionalist, there are indefinitely ways to be a hyperintensionalist, since there are indefinitely many alternative principles of individuation one might propose in place of intensionalism. Therefore in this essay I cannot consider exhaustively all of the versions of modalism that would result from adopting alternative hyperintensionalist views on propositional individuation. However, I will still try to be comprehensive within some well-defined bounds.

[2] As I will state in more detail in §2.2., the standard picture of things is that necessities are properties of propositions and that logical truth is a property of sentences. Any development of the identity thesis requires a revision to this picture, but insofar as modalism preserves an ordinary understanding of necessity, the status of necessity as a property of propositions will be preserved within modalist views.

Now onto the logicist camp. Logicists attempt to understand absolute necessity by identifying it with some prior understanding of logical truth. But this is not sufficiently informative by itself. There are many varieties of logical truth; the beginning logic student knows of at least two, these being semantic validity and syntactic derivability. A formula ϕ is a semantic logical truth iff it is true in all models (or 'on all interpretations'); a formula ϕ is syntactically derivable iff it is a theorem of a certain logical calculus. But there are other varieties of logical truth. For example, Bacon and Fine [4] distinguish the semantic and derivational notions of logical truth from a *formal/substitutional* kind of logical truth and additionally a *quantificational* kind of logical truth. A formula ϕ is logically true in the formal or substitutional sense iff all of its admissible substitution instances are true. A formula ϕ is logically true in the quantificational sense iff the universal generalisation of that formula $\forall(\phi)$ is true, where the universal generalisation $\forall(\phi)$ of ϕ is the result of replacing all of ϕ's non-logical vocabulary with appropriate variables and prefixing with the relevant universal quantifiers.[3] Of these notions of logical truth, we will be considering especially the semantic, the substitutional, and the syntactic. To these there therefore correspond three prospects for developing logicism. To fix some terminology, let 'semantic logicists' be those who identify absolute necessity with semantic validity, let 'substitutional logicists' be those who identify absolute necessity with logical truth in the substitutional sense, and finally let 'syntactic logicists' be those who identify absolute necessity with logical truth in the syntactic or derivational sense. I cover these kinds of logicism in chapters 6-8. The semantic view will be discussed in chapter 6, and the substitutional and syntactic accounts will be discussed across Chapters 7 and 8.

To sum up, there are modalist theories of logic and there are logicist theories of necessity. I am going to try to show that both kinds of view face fundamental difficulties. I divide modalist theories into intensionalist and hyperintensionalist modalisms, based on whether they take propositions to be individuated by metaphysically necessary equivalence or not. For present purposes I shall divide logicist theories into semantic, substitutional and syntactic logicisms, based on which particular kind of logical truth each theory takes absolute necessity to be. Now that I have spelled this taxonomy out, in the next sections I want to mention some final points of interest that apply to both modalism and logicism, before providing a more detailed overview of the discussion to come.

[3]Again, this concept of logical truth is identified and put to use especially in [68]. We will not be investigating it in this essay due to constraints on space.

2.2 Sentences and Propositions

A more or less standard picture of things has it that the bearers of logical properties (logical truth, consistency, and consequence) are declarative sentences and other linguistic items like formulas, and that the bearers of necessity are propositions, conceived of as the distinct, extralinguistic contents of sentences and formulas. If this picture is right, then necessity and logic are not identical, and the identity thesis is simply false. Therefore any development of the identity thesis, be it logicist or modalist must invariably go against this standard picture in some way. We have seen this hinted at already; modalisms as characterised above retain a concept of necessities as properties of propositions. Due to the identity thesis, then, they must also take logical truth to be a property of propositions, rather than of sentences. Likewise, any version of logicism which understands logical truth to be a particular property of formulas and sentences must conclude that necessity is a property of sentences.

In general there seem to be two basic ways of revising the 'standard picture' just characterised: either one makes propositions into the bearers of both necessities and logical properties, or else one does the same thing for sentences instead. To be clear however, this distinction—between what could be called 'propositionalist' and 'sententialist' views of the bearers of modal and logical properties—is not the same distinction as that between modalism and logicism. Even though it seems that modalists must be propositionalists, and that logicists will tend to be sententialists, there are conceivably logicist views which are in fact propositionalist in character, in that they take logical truth and necessity to be properties of propositions. For example, if one holds to some form of structured theory of propositions, according to which propositions have some kind of quasi-syntactic structure, it might be possible to formulate a conception of logical truth as a property of propositions which is analogous to either the semantic, substitutional, or derivational varieties of logical truth recognised for formulas and sentences. It is important to mention this possibility since some of the most prominent versions of logicism present in the literature seem to fall into precisely this camp. For example, Sider [62] develops a theory of necessity which is essentially a form of syntactic or derivational logicism. For Sider A is necessary if and only if A can be derived from a conventionally decided-upon stock of modal axioms, making use of a conventionally decided-upon stock of modal inference rules. This much is clearly a view of necessity as identical with syntactic derivability. However, in Sider's theory, necessity is still a property of propositions: A and all of the 'modal axioms' are propositions, and the 'modal rules' take propositions for their inputs and outputs. Therefore Sider's view can be thought of as a syntactic form of logicism which is nevertheless propositionalist in character, since it identifies necessity with derivability understood as a property of propositions. (Moreover, the positing of propositional inference rules seems to presuppose an underlying picture of propositions as structured entities as mentioned before.)

The fact that any development of the logicist thesis requires a certain amount of revisionism to the standard picture will already convince some that the identity thesis in untenable. Such people would agree with my basic contention already, and would need no further convincing. Nevertheless, I actually do not think that loyalty to the standard picture is good enough reason to reject the identity thesis. There have been prominent philosophers who have disagreed with this picture, and there are ways of arguing that the revisionism involved is not as drastic as it might appear.

Take the propositionalist line, which says that propositions bear both modal and logical properties. On this view it is propositions which are necessary and possible, and which necessitate one another. And it is also the propositions which enter into relations of consequence, and which are (in)consistent or else logically true. Now, this sort of picture does not need to wholly undermine our practice of identifying sentences as logically true or consistent, or sententially-composed arguments as valid or invalid, though it does require that any talk of logical statuses in the case of sentences and sentential arguments be demoted to the level of analogy. That is: sentences can perhaps be called logically true insofar as they express a logically true proposition, and a sentential argument can be called valid perhaps insofar as the proposition expressed by the conclusion is a consequence of the propositions expressed by the premises. In fact, this picture seems not so different from the common view that propositions are the primary bearers of truth and falsity, and that sentences are only called true or false by courtesy, insofar as they express propositions (e.g. Soames [63], Ch. 1).

Or take the sententialist line, which says that sentences and formulas bear both modal and logical properties. This view is also decently represented in the history of analytic philosophy, most notably in the particular brand of modal skepticism promoted by Quine. In "Three Grades of Modal Involvement", Quine distinguishes three ways in which necessity can be introduced into logic (Quine [50]). The first of these involves admitting a unary predicate N into one's language which attaches to names of sentences,[4] producing modal claims of subject-predicate form 'Ns', to be read as something like 'the sentence s is necessary'. The second and third grades of involvement both involve introducing a connective $\Box$ which expresses the English phrase 'necessarily', which is the far more familiar way of doing modal logic. In particular, the second grade of involvement limits the use of $\Box$ so that it may only be applied to sentences (in the technical sense of 'sentence' used in logic, that is, formulas without free variables), while the third grade allows $\Box$ to occur in front of formulas with free variables too. Importantly for the point I am making here, Quine sees

[4] Quine uses 'Nec' rather than 'N', and 'nec' rather than $\Box$.

only the *first* grade of modal involvement as fully legitimate, and sees the second grade as legitimate only insofar as it can be read as an alternate mode of notation for the first grade. Quine finds the third grade of modal involvement, in which $\Box$ can apply to open formulas, entirely illegitimate for reasons that do not immediately concern us here. All I mean to show by referencing Quine at this point is to point out that, for him, necessity can be spoken about legitimately when, but only when, it is referenced as a property of sentences. This way of thinking can be thought to fall on the 'sententialist' end of the division when it comes to disagreeing with what we originally called the 'standard' picture of modal and logical properties.

In this section I began by drawing attention to the fact that any development of the identity thesis must revise a standard picture according to which logical truth is a property of sentences and formulas while necessity is a property of propositions. The distinction between the two main ways of revising this standard picture, which I have called propositionalist and sententialist respectively, approximates the distinction between modalism and logicism. Modalists must be propositionalists, it seems, and most logicists will tend towards sententialism. Nevertheless, importantly, it is possible to have versions of logicism which take logical truth and necessity to be borne by propositions. Some prominent versions of logicism in the literature, such as Sider's, are like this. I then tried to respond to the thought—friendly to my own position—that the identity thesis is untenable simply because it requires revising the aforementioned 'standard picture' of modal and logical properties. This reasoning is unsatisfactory because the standard picture is not itself so universal; there have already been good and prominent philosophers who have either gone for a more propositionalist or sententialist view of things. More substantial argument is needed to rebut the identity thesis.

2.3 Contingent Logical Truths

As just mentioned, there will be those who reject the identity thesis out of hand simply because it involves a revision to the picture of logical truth as a property of sentences and necessity as a property of propositions. Another group that will quickly reject the identity thesis will be those who countenance the existence of contingent logical truths. Examples of contingent logical truths are best found in the logic of actuality. Take a simple language PML of propositional modal logic supplemented with a unary operator '@' read as 'actually'. A standard semantics for this language can be provided as follows. Let a model be a triple $M = (W, V, w_@)$ where W is a nonempty set of 'worlds', $w_@ \in W$ is the 'actual world', and V is a function which takes each sentence letter p to a set of worlds $V(p) \subseteq W$. The semantic clauses for this semantics are then as follows, with formulas being satisfied at a model M with respect to a world w from that model:

$M, w \nvDash \bot$
$M, w \vDash p$ iff $w \in V(p)$
$M, w \vDash \Box\phi$ iff for all $v \in W$, $M, v \vDash \phi$
$M, w \vDash @\phi$ iff $M, w_@ \vDash \phi$
$M, w \vDash (\phi \to \psi)$ iff $M, w \nvDash \phi$ or $M, w \vDash \psi$.

Call a formula ϕ a 'real-world' consequence of the formulas Γ ($\Gamma \vDash_r \phi$) iff for all models M, if $M, w_@ \vDash \gamma$ for all $\gamma \in \Gamma$ then we have $M, w_@ \vDash \phi$. Call ϕ 'real-world' valid ($\vDash_r \phi$) iff ϕ is a real-world consequence of the empty set. Additionally, call a formula ϕ a 'general' consequence of the formulas Γ ($\Gamma \vDash_g \phi$) iff for all models M and all worlds w in W, if $M, w \vDash \gamma$ for all $\gamma \in \Gamma$ then we have $M, w \vDash \phi$. Call ϕ 'generally' valid ($\vDash_g \phi$) iff ϕ is a real-world consequence of the empty set.

If real-world validity characterises the logical truths of PML$_@$, then there are contingent logical truths. That is because, for instance, the formula $@p \to p$ is real-world valid, but almost never necessarily true. To see this, take any model $M = (W, V, w_@)$ where W contains some world $v \neq w_@$, and where $w_@ \in V(p)$ but $v \notin V(p)$. In this case we have $M, v \nvDash @p \to p$, meaning that $M, w_@ \nvDash \Box(@p \to p)$, so that $@p \to p$ is only contingently true at the actual world $w_@$, even though it is real-world valid. By the same token, real-world validity is not closed under the rule of necessitation. By contrast, if general validity characterises the logical truths of PML$_@$, then there are no contingent logical truths in the language, since by the semantic clause we can see that if ϕ is generally valid then for any model M and world w from w, we have $M, w \vDash \Box\phi$. There has been some dispute as to whether real-world validity or general validity is the better characterisation of logical truth for this propositional language. Some (including Hanson [28]) have rejected real-world validity in favour of general validity on the grounds that logical truths must be necessary. Others (including Zalta [71]) have insisted that, after all, we should accept the real-world validities as logical truths and embrace the existence of contingent logical truths.

If there are contingent logical truths, then the identity thesis is simply false. If the identity thesis is true, a contingent logical truth is a truth which is both necessary (being a logical truth) and contingent, which is absurd. Therefore any development of the identity thesis must deny contingent logical truths in general, and in particular cannot countenance real-world validity as a characterisation of the logical truths of PML$_@$. Some will see this as an unacceptable cost of the identity thesis, and indeed this would be another way to give a short, pithy argument against the identity thesis. As with the objection of the previous section, however, I feel that this argument against the identity thesis is not completely persuasive by itself. As mentioned, there are already philosophers who reject real-world validity in particular and contingent logical truths in general, and to these writers it will be necessary to offer more sustained arguments against particular, developed understandings of the identity thesis.

2.4 Roadmap

In this section I will give a detailed outline of the discussion to come. These first two chapters have served to introduce our topic, our key terms, and our assumptions. They have also served to introduce the various ways in which the identity thesis can be developed into either modalist theories of logic or logicist theories of modality. In the following chapters I will start to examine these ways of developing the identity thesis. Modalist views are discussed in Chapters 3-5, while logicist views are discussed in Chapters 6-8, and Chapter 9 concludes.

Chapter 3. Beginning with modalism, we start in Chapter 3 by setting out the view in some detail, and the basic problems which face it as an account of logical consequence. Modalist views of logic in general have a decent pedigree in the history of philosophy, especially among the medievals, but they are in some ways pathological. Distinctively, they obliterate any relevance of form to the validity of arguments. For a modalist, 'a man runs ∴ an animal runs' is logically valid simply because the proposition that a man runs necessitates the proposition that an animal runs. Any argument with a necessary conclusion or an impossible premise will be logically valid for the same reason. Though this applies to any form of modalism, for intensionalistic modalism the problem is especially pronounced by the fact that absolute and metaphysical necessity can be shown to be identical, so that questions of validity become questions of metaphysical possibility and impossibility. The aim of this chapter is basically to consolidate the case against views of logical consequence as mere necessary truth preservation.

Chapters 4 and 5. These chapters form a mathematical excursus that follows on from the previous discussion of modalism. Here I set aside the problematic aspects of modalist views on logic and consider the consequences of modalism for the behaviour of logical necessity. In order to do this, I develop a general non-reductive account of necessities as properties of propositions in an algebraic framework. In Chapter 4 I do this while assuming an intensionalist's coarse-grained view of propositions; this allows for a focus on complete atomic Boolean algebras and results in a very nice and simple framework. In Chapter 5 I then consider variants of this framework which allow for hyperintensionalist individuation and non-atomic algebras, noting that such allowances result in some interesting complications. Besides offering insight into the consequences of modalism for the logic of logical necessity, this example of a non-reductive account of necessity contrasts nicely with those reductive accounts of necessity which are presented in later chapters.

Chapter 6. The following three chapters discuss the various kinds of logicism introduced earlier in this chapter. Chapter 6 discusses the semantic variant of logicism, according to which absolute necessity is semantic validity. The methodology here, as in subsequent chapters, is to develop multiple systems of semantics for modal logic which model the respective logicist theories. We can then assess the consequences of each theory by looking to properties of the semantic systems themselves. Carnap's semantics for modal logic give us a paradigm case here. I examine both the case of propositional modal logic and the case of first-order modal logic. For propositional logic: I note that by a theorem of Carnap, semantic logicism implies that the logic of absolute necessity is S5. For first-order logic: I show that a substitutional reading of the quantifiers must be assumed in order to preserve classical quantification theory, and that in order to preserve Leibniz's Law a restriction must be imposed according to which no object may have two names. I will argue that both of these impositions are philosophically problematic in Chapter 9, since they also crop up for the other two versions of logicism studied in the following two chapters.

Chapter 7. The treatments of substitutional and syntactic logicism are much more technical than the treatment of semantic logicism. This is because the systems of semantics required to model substitutional and syntactic logicism are much less familiar than the semantics of Carnap and Kripke. In Chapter 7 I consider substitutional logicism by looking at the McKinsey semantics for modal logic, which are not nearly so well-understood as the Kripke semantics. Here we are able to draw on the recent work of Bacon and Fine [4] in order to draw some initial conclusions. On the assumption that $\Box$ is a logical constant, substitutional logicism implies that the logic of absolute necessity extends the familiar logic S4M and that it is included in the less familiar logic known as Med. At present, however, an exact answer is not possible. However, if a certain conjecture known as the Uniqueness Conjecture is true, then the logic is just Med itself. Next, in order to consider the case where $\Box$ is not considered as a logical constant, I construct an augmented version of the McKinsey semantics. I use this augmented semantics to investigate the logic of substitutional validity if $\Box$ is not treated as a logical constant—this turns out to be an interesting logic which Urquhart [67] has called TS. I then consider some remarks of Bacon and Fine [4] with respect to extending the McKinsey semantics to first-order logic. Once again, as in Chapter 6, a substitutional reading of the quantifiers must be assumed in order to preserve classical quantification theory, and in order to preserve Leibniz's Law a restriction must be imposed according to which no object in the domain may have two names.

Chapter 8. The system of semantics I use to model syntactic logicism has needed to be developed essentially from scratch. Therefore, before using it to discern the logic(s) of derivational logical truth, I have devoted substantial space in Chapter 8 to the general development of the correspondence theory of the semantics, which is of independent interest. I then consider the logics of derivational logical truth. I define a function L_α on propositional logics and argue that if $\Box$ is treated as a logical constant, then syntactic logicism implies that the logic of absolute necessity is a fixed point of L_α. I also argue that, if $\Box$ is not treated as a logical constant, then syntactic logicism implies that the logic of absolute necessity is $L_\alpha(\mathrm{PC})$, which turns out to be equal to TS. The function L_α and its fixed points turn out to have a highly interesting relationship back to the McKinsey semantics. Namely: it can be shown that Med is a fixed-point of L_α, and that the logic of formal validity on the McKinsey semantics is equal to the intersection of all of L_α's fixed points. That also means that if the Uniqueness Conjecture from Chapter 7 is true, Med is the unique fixed point of L_α and hence the sole candidate for the logic of absolute necessity on syntactic logicism. I also consider the case of first-order logic, and show that, as in previous chapters, substitutional readings of the quantifiers and requiring that no object have two names are necessary to validate classical quantification theory and Leibniz's Law.

Chapter 9. In this chapter, I conclude our study of logicism by providing general critiques of the various forms of logicism investigated in previous three chapters. For all three theories, it turned out that substitutional readings of the quantifiers and a requirement that no object have two names were necessary to validate classical quantification theory and Leibniz's Law. I argue that the restrictions are insufficiently motivated and philosophically problematic.

Chapter 10. I then sum up my general thoughts and observations taken from the essay as a whole, concluding ultimately that the identity thesis should be rejected.[5]

[5] Also included after the main discussion are several appendices. Appendix A sets out some of the algebraic and logical background that I draw upon freely in the main discussion. Appendices B and C, which are largely technical, then consider two generalisations of the syntactical semantics introduced in Chapter 8.

Part II

Identifying Logic and Necessity

3
Modalism about Logic

Modalists about logic are those who accept the identity thesis, providing an account of logic in terms of a prior understanding of necessity. In this chapter I discuss some of the main general features of such a position, including its anti-formalistic view of logical validity, and the consequences of pairing it with either intensionalist or hyperintensionalist views of propositional individuation.

3.1 Logical Materialism

In recent writing on the philosophy of logic, the term 'logical hylomorphism' has sometimes been used to denote the family of views which describe logical validity as something dependant on *form* (Novaes [44]). We will see such conceptions of logical validity at play in later chapters; the idea that logical truth and validity are matters of form is very intuitive, and generally in keeping with the thought that questions of logical truth and validity are *a priori*.

Modalism is committed to a much less common view of logical consequence, namely, that logical validity is nothing more than necessary truth-preservation. For the modalist, an argument $\Gamma \therefore \phi$ is logically valid if and only if the propositions expressed by the formulas in Γ absolutely necessitate the proposition expressed by ϕ.[1] Beyond this, the form of these formulas is irrelevant; really all that matters is necessitation. The idea that a modal connection between premises and conclusion is all there is to logical validity has been called 'Aristotle's Thesis' by Rumfitt [56]. But

[1] Strictly, modalists actually think that 'logical validity' only applies to arguments composed of propositions; but as mentioned previously (§2.2.), modalists may still apply their account to arguments composed of formulas and sentences by analogy.

I prefer to call it 'logical materialism', since the name better highlights the contrast with logical hylomorphism, as well as the connection to the idea of 'material validity'. A logical materialist is someone who believes that necessary truth-preservation is the whole of logical validity; all modalists are materialists.[2]

Though logical materialism was well-represented among the ancients and especially the medievals, today it is a heterodox view. Thus, while arguments like 'a man runs $\therefore$ a man moves' could have been seen as valid among many medieval logicians,[3] today this would be seen by most as a logically invalid (because formally invalid) argument, even though it is necessarily truth-preserving. The historical reasons for this are of interest, insofar as they can be identified. First, as Shieh [60] details, the early history of analytic philosophy was dominated by various forms of skepticism about modality. In the early pioneers of modern logic, most importantly Russell and Frege, this prompted a move toward non-modal understandings of logical truth and consequence. To be sure, previous writers had identified purely formal notions of consequence—Bolzano [6] most notably. But given their prominent place at the head of the analytic tradition, it is plausible that the rejection of modal understandings of consequence in Frege and Russell had more effect in setting the tone of later thinking about logic. Second, the profound influence of Tarski's [66] model-theoretic definition of consequence has cemented a fundamentally non-modal characterisation of consequence at the heart of discipline. Semantic validity in the style of Tarski makes for something of a benchmark in characterising logics and their consequence relations, and is entirely non-modal.

It is not completely clear to what extent these and other points connect—e.g. whether and to what degree Tarski was influenced by early analytic skepticism about the modal in constructing his definition of consequence, or how the reception of Tarski's account might in turn have bolstered anti-modal sentiments among philosophers. But the result of these and other factors seems clear enough. Today it is the case that two students studying separate introductory logic courses at separate universities under separate teachers and working from separate textbooks will nevertheless cover broadly the same material. They will learn some propositional logic and some first-order logic, they will learn how to handle a proof system, and they will learn about models and interpretations

[2] Are all materialists also modalists? In practice, potentially, but one minute difference between the camps as I have described them would be that the materialist is not committed to the view that logical validity *just is* necessitation. Because of her commitment to the identity thesis, the modalist must speak of valid arguments composed of formulas through use of analogy. The materialist, who is not necessarily committed to the identity thesis, is able to keep formulas as the bearers of logical properties and propositions as the bearers of modal properties. Even though validity and necessitation go hand-in-hand for the materialist, for her they are not necessarily the same property.

[3] Read [53] notes that it was mainly in Oxford that such inferences were counted as valid; in Paris, by contrast, a recognisable condition of formality was indeed enforced. As examples: Robert Fland counts 'There is a man, so there is an animal' as valid, while Jean Buridan counts 'A man runs, so an animal runs' as invalid because the uniform substitution instance 'A horse walks, so wood walks' is non-truth-preserving ([53], 904).

in the course of learning how to provide countermodels to invalid arguments. Even if they do not learn how to prove soundness and completeness, they will learn that there is such a thing as logical truth understood alternately as truth in all models and as provability within a logical calculus, and they will learn that these are known to coincide in the elementary case. Though the class may begin with the familiar intuitive modal gloss on logical validity ('it is impossible for the conclusion to be false if the premises are true'), the class will eventually proceed to replace this with a mathematically precise characterisation of propositional and first-order validities in terms of either models or a proof system. One consequence of this—of the fact that this is now just the standard way to be introduced to logic—is that the characterisations of logical truth in either proof- or model-theoretic terms are now the norm, both for experts and for the ordinary end-user. If these were ever more recherché than the modal characterisation of validity, at any rate this can no longer be plausibly claimed. For students, teachers, and researchers—meaning, for the logical community as a whole—'logical consequence', say for classical first-order logic, just means that well-defined relation between formulas which can be characterised in terms of models or in terms of a certain adequate proof system. The modal gloss is nothing more than an early teaching aid; to insist on it as being the whole of consequence is really just to be making a student's mistake. Worse, perhaps. If, in the logical community, 'logical consequence' just means a certain formal relation between formulas, the logical materialist might just be accused of trying to change the subject altogether.

To explain what I mean: the logical materialist recommends that the whole of logic should be seen as the study of necessary truth-preservation. On this view formal validity is, at best, incidental to the concerns of the logician as such. But given that, for the ordinary logician, 'logical consequence' just means a certain non-modal, proof- or model-theoretic relation between formulas, it is not clear how the ordinary logician ought to even take this recommendation. Is it (a) just a recommendation to give up doing what is currently called 'logic' and to take up the pure and dedicated study of necessary truth-preservation instead? Or (b) is there meant to be some indication that really, necessary truth-preservation is what logicians were getting at all along, and that they were somehow deceived by their own art into thinking that they were really interested in something else, like semantic or syntactic validity? The first of these alternatives really does amount to an attempt to change the subject. As a recommendation, it could only be followed on its own terms if logicians were to collectively cease all current operations and start investigating something basically different instead. In that sense it seems more like an economic recommendation than a philosophical one: a recommendation to re-allocate the academy's resources away from the study of formal consequence relations to the study of necessary truth-preservation. But such a recommendation just seems misconceived.

The more intelligible way of construing the materialist's recommendation is (b). The idea here is that really, even though logicians do indeed study various non-modal relations between formulas, such as semantic and syntactic validity, the proper aim of such study is a better understanding of various patterns of necessary truth-preservation. We care about the formal relation of semantic consequence only because we can apply it to questions of necessary truth-preservation. For an example, consider this sequent from elementary logic.

$$\forall x(Fx \to Gx), Fa \vDash Ga$$

In this case, the purely mathematical fact that these particular formulas stand in this particular relationship of semantic consequence codifies a lot of information about what the materialist cares about, which is necessary truth-preservation. Any argument which is faithfully formalised in this way is necessarily truth-preserving. For example:

All men are mortal, Socrates is a man $\therefore$ Socrates is mortal

For the logical hylomorphist, the validity of this argument actually consists in the fact that it has the form of the sequent given above in first-order logic, and that this sequent holds as a true semantic consequence. For the materialist, on the other hand, the validity of the latter argument consists merely in the fact that, necessarily, if all men are mortal and Socrates is a man, then Socrates is mortal. The real interest in the first-order sequent is not as the *source* of logical validity, but as a schema which codifies a pattern of valid inference. Now the suggestion that semantic consequence (for example) is of merely instrumental interest to the logician is still heterodox. However, at the very least, (b) cannot be accused of trying to change the subject; on this view, logic as it currently stands may continue, and a logician can study formal validity *qua* logician.

But if the materialist holds that formal validity is only of instrumental interest to the logician, then she must accept that it should cease to be of interest whenever it does not happen to align with what her logician *really* cares about, which is necessary truth preservation. This means that any kind of formal consequence relation which does not guarantee necessary truth preservation is once again of limited interest to logicians, as far as the materialist is concerned. Any argument which is valid by the lights of elementary (non-modal, first-order) logic is necessarily truth preserving. But in previous chapters we have already seen that there are other consequence relations studied by logicians that do not guarantee necessary truth preservation. For example, take the relation of real-world consequence ($\vDash_r$) defined in §2.3. over the language of propositional modal logic with an actuality operator. Here the sequent $@p \vDash_r p$ holds, even though the argument 'actually Socrates is bearded $\therefore$ Socrates is bearded', which has the form of the real-world valid sequent, is not necessarily truth preserving. After all, Socrates could have been clean shaven though he is actually bearded. So it is not necessarily the case that, if Socrates is actually bearded, then Socrates is bearded. The logical materialist must conclude that consequence relations like $\vDash_r$ should be of limited (if any) interest to the logician, even though they are of actual interest to practicing logicians. So even on this more moderate suggestion, the materialist is advocating for the reformation of logic as a discipline: a narrowing of focus to just those consequence relations which encode information about necessary truth preservation. This again reveals the basic cost of materialism, which is that it proposes to constrain logic as a discipline in a way that is more or less alien to the way that logic is actually practiced. In this way, the materialist must presume on some level that working logicians are mistaken about the identity of their subject matter. Logicians think that the subjects of their investigations are formal consequence relations, but the materialist thinks that this is wrong.

Claims to the effect that the practitioners of a science are mistaken about the identity of their own subject matter are, in fact, not unheard of. Such claims are common enough in the philosophy of science. Nominalists about numbers believe that platonist mathematicians are fundamentally mistaken about the subject of their inquiries; such mathematicians take themselves to investigate numbers (among other things), whereas the nomimalist says they do not because there are no numbers. Again, some eliminative materialists about the mind have claimed that psychologists are fundamentally wrong about what it is they study; psychologists take themselves to study the mind and mental states, but some versions of eliminative materialism imply that this is wrong because there are no mental states. As applied to the history of science, sometimes such claims are simply and uncontroversially true. At one time, scientists working on heat took themselves to be studying the behaviour of caloric fluid; now we know that they were wrong, because there is no such thing as caloric fluid.

Nevertheless, the materialist's claim that logicians are mistaken about their own subject matter differs in some important respects from these other claims. Chiefly, the logical materialist does not deny the *existence* of formal consequence relations. (It is hard to see how one could; so long as one believes that there are such things as proofs or models, there must be formal consequence relations.) Because of this, the materialist is unable to provide a convincing explanation as to *how* logicians could be mistaken about their subject matter. The mathematical nominalist can provide a story as to how mathematicians could be wrong about their subject matter. Since there are no abstracta, the nominalist says, mathematical theories simply cannot be literally true in the way that the platonist imagines; there is simply nothing appropriate there for the theories to be correct about. The logical materialist, on the other hand, must admit that formal consequence relations *are* there—including ones which do not guarantee necessary truth-preservation—and that logicians ostensibly do primarily study them. That leaves the materalist's claim, that logicians are wrong about formal consequence relations being of primary concern in logic, unexplained—and not clearly explicable.

Putting it another way: experts of a mature scientific discipline are in a position of authority with respect to what they study. Such authority does not amount to infallibility, but in general if there is a point of broad agreement among experts, then a burden of proof falls upon those who would challenge that agreement to show why the experts are wrong. It is a point of agreement among logicians, both expert and not, that their subject matter is primarily the study of formal consequence relations like semantic and syntactic consequence. Therefore the burden of proof falls to the materialist to show why this cannot be right. We have not found any clear story that the materialist can tell towards meeting this burden. It is not open to the materialist to deny the existence of formal consequence relations outright. But, given the existence of formal consequence relations, the claim that logicians are mistaken in thinking that they study those relations seems mysterious. To state it simply: what is stopping them from being right?

None of this amounts to a decisive rebuttal of materialism, but it does indicate some of the broad problems that the materialist needs to reckon with, and some of the basic defects of the view itself. It is a cost of any view in the philosophy of science if that view entails that practitioners of a science are systematically mistaken about their subject in some deep way; views that imply this much seem to flout a basic principle of intellectual charity. They fail to take the science in question on its own terms, as understood by its practitioners. Though the cost of such an implication can in principle be offset, it is not clear that the logical materialist has enough to offer in this regard. They claim that logicians are systematically mistaken in thinking that the primary objects of their study are formal consequence relations, but they give no story as to why this cannot be so.

In the following sections I will talk about some further heterodox features of materialism when paired with intensionalism about propositional individuation, as well as how these further features can be avoided if intensionalism is denied.

3.2 Intensionalism and Metaphysical Necessity

Logical materialism paired with intensionalism about propositional individuation lies in particularly stark contrast to orthodox pictures of logical consequence. This is because, if intensionalism is true, absolute necessity and metaphysical necessity coincide.[4] This means that absolute necessity

[4] Take any proposition x. If x is absolutely necessary then x is metaphysically necessary simply because absolute necessity is as extensionally broad as every other necessity (cf. §1.2.). Conversely, if x is metaphysically necessary, then it is metaphysically necessarily equivalent with any proposition t expressed by a tautology of the propositional calculus. By intensionalism $x = t$, but then by the fact that t is absolutely necessary, being a logical truth, x is absolutely

will inherit properties of objective necessities—in particular, the substitution of co-referring terms. Since water = H2O, and it is metaphysically necessary that water = water, it is also metaphysically necessary that water = H2O, and hence *absolutely* necessary that water = H2O.[5] Hence, by materialism, it is a logical truth that water = H2O. In this way, the combination of intensionalism and modalism (which implies materialism) obliterates the connection between logical truth and the *a priori*.

To be clear, the intensionalist modalist can still say that much about necessary truth-preservation can be known *a priori*. Arguments which take the form of a valid sequent of elementary logic are all necessarily truth-preserving; so it can be known *a priori* that arguments like 'all men are mortal, Socrates is a man ∴ Socrates is mortal' are valid. And it can be known that 'water = water' is a logical truth since it has the form of the law of reflexivity ('$x = x$'). However, given intensionalism and modalism, *a priori* truth-preservation is no longer any kind of necessary condition on logical validity. Logical truth coincides with metaphysical necessity, and many sentences which express metaphysical necessities are simply not *a priori*. The property of *a priority* is less central to ordinary conceptions of logical consequence than is formality. There are other positions besides materialism which might imply that some claims are logical truths and *a posteriori*. For a simple example, the substitutional account of logical truth (studied in Chapter 7 below) has the consequence that true statements are logically true if they contain no non-logical vocabulary. This means that, for instance, '$\exists x \exists y(x \neq y)$' is logically true because it is true and contains only logical symbols, even though it is not clearly *a priori* that there are at least two individuals in existence.[6] Nevertheless, the loss of *a prioricity* is an interesting consequence of intensionalist modalism, and the extent to which the view makes logical truth *a posteriori* is far starker than the substitutional view.

3.3 Varieties of Hyperintensionalism

A hyperintensionalist view of propositional individuation is just a view which denies intensionalism: hyperintensionalists say it is metaphysically possible for there to be propositions that are distinct and yet metaphysically necessarily equivalent. Hyperintensionalist views are internally diverse, so there are many possible combinations of modalism and materialism with versions of hyperintensionalism. In general, hyperintensionalist versions of modalism will not have the

necessary. So x is absolutely necessary iff it is metaphysically necessary.

[5] We will assume here that 'H2O' really is rigid, and not some kind of abbreviated chemical description.

[6] This is a controversial point about the substitutional account as we state it. Some versions of the substitutional theory of logical truth, including Halbach's [26], avoid theorems like $\exists x \exists y(x \neq y)$ by allowing for restrictions on the domain of quantification.

consequence that metaphysical and absolute necessity coincide.[7] On hyperintensionalism, absolute necessity might outstrip metaphysical necessity.

For some versions of hyperintensionalist modalism, the very same loss of *a prioricity* results as in the case of intensionalist modalism. Take a typical Russellian view of structured propositions. On this view, the proposition expressed by an atomic sentence like 'Rab' is a structured whole consisting of the individuals denoted by 'a' and 'b' and the relation denoted by the predicate 'R'.[8] Since the entities named by terms are themselves the constituents of propositions, then we still get a property of substitution of rigidly co-referring terms. For example, since 'Hesperus' and 'Phosphorus' denote the very same thing, 'Hesperus = Hesperus' and 'Hesperus = Phosphorus' express the very same proposition. Therefore, since the proposition expressed by the former is presumably logically true, so must be the latter, even though it is not *a priori*.

On the other hand, for other versions of hyperintensionalism, this is not so. If one takes propositions to be sufficiently fine-grained, it might be possible to retain some amount of *a prioricity* for logical validity while remaining a materialist. For example, take a typical Fregean view of structured propositions. On this view, the proposition expressed by an atomic sentence like 'Rab' is a structured whole consisting of the *senses* of the individual terms 'a' and 'b' and the sense of the predicate 'R'. Even if 'a' and 'b' turn out to refer to the very same thing, if they differ in their sense then the Fregean view denies that they can always be substituted to yield sentences expressing the same proposition. If 'Hesperus' and 'Phosphorus' have distinct senses, then 'Hesperus = Hesperus' and 'Hesperus = Phosphorus' express distinct propositions for the Fregean. Therefore for the Fregean modalist, it might be that only the former is absolutely necessary / logically true, while the latter is not. And in general, to the extent that a Fregean allows distinctions of sense to track distinctions of cognitive value, it should be possible for her to insist that all absolutely necessary propositions are *a priori*, since she can always insist that a substitution that would take an *a priori* claim like 'water = water' to an *a posteriori* claim like 'water = H2O' alters the proposition expressed by substituting terms which differ in sense. Therefore she could insist that all such substitutions take us from propositions which are absolutely necessary to propositions which are not.

Next, let us take a version of hyperintensionalism which does not go via the route of structured propositions. Hyperintensionalisms based on the idea of *impossible worlds* can also retain the *a*

[7] This is not to say that a hyperintensionalist *cannot* say that absolute and metaphysical necessity coincide. Only that on hyperintensionalism, it is consistent to suppose that the two come apart. For on hyperintensionalism it is consistent to suppose that metaphysically necessarily equivalent propositions are distinct, so that arguments such as those given in footnote 6 fail.

[8] This view is hyperintensionalist since, for example, the propositions expressed by '$2 + 2 = 4$' and '$1 + 1 = 2$' are distinct despite being metaphysically necessarily equivalent propositions, since one contains the number 1 as a constituent while the other does not.

prioricity of consequence when paired with modalism. On such views, the proposition expressed by a sentence is identified with the class of worlds, both possible and not, in which the sentence is satisfied. Now, importantly, there is not a unified concept of an impossible world on offer in the literature. Different authors often work with very different conceptions of, not only the nature of impossible worlds, but also of their strictly formal properties within well-defined systems of semantics. For example, Jago and Berto [5] distinguish several alternative notions of an impossible world. One example of a choice point: some theorists may take their impossible worlds to violate the laws of logic, and others may not. Now impossible world theorists are, in principle, able to carve up logical space as finely as they please. To see this, suppose we have an impossible worlds theorist of the 'linguistic ersatzer' variety as discussed in ([5], §3.6). For this theorist, a world is just a set of sentences—of English, let us suppose, even though Jago and Berto discuss positing a distinguished 'worldmaking' language ([5], 85) for this purpose. For our ersatzer, it may be that only some sets of sentences count as *possible* worlds; but any set of sentences which fails to be a possible world still succeeds in being a world, albeit an impossible one. In addition, let us suppose that our ersatzer accepts the claim that the proposition expressed by a sentence is identical with the class of possible *and* impossible worlds at which that sentence is true (where a sentence is true at a world iff it is a member of that world). In this case, our toy example of an impossible world theorist would be committed to the view that no two distinct sentences of English express the same proposition. For if ϕ and ψ are two such sentences, ϕ is true in the world $\{\phi\}$ while ψ is not. Given the freedom that impossible world theory has to divide up logical space as finely as desired—in principle, if not necessarily in practice—it is also possible in principle for a materialist impossible world theorist to maintain the *a prioricity* of logical truth. For she has the resources to maintain that every *a priori* sentence expresses a different proposition from every *a posteriori* sentence, just like the materialist Fregean.

To sum up the main point: the realm of hyperintensionalisms is internally diverse, and the ways in which modalism can combine with hyperintensionalisms are also diverse. This leads to different forms of modalism having alternative things to say about the *a prioricity* of consequence in particular. But the internal diversity of modalist hyperintensionalisms also presents a problem about how to constrain the discussion of the following two chapters. I will state and offer my solution to this problem in the next section.

3.4 Approaching Algebraic Treatments of Necessity

Modalism about logic has the following points against it which we have already mentioned. Via its commitment to materialism about consequence it is essentially revisionary, contradicting the understanding of working logicians of their own subject as interested in formal consequence relations. Consequently, and like all views which incorporate the identity thesis, it discounts certain areas of current logical investigation as illegitimate—e.g. the logic of actuality and the way in which it allows for contingent logical truths. When coupled with intensionalism, modalism (like materialism) has counter-intuitive consequences regarding the *a priority* of logic and hence the dependence of logical investigation on metaphysics and empirical science.[9]

But I want to put these points to one side for the time being. There are still important and interesting questions regarding the consequences of modalism which remain unanswered if we only focus on the negatives. In the following two chapters I want to change tack and go in a slightly different direction. The modalist identifies logical consequence with a prior conception of necessity; but what should this prior, non-reductive conception of necessity consist in? This question should not be put off for future study, but pertains directly to the present discussion. For alternate answers to this question yield alternate consequences with respect to, for instance, the behaviour of absolute necessity (and *ipso facto* for logical truth, if one follows the modalist). Therefore we need to construct and consider a general non-reductive treatment of necessities. As I argue in the next chapter, this can be carried out on the basis of the general assumptions made back in Chapter 1 (e.g. monism, the necessity of classical logic) plus the assumption of intensionalism about propositional individuation. This framework is algebraic, in the sense that it centers on the idea that necessities are operators on propositions, which are taken to form a certain kind of Boolean algebra. The account of necessities is non-reductive in the sense that it does not attempt to analyse modal concepts into non-modal concepts, and also in the sense that (unlike views considered later) does not attempt to understand absolute necessity in terms of a prior conception of logical consequence. This framework allows us to precisely define important ideas like relative breadth and the logic of a given necessity, and allows us to give a detailed argument that the logic of absolute necessity must be S5. Along the way I will remark what this means for modalists. In Chapter 5, I then consider some variants of the framework offered in Chapter 4 which proceed on the basis of various hyperintensionalist views.

[9] For example, if intensionalism and materialism are true: to know that 'water is wet ∴ H_2O is wet' is valid we need to know something about the chemistry of water. To know whether 'Smith's C-fibres are firing ∴ Smith is in pain' is valid we need to know something about the metaphysics of mind.

Now, as mentioned above, hyperintensionalism comes in indefinitely many varieties. Therefore it would be impossible for me to consider how a hyperintensionalist should proceed to construct a theory of necessities in complete generality. Instead, what I will do is consider just two varieties of hyperintensionalism which are both very close to intensionalism. For example, in contrast to most structured views of propositions, our versions of hyperintensionalism will both accept the thesis that propositions form a Boolean algebra under conjunction, disjunction and negation (so, for instance, they will both admit that tautologically equivalent sentences express the same proposition). But beyond this, each kind of hyperintensionalism we consider will hold additional theses in common with intensionalism. I will explain these here, admittedly well in advance, since doing so will allow me to introduce some ideas that are also helpful to flag for Chapter 4.

The following thesis is commonly accepted, and I will be accepting it throughout the following two chapters.

> LEIBNIZ'S THESIS (METAPHYSICAL)
>
> Metaphysical necessity is truth in all possible worlds.[10]

In Chapters 4 and 5 I am going to adopt a familiar construal of possible-world talk in an algebraic context. Namely: a possible world is an atomic proposition, a proposition a such that for every proposition x, a either entails x or it entails x's negation. Once I have done this I will put Leibniz's thesis to work and argue that: intensionalism is true iff the algebra of propositions is an atomic Boolean algebra (I define all relevant terms in due course). This means that in Chapter 4 we mainly focus on the properties of atomic Boolean algebras. It also means that in Chapter 5 we mainly focus on non-atomic Boolean algebras.

Now, we are already familiar with the point that if intensionalism is true, then absolute necessity and metaphysical necessity coincide. Hence intensionalism and the metaphysical version of Leibniz's thesis suggest the following claims.[11]

[10] In working with Leibniz's thesis for metaphysical necessity, I am not departing from the gloss on metaphysical necessity as the 'broadest objective necessity' introduced back in Chapter 1. Indeed, it is a natural thought that truth in all possible worlds coincides with the broadest objective necessity. If there is a possible world in which x is true, it seems that x should be objectively possible in some sense (hence in the broadest sense). Conversely, if there is no world in which x is true, meaning there is no way the world could have been such as to make x true, it seems wrong to say in any objective sense that x is possible.

[11] Strictly speaking, as we have defined our terms, the absolute version of Leibniz's thesis is not *entailed* by the metaphysical version of Leibniz's thesis and intensionalism. It is consistent with intensionalism about propositions that *properties* of propositions may necessarily coincide without being identical. In practice, I do not know of any intensionalists about propositions who do not also assert that metaphysically necessarily equivalent properties are identical. Moreover, to accept coarse-grained propositions and fine-grained properties of propositions seems like a strange position to take.

LEIBNIZ'S THESIS (ABSOLUTE)

Absolute necessity is truth in all possible worlds.

GENERAL MODAL INDIVIDUATION

Propositions that are absolutely necessarily equivalent are identical.

These are the theses which our versions of hyperintensionalism will be based around. No hyperintensionalist theory can adopt both. If we accept the absolute version of Leibniz's thesis *and* general modal individuation, with the metaphysical version of Leibniz's thesis still in the background, we end up with intensionalism. Accordingly we will consider one variety of hyperintensionalism which accepts general modal individuation while denying the absolute version of Leibniz's thesis, and we will consider a second version of hyperintensionalism which does the converse, denying general modal individuation while accepting the absolute version of Leibniz's thesis.

This suffices to set the stage. Now to sum up and conclcude. In this chapter I have tried to make a basic case against modalism about logical consequence, and I have started to introduce the discussion of the following two chapters. Modalism is not an attractive thesis. Mainly (not solely) because it is implausibly revisionary to the scope of logic as actually practiced, without offering sufficient motivation for said revisionism. Ultimately, despite whatever historical pedigree it might have, we should reject it. But before moving on from modalism, there still remain a number of interesting questions about how the modalist should understand, for instance, the behaviour of absolute-cum-logical necessity. This invites us to consider non-reductive theories of necessity in general, which we begin to do in the next chapter.[12] Now, by way of setting the stage, I have already flagged some of the methodological and dialectical choices to be made in these chapters. I adopt a broadly mathematical methodology and an algebraic setting. By adopting the metaphysical version of which I call 'Leibniz's thesis', and by identifying possible worlds with atomic propositions, I will try to correlate the thesis of intensionalism with a certain property (atomicity) of Boolean algebras in order to compare certain aspects of intensionalism and hyperintensionalism in this formal setting. I will investigate notions of breadth and of the logic of a given necessity. And along the way I will comment on important implications for the modalist.

12. An additional motivation, worth restatement, is to contrast a non-reductive treatment of necessity with reductive treatments as they arise from Chapter 6 onward.

4

Algebra and Necessity I: Intensionalism

In this chapter I develop a general theory of necessities on the basis of some of the general assumptions adopted in Chapter 1 (namely: monism, absolutism, the absolute necessity of classical propositional logic, and closure for absolute necessity). The theory is non-reductive: it argues for a certain theory of absolute necessity, but does not presume this theory to explain the modal in terms of the non-modal. It is also given and elaborated in mathematical terms.

First I will give an outline of the theory. Intensionalism implies Booleanism, which says that the propositions form a Boolean algebra under conjunction, disjunction, and negation. In §4.1 I explain this point in detail. I also argue that Booleans generally should think that the algebra of propositions is complete and that, if intensionalism is true, we should think that the algebra is atomic (relevant technical terms are defined as we proceed). In §4.2 I invoke monism to motivate a formal characterisation of what a necessity is, understood as a property of (meaning, an operator on) propositions. I then proceed to establish some results about necessity operators understood in the proposed way. One particularly welcome result is that the necessity operators on a given complete atomic Boolean algebra turn out to be precisely those operators which distribute over infima (which can be understood as arbitrary conjunctions). Finally, in §4.3 I discuss the concept of relative 'breadth' between necessities. I motivate a definition of breadth in this algebraic context, and I show on this framework that the broadest necessity (which I argue must be absolute necessity, which coincides with metaphysical necessity for the intensionalist and logical truth for the modalist) obeys S5.

4.1 Intensionalism and the Algebra of Propositions

4.1.1 Booleanism and Boolean Algebra

In this and the following chapter, we will be talking about Boolean algebras. The definition of Boolean algebras I will be working with is the following.

Definition 4.1.1. A Boolean algebra is a tuple $B = (|B|, \cdot_B, +_B, -_B, 0_B, 1_B)$ with $0 \neq 1$ such that the following equations hold for all $x, y, z \in |B|$.[1]

$x + 0 = 0$	$x \cdot 1 = x$
$x + -x = 1$	$x \cdot -x = 0$
$x + y = y + x$	$x \cdot y = y \cdot x$
$x + (y \cdot z) = (x + y) \cdot (x + z)$	$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

Here 0_B, and 1_B are the bottom and top elements of the algebra B, $+_B$ and $\cdot_B$ are the meet and join operations on the algebra, which are both binary and written infix, and $-_B$ is the compliment operation on the algebra, which is unary. Where context makes it clear which algebra is being discussed, we may omit identifying subscripts.

Now, there are propositions, and there are operations on propositions which correspond to the usual truth-functional connectives of conjunction, disjunction, and negation. Let the 'algebra of propositions' $\mathbb{P} = (|\mathbb{P}|, \sqcap, \sqcup, \sim, \mathbf{F}, \mathbf{T})$ be the algebra consisting of the domain $|\mathbb{P}|$ of all propositions,[2] the operations of conjunction ($\sqcap$), disjunction ($\sqcup$) and negation ($\sim$), and distinguished elements **F** and **T** where for some $x \in \mathbb{P}$, $\mathbf{F} = (x \sqcap -x)$ and $\mathbf{T} = (x \sqcup \sim x)$. Booleanism is the thesis that $\mathbb{P}$ satisfies the definition of a Boolean algebra. Given that all classical tautologies are indeed metaphysically necessary (which follows from the assumption that they are all absolutely necessary), intensionalism implies Booleanism. To see this, consider a language of propositional logic $\mathrm{PL}_{\neg\vee\wedge}$ built-up from the stock of sentence letters $p, q, r, \ldots$ from the connectives $\wedge, \vee, \neg$ in the usual way, introducing the further connectives $\rightarrow$ and $\leftrightarrow$ by the standard abbreviation schemes $(\phi \rightarrow \psi) := (\neg\phi \vee \psi)$ and $(\phi \leftrightarrow \psi) := (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. Let a propositional interpretation of $\mathrm{PL}_{\neg\vee\wedge}$ be any function $[\cdot] : \mathrm{PL}_{\neg\vee\wedge} \rightarrow |\mathbb{P}|$ such that for all formulas ϕ and ψ of PL:

[1] There are many equivalent ways of defining Boolean algebras; this one is due to Huntington [29]; associativity and the De Morgan laws can be derived from these. The condition that $0 \neq 1$ is added to rule out degenerate (i.e. single-element) algebras, which otherwise complicate the statement of certain results.

[2] There are good reasons to think that there cannot be a set of all propositions. For example, there is at least one proposition for every cardinal number κ, such as the proposition that there exist at least κ horses. Since the cardinals do not form a set, neither can the propositions. We will set this point aside in the present discussion and proceed *as if* all propositions form a set. The alternative approach would have been to allow for proper class sized Boolean algebras, modalities, and so on, but this would have needlessly complicated exposition.

$$[\neg\phi] = \sim [\phi]$$
$$[(\phi \wedge \psi)] = [\phi] \sqcap [\psi]$$
$$[(\phi \vee \psi)] = [\phi] \sqcup [\psi]$$

Our assumption that the theorems of classical propositional logic are all absolutely (and hence metaphysically) necessary implies that for any formula ϕ and any propositional interpretation $[\cdot]$, if ϕ is a classical tautology then $[\phi]$ is metaphysically necessary. Therefore for any formulas ϕ and ψ and any propositional interpretation $[\cdot]$, if $(\phi \leftrightarrow \psi)$ is a tautology then $[(\phi \leftrightarrow \psi)]$ is metaphysically necessary. This is just what it means to say that $[\phi]$ and $[\psi]$ are metaphysically necessarily equivalent. So by intensionalism, $[\phi] = [\psi]$. Summing this up, intensionalism implies that for any propositional interpretation $[\cdot]$ and formulas ϕ and ψ: if $(\phi \leftrightarrow \psi)$ is a tautology then $[\phi] = [\psi]$. Next we note that all of the following are tautologies:[3]

$$\begin{array}{ll} p \vee (s \wedge \neg s) \leftrightarrow (s \wedge \neg s) & p \wedge (s \vee \neg s) \leftrightarrow p \\ p \vee q \leftrightarrow q \vee p & p \wedge q \leftrightarrow q \wedge p \\ p \vee (p \wedge r) \leftrightarrow (p \vee p) \wedge (p \vee r) & p \wedge (p \vee r) \leftrightarrow (p \wedge p) \vee (p \wedge r) \\ p \vee \neg p \leftrightarrow s \vee \neg s & p \wedge \neg p \leftrightarrow s \wedge \neg s \end{array}$$

Using these we can show that, given intensionalism, $\mathbb{P}$ satisfies each of the axioms characteristic of Boolean algebras. Since none of the axioms presents special difficulty, I will just give two examples for the sake of brevity. These are (i) the principle that $x \sqcup y = y \sqcup x$ for all x and y and (ii) the principle that $x \sqcup \sim x = \mathbf{T}$ for all x.

> (i) Since $p \vee q \leftrightarrow q \vee p$ is a tautology, we have that for any propositional interpretation $[\cdot]$, $[p \vee q] = [q \vee p]$. Take any propositions x and y and any propositional interpretation $[\cdot]$ such that $[p] = x$ and $[q] = y$. Then we have that $x \sqcup y = [p \vee q] = [q \vee p] = y \sqcup x$.
>
> (ii) Since $p \vee \neg p \leftrightarrow s \vee \neg s$ is a tautology, we have that for any propositional interpretation $[\cdot]$, $[p \vee \neg p] = [s \vee \neg s]$. We know that there is some proposition z such that $z \sqcup \sim z = \mathbf{T}$. Therefore for any y, take some propositional interpretation such that $[p] = y$ and $[s] = z$. Then $y \sqcup -y = [p \vee \neg p] = [s \vee \neg s] = z \sqcup \sim z = \mathbf{T}$.

Having shown how intensionalism implies that $\mathbb{P}$ is a Boolean algebra[4] I will now allow the use of Boolean notation with respect to $\mathbb{P}$. For example, we may use '$+_{\mathbb{P}}$' as alternate notation for $\sqcup$, and '0' for **F**. Now in the remainder of this section I want to mention some further points

[3] I follow standard conventions for omitting parentheses in $\mathrm{PL}_{\neg\vee\wedge}$.

[4] The final further condition that $\mathbf{T} \neq \mathbf{F}$ is ensured by the fact that there are at least two propositions (for example, the proposition that $0 = 0$ and the proposition that $0 = 1$).

of philosophical interest (especially concerning propositional entailment) and to motivate some further properties of $\mathbb{P}$. I will argue that all Booleans should think that $\mathbb{P}$ is infinite and complete, and I will argue that if intensionalism is true, we can infer that $\mathbb{P}$ is atomic. Thus I argue that if intensionalism is right, $\mathbb{P}$ is a complete atomic Boolean algebra.

4.1.2 Boolean Entailment as Absolute Necessitation

But first, in any Boolean algebra B one can define a canonical ordering $\leq_B$ as follows:

Definition 4.1.2. $x \leq y$ iff $(x \cdot y) = x$.

Definition 4.1.3. $x < y$ iff $x \leq y$ and $y \not\leq x$.

Since $\mathbb{P}$ is a Boolean algebra, there is also a relation $\leq_{\mathbb{P}}$ between propositions. Where x and y are propositions, $x \leq_{\mathbb{P}} y$ iff the conjunction of x and y is the very same proposition as x itself. I will sometimes refer to $\leq$ as (one-one) Boolean entailment.

Now recall that the modalist believes there is such a thing as logical consequence as a relation between propositions. For the modalist, logical consequence is the very same thing as absolute necessitation, and it would seem that $\leq_{\mathbb{P}}$ is a natural candidate to be identified with this notion of propositional logical consequence. So the modalist might be able to offer a novel argument that $\leq_{\mathbb{P}}$ is absolute necessitation. Even so, such an argument would be redundant. Since it is possible to provide an argument that $\leq_{\mathbb{P}}$ is identical with absolute necessitation which does not proceed on the basis of modalism, but only on the basis of the general assumptions listed at the start of the chapter, including monism and intensionalism.[5]

For the purposes of this argument let us define a new operation $\Rightarrow$. On any Boolean algebra, let $(x \Rightarrow y) := (-x + y)$. Then in $\mathbb{P}$, $\Rightarrow$ functions as the propositional counterpart of material implication. Moreover we have the derived rule $[\phi \to \psi] = [\phi] \Rightarrow [\psi]$ for any propositional interpretation $[\cdot]$. Given this new operation, we can characterise (one-one) absolute necessitation. A proposition x absolutely necessitates another proposition y iff the material conditional $x \Rightarrow y$ is absolutely necessary. Using this, we give an argument to the effect that for all propositions x and y, x absolutely necessitates y iff $x \leq y$.

[5]Strictly speaking, it is not right to say that $\leq_{\mathbb{P}}$ *is* absolute necessitation, since necessitation is many-one and $\leq_{\mathbb{P}}$ is one-one. Rather, the idea is that x is absolutely necessitated by a set of propositions S iff $\prod S \leq_{\mathbb{P}} x$, where $\prod S$ is the conjunction of the propositions in S (see §4.1.4 below). This idea will be captured in the many-one relation '$\Rrightarrow$' defined in §4.2.

(Right to left.) Suppose that $x \leq y$. Take some propositional interpretation $[\cdot]$ such that $[p] = x$ and $[q] = y$. Then since $(p \wedge q) \rightarrow q$ is a tautology, we have that the proposition $[(p \wedge q) \rightarrow q] = (x \cdot y) \Rightarrow y$ is absolutely necessary. But then since $(x \cdot y) = x$, it follows by substitution that $x \Rightarrow y$ is absolutely necessary, meaning that x absolutely necessitates y.

(Left to right.) Suppose that $x \Rightarrow y$ is absolutely necessary. From this we will show that x is absolutely necessarily equivalent with $x \cdot y$. From this it follows by either intensionalism or by the broader principle of modal individuation mentioned in §3.3 that $x = x \cdot y$ so that $x \leq y$. Now, one can see that $x \cdot y$ absolutely necessitates x simply by the fact that $p \wedge q \rightarrow p$ is a tautology. Hence for any propositional interpretation $[\cdot]$ with $[p] = x$ and $[q] = y$, the proposition $[p \wedge q \rightarrow p] = (x \cdot y) \Rightarrow x$ is absolutely necessary. In order to show that x also absolutely necessitates $x \cdot y$, we first note that the formula $(p \rightarrow q) \rightarrow (p \rightarrow p \wedge q)$ is a tautology, meaning that where $[p] = x$ and $[q] = y$, the proposition $[(p \rightarrow q) \rightarrow (p \rightarrow p \wedge q)] = (x \Rightarrow y) \Rightarrow (x \Rightarrow (x \cdot y))$ is absolutely necessary. Now, a propositional version of the principle (Closure) mentioned back in §1.3.1 says that for any propositions z and z': if z is absolutely necessary and $z \Rightarrow z'$ is absolutely necessary, then z' is absolutely necessary. Applying this here allows us to infer that since $x \Rightarrow y$ is absolutely necessary, $(x \Rightarrow (x \cdot y))$ is absolutely necessary, completing the argument.

Note that the this argument can also proceed under the weaker principle of general modal individuation accepted by certain forms of hyperintensionalism (§3.4). For this reason, we will return to this argument and its conclusion in Chapter 5. Next, we return to consider some further properties of $\mathbb{P}$.

4.1.3 Infinitude

It is generally plausible that there are infinitely many propositions, meaning that $|\mathbb{P}|$ is infinite so that $\mathbb{P}$ is an infinite algebra. In fact, there is a standard way of arguing that there must be infinitely many propositions. Consider each of the propositions expressed by the sequence of claims "there is a horse", "there are two horses", "there are three horses", and so on to infinity. The idea is that each of these propositions is distinct, because for any natural numbers $n < k$, it is metaphysically possible that there are exactly n horses—hence, that there are not k horses. Therefore there are infinitely many distinct propositions expressed in this sequence of claims; therefore there are infinitely many propositions.

I do not know of any good way to block this argument. The only way available seems to be to deny the general point that for any natural numbers $n < k$, it is metaphysically possible that there are n horses and that there are not k horses. If this point is accepted, then the argument goes through, just because in this case "there are n horses" and "there are k horses" express propositions that are distinct because they have distinct properties (e.g. the proposition $\pi(n)$ that there are n horses is compossible with the negation of the proposition $\pi(k)$ that there are k horses, but $\pi(n)$ is not compossible with the negation of $\pi(n)$ itself). But the only way I can think to deny that it is metaphysically possible that there are n horses and that there are not k horses if $n < k$, is to embrace a kind of necessitarianism. That is: if one thinks that all true propositions are metaphysically necessary, then it is metaphysically impossible for there to be two horses and not three horses simply because there are at least three horses. But I will not be considering views like necessitarianism to be live options here. And I accept the conclusion that $\mathbb{P}$ is infinite.

We note also that this way of establishing the infinitude of propositions in fact establishes something stronger, which is that there is an infinitely descending chain of propositions. Since for any n we have $\pi(n+1) <_{\mathbb{P}} \pi(n)$, we have the infinite descending chain $\pi(1) >_{\mathbb{P}} \pi(2) >_{\mathbb{P}} \pi(3) >_{\mathbb{P}} \ldots$. I will use this property later to streamline an argument that the logic of absolute necessity is exactly S5.

4.1.4 Completeness

In any Boolean algebra, given the canonical ordering $\leq$ defined above, we can define concepts of a least upper bound (supremum) and greatest lower bound (infimum) as follows.

Definition 4.1.4. Where $A \subseteq |B|$, x is an upper (lower) bound for A iff $y \leq x$ ($x \leq y$) for all $y \in A$. Additionally, x is a least upper bound (greatest lower bound) for A iff x is an upper bound (lower bound) for A and $x \leq z$ for every z an upper bound of A ($z \leq x$ for every z a lower bound of A). If it exists, A's greatest lower bound (least upper bound) is unique and so is denoted $\prod A$ ($\sum A$).

Definition 4.1.5. A Boolean algebra B is complete iff for all $A \subseteq |B|$, $\sum A, \prod A \in |B|$.

So, to claim that $\mathbb{P}$ is complete is to claim that for any set of propositions $A \subseteq |\mathbb{P}|$, there is a supremum $\sum A \in |\mathbb{P}|$ and an infimum $\prod A \in |\mathbb{P}|$. Every finite Boolean algebra is complete, but because $\mathbb{P}$ is infinite, its completeness is non-trivial. However, I take it to be eminently plausible. To show this, I will first show that the completeness of $\mathbb{P}$ amounts to the assertion that every collection of propositions has a conjunction and disjunction. That is, the conjunction of a collection of propositions A will be identical to its greatest lower bound $\prod A$, and the disjunction of a collection of propositions A will be identical to its least upper bound $\sum A$. The identification of lower and upper bounds with conjunctions and disjunctions is easily justified. Where $\bigwedge A$ denotes the conjunction of A in a more neutral way, recalling that $\leq$ in $\mathbb{P}$ encodes a natural notion of entailment, it should clearly be the case that $\bigwedge A \leq x$ for all $x \in A$ (conjunctions entail each of their conjuncts), and that for any y, if $y \leq x$ for all $x \in A$ then $y \leq \bigwedge A$ (any proposition which entails every conjunct of a conjunction also entails the conjunction itself). But this just means that $\bigwedge A = \prod A$. Likewise, where $\bigvee A$ denotes the disjunction of A in a more neutral way, it should clearly be the case that $x \leq \bigvee A$ for all $x \in A$ (disjunctions are entailed by each of their disjuncts), and that for any y, if $x \leq y$ for all $x \in A$ then $\bigvee A \leq y$ (any proposition entailed by each disjunct of a disjunction is also entailed by the disjunction itself). But then $\bigvee A = \sum A$.

Now, given that completeness amounts the assertion that every collection of propositions has a conjunction and disjunction, the denial of completeness for $\mathbb{P}$ amounts to the assertion that there are some infinite collections of propositions without a conjunction or without a disjunction. Such an assertion seems difficult to motivate in the present setting. It might have been less so in a setting where propositions are more like sentences of natural language, and so cannot be of infinite 'length'. But under intensionalism, propositions have no form, and there is nothing *intrinsically* infinite about a proposition which just happens to be the conjunction of some infinite collection of propositions. Hence there is no intrinsic difference between the conjunction of infinitely many propositions and the conjunction of finitely many propositions. For example, there is clearly nothing intrinsically infinite about the proposition that a particular line l is no greater than one inch long. However, given intentionalism it is identical to the conjunction of the infinite collection of propositions:

$$\left\{\text{That } l \text{ is no greater than } 1 + \frac{1}{n} \text{ inches long} : n \in \omega\right\}$$

Given that every finite collection does and (as we just saw) some infinite collections do as well, the intensionalist should say that every collection of propositions has a conjunction and disjunction. For an intensionalist otherwise does not seem to have the resources to distinguish between legitimate and illegitimate cases of infinite conjunction.

4.1.5 Atomicity

In a Boolean algebra, 'atoms' are least nonzero elements. More specifically:

Definition 4.1.6. x is an atom iff for all y, either $x \leq y$ or $x \leq -y$ and not both.[6]

In this and the next chapter I reserve 'a' to stand for atoms. Not all Boolean algebras have atoms. Even among those with atoms, not all Boolean algebras have the following property:

Definition 4.1.7. B is atomic iff for all $y \in |B|$: if $y \neq 0$, there is some a where $a \leq y$.

[6] Equivalently: for all y, either $x \cdot y = x$ or $x \cdot y = 0$.

To claim that $\mathbb{P}$ is atomic is to claim that for every proposition x that is distinct from 0, there is some atomic proposition a such that $a \leq x$. I want to argue that if intensionalism is true, then $\mathbb{P}$ must be atomic. To do this, I invoke two plausible identifications. The first is to identify atomic propositions with possible worlds. Under this conception, a possible world is a proposition which entails exactly one of x and $-x$ for each proposition x, and for a proposition to be true at a world is just for it to be entailed by that world. The set of propositions true at a world a, which is simply the set $\uparrow(a) = \{x : a \leq x\}$, has the desirable property that for all sets of propositions S, $S \subseteq \uparrow(a)$ iff $\prod S \in \uparrow(a)$. Now, with the notion of a possible world in hand, the second identification I adopt is the following, mentioned in the previous chapter:

> LEIBNIZ'S THESIS (METAPHYSICAL)
> Metaphysical necessity is truth-in-all-possible-worlds.

This identification needs little argument given its pervasiveness in the literature. Nevertheless we can motivate it as we noted in passing in the previous chapter.[7] Namely, given that metaphysical necessity is the broadest objective necessity, it ought to coincide with truth at all possible worlds. For if there is a possible world in which x is true, it seems that x should be objectively possible in some sense (hence in the broadest sense). Conversely, if there is no world in which x is true, meaning there is no way the world could have been such as to make x true, it seems wrong to say in any objective sense that x is possible. At any rate, from this identification it follows that metaphysically necessary equivalence is *truth at precisely the same possible worlds*. If intensionalism is true, then any propositions that are true in precicely the same possible worlds ought to be identical. And it turns out that, on the assumption that $\mathbb{P}$ is a complete Boolean algebra, this last property is equivalent to atomicity:

Definition 4.1.8. Elements x, y of a Boolean algebra B are atomically equivalent ($x \approx y$) iff for all atoms a in B: $a \leq x$ iff $a \leq y$.

Proposition 4.1.9. Suppose B is a complete Boolean algebra. Then B is atomic iff for all $x, y \in |B|$: if x and y are atomically equivalent then $x = y$.

Proof. (Left to right.) Suppose that B is atomic. Then taking any distinct elements x and y of B, we show that $x \not\approx y$. Since $x \neq y$, we know that either $x \not\leq y$ or $y \not\leq x$. Without loss of generality suppose that $x \not\leq y$. In this case, $x \cdot -y \neq 0$, so by atomicity there is some atom a such that $a \leq x \cdot -y$, meaning that $a \leq x$ and $a \not\leq y$, so that x and y are not atomically equivalent.

[7] See §3.4, footnote 8.

(Right to left.) Suppose that for any x and y, if $x \approx y$ then $x = y$. In that case, wherever $x \neq 0$, it must be the case that $x \not\approx 0$. But $a \not\lesssim 0$ for all atoms a, meaning we must have $a \leq x$ for some a. But this just means that B is atomic. □

I therefore claim that intensionalism implies that the Boolean algebra $\mathbb{P}$ is atomic, given the plausible assumption that $\mathbb{P}$ is a complete Boolean algebra. In fact, I also claim the converse: that on the assumption that $\mathbb{P}$ is a complete Boolean algebra, the atomicity of $\mathbb{P}$ implies intensionalism. Since atomic equivalence in $\mathbb{P}$ amounts to metaphysically necessary equivalence, on the assumption that $\mathbb{P}$ is a complete Boolean algebra, atomicity implies that all metaphysically necessarily equivalent propositions are identical. But this is just what intensionalism says. This means that if some version of *hyperintensionalism* is true, then so long as $\mathbb{P}$ is a complete Boolean algebra, $\mathbb{P}$ must be non-atomic. This point will be important in Chapter 5, when we consider hyperintensionalism in conjunction with non-atomic algebras.

This completes my preliminary remarks concerning the nature of $\mathbb{P}$ under intensionalism. In the following section I will motivate a certain definition of necessities, understood as operators on $\mathbb{P}$.

4.2 Modalities and Necessities

Necessities are properties of propositions. Taking a cue from type theory, a natural way to conceive of properties of a given type of entity τ is as unary functions $F : \tau \to |\mathbb{P}|$. For example, properties of individuals can be understood as functions F which take an individual c and output a proposition $F(c)$ — intuitively, the proposition that c is F. If we adopt this helpful way of thinking, properties of propositions are seen to be functions $F : |\mathbb{P}| \to |\mathbb{P}|$. In other words, properties of propositions are unary propositional operators. But if properties of propositions are simply propositional operators, then the necessities are somewhere among the propositional operators. This clarifies our task. To define the necessities, we must provide a characterisation of what it is for a propositional operator $F : |\mathbb{P}| \to |\mathbb{P}|$ to be a necessity. In order to do this, I want to provide a more general characterisation of 'necessities' which will apply to operators of any Boolean algebra. And to do *this*, I want to recall some ideas first set out back in Chapter 1.

Recall our assumptions of monism and absolutism as theses about the structure of the space of necessities under the relation of 'relativisation'. Monism says that there is only one necessity which is not a relative necessity (we have been calling this 'absolute necessity'); absolutism says that every relative necessity is a relativisation of some absolute necessity. Together these theses imply that any necessity apart from absolute necessity is a relativisation of absolute necessity. This means that every necessity N besides absolute necessity can be characterised in the following way. N is associated somehow with some propositions—it is possible for one and the same necessity to be associated with a different set of propositions in different possible worlds—and for a proposition x to be N-necessary at a world is simply for x to be *absolutely* necessitated by the propositions associated with N at that world. This is what it means to say that N is a relativisation of absolute necessity. Now, as we argued above in §4.1.2, and seeing as we are intensionalists in this chapter, we additionally have an argument to the effect that absolute necessitation can be identified with Boolean entailment $\leq_{\mathbb{P}}$ in the algebra of propositions. Putting all of this together we get that every necessity N besides absolute necessity holds fixed some propositions $\mu_N(w)$ at each world w, and for a proposition x to be N-necessary at a world w is just for the propositions $\mu_N(w)$ to entail x. We have not spoken about what many-one propositional entailment means yet, as the relation $\leq$ is only one-one. However, that can be amended:

Definition 4.2.1. $A \Rrightarrow x$ iff for all y, if y is a lower bound for A then $y \leq x$.

Where the property of completeness is present, the relation $\Rrightarrow$ can be reduced to simpler terms. In what follows, propositions that hold specifically in *complete* Boolean algebras will be marked 'C'. Thus the following holds in complete algebras:

Proposition 4.2.2 (C). $A \Rrightarrow x$ iff $\prod A \leq x$.

In the algebra of propositions, $A \Rrightarrow x$ iff every proposition that entails everything in A also entails x. This is a natural notion of many-one entailment. Therefore for a proposition x to be N-necessary at a world w is just for it to be the case that $\mu_N(w) \Rrightarrow x$—meaning, $\prod \mu_N(w) \leq x$, given that the algebra of propositions itself is complete. This characterisation of necessities applies primarily to the the relative necessities, but it can also be extended to absolute necessity. Absolute necessity is unique in that, unlike relative necessities, it holds nothing fixed at any world. In other words, if $\mu_N(w)$ denotes those propositions held fixed by N at w, we can extend this notation to define a similar symbol μ_{Abs} for absolute necessity by simply fixing it so that $\mu_{Abs}(w) = \varnothing$ for all w. So putting all of *this* together gets us the fact that for *any* necessity N, for a proposition x to be N-necessary at a world w is simply for x to be entailed by $\mu_N(w)$, the propositions held fixed by N at w. In the case of absolute necessity, this just means that x is absolutely necessary at w iff $\varnothing \Rrightarrow x$ (iff $x = 1$).

This gives us a feature that all necessities share; every necessity is associated in the requisite way with a function that selects some propositions at each world. In order to turn this into a definition of the necessities in any given Boolean algebra, we need only go in the other direction. That is, we will start by defining a certain kind of function on algebras which assigns atoms to elements (worlds to propositions, in the case of $\mathbb{P}$), and then we will define necessities as operators which relate to these functions in the right way. So to start we define:

Definition 4.2.3. A modality on a Boolean algebra B is a function $\mu : Atom(B) \to \mathcal{P}|B|$.

In particular, a modality on $\mathbb{P}$ is a function which takes every possible world a to a set of propositions $\mu(a)$. Next, we offer the following general characterisation of necessities in terms of these 'modalities'. The following definitions apply to operators N on any Boolean algebra, and so are stated in the most general terms.

Definition 4.2.4. N is a μ-necessity iff for all x and all atoms a: $a \leq Nx$ iff $\mu(a) \Rrightarrow x$.

Definition 4.2.5. N is a necessity iff N is a μ-necessity for some modality μ.

The idea here is straightforward. A necessity is an operator which is associated in the right sort of way with a modality μ on an underlying Boolean algebra. For N to be associated in the right sort of way with μ means that for any atom a and any element x, $a \leq Nx$ iff $\mu(a) \Rrightarrow x$. In the algebra of propositions, this means a necessity is a propositional operator which is associated in the right sort of way with a modality—a function from worlds to sets of propositions—meaning that for any world a and proposition x, Nx is true at a iff x is entailed by $\mu(a)$. Or again, a necessity in the algebra of propositions is an operator N associated with certain propositions—potentially different propositions at different world—such that for a proposition x to be N at a given world a is just for x to be entailed by the propositions associated with N at a. Thus all of this cashes out the ideas expressed less formally at the start of this section. And once again, these definitions can be applied to any Boolean algebra whatsoever. We can consider some examples to get a better handle on the concepts involved.

Example 4.2.6. Take the two-element Boolean algebra **2** with domain $\{0, 1\}$. The only atom in this algebra is 1, and so there are only four modalities in this algebra which set $\mu_a(1) = \varnothing$, $\mu_b(1) = \{0\}$, $\mu_c(1) = \{1\}$, and $\mu_d(1) = \{0, 1\}$ respectively. The identity operator $id : x \mapsto x$ on **2** is a μ_a-necessity (as well as a μ_c-necessity). Since for $x = 0, 1$ we have $1 \leq id(x)$ iff $\mu_a(1) \Rrightarrow x$. Meanwhile the constant operator with $C_1(1) = 1$ and $C_1(0) = 1$ is a μ_b necessity and a μ_d necessity. And one can see that id and C_1 are the only necessities on **2**.

Example 4.2.7. Take any powerset algebra $B_X = (\mathcal{P}(X), \cap, \cup, (X - \cdot), X, \varnothing)$. The atoms of B_X are the singletons $\{x\}$ for $x \in X$. The modalities on B_X are those functions which map singletons of X's elements to subsets of X. An operator $N : \mathcal{P}(X) \to \mathcal{P}(X)$ is a μ-necessity for a modality μ iff for all $x \in X$ and $Y \subseteq X$: $\{x\} \subseteq NY$ iff $\bigcap \mu(\{x\}) \subseteq Y$. Equivalently, N is a μ-necessity iff for all $Y \subseteq X$, $NY = \{x : \bigcap \mu(\{x\}) \subseteq Y\}$. From this it follows that for any modality μ on B_X, there is exactly one μ-necessity.

Example 4.2.8. Take any Boolean algebra B which is atomless (i.e. has no atomic elements). The only modality in this algebra is the empty function $\varnothing$. Vacuously, every operator on B is a $\varnothing$-necessity.[8]

Now that we have a definition of necessities, we can establish some results. To start with, we will prove some correspondences between modalities and necessities, both generally and more specifically in the context of complete and complete-atomic algebras. So to begin: some modalities, even though they are distinct, are equivalent in the following strong sense.

Definition 4.2.9 (Similarity)**.** $\mu \sim \mu'$ iff for all x and all atoms a, $\mu(a) \Rrightarrow x$ iff $\mu'(a) \Rrightarrow x$.

Similarity between modalties is an equivalence relation. Moreover, it is clear by our definitions that any μ-necessity will also be a μ'-necessity whenever $\mu \sim \mu'$. That is:

Proposition 4.2.10. If N is a μ-necessity and $\mu \sim \mu'$, then N is a μ'-necessity.

The proof is immediate. Where completeness holds, similarity can be stated in simpler terms.

Proposition 4.2.11 (C)**.** $\mu \sim \mu'$ iff for all atoms a, $\prod \mu(a) = \prod \mu'(a)$.

Here the proof is also immediate. Now, is it always the case that for a given modality μ, there exists a μ-necessity? In fact the answer in general is *no*. In Boolean algebras wherein completeness fails, one can find examples of modalities for which there are no necessities.

Example 4.2.12. Take the algebra of finite and cofinite sets of integers under the usual set operations. This algebra is atomic (its atoms are the singletons $\{z\}$ for $z \in \mathbb{Z}$) but it is not complete (for example, the set $\{\{z\} : z \text{ is even}\}$ has no supremum since the set of even integers is not finite or cofinite). Consider the modality μ such that $\mu(\{z\}) = \varnothing$ where z is odd and $\mu(\{z\}) = \{\{x\} : x \in \mathbb{Z}\}$ where z is even. For reductio, suppose that there is some operator N on the algebra which is a μ-necessity. Therefore for any integer z and any finite or cofinite set of integers X we have $\{z\} \subseteq NX$ iff $\mu(\{z\}) \Rrightarrow X$. Equivalently, for any such z and X we have $z \in NX$ iff $\mu(\{z\}) \Rrightarrow X$, so that for any co/finite X we have $NX = \{z : \mu(\{z\}) \Rrightarrow X\}$. Now take for example the finite set $\{1, 2, 3\}$. Note that $\varnothing \not\Rrightarrow \{1, 2, 3\}$ but that $\{\{x\} : x \in \mathbb{Z}\} \Rrightarrow \{1, 2, 3\}$. Therefore $\mu(\{z\}) \Rrightarrow \{1, 2, 3\}$ if and only if z is even, meaning that $N\{1, 2, 3\} = \{z : z \text{ is even}\}$. But this means that the underlying algebra is not closed under N, so N cannot in fact be a μ-necessity.

[8] This may be an ecumenical defect of the present account, in that it does not play nicely with opposing views on which $\mathbb{P}$ itself is atomless, so that there are propositions but no worlds (e.g. suppose with Rayo [51] that logical space is 'open'). It is possible to generalise the account to deal with such cases (i.e. by defining modalities as functions from *propositions* to sets of propositions), but given that we are not considering gunky views of logical space in any detail, since they are relatively heterodox, I have opted for greater simplicity, which the present definition provides.

On the other hand, one can quickly show that in any complete algebra, there always exists a μ-necessity for any modality μ. To introduce some new notation:

Definition 4.2.13. For any modality μ, let $\Box_\mu : x \mapsto \sum\{a : \mu(a) \Rrightarrow x\}$.

For a modality μ, the operation $\Box_\mu$ is defined (i.e. exists) iff for all x, the set of atoms $\{a : \mu(a) \Rrightarrow x\}$ has a supremum. (For example, in Example 4.2.12, $\Box_\mu$ does not exist.) Whenever this operation exists, we can show that it is a μ-necessity. First an easy lemma.

Proposition 4.2.14. Suppose A is a set of atoms and $\sum A$ exists. Then $a \in A$ iff $a \leq \sum A$.

Proof. Trivially, if $a \in A$ then $a \leq \sum A$. Conversely, if $a \notin A$ then we have $a' \leq -a$ for any $a' \in A$. In this case we have $\sum A \leq -a$, so if we had $a \leq \sum A$ we would have $a \leq -a$, which would require that $a = (a \cdot -a) = 0$, which is absurd, so $a \not\leq \sum A$. $\Box$

Proposition 4.2.15. If $\Box_\mu$ exists, then it is a μ-necessity.

Proof. Suppose that we have the operation $\Box_\mu$. Then for any x and any atom a, we have $a \leq \Box_\mu x = \sum\{a : \mu(a) \Rrightarrow x\}$. By Proposition 4.2.14, then, $a \leq \Box_\mu x$ iff $\mu(a) \Rrightarrow x$. $\Box$

Of course, in any complete algebra, $\Box_\mu$ will be defined for any modality μ, since then all sets of atoms will have suprema. From this we get:

Corollary 4.2.16 (C)**.** For all μ, there is a μ-necessity.

So far we have established that in any complete algebra, every modality has at least one necessity associated with it. Sometimes, modalities will have many necessities associated with them. However, in *atomic* algebras it turns out that every modality has at most one necessity. This is because the following holds in general.

Proposition 4.2.17. If N is a μ-necessity and N' is a μ-necessity then $\forall x : Nx \approx N'x$.

Proof. Suppose N and N' are μ-necessities. Then for any x we have for all a that $a \leq Nx$ iff $\mu(a) \Rrightarrow x$ iff $a \leq N'x$. Hence $Nx \approx N'x$. $\Box$

Hence the following holds in atomic algebras. Propositions that hold specifically in *atomic* Boolean algebras will be marked 'A'. Thus:

Proposition 4.2.18 (A)**.** If N is a μ-necessity and N' is a μ-necessity then $N = N'$.

Proof. Immediate by Propositions 4.2.17 and 4.1.8. $\Box$

This means that in algebras that are both complete and atomic we have the following. Propositions that hold specifically in *complete atomic* Boolean algebras will be marked 'CA'. Thus:

Proposition 4.2.19 (CA)**.** For any modality μ there is exactly one μ-necessity.

Proof. Immediate by Corollary 4.2.16 and Proposition 4.2.18. $\square$

In particular we have the following, putting together the previous results:

Proposition 4.2.20 (CA)**.** N is a necessity iff $N = \Box_\mu$ for some modality μ.

Proof. Immediate by Proposition 4.2.15 and Proposition 4.2.19. $\square$

Therefore in complete atomic Boolean algebras we have a particularly tight connection between modalities and necessities. This actually goes both ways. Not only does every modality have a unique necessity, but generally, every necessity turns out to correspond to at most one modality up to similarity:

Proposition 4.2.21. If N is a μ-necessity and a μ'-necessity then $\mu \sim \mu'$.

Proof. Suppose N is a μ-necessity and a μ'-necessity. Then for all x and a, we have $a \leq Nx$ iff $\mu(a) \Rrightarrow x$ and $a \leq Nx$ iff $\mu'(a) \Rrightarrow x$. So for all a and x we have $\mu(a) \Rrightarrow x$ iff $\mu'(a) \Rrightarrow x$. $\square$

Thus on the assumption that $\mathbb{P}$ itself is complete and atomic, such correspondences also apply to those necessities that are properties of propositions and their respective modalities.

The next thing I would like to establish is a particularly interesting result which holds in complete atomic algebras. This result says that the necessities are precisely those operators which commute with infima (meaning, in the case of $\mathbb{P}$, that the necessities are precisely those operators which commute with conjunctions). The latter property can be expressed as follows. Where N is an operator and A a set of elements of the underlying algebra, we adopt the notational convention that $NA := \{Nx : x \in A\}$.

Definition 4.2.22. N is $\prod$-commutative iff for all A, if $\prod A$ exists then $N \prod A = \prod NA$.

Now, one half of the promised result happens to hold in all complete algebras. That is:

Proposition 4.2.23 (C)**.** Every $\prod$-commutative operator is a necessity.

Proof. Suppose that N is $\prod$-commutative. Then consider the modality $\mu_N : a \mapsto \{x : a \leq Nx\}$. Then we can show that N is a μ_N-necessity. For all x and a, if $a \leq Nx$ then $x \in \mu_N(a)$ so that $\mu_N(a) \Rrightarrow x$. Conversely, for all x and a, if $\mu_N(a) \Rrightarrow x$ then by the completeness property we have $\prod \mu_N(a) \leq x$, so that $\prod \mu_N(a) \cdot x = \prod \mu_N(a)$. Therefore we have:

$$N(\prod \mu_N(a) \cdot x) = (N \prod \mu_N(a) \cdot Nx) = N(\prod \mu_N(a))$$

Meaning that $N \prod \mu_N(a) \leq Nx$. But again since N is $\prod$-commutative we get

$$N \prod \mu_N(a) = \prod N\mu_n(a) = \prod\{Nx : x \in \mu_N(a)\} = \prod\{Nx : a \leq Nx\}$$

But since a is a lower bound for $\{Nx : a \leq Nx\}$, we have $a \leq \prod\{Nx : x \leq Nx\} = N \prod \mu_N(a) \leq Nx$ so that $a \leq Nx$ by transitivity. □

Meanwhile the converse of this result requires an assumption of atomicity. First, two lemmas.

Proposition 4.2.24. If N is a necessity and A is a set of elements such that $\prod A$ and $\prod NA$ both exist, then $N \prod A \approx \prod NA$.

Proof. Suppose that N is a necessity. So for some μ it holds that for all x and a: $a \leq Nx$ iff $\mu(a) \Rrightarrow x$. For any set A of elements of the algebra where $\prod A$ and $\prod NA$ exist, for any a we therefore have $a \leq N \prod A$ iff $\mu(a) \Rrightarrow \prod A$. But it follows from our definitions that $\mu(a) \Rrightarrow \prod A$ iff $\mu(a) \Rrightarrow y$ for all $y \in A$. Therefore we have $a \leq N \prod A$ iff $a \leq Ny$ for all $y \in A$, which is the case iff $a \leq \prod NA$. Therefore $N \prod A \approx \prod NA$. □

Corollary 4.2.25 (C)**.** If N is a necessity then for all A: $N \prod A \approx \prod NA$

From which we get:

Proposition 4.2.26 (CA)**.** Every necessity is $\prod$-commutative.

Proof. Immediate by Corollary 4.2.25 and Proposition 4.1.8. □

Combining this result with its converse above finally gets us:

Proposition 4.2.27 (CA)**.** The necessities are exactly the $\prod$-commutative operators.

Proof. Immediate by Propositions 4.2.23 and 4.2.26. □

At this point we should pause and take stock. In this section we have motivated a definition of necessities as propositional operators. To do this, we articulated a more abstract definition of 'necessities' as operators on Boolean algebras more generally. We investigated some features of necessities so defined, and we have just shown that in complete atomic algebras, the necessities are just those operators which commute with infima. The implications of this result for intensionalism are clear enough. If our arguments from the previous section (§4.1) hold, intensionalism has the consequence that the algebra of propositions $\mathbb{P}$ itself is a complete atomic algebra. Therefore we have shown in this section that on this general understanding of necessities, intensionalism has the consequence that the necessities on $\mathbb{P}$ are just those propositional operators that commute with conjunction.

This result is somewhat surprising, but very welcome. Besides providing the intensionalist with a neat definition the necessities, the property of commuting with conjunction is one that is already well-associated with characterisations of necessity thanks to a long tradition of work in algebraic semantics for modal logic. In such semantic frameworks (cf. Lemmon [38, 39]) *modal algebras* make for the basis of interpretatations of propositional modal logic. A modal algebra is just a pair (B, N) where B is a Boolean algebra and N is an operator on B such that $N1 = 1$ and $N(x \cdot y) = (Nx \cdot Ny)$. Another equivalent way to state this is simply to say that for all finite $A \subseteq |B|$, $N \prod A = \prod NA$ (we get $N1 = 1$ from this since $\prod \varnothing = 1$, so that $N \prod \varnothing = \prod N\varnothing = \prod \varnothing = 1$). The operator N essentially serves as the semantic value for the modal operator in the language of modal logic. So from the algebraic tradition of modal semantics we also get the idea, even if only implicitly, that necessities are primarily defined by their commutative relationship with conjunction. Here we have seen some independent vindication of that thought.

In the next section I will establish some further properties of necessities in complete atomic algebras, and I will also re-introduce the concept of breadth first mentioned in Chapter 1. I will establish the identity of the broadest necessity in $\mathbb{P}$, and argue that its logic must be $\mathsf{S5}$ by making use of an algebraic semantics for modal logic.

4.3 The Broadest Necessity

In Chapter 1 I mentioned a concept called 'breadth'. The basic idea is that some necessities are broader than others, meaning that some necessities are stronger than others in some non-contingent way. I propose to adopt the following sharpening of this idea:

Definition 4.3.1. N is at least as broad as N' iff $\forall x : Nx \leq N'x$.

My reasoning is as follows. In the algebra of propositions, recall that $\leq$ represents absolute necessitation according the argument of §4.1.2. Therefore this definition implies that in the algebra of propositions, an operator N is as broad as N' iff for all x, Nx absolutely necessitates $N'x$. This is essentially the same characterisation of breadth as that given in Bacon [2]. I say that N is broader than N' iff N is at least as broad as N' but not vice versa. Note that if N and N' are at least as broad as one another, then $N = N'$. Therefore an operator N of a certain class of operators is the *broadest* member of that class iff it is at least as broad as every operator in that class. The broadest necessity, if it exists, is the unique necessity at least as broad as all other necessities.

4.3.1 Existence and Identity

Now that we have a definition of breadth, we can establish the existence of a broadest necessity in a given Boolean algebra. First, note that in any Boolean algebra we have a unique modality which maps every atom to the empty set. In this and the following chapter I reserve 'α' to denote this modality in any given algebra:[9]

Definition 4.3.2. $\alpha : a \mapsto \varnothing$.

Associated with this modality is the operator $\Box_\alpha$ defined as per Definition 4.2.13. Now this operator is not guaranteed to exist in an algebra. In general we have:

Proposition 4.3.3. $\Box_\alpha$ exists iff the set of atoms has a supremum.

Proof. By definition. $\Box$

The behaviour of $\Box_\alpha$ is as follows. If $x \neq 1$, then $\Box_\alpha x = \sum\{a : \varnothing \Rrightarrow x\} = \sum \varnothing = 0$. Otherwise, $\Box_\alpha$ maps 1 to the supremum of the set of all atoms. Therefore without this supremum, $\Box_\alpha$ is undefined. However, on the condition that such a supremum exists it is not difficult to show that $\Box_\alpha$ is as broad as any necessity. That is, we have:

Proposition 4.3.4. If it exists, $\Box_\alpha$ is the broadest necessity.

Proof. It suffices to show that $\Box_\alpha$ is at least as broad as any necessity. Suppose $\Box_\alpha$ is defined, and that N is a necessity, being specifically a μ-necessity for some modality μ. We first note that $N1 \approx 1$, since for all a we have $a \leq N1$ iff $\mu(a) \Rrightarrow 1$, the right-hand side of which is always true. But given that $\Box_\alpha 1 = \sum\{a : a \text{ is an atom}\}$, we have $\Box_\alpha 1 \leq N1$. And for all elements x besides 1, we have $\Box_\alpha x = 0 \leq Nx$. Therefore $\Box_\alpha$ is at least as broad as N. $\Box$

[9] In atomless algebras, α is identical with the empty function.

Now of course, given that completeness yeilds a supremum for the set of atoms, we get:

Corollary 4.3.5 (C). $\Box_\alpha$ is the broadest necessity.

This is what we get under completeness, but what of the case where we have both completeness and atomicity, as in intensionalism? Well in this case we can establish the identity of $\Box_\alpha$ and the following operator:

Definition 4.3.6. $Lx := \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise.} \end{cases}$

In any Boolean algebra, the operation L takes the top element to itself, and everything else to the bottom element. Now it is easy to show that L is always a necessity.

Proposition 4.3.7. L is an α-necessity (hence a necessity).

Proof. For any atom a, we have $a \leq Lx$ iff $x = 1$ iff $\alpha(x) = \varnothing \Rrightarrow x$. $\Box$

But this is not the same as saying that L and $\Box_\alpha$ are the very same operator. As it turns out, the strength of this latter claim is equivalent to atomicity, modulo completeness:

Proposition 4.3.8 (C). $L = \Box_\alpha$ iff atomicity holds.

Proof. Given completeness and atomicity, we have $L = \Box_\alpha$ immediately by Propositions 4.3.7, 4.2.15, and 4.2.19. Conversely, if $L = \Box_\alpha$ then since $L1 = 1$ and $\Box_\alpha 1 = \sum\{a : a \text{ is an atom}\}$, we have that $1 = \sum\{a : a \text{ is an atom}\}$. But it is known that the latter property is equivalent with atomicity ([13], 118). $\Box$

Corollary 4.3.9 (CA). $L = \Box_\alpha$

To finish this section I want to take stock, and recap some of the main results we have established so far; then I want to consider a final point of technical interest, which is the question of the existence of a broadest necessity in algebras in which the operator $\Box_\alpha$ does not exist (i.e. is undefined). The latter point will help us see in general the relationship between $\Box_\alpha$ and the concept of a 'broadest necessity' in more generality.

In this chapter so far I have developed a formal theory of necessities in algebraic terms. I have tried to establish some results under the most general conditions possible, but I have also wanted to highlight those results which are solely afforded under the conditions of completeness and atomicity accepted by the intensionalist. In general, there is exactly one modality (up to similarity) for every necessity; given completeness, there is at least one necessity for every modality. But only under the further condition of atomicity is it the case that there is a *unique* necessity for every modality:

Proposition 4.3.10 (C). There is a unique μ-necessity for all μ iff atomicity holds.

Proof. (Right to left.) Immediate by Proposition 4.2.19. (Left to right.) Follows by Propositions 4.2.15, 4.3.7, and 4.3.8. □

Again: in general, necessities N are such that $N \prod A \approx \prod NA$ whenever $\prod A$ and $\prod NA$ exist; given completeness, the latter holds for all A and we additionally have that every $\prod$-commutative operator is a necessity. But only under the further condition of atomicity is it the case that every necessity commutes with $\prod$:

Proposition 4.3.11 (C). All necessities are $\prod$-commutative only if atomicity holds.

Proof. If all necessities are $\prod$-commutative then we have $\Box_\alpha 1 = \Box_\alpha \prod \varnothing = \prod \Box_\alpha \varnothing = \prod \varnothing = 1$. But in this case we have that $1 = \sum\{a : a \text{ is an atom}\}$, which as mentioned previously is equivalent with atomicity ([13], 118). □

For the sake of completeness, I want to round out this section by very briefly dealing with cases in which $\Box_\alpha$ does not exist. What becomes of the broadest necessity in such cases? In fact, it turns out that if $\Box_\alpha$ is undefined, there is no broadest necessity at all. For every necessity N there will be a necessity N' which is strictly broader.

Definition 4.3.12. $x < y$ iff $x \leq y$ and $y \not\leq x$.

Proposition 4.3.13. If $\Box_\alpha$ does not exist, then there is no broadest necessity: for every necessity N there is some necessity N' strictly broader than N.

Proof. If $\Box_\alpha$ does not exist then the set of atoms has no supremum (by Proposition 4.3.3). Suppose N is a necessity, being specifically a μ-necessity for some modality μ. We note that $N1 \approx 1$, since for all a we have $a \leq N1$ iff $\mu(a) \Rrightarrow 1$, the right-hand side of which is always true. So $N1$ is some upper bound for the set of atoms. It cannot be the least upper bound by hypothesis; there must be some other upper bound z of the set of atoms such that $N1 \not\leq z$. So $z' = (N1 \cdot z) < N1$ is also an upper bound for the set of atoms. Let us define the operator N' such that $N'x = Nx$ for $x \neq 1$ and $N'1 = z'$. Clearly, N' is strictly broader than N. To see that N' is also a necessity we need only note that by our definitions we have $a \leq N'x$ iff $\mu(a) \Rrightarrow x$ for all x and a, so that N' is a μ-necessity in particular. □

4.3.2 Logics and Necessities

Our intensionalist can identify the unique absolute necessity with the broadest necessity $\Box_\alpha$. Having done so, she can then start to ask questions about the behaviour of this necessity. What is the logic of $\Box_\alpha$? For our intensionalist, who holds that $\Box_\alpha = L$, the question is equivalent to that of the logic of L. But more generally: what logical principles are shared among all necessities? Is it possible to show, for instance, that every necessity obeys a normal modal logic? In order to answer these questions, we need to provide a semantics for propositional modal logic which makes use of the above algebraic apparatus in interpreting sentences of the modal language. There is a long tradition of employing Boolean algebras to provide semantics for modal logics. The details of our particular semantics are as follows. (In what follows, the formal language under consideration is 'PML' as defined in Appendix A; I refer the reader there for details on notation and terminology concerning modal logics.)

Definition 4.3.14. A frame is a pair $(B, *)$ with B a Boolean algebra and $* : |B| \to |B|$.

We will, of course, be particularly interested in cases of frames $(B, *)$ where $*$ is a necessity in B. Now, we construct models from frames as follows.

Definition 4.3.15. A model structure is a pair (F, a) where F is a frame and a is an atom.

Here a must be an atom from F's underlying algebra. That means that where F is atomless, there are no model structures based on F. So we constrain attention to frames which contain at least one atom. Models result from adjoining interpretations to model structures:

Definition 4.3.16. A model is a tuple $M = (F, a, [\cdot])$ where (F, a) is a model structure and $[\cdot]$ is a function from PML to the underlying algebra where for all ϕ, ψ we have:

$$[\bot] = 0$$
$$[\phi \to \psi] = [\phi] \Rightarrow [\psi]$$
$$[\Box\phi] = *[\phi]$$

Where recall $(x \Rightarrow y) := (-x + y)$. Truth in a model is defined like so:

Definition 4.3.17. Where $M = (F, a, [\cdot])$: $M \vDash \phi$ iff $a \leq [\phi]$.

By the properties of the set $\{x : a \leq x\}$,[10] we can infer immediately that all tautologies of PML will be true in all models. We define validity in model structures and frames as follows:

[10] In particular, it is easy to verify that $a \not\leq [\bot]$ and $a \leq [\phi \to \psi]$ iff $a \not\leq [\phi]$ or $a \leq [\psi]$.

Definition 4.3.18. ϕ is valid in (F, a) iff ϕ is true in all models $(F, a, [\cdot])$ based on (F, a).

Definition 4.3.19. ϕ is F-valid iff ϕ is valid on all model structures (F, a) based on F.

Importantly, the set of formulas valid in a particular frame always constitutes a logic, where in PML a *logic* (Def. A.1.5) is a set of formulas Λ which (i) contains all tautologies, (ii) is closed under modus ponens, and (iii) is closed under uniform substitution of sentence letters for formulas.[11]

Proposition 4.3.20. For any frame F, $\{\phi : \phi \text{ is } F\text{-valid}\}$ is a logic.

Proof. We can show that the set of F-valid formulas meets the requirements of a logic as follows. (i) The fact that $\{\phi : \phi \text{ is } F\text{-valid}\}$ contains all tautologies follows just by the definition of a Boolean algebra: in particular, whenever ϕ is a tautology it can be noted that $[\phi] = 1$ in any model $(F, a, [\cdot])$. (ii) Closure under modus ponens is also trivial, which leaves us to establish (iii) closure under uniform substitutions. To show this, Suppose that a uniform substitution instance ϕ' of ϕ is not F-valid. In particular, where $p_1, ..., p_n$ are the sentence letters which occur at least once in ϕ, let $\theta_1, ..., \theta_n$ be the formulas which uniformly replace $p_1, ..., p_n$ respectively in ϕ to yield ϕ'. Now since ϕ' is not F-valid, there is some model $(F, a, [\cdot])$ such that $a \not\leq [\phi']$. Now consider the model $(F, a, [\cdot]')$ such that $[q] = [q]'$ for every sentence letter q not in ϕ, but where $[p_i]' = [\theta_i]$ for each $i \leq n$. By a routine induction on formula complexity for ϕ's subformulas, one may establish that $[\phi]' = [\phi']$. Hence since $a \not\leq [\phi']$ we have $a \not\leq [\phi]'$, meaning that ϕ is also not F-valid. By contraposition, if ϕ is F-valid then every substitution instance of ϕ is F-valid. $\square$

Therefore we adopt the following notation:

Definition 4.3.21. $\Lambda_F = \{\phi : \phi \text{ is valid on } F\}$.

We can bring this semantics to bear on questions concerning the logic of absolute necessity, as well as concerning logics of necessity in general, as follows. Let a *propositional frame* be any frame whose underlying algebra is $\mathbb{P}$, the algebra of propositions. Then the logic of a propositional operator $N : |\mathbb{P}| \to |\mathbb{P}|$ can be identified with the logic of the propositional frame $(\mathbb{P}, N)$, meaning $\Lambda_{(\mathbb{P},N)}$. On the assumption that $\mathbb{P}$ is complete and atomic, as our intensionalist believes, the operator on $\mathbb{P}$ which can be identified with absolute necessity in the sense of the monist is $\Box_\alpha = L$. Therefore our intensionalist can identify the logic of absolute necessity with $\Lambda_{(\mathbb{P},L)}$. Moreover, she can generally identify the logic of any necessity N on $\mathbb{P}$ with the logic $\Lambda_{(\mathbb{P},N)}$ of the propositional frame $(\mathbb{P}, N)$.

[11] Here the reader may consult Appendix A (§A.1) for definitions of all basic notions relating to propositional modal logic.

Therefore the initial questions of this section can now be recast: in general, what are the logics $\Lambda_{(\mathbb{P},N)}$ where N is a necessity and, specifically, what is the logic $\Lambda_{(\mathbb{P},L)}$?

Now to give an overview of the answers to these questions. Firstly I will show in general that wherever we have an algebra B which is atomic, $\Lambda_{(B,N)}$ is normal for any N a necessity on B. Thus given the atomicity of $\mathbb{P}$, we have the result that every necessity on the algebra of propositions obeys a normal modal logic. Interestingly, in cases where B can be non-atomic, I show that one can find necessities N on B such that $\Lambda_{(B,N)}$ is *not* normal (in particular, failing to admit necessitation). I explain the significance of the latter point later on. Secondly I will show that, for any infinitely large Boolean algebra B, $\Lambda_{(B,L)} = \mathsf{S5}$. This means that, on the back of the argument given in §4.1.3 for the infinitude of $\mathbb{P}$, we have that $\Lambda_{(\mathbb{P},L)} = \mathsf{S5}$, which provides our intensionalist with S5 as the logic of absolute necessity.

To take these points in order, first we have:

Proposition 4.3.22. Suppose B is atomic. Then if N is a necessity on B, $\Lambda_{(B,N)}$ is normal.

Proof. Let $F = (B, N)$. It suffices to show that (i) Λ_F contains $\Box(p \to q) \to (\Box p \to \Box q)$ and that (ii) if $\phi \in \Lambda_F$ then $\Box\phi \in \Lambda_F$. Take any model $M = (F, a, [.])$ based on F. Then:

> (i) Suppose for reductio that $M \not\models \Box(p \to q) \to (\Box p \to \Box q)$. Then $a \not\leq [\Box(p \to q) \to (\Box p \to \Box q)]$, meaning $a \leq -[\Box(p \to q) \to (\Box p \to \Box q)] = N([p] \Rightarrow [q]) \cdot (N[p] \cdot -N[q])$. Now, by associativity and by Proposition 4.2.24 we have that $N([p] \Rightarrow [q]) \cdot (N[p] \cdot -N[q]) \approx N(([p] \Rightarrow [q]) \cdot [p]) \cdot -N[q]$. Therefore $a \leq N(([p] \Rightarrow [q]) \cdot [p]) \cdot -N[q]$, which means that $a \leq N(([p] \Rightarrow [q]) \cdot [p])$ and $a \not\leq N[q]$. Now let μ be some modality on B such that N is a μ-necessity. By the latter observations we have that $\mu(a) \Rrightarrow (([p \Rightarrow q]) \cdot [p])$ but $\mu(a) \not\Rrightarrow [q]$, which is absurd since $[p \Rightarrow q] \cdot [p] \leq [q]$.
>
> (ii) Suppose that $\phi \in \Lambda_F$. For any model $M' = (F, a', [\cdot])$ based on F and with the same interpretation function as M, we have $M' \models \phi$ and hence $a' \leq [\phi]$ for all atoms a' in F. Therefore $[\phi] \approx 1$. Since B is atomic this means that $[\phi] = 1$, and so we have $a \leq N[\phi] = N1$ iff $\mu(a) \Rrightarrow 1$, where μ is some modality corresponding to N. Since $\mu(a) \Rrightarrow 1$ is guaranteed to be true we have $a \leq N[\phi] = [\Box\phi]$ so that $M \models \Box\phi$. Since M was picked arbitrarily we have that this holds for all models based on F. Hence $\Box\phi \in \Lambda_F$.

Which completes the proof. □

Next, going by the order of our outline, we will show that in cases of non-atomic algebras, one can find necessities which obey a non-normal modal logic. In fact, an important case of this phenomenon is the following.

Proposition 4.3.23 (C)**.** Atomicity holds iff the logic of the broadest necessity is normal.

Proof. (Left to right.) Immediate by Corollary 4.3.5 and Proposition 4.3.22. (Right to left.) Suppose that B is non-atomic (but still complete). Here the operator $\Box_\alpha$ will be defined and will be distinct from L, and will be the broadest necessity on B. We can show in this case that the logic $\Lambda_{(B,\Box_\alpha)}$ is non-normal as follows. Since $\Lambda_{(B,\Box_\alpha)}$ is a logic, it contains the formula $(p \vee \neg p)$; if it were normal, then by necessitation it should also contain $\Box\Box(p \vee \neg p)$. However, let $M = (B, \Box_\alpha, a, [\cdot])$ be some model based on $(B, \Box_\alpha)$. We have $M \vDash \Box\Box(p \vee \neg p)$ iff $a \leq \Box_\alpha\Box_\alpha([p] + -[p]) = \Box_\alpha\Box_\alpha 1$. And $a \leq \Box_\alpha\Box_\alpha 1$ iff $\alpha(a) \Rrightarrow \Box_\alpha 1$. But $\alpha(a) \Rrightarrow \Box_\alpha 1$ just means $1 \leq \Box_\alpha 1$. In absence of atomicity, $\Box_\alpha 1$, which is the supremum of B's atoms, is distinct from 1 so that $1 \not\leq \Box_\alpha 1$. So $M \nvDash \Box_\alpha\Box_\alpha(p \vee \neg p)$. So $\Lambda_{(B,\Box_\alpha)}$ cannot be normal. $\Box$

Corollary 4.3.24 (C)**.** Atomicity holds iff all necessities have normal logics.

In other words, atomicity is required to ensure that all necessities obey normal logics (at least given completeness, going by this previous result), and if atomicity fails then some necessities will have non-normal logics. We will return to the topic of logics of necessities in non-atomic algebras in the next chapter. Narrowing in now to consider logics of the form $\Lambda_{(B,L)}$, note that in fact it is not necessary to rely on atomicity to show that $\Lambda_{(B,L)}$ must be normal, or that $\Lambda_{(B,L)}$ must be an extension of the strong logic S5. We can argue in a more general way as follows.

Proposition 4.3.25. For any algebra B and any model $(B, L, a, [\cdot])$ based on the frame (B, L), and any formula ϕ: if $\vdash_{\mathsf{S5}} \phi$ then $[\phi] = 1$.

Proof. Suppose we have some model $(B, L, a, [\cdot])$. We proceed by induction on length of proof with respect to S5. It is known that S5 can be characterised as the smallest set of formulas which contains all tautologies, all instances of the schemas (i) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ (ii) $\Box\phi \to \phi$, (iii) $\Diamond\phi \to \Box\Diamond\phi$, and which is closed under modus ponens and necessitation. So it suffices to show that the set $\{\psi : [\psi] = 1\}$ contains all instance of (i)-(iii), and is closed under MP and Nec. For the sake of brevity I will just show the cases of (ii), (iii) and (MP), as the other steps present no special difficulty.

(ii) $[\Box\phi \to \phi] = L[\phi] \Rightarrow [\phi]$. If $[\phi] = 1$ then $(L[\phi] \Rightarrow [\phi]) = (1 \Rightarrow 1) = 1$. Otherwise, $(L[\phi] \Rightarrow [\phi]) = (0 \Rightarrow [\phi]) = 1$.

(iii) $[\Diamond\phi \to \Box\Diamond\phi] = -L - [\phi] \Rightarrow L - L - [\phi]$. If $[\phi] = 0$ then $-L - [\phi] \Rightarrow L - L - [\phi] = (0 \Rightarrow 0) = 1$. Else, $-L - [\phi] \Rightarrow L - L - [\phi] = (1 \Rightarrow 1) = 1$.

(MP) Suppose $[\phi \to \psi] = ([\phi] \Rightarrow [\psi]) = 1$. Then $[\phi] \leq [\psi]$. So if additionally $[\phi] = 1$ we have $[\psi] = 1$.

Which completes the proof. □

Corollary 4.3.26. For any B, $\Lambda_{(B,L)}$ is an extension of $\mathsf{S5}$.

A well-known theorem owed to Scroggs [59] classifies the extensions of $\mathsf{S5}$ in a nice way. Namely, the extensions of $\mathsf{S5}$ are exactly the logics: $\mathsf{S5}$, Triv, the inconsistent logic In, and the sequence of logics $\mathsf{S5Alt}_n = \mathsf{S5} + \mathrm{Alt}_n$ where for any positive integer n, Alt_n is:

$$\mathrm{Alt}_n \quad \Box(p_1 \to p_n) \vee \Box((p_1 \wedge p_2) \to p_n) \vee ... \vee \Box((p_1 \wedge ... \wedge p_{n-1}) \to p_n)$$

Or, using standard abbreviations:

$$\mathrm{Alt}_n \quad \bigvee_{k=1}^{n} \Box((\bigwedge_{i=1}^{k-1} p_i) \to p_k)$$

Therefore it suffices to proceed by process of elimination. We can show that a certain logic $\Lambda_{(B,L)}$ is exactly $\mathsf{S5}$ if we show that it is not any of $\mathsf{S5}$'s proper extensions. We do this now for a class of Boolean algebras broad enough for our purposes.

Proposition 4.3.27. Suppose B is infinite and contains an atom as well as an infinitely descending sequence $x_1 > x_2 > x_3 > ....$ Then $\Lambda_{(B,L)} = \mathsf{S5}$.[12]

Proof. By elimination, we show that if B is infinite and not atomless then $\Lambda_{(B,L)}$ cannot be Triv, In, or any of the logics $\mathsf{S5Alt}_n$. From this it will follow by Scroggs' theorem and Corollary 4.3.26 that $\Lambda_{(B,L)} = \mathsf{S5}$. First of all, since B contains at least one atom, there is some model $M = (B, L, a, [\cdot])$ based on (B, L), and in which we have $M \nvDash \bot$. Therefore $\Lambda_{(B,L)} \neq \mathsf{In}$. Secondly, since B is infinite we must have it that every atom in B is distinct from 1. Now consider the a model $M = (B, L, a, [\cdot])$ in which $[p] = a$. Then we have $M \nvDash p \to \Box p$ since $[p \to \Box p] = (a \Rightarrow 0) = -a$, and $a \nleq -a$. Therefore $\Lambda_{(B,L)} \neq \mathsf{Triv}$. Finally, for any positive integer n, consider a model $M = (B, L, a, [\cdot])$ in which for the sentence letters $p_1, ..., p_n$ we have the system of inequalities $0 < [p_n] < ... < [p_1] < 1$.[13] In this case we have that for all $k \leq n$, $\prod_{j=1}^{k-1}[p_j] = [p_{k-1}] \nleq [p_k]$. Now it is clear that

$$\left[\bigvee_{k=1}^{n} \Box\left(\left(\bigwedge_{i=1}^{k-1} p_i\right) \to p_k\right)\right] = \sum_{k=1}^{n} L\left(\left(\prod_{j=1}^{k-1}[p_j]\right) \Rightarrow [p_k]\right)$$

[12] The existence of an atom is only required to ensure the existence of a model structure (B, L, a) based on (B, L); in cases where this fails, $\Lambda_{(B,L)}$ is the inconsistent logic. The assumption of an infinite chain is not strictly necessary, but allows for a quicker argument.

[13] This is allowed by the existence of an infinite chain in B.

But as we have $\prod_{j=1}^{k-1}[p_j] \not\leq [p_k]$, we have $((\prod_{j=1}^{k-1}[p_j]) \Rightarrow [p_k]) \neq 1$, so that $L((\prod_{j=1}^{k-1}[p_j]) \Rightarrow [p_k]) = 0$ for all $k \leq n$. So the whole term on the right-hand side just evaluates to 0, so that:

$$\left[\bigvee_{k=1}^{n} \Box\left(\left(\bigwedge_{i=1}^{k-1} p_i\right) \to p_k\right)\right] = 0$$

Meaning that $M \not\models \bigvee_{k=1}^{n} \Box((\bigwedge_{i=1}^{k-1} p_i) \to p_k)$, so that we have $\Lambda_{(B,L)} \neq \mathsf{S5Alt}_n$. $\Box$

Now by our arguments that the algebra of propositions is atomic and contains an infinite descending chain (§4.1.3), we can infer that this holds in particular for $\Lambda_{(\mathbb{P},L)}$. In addition it is not difficult to show the following:

Proposition 4.3.28. If B is complete, atomic, and infinite, it contains an infinitely descending sequence $x_1 > x_2 > x_3 > ....$

Proof. Observe that if B is complete and atomic, B is infinite only if B contains infinitely many atoms. (Since every element in B can be uniquely decomposed into atoms by Proposition 4.1.9, finitely many atoms would result in only finitely many elements in B.) So we have a set of pairwise disjoint atoms $\{a_n\}$. Now if we define the sequence $x_n = \sum_{k \geq n} a_k$, we can see that $x_{n+1} > x_n$ for each n, so $\{x_n\}$ is an infinitely descending chain in B. $\Box$

Corollary 4.3.29. If B is infinite, complete, and atomic, then $\Lambda_{B,L} = \mathsf{S5}$.

Of course we can then combine this with Proposition 4.3.8. to obtain the same for $\Box_\alpha$:

Corollary 4.3.30. If B is infinite, complete, and atomic, $\Lambda_{(B,\Box_\alpha)} = \Lambda_{(B,L)} = \mathsf{S5}$.

To sum up: for our intensionalist, every necessity on the algebra of propositions obeys a normal logic, and absolute necessity, which is also the broadest necessity, obeys the logic S5. This is because our intensionalist takes the algebra of propositions $\mathbb{P}$ to satisfy the relevant properties utilised in these results: infinitude, completeness and, importantly, atomicity. And to sum up the implications for any modalist who wants to adopt intensionalism. If modalism and intensionalism are both true then (i) logical consequence is Boolean entailment and (ii) logical truth obeys the system S5.

4.4 Conclusion

In this chapter I have tried to accomplish three main goals on behalf of our intensionalist. First, to motivate a definition of necessities in an algebraic context. Second, to develop some key aspects of the resulting theory of necessities in general terms, introducing the topics of correspondences between modalities and necessities, the algebraic properties of necessities in general (such as their relationship with $\prod$-commutativity), and ways of discerning the logic of a given necessity via algebraic semantics. Third and finally, I have wanted to highlight the advantage of adopting atomicity in the way that our intensionalist does distinctively; not only does it tidy up the correspondances between modalities and necessities, and provide for a neat characterisation of the necessities in terms of commutativity with $\prod$, but atomicity also allows us the result that all necessities obey a normal modal logic, and that the broadest necessity satisfies the normal modal logic $\mathsf{S5}$. Additionally I have wanted to hint at some of the stranger results which come by denying atomicity. In the next chapter, we will look to these in more detail as part of our study of some varieties of hyperintensionalism.

5

Algebra and Necessity II: Hyperintensionalism

The previous chapter developed an algebraic treatment of necessities from the perspective of intensionalism. In the present chapter, by contrast, I discuss how certain forms of hyperintensionalism might be able to develop a more or less similar treatment. To this end, as discussed at the end of Chapter 3, I will be confining attention to just two versions of hyperintensionalism. Both of these theories accept Booleanism, and they also accept the metaphysical version of what I termed Leibniz's thesis (which says that metaphysical necessity is truth-in-all-possible-worlds, see §3.4 above). They also accept the identification of possible worlds with atomic propositions. But they differ with respect to the following two theses.

> LEIBNIZ'S THESIS (ABSOLUTE)
>
> Absolute necessity is truth-in-all-possible-worlds.
>
> GENERAL MODAL INDIVIDUATION
>
> Propositions that are absolutely necessarily equivalent are identical.

An intensionalist can (and is likely to) accept both of these theses; a hyperintensionalist cannot, since these in conjunction with the metaphysical version of Leibniz's thesis yield intensionalism. So our two versions of hyperintensionalism are arrived at by rejecting either one or the other of these. In §5.1 I consider the view which accepts general modal individuation while rejecting the absolute version of Leibniz's thesis. In §5.2 I consider the view which accepts the absolute version of Leibniz's thesis while rejecting general modal individuation.

The view considered in §5.1 turns out to have some very strange consequences. It implies that absolute necessity is not the broadest necessity, that many necessities obey non-normal modal logics, and that necessities do not compose. Meanwhile, the view considered in §5.2 turns out to require a revision to the definition of a necessity offered in the previous chapter, though by doing so it is able to avoid many of the problematic consequence of the first view, and much better able to mimic the results of the intensionalist established in the previous chapter. In §5.3 I take a step back to compare and contrast the three theories so-far constructed.

5.1 General Modal Individuation

Consider the following view. Metaphysical necessity is truth-in-all-possible worlds, but absolute necessity is not. Possible worlds are atomic propositions. Metaphysically necessarily equivalent propositions can be distinct, but any propositions that are *absolutely* necessarily equivalent are identical. Moreover, the assumptions of Chapter 1 are retained, so we have monism and absolutism, closure for absolute necessity, and we have the absolute necessity of classical propositional logic. On these assumptions, note that we can run essentially identical arguments as in the previous chapter to show that the algebra of propositions $\mathbb{P}$ is a Boolean algebra, that it is complete, and that it is infinite. However, given the identification of metaphysical necessity with truth at all possible worlds, on the present view we must infer that $\mathbb{P}$ is *non-atomic*. For if intensionalism is false, there must be some propositions x and y which are distinct while being atomically equivalent (true at all the same possible worlds); by Proposition 4.1.9., this is equivalent to denying atomicity.

There is one further, central aspect of the intensionalist treatment that we can carry over. This is the identification of absolute necessitation with Boolean entailment. We noted in §4.1.2 that the argument for identifying absolute necessitation with $\leq$ (or, strictly speaking, with $\Rrightarrow$) goes through on general modal individuation even if one denies intensionalism. Hence this version of hyperintensionalism motivates the very same characterisation of a necessity as that given by the intensionalist (Defs. 4.2.4 and 4.2.5). But precisely because of this, she must bear the brunt of the consequences of denying atomicity also noted in the previous chapter. For instance since $\mathbb{P}$ is non-atomic, it is not the case that every modality has a unique necessity in $\mathbb{P}$ (Prop. 4.3.10); nor will every necessity commute with $\prod$ (Prop. 4.3.11). Additionally, some necessities in $\mathbb{P}$ will obey non-normal logics (Corollary 4.3.24); in particular, $\Box_\alpha$, which remains the broadest necessity (Prop. 4.3.4), does not obey a normal logic (Prop. 4.3.23). We will see this now.

5.1.1 Necessities with Non-Normal Logics

As was established in the previous chapter (Prop. 4.3.23), if atomicity fails in a complete algebra B then the logic $\Lambda_{(B,\Box_\alpha)}$ of its broadest necessity is non-normal. In particular, the rule of necessitation fails. The argument for that result hinted at the following stronger result.

Proposition 5.1.1. If B is complete and non-atomic, $\vdash_{\Lambda_{(B,\Box_\alpha)}} \neg\Box\Box p$.

Proof. Take B complete and non-atomic and recall that for all x, $\Box_\alpha x = \sum Atom(B)$ if $x = 1$ and $\Box_\alpha x = 0$ otherwise. Since B is non-atomic we have $\sum Atom(B) \neq 1$, hence for all x we have $\Box_\alpha\Box_\alpha x = 0$. Therefore if we take any model $M = (B, \Box_\alpha, a, [\cdot])$ based on $(B, \Box_\alpha)$, we will have $a \leq -\Box_\alpha\Box_\alpha[p] = [\neg\Box\Box p]$, meaning $M \vDash \neg\Box\Box p$. $\Box$

In complete non-atomic algebras, that is, the logic of the broadest necessity satisfies the principle that: everything is possibly possible. This is a radical proposal; it implies for instance that even the necessity of tautologies is not itself necessary, so that we have $\neg\Box\Box\top$. Therefore this is one more point that many will find unacceptable about the present view. Note that by an obvious modification of the above argument one can show that where B is complete and non-atomic, we have $\vdash_{\Lambda_{(B,\Box_\alpha)}} \Box\top \wedge \neg\Box\Box\top$.[1] This means that on such algebras, the logic of the broadest necessity does not obey the 4 axiom. We will next see one further consequence of this.

5.1.2 Composition Failure

On the present view, the broadest necessity does not satisfy the characteristic principles of S4—especially the 4 axiom, $\Box p \rightarrow \Box\Box p$. This is of interest, for there is a well-known and simple argument for the thesis that the broadest necessity must satisfy 4 (cf. Williamson [69]). Consider: if $\Box$ expresses the broadest necessity, then $\Box p$ should imply that every necessity holds of p. But, again plausibly, the self-iteration of the broadest necessity $\Box\Box$ is also a necessity, so in particular $\Box p$ should imply $\Box\Box p$. From this argument we can guess that, if the broadest necessity does not satisfy 4, this must mean that the self-iteration of the broadest necessity $\Box_\alpha\Box_\alpha$ is not in fact a necessity. More generally, this must mean that necessities in general do not compose. So it must not be the case that, generally, if N and N' are necessities then NN' is also a necessity. This latter property does hold in the context of complete atomic algebras. Thus:

Proposition 5.1.2 (CA)**.** If N is a necessity and N' is a necessity then NN' is a necessity.

[1] We have $\vdash_{\Lambda_{(B,\Box_\alpha)}} \Box\top$ via a trivial argument since $[\top] = 1$ for any interpretation function $[\cdot]$ on B.

Proof. N and N' are necessities iff they are $\prod$-commutative. We can show that NN' is also $\prod$-commutative by noting that for any A:

$$NN'\prod A = N\prod N'A = \prod NN'A$$

Therefore by Proposition 4.2.23, NN' is a necessity. $\square$

But indeed, without atomicity, we can show that necessities need not compose. Indeed, it turns out that if we assume completeness and the existence of an atom[2] then the general composition of necessities is equivalent to atomicity:

Proposition 5.1.3 (C). If there is an atom, then necessities compose iff atomicity holds.

Proof. Suppose there is an atom. (Right to left.) By Proposition 5.1.2. (Left to right.) We will suppose that necessities compose and we will show that $\Box_\alpha 1 = \sum\{a : a \text{ is an atom}\} = 1$, which is known to be equivalent to atomicity given completeness ([13], 118). Now, if necessities compose then $\Box_\alpha\Box_\alpha$ is a necessity. Since $\Box_\alpha$ is the broadest necessity we have $\Box_\alpha 1 \leq \Box_\alpha\Box_\alpha 1$, where $\Box_\alpha 1 \neq 0$ since there is an atom. But if $\Box_\alpha 1 \neq 1$ then $\Box_\alpha\Box_\alpha 1 = \sum\{a : \varnothing \Rrightarrow \Box_\alpha 1\} = \sum\varnothing = 0$, which is false. So $\Box_\alpha 1 = 1$, and hence atomicity holds. $\square$

Failure of composition is another unorthodox consequence of the present view. It is one further point that many will find unacceptable.

5.2 Leibniz's Thesis

To avoid the consequences of the version of hyperintensionalism just considered, we need to re-examine its basic assumptions. The problems of the previous view arose by the fact that, as a variety of hyperintensionalism, it needed to deny the atomicity of the algebra of propositions. But by accepting general modal individuation, it was led to a characterisation of absolute necessitation as Boolean entailment, and so to the same general characterisation of necessities as the intensionalist. This forced them to accept some unwelcome consequences including composition failure, non-normal logics for necessities, and so on. To avoid all of these problems, we need a view which rejects the intensionalist's characterisation of the necessities given in Definitions 4.2.4 and 4.2.5, and which more generally rejects the identification of absolute necessitation with Boolean entailment, and general modal individuation along with it.

[2]Recall that as in Example 4.2.8, in atomless algebras all operators are necessities and so necessities trivially compose. This is the only case in which the necessities of a non-atomic algebra compose.

5.2.1 Leibniz Necessities

In rejecting the identification of absolute necessitation and Boolean entailment, this second kind of hyperintensionalist owes us an alternative account of what absolute necessity is in algebraic terms. The most obvious substantive account in the vicinity would be to identify absolute necessity with metaphysical necessity (truth-in-all-possible-worlds), and absolute necessitation with metaphysical necessitation. So consider the following view. Absolute necessity is metaphysical necessity, which is truth-in-all-possible worlds. Possible worlds are atomic propositions. Metaphysically (and hence absolutely) necessarily equivalent propositions can be distinct. Moreover, the assumptions of Chapter 1 are retained, so we have monism and absolutism, closure for absolute necessity, and we have the absolute necessity of classical propositional logic. On these assumptions, unlike with intensionalism and the view considered in §5.1, we cannot offer an argument for Booleanism—that $\mathbb{P}$ is a Boolean algebra—so we will also add this as an assumption. Given this, we can run essentially identical arguments as in the previous chapter to show that the algebra of propositions $\mathbb{P}$ is complete, and that it is infinite. And as in §5.1, given the identification of metaphysical necessity with truth at all possible worlds, on the present view we must infer that $\mathbb{P}$ is *non-atomic*. For if intensionalism is false, there must be some propositions x and y which are distinct while being atomically equivalent (true at all the same possible worlds); by Proposition 4.1.9 this is equivalent to denying atomicity.

So this view accepts the absolute version of Leibniz's thesis and denies general modal invididuation along with intensionalism. Importantly, it also leads one to reject the characterisation of necessities we have used until now. To see this, we need to get clearer about how to think about metaphysical necessitation. Consider the following relation.

Definition 5.2.1. $A \rightsquigarrow x$ iff for all atoms a : if a is a lower bound for A then $a \leq y$.

In the algebra of propositions $A \rightsquigarrow x$ iff any world at which every proposition in A is true is also a world in which x is true. Therefore, assuming the metaphysical version of Leibniz's thesis, $\rightsquigarrow$ in the algebra of propositions just is metaphysical necessitation. The present view therefore identifies $\rightsquigarrow$ with absolute necessitation. And it is this point which leads us to an alternative conception of necessities. Substituting $\rightsquigarrow$ in for $\Rrightarrow$ in Definitions 4.2.4 and 4.2.5 yield the following notions:

Definition 5.2.2. N is a Leibniz μ-necessity iff for all a, x: $a \leq Nx$ iff $\mu(a) \rightsquigarrow x$.

Definition 5.2.3. N is a Leibniz necessity iff N is a Leibniz μ-necessity for some μ.

How do Leibniz necessities relate to the necessities discussed up to now? And how does $\rightsquigarrow$ relate to $\Rrightarrow$. In general, the answer is that these pairs of notions converge where atomicity holds, and diverge when atomicity fails. We can see first that:

Proposition 5.2.4. If $A \Rrightarrow x$ then $A \rightsquigarrow x$.

Proof. Immediate. □

Proposition 5.2.5. Atomicity holds iff for all A and x, if $A \rightsquigarrow x$ then $A \Rrightarrow x$.

Proof. (Left to right.) Suppose atomicity holds. If $A \not\Rrightarrow x$ then there is some y a lower bound for A such that $y \not\leq x$. Hence $y \cdot -x \neq 0$, so by atomicity there is some a with $a \leq y \cdot -x$. Since $a \leq y$ we have $a \leq z$ for all $z \in A$ while $a \not\leq x$. So $A \not\rightsquigarrow x$. (Right to left.) Suppose atomicity fails. Then by Proposition 4.1.9 there are $x \neq y$ such that $x \approx y$. Since $x \neq y$, either $x \not\leq y$ or $y \not\leq x$. So suppose without loss of generality that $x \not\leq y$. Then $\{x\} \not\Rrightarrow y$ but $\{x\} \rightsquigarrow y$. □

Hence in atomic algebras we have:

Corollary 5.2.6. (A) All Leibniz necessities are necessities and vice versa.

But in non-atomic algebras the two classes diverge. For example:

Proposition 5.2.7. (C) L is a Leibniz necessity only if there are no atoms or atomicity holds.

Proof. Suppose L is a Leibniz necessity. Then there is some modality μ such that for all x, a: $a \leq Lx$ iff $\mu(a) \rightsquigarrow x$. Now consider the sum $\sum a$ of all atoms. For all atoms a we have $a \leq L\big(\sum a\big)$ iff $\mu(a) \rightsquigarrow \sum a$, which is always vacuously true by the definition of $\rightsquigarrow$. So $L \sum a \approx 1$, requiring that either $\sum a = 1$ or $0 \approx 1$. But $0 \approx 1$ only if there are no atoms, and as noted previously, $\sum a = 1$ only if atomicity holds ([13], 118). □

Now we will look at how the present variety of hyperintensionalism can recover some of the main results of the previous chapter. But before moving on we should show note another basic property of Leibniz necessities:

Definition 5.2.8. $x \lesssim y$ iff for all atoms a, if $a \leq x$ then $a \leq y$.

Proposition 5.2.9. If N is a Leibniz necessity and $x \lesssim y$, then $Nx \lesssim Ny$.

Proof. Suppose N is Leibniz necessity and that $x \lesssim y$. Then there is some modality μ such that for all a and z, $a \leq Nz$ iff $\mu(a) \rightsquigarrow z$. Now for any atom a', if $a' \leq Nx$ then $\mu(a') \rightsquigarrow x$. Since $x \lesssim y$, this implies that $\mu(a') \rightsquigarrow y$, so that $a' \leq Ny$ too. □

Corollary 5.2.10. If N is a Leibniz necessity and $x \approx y$, then $Nx \approx Ny$.

5.2.2 Emulating Intensionalism

There are four important results about necessities in complete atomic algebras I want to recall here, for which we will provide analogues for Leibniz necessities in complete algebras. These are (i) the existence of a necessity correspondent to every modality, (ii) the $\prod$-commutativity of necessities, (iii) the composition of necessities and (iv) that every necessity obeys a normal logic. The first and most basic is the result that in complete algebras, there is a necessity correspondent to every modality. We can establish the same for Leibniz necessities in complete algebras. To do this, I first introduce the following notation.

Definition 5.2.11. For any modality μ, let $\widetilde{\mu} : a \mapsto \sum\{a' : a' \text{ is a lower bound of } \mu(a)\}$.

The modality $\widetilde{\mu}$ will always be defined in complete algebras. With this notation in hand we can consider the operator $\Box_{\widetilde{\mu}} : x \mapsto \sum\{a : \widetilde{\mu}(a) \Rrightarrow a\}$. We can then show that in complete algebras, for each modality μ, $\Box_{\widetilde{\mu}}$ is a Leibniz μ-necessity. First an observation.

Proposition 5.2.12 (C)**.** $A \rightsquigarrow x$ iff $\sum\{a : a \text{ a lower bound of } A\} \Rrightarrow x$.

Proof. Immediate. □

Corollary 5.2.13 (C)**.** $\mu(a) \rightsquigarrow x$ iff $\widetilde{\mu}(a) \Rrightarrow x$.

Proposition 5.2.14 (C)**.** $\Box_{\widetilde{\mu}}$ is a Leibniz μ-necessity.

Proof. For any a and x: $a \leq \Box_{\widetilde{\mu}} x$ iff $\widetilde{\mu}(a) \Rrightarrow x$ since $\Box_{\widetilde{\mu}}$ is a $\widetilde{\mu}$-necessity (Prop 4.2.15). And $\widetilde{\mu}(a) \Rrightarrow x$ iff $\mu(a) \rightsquigarrow x$ by Corollary 5.2.10. □

Now in the previous chapter, we were also able to show that in complete algebras, every modality corresponds to a unique necessity up to atomic equivalence (meaning that if N and N' are both μ-necessities then $Nx \approx N'x$ for all x). The same can be noted here:

Proposition 5.2.15. If N and N' are Leibniz μ-necessities, then $Nx \approx N'x$ for all x.

Proof. In this case for all a and x we have $a \leq Nx$ iff $\mu(a) \rightsquigarrow x$ iff $a \leq N'x$. □

Moving on, our next result is concerns $\prod$-commutativity. We showed in the previous chapter that all necessities possess a form of commutativity property with $\prod$. We can show the same here for Leibniz necessities:

Proposition 5.2.16. If N is a Leibniz necessity and A is a set of elements such that $\prod NA$ and $N \prod A$ both exist, then $\prod NA \approx N \prod A$.

Proof. Suppose that N is a Leibniz necessity. So for some μ it holds that for all x and a: $a \leq Nx$ iff $\mu(a) \rightsquigarrow x$. For any set A of elements of the algebra where $\prod A$ and $\prod NA$ exist, for any a we therefore have $a \leq N \prod A$ iff $\mu(a) \rightsquigarrow \prod A$. But it follows from our definitions that $\mu(a) \rightsquigarrow \prod A$ iff $\mu(a) \rightsquigarrow y$ for all $y \in A$. Therefore we have $a \leq N \prod A$ iff $a \leq Ny$ for all $y \in A$, which is the case iff $a \leq \prod NA$. Therefore $N \prod A \approx \prod NA$. □

Corollary 5.2.17 (C). If N is a Leibniz necessity then $\prod NA \approx N \prod A$ for all A.

Corollary 5.2.18. If N is a Leibniz necessity then $N1 \approx 1$.

Proof. $N1 = N \prod \varnothing \approx \prod\{Nx : x \in \varnothing\} = 1$. □

Then using this result we can also provide an analogue for Propositions 4.2.23 and 4.2.27:

Proposition 5.2.19. (C) If $\prod NA \approx N \prod A$ for all A then N is a Leibniz necessity.

Proof. Suppose $\prod NA \approx N \prod A$ for all A. Then consider the modality $\mu_N : a \mapsto \{x : a \leq Nx\}$. We show that for all atoms a: $a \leq Na$ iff $\mu_N(a) \rightsquigarrow x$, so that N is a Leibniz necessity. First: if $a \leq Nx$ then $x \in \mu_N(a)$, so that immediately $\mu_N(a) \rightsquigarrow x$. Second: if $\mu_N(a) \rightsquigarrow x$ then by completeness we have $\prod \mu_N(a) \lesssim x$. By Proposition 5.2.9 we then have $N \prod \mu_N(a) \lesssim Nx$, and by Proposition 5.2.16 we have

$$\prod\{Ny : a \leq Ny\} = \prod N\mu_N(a) \approx N \prod \mu_N(a) \lesssim Nx$$

Now since $x \approx y$ implies $x \lesssim y$ and $\lesssim$ is transitive, we get $\prod\{Ny : a \leq Ny\} \lesssim Nx$. But since $a \leq \prod\{Ny : a \leq Ny\}$ we infer that $a \leq Nx$ as required. □

Corollary 5.2.20 (C). N is a Leibniz necessity iff $\prod NA \approx N \prod A$ for all A.

We may use commutativity to establish the composition of Leibniz necessities like so:

Proposition 5.2.21 (C). If N and N' are Leibniz necessities, NN' is a Leibniz necessity.

Proof. For any subset of the domain A we have $N' \prod A \approx \prod N'A$ since N' is a Leibniz necessity by Prop 5.2.16. Hence by Corollary 5.2.10 we have $NN' \prod A \approx N \prod N'A$. But since N is also a Leibniz necessity we have $N \prod N'A \approx \prod NN'A$, meaning altogether that $NN' \prod A \approx \prod NN'A$, so that by Prop 5.2.19, NN' is also a Leibniz necessity. □

Finally we consider normality for the modal logics of Leibniz necessities. In the last chapter we showed that in complete atomic algebras, all necessities obey normal modal logics. We also showed that in complete non-atomic algebras, some necessities do not obey normal logics. Here we will show that in any algebra, even where they are non-complete, all *Leibniz* necessities obey normal logics.

Proposition 5.2.22. If N is a Leibniz necessity on B, $\Lambda_{(B,N)}$ is normal.

Proof. Suppose that N is a Leibniz necessity on B, and that B is complete. From this we show that $\Lambda_{(B,N)}$ is closed under necessitation and contains K.

> (Necessitation) Suppose for reductio that for some ϕ we have $\vdash_{\Lambda_{(B,N)}} \phi$ but $\nvdash_{\Lambda_{(B,N)}} \Box\phi$. Then there is some model $M = (B, N, a, [\cdot])$ based on (B, N) such that $M \nvDash \Box\phi$. Hence $a \nleq [\Box\phi] = N[\phi]$. So $N[\phi] \not\approx 1$. By Corollary 5.2.18 this means that $N[\phi] \not\approx N1$, which by Corollary 5.2.10 implies that $[\phi] \not\approx 1$. But that means that there is some atom a' in B such that $a' \nleq [\phi]$, meaning that where $M' = (B, N, a', [\cdot])$ is the model based on (B, N) which we get just by switching a' for a, $M' \nvDash \phi$. Hence in fact $\vdash_{\Lambda_{(B,N)}} \phi$, which is absurd.
>
> (K) Suppose for reductio that $\nvdash_{(B,N)} \Box(p \to q) \to (\Box p \to \Box q)$. Then there is some model $M = (B, N, a, [\cdot])$ based on (B, N) such that $M \nvDash \Box(p \to q) \to (\Box p \to \Box q)$. Hence by calculation we have $a \leq N([p] \Rightarrow [q]) \cdot N[p] \cdot -N[q]$. But via Proposition 5.2.16 this means that $a \leq N(([p] \Rightarrow [q]) \cdot [p]) \cdot -N[q]$, and since generally $(x \Rightarrow y) \cdot x = y$, this means that $a \leq N[q] \cdot -N[q]$. So $a \leq 0$, which is absurd.

Which suffices to show that $\Lambda_{(B,N)}$ is normal. $\Box$

Here then we have shown how to recover some of the main results of the previous chapter in non-atomic algebras by working with what we have termed Leibniz necessities. Before moving on, we should also consider what the logic of the broadest Leibniz necessity is, in analogy to the discussion of the logic of the broadest necessity in the previous chapter. For this purpose one might think it necessary to introduce a new notion of breadth (perhaps that N is as 'Leibniz' broad as N' iff $Nx \lesssim N'x$ for all x) but in fact this is not necessary. Just using the previous definition of breadth (Def. 4.3.1), in complete algebras we have:

Proposition 5.2.23 (C). $\Box_{\widetilde{\alpha}}$ is the broadest Leibniz necessity.

Proof. By Def. 5.2.11, we infer that for all a, $\widetilde{\alpha}(a)$ is the supremum of all atoms. Hence $\Box_{\widetilde{\alpha}}x = \sum\{a : a \text{ is an atom}\}$ if $x \approx 1$ and $\Box_{\widetilde{\alpha}}x = 0$ otherwise. Let N be a Leibniz necessity. If $x \not\approx 1$ then $\Box_{\widetilde{\alpha}}x = 0 \leq Nx$, and if $x \approx 1$ then $Nx \approx 1$, meaning $\Box_{\widetilde{\alpha}}x \leq Nx$. Hence $\Box_{\widetilde{\alpha}}x \leq Nx$ for all x. $\Box$

Therefore our question becomes: what is the identity of the logic $\Lambda_{(B,\Box_{\widetilde{\alpha}})}$ (for complete B where $\Box_{\widetilde{\alpha}}$ is guaranteed to exist)? We may answer this question as follows.

Proposition 5.2.24. If B is complete, then $\Lambda_{(B,\Box_{\widetilde{\alpha}})}$ is a normal extension of $\mathsf{S5}$.

Proof. We know by Prop. 5.2.22 that $\Lambda_{(B,\Box_{\underset{\alpha}{\sim}})}$ is normal. So it suffices to show that this logic also contains the T axiom $\Box p \to p$ and the 5 axiom $\Diamond p \to \Box\Diamond p$:

(T) Take any model $M = (B, \Box_{\underset{\alpha}{\sim}}, a, [\cdot])$ based on the frame $(B, \Box_{\underset{\alpha}{\sim}})$. For all x in B we clearly have $\Box_{\underset{\alpha}{\sim}} x \lesssim x$. Hence in particular: if $a \leq \Box_{\underset{\alpha}{\sim}}[p]$ then $a \leq [p]$, so that $a \leq \Box_{\underset{\alpha}{\sim}}[p] \Rightarrow [p] = [\Box p \to p]$. So $M \vDash \Box p \to p$.

(5) Take any model $M = (B, \Box_{\underset{\alpha}{\sim}}, a, [\cdot])$ based on the frame $(B, \Box_{\underset{\alpha}{\sim}})$. If $a \leq [\Diamond p] = -\Box_{\underset{\alpha}{\sim}} - [p]$ then $a \not\lesssim \Box_{\underset{\alpha}{\sim}} - [p]$, meaning that $\alpha(a) = \varnothing \not\rightarrowtail -[p]$. In other words, $-[p] \not\approx 1$, from which it follows that $\Box_{\underset{\alpha}{\sim}} - [p] = 0$, so that $-\Box_{\underset{\alpha}{\sim}} - [p] = 1$ and, finally, $\Box_{\underset{\alpha}{\sim}} - \Box_{\underset{\alpha}{\sim}} - [p] = \sum\{a : a \text{ is an atom}\}$. Therefore $a \leq \Box_{\underset{\alpha}{\sim}} - \Box_{\underset{\alpha}{\sim}} - [p] = [\Box\Diamond p]$. Hence $a \leq [\Diamond p] \Rightarrow [\Box\Diamond p] = [\Diamond p \to \Box\Diamond p]$. So $M \vDash \Diamond p \to \Box\Diamond p$.

Which completes the argument. $\Box$

From this we can infer that for our hyperintensionalist, who holds that $\mathbb{P}$ is complete, the propositional logic of the broadest Leibniz necessity $\Lambda_{(\mathbb{P},\Box_{\underset{\alpha}{\sim}})}$ is some extension of $\mathsf{S5}$.

5.3 General Comments

In this and the previous chapter we have shown how to develop non-reductive theories of necessities in an algebraic (specifically Boolean) setting—as well as how to use these theories to supply algebraic semantics for modal logic. In the following chapters we will begin to consider varieties of logicism, which are reductive theories of necessity. These theories will provide us with very different pictures of modal reality, and very different systems of modal semantics.

6

Logicism I: Semantic Validity

In the following three chapters we will be discussing various forms of logicism about necessity. Recall that a logicist theory is one which develops the identity thesis in the form of a reductive account of necessity in terms of some particular notion of logical truth. In Chapter 1 I specified three notions of logical truth: the semantic, the substitutional, and the syntactic. Hence we will consider three corresponding varieties of logicism. In the present chapter I will consider semantic logicism. In §6.1 I detail how the discussion will proceed, and in particular talk about the method of constructing systems of semantics for modal logic in order to model the consequences of these views (this method will also be used for the remaining forms of logicism in the following two chapters). I also introduce the matter of *constancy*—whether or not to treat the connective 'it is a logical truth that...' as a logical constant in its own right. The question of constancy is not so pronounced in the case of substitutional logicism, but comes to prominance in the discussion of the other kinds of logicism later on. In §6.2 I discuss semantic logicism and the modal semantics of Carnap. I show that in order to preserve classical behaviour for quantification and identity on the semantics, one must adopt a substitutional reading of the quantifiers and a particular idiosyncratic way of evaluating identity statements—both of which Carnap himself adopted.

6.1 Methodological Preliminaries

6.1.1 Applied Semantics

We are going to be modelling various forms of logicism by considering various systems of semantics for languages of propositional and first-order modal logic. This methodology could be referred to as the method of applied semantics.[1] A semantics for a formal language[2] can be approached and treated in at least two ways. Firstly, as purely a mathematical construction which may some shed insight on the language to which it corresponds. Secondly, as a representation of how expressions of a certain relevant kind come to be true or false; a representation with philosophical cash-value. For example, the Kripke semantics can be studied from a purely mathematical perspective—in order to gain insight into the structure of the lattice of modal logics, for instance. But the Kripke semantics can also, and has often been taken to, present a *bona fide* picture of the modal realm. The modal realm according to this reading of the Kripke semantics is centrally a realm of possible worlds and accessibility relations. The modal realm, according to the semantics we will consider here, is a far less metaphysically rich place than that. In the Kripke semantics, read philosophically, we see that modal statements are determined as true or false depending on the facts about which possible worlds are accessible from which. In each of the semantics to be considered in the following chapters, on the other hand, we get a picture according to which modal statements are determined as true or false merely depending on the facts about which formulas are logically true (or which formulas logically follow from a designated set of other formulas). It is in this way that the various semantic frameworks to be considered 'cash out' each kind of logicism as a form of reductive thesis about necessity. By setting out each in detail, we can get a better idea about what each version of logicism implies about necessity in detail. These results can then be used to mount a critique on the logicist's general project.

Often when investigating the semantics of formal languages from a philosophical angle, the notion of an *intended model* or structure plays a central role. Indeed, in previous chapters when investigating algebraic modal semantics, we singled out an intended structure in the form of a distinguished Boolean algebra of propositions $\mathbb{P}$ (we did not single out a distinguished actual world in this algebra, but we could have done so by defining the actual world a as the uniquely true

[1] Following the distinction between pure and applied semantics as described for example by Plantinga [46]

[2] Importantly, we need to distinguish the practice of providing semantics for formal languages from the practice (initiated by Montague) of providing formal semantics for fragments of natural language. Although formal semantics for natural language is created in the image of semantics for formal languages, we will not be doing formal semantics in the style of Montague for some fragment of English, for example. We also will not have reason to use any of the type-theoretic resources standardly utilised in that field.

atomic proposition). The purpose of fixing an intended structure is to tie the pure formalism of the semantics back to philosophical theory; to provide a way of deriving a clear philosophical cash-value from from the mathematics. Given an intended structure or model, one can refer to a particular formula as simply *true* (rather than true-with-respect-to some interpretation) or simply *valid* (rather than valid-with-respect-to some restriction of the semantics). Thus in Chapter 4 we used $\mathbb{P}$ to argue that if intensionalism is true, the logic of the broadest necessity is $\mathsf{S5}$. Now, in the following discussion, designating an intended model will not in fact be so relevant to our purposes; the main philosophical results we will derive from our discussion have more to do with what maneuvers one must make within a particular framework to ensure that classical logic remains valid. And although we will be interested in finding out what the logics of various kinds of logical truth are as we continue, to do so it will not be necessary to designate an intended Tarskian model or state description (terminology explained in Appendix A).[3] The central technical results will in the end be seen to have a philosophical cash value—which has to do with the critique of logicism—and this will be shown most fully in Chapter 9.

6.1.2 Constancy and Inconstancy

Is $\Box$ a logical constant? Appropriating the terminology of Paseau and Griffiths in [45], let *the constancy hypothesis* ('constancy' for short) be the assertion that $\Box$ is a logical constant. It turns out that in the context of investigating the modal logic of a certain kind of logical truth, the assumption of constancy (or alternatively its denial, which we will call *inconstancy*) can have a big impact on the resulting system. In Chapter 7 we see that the propositional logic of substitutional logical truth under constancy is a certain normal extension of $\mathsf{S4}$ which we call McK. While the propositional logic of substitutional logical truth under *in*constancy is a rather different non-normal logic by the name of TS. In Chapter 8 I argue that a similar phenomenon occurs in the context of syntactic logical truth.

Therefore for each variety of logical truth we consider, we can ask: what is the logic of this sort of logical truth under constancy, and what is it under inconstancy? By the end of the next few chapters we shall have answers to this question in each case. Interestingly, semantic logical truth seems to be something of an exception to this scheme. I discuss this and suggest a tentative explanation in §6.2.2. A table of propositional logics of logical truth is collected together in Chapter 10.

[3] It is easy to see how one might: pick some relevant fragment of English, formalise it into the language of predicate logic, then in the Tarskian case fix an intended domain and interpretation function as usual. Or in the state description case, pick that state description S whose members are precisely those basic sentences that formalise true sentences from the fragment of English in question. Additional parameters such as substitution monoids (Chapter 7) or logics- and modalities-of-evaluation (Chapter 8) can then be fixed as desired to model one's chosen situation.

6.2 Semantic Logicism and Carnap's Modal Semantics

We will begin with semantic logicism. The semantic conception of logical truth has it that ϕ is a logical truth iff ϕ is true under every admissible interpretation. What counts as an 'admissible' interpretation depends on the semantics one provides for the language in which ϕ is expressed. For instance, if ϕ is taken from a propositional language, the interpretations will be truth-assignments or something equivalent. If ϕ is a first-order formula, the admissible interpretations may be Tarskian models or, alternatively, state descriptions. The semantic logicist identifies absolute necessity with semantic logical truth.

We will use the method of applied semantics described in §6.1 to investigate the implications of semantic logicism for the behaviour of absolute necessity. The semantic logicist must, if their view is to be adopted seriously, be able to develop an account of the interactions between absolute necessity and logical vocabulary—especially truth-functors, quantifiers, and the identity predicate. Therefore we will consider systems of semantics in which $\Box\phi$ is interpreted as 'it is a semantic logical truth that ϕ', and investigate the behaviour of $\Box$, the quantifiers, and identity under these systems. We start with the propositional case, and build to Carnap's classic result that the propositional logic of semantic validity is $\mathsf{S5}$. In the quantificational case, we note an important choice point. Both of Carnap's treatments of quantified modal logic involve a semantics based on state descriptions, incorporating a substitutional reading of the universal quantifier and a clause for the identity predicate according to which a sentence $c = d$ is true iff c and d are the very same constant. Both of these aspects of Carnap's treatment seem unnatural: in natural language, for instance, the standard type of quantification appears to be objectual rather than substitutional. Also in natural language, it is possible to have true identity claims in which the same name does not just occur twice (e.g. 'Hesperus = Phosphorus'). However, it turns out that these wrinkles in Carnap's account are essential. In §6.2.4 I show that if we attempt to incorporate Carnap's clause for $\Box$ into a more familiar semantics for quantification and identity, the result is a slew of basic failures of classical principles for the quantifier and identity: universal instantiation and Leibniz's Law both suffer from general counterexamples. Hence, in order for the semantic logicist to account for the interactions between necessity and quantification/identity, they must adopt Carnaps' conventions for evaluating claims involving the latter. I return to this point in Chapter 9.

6.2.1 The Propositional Case

In two classic studies ([9] and [10]), Carnap provides alternate systems of semantics for modal logic in which the necessity operator is interpreted as expressing semantic validity (or 'L-truth', in

Carnap's vocabulary). An attractive feature of Carnap's general strategy is that, unlike the various other systems of modal semantics (e.g. Kripke's, or McKinsey's), Carnap does not introduce any new piece of formal machinery for interpreting modal formulas over and above the machinery provided in the semantics for non-modal logic. For example, we can provide a very simple semantics for nonmodal propositional logic as follows. Let PL be the fragment of PML in which $\Box$ does not occur. Interpretations of this language can be especially simple.

Definition 6.2.1. A (propositional) model is a set of sentence letters $m \subseteq \mathrm{At}$.

We can provide a definition of satisfaction of a formula ϕ at a model m ($m \vDash \phi$) via the following semantic clauses.

$m \nvDash \bot$
$m \vDash p$ iff $p \in m$ for all sentence letters p.
$m \vDash (\phi \to \psi)$ iff $m \nvDash \phi$ or $m \vDash \psi$.

We may write $\Gamma \vDash \phi$ if any model m at which every formula $\gamma \in \Gamma$ is true is also a model in which ϕ is true. Then a formula is (semantically) valid iff $\varnothing \vDash \phi$, meaning that ϕ is true at all models m. We can write this as $\vDash \phi$ for short. So far, all of this constitutes a semantics for non-modal propositional logic. Employing Carnap's strategy, we can expand it into a semantics for PML as follows. Keep all of the above clauses as they are, and keep the definition of semantic validity as it is—though now allow ϕ to range over all PML formulas. Then simply add the following semantic clause for modal formulas:

$m \vDash \Box\phi$ iff $\vDash \phi$

Here we have a clause for the $\Box$ which, once again, only makes use of the tools provided by the semantics of non-modal propositional logic—no need for talk of worlds and accessibility relations, or substitutions, etc. And in fact this semantics for PML leads to a nice account of the logic $\Box$. Let C be the set of formulas that are semantically valid on this semantics. C is not in fact a logic in PML, because it is not closed under uniform substitution: note that we have $\neg\Box p \in C$ for every sentence letter p, even though clearly $\neg\Box\top \notin C$. Nevertheless, C includes PC and is closed under modus ponens. Hence we can consider the set $\ell C \subseteq C$ of all formulas that have all their substitution instances in C (see Def A.1.8). We know (Prop A.1.9) that ℓC is a logic, and that it is the largest logic with respect to inclusion that is included in C. Therefore we can think of ℓC as the (propositional) logic of semantic logical truth.[4] Now, Carnap famously showed:

[4] Even though C is the set of all validities on Carnap's semantics, it is not the 'logic of semantic logical truth' because it is not a logic. ℓC is the most eligible logic to receive this title because it is the unique logic which is correct and as complete as possible for the interpretation of $\Box$ as 'it is a semantic logical truth that...'.

Proposition 6.2.2 (Carnap). $\ell C = \mathsf{S5}$

Therefore we can say that the propositional logic of semantic logical truth is $\mathsf{S5}$. Hence also, for the semantic logicist, this is the logic of absolute necessity (putting the semantic logicist, on this small point, in agreement with the non-reductive intensionalist discussed in Ch 4). Now, besides the minor wrinkle of C itself not being a logic, this is clearly a very simple and natural way to establish a semantics for propositional modal logic. It elegantly builds on top of the semantics of non-modal propositional logic and, importantly, it does no violence whatsoever to the interpretation and behaviour of the underlying non-modal logic (i.e. the truth-functional connectives). We will next consider extensions of Carnap's strategy to first-order modal logic, and see how some of these nice features of the propositional case start to break down with the introduction of quantifiers and identity. Though first I want to make a few points about the question of constancy as it relates to what we have just say about the logic of semantic logical truth.

6.2.2 The Logical Invariance of Semantic Logical Truth

Interestingly, the above argument for $\mathsf{S5}$ as the logic of semantic logical truth does not turn on any explicit assumption of either constancy or inconstancy for $\Box$ (cf. §6.1.2). As the reader will see, this is is decidedly unlike the situation we find in later chapters, where the choice between constancy and inconstancy results in a drastically different logic for the variety of logical truth in question (for example, in the case of substitutional logical truth (Ch 7), the assumption of constancy yields the logic McK, while the assumption of inconstancy yields the logic TS). What makes semantic logical truth special in this way, in that its logic is invariant under considerations of constancy? I suspect that the answer to this has something to do with the general concept of a logical constant, and how it may relate to (and sometimes vary upon) the particular notion of logical truth being discussed. Here I want to offer some tentative remarks along these lines to try and explain this special invariance exhibited by semantic logical truth. Some of what I say here depends on later material (especially Chapter 7's discussion of substitutional logical truth, and Chapter 8's discussion of syntactic logical truth); the reader should feel free to skip this section for the time being until having seen what I have to say there.

My idea here, tentatively stated, is that associated with each standard variety of logical truth (substitutional, semantic, syntactic) is some different notion of logical constanthood—some different conception of what it means to treat a symbol as a logical constant. Being a notion of constanthood, each of these alternative concepts involves some idea of invariance. Being a notion of *logical* constanthood, each of these alternative concepts also relates to and is conditioned by one's operative idea of logical truth. Let us take a concrete example. In relation to the idea of substitutional logical truth, is there also an idea of a substitutional logical constant? I think so. A formula ϕ is a substitutional logical truth iff wherever ς is an admissible substitution mapping, the substitution instance $\varsigma\phi$ of ϕ is true. Accordingly, a symbol ε is a substitutional logical constant iff $\varsigma(\varepsilon) = \varepsilon$ for all admissible substitution mappings ς. This is a notion of constanthood since in this case ε is *invariant* under all substitution mappings. Moreover, it is a notion of logical constanthood because the way in which ε is invariant correlates to the idea of substitutional logical truth, which also involves a kind of invariance. A substitututional logical truth is a true formula ϕ whose truth-value is invariant under our choice of substitution ς. For any admissible substitutions ς and ς', the formulas $\varsigma\phi$ and $\varsigma'\phi$ have the same truth-value. And since the identity mapping is an admissible substitution, this truth-value is the same as ϕ, which is truth. Similarly, a substitutional logical constant is a symbol ε such that for all admissible substitutions ς and ς', one has $\varsigma\varepsilon = \varsigma'\varepsilon$. And since the identity mapping is an admissible substitution, we must have $\varsigma\varepsilon = id\varepsilon = \varepsilon$ for any admissible ς. This analogy is clearly not perfect: it is the *truth* of ϕ which is invariant under substitution, but the *identity* of ε which is invariant under the same. But the similarity is suggestive nonetheless, and might allow us to provide a concept of a semantic logical constants along like lines. So let us try to draw an analogy here to semantic logical truth.

A semantic logical truth is a formula ϕ whose truth is invariant under the choice of interpretation of ϕ's non-logical symbols (in the propositional case, invariant under our choice of propositional model m). From this we surmise that interpretations (models, structures, etc.) might therefore draw the distinction between the logical constants and other symbols, in the same way that substitutions did on the substitutional understanding of logical truth. Given a range of *admissible interpretations* i of ϕ, the semantic logical constants are just those symbols ε which are interpreted the same way at each i, while the non-logical constants and variables are those which are interpreted differently in various of the is.

This account might have an implication which is pertinent to our question concerning the logic of semantic logical truth. It seems plausible that the semantic clauses we provided for PML already ensure that $\Box$ is a semantic logical constant, because they ensure that $\Box$ has an invariant interpretation at all models. By this I do not mean that for any ϕ, either $\Box\phi$ is true at all models or false at all models. Though this is true, if this were the criterion, then $\rightarrow$ would not count as a constant, since implications do not receive the same truth-value at all models (take $p \rightarrow q$, for instance, which is true at some models and false others). Rather, the idea is this. Atomic sentences are interpreted in a model by being given a truth-value. So the sentence letters are not logical constants: each model m gives different truth-values to sentence letters, which amount to different interpretations. By the same token, the sentence $\bot$ is a constant because it is always interpreted as falsehood. Operators like $\Box$ and $\to$ are not given truth-values at models; that is not the mode in which they are interpreted. Rather, we say that the symbol '$\to$' represents material implication because of the way in which the truth-value of a compound $(\phi \to \psi)$ is determined by the truth-values of ϕ and ψ—namely, according to the truth-table of material implication. By contrast, suppose we added a new connective $\to'$ to PML with the clauses:

If $p_0 \in m$, then $m \vDash (\phi \to' \psi)$ iff $m \vDash (\phi \to \psi)$
If $p_0 \notin m$, then $m \vDash (\phi \to' \psi)$ iff $m \vDash (\phi \wedge \psi)$

In this case, formulas containing $\to'$ are evaluated differently dependent on whether the sentence letter p_0 is present in the model. So $\to'$ is not interpreted in an invariant way across all models. So $\to'$ is not a semantic logical constant.[5] On the other hand, $\to$ and $\Box$ are interpreted in the same way at every model. Hence they are semantic logical constants.

This allows us the following explanation of the apparent 'invariance' of the logic of semantic logical truth under the question of constancy, which we noted at the start of the section. The explanation is that, in fact, this invariance is illusory. Carnap's semantics implicitly affirms constancy by providing an invariant interpretation of $\Box$ across all models. Hence S5 is the the logic of semantic logical validity under constancy, not *tout court*. Interestingly, this explanation also seems to imply that it would be impossible to obtain a propositional logic of semantic logical truth given *inconstancy*. To do that we would need a semantics similar to Carnap's in which $\Box$ is interpreted as expressing semantic validity and $\Box$ is not treated as a logical constant. But if the previous explanation is right: if we give such an interpretation to $\Box$, *ipso facto* we are interpreting $\Box$ as a logical constant. So there is no way to investigate, and perhaps it makes no sense to even talk about, a propositional logic of semantic logical truth under inconstancy.

With this short digression complete, we can proceed to consider some problems involving quantified modal logic.

[5] A problem: one can combine these clauses into a single clause like so

$$m \vDash (\phi \to' \psi) \text{ iff } (p_0 \in m \text{ and } m \vDash (\phi \to \psi)) \text{ or } (p_0 \notin m \text{ and } m \vDash (\phi \wedge \psi)).$$

In this case, is $\to'$ interpreted 'in the same way' in all models? I am not entirely sure what one could say in response here. Perhaps this shows that $\to'$ is bad example to contrast, but I do not think it invalidates my argument that $\to$ and $\Box$ are constants on this semantics.

6.2.3 Quantification and Identity

In Carnap's two studies of modal semantics mentioned previously, two alternative systems of semantics are provided for quantified modal logic. In this section we will be providing a semantics that is closer to Carnap's earlier semantics,[6] in which Leibniz's Law is preserved in full generality.[7] Carnap [9] provides a semantics for quantified modal logic in terms of state descriptions, applying a substitutional reading to the quantifers. Hence I will render these clauses by using our language QMLΠ of substitutional quantification. We define truth of a sentence ϕ at a state description S ($S \vDash \phi$) by the following clauses.

$S \nvDash \bot$
$S \vDash \phi$ iff $\phi \in S$ for all basic ϕ.
$S \vDash c = d$ iff c and d are the same constant.
$S \vDash (\phi \to \psi)$ iff $S \nvDash \phi$ or $S \vDash \psi$.
$S \vDash \Pi v \phi$ iff $S \vDash \phi[c/v]$ for all constants c.
$S \vDash \Box\phi$ iff $S' \vDash \phi$ for all state descriptions S'.

The differences between Carnap's own semantics and ours are worth noting, though they are incidental. Carnap defines state descriptions differently from us (Def A.2.13), as sets which include, for every basic sentence ϕ,[8] either ϕ or $\neg\phi$. We mimic the inclusion of $\neg\phi$ by simply omitting ϕ from S. Also, Carnap does not use $\bot$ and $\to$ as truth-functional primitives, preferring $\neg$, $\vee$, and $\wedge$ as primitives.

The clauses for $\bot$ and $\to$ are as in propositional logic, and the clause for basic sentences also mimics the clause for sentence letters from the propositional semantics given above. The clause for $\Box$ is clearly the natural extension of Carnap's clause to this extended setting. But the two remaining clauses, for the quantifier and for identity, are much less intuitive. Substitutional quantification itself is not a problematic idea taken on its own terms, but it is not the most common, nor the most natural, way of dealing with quantification in a formal setting—that would be the more standard objectual, or Tarskian reading of the quantifier. The clause for identity is stranger still, ruling out the truth of any sentence $c = d$ where c and d happen to be different names. Surely in natural

[6]Though especially to the more recent and perspicuous presentation of this semantics by Max Cresswell [16].
[7]The differences between the semantics of [9] and [10] are surveyed in Williamson ([68], Ch. 2) as well as in Cresswell [16]. The novel developments of the later semantics came as the result of Carnap's increased emphasis on the distinction between intension and extension, and involved several important changes. One such change, unacceptable from our point of view (§1.2), is that the 1947 semantics does not fully validate Leibniz's Law, owing to the way in which variable assignments are relativised to state descriptions.
[8]'Basic sentence' is also our term, not used by Carnap.

language, where one can have many names for the same object, identity claims involving distinct names are ubiquitous (indeed, it is difficult to see what the *use* of identity would be if we could not make true identity claims using distinct names). Let us put these questions to one side for a moment—only a moment. The above semantics results in a set of valid sentences which is not easily characterisable—for instance, it is not recursively axiomatisable.[9] The principles of S5 are all validated, as are all sentential instances of the Barcan formula and its converse:

BFΠ	$\Pi v \Box \phi \to \Box \Pi v \phi$
CBFΠ	$\Box \Pi v \phi \to \Pi v \Box \phi$

In addition to these Carnap notes several other, somewhat more exotic principles that turn out to be generally valid. Consider the schema ([9], 54; [16], §4):

Assimilation $\quad \Box \Pi x \Pi y \Pi z_1 ... \Pi z_n (\bigwedge_{i \leq n} (x \neq z_i) \to (\Box \phi \to \Box \phi[y/x]))$

Where ϕ contains no free variables other than $x, y, z_1, ..., z_n$, no constants, and no occurances of '$=$', and $\bigwedge_{i \leq n}(x \neq z_i)$ abbreviates $(x \neq z_1 \wedge ... \wedge x \neq z_n)$. As a special case of Assimilation we have:

Assimilation$_0$ $\quad \Box \Pi x \Pi y (\Box \phi \to \Box \phi[y/x])$

Wherever ϕ has at most x and y free, and contains no constants and no occurrences of '$=$'. As a rough informal gloss: necessarily for any x and y, if x is some way necessarily then y is also that way necessarily. In other words, very roughly speaking, the only necessary properties anything can have are those necessary properties everything has. Now, there is not much more that we can do with this semantics beyond noting some interesting valid schemas like this one. It has already been much-investigated, by Carnap himself and several others. So at this point, in order not to stray too far from our main discussion, we will move on to reconsider the matter of interpreting quantification and identity.

6.2.4 Carnap's Clause in Tarskian Semantics

Now that we have the flavour of Carnap's semantics for quantified modal logic, let us return to the questions raised previously about the treatment of quantification and identity in (our rendering of) Carnap's modal semantics. We ask: when employing Carnap's semantic strategy, is it essential

[9] *Proof.* If we could recursively axiomatise $C_\Pi = \{\phi : S \models \phi \text{ for all } S\}$, we would have a decision procedure for first-order logic. Since where ϕ is just a formula of (substitutional) non-modal first order logic, we have $\neg\Box\phi \in C_\Pi$ if ϕ is not a theorem of first-order logic, and $\Box\phi \in C_\Pi$ otherwise. So to decide whether ϕ is a theorem of first-order logic, we could just recursively output all theorems of C_Π until either we proved $\Box\phi$ or $\neg\Box\phi$. Since first-order logic is undecidable, this is impossible. So C_Π is not recursively axiomatisable.

to treat quantification substitutionally, and to disallow true identity claims containing distinct names? Or can one construct an extension of Carnap's semantic strategy for first-order logic given more a more natural and common semantics for identity and quantification?

To answer this question, we shall now trade state descriptions S for Tarskian structures M (Def A.2.10). And we shall consider the following semantics for our language QML$\forall$ of objectual quantification. Where M is a structure and a is an assignment on M, truth of a formula ϕ at M according to a ($M, a \vDash \phi$) is defined as follows.

$M, a \nvDash \bot$
$M, a \vDash P^n t_1 ... t_n$ iff $([t_1]_{M,a}, ..., [t_n]_{M,a}) \in [P^n]_M$ for any c_i and P^n
$M, a \vDash (\phi \to \psi)$ iff $M, a \nvDash \phi$ or $M, a \vDash \psi$
$M, a \vDash \forall v \phi$ iff for all $u \in |M|$, $M, a^v_u \vDash \phi$
$M, a \vDash \Box \phi$ iff for all structures M' and all a' on M': $M', a' \vDash \phi$

Again, for now we can abbreviate 'for all structures M' and all a' on M': $M', a' \vDash \phi$' as simply $\vDash \phi$. This semantics incorporates Carnap's modal clause into a more familiar semantics for first-order logic with objectual quantification. The universal quantifier $\forall$ is interpreted by appeal to variable assignments, and the identity predicate is interpreted like any other predicate. The definition of a structure is such that M always interprets $=$ as the identity relation $[=]_M = \{(u, u) : u \in |M|\}$ on its domain. Hence for constants c and d we have the more familiar derived clause:

$M, a \vDash c = d$ iff $[c]_M = [d]_M$.

What is the result of combining Carnap's modal clause with this more conventional underlying semantics for first-order logic? It turns out, nothing good. Two essential classical principles involving $\forall$ and $=$ turn out to exhibit broad failures, these being universal instantiation and Leibniz's Law. Let us take Leibniz's Law first:

Example 6.2.3. Let t and t' be any distinct terms (constants or variables) and pick some M and a such that $[t]_{M,a} = [t']_{M,a}$. We have $M, a \vDash t = t'$, and since $\vDash t = t$ we also have $M, a \vDash \Box t = t$. However, because t and t' are distinct terms we have $\nvDash t = t'$. This means that $M, a \vDash \neg\Box t = t'$. So we have $M, a \nvDash t = t' \to (\Box t = t \to \Box t = t')$ as a failure of Leibniz's Law.

A natural response to this would be to try to save Leibniz's Law in this semantics by placing some restrictions on what models and/or assignments are considered admissible. If we only allow consideration of models fulfilling some side-constraint, then perhaps we can rule out all of the potential counter-examples to Leibniz's Law, so that Leibniz's Law is valid in the restricted semantics with said side-constraint applied. It turns out that this can be done. But the result is a vindication of Carnap's clause for identity.

Proposition 6.2.4. Suppose that $M, a \vDash \phi$ wherever ϕ is an instance of Leibniz's Law. Then for all terms t, t' we have $M, a \vDash t = t'$ iff t and t' are the very same term.

Proof. By contraposition. If there are distinct t, t' such that $M, a \vDash t = t'$, one can proceed exactly as in Example 6.2.3 to find a instance of Leibniz's Law which fails at M, a. □

(The converse, that any such pair M, a does validate Leibniz's Law, is trivial.) What this means is that we cannot move away from Carnap's clause for identity to a more familiar interpretation of identity without invalidating the logic of identity's central principle. Carnap's clause is a necessary evil. Now, when it comes to universal instantiation, things are even worse. Consider the following counterexample

Example 6.2.5. Since $\nvDash x = c$, we have $M, a \vDash \Diamond x = c$ for any M and any a on M—in particular, this also goes for any assignment a^x_u with $u \in |M|$. From this it follows also that $M, a \vDash \forall x \Diamond x = c$, meaning that $\vDash \forall x \Diamond x = c$. However, since $\vDash c = c$, we also have $\vDash \neg \Diamond c \neq c$. So in fact we have $\vDash \neg(\forall x \Diamond (x \neq c) \to \Diamond c \neq c)$.

So not only is universal instantiation invalid, but some instances of universal instantiation are actually *unsatisfiable*. This means that there is no structure and assignment which satisfies every instance of universal instantiation. Unlike with Leibniz's Law, it is not even possible to validate universal instantiation by considering some restricted class of models.

I take this to indicate, similarly to Carnap's clause for identity, that objectual readings of the quantifier are a no-go in any attempt to provide a Carnapian semantics for quantified modal logic. This leaves a substitutional reading of the quantifer as the best remaining option

6.3 Conclusion

The semantic logicist wants to identify absolute necessity with semantic logical truth. In order to study the behaviour of the latter we have looked to Carnap's modal semantics. In the propositional case we saw that S5 is the logic of semantic logical truth under constancy, and we argued moreover that no sense could be made of a logical of semantic logical truth under inconstancy. We then considered extensions of Carnap's semantic strategy to quantified modal logic. I argued that Carnap's own substitutional reading of the quantifier and non-standard reading of the identity predicate, while off-putting at first glance, are essential to ensuring the validity of classical logic.

In particular, an objectual reading of the quantifier results in failure of universal instantiation, and any alternate reading of the identity predicate results in failures of Leibniz's Law. This spells trouble for the semantic logicist, who ultimately must be able to make sense of necessity *as* semantic logical truth in its interactions with quantification and identity. I will press these concerns more sharply in Chapter 9. In the meantime, in the following two chapters we shall see that these basic problems generalise to our other forms of logicism.

7
Logicism II: Substitutions

The substitutional conception of logical truth has it that ϕ is a logical truth iff every admissible substitution instance of ϕ is true. What counts as an 'admissible' substitution instance has much to do with what one's operative understanding of logical form happens to be. For this reason I will also sometimes call this the conception of logical truth as formal validity.

Substitutional logicism identifies absolute necessity with substitutional logical truth. So in order to investigate the implications of this view for the logic of absolute necessity, we will once again consider systems of semantics in which $\Box$ represents the relevant kind of logical truth—in this case, the substitutional. Systems of semantics in which the $\Box$ represents substitutional logical truth were first introduced by McKinsey in [42], and subsequently much neglected. The recent work of Bacon and Fine [4] has remedied this situation, and in §7.1 I will be covering some of their main results on the logic of subsitutional validity in the propositional case. Their work does much to answer the question of the logic of substitutional validity given constancy—i.e. given that $\Box$ is treated as a logical constant. In §7.2 I consider the case in which constancy fails.

7.1 McKinsey Semantics and the Uniqueness Conjecture

For reasons I will explain shortly, the McKinsey semantics is not set out in terms of a satisfaction relation ($\models$) between models and formulas. Instead, the McKinsey semantics in its contemporary form is expressed in terms of valuations, which are functions v from formulas to the set of truth values $\{0, 1\}$. In general we have:

Definition 7.1.1. A PML valuation is a function v : PML $\to \{0,1\}$ such that $v(\bot) = 0$ and for all ϕ and ψ we have:

$v(\phi \to \psi) = 1$ iff $v(\phi) = 0$ or $v(\psi) = 1$.

It will be useful to also have the following notion:

Definition 7.1.2. A PML subvaluation is a function v^- : At $\to \{0,1\}$. A PML valuation v extends v^- iff $v(p) = v^-(p)$ for all sentence letters p.

Now, the McKinsey semantics considers all valuations of the following kind:

Definition 7.1.3. A McKinsey valuation for PML is a PML valuation v where for all ϕ:

$v(\Box\phi) = 1$ iff $v(s\phi) = 1$ for all substitutions s.

Where we recall that a substitution s in PML is a mapping s : PML $\to$ PML such that $s\bot = \bot$, and such that for all ϕ and ψ we have $s(\phi \to \psi) = (s\phi \to s\psi)$ and $s\Box\phi = \Box s\phi$. The defining condition on McKinsey evaluations encodes the idea that $\Box\phi$ should be true iff all substitution instances of ϕ are true. As mentioned at the start of this chapter, this in turn encodes a particular notion of formal validity. A substitution instance $s\phi$ of ϕ leaves all of the connectives in ϕ as they were, but may replace the sentence letters in ϕ with any formulas whatever. In this way, the sentence letters of ϕ act as variables representing arbitrary formulas, and so each formula can also be seen as representing a certain logical form which other formulas may also possess. For example, we can say that the formula $(q \lor r) \to (q \lor r)$ has the logical form represented by $p \to p$, since the former is a substitution instance of the latter. Then to say of a formula ϕ that all its substitution instances are true is the same as saying that every formula which has ϕ's logical form is true. We say '*logical* form' because what remains constant through any substitution is the distribution of ϕ's connectives, which for now we are assuming are all logical constants. We will consider how to modify this picture when $\Box$ is not treated as a logical constant in the following section.

In his original article on the topic, McKinsey set himself what amounts to the following question: what is the logic of formal validity? In other words: what is the identity of the set of formulas ϕ such that for any substitution instance $s\phi$ of ϕ, $s\phi$ is true in all McKinsey valuations v? Before saying any more, we will give this set a name.

Definition 7.1.4. Let $\mathsf{McK} = \{\phi : v(s\phi) = 1$ for all s and all McKinsey $v\}$.[1]

[1] Burgess [7] uses 'McK' as an alternative name for $\mathsf{S4M}$. But given that $\mathsf{S4M}$ is not itself identical with the full logic of substitutional validity, but only the lower bound McKinsey established for that logic, I think that ours is the more appropriate use of the name.

Other terms we will use for McK are: the logic of formal validity, or the logic of substitutional validity. Bacon and Fine [4] refer to McK as the logic of the full substitution class, because we are considering all substitutions s as opposed to some restricted class of them. What does concern us here is the question of which set McK *is*. Although the matter presently remains open, Bacon and Fine [4] have established important bounds on McK we which shall explain.

7.1.1 Bounds on McK

The first bound to place on McK is one of consistency. Prior to investigation, McK may turn out to be the inconsistent logic—for example, if there are no McKinsey valuations. And the existence of a McKinsey valuation is non-trivial. If we look again at the definition of a McKinsey valuation we note that the clause itself contains an element of impredicativity. Usually it is the case with semantic clauses that the truth value of a formula ϕ in an interpretation depends only the on the treatment of formulas of strictly less complexity. For example, in the Kripke semantics the truth value of $\Box\phi$ at a world is a function of the truth value of ϕ at all accessible worlds. The same basic point clearly goes for Carnap's semantics, and it also goes for the algebraic semantics of Chapters 4 and 5, where the interpretation $[\Box\phi] = N[\phi]$ of the boxed formula depends only on the interpretation $[\phi]$ of ϕ as well as the identity of the designated operator N used to interpret the $\Box$. But a consequence of McKinsey's clause is that $\Box\phi$ is true in a McKinsey valuation v only if *every* substitution instance $s\phi$ of ϕ is true in v, where $s\phi$ might be a far more complex formula than ϕ itself.[2]

Nevertheless, it has been shown that there are McKinsey valuations; the impredicativity does not lead to inconsistency. In general:

Proposition 7.1.5 (Existence)**.** Every subvaluation is extended by some McKinsey valuation.

This result was established independently by Fine and by Prucnal ([4], 10). Now, besides establishing the consistency of McK—which comes as an immediate corollary of the existence theorem—Bacon and Fine [4] have been able to establish some tight bounds on McK. To express these bounds I must briefly assume familiarity with the Kripke semantics in order to define a logic called Med.

Definition 7.1.6. A Medvedev frame is a Kripke frame of the form $((\mathcal{P}(X) - \varnothing), \supseteq)$ where X is a finite nonempty set. Let Med be the class of all Kripke frames isomorphic to some Medvedev frame.

[2] Cresswell [15] has investigated a version of the McKinsey semantics in which this impredicativity is eliminated by restricting McKinsey's clause to only those substitutions s in which sp is nonmodal for every sentence letter p. It has not yet been determined whether the logic of substitutional validity on this restricted semantics is identical to McK (cf. [4], 9). But this point is to the side of our concerns, since we are only interested in the logic of formal validity embodied in the full, impredicative McKinsey semantics.

Definition 7.1.7. $\mathsf{Med} = \{\phi : \ \phi$ is valid on all Medvedev frames in Kripke semantics.$\}$

It follows that Med is a normal modal logic. Moreover we have that Med is a normal extension of the logic S4M. Now Bacon and Fine have been able to establish the following:

Proposition 7.1.8 ([4], Prop. 29)**.** $\mathsf{S4M} \subseteq \mathsf{McK} \subseteq \mathsf{Med}$.[3]

However, most striking of all is the following. Recall that it was a non-trivial matter whether any McKinsey valuations existed. It is also non-trivial, and remains an open question, whether McKinsey extensions are always unique. That is:

Conjecture 7.1.9 (Uniqueness)**.** Any subvaluation is extended by one McKinsey valuation.

A version of this conjecture can be attributed to Friedman [24] in the context of intermediate propositional logics (cf. [4], 10). It seems plausible that the conjecture should be true. If it is, and this can be proven, then it follows that McK can be characterised fully. For we have:

Proposition 7.1.10 ([4], Prop. 28)**.** If the Uniqueness Conjecture holds, $\mathsf{McK} = \mathsf{Med}$.

Let us now take stock. Provided constancy holds, the propositional logic of substitutional logical truth is McK. From the perspective of the substitutional logicist this means that, provided constancy holds, the propositional logic of absolute necessity is McK. Now, in dicsussing the very recent work of Bacon and Fine, we have seen two things. First, tight bounds can be placed on McK, allowing us a decent understanding of what the logic is like. It is a normal extension of S4M, meaning that it contains principles like $\Box\Diamond p \to \Diamond\Box p$. For this reason, it is not contained in S5, for instance. Rather, it is contained in (and may be identical to) the logic of Medvedev frames Med. The second thing we have seen is that, despite recent progress, the question of the logic of substitutional validity remains open. We do not know whether $\mathsf{McK} = \mathsf{Med}$, or whether McK is only properly included in Med. This is because the Uniqueness Conjecture has so far resisted proof (as well as refutation). Therefore we have reached the limit of what we can presently say. Hopefully, future work in the area will provide us with the complete picture of things. This much will do for our purposes.

[3] In fact, they are able to establish a slightly tighter bound, using a logic they term 'S4MSub' as the lower bound ([4], 30). But the definition of S4Sub is relatively complicated, and the bound of S4M suffices for our discussion.

7.2 A Uniqueness Theorem for Inconstancy

So much for the case of constancy. Next I will consider the case of inconstancy. It turns out that where $\Box$ is not treated as a logical constant—that is, as not being part of the logical form of any formula—it is in fact possible to pin down a definite logic for substitutional logical truth. Moreover, this logic turns out to be a logic known as TS. In order to arrive at this result, we will need to introduce a modified version of the McKinsey semantics. Rather than working with substitutions, we will need to work with the following generalisation.

Definition 7.2.1. A presubstitution is a function $h : \text{PML} \to \text{PML}$ such that $h\bot = \bot$ and $h(\phi \to \psi) = (h\phi \to h\psi)$ for all ϕ and ψ.

Note then that a presubstitution is like a substitution except that we do not necessarily have $h\Box\phi = \Box h\phi$ for all ϕ. Indeed, $h\Box\phi$ can be any formula whatever, so that boxed formulas are treated essentially like sentence letters. This makes presubstitutions adequate to represent a conception of logical form in which $\Box$ is *not* a logical constant. If $\Box$ is not a logical constant, that is, it should be possible to substitute $\Box$ with any unary sentential operator whatever while preserving logical form. I claim that this idea is adequately captured by allowing the substitution of any boxed formula $\Box\phi$ with any formula ψ while preserving logical form, because (for example) if we are substituting operators for operators, $\Box$ might be substituted for some operator O which when applied to the relevant substitution instance ϕ, could return a formula $O\phi$ equivalent to ψ. At any rate, we see that presubstitutions characterise a notion of logical form which is very different from that characterised by substitutions—one in which, crucially, $\Box$ is entirely irrelevant to logical form. For example, take the formulas $\Box p \to p$ and $q \to p$. The former is a substitution instance of the latter, but not vice versa. However, both formulas are presubstitution instances of one another. Therefore by the lights of presubstitutions, $\Box p \to p$ and $q \to p$ have precisely the same logical form. Or again, take any formula ϕ and consider the formula $\phi^{-\Box}$ which results from deleting all instances of $\Box$. Except in the limiting case where ϕ itself contains no instance of $\Box$, $\phi^{-\Box}$ is otherwise never a substitution instance of ϕ.[4] This makes sense when we think of the notion of logical form characterised by substitutions: on this conception $\Box$ is a logical constant, hence deleting instances of $\Box$ in ϕ amounts to deleting a part of ϕ's logical form. By contrast, $\phi^{-\Box}$ is always a presubstitution instance of ϕ.[5] With this broader notion of logical form in hand, we can introduce a modified version of McKinsey's semantics as follows.

[4] *Proof.* For any substitution s, $s\phi$ must contain at least as many instances of $\Box$ as ϕ does. Hence if ϕ contains at least one instance of $\Box$, $\phi^{-\Box}$, which contains no instance of $\Box$, cannot be a substitution instance of ϕ.

[5] Note that the mapping $\phi \mapsto \phi^{-\Box}$ is a presubstitution, since $\bot^{-\Box} = \bot$ and $(\phi \to \psi)^{-\Box} = (\phi^{-\Box} \to \psi^{-\Box})$.

Definition 7.2.2. A pre-McKinsey valuation for PML is a PML valuation v where for all ϕ:

$$v(\Box\phi) = 1 \text{ iff } v(h\phi) = 1 \text{ for all presubstitutions } h.$$

I will use the phrase 'inconstant McKinsey semantics' to refer to the semantics dealing with the totality of pre-McKinsey valuations. We can see that the semantic clause for $\Box$ in the inconstant McKinsey semantics exhibits exactly the same kind of impredicativity as the clause for $\Box$ in the McKinsey semantics, since $h\phi$ may be of much greater complexity than ϕ. Therefore it is nontrivial whether there are any pre-McKinsey valuations, and nontrivial whether every subvaluation is extended by a unique pre-McKinsey valuation. In addition, we are faced with the analogue of McKinsey's question: what is the logic of formal validity as understood from the perspective of presubstitutions? Again, let us give this set a name:

Definition 7.2.3. Let $\mathsf{PMcK} = \{\phi : v(s\phi) = 1 \text{ for all } s \text{ and all pre-McKinsey } v\}$.[6]

In the remainder of this section I will show that, as a matter of fact, every subvaluation is extended by a unique pre-McKinsey valuation, and that the logic PMcK is identical with a systems known as TS in the literature. First some preliminary results.

Proposition 7.2.4. If ϕ is a tautology and h is a presubstitution, $h\phi$ is a tautology.

Proof. We first consider the case of non-modal tautologies (tautologies which contain no instance of $\Box$), and then consider tautologies in general. So, suppose ϕ is a non-modal tautology and h is a presubstitution. Find some substitution s such that $hp = sp$ for all sentence letters p, then $h\phi = s\phi$ since ϕ contains no instance of $\Box$. But $s\phi$ is a tautology since the tautologies are closed uniform substitution. This establishes our result for non-modal tautologies. Next suppose that ψ is a tautology. Then there is some non-modal tautology χ and some substitution s such that $s\chi = \psi$. Note that every substitution is a presubstitution and presubstitutions are closed under composition. Therefore hs is a presubstitution for any presubstitution h, but that means that for any h, $hs\chi = h\psi$ is a tautology. $\Box$

Proposition 7.2.5. If ϕ is not a tautology, for some presubstitution h, $\neg h\phi$ is a tautology.

Proof. Let h_0 be a presubstitution which takes each subformula $\Box\psi_i$ in ϕ to a sentence letter q_i non-occurring in ϕ such that $q_i \neq q_j$ wherever $i \neq j$. Also suppose that $h_0 p = p$ for all sentence letters p. $h_0\phi$ is a non-modal formula, and is importantly not a tautology. To see this, take

[6] Note that we still use substitutions s in this definition. That might seem incongruous with the idea that presubstitutions characterise the salient notion of logical form in this case. I will return to this point in §7.2.1 below.

the unique substitution s such that $sq_i = \Box\psi_i$ for all of the new sentence letters q_i introduced previously, and such that $sp = p$ for all other sentence letters p, including those which occur in ϕ. We can see via induction on complexity of ϕ's subformulas that $sh_0\phi = \phi$.[7] But then by Proposition 3, if $h_0\phi$ were a tautology then $sh_0\phi = \phi$ would be a tautology, which is ruled out by hypothesis. Now since $h_0\phi$ is a non-modal formula which is not a tautology, there is some valuation v such that $v(h_0\phi) = 0$. Now let t be some substitution such that for all sentence letters p in $h_0\phi$, $tp = \bot$ iff $v(p) = 0$ and $tp = (\bot \to \bot)$ iff $v(p) = 1$. Then $th_0\phi$ contains no sentence letters or $\Box$, so either $th_0\phi$ is a non-modal tautology or $\neg th_0\phi$ is. But the former cannot be since $v(th_0\phi) = 0$, so we have that $\neg th_0\phi$ is a (non-modal) tautology. □

Proposition 7.2.6. v is pre-McKinsey iff for all ϕ: $v(\Box\phi) = 1$ iff ϕ is a tautology.

Proof. (Right to left) Suppose that $v(\Box\phi) = 1$ iff ϕ is a tautology. By our previous two results, we see that if ϕ is a tautology then every presubstitution instance $h\phi$ of ϕ is a tautology, and that if ϕ is not a tautology then some presubstitution instance of ϕ is equivalent to the negation of a tautology. Hence we have that $v(\Box\phi) = 1$ iff $v(h\phi) = 1$ for all presubstitutions h, so v is pre-McKinsey. (Left to right) Suppose v is pre-McKinsey. Then if ϕ is a tautology then for all presubstitutions h, $h\phi$ is a tautology so that $v(h\phi) = 1$. Conversely, if ϕ is not a tautology then for some presubstitution h_0, we have $h_0\phi$ as a non-modal contradiction meaning that $s(h_0\phi) = 0$ so that $v(\Box\phi) = 0$. So $v(\Box\phi) = 1$ iff ϕ is a tautology. □

These results now allow us to establish a uniqueness property for pre-McKinsey valuations.

Proposition 7.2.7. Every subvaluation is extended by exactly one pre-McKinsey valuation.

Proof. By routine induction on formula complexity, show that $v(\phi) = v'(\phi)$ for any pre-McKinsey valuations v and v' extending a given subvaluation v^-. In the induction step for $\Box$ we see that $v(\Box\phi) = 1$ iff $\vdash_{\text{PC}} \phi$ iff $v'(\Box\phi) = 1$. □

Now that we have our uniqueness theorem, it is possible to set out our inconstant McKinsey semantics in the form of a number of semantic clauses. So, let us define truth of a formula ϕ at a propositional model m ($m \vDash \phi$) in this semantics as follows.

[7] (Base case.) If p is a sentence letter in ϕ, then $sh_0p = p$ by definition. Additionally, $sh_0\bot = \bot$. ($\to$-Induction) Suppose $sh_0\psi = \psi$ and $sh_0\chi = \chi$; then $sh_0(\psi \to \chi) = (sh_0\psi \to sh_0\chi) = (sh_0\psi \to sh_0\chi)$. ($\Box$-Induction) for any $\Box\psi_i$ a subformula in ϕ, we have $h_0\Box\psi = q_i$ and $sq_i = \Box\psi_i$ so that $sh_0\Box\psi_i = \Box\psi_i$ by definition.

$m \nvDash \bot$
$m \vDash p$ iff $p \in m$ for all sentence letters p.
$m \vDash (\phi \rightarrow \psi)$ iff $m \nvDash \phi$ or $m \vDash \psi$.
$m \vDash \Box\phi$ iff for all presubstitutions h, $m \vDash h\phi$.

Our uniqueness theorem (Prop 7.2.7) ensures that these clauses define a unique satisfaction $\vDash$ relation between models and formulas. If we write $\vDash \phi$ to indicate that ϕ is true at all models on this semantics, then we know by definition that PMcK the set of all formulas ϕ such that $\vDash s\phi$ for all substitutions s.

At this point, we have enough information to identify PMcK with another system named in the literature. In [67], Urquhart gives a semantics (the 'τ-semantics') for PML along the following lines. I will indicate Urquhart's satisfaction relation $\vDash_\tau$. Then:

$m \nvDash_\tau \bot$
$m \vDash_\tau p$ iff $p \in m$ for all sentence letters p.
$m \vDash_\tau (\phi \rightarrow \psi)$ iff $m \nvDash_\tau \phi$ or $m \vDash_\tau \psi$.
$m \vDash_\tau \Box\phi$ iff ϕ is a tautology.

We write $\vDash_\tau \phi$ to indicate that ϕ is true at all models on this τ-semantics. Then Urquhart defines the logic TS, the logic of tautological validity, as the set of formulas ϕ such that $\vDash_\tau s\phi$ for all substitutions s. Now note, by Proposition 7.2.6 it follows that in fact, the satisfaction relation $\vDash_\tau$ defined by Urquhart is exactly equivalent to the satisfaction relation of our inconstant McKinsey semantics. That is: for any m we have that ϕ is a tautology iff for all presubstitutions h, $m \vDash h\phi$. Hence a trivial induction shows that $m \vDash \phi$ iff $m \vDash_\tau \phi$ for all ϕ. And from this it immediately follows that:

Proposition 7.2.8. $\mathsf{PMcK} = \mathsf{TS}$.

This completes our treatment of the propositional logic of substitutional logical truth. Next we will consider problems involving the introduction of quantifiers. But first a digression.

7.2.1 Presubstitutions and the Definition of a Logic

Before moving on to consider points about quantification and identity, I want to address a question which arises with our definition of the logic PMcK of formal validity as understood with respect to presubstitutions. Throughout this essay I maintain a definition of a (propositional) modal logic as a set of formulas which contains all tautologies, which is closed under modus ponens, and which is closed under all substitutions. This definition is standard. But plausibly, the condition about closure under substitution is ultimately based in the idea that any formula which has the same logical form as a theorem of some logic should also be a theorem of that logic. But then it could be argued that, if we are to consider an alternative conception of logical form in terms of presubstitutions, we should accordingly work with an alternative definition of a 'logic': one which insists on closure under all presubstitutions. The definition of the logic of formal validity under inconstancy as PMcK then seems problematic; it refers to pre-McKinsey valuations on the one hand, but on the other hand it only requires that PMcK itself be closed under substitutions. What happens if we follow this line of criticism? Suppose we introduce the following:

Definition 7.2.9. A prelogic is a set of formulas which contains all tautologies, and is closed under modus ponens and presubstitutions.

And then suppose we define

Definition 7.2.10. Let $\mathsf{PMcK}' = \{\phi : v(h\phi) = 1$ for all h and all pre-McKinsey $v\}$.

Here, 'h' ranges over presubstitutions. PMcK' is clearly a prelogic, so this seems like it should make for a better definition of the 'logic of formal validity' under inconstancy. But the result of this definition is that $\mathsf{PMcK}' = \mathrm{PC}$. To see this we note that:

Proposition 7.2.11. The only prelogics are PC and the inconsistent logic.

Proof. The inconsistent logic is trivialy a prelogic since it contains all formulas. Now suppose we have a prelogic Λ which is not the inconsistent logic (we may use 'Λ' because every prelogic is trivially also a logic). Note that by Proposition 7.2.5, every member ϕ of Λ must be a tautology, since otherwise there would be some presubstitution instance $h\phi$ of Λ that is equivalent to a contradiction. Hence $\Lambda = \mathrm{PC}$. $\square$

Hence, since $\bot \notin \mathsf{PMcK}'$ we have:

Proposition 7.2.12. $\mathsf{PMcK}' = \mathrm{PC}$

In fact this is not very surprising: if one thinks that $\Box$ is not a logical constant, then it should be impossible to have a system worth calling a 'logic' which contains any non-tautological principles for $\Box$. But then this means that the project of finding a 'logic of formal validity', a logic in which $\Box$ satisfies some distinctive non-tautological principles reflecting properties of formal validity, is misconceived under inconstancy and under the understanding of 'logics' as meaning our prelogics. Nevertheless, this does not mean that our result identifying PMcK with TS is without import. Even if inconstancy is taken to hold, and even if one thinks that the term 'logic' should be reserved for something like a prelogic, one can still recognise sets of formulas as embodying a *theory*, in some weaker sense, of the connective 'it is a (substitutional) logical truth that...' (or indeed, for our logicist, of the connective 'it is absolutely necessary that'). Therefore our substitutional logicist who denies constancy can still make use of TS, even if they do not call it a logic. Now, because these points about inconstancy relate most directly to the concept of a presubstitution, I have included them in this chapter. However the same points apply to treatments of the logic of syntactic logical truth under the assumption of inconstancy.

7.3 Bacon and Fine on Quantification and Identity

In discussing extensions of McKinsey semantics to first-order modal logic with identity, we can look to the remarks of Bacon and Fine ([4], §6), who have considered this already. Near to the end of their paper they consider how to define valuations for the language of first-order modal logic with identity. In the first place, Bacon and Fine characterise and use a notion of a substitution in such a language which is much the same as our QML∀- and QMLΠ-substitutions. So I will make free use of our substitutions in discussing Bacon and Fine's proposals. Now, let us define an extended notion of valuation to use for quantified languages.

Definition 7.3.1. A QML∀-valuation is a function v : QML∀ $\to \{0, 1\}$ such that:

$v(\bot) = 0$
$v(\phi \to \psi) = 1$ iff $v(\phi) = 0$ or $v(\psi) = 1$.
$v(t = t) = 1$ for all terms t.

For the sake of brevity I refer to QML∀-valuations as 'valuations' in the present section. Note that I am not yet specifying any general rules for the evaluation of atomic formulas, or formulas of the form $\Box\phi$ or $\forall v\phi$. That is so we can consider some alternatives in turn. First let us consider the $\Box$, since the suggestion here remains the most consistent throughout.

Proposition 7.3.2. A substitution monoid is a set σ of QML∀-substitutions σ which is closed under composition and contains the identity map.

Definition 7.3.3. A valuation v is McKinsey$_\sigma$, where σ is a substitution monoid, iff:

$$v(\Box\phi) = 1 \text{ iff } v(\varsigma\phi) = 1 \text{ for all } \varsigma \in \sigma.$$

Where $\overline{\sigma}$ is the set (monoid) of all QML$\forall$-substitutions, we can just refer to McKinsey$_{\overline{\sigma}}$ valuations as 'full McKinsey' valuations. A problem immediately arises when considering full McKinsey valuations, even before we consider quantification. Consider the following instance of Leibniz's Law:

$$c = d \rightarrow (\Box c = c \rightarrow \Box c = d)$$

We can construct a general style of countermodel for this sentence. Suppose we have constants c, d, d' that are all distinct from one another, and suppose we have a valuation v such that $v(c = d) = 1$ and $v(c = d') = 0$. If v is a full McKinsey valuation, then we have $v(\Box c = c) = 1$ since for any substitution ς we will have $v(\varsigma c = \varsigma c) = 1$. Nevertheless we have $v(\Box c = d) = 0$ since $v(\varsigma_{d \rightarrow d'}(c = d)) = v(c = d') = 0$.

As Bacon and Fine note, this kind of counterexample can be avoided in one of two ways. The first and most obvious is to retreat from the use of full McKinsey valuations, and to instead restrict attention to McKinsey$_\sigma$ valuations for some restricted class of substitutions σ, such that at least $\varsigma_{d \rightarrow d'} \notin \sigma$. Here we need to be careful, however. Recall the discussion of §7.2.1: when interpreting $\Box$ as expressing formal validity, to designate a certain class of substitutions σ as the class of 'admissible' substitutions is to endorse a certain view of logical form. If we restrict the class of admissible substitutions from $\overline{\sigma}$ to some class σ which doesn't contain $\varsigma_{d \rightarrow d'}$, we deny that $c = d'$ is an admissible substitution instance of $c = d$. In doing so, and especially if we insist that $\Box$ expresses substitutional logical truth, we deny that the formula $c = d'$ shares the logical form of $c = d$. That seems highly implausible, and so this way out of the counterexample seems suspect. It is worth noting that, following this line of thought, Bacon and Fine do briefly entertain the idea that $c = d'$ might indeed differ from $c = d$ with respect to logical form. The idea is to adopt an essentially heterodox view of logical form according to which the patterns of denotation in a formula might factor into that form [4]. Suppose we fix a nonempty domain D and have our valuations assign each constant c_i some object $v(c_i) \in D$, then evaluating $v(c_1 = c_2) = 1$ iff $v(c_1) = v(c_2)$. Then we might stipulate: a substitution ς is only admissible at v (or at v and D, perhaps) iff for all c_i and c_j, if $v(c_i) = v(c_j)$ then $v(\varsigma c_i) = v(\varsigma c_j)$. In this way, $\varsigma_{d \rightarrow d'}$ would be inadmissible in our countermodel since it does not preserve patterns of co-denotation. The idea of building-in denotational information into the logical form of a formula is highly unusual. Probably it stretches the sense of the word 'form' to breaking point. If we accept the idea that admissibility of substitutions correlates with our understanding of logical form, it follows that relativising the admissibility of substitutions to extra-linguistic matters like denotation and choice of domain correlates with relativising logical form to these things. Whatever the logical form of a formula ϕ is, it seems that it ought to be invariant over the ways in which ϕ's nonlogical vocabulary is interpreted. That is, ϕ's logical form just shouldn't be the sort of thing that is contingent on the interpretation of its non-logical symbols—e.g. on what the names in ϕ happen to denote. Therefore this way out of the counterexample seems drastic, involving insufficiently motivated revisions to the concept of logical form.

As I say, this first way out of the counterexample is only considered briefly by Bacon and Fine. The second way out, which they prioritise, is to make use of the idea of a *logically perfect language* as it is found in Russell.[8] The phrase is 'logically perfect language', but to call a language logically perfect is to state something about its relationship to the world. Accordingly, the language of first-order modal logic is called logically perfect, not by it self, but in relation to a given interpretation. At this point, it will be good to introduce the use of Tarskian structures M (Def A.2.10) in order to restate Bacon and Fine's idea.

Definition 7.3.4. The language QML$\forall$ is logically perfect with respect to the structure $M = (|M|, [\cdot]_M)$ (or for brevity: M is perfect) iff the following two conditions hold.

(i) For every $u \in |M|$, there is some constant c with $[c]_M = u$.
(ii) For any distinct constants c and d, $[c]_M \neq [d]_M$.

How does this idea help one to avoid the previous counterexamples to Leibniz's Law? Suppose that we have a valuation that is associated to a Tarskian structure in the following sense.

Definition 7.3.5. A valuation v corresponds to a structure M iff

$$v(P^n c_1 ... c_n) = 1 \text{ iff } ([c_1]_M, ..., [c_n]_M) \in [P^n]_M \text{ for any } c_i \text{ and } P^n.$$

In this case, if M is perfect then for any valuation v correspondent to M will have $v(c = d) = 1$ iff c and d are the very same constant. Let us name this property and state this observation formally.

Definition 7.3.6. v is discerning iff $v(c = d) = 1$ iff c and d are the same constant.

Proposition 7.3.7. If M is perfect and v corresponds to M then v is discerning.

Proof. Immediate. □

[8] "In a logically perfect language the words in a proposition would correspond one by one with the components of the corresponding fact, with the exception of such words as "or", "not", "if", "then", which have a different function. In a logically perfect language, there will be one word and no more for every simple object, and everything that is not simple will be expressed by a combination of words, by a combination derived, of course, from the words for the simple things that enter in, one word for each simple component." Russell [57], 25.

Discerning valuations do not suffer from counterexamples to Leibniz's Law. In fact, it is easy to see that all instances of Leibniz's Law will be fully validated at *any* discerning valuation (even one which happens to be be fully McKinsey). But the reason why this is so is of interest:

Proposition 7.3.8. If v is discerning and ϕ is a sentence and an instance of Leibniz's Law, then $v(\phi) = 1$.

Proof. If ϕ is a sentence and an instance of Leibniz's Law then it is of the form $c = c' \rightarrow (\psi \rightarrow \psi[c'//c])$ for some ψ, c and c'. If c and c' are not the same constant then $v(c = c') = 0$ so that $v(\phi) = 1$. Else, if c and c' are the same constant, then $\psi[c'//c]$ and ψ are the same formula. So $v(\psi \rightarrow \psi[c'//c]) = 1$, meaning $v(\phi) = 1$. $\square$

In discerning valuations, instances of Leibniz's Law divide into two kinds: those that are trivially true because of a false antecedent, and those that are trivially true because of a tautological consequent. There seems to be something of a Pyrrhic victory to this strategy of saving Leibniz's Law, but I will postpone extended critique until Chapter 9.

It is important to recognise that, so long as we do not want to do too much violence to the idea of logical form, the use of discerning valuations is the only viable way to fully preserve Leibniz's Law in the first-order McKinsey semantics. For example, we can establish results like the following, on weak assumptions:

Proposition 7.3.9. Suppose that v is...

(a) Non-discerning.

(b) McKinsey$_\sigma$, where σ at least contains $\varsigma_{c \rightarrow t}$ for all c and t.

(c) such that $v(\phi) = 1$ for every sentential instance of Leibniz's Law.

Then $v(c = d) = 1$ for all constants c, d.

Proof. By (a), since v is non-discerning there are distinct constants c_1 and c_2 with $v(c_1 = c_2) = 1$. Now by (c) we have: $v(c_1 = c_2 \rightarrow (\Box c_1 = c_1 \rightarrow \Box c_1 = c_2)) = 1$, so that $v(\Box c_1 = c_1 \rightarrow \Box c_1 = c_2) = 1$. But $v(\Box c_1 = c_1) = 1$ since v is McKinsey$_\sigma$ and for all substitutions ς (hence for any substitution in σ) we have $v(\varsigma c_1 = \varsigma c_1) = 1$. Therefore we have $v(\Box c_1 = c_2) = 1$. Now take any constants c and d. By (b), in σ we have $\varsigma_{c_1 \rightarrow c}$ and $\varsigma_{c_2 \rightarrow d}$. So we have $v(\Box c_1 = c_2) = v(\varsigma_{c_1 \rightarrow c} \varsigma_{c_2 \rightarrow d} c_1 = c_2) = v(c = d) = 1$. $\square$

In other words, if we want a valuation v that is McKinsey for even a minimal collection of substitutions, and which validates Leibniz's Law, our two options are to let v validate every identity claim or to let v be discerning. Note that we could also have replaced (b) in the above with many alternate monoids. For instance, the set of 'distinctness-preserving substitutions'—substitutions ς such that $\varsigma(c) \neq \varsigma(d)$ whenever c and d are distinct constants (the argument here is essentially unchanged).

To take stock: if we want to preserve Leibniz's Law then we need to confine attention to discerning valuations. A valuation will be discerning if it corresponds to a perfect structure M, so in particular we can confine our attention to valuations which correspond to perfect structures. Insisting on a correspondence to Tarskian structures provides us with a way to evaluate atomic formulas; McKinsey's clause provides us with a way to evaluate modal statements, and the definition of a valuation allows us to evaluate truth-functional compounds. Therefore, all that remains is to consider how to evaluate the quantifiers. An interesting feature of Bacon and Fine's treatment of the quantifier is that it is substitutional. Although they discuss a domain and denotation function (which we have naturally rendered in the language of Tarskian structures), Bacon and Fine do not propose to interpret quantification in the usual Tarskian way—that is, by way of variable assignments and relationships of variance between such assignments. Rather, they propose to use the following:

Definition 7.3.10. v is substitutional iff: $v(\forall x\phi) = 1$ iff $v(\phi[c/x]) = 1$ for all c.[9]

This substitutional reading of the quantifier seems like it might be licensed by the presumption of logical perfection: given that there is a one-one correspondence between objects and constants, this would seem to imply that quantification over objects can be carried out via substitutional quantification over names. I will argue in a moment (§7.3.1) that this thought, expressed in general terms, is mistaken. But Bacon and Fine do seem to be aware of the differences between this substitutional reading of the quantifier and standard objectual interpretations, even assuming logical perfection. Their preemptive solution is to make use of what they call *rigid* terms. I render this idea as follows.

Definition 7.3.11. An expression ε (term or predicate) is rigid with respect to a substitution monoid σ (or σ-rigid) iff $\varsigma(\varepsilon) = \varepsilon$ for every $\varsigma \in \sigma$.

[9] Here I use 'x' to ambiguously designate a variable, rather than 'v' which I use elsewhere. This is to prevent notational confusion when using valuations.

(Note that logical symbols are rigid for any substitution monoid, and that no non-logical symbols are $\overline{\sigma}$-rigid.) Rigid expressions can facilitate a number of things. For example: even though the necessity of identity is validated in discerning valuations along with Leibniz's Law, the necessity of distinctness fails badly in any full McKinsey valuation. In fact:

Proposition 7.3.12. If v is fully McKinsey, then for any constants c, d: $v(\Diamond c = d) = 1$.

Proof. We have $v(\varsigma_{c \to d} c = d) = v(d = d) = 1$. □

So it is only by the use of rigid constants that we can find true instances of the necessity of distinctness. For instance:

Proposition 7.3.13. If v is McKinsey$_\sigma$ and c and d are σ-rigid: $v(c \neq d \to \Box c \neq d) = 1$.

Proof. If c and d are rigid, then $\varsigma(c \neq d) = c \neq d$ for all $\varsigma \in \sigma$. □

The other thing that Bacon and Fine claim rigid terms can do is provide a notion of quantification that is closer to mimicking the ordinary objectual reading. Suppose:

Definition 7.3.14. A McKinsey$_\sigma$ valuation v is rigid-substitutional iff for all ϕ and x: $v(\forall x \phi) = 1$ iff $v(\phi[c/x])$ for all σ-rigid c.

Note that for this suggestion to work as a rendering of universal quantification, we need every term to be rigid in the relevant sense. If v is McKinsey$_\sigma$ and rigid-substitutional, but a particular constant c is not σ-rigid, then we might have it that $v(\forall x \phi \to \phi[c/x]) = 0$. So for such valuations to be viable, one would need to pick σ such that $\varsigma(c) = c$ for every constant c and every $\varsigma \in \sigma$. The suggestion from Bacon and Fine that either of these *prima facie* substitutional clauses might be able to capture an objectual reading of the quantifier is intriguing. In the next section I am going to consider the possibility of incorporating McKinsey's clause into a more familiar Tarskian semantics for first-order logic, with a straightforwardly objectual reading of the universal quantifier. This will give us a chance to test Bacon and Fine's suggestion.

7.3.1 Really Objectual Quantification

The reader should consider this section in a conditional way. I am going to briefly sketch a more thoroughly Tarskian semantics for QML∀ to incorporate McKinsey's clause. In doing so, and for the sake of brevity, I am momentarily going to suspend all concerns of impredicativity.[10] I will presume that it is possible to provide first-order versions of the 'existence and uniqueness' conjectures, so that the following clauses succeed in picking out some determinate satisfaction relation. My aim is twofold. First, to show that it is a mistake to think that a substitutional reading of the quantifiers can fully mimic the objectual reading—crucially, even in perfect structures. The way I do this is to show that, in perfect structures, we have violations of universal instantiation. This also serves my second aim. In the same way that we say above that the assumption of logical perfection (diserning valuations and perfect structures) was not only convenient but necessary for the preservation of Leibniz's Law, I want to show here that a substitutional reading of the quantifiers is necessary to preserve universal instantiation on the McKinsey semantics. This will then provide material for the extended critique of logicism in Chapter 9.

Here are the clauses for our proposed, more thoroughly Tarskian McKinsey semantics for first-order modal logic. Where M is a structure and a is an assignment on M:

$M, a \not\models \bot$

$M, a \models P^n t_1 ... t_n$ iff $([t_1]_{M,a}, ..., [t_n]_{M,a}) \in [P^n]_M$ for any c_i and P^n

$M, a \models (\phi \to \psi)$ iff $M, a \not\models \phi$ or $M, a \models \psi$

$M, a \models \forall v \phi$ iff for all $u \in |M|$, $M, a^v_u \models \phi$

$M, a \models \Box\phi$ iff for all substitutions ς: $M, a \models \varsigma\phi$

We do not require that M be perfect for these clauses to be defined. But we can for now restrict attention just to perfect structures. The first thing to note is that universal instantiation ($\forall v \phi \to \phi[t/v]$) is not valid even over perfect structures:

Example 7.3.15. Let M be a perfect structure, and a be some assignment on M. Note that the domain $|M|$ must be infinite since there are infinitely many constants. Now for any assignment a' on M, let d be some constant such that $a'(x) \neq [d]_M$. Then we have $M, a' \not\models \varsigma_{c \to d}(x = c)$. Therefore for any assignment a' we have $M, a' \models \neg\Box x = c$. So in particular, for all x-variants of a this holds. Hence $M, a \models \forall x \neg \Box x = c$. Nevertheless, we can also see that $M, a \models \Box c = c$, since for any substitution $M, a \models \varsigma c = \varsigma c$. So we have $M, a \not\models \forall x \neg\Box x = c \to \neg\Box c = c$ as a failure of universal instantiation.

[10] As a matter of fact, however, we do not make essential use of this impredicativity in any of the discussion. Therefore if the reader prefers, they can understand this section to be carried out in a predicative fragment of the full McKinsey semantics. To do this, one can follow Cresswell's [15] strategy and stipulate that for all sentence letters p, $\varsigma(p)$ is nonmodal. Alternatively, we can restrict attention to substitutions that are predicative in the sense of Definition 7.3.18. Either of these implicit restrictions suffices by itself to make our discussion and semantic clauses determinate, since our language does not contain complex terms or predicates.

In fact then, not only do we have failures of universal instantiation at some perfect structures, but at all perfect structures:

Proposition 7.3.16. If M is perfect then for all a: $M, a \vDash \neg(\forall x \neg \Box x = c \rightarrow \neg \Box c = c)$.

What this shows is that the proposed substitutional clauses for $\forall$, both rigid and not (Defs 7.3.10 and 7.3.14) do not truly mimic the behaviour of genuine objectual quantification under the McKinsey clause, even granting logical perfection. A substitutional reading of the quantifier is trivially able to satisfy some form of universal instantiation. If v is substitutional then $v(\forall x\phi \rightarrow \phi[c/x]) = 1$ for all c; if v is McKinsey$_\sigma$ and rigid-substitutional then $v(\forall x\phi \rightarrow \phi[c/x]) = 1$ trivially for all σ-rigid c. But once we implement a genuinely objectual reading for the quantifier, we see that universal instantiation actually fails for the objectual quantifer. Now, this also means that the substitutional readings for the quantifiers are non-dispensible for the McKinsey semantics: it is only by moving away from an objectual reading and towards a substitutional reading of the quantifiers that we are able to validate universal instantiation in a first-order McKinsey semantics.

There is one line of response to this argument which we need to consider. Might it be possible to revise the present objectual semantics for QML$\forall$ in order to validate universal instantiation? The only possibility seems to somehow prohibit the use of the substitution $\varsigma_{c \rightarrow d}$ in Example 7.3.15. But note that any substitution ς which took c to some d with $a'(x) \neq [d]_M$ would have worked equally well. And we cannot rule out all and only such substitutions without building in information about denotation into the admissibilty of substitutions, which we objected to previously.[11] So we need to make a more drastic restriction to the set of admissible substitutions. As a quite natural suggestion: perhaps we can rule out the use of $\varsigma_{c \rightarrow d}$ by requiring that c be rigid—and so requiring that the class of admissible substitutions be restricted accordingly. Of course, this move can work only if we insist further that *all* individual constants are rigid. I have found there to be at least two serious problems with this suggestion, but to explain them we must set the idea out more fully, and show what it gets right. Suppose we recast our semantic clauses so that satisfaction is now relativised to a structure, assignment, *and* a substitution monoid ($M, a \vDash_\sigma \phi$). Then leaving all of the other clauses essentially as they were before, suppose we change our clause for $\Box$ to:

[11] One might try to avoid the counterexample by excluding the identity predicate from the class of logical symbols. Let a quasi-substitution be a mapping π which is defined just like a substitution except for the fact that $\pi(=)$ can now be any binary predicate. Set up the semantics with quasi-substitutions in place of substitutions. Then supposing there are P^2 and t such that $M, a \nvDash P^2tt$, we avoid the counterexample to Leibniz's Law since, where $\pi(c) = t$ and $\pi(=) = P^2$ then $\pi(c = c) = P^2tt$, so $M, a \nvDash \Box c = c$ after all. I do not know if this works in full generality. But at any rate, it is unacceptable by our assumptions (and from the point of view of the substitutional logicist). $\Box c = c$ must valid because we assume that $c = c$ is a logical truth.

$M, a \vDash_\sigma \Box\phi$ iff for all substitutions $\varsigma \in \sigma$: $M, a \vDash \varsigma\phi$

In this case, our original semantics is what we get when we fix $\sigma = \overline{\sigma}$. But suppose we pick a more restrictive substitution monoid. For instance:

Definition 7.3.17. ς is first-order fixed iff $\varsigma(c) = c$ for all individual constants c.

We note that all constants are rigid with respect to the monoid of first-order fixed substitutions—let σ_1 denote this set. Therefore if we just consider satisfaction relative to σ_1, we can see that the sort of substitution needed in Example 7.3.15 is not available. In fact, we can show that universal instantiation *is* validated on this restriction. Strictly speaking, we will prove this for the following restriction of σ_1.[12]

Definition 7.3.18. ς is predicative iff $\varsigma\phi$ is never of greater complexity than ϕ itself.

Let $\pi\sigma_1$ be the monoid of predicative first-order fixed substitutions. Now some lemmas.

Proposition 7.3.19. If $\varsigma \in \pi\sigma_1$ then we always have: $\varsigma(\phi[c/x]) = (\varsigma\phi)[c/x]$.

Proof. By induction on complexity of ϕ.

(Base case) If x does not occur in ϕ then since $\varsigma \in \pi\sigma_1$, x also does not occur in $\varsigma\phi$, so the proposition trivially holds. If x does occur in ϕ then ϕ is of the form $P^n t_1 \ldots t_n$ where P^n is some n-ary predicate constant or $=$, and some t_i is x. In this case it is easily verified that

$$\varsigma(P^n t_1 \ldots x \ldots t_n[c/x]) = (\varsigma(P^n t_1 \ldots x \ldots t_n))[c/x] = \varsigma(P^n) t_1 \ldots c \ldots t_n.$$

($\rightarrow$) By induction hypothesis we have

$$((\varsigma\phi)[c/x] \rightarrow (\varsigma\psi)[c/x]) = (\varsigma(\phi[c/x]) \rightarrow \varsigma(\psi[c/x]))$$

And by calculation we can see that

$$(\varsigma(\phi \rightarrow \psi))[c/x] = ((\varsigma\phi)[c/x] \rightarrow (\varsigma\psi)[c/x])$$
$$\varsigma((\phi \rightarrow \psi)[c/x]) = (\varsigma(\phi[c/x]) \rightarrow \varsigma(\psi[c/x]))$$

From which we get $(\varsigma(\phi \rightarrow \psi))[c/x] = \varsigma((\phi \rightarrow \psi)[c/x])$.

($\forall$) By induction hypothesis we have (for all y):

$$\forall y((\varsigma\phi)[c/x]) = \forall y(\varsigma(\phi[c/x]))$$

[12] The restriction to predicative substitutions is needed explicitly here in order to allow for proof by induction on formula complexity.

And by calculation we can see that

$$(\varsigma(\forall y\phi))[c/x] = \forall y((\varsigma\phi)[c/x])$$
$$\varsigma((\forall y\phi)[c/x]) = \forall y(\varsigma(\phi[c/x]))$$

From which it follows that $(\varsigma(\forall y\phi))[c/x] = \varsigma((\forall y\phi)[c/x])$

($\Box$) By induction hypothesis we have

$$\Box\varsigma(\phi[c/x]) = \Box(\varsigma\phi)[c/x]$$

And by calculation we can see that

$$\varsigma((\Box\phi)[c/x]) = \Box\varsigma(\phi[c/x])$$
$$(\varsigma\Box\phi)[c/x] = \Box(\varsigma\phi)[c/x]$$

From which it follows that $\varsigma((\Box\phi)[c/x]) = (\varsigma\Box\phi)[c/x]$ $\Box$

Proposition 7.3.20. $M, a \vDash_{\pi\sigma_1} \phi[c/x]$ iff $M, a^x_{[c]_M} \vDash_{\pi\sigma_1} \phi$.

Proof. By induction on complexity of ϕ. The base case is trivial, as is the induction step for $\rightarrow$. The induction step for $\forall$ is the same as when establishing the same proposition in non-modal first-order logic. The $\Box$ step is as follows. Let ϕ be $\Box\psi$ for some ψ. Assume the proposition holds for all formulas of less complexity than ϕ. $M, a^x_{[c]_M} \vDash_{\pi\sigma_1} \Box\psi$ iff for all $\varsigma \in \pi\sigma_1$ we have $M, a^x_{[c]_M} \vDash_{\pi\sigma_1} \varsigma\psi$. By assumption this holds iff $M, a \vDash_{\pi\sigma_1} (\varsigma\psi)[c/x]$ for all $\varsigma \in \pi\sigma_1$. By Prop 7.3.19 then this holds iff $M, a \vDash_{\pi\sigma_1} \varsigma(\psi[c/x])$, which holds iff $M, a \vDash_{\pi\sigma_1} \Box(\psi[c/x])$. But $\Box(\psi[c/x])$ is $(\Box\psi)[c/x]$ so we are done. $\Box$

Proposition 7.3.21. $\vDash_{\pi\sigma_1} \forall x\phi \rightarrow \phi[c/x]$

Proof. Suppose that $M, a \vDash_{\pi\sigma_1} \forall x\phi$. Then $M, a^x_{[c]_M} \vDash_{\pi\sigma_1} \phi$ so by Prop 7.3.20 we have $M, a \vDash_{\pi\sigma_1} \phi[c/x]$. So $M, a \vDash_{\pi\sigma_1} \forall x\phi \rightarrow \phi[c/x]$. $\Box$

Not only does the restriction to $\pi\sigma_1$ allow one to validate universal instantiation. In perfect structures it actually allows one to validate the substitutional reading of the quantifiers suggested by Bacon and Fine:

Proposition 7.3.22. If M is perfect, then $M, a \vDash_{\pi\sigma_1} \forall x\phi$ iff $M, a \vDash_{\pi\sigma_1} \phi[c/x]$ for all c.

Proof. Since M is perfect, every $u \in |M|$ has exactly one constant c_u such that $[c_u]_M = M$, and each constant names something in $|M|$. So by the semantic clause we have $M, a \vDash_{\pi\sigma_1} \forall x\phi$ iff for all $u \in |M|$ $M, a^x_u \vDash_{\pi\sigma_1} \phi$, which is equivalent to having $M, a^x_{[c]} \vDash_{\pi\sigma_1} \phi$ for each constant c. But by Prop 7.3.20 this is equivalent to: $M, a \vDash_{\pi\sigma_1} \phi[c/x]$ for all c. $\Box$

So by restricting the range of admissible substitutions, it seems one *can* have a thoroughly objectual reading of the quantifiers in McKinsey semantics—contrary to our previous claim.

But there are problems with this approach. In fact, as I mentioned in passing above, there are at least two serious ones. The first is the most obvious: this approach requires that all individual constants be rigid under admissible substitutions. This is necessary, since the *only* thing that was required to generate the counterexample to universal instantiation above (Example 7.3.15) was the existence of a substitution which took some constant c to some other term t. From here we only needed a model M and assignment a such that $M, a \nvDash x = t$. So all constants need to be rigid for the objectual quantifier to obey universal instantiation. But then if we are continuing to read $\Box$ as expressing substitutional logical truth, we must say on this approach that all individual constants are logical constants. And that seems drastic, in the same way that requiring logical perfection seemed drastic.

The second problem, more technical, involves the generalisability of this strategy in systems of second-order quantification. Suppose we extend our language with a supply of predicate variables $X^n, Y^n, Z^n, ...$ and allow for quantification over these variables.[13] We can easily supplement our grammar as needed, and provide a semantics for this modestly extended language by (i) requiring that each assignment a on M now assign each predicate variable X^n with a set of tuples $a(X^n) \subseteq |M|^n$ as with predicate constants, and (ii) providing the following clause for second-order quantification.[14]

$$M, a \vDash_\sigma \forall X^n \phi \text{ iff for all } U \subseteq |M|^n\text{: } M, a^X_U \vDash_\sigma \forall X^n \phi.$$

Finally we need to have a slightly broader notion of a substitution. This can be achieved just by adding the following rules for applying substitutions to formulas with predicate variables.

$$\varsigma \forall X \phi = \forall X \varsigma \phi$$
$$\varsigma X t_1 ... t_n = \varsigma(X)\varsigma(t_1)...\varsigma(t_n)$$
$$\varsigma(X) = X$$

The first and third rule show that X is treated like a variable. The second shows that X is treated like a predicate. Now, in this extended context, even if we restrict attention to $\pi\sigma_1$ we find that the *second*-order version of universal instantiation ($\forall X^n \phi \to \phi[P^n/X^n]$) is invalid. Here is an example:

[13] As in Appendix C, though we may exclude λ-terms in order to avoid complications of impredicativity again, as well as to simplify our extended semantics.

[14] We might also extend the definition of a substitution so that where P is an n-adic constant, $\varsigma(P)$ might be some variable X. But this actually makes no difference to our discussion, and it is simpler to leave the definition of a substitution as it is.

Definition 7.3.23. Where P and Q are n-ary predicate constants, let $\varsigma_{P\to Q}$ be the unique substitution which sends P to Q and leaves everything else as it is.

Example 7.3.24. Let M be a structure and let a be any assignment on M. Now for an arbitrary assignment a', let Q be some unary predicate constant such that $a'(X) \neq [Q]_M$. In this case we have $M, a' \nvDash_{\pi\sigma_1} \Box\forall x(Xx \leftrightarrow Px)$ because $M, a' \nvDash_{\pi\sigma_1} \varsigma_{P\to Q}(\forall x(Xx \leftrightarrow Px))$, since $\varsigma_{P\to Q}(\forall x(Xx \leftrightarrow Px)) = \forall x(Xx \leftrightarrow Qx)$. Since a' here was arbitrary we have this for all assignments. Hence in particular for all $U \subseteq |M|$ we have $M, a^X_U \vDash_{\pi\sigma_1} \neg\Box\forall x(Xx \leftrightarrow Px)$. So $M, a \vDash_{\pi\sigma_1} \forall X\neg\Box\forall x(Xx \leftrightarrow Px)$. Nevertheless, it is quick to see that we have $M, a \vDash_{\pi\sigma_1} \Box\forall x(Px \leftrightarrow Px)$. Hence we have $M, a \nvDash_{\pi\sigma_1} \forall X\neg\Box\forall x(Xx \leftrightarrow Px) \to \neg\Box\forall x(Px \leftrightarrow Px)$ as a failure of universal instantiation.

No problem, one might think. To avoid this counterexample, which is exactly analogous to Example 7.3.15, we just need to ensure that *predicate* constants are rigid, like the individual constants. We can do this like so.

Proposition 7.3.25. ς is second-order fixed iff $\varsigma(P^n) = P^n$ for all predicate constants P^n.

We can let σ_2 be the set of second-order rigid substitutions, and then let $\sigma_{1,2} = \sigma_1 \cap \sigma_2$ be the set of all substitutions that are *both* first- and second-order fixed (we need to retain the restriction to first-order fixed substitutions in order to still fend off Example 7.3.15). But here is the problem. By making both of these restrictions, what we have done is effectively declared *all* symbols to be logical constants. And this has the following consequence:

Proposition 7.3.26. $\sigma_{1,2}$ just contains the identity map.

Proof. Clearly the identity map is both first- and second-order fixed. Now suppose that ς is first- and second-order fixed. In that case we have $\varsigma(\varepsilon) = \varepsilon$ where ε is any term or predicate. In this case, by a trivial induction on formula complexity we can show that $\varsigma\phi = \phi$ for all ϕ, so ς is the identity map. □

Corollary 7.3.27. $\vDash_{\sigma_{1,2}} \phi \leftrightarrow \Box\phi$

This is unacceptable. And to my mind it means that we cannot accept the strategy we have been exploring here. To recap, I gave an argument above that the McKinsey semantics cannot be satisfactorily combined with an objectual reading of the quantifiers; only a substitutional reading (as suggested by Bacon and Fine) can successfully preserve the classical behaviour of the quantifiers in the context of McKinsey's clause for modal formulas. Just now we have been considering one line of resistance to this conclusion. It is possible at least to recover universal instantiation for the first-order quantifier if we restrict attention to those substitutions which leave all individual terms alone—the 'first-order fixed' substitutions. Although this may seem promising at first glance, philosophically it requires us to adopt the drastic tactic of treating all names as logical constants, and moreover, it is not generalisable to systems of higher-order quantification. If one allows quantification over predicates, then violations of universal instantiation reemerge. Trying to solve these higher-order violations by this strategy then results in a collapse of the distinction between truth and substitutional logical validity, since now *all* symbols are logical constants.

So to conclude, the moral I have wanted to draw in this section is that when we extend the McKinsey semantics to first-order modal logic, the only way to satisfactorily ensure that identity and the quantifiers retain their classical behaviour is to make two big concessions. First, that we refrain from an objectual reading of the universal quantifier. Second, that we evaluate an identity statement $c = d$ between two names as true iff c and d are the very same name. We will return to these concessions in Chapter 9.

7.4 Conclusion

The substitutional logicist wants to identify absolute necessity with substitutional logical truth. In order to study the behaviour of the latter we have looked to the McKinsey semantics, and the the recent work of Bacon and Fine. In the propositional case we see that, while work is still ongoing, tight bounds may be placed on the logic of substitutional validity. If the so-called Uniqueness Conjecture is true, then the logic of substitutional validity (assuming constancy) is exactly the logic Med of Medvedev frames. On the other hand, if constancy fails, we can infer that the propositional logic of substitutional logical validity is Urquhart's logic TS of tautological validity.

But ultimately, the substitutional logicist needs to make sense of necessity in its interactions with quantifiers and identity. We saw that in extending the McKinsey semantics to deal with first-order modal logic with identity, a number of problems arose and number of concessions needed to be made in relation to the interpretation of quantification and identity. Importantly, as I have argued, when applying the McKinsey semantics to first-order logic we cannot satisfactorily retain objectual readings of the quantifiers or non-trivial readings of the identity predicate. Quantifiers must be substitutional and identity statements between names are only true if the same name appears on both sides.

8

Logicism III: Syntactic Logicism

The syntactic conception of logical truth has it that ϕ is a logical truth iff it is a theorem of a given logical system. Syntactic logicism identifies absolute necessity with syntactic logical truth. In order to investigate the implications of this view for the behaviour of absolute necessity, I will develop a semantics in which $\Box$ can represent this kind of logical truth. Here, unlike the case of semantic and substitutional logicism, where we had Carnap's and McKinsey's ideas to draw upon, to consider the case of syntactic logicism we need to start essentially from scratch. Therefore the first part of this chapter (§8.1) will be devoted to developing an appropriate system of semantics, and investigating its general features, which are of independent interest. Then in the second part of the chapter (§8.2) I apply this semantics to the question of the logic of syntactic logical truth. Then in the third part (§8.3), mirroring previous chapters, I consider the case of quantifcational languages with identity.

Before getting started, I want to draw attention to an important problem for the syntactic logicist. The definition of syntactic logical truth is *prima facie* ambiguous. We say that ϕ is such a logical truth iff it is a theorem of a 'given' system of logic. But which logic do we mean? There are many systems of logic, even in a simple language like PML. This indicates that there ought to be many varieties of syntactic logical truth: one for each logic. But then how can our syntactic logicist (who accepts monism, recall) identify the unique absolute necessity with syntactic logical truth? Clearly she must choose some particular logic Λ and claim that for a formula to be absolutely necessary is for it to be a theorem of Λ. But what principle could justify such a choice? The proper solution to this puzzle must wait until §8.2. But to state the idea in outline: given some particularly weak assumptions about the behaviour of absolute necessity, it is possible to place very tight bounds on candidates for Λ, and it is moreover possible to identify a unique maximal logic satisfying all relevant conditions. Surprisingly, this logic is Med, and the results we establish here accordingly link back to Bacon and Fine's work on the McKinsey semantics outlined in the previous chapter.

8.1 Modal Semantics Based on Syntactic Consequence

8.1.1 General Concepts

First of all, we are going to need a semantics for propositional modal logic which will encode the syntactic logicist's intended range of meanings for $\Box$. We can assume that the syntactic logicist recognises both absolute and relative necessities as outlined in Chapter 1. We will also not prejudge the question just raised about *which* logic Λ the syntactic logicist identifies as the set of absolute necessities. So in this semantics an interpretation will consist of three components: an underlying propositional model m specifying which sentence letters are true, a particular designated logic Λ (which I will call the logic of evaluation), and a function μ which will designate a set of formulas $\mu(m)$ at m. The idea is that a formula $\Box\phi$ is true at an interpretation ($m \vDash_{\Lambda,\mu} \Box\phi$) if and only if ϕ is a consequence according to Λ of the formulas $\mu(m)$. The inclusion of the designated formulas $\mu(m)$ (rather than just calling $\Box\phi$ true iff ϕ is a theorem of the logic of evaluation) is important because it gives the syntactic logicist range to read $\Box$ as representing a variety of relative necessity which 'holds fixed' $\mu(m)$.

More precisely now. Our semantics is populated by three kinds of objects: propostional models, functions called 'modalities' (which play a similar role to the 'modalities' over Boolean algebras considered in earlier chapters), and logics. We have defined logics and propositional models already; modalities are defined as follows.

Definition 8.1.1. A modality (in this semantics) is a function $\mu : \mathcal{P}(\mathrm{At}) \to \mathcal{P}(\mathrm{PML})$ from models to sets of formulas. The set of modalities is $\mathbf{Mod}$.

Additionally, let $\mathbf{Log}$ be the set of PML logics. Now, modalities are functions which assign to every model m a set of formulas $\mu(m)$. We will speak of the formulas in $\mu(m)$ as being either 'designated' by μ in m or 'held fixed' by μ in m. In the present semantics, formulas ϕ are evaluated true or false in models m with respect to a modality μ and a logic of evaluation Λ according to the following clauses.

$m \nvDash_{\Lambda,\mu} \bot$.
$m \vDash_{\Lambda,\mu} p$ iff $p \in m$ for all sentence letters p.
$m \vDash_{\Lambda,\mu} (\phi \to \psi)$ iff $m \nvDash_{\Lambda,\mu} \phi$ or $m \vDash_{\Lambda,\mu} \psi$.
$m \vDash_{\Lambda,\mu} \Box\phi$ iff $\mu(m) \vdash_{\Lambda} \phi$.

Right away, we see that many kinds of semantic consequence relation and validity are definable in this semantics. Regarding semantic consequence relations, we go by the following naming scheme:

Definition 8.1.2. We write:

$\Gamma \vDash_{\Lambda,\mu} \phi$ iff for all models m, if $m \vDash_{\Lambda,\mu} \psi$ for all $\psi \in \Gamma$ then $m \vDash_{\Lambda,\mu} \phi$.
$\Gamma \vDash_{\mathbf{L},\mathbf{M}} \phi$ iff $\Gamma \vDash_{\Lambda,\mu} \phi$ for all $\Lambda \in \mathbf{L}$ and $\mu \in \mathbf{M}$.
$\Gamma \vDash \phi$ iff $\Gamma \vDash_{\text{Log, Mod}} \phi$.

As a notational convention, when Γ is the empty set we may omit it. So for example, '$\vDash_{\Lambda,\mu} \phi$' abbreviates '$\varnothing \vDash_{\Lambda,\mu} \phi$'. Using this we name various kinds of validity as follows.

Definition 8.1.3. $\text{Va}(\mathbf{L}, \mathbf{M}) := \{\phi : \vDash_{\mathbf{L},\mathbf{M}} \phi\}$. We then say that:
ϕ is $(\mathbf{L}, \mathbf{M})$-valid iff $\phi \in \text{Va}(\mathbf{L}, \mathbf{M})$.
ϕ is (Λ, μ)-valid iff ϕ is $(\{\Lambda\}, \{\mu\})$-valid.
ϕ is generally valid iff ϕ is $(\mathbf{Log}, \mathbf{Mod})$-valid.

We will now begin to develop some of the correspondence theory of this semantics—which is to say, the theory of soundness and (strong and weak) completeness of various logics with respect to these notions of validity and semantic consequence. Given the number of different kinds of validity and semantic consequence definable in this semantics, its correspondence theory is quite rich and complex. Therefore I will not try to be exhaustive in this development, but will just present those of its features which I have found the most interesting.

For any set of logics $\mathbf{L}$ and set of modalities $\mathbf{M}$, we have in our semantics a consequence relation $\vDash_{\mathbf{L},\mathbf{M}}$ and a set of validities $\text{Va}(\mathbf{L}, \mathbf{M})$. Using these we define notions of weak completeness, strong completeness, and soundness.

Definition 8.1.4. A logic Λ is...

...strongly complete for $(\mathbf{L}, \mathbf{M})$ iff $\vDash_{\mathbf{L},\mathbf{M}} \subseteq \vdash_{\Lambda}$.
...weakly complete for $(\mathbf{L}, \mathbf{M})$ iff $\text{Va}(\mathbf{L}, \mathbf{M}) \subseteq \Lambda$.
...sound for $(\mathbf{L}, \mathbf{M})$ iff $\Lambda \subseteq \text{Va}(\mathbf{L}, \mathbf{M})$.

When we say that one consequence relation is included in another, as in $\vDash_{\mathbf{L},\mathbf{M}} \subseteq \vdash_{\Lambda}$, we mean that every implication validated by the first is validated by the second, so in this case: for all formulas ϕ and sets of formulas Γ, if $\Gamma \vDash_{\mathbf{L},\mathbf{M}} \phi$, then $\Gamma \vdash_{\Lambda} \phi$. The following can be proved fairly immediately from the definitions, and are useful to note.

Proposition 8.1.5. For any logic Λ and $\mathbf{L} \subseteq \mathbf{Log}$, $\mathbf{M} \subseteq \mathbf{Mod}$:

(i) If Λ is sound for $(\mathbf{L}, \mathbf{M})$, then $\vdash_\Lambda \subseteq \vDash_{\mathbf{L},\mathbf{M}}$.

(ii) Λ is weakly complete for $(\mathbf{L}, \mathbf{M})$ iff for all ϕ and finite Γ: $\Gamma \vDash_{\mathbf{L},\mathbf{M}} \phi \Rightarrow \Gamma \vdash_\Lambda \phi$.

(iii) If Λ is strongly complete for $(\mathbf{L}, \mathbf{M})$ then Λ is weakly complete for $(\mathbf{L}, \mathbf{M})$.

(iv) Λ is sound and strongly complete for $(\mathbf{L}, \mathbf{M})$ iff $\vDash_{\mathbf{L},\mathbf{M}} = \vdash_\Lambda$.

(v) Λ is sound and weakly complete for $(\mathbf{L}, \mathbf{M})$ iff $\mathrm{Va}(\mathbf{L}, \mathbf{M}) = \Lambda$.

In beginning a treatment of correspondence theory within a semantic framework, particularly one in which multiple notions of semantic consequence validity are definable, a natural place to start is with the most general variety of each. In our case, this will mean characterising the relation of general semantic consequence $\vDash$ $(= \vDash_{\mathbf{Log},\mathbf{Mod}})$ as well as general validity, meaning the identity of the set $\mathrm{Va}(\mathbf{Log}, \mathbf{Mod})$ of generally valid formulas.

Unsurprisingly, these notions of consequence and validity are very weak. In fact, we will show that the logic $\mathsf{S0.5}^0$ is sound and strongly complete for $(\mathbf{Log}, \mathbf{Mod})$. $\mathsf{S0.5}^0$ is defined as the smallest logic which contains the K axiom as well as every formula $\Box\phi$ where ϕ is a tautology. This logic is non-normal, and even non-congruential.[1] The proof of soundness is straightforward and goes as follows.

Proposition 8.1.6. $\mathsf{S0.5}^0$ is sound for $(\mathbf{Log}, \mathbf{Mod})$.

Proof. By induction on length of proof. $\mathsf{S0.5}^0$ can be axiomatised with modus ponens as its sole inference rule if we include all substitution instances of K as axioms, alongside all tautologies ϕ and all necessitated tautologies $\Box\phi$ where ϕ is a tautology.[2] In this case it suffices to show that (i) all tautologies and necessitated tautologies are generally valid, (ii) that all instances of K are generally valid, and (iii) that modus ponens preserves general validity. (i) and (iii) are both fairly immediate, as the validity of tautologies and the admittence of modus ponens is ensured by the semantic clause for the truth-functional constants $\bot$ and $\to$. Additionally, the necessitations $\Box\phi$ of tautologies are generally valid since if ϕ is a tautology then $\vdash_\Lambda \phi$ for any logic Λ. Thus in such cases we will have $m \vDash_{\Lambda,\mu} \Box\phi$ for any model m, modality μ, and logic Λ. (ii) is also easy to show. Consider any uniform substitution instance of K, $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. Then where m is any model, Λ any logic, and μ any modality, suppose that $m \vDash_{\Lambda,\mu} \Box(\phi \to \psi)$ and $m \vDash_{\Lambda,\mu} \Box\phi$. In this case we

[1] A normal logic is a logic containing K and admitting the rule of Necessitation. A congruential or 'classical' modal logic is one admitting the rule: if $\phi \leftrightarrow \psi$ is a theorem then so is $\Box\phi \leftrightarrow \Box\psi$. $\mathsf{S0.5}^0$ admits only a restricted version of either rule. Namely: if ϕ is a tautology then $\Box\phi$ is a theorem, and if $\phi \leftrightarrow \psi$ is a tautology then $\Box\phi \leftrightarrow \Box\psi$ is a theorem.

[2] This is similarly the case for the other logics we consider in §3, and so usually I will not give direct proofs that uniform substitution itself is valid, instead proving that all substitution instances of the characteristic axioms of each logic are valid, which suffices to imply uniform substitution in this context.

have $\mu(m) \vdash_\Lambda \phi \to \psi$ and $\mu(m) \vdash_\Lambda \phi$. From this we get $\mu(m) \vdash_\Lambda \psi$ via the basic properties of consequence relations, so that $m \vDash_{\Lambda,\mu} \psi$. Therefore we have $m \vDash_{\Lambda,\mu} \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ for any m, μ, and Λ, so that any instance of K is generally valid. $\Box$

The proof of strong completeness is only slightly more complicated. It follows a standard strategy of completeness result, which is to show that any $\mathsf{S0.5}^0$-consistent set of formulas is satisfiable. First we introduce the following useful notation.

Definition 8.1.7. For any set Γ of formulas, let $m_\Gamma = (\Gamma \cap \mathrm{At})$.

m_Γ is the model which contains exactly those sentence letters contained in Γ. Importantly, via the semantic clauses we can see that for any Λ, μ, and Γ, we always have $m_\Gamma \vDash_{\Lambda,\mu} p$ iff $p \in \Gamma$ where p is any sentence letter. Now we may continue.

Proposition 8.1.8. $\mathsf{S0.5}^0$ is strongly complete for $(\mathbf{Log}, \mathbf{Mod})$.

Proof. By contraposition; that is, we will show that if $\Gamma \nvdash_{\mathsf{S0.5}^0} \phi$, then $\Gamma \nvDash \phi$. First then suppose that $\Gamma \nvdash_{\mathsf{S0.5}^0} \phi$. In this case, it follows that the set $\Gamma \cup \{\neg\phi\}$ is $\mathsf{S0.5}^0$-consistent. So by Lindenbaum's lemma,[3] there is a maximal $\mathsf{S0.5}^0$-consistent set Δ which includes $\Gamma \cup \{\neg\phi\}$. We can show that the formulas in Δ are jointly satisfiable as follows. Let μ be some modality such that $\mu(m_\Delta) = \Box^-(\Delta)$. Then by induction on formula complexity, one can show that $\psi \in \Delta$ iff $m_\Delta \vDash_{\mathrm{PC},\mu} \psi$ for all formulas ψ. The base case (i.e. $\bot$ and the sentence letters), as well as the induction step for $\to$, are routine. The induction step for $\Box$ goes as follows. Assume for induction that $\psi \in \Delta$ iff $m_\Delta \vDash_{\mathrm{PC},\mu} \psi$ for an arbitrary ψ. Then if $\Box\psi \in \Delta$, we have $\psi \in \Box^-(\Delta) = \mu(m_\Delta)$ so that $\mu(m_\Delta) \vdash_{\mathrm{PC}} \psi$, meaning $m_\Delta \vDash_{\mathrm{PC},\mu} \Box\psi$. Conversely, if $m_\Delta \vDash_{\mathrm{PC},\mu} \Box\psi$, then there are $\delta_1, ..., \delta_n \in \mu(m_\Delta)$ such that $\vdash_{\mathrm{PC}} (\delta_1 \wedge ... \wedge \delta_n) \to \psi$. Now by the definition of $\mathsf{S0.5}^0$, we therefore have $\vdash_{\mathsf{S0.5}^0} \Box((\delta_1 \wedge ... \wedge \delta_n) \to \psi)$, so that by manipulations allowed by the logic, $\vdash_{\mathsf{S0.5}^0} (\Box\delta_1 \wedge ... \wedge \Box\delta_n) \to \Box\psi$.[4] Now, since Δ is maximally $\mathsf{S0.5}^0$ consistent, $(\Box\delta_1 \wedge ... \wedge \Box\delta_n) \to \Box\psi \in \Delta$. But by hypothesis, since the δ_i were taken from $\Box^-(\Delta)$, $\Box\delta_1, ..., \Box\delta_n \in \Delta$. So by maximal consistency, $\Box\psi \in \Delta$, completing the induction. Since Δ is satisfied at m_Δ according to PC and μ, we have it that $m_\Delta \vDash_{\mathrm{PC},\mu} \gamma$ for every $\gamma \in \Gamma$, but that $m_\Delta \nvDash_{\mathrm{PC},\mu} \phi$ since $m_\Delta \vDash_{\mathrm{PC},\mu} \neg\phi$. Thus $\Gamma \nvDash \phi$. $\Box$

The argument here can be generalised quite broadly, as we will see shortly.

[3] Lindenbaum's lemma is valid for all logical consequence relations $\vdash_\Lambda$ via compactness and a standard argument.

[4] In particular, $\mathsf{S0.5}^0$ contains $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q)$.

Now that we have a grasp on the most general kind of validity and semantic consequence definable on the semantics, we may begin to consider some more specific kinds. It turns out the most interesting kinds of restriction on general validity and consequence are of quite a special sort. So first I will spend some time motivating this type of restriction. Let us suppose that we want to find a kind of validity in this semantics which validates every substitution instance of the D axiom ($\Box p \to \Diamond p$). To do this, one might think to place a condition of *consistency* on the set of formulas held fixed at models by modalities. An instance of the D axiom is only invalidated if for some ϕ, we have $\Box\phi$ and $\Box\neg\phi$ true at a model, so we will validate every instance of D so long as we ensure it is never the case that the set of designated formulas at a model entails some formula and its negation according to the logic of evaluation. The mentioning of the logic of evaluation is essential here, since whether or not the set $\mu(m)$ of formulas designated at a model entails ϕ and $\neg\phi$ according to the logic of evaluation usually will depend on what the logic of evaluation is. For example, the set $\{p, \neg\Diamond p\}$ is PC-consistent but it is inconsistent according to many stronger logics. This means that the kind of condition on modalities needed to validate D is one which is relativised to the logic of evaluation itself.[5] In order to have a notion of a condition on modalities which may be relativised to the logic of evaluation, we will introduce the following.

Definition 8.1.9. A condition on modalities is a function $C : \mathbf{Log} \to \mathcal{P}(\mathbf{Mod})$.

To illustrate with an example, let us consider the case of consistency which we proposed above. We may identify the condition of consistency with the function $Con : \mathbf{Log} \to \mathcal{P}(\mathbf{Mod})$ where, for each logic Λ, $Con(\Lambda)$ is the set of modalities μ such that for all models m, $\mu(m)$ is Λ-consistent. With Con, we can define the kinds of validity and semantic consequence which we proposed above to validate all instances of the D axiom. In particular, by considering those pairs $(\mathbf{L}, \mathbf{M}) = (\{\Lambda\}, C(\Lambda))$, where $\mathbf{L}$ is a singleton $\{\Lambda\}$ and where $\mathbf{M}$ is the result $Con(\Lambda)$ of applying the condition of consistency to Λ to yield the relevant set of modalities. We will see shortly that, indeed, every instance of D is $(\{\Lambda\}, Con(\Lambda))$-valid for any logic Λ. Such pairs of the general form $(\{\Lambda\}, C(\Lambda))$, where C is a condition on modalities and Λ is a particular logic, make up the 'special sort' of restriction on general validity I mentioned just above. Their ubiquity in the following discussion demands a cleaner notation, and so we introduce the following.

Definition 8.1.10. Where C is a condition, we define the function $V_C : \Lambda \mapsto \mathrm{Va}(\{\Lambda\}, C(\Lambda))$.

[5] There *is* a property of absolute consistency, wherein a set Γ is absolutely consistent iff it does not contain both ϕ and $\neg\phi$ for any ϕ, but this property is far too weak to be used in the above way to validate D. For example, $\{p, p \to q, \neg q\}$ is absolutely consistent in this sense, but it is PC-inconsistent, so if $\mu(m) = \{p, p \to q, \neg q\}$ then we will have $m \vDash_{\Lambda,\mu} \Box\phi \wedge \Box\neg\phi$ for any Λ and any ϕ.

So we will speak of the 'V-function' of the condition C as the function V_C which maps every logic Λ to the set $V_C(\Lambda) = \text{Va}(\{\Lambda\}, C(\Lambda))$ of formulas which are $(\{\Lambda\}, C(\Lambda))$-valid. Similarly for semantic consequence relations:

Definition 8.1.11. Where C is a condition, let '$\vDash_{\Lambda,C}$' abbreviate '$\vDash_{\{\Lambda\},C(\Lambda)}$'.

As a final piece of notation before continuing to more substantial points, when C is a constant function, in the sense that for some $\mathbf{M} \subseteq \mathbf{Mod}$, $C(\Lambda) = \mathbf{M}$ for all Λ, then we will use the following notation:

Definition 8.1.12. If $C(\Lambda) = \mathbf{M}$ for all logics Λ, we write V_C as $V_{\mathbf{M}}$, and $\vDash_{\Lambda,C}$ as $\vDash_{\Lambda,\mathbf{M}}$.

Therefore note that $\vDash_{\Lambda,\mathbf{M}}$ also serves as a general abbreviation for the notation $\vDash_{\{\Lambda\},\mathbf{M}}$ as defined in Definition 8.1.2. As an example of such a condition, we can consider the function which maps every logic Λ to the set $\mathbf{Mod}$ of all modalities. Call this function the *trivial condition* on modalities. Then the V-function of the trivial condition is $V_{\mathbf{Mod}}$, and the kinds of semantic consequence definable in terms of this condition are of the form $\vDash_{\Lambda,\mathbf{Mod}}$, while the types of validity are those of the form $V_{\mathbf{Mod}}(\Lambda)$.

Now, I have just motivated and introduced the concept of a condition on modalities, considered as a function, and I have introduced some notation to aid in the use of these conditions. Additionally I have introduced as examples the condition of Consistency, Con, and the trivial condition, which maps each logic to the set of all modalities. Next I will show how to provide completeness results for the varieties of semantic consequence and validity definable in terms of these conditions, and also introduce some further conditions on modalities.

As mentioned above, the simple style of argument used to prove completeness in Proposition 8.1.8 can be generalised quite broadly. In the first part of this section I will show exactly how this can be done, and then apply this general form of argument to the types of semantic consequence and validity relating to the two conditions introduced in the previous section (i.e. Con and the trivial condition). By this I mean those semantic consequence relations of the form $\vDash_{\Lambda,Con}$ and $\vDash_{\Lambda,\mathbf{Mod}}$, as well as those varieties of validity of the form $V_{\mathbf{Mod}}(\Lambda)$ and $V_{Con}(\Lambda)$. I will then introduce a larger family of conditions on modalities, and apply the same techniques to these.

First, in order to to sufficiently generalise the argument for Proposition 8.1.8, we will need to set out the following two definitions.

Definition 8.1.13. A modality μ is a Λ-modality iff for every model m, there is some maximally Λ-consistent set Δ such that $m = m_\Delta$ and $\mu(m) = \Box^-(\Delta)$.

Definition 8.1.14. Suppose that Γ is maximally Λ-consistent. Then μ is a (Γ, Λ)-modality iff μ is a Λ-modality and $\mu(m_\Gamma) = \Box^-(\Gamma)$.

The following existence result for such modalities will come in handy shortly.

Proposition 8.1.15. If Λ is consistent, then (i) there is a Λ-modality and (ii) for any maximally Λ-consistent set Γ, there is a (Γ, Λ)-modality.

Proof. Assume Λ is consistent. (i) Wherever m is a model, let $\overline{m} = m \cup \{\neg p : p \notin m\}$. Every $\overline{m}$ is Λ-consistent. For otherwise, if some $\overline{m_0}$ is Λ-inconsistent, there are sentence letters $p_1, ..., p_n \in m$ and $q_1, ..., q_m \notin m$ such that $(p_1 \wedge ...p_n) \wedge (\neg q_1 \wedge ... \wedge q_m) \vdash_\Lambda \bot$, meaning that $\vdash_\Lambda \neg((p_1 \wedge ...p_n) \wedge (\neg q_1 \wedge ... \wedge q_m))$. By uniform substitution, replacing each p_i with $\top$ and each q_i with $\bot$, we have $\vdash_\Lambda \neg((\top \wedge ...\top) \wedge (\neg\bot \wedge ... \wedge \neg\bot))$, which is absurd since Λ is consistent. Therefore, for any model m, which is just a set of sentence letters, there is some maximally Λ-consistent set Δ such that $\overline{m} \subseteq \Delta$, meaning that $m = m_\Delta$. Let μ be any modality such that for each m, $\mu(m) = \Box^-(\Delta)$ for one of these maximally consistent extensions Δ. Then μ is a Λ-modality. (ii) Consider a maximally Λ-consistent Γ. Where μ is any Λ-modality, let μ' be the unique modality such that $\mu'(m) = \mu(m)$ for all $m \neq m_\Gamma$, and where $\mu'(m_\Gamma) = \Box^-(\Gamma)$. Since Γ is maximally Λ-consistent, and μ' and μ agree everywhere besides possibly m_Γ, μ' is also a Λ-modality, and is in fact a (Γ, Λ)-modality. $\Box$

Now, the way that we established completeness in Proposition 8.1.8 was first of all to establish that any maximally $\mathsf{S0.5}^0$-consistent set is satisfied in some model with respect to some modality and logic of evaluation. From this completeness follows quickly, since if $\Gamma \nvdash_{\mathsf{S0.5}^0} \phi$, then $\Gamma \cup \{\neg\phi\}$ is $\mathsf{S0.5}^0$-consistent, hence included in some maximally $\mathsf{S0.5}^0$-consistent set Γ', hence satisfied in some model. Since all of Γ is true but ϕ false in some model (etc.), we have it that $\Gamma \nvDash \phi$.

The following result can be seen to generalise the first part of this argument. Showing conditions under which certain maximally consistent sets of formulas are satisfiable with respect to certain classes of modalities.

Proposition 8.1.16. Suppose Λ and Λ_0 are consistent logics where Λ contains K and includes $\Box(\Lambda_0)$. Suppose also that μ is a (Γ, Λ)-modality. Then for all ϕ, we have $m_\Gamma \vDash_{\Lambda_0,\mu} \phi$ if and only if $\phi \in \Gamma$.

Proof. The argument here is by induction on formula complexity as in the main part of Proposition 8.1.8. The only alterations are to replace 'PC' with 'Λ_0' throughout, and '$\mathsf{S0.5}^0$' with 'Λ' throughout. Note that Λ is an extension of $\mathsf{S0.5}^0$, and therefore will admit the same manipulations used in that proof that were licensed by $\mathsf{S0.5}^0$. $\Box$

As an easy consequence of this proposition we have the following general result.

Proposition 8.1.17. Suppose Λ and Λ_0 are consistent logics where Λ contains K and includes $\Box(\Lambda_0)$, and that $\mathbf{M} \subseteq \mathbf{Mod}$ contains every Λ-modality. Then Λ is strongly complete for $(\{\Lambda_0\}, \mathbf{M})$.

Proof. By contraposition. If $\Gamma \nvdash_\Lambda \phi$ then $\Gamma \cup \{\neg\phi\}$ is Λ-consistent, and is therefore included in some maximally Λ-consistent set Γ'. Let μ be some (Γ', Λ)-modality, which exists by Proposition 8.1.15. Then by Proposition 8.1.16, every formula in Γ' is satisfied at $m_{\Gamma'}$ according to Λ_0 and μ. So in particular, $m \vDash_{\Lambda_0,\mu} \psi$ for every $\psi \in \Gamma$ while $m \vDash_{\Lambda_0,\mu} \neg\phi$. This means that $\Gamma \nvDash_{\{\Lambda_0\},\mathbf{M}} \phi$. So by contraposition, if $\Gamma \vDash_{\{\Lambda_0\},\mathbf{M}} \phi$ then $\Gamma \vdash_\Lambda \phi$. $\Box$

As a special case of this we have:

Proposition 8.1.18. Suppose that Λ contains K and includes $\Box(\Lambda_0)$, and that $C(\Lambda_0)$ contains every Λ-modality. Then $\vDash_{\Lambda_0,C} \subseteq \vdash_\Lambda$, meaning also that $V_C(\Lambda_0) \subseteq \Lambda$.

In this way we have now generalised the argument from Proposition 8.1.8 into a schematic completeness result which can be applied in a large variety of cases. So in the first place, let us apply this result to the two conditions on modalities we have introduced so far. Taking the trivial condition first, we can establish soundness and strong completeness for all pairs $(\{\Lambda\}, \mathbf{Mod})$ in the following way.

Proposition 8.1.19. For all Λ, $\vDash_{\Lambda,\mathbf{Mod}} = \vdash_{\mathsf{S0.5^0}+\Box(\Lambda)}$.

Proof. ($\supseteq$) To show that $\mathsf{S0.5}^0 + \Box(\Lambda)$ is sound for $(\{\Lambda\}, \mathbf{Mod})$, it suffices to show that every theorem of this logic is $(\{\Lambda\}, \mathbf{Mod})$-valid. For this it in turn suffices to show that every tautology, every substitution instance of K, and every formula of the form $\Box\phi$ where $\phi \in \Lambda$, is $(\{\Lambda\}, \mathbf{Mod})$-valid, and that $(\{\Lambda\}, \mathbf{Mod})$-valid is closed under modus ponens. Here the argument is unchanged from Proposition 8.1.6, except for the addition of showing that all of $\Box(\Lambda)$ is $(\{\Lambda\}, \mathbf{Mod})$-valid, which is immediate since for any model m and $\mu \in \mathbf{Mod}$, we will have $m \vDash_{\Lambda,\mu} \Box\phi$ if $\vdash_\Lambda \phi$. ($\subseteq$) That $\mathsf{S0.5}^0 + \Box(\Lambda)$ is strongly complete for $(\{\Lambda\}, \mathbf{Mod})$ follows immediately from Proposition 8.1.17, since trivially every $(\mathsf{S0.5}^0 + \Box(\Lambda))$-modality is in $\mathbf{Mod}$. $\Box$

Next we move onto the Consistency condition Con, where recall that for each Λ, $Con(\Lambda) = \{\mu : \forall m \subseteq \text{At}, \mu(m) \nvdash_\Lambda \bot\}$ is the set of modalities which assign Λ-consistent sets of formulas to each model. In the following, the logic $\mathsf{S0.5^0D}$ is defined as the smallest logic which extends $\mathsf{S0.5}^0$ and which contains the D axiom. Just as in the previous result, we can establish soundness and strong completeness for all pairs $(\{\Lambda\}, Con(\Lambda))$ in the following way.

Proposition 8.1.20. For all Λ, $\vDash_{\Lambda,Con} = \vdash_{\mathsf{S0.5^0D}+\Box(\Lambda)}$.

Proof. ($\supseteq$) To show that $\mathsf{S0.5^0D} + \Box(\Lambda)$ is sound for $(\{\Lambda\}, Con(\Lambda))$, it suffices to show that every theorem of this logic is $(\{\Lambda\}, Con(\Lambda))$-valid. For this it in turn suffices to show that every tautology, every substitution instance of K and D, and every formula of the form $\Box\phi$ where $\phi \in \Lambda$, is $(\{\Lambda\}, Con(\Lambda))$-valid, and that $(\{\Lambda\}, Con(\Lambda))$-validity is closed under modus ponens. Here the argument is unchanged from Propositions 8.1.6 and 8.1.19, except for the addition of showing that all instances of D are $(\{\Lambda\}, Con(\Lambda))$-valid. So, for any ϕ, m, and $\mu \in Con(\Lambda)$ if $m \vDash_{\Lambda,\mu} \Box\phi$ then $\mu(m) \vdash_\Lambda \phi$ so by the definition of Con, $\mu(m) \nvdash_\Lambda \neg\phi$, so that $m \vDash_{\Lambda,\mu} \Diamond\phi$. Therefore $m \vDash_{\Lambda,\mu} \Box\phi \to \Diamond\phi$. ($\subseteq$) By Proposition 8.1.17, to show that $\mathsf{S0.5^0D} + \Box(\Lambda)$ is strongly complete for $(\{\Lambda\}, Con(\Lambda))$, it suffices to show that all $(\mathsf{S0.5^0D} + \Box(\Lambda))$-modalities are in $Con(\Lambda)$. So, suppose that μ is some $(\mathsf{S0.5^0D}+\Box(\Lambda))$-modality. Then for all models m, there is some maximally $(\mathsf{S0.5^0D} + \Box(\Lambda))$-consistent set Δ where $m = m_\Delta$ and $\mu(m) = \Box^-(\Delta)$. Now, suppose for reductio that $\mu \notin Con(\Lambda)$, so that for some model m_0, $\mu(m_0) \vdash_\Lambda \bot$, so that $m_0 \vDash_{\Lambda,\mu} \Box\bot$. Since $\mu(m_0) = \Box^-(\Delta_0)$ and $m_0 = m_{\Delta_0}$ for some some maximally $(\mathsf{S0.5^0D} + \Box(\Lambda))$-consistent set Δ_0, we have it that μ is a (Δ_0, Λ)-modality. Hence by Proposition 8.1.16, $\Box\bot \in \Delta_0$, which is absurd.[6] So we must reject the assumption that there is some m such that $\mu(m) \vdash_\Lambda \bot$. Thus μ is in $Con(\Lambda)$. □

Of course, strong completeness implies weak completeness. So we get the following as an immediate corollary of the above.

Proposition 8.1.21. For all Λ,

$V_{\mathbf{Mod}}(\Lambda) = \mathsf{S0.5^0} + \Box(\Lambda)$

$V_{Con}(\Lambda) = \mathsf{S0.5^0D} + \Box(\Lambda)$

We will now introduce a family of further conditions on modalities, and apply the same techniques as above. Here is a short list of some naturally formulated conditions, including the condition of Consistency investigated above:

Consistency	$Con(\Lambda) := \{\mu \mid \forall m \subseteq \mathrm{At} : \mu(m) \nvdash_\Lambda \bot\}$
Factivity	$Fac(\Lambda) := \{\mu \mid \forall m \subseteq \mathrm{At} :$ if $\mu(m) \vdash_\Lambda \phi$ then $m \vDash_{\Lambda,\mu} \phi\}$
Brouwer Closure	$BCl(\Lambda) := \{\mu \mid \forall m \subseteq \mathrm{At} :$ if $m \vDash_{\Lambda,\mu} \phi$ then $\Diamond\phi \in \mu(m)\}$
$\Box$-Closure	$\Box Cl(\Lambda) := \{\mu \mid \forall m \subseteq \mathrm{At} :$ if $\mu(m) \vdash_\Lambda \phi$ then $\Box\phi \in \mu(m)\}$
$\Diamond$-Closure	$\Diamond Cl(\Lambda) := \{\mu \mid \forall m \subseteq \mathrm{At} :$ if $\mu(m) \nvdash_\Lambda \neg\phi$ then $\Diamond\phi \in \mu(m)\}$

Conditions can also be combined in a natural way:

[6] That is, $\neg\Box\bot$ is a theorem of $\mathsf{S0.5^0D}$, and so must also be in Δ_0.

Definition 8.1.22. Where C and C' are conditions, let $(C \oplus C')$ be the unique condition such that for all Λ, $(C \oplus C')(\Lambda) = (C(\Lambda) \cap C'(\Lambda))$.[7]

Here are some further conditions defined from the above by the use of combination:

Dual Closure	$DCl := \Box Cl \oplus \Diamond Cl$
Consistent X-Closure	$XCC := XCl \oplus Con$
Factive X-Closure	$XCF := XCl \oplus Fac$
	(Where 'X' $= \Box/\Diamond/$Brouwer/Dual.)

The techniques we applied above to the Consistency condition and the trivial condition can be applied to these conditions in an essentially similar way to derive soundness and completeness results. This is what we shall do now. First we will establish strong completeness results for each, and then from these derive the corresponding weak completeness results.

Proposition 8.1.23 (Soundness and Strong Completeness)**.** For all Λ,

(i) $\vDash_{\Lambda,Fac} = \vdash_{\mathsf{S0.5}+\Box(\Lambda)}$.
(ii) $\vDash_{\Lambda,\Box Cl} = \vdash_{\mathsf{S0.5}^0+4+\Box(\Lambda)}$.
(iii) $\vDash_{\Lambda,BCl} = \vdash_{\mathsf{S0.5}^0+\mathrm{B}+\Box(\Lambda)}$.
(iv) $\vDash_{\Lambda,\Diamond Cl} = \vdash_{\mathsf{S0.5}^0+5+\Box(\Lambda)}$.
(v) $\vDash_{\Lambda,DCl} = \vdash_{\mathsf{S0.5}^0+4+5+\Box(\Lambda)}$.
(vi) $\vDash_{\Lambda,\Box CC} = \vdash_{\mathsf{S0.5}^0\mathsf{D}+4+\Box(\Lambda)}$.
(vii) $\vDash_{\Lambda,BCC} = \vdash_{\mathsf{S0.5}^0\mathsf{D}+\mathrm{B}+\Box(\Lambda)}$.
(viii) $\vDash_{\Lambda,\Diamond CC} = \vdash_{\mathsf{S0.5}^0\mathsf{D}+5+\Box(\Lambda)}$.
(ix) $\vDash_{\Lambda,DCC} = \vdash_{\mathsf{S0.5}^0\mathsf{D}+4+5+\Box(\Lambda)}$.
(x) $\vDash_{\Lambda,\Box CF} = \vdash_{\mathsf{S0.5}+4+\Box(\Lambda)}$.
(xi) $\vDash_{\Lambda,BCF} = \vdash_{\mathsf{S0.5}+\mathrm{B}+\Box(\Lambda)}$.
(xii) $\vDash_{\Lambda,\Diamond CF} = \vdash_{\mathsf{S0.5}+5+\Box(\Lambda)}$.
(xiii) $\vDash_{\Lambda,DCF} = \vdash_{\mathsf{S0.5}+4+5+\Box(\Lambda)}$.

Proof. We will prove the cases (i)-(iv) individually. The cases of (v)-(xiii) can be established by combining previous arguments in a natural way, so it will suffice to demonstrate (v) as an example of such combination. Note that for every condition on the list, $V_C(\Lambda)$ contains $\mathsf{S0.5}^0$ and includes $\Box(\Lambda)$ by the fact that $V_C(\Lambda) \supseteq V_{\mathbf{Mod}}(\Lambda)$. Therefore in each case, to show soundness (that the set on the right is included in the set on the left) we need only focus on showing the validity of the new axioms.

(i) ($\supseteq$) In addition to including $\mathsf{S0.5}^0$ and $\Box(\Lambda)$, $V_{Fac}(\Lambda)$ contains every instance of the T axiom: if $m \vDash_{\Lambda,\mu} \Box\phi$ and $\mu \in Fac(\Lambda)$ then $\mu(m) \vdash_\Lambda \phi$ so that $m \vDash_{\Lambda,\mu} \phi$. Hence $m \vDash_{\Lambda,\mu} \Box\phi \rightarrow \phi$. Therefore $\mathsf{S0.5} + \Box(\Lambda)$ is included in $V_{Fac}(\Lambda)$, so that $\vdash_{\mathsf{S0.5}+\Box(\Lambda)}$ is contained in $\vDash_{\Lambda,Fac}$. ($\subseteq$) By Proposition 8.1.17, it suffices to show that every $(\mathsf{S0.5} + \Box(\Lambda))$-modality is in $Fac(\Lambda)$. If μ is a $(\mathsf{S0.5} + \Box(\Lambda))$-modality, then for any model m, $\mu(m) = \Box^-(\Delta)$ for some maximally $(\mathsf{S0.5} + \Box(\Lambda))$-consistent

[7] Note that, owing to the properties that $\oplus$ inherits from $\cap$, the conditions on modalities form a commutative monoid under $\oplus$, with the trivial condition serving as the identity element.

set Δ where by Proposition 8.1.16 we have $\psi \in \Delta$ iff $m \vDash_{\Lambda,\mu} \psi$ for all ψ. Therefore, if $\mu(m) \vdash_\Lambda \phi$ then $m \vDash_{\Lambda,\mu} \Box\phi$, so $\Box\phi \in \Gamma$. But $\Box\phi \to \phi \in \Gamma$ as well, so $\phi \in \Gamma$ and thus $m \vDash_{\Lambda,\mu} \phi$. Since m was arbitrary, this holds for all models, meaning that $\mu \in Fac(\Lambda)$.

(ii) ($\supseteq$) $V_{\Box Cl}(\Lambda)$ every instance of the 4 axiom: if $m \vDash_{\Lambda,\mu} \Box\phi$ then $\mu(m) \vdash_\Lambda \phi$, so by $\Box$-closure $\Box\phi \in \mu(m)$ meaning that $m \vDash_{\Lambda,\mu} \Box\Box\phi$. ($\subseteq$) Again it suffices to show that every $(\mathsf{S0.5}^0 + 4 + \Box(\Lambda))$-modality is in $\Box Cl(\Lambda)$. If μ is a $(\mathsf{S0.5}^0 + 4 + \Box(\Lambda))$-modality, then for any model m, $\mu(m) = \Box^-(\Delta)$ for some maximally $(\mathsf{S0.5}^0 + 4 + \Box(\Lambda))$-consistent set Δ where by Proposition 8.1.16 we have $\psi \in \Delta$ iff $m \vDash_{\Lambda,\mu} \psi$ for all ψ. Therefore, if $\mu(m) \vdash_\Lambda \phi$ then $\Box\phi \in \Delta$. But by hypothesis, Δ also contains the relevant instance of 4, $\Box\phi \to \Box\Box\phi$. So $\Box\Box\phi \in \Delta$ meaning that $\Box\phi \in \mu(m)$. Since m was arbitrary here, this holds for all models, meaning that $\mu \in \Box Cl(\Lambda)$.

(iii) ($\supseteq$) $V_{BCl}(\Lambda)$ contains every instance of the B axiom: if $m \vDash_{\Lambda,\mu} \phi$ then $\Diamond\phi \in \mu(m)$ by Brouwer closure, so $\mu(m) \vdash_\Lambda \Diamond\phi$ and $m \vDash_{\Lambda,\mu} \Box\Diamond\phi$. ($\subseteq$) It suffices to show that every $(\mathsf{S0.5}^0 + \mathrm{B} + \Box(\Lambda))$-modality is in $BCl(\Lambda)$. If μ is a $(\mathsf{S0.5}^0 + B + \Box(\Lambda))$-modality, then for any model m, $\mu(m) = \Box^-(\Delta)$ for some maximally $(\mathsf{S0.5}^0 + B + \Box(\Lambda))$-consistent set Δ where by Proposition 8.1.16 we have $\psi \in \Delta$ iff $m \vDash_{\Lambda,\mu} \psi$ for all ψ. Therefore, if $m \vDash_{\Lambda,\mu} \phi$ then $\phi \in \Delta$. But by hypothesis, Δ also contains the relevant instance of B, $\phi \to \Box\Diamond\phi$. So $\Box\Diamond\phi \in \Delta$ meaning that $\Diamond\phi \in \mu(m)$ so that $m \vdash_\Lambda \Diamond\phi$. Since m was arbitrary here, this holds for all models, meaning that $\mu \in BCl(\Lambda)$.

(iv) ($\supseteq$) $V_{\Diamond Cl}(\Lambda)$ contains every instance of the 5 axiom: if $m \vDash_{\Lambda,\mu} \Diamond\phi$ then $\mu(m) \nvdash_\Lambda \neg\phi$, meaning that then $\Diamond\phi \in \mu(m)$ by $\Diamond$-closure. In this case, $m \vDash_{\Lambda,\mu} \Box\Diamond\phi$. ($\subseteq$) It suffices to show that every $(\mathsf{S0.5}^0 + 5 + \Box(\Lambda))$-modality is in $\Diamond Cl(\Lambda)$. If μ is a $(\mathsf{S0.5}^0 + 5 + \Box(\Lambda))$-modality, then for any model m, $\mu(m) = \Box^-(\Delta)$ for some maximally $(\mathsf{S0.5}^0 + 5 + \Box(\Lambda))$-consistent set Δ where by Proposition 8.1.16 we have $\psi \in \Delta$ iff $m \vDash_{\Lambda,\mu} \psi$ for all ψ. Therefore, if $\mu(m) \nvdash_\Lambda \neg\phi$, then $m \vDash_{\Lambda,\mu} \Diamond\phi$ so that $\Diamond\phi \in \Delta$. But by hypothesis, Δ also contains the relevant instance of 5, $\Diamond\phi \to \Box\Diamond\phi$. So $\Box\Diamond\phi \in \Delta$ meaning that $\Diamond\phi \in \mu(m)$. Since m was arbitrary, this holds for all models, so $\mu \in \Diamond Cl(\Lambda)$.

In the case of (v)-(xiii), the previous arguments can be reused and combined in natural ways. I will prove (x) as an example.

(x) ($\supseteq$) $V_{\Box CF}(\Lambda)$ contains every instance of the T and 4 axioms since it includes $V_{\Box Cl}(\Lambda)$ and $V_{Fac}(\Lambda)$. ($\subseteq$) It suffices to show that every $(\mathsf{S0.5} + 4 + \Box(\Lambda))$-modality is in $\Box CF(\Lambda)$. Here the argument that $\mu \in \Box Cl(\Lambda)$ can be repeated from (ii), and the argument that $\mu \in Fac(\Lambda)$ can be repeated from (i), with obvious relabelling.

None of the other cases involve new difficulties. $\Box$

Since strong completeness implies weak completeness, we get the following as a corollary.

Proposition 8.1.24 (Soundness and Weak Completeness)**.** For all Λ,

(i) $V_{Fac}(\Lambda) = \mathsf{S0.5} + \Box(\Lambda)$
(ii) $V_{\Box Cl}(\Lambda) = \mathsf{S0.5}^0 + 4 + \Box(\Lambda)$.
(iii) $V_{BCl}(\Lambda) = \mathsf{S0.5}^0 + \mathsf{B} + \Box(\Lambda)$
(iv) $V_{\Diamond Cl}(\Lambda) = \mathsf{S0.5}^0 + 5 + \Box(\Lambda)$.
(v) $V_{DCl}(\Lambda) = \mathsf{S0.5}^0 + 4 + 5 + \Box(\Lambda)$.
(vi) $V_{\Box CC}(\Lambda) = \mathsf{S0.5}^0\mathsf{D} + 4 + \Box(\Lambda)$.
(vii) $V_{BCC}(\Lambda) = \mathsf{S0.5}^0\mathsf{D} + \mathsf{B} + \Box(\Lambda)$.
(viii) $V_{\Diamond CC}(\Lambda) = \mathsf{S0.5}^0\mathsf{D} + 5 + \Box(\Lambda)$.
(ix) $V_{DCC}(\Lambda) = \mathsf{S0.5}^0\mathsf{D} + 4 + 5 + \Box(\Lambda)$.
(x) $V_{\Box CF}(\Lambda) = \mathsf{S0.5} + 4 + \Box(\Lambda)$.
(xi) $V_{BCF}(\Lambda) = \mathsf{S0.5} + \mathsf{B} + \Box(\Lambda)$.
(xii) $V_{\Diamond CF}(\Lambda) = \mathsf{S0.5} + 5 + \Box(\Lambda)$.
(xiii) $V_{DCF}(\Lambda) = \mathsf{S0.5} + 4 + 5 + \Box(\Lambda)$.

Williamson has shown ([68], 110) that $(\mathsf{S0.5} + 4 + 5) = \mathsf{S5}$, hence the cases marked (xiii) in the previous two propositions may be rewritten as $\vDash_{\Lambda,DCF} = \vdash_{\mathsf{S5}+\Box(\Lambda)}$ and $V_{DCF}(\Lambda) = \mathsf{S5} + \Box(\Lambda)$ respectively. Now that we have a handle on how to establish completeness results in the semantics, and an inventory of such results for a range of conditions, next we will move on to discuss some important properties of conditions, relating to the idea of a condition *determining* a logic or set of logics.

8.1.2 Determinacy

In this section I introduce the class of *standard conditions* on modalities, which includes all of the conditions discussed previously. I will also motivate a stricter notion of correspondence between conditions and logics.

In investigating the correspondence theory of a modal semantics, there is interest in demonstrating especially tight connections between certain logics and certain properties of the objects populating the framework. That is to put it very generally but, for a concrete example, in the Kripke semantics we are able to speak of certain properties of accessibility relations as determining certain logics. The property of reflexivity determines the logic T, in that T is sound and complete over the reflexive frames; adding transitivity on top of reflexivity determines the stronger logic $\mathsf{S4}$, and so on. Now in the present semantics, we have seen how a condition on modalities can ensure the validity of a desired axiom. (For example, the condition Fac ensures the validity of the T axiom, and $\Box Cl$ ensures the validity of 4). But, unlike for accessibility relations in the Kripke semantics, it is not the case that our conditions on modalities determine a single logic. Instead, our placing the condition of, say, Factivity on modalities determines a family of logics of the form $\mathsf{S0.5} + \Box(\Lambda)$. Such families can be quite disunified. In the case of Fac, the family of logics of the form $\mathsf{S0.5} + \Box(\Lambda)$ includes all normal extensions of the logic T (since if Λ is a normal extension of T, then $\mathsf{S0.5} + \Box(\Lambda) = \Lambda$). This is disappointing from the point of view just mentioned—that in modal semantics we should be finding tight relation of determination between conditions and logics. Naturally, we may ask whether *any* conditions on modalities are able to fulfil the task of determining just one logic. This is to ask whether there are conditions of the following kind.

Definition 8.1.25. A condition C is determinate iff $\exists \Lambda' \, \forall \Lambda : V_C(\Lambda) = \Lambda'$.

To parse this definition: we are saying that a determinate condition C is one such that the set of valid formulas $\mathrm{Va}(\{\Lambda\}, C(\Lambda))$ is the same for any input Λ. Clearly, none of the conditions considered in the previous section are determinate, owing to the '$+ \Box(\Lambda)$' which features in each case of Propositions 8.1.24 and 8.1.23. The question is then whether any conditions *are* determinate in this sense. The answer turns out to be yes, but only for a quite narrow selection. In particular:

Definition 8.1.26. A condition C is an *inconsistency* condition iff for all Λ, every $\mu \in C(\Lambda)$ is such that $\mu(m)$ is Λ-inconsistent for every model m.

Proposition 8.1.27. A condition C is determinate only if it is an inconsistency condition.[8]

Proof. If C is determinate, then for all Λ, $V_C(\Lambda) = V_C(\mathrm{PML})$ where, recall, PML is the inconsistent logic. But $V_C(\mathrm{PML}) = \mathrm{Va}(\{\mathrm{PML}\}, C(\mathrm{PML}))$ contains the formula $\Box\bot$, as for every model m and any modality μ, $m \vDash_{\mathrm{PML},\mu} \bot$ by the fact that $\mu(m) \vdash_{\mathrm{PML}} \bot$. Therefore for all Λ, if C is determinate then $\Box\bot \in V_C(\Lambda)$. But in this case, for all $\mu \in C(\Lambda)$ and all models m, $\mu(m) \vdash_\Lambda \bot$, so that $m \vDash_{\Lambda,\mu} \Box\bot$, so that $\mu(m) \vdash_\Lambda \bot$. This means that C must be an inconsistency condition. $\Box$

Clearly then, in practice, the class of determinate conditions is not so interesting. The basic problem is the fact that no matter what condition C one picks, for any Λ, $V_C(\Lambda)$ will always include the set $\Box(\Lambda)$. This is immediate by the semantic clause for $\Box$: every theorem of the logic of evaluation will be valid according to said logic, no matter what modalities one considers. Therefore in particular, $\Box(\mathrm{PML}) \subseteq V_C(\mathrm{PML})$, so that if C is determinate, $\Box(\mathrm{PML}) \subseteq V_C(\Lambda)$ for all Λ. Since $\Box\bot \in \Box(\mathrm{PML})$, we therefore have as in the latter proof that $m \vDash_{\Lambda,\mu} \Box\bot$ for all models m and all $\mu \in C(\Lambda)$, so that $\mu(m) \vdash_\Lambda \bot$ for all m and for all $\mu \in C(\Lambda)$. The fact that $\Box(\Lambda)$ is included in any set $V_C(\Lambda)$ indicates that the best interesting approximation of the property of being determinate must be something like the following.

[8] Note that the converse of this result—that every inconsistency condition is determinate—does not generally hold. Suppose C is an inconsistency condition such that for some Λ, Λ', $C(\Lambda) = \varnothing$ and $C(\Lambda') \neq \varnothing$. Then $V_C(\Lambda) =$ PML while $V_C(\Lambda') = \mathsf{Ver}$ (the proofs are trivial). But from this we can see that a weakening of the converse does hold: if C is an inconsistency condition and either $C(\Lambda) = \varnothing$ for all Λ *or* $C(\Lambda) \neq \varnothing$ for all Λ, then C is determinate, and in particular determines PML or Ver respectively.

Definition 8.1.28. C is standard iff $\exists \Lambda' \, \forall \Lambda : V_C(\Lambda) = \Lambda' + \Box(\Lambda)$.

And a quick inspection will show that all of the conditions considered in the previous section are standard in this sense. Given this, we are left with the following problem. On the one hand, naturally specified standard conditions like Fac determine quite wide and disunified families of logics. On the other hand, the property of a condition being standard seems to be the closest we can get to the property of being determinate without lapsing into triviality. If we want to arrive at tighter correspondences between conditions and logics, we cannot rely on defining some property of conditions which is stricter than standardness. Rather, what is needed are tighter notions of correspondence altogether—i.e. ones that are stricter than Λ corresponding to C iff Λ is sound and (strongly/weakly) complete for $(\{\Lambda_0\}, C(\Lambda_0))$ for some logic of evaluation Λ_0. In the next section I will discuss one example of such a stricter notion.

8.1.3 Generation

To motivate the particular notion of correspondence that I have in mind, we can start with the following observation. In modal semantics generally, it is natural to talk of a certain property (of accessibility relations, neighborhood functions, etc.) as *giving rise* to a particular logic, or *yielding* it, etc. Hence, it is natural enough to say that the property of reflexivity for accessibility relations gives rise to or *generates* the logic T in the Kripke semantics. Now in the present semantics, such talk of generation can be appropriated and employed in an especially interesting sort of way. Note that for every standard condition C, the function V_C is a function from logics to logics.[9] Hence in each case where we have an equation of the form $V_C(\Lambda) = \Lambda'$, we might want to talk of the condition C as 'generating' the logic Λ' from the input logic Λ. Adopting such talk, we can naturally extend it to cases in which we arrive at one logic from another via multiple applications of a function V_C. Hence it seems equally natural to talk about C generating Λ' from Λ if it does so in two steps—that is, if $V_C^2(\Lambda) = \Lambda'$, where we adopt the standard convention of writing $V_C^{\beta+1}(\Lambda) = V_C(V_C^{\beta}(\Lambda))$ where β is an ordinal—and so on for any finite iterations of V_C. Extending this point even further, we can even give a natural definition of transfinite iterations of V-functions in certain cases. Consider the following:

Definition 8.1.29. Where C is a condition, Λ is a logic, and λ is a limit ordinal, C is λ-cumulative at Λ iff for all ordinals $\beta < \gamma < \lambda$ we have $V_C^{\beta}(\Lambda) \subseteq V_C^{\gamma}(\Lambda)$.

[9] This is not the case for all V-functions whatever, but rather a consequence of the definition of standard conditions. For example, the function V_α introduced in the following section does not always output a logic.

Definition 8.1.30. Where λ is a limit ordinal and C a condition such that C is λ-cumulative at Λ, then $V_C^\lambda(\Lambda) := \bigcup_{\beta<\lambda} V_C^\beta(\Lambda)$. If C is not λ-cumulative at Λ, $V_C^\lambda(\Lambda)$ is undefined.

To my mind, it seems no less natural to talk of C generating a logic Λ' from an input Λ after some transfinite application of V_C than it is to say the same for finite iterations. Therefore I will make our talk of generation precise as follows:

Definition 8.1.31. C generates Λ' from Λ iff $\exists\beta : V_C^\beta(\Lambda) = \Lambda'$.

Given how we have been speaking so far, it is clear that our proposed stricter notion of correspondence is something like this: a condition corresponds to a logic in the present sense just in case the condition generates the logic. But this proposal is not entirely clear, since our present notion of generation is relative: a logic is only ever generated by a condition with respect to some input logic. Therefore we need a further notion of generation *simpliciter*. Now it seems to me that a natural way to eliminate the additional parameter is to fix the input logic as the smallest logic, PC. In other words, we can say:

Definition 8.1.32. C generates Λ iff C generates Λ from PC.

The intuitive advantage of this strategy is that it isolates the 'work' done by the condition C in generating logics. Since PC contains no non-tautological principles about the behaviour of modal operators, if a condition C can generate Λ from PC then it is clearly the condition itself doing the work of generation as opposed to the input logic. To take an example: the condition Fac was defined with an eye to validating the T axiom ($\Box p \to p$). Now since $V_{Fac}(\Lambda) = \Lambda$ whenever Λ is a normal extension of T, it is technically the case that Fac can generate any normal extension Λ of T *from* Λ itself. Such extensions may contain all sorts of strong principles which, unlike the T axiom, bear no obvious connection to the definition of Fac itself. By contrast, we shall see below that the strongest logic Fac is able to generate from PC is T, which recall is just the smallest normal logic containing the T axiom. Intuitively this indicates that T marks the upper bound of what Fac is capable of contributing to the process of generation all by itself.

In general, there is a lot to be said about generation with respect to standard conditions, since the V-functions of such conditions turn out to be very well-behaved. For example, in the remainder of this section I will show that whenever C is a standard condition:

$V_C^\omega(\Lambda) = V_C(\Lambda) + \text{Nec}$ where defined.
$V_C^\omega(\Lambda)$ is a V_C-fixed-point if defined (so that $V_C^\omega(\Lambda) = V_C^\beta(\Lambda)$ for all $\beta > \omega$).

These amount to giving a full characterisation of the transfinite applications of the V-function of any standard condition. From these, by setting Λ to PC, we get that the largest and final logic generated by any standard condition C is just the logic $V_C^\omega(\mathrm{PC}) = V_C(\mathrm{PC}) + \mathrm{Nec}$. For example, $V_{Fac}^\omega(\mathrm{PC}) = \mathsf{S0.5} + \mathrm{Nec} = \mathsf{T}$. Once we have proven the above statements, we will apply them in this way to all of the standard conditions introduced previously. Now, the first of the above offset statements, which simply describes the behaviour of ω-fold applications of V-functions for standard conditions, can be proved as follows. First a lemma.

Proposition 8.1.33. Suppose C is ω-cumulative at Λ. Then $V_C^\omega(\Lambda)$ is a logic.

Proof. Straightforward. $\square$

Proposition 8.1.34. If C is standard, then where defined, $V_C^\omega(\Lambda) = V_C(\Lambda) + \mathrm{Nec}$.

Proof. ($\supseteq$) Since $V_C^\omega(\Lambda)$ is defined, C is ω-cumulative at Λ, so that $\bigcup_n V_C^n(\Lambda) = V_C^\omega(\Lambda)$ is a logic. From this, it is simple to show that it includes $V_C(\Lambda) + \mathrm{Nec}$. For $V_C^\omega(\Lambda)$ includes $V_C(\Lambda)$ by definition, and can be shown to admit Necessitation by an argument similar to the previous arguments for modus ponens and substitution. That is, if $\phi \in V_C^\omega(\Lambda)$, then there is some finite n such that $\phi \in V_C^n(\Lambda)$, so that therefore $\Box\phi \in \Box(V_C^n(\Lambda)) \subseteq V_C^{n+1}(\Lambda) \subseteq V_C^\omega(\Lambda)$. ($\subseteq$) The logic $V_C(\Lambda) + \mathrm{Nec}$ includes every $V_C^n(\Lambda)$ for $n \in \omega$. For it includes $V_C^1(\Lambda) = V_C(\Lambda) = V_C(\mathrm{PC}) + \Box(\Lambda)$, and for every $k \in \omega$, if it includes $V_C^k(\Lambda)$, it must also include $V_C^{k+1}(\Lambda) = V_C(\mathrm{PC}) + \Box(V_C^k(\Lambda))$ thanks to admitting Necessitation. Therefore where defined, $\bigcup_{n\in\omega} V_C^n(\Lambda) = V_C^\omega(\Lambda)$, is included in $V_C(\Lambda) + \mathrm{Nec}$. $\square$

Since this result only describes the behaviour of ω-fold applications of V_C for standard C, there remains a question about the behaviour of β-fold applications for transfinite $\beta > \omega$. This is a question settled by the second offset statement given above; if $V_C^\omega(\Lambda)$ is a V_C-fixed-point where defined, this means that β-fold applications of V_C are always redundant when $\beta > \omega$, since in such cases we simply have $V_C^\beta(\Lambda) = V_C^\omega(\Lambda)$. In the case of generation, this just means that no new logics are generated by a condition C after $V_C^\omega(\mathrm{PC})$. In order to prove this, we will make use of the following two lemmas.

Proposition 8.1.35. If C is standard then: for all Λ, $V_C(\Lambda) = V_C(\mathrm{PC}) + \Box(\Lambda)$.

Proof. For some Λ_0, $V_C(\Lambda) = \Lambda_0 + \Box(\Lambda)$ and $V_C(\mathrm{PC}) = \Lambda_0 + \Box(\mathrm{PC})$. So since $\Box(\mathrm{PC}) \subseteq \Box(\Lambda)$, we have $V_C(\mathrm{PC}) + \Box(\Lambda) = \Lambda_0 + \Box(\mathrm{PC}) + \Box(\Lambda) = \Lambda_0 + \Box(\Lambda) = V_C(\Lambda)$. $\square$

Proposition 8.1.36. Suppose that Λ' is a logic and that $\{\Lambda_n\}_{n\in\omega}$ a set of logics such that $\Lambda_n \subseteq \Lambda_m$ whenever $n \leq m$. Then we have:

$$\bigcup_n(\Lambda' + \Box(\Lambda_n)) = \Lambda' + \Box(\bigcup_n \Lambda_n)$$

Proof. ($\supseteq$) By the fact that $\{\Lambda_n\}_{n\in\omega}$ a set of logics such that $\Lambda_n \subseteq \Lambda_m$ whenever $n \leq m$, it follows that $\{\Lambda' + \Box(\Lambda_n)\}_{n\in\omega}$ is a logic. Additionally, it is clear that $\bigcup_n(\Lambda' + \Box(\Lambda_n))$ includes Λ', and it can be shown to also include $\Box(\bigcup_n \Lambda_n)$. For if $\phi \in \bigcup_n \Lambda_n$, then for some $k \in \omega$, $\phi \in \Lambda_k$, which means that $\Box\phi \in \Box(\Lambda_k) \subseteq (\Lambda' + \Box(\Lambda_k)) \subseteq \bigcup_n(\Lambda' + \Box(\Lambda_n))$. ($\supseteq$) It is clear that for each n, $(\Lambda' + \Box(\Lambda_n))$ will be included in $\Lambda' + \Box(\bigcup_n \Lambda_n)$, since the latter includes both Λ' and $\Box(\Lambda_n)$ for each n. Therefore $\bigcup_n(\Lambda' + \Box(\Lambda_n)) \subseteq \Lambda' + \Box(\bigcup_n \Lambda_n)$. ☐

With these in hand we may prove our desired result as follows.

Proposition 8.1.37. For standard C and all Λ, $V_C^\omega(\Lambda)$ is a V_C-fixed-point if defined.

Proof. Assuming that $V_C^\omega(\Lambda)$ is defined we proceed as follows. Where we fix n as a variable ranging over the finite ordinals:

$$V_C^\omega(\Lambda) = \bigcup_n V_C^n(\Lambda) = V_C^0(\Lambda) \cup \bigcup_{n\geq 1} V_C^n(\Lambda) = \bigcup_{n\geq 1} V_C^n(\Lambda)$$

The final step in this manipulation comes by the fact that, by ω-cumulativity, $V_C^0(\Lambda) = \Lambda \subseteq \bigcup_{n\geq 1} V_C^n(\Lambda)$. Next we have, by simple manipulations and an application of Prop. 8.1.35,

$$\bigcup_{n\geq 1} V_C^n(\Lambda) = \bigcup_n V_C^{n+1}(\Lambda) = \bigcup_n V_C(V_C^n(\Lambda)) = \bigcup_n(V_C(\mathrm{PC}) + \Box(V_C^n(\Lambda)))$$

Then by applying Prop. 8.1.36, we have

$$\bigcup_n(V_C(\mathrm{PC}) + \Box(V_C^n(\Lambda))) = V_C(\mathrm{PC}) + \Box(\bigcup_n V_C^n(\Lambda))$$

But since $\bigcup_n V_C^n(\Lambda)$ just is $V_C^\omega(\Lambda)$, this means that:

$$V_C(\mathrm{PC}) + \Box(\bigcup_n V_C^n(\Lambda)) = V_C(\mathrm{PC}) + \Box(V_C^\omega(\Lambda)) = V_C^{\omega+1}(\Lambda)$$

So that, putting this all together, we get

$$V_C^\omega(\Lambda) = V_C^{\omega+1}(\Lambda)$$

Which means that $V_C^\omega(\Lambda)$ is a V_C-fixed-point. ☐

In order to apply these results in the desired way to the standard conditions introduced previously, we must first show that for each condition C, the transfinite application $V_C^\omega(\mathrm{PC})$ exists:

Proposition 8.1.38. If C is standard, then C is ω-cumulative at PC.

Proof. It can be seen immediately from the definition of a standard condition, or alternatively from Prop. 8.1.35, that whenever we have $\Lambda \subseteq \Lambda'$ we also have $V_C(\Lambda) \subseteq V_C(\Lambda')$. In other words, when C is standard, V_C is monotonic. From this, since $\mathrm{PC} \subseteq V_C(\mathrm{PC})$, we have $V_C^n(\Lambda) \subseteq V_C^{n+1}(\mathrm{PC})$ for every $n \in \omega$, meaning that C is ω-cumulative at PC. $\square$

Finally then, we can apply these results to the standard conditions introduced previously:

Proposition 8.1.39. Combining Propositions 8.1.21 and 8.1.34:

$V^\omega_{\mathbf{Mod}}(\mathrm{PC}) = \mathsf{K} \quad V^\omega_{Con}(\mathrm{PC}) = \mathsf{D}$

Then, combining Propositions 8.1.24 and 8.1.34:

(i) $V^\omega_{Fac}(\mathrm{PC}) = \mathsf{T}$ (ii) $V^\omega_{\Box Cl}(\mathrm{PC}) = \mathsf{K4}$
(iii) $V^\omega_{BCl}(\mathrm{PC}) = \mathsf{KB}$ (iv) $V^\omega_{\Diamond Cl}(\mathrm{PC}) = \mathsf{K5}$
(v) $V^\omega_{DCl}(\mathrm{PC}) = \mathsf{K45}$ (vi) $V^\omega_{\Box CC}(\mathrm{PC}) = \mathsf{KD4}$
(vii) $V^\omega_{BCC}(\mathrm{PC}) = \mathsf{KDB}$ (viii) $V^\omega_{\Diamond CC}(\mathrm{PC}) = \mathsf{KD5}$
(ix) $V^\omega_{DCC}(\mathrm{PC}) = \mathsf{KD45}$ (x) $V^\omega_{\Box CF}(\mathrm{PC}) = \mathsf{S4}$
(xi) $V^\omega_{BCF}(\mathrm{PC}) = \mathsf{B}$ (xii) $V^\omega_{\Diamond CF}(\mathrm{PC}) = \mathsf{S5}$
(xiii) $V^\omega_{DCF}(\mathrm{PC}) = \mathsf{S5}$

Also, by Prop 8.1.37, $V_C^\beta(\mathrm{PC}) = V_C^\omega(\mathrm{PC})$ for all $\beta > \omega$ for each of these conditions C.

I will conclude the discussion of correspondence theory here. To recap the main line of discussion: many naturally specified conditions C on modalities turn out to be what we have termed 'standard', in the sense that there is some logic Λ_0 such that for any Λ, the set of $(\{\Lambda\}, C(\Lambda))$-valid formulas $V_C(\Lambda) = \Lambda_0 + \Box(\Lambda)$. If 'correspondance' between a logic Λ and a condition C is only understood as Λ being sound and complete for $(\{\Lambda'\}, C(\Lambda'))$-validity for some logic of evaluation Λ'—in other words, as Λ simply being some value of the function V_C—then standard conditions often correspond to very broad and disunified classes of logics. One solution to this problem which I have just explored is to consider stricter notions of correspondance, and to this end I have proposed generation as one such notion. As we have seen, the use of generation as a tighter notion of correspondence between conditions on modalities and logics allows us to specify much narrower and more unified classes of logics as corresponding to standard conditions. For example, the logics generated by the condition Fac are all extensions of $\mathsf{S0.5}$ and are all included in T—the only differences between successive logics generated by Fac concern the degree to which Necessitation is admitted, in that each $V_{Fac}^{n+1}(\mathrm{PC})$ only adds the necessitations $\Box(V_{Fac}^{n}(\mathrm{PC}))$ of every theorem from the previous logic $V_{Fac}^{n}(\mathrm{PC})$.

There is no need to think that generation is the only interesting, stricter notion of correspondence between logics and conditions available in this semantics. Generation seems an especially natural kind of correspondence in the context of discussing standard conditions, but there may be other equally natural notions of correspondence to be identified. In addition, in analysing conditions that are non-standard, or more specifically which fail to be ω-cumulative at PC,[10] it seems possible that alternative notions of correspondence will prove more appropriate and interesting.

8.2 The Logic of Syntactic Logical Truth

Now that we have developed a semantic framework adequate for the syntactic logicist, we can apply it to the question of the logic of syntactic logical truth—the logic of absolute necessity, according to the syntactic logicist. It is at this point that we must recall a basic interpretive difficulty raised at the start of this chapter. To be a syntactic logical truth is to be a theorem of a given logical system. But there there are many logical systems, even in a simple language like PML; hence there are, in a sense, many distinct properties of syntactic logical truth. For instance, we have the property of being a logical truth with respect to PC, to S5, and in general the property of being a logical truth with respect to Λ for each logic Λ. Here then is the difficulty for our syntactic logicist. Since she is a monist she says that there is just one absolute necessity, and she says that absolute necessity is syntactic logical truth—without any further qualification, *full stop* as it were. So she must think that, apart from the concept of logical-truth-with-respect-to-Λ for a specified logic Λ, there is a concept of some unique privileged variety of syntactic logical truth, which turns out to be identical to absolute necessity. We might call this syntactic logical truth *tout court*.[11] In PML, for instance, she must say that there is some unique logic Λ such that for a formula ϕ to be a theorem of Λ is for ϕ to be absolutely necessary—which is to say, a syntactic logical truth *tout court*. But how can the syntactic logicist make a principled choice here? What reasons can she give as to why Λ is special in this regard? We will detail the answer as we progress.

First, though, we must focus in a general way on modal logics which encode a reading of $\Box\phi$ merely as "ϕ is a theorem of Λ" for some given logic Λ. We will see that for any logic Λ, it is possible to define a logic of logical-truth-with-respect-to-Λ, or a logic of Λ-theoremhood. For instance it is possible to define a logic of PC-theoremhood, which encodes a reading of $\Box\phi$ as

[10] Which includes functions like L_α defined in the next section.

[11] It seems reasonable to identify this idea with that of 'demonstrability' as discussed in Burgess [7].

"ϕ is a tautology"—this logic turns out to be TS, which we have already encountered. Once we have shown generally how to define the logic of Λ-theoremhood for each Λ, we will show how a syntactic logicist can use this to try to pin down the logic of syntactic logical truth *tout court*. Of particular interest here will be logics that are 'fixed points', in the sense that they serve as their own logic of theoremhood. I claim that if any sense can be made of syntactic logical truth *tout court* then, given the constancy hypothesis, such fixed points provide the best way of pinning down the characteristics of this kind of logical truth. Interestingly, it turns out that such fixed points also provide us a link back to the McKinsey semantics, which shall be explained as we proceed. Now, to characterise logics of syntactic logical truth we consider a particular modality α:

Definition 8.2.1. α is the modality such that $\alpha(m) = \varnothing$ for all models m.

I use the symbol 'α' in analogy to the modality on Boolean algebras discussed in earlier chapters. In considering the types of validity $V_\alpha(\Lambda)$ which result from fixing α as the modality of evaluation, one can see that we get the following for all m, Λ, and ϕ:

$m \vDash_{\Lambda,\alpha} \Box\phi$ if and only if $\vdash_\Lambda \phi$.

Hence for any logic Λ, we have $\Box\phi \in V_\alpha(\Lambda)$ iff $\vdash_\Lambda \phi$ and $\neg\Box\phi \in V_\alpha(\Lambda)$ iff $\nvdash_\Lambda \phi$. The set $V_\alpha(\Lambda)$ contains precisely those formulas which are true in all models given the reading of $\Box$ as 'is is a theorem of Λ that…', but in general, $V_\alpha(\Lambda)$ cannot itself be considered the logic of logical-truth-in-Λ. This is simply because if Λ is consistent, $V_\alpha(\Lambda)$ is not a logic: in such cases, $\nvdash_\Lambda p$ but $\vdash_\Lambda \top$, so that $\neg\Box p \in V_\alpha(\Lambda)$ even though $\neg\Box\top \notin V_\alpha(\Lambda)$, which means a failure of uniform substitution. On the other hand, $V_\alpha(\Lambda)$ contains all of PC and is closed under modus ponens (proofs trivial), so that we can plausibly arrive at the correct logic of logical truth for Λ from $V_\alpha(\Lambda)$ as follows. Using the notation ℓ (Def. A.1.8) we have $\ell V_\alpha(\Lambda)$ as the largest logic with respect to inclusion that is included in $V_\alpha(\Lambda)$ (by Prop. A.1.9). Hence, $\ell V_\alpha(\Lambda)$ is the logic which is correct and *as complete as possible* for the intended interpretation of $\Box$ as 'it is a theorem of Λ that…'. We introduce some cleaner notation at this point:

Definition 8.2.2. Where C is a condition on modalities, let $L_C(\Lambda) := \ell V_C(\Lambda)$.

Thus we arrive at the function L_α, and the logics $L_\alpha(\Lambda)$ for each system of propositional modal logic Λ. For any logic Λ, I claim that $L_\alpha(\Lambda)$ is the propositional modal logic of logical truth for Λ—because it is the logic which encodes, correctly and as fully as possible, the reading of $\Box$ as 'it is a theorem of Λ that…'.

With these logics in hand I wish now to show how a syntactic logicist can try to bridge the gap between our understanding of logical truth for a given logic Λ and the more obscure notion of syntactic logical truth *tout court*. In the following section I will argue for two theses which jointly determine how the behaviour of L_α bears on how the syntactic logicist ought to characterise syntactic logical truth *tout court*.

8.2.1 Constancy

I claim that, for the syntactic logicist, the identity of the logic of syntactic logical truth *tout court* depends crucially on whether the connective 'it is logically necessary that...' is taken as a logical constant. In other words, whether constancy holds. In this section I will argue for the following pair of theses, which relate constancy, the logic of (syntactic) logical truth *tout court*, and L_α.

> (1) If constancy fails, the logic of logical truth *tout court* is $L_\alpha(\mathrm{PC})$.
> (2) If constancy holds, the logic of logical truth *tout court* is a fixed point of L_α.

My arguments in each case rely on the following hypothesis:

> $(*)$ The (propositional) logic of logical truth is $L_\alpha(\Lambda)$ for some logic Λ.

This thesis establishes the requisite link between particular logics of logical truth on the one hand, and the logic of syntactic logical truth *tout court* on the other. I will argue for $(*)$ first, and then use this hypothesis to argue for (1) and (2). The argument for $(*)$ relies on the observation that for each PML logic Λ, $L_\alpha(\Lambda)$ is the logic of logical truth for the logic Λ, as we said before. This being the case, we can reason as follows. First, assuming an intended reading of $\Box$ as 'it is logically true (*tout court*) that...' the set of PML formulas which are logical truths *tout court* under this reading must be a logic—for all classical tautologies are logical truths, and logical truth *tout court* is plausibly closed under both modus ponens and uniform substitution.[12] Therefore, if Λ is the set of logical truths *tout court*, then for a formula ϕ to be a logical truth *tout court* is for it to be a theorem of Λ. But this means that the logic of logical truth for Λ, $L_\alpha(\Lambda)$, is also the propositional logic of logical truth *tout court*.

[12] That uniform substitution obtains here is perhaps less obvious than for modus ponens, but we can argue for it in a couple of ways. First, the intended role of sentence letters $p, q, r, \ldots$ is to act as variables, ranging over all formulas in the language. By this role, to deny that uniform substitution holds for logical truth is simply to give the sentence letters an alternate role—in a way, to change the subject. As a more substantial argument: even if one is not working with a strictly substitutional definition of logical truth, many will want to say that for any ϕ, ϕ is logically truth *only if* all admissible substitution instances of ϕ are true. Suppose that this is so. Then assuming that all uniform substitution instances of a formula are admissible, it follows that if ϕ is logically true, then every uniform substitution instances of ϕ is logically true.

With ($*$) in hand, theses (1) and (2) are not difficult to justify. Taking the first of these first, (1) says that if constancy fails, then the propositional logic of logical truth coincides with the logic of logical truth for PC. Since PC is simply the set of tautologies formulable in PML, this logic is also the logic of tautological validity—i.e. the logic which encodes the reading of $\Box$ as 'it is a tautology that...'. The argument here is again fairly simple: if $\Box$ is not a logical constant, then the only logical constants of PML are the truth-functional connectives (including $\to$ and $\bot$). Now, logical truths should be true as a matter of logic alone, meaning that the only logical necessities of PML should be tautologies: formulas which are true in virtue of the behaviour of the remaining, truth-functional logical constants. So if constancy fails, the set of tautologies PC is the set of logically true formulas, meaning that $L_\alpha(\mathrm{PC})$, as the logic of logical truth for PC, is also the propositional logic of syntactic logical truth *tout court*. As for thesis (2), the argument here is slightly more complicated. It relies on two additional premises, plus some elementary results about L_α. The premises are, first, that the logic of logical truth *tout court* should contain the T axiom and, second, that if constancy holds, it should admit Necessitation. The first premise needs little justification: presumably, whatever is logically true *tout court* is true, so the T axiom $\Box p \to p$ should feature in the logic of logical truth *tout court*. The second premise is also very intuitive. The propositional logic of logical truth *tout court* is the PML logic which is correct and complete for the reading of $\Box$ as 'it is logically true *tout court* that...'. If constancy holds, then theorems of the logic of logical truth are themselves logical truths, since they involve only the behaviour of logical constants. Therefore if ϕ is a theorem of the logic of logical truth, then ϕ is a logical truth, so that $\Box\phi$ should also be a theorem of the logic of logical truth. Given that T and Necessitation are admitted by the logic of logical truth *tout court*, we can reason to (2) by ($*$) and by the following results.

Proposition 8.2.3. $\phi \in \Lambda$ iff $\Box\phi \in L_\alpha(\Lambda)$.

Proof. If $\phi \in \Lambda$, then for all models m, $m \vDash_{\Lambda,\alpha} \Box\phi$, so that $\Box\phi \in V_\alpha(\Lambda)$. By the same token, since Λ is closed under uniform substitution then, if $\phi \in \Lambda$ then for all uniform substitution instances ϕ' of ϕ, $\Box\phi' \in V_\alpha(\Lambda)$. So: $\Box\phi \in L_\alpha(\Lambda)$. Conversely, if $\Box\phi \in L_\alpha(\Lambda) \subseteq V_\alpha(\Lambda)$ then for all models m, $m \vDash_{\Lambda,\alpha} \Box\phi$, which is so iff $\vdash_\Lambda \phi$, iff $\phi \in \Lambda$. $\Box$

Proposition 8.2.4. If $L_\alpha(\Lambda)$ contains T and admits Necessitation, $\Lambda = L_\alpha(\Lambda)$.

Proof. By Proposition 8.2.5, $\phi \in \Lambda$ iff $\Box\phi \in L_\alpha(\Lambda)$. If $L_\alpha(\Lambda)$ contains T and admits Necessitation, then for all ϕ, $\phi \in L_\alpha(\Lambda)$ iff $\Box\phi \in L_\alpha(\Lambda)$, from which we get $\phi \in \Lambda$ iff $\phi \in L_\alpha(\Lambda)$. $\Box$

On its face, (2) might look like a far weaker claim than (1), which pins down a particular logic. For all it says is that if constancy holds, the logic of logical truth is just one member of a certain class of logics. But in fact, the property of being a fixed point of L_α is highly interesting and rare. As I will show below, a lot can be derived from the assumption that a certain logic is a fixed point of L_α and, though we can show that there is at least one such fixed point (in particular, Med is such a logic), it is a nontrivial question whether there is more than one.[13]

8.2.2 The Logic of Tautological Validity

Before I say more about fixed points of L_α, however, let us say a little more about the logic $L_\alpha(\mathrm{PC})$ singled out by thesis (1). On inspection we can quickly see that this logic is identical with the system TS of Urquhart [67], which we have already encountered.[14]

We could move on at this point, though there is an interesting point of contention here that I do not want to gloss over. I have claimed that for any logic Λ, $L_\alpha(\Lambda)$ is the logic of Λ-theoremhood, or logical-truth-according-to-Λ. So in particular I claim that $L_\alpha(\mathrm{PC}) = \mathsf{TS}$ is the logic of PC-theoremhood, meaning that it is the logic which best encodes the reading of $\Box$ as 'it is a tautology that...'. This places me in a debate between Sylvan and Cresswell on the logic of tautological validity. In [37], E. J. Lemmon introduced the system $\mathsf{S0.5}$, with his own intended reading for it as the logic of the connective 'it is a tautology that...'. Later Sylvan (then Routley, in [55] for instance) cast doubt on $\mathsf{S0.5}$ being the true logic of tautological validity. He provided a semantics for a system very much like TS—though in fact his system was equal to our set $V_\alpha(\mathrm{PC})$, which is not a logic by our definition, since it does not admit uniform substitution. Therefore among other things, Sylvan took $\Diamond\Box p$ and $\Diamond\Diamond p$ to be theorems of this logic. Now, in response to this and to others, Cresswell [17] tried to give arguments that in fact systems like Sylvan's were far too strong, and that $\Diamond\Diamond p$ and other such formulas should *not* be part of the logic of tautological validity. Nevertheless, in [67], we find Urquhart providing a system of tautological validity which validates the same controversial theses; his τ-semantics is essentially like Sylvan's, but ensures closure under

[13] Despite our arguments one might instead entertain the idea that the true object of talk of the logic of logical truth *tout court* must just the common part or intersection $\bigcap_\Lambda L_\alpha(\Lambda)$ of all the logics of logical truth. But this will not do; all writers seem agreed that the logic of derivability, or syntactic logical truth *tout court*, should contain the T axiom $\Box p \to p$, which is not contained in $\bigcap_\Lambda L_\alpha(\Lambda)$, since for instance it is not contained in $L_\alpha(\mathrm{PML}) = \mathsf{Ver}$.

[14] Urquhart's characterises TS as follows. Let a τ-assignment be a truth-value assignment which gives the normal valuations to truth-functional compounds and values $\Box\phi$ as true iff $\phi \in \mathrm{PC}$. Then ϕ is in TS iff every substitution instance of ϕ is true at all τ-assignments. Clearly Urquhart's τ-semantics is identical to the special case of our semantics where PC is fixed as logic of evaluation and α is fixed as modality of evaluation; being true at all τ-assignments is therefore the same as being a member of $V_\alpha(\mathrm{PC})$ and so being a member of TS is the same as being a member of $L_\alpha(\mathrm{PC})$.

uniform substitution, which is why his system is equal to $L_\alpha(\mathrm{PC})$. Now, in this disagreement, I clearly come down on the side of Sylvan and Urquhart. Though I think that Cresswell's argument is an interesting one, I do not think that it establishes that $\mathsf{S0.5}$ is really the logic of tautological validity. Simply put, if we are to discern the correct logic of tautological validity, the best that we can do is do define and investigate a semantics for PML in which $\Box\phi$ is true iff ϕ is in fact a tautology (indeed, to do anything *else* would seem in need of strong justification). But doing so, as we see from the above, we find TS to be vindicated rather than $\mathsf{S0.5}$. Therefore TS has the better systematic backing, and until a similarly natural and developed semantic argument can be made for $\mathsf{S0.5}$, then the claim that TS is the true logic of 'it is a tautology that...' should be accepted. Cresswell's arguments in [17] hint at the possibility of such an alternative argument, but are in need of development before they can be properly evaluated.

8.2.3 Fixed Points of L_α

If constancy holds, then by thesis (2) above we have that the propositional logic of syntactic logical truth *tout court* (and of absolute necessity, if the syntactic logicist is right) is some fixed point of L_α. This means that we must now investigate these fixed points. This topic turns out to connect back to the McKinsey semantics as described in the previous chapter in a substantial way. For it turns out that the fixed points of L_α are exactly what we might naturally term 'McKinsey logics' in the following sense.

Definition 8.2.5. For $\Delta \subseteq \mathrm{PML}$, a Δ-valuation is a valuation v such that for all ϕ:

$$v(\Box\phi) = 1 \text{ iff } \phi \in \Delta.$$

Definition 8.2.6. Λ is a McKinsey logic iff every Λ-valuation is McKinsey.

In what follows I will proceed to show that a logic is a fixed point of L_α iff it is McKinsey. On the path to this result I will show that any fixed point of L_α must exhibit a number of very strong properties. These include being normal extensions of $\mathsf{S4M}$, being structurally complete in the sense of Rybakov [58], and admitting a strong disjunction rule. Through this connection, it is possible to show the following: that there is at least one fixed point of L_α, that Med is the greatest fixed point of L_α, and that if the Uniqueness Conjecture (Conj 7.1.9) mentioned in the previous chapter is correct, then Med is the only fixed point of L_α.

So to start, note that we can provide an alternative characterisation of L_α in terms of Λ-valuations for a logic Λ. This will be helpful in shortening certain arguments later on:

Proposition 8.2.7. $\phi \in L_\alpha(\Lambda)$ iff for all Λ-valuations v and substitutions s, $v(s\phi) = 1$.

Proof. $\phi \in L_\alpha(\Lambda)$ iff for all substitutions s and all models m, $m \vDash_{\Lambda,\alpha} s\phi$. Now consider that we can draw a one-one correspondance between models and subvaluations as follows. Where m is a model, let v_m^- be the unique subvaluation such that $v_m^-(p) = 1$ iff $p \in m$ for all $p \in \text{At}$. Then let v_m be the unique Λ-valuation extending v_m^-. It should be clear that the valuations v_m make up all of the Λ-valuations, since the subvaluations v_m^- make up all of the subvaluations and only one Λ-valuations extends any given subvalution. In this case we may note (by comparing the semantic clauses in the case of $\vDash_{\Lambda,\alpha}$ with the properties of Λ-valuations) that for all m and ϕ, $m \vDash_{\Lambda,\alpha} \phi$ iff $v_m(\phi) = 1$. So in particular, for any ϕ and any substitution s, we have $m \vDash_{\Lambda,\alpha} s\phi$ iff $v_m(s\phi) = 1$. Therefore we have $m \vDash_{\Lambda,\alpha} s\phi$ for all m s and ϕ iff we have $v_m(s\phi) = 1$ for all m, s and ϕ. But since the valuations v_m make up all of the Λ-valuations, we have our result. $\square$

With this hand, we can immediately show the following:

Proposition 8.2.8. Every McKinsey logic is a fixed point of L_α.

Proof. Suppose that Λ is a McKinsey logic. If $\phi \in \Lambda$, then for all Λ-valuations v we have $v(\Box\phi) = 1$. Since each Λ-valuation is a Mckinsey valuation, then for all Λ-valuations v and for all substitutions s, $v(s\phi) = 1$. Thus $\phi \in L_\alpha(\Lambda)$. Conversely, if $\phi \in L_\alpha(\Lambda)$ then for all substitutions s and all Λ-valuations v we have $v(s\phi) = 1$. Since each v here is a McKinsey valuation, $v(\Box\phi) = 1$, and so $\phi \in \Lambda$ since each v is a Λ-valuation. So $\Lambda = L_\alpha(\Lambda)$. $\square$

This implies that $L_\alpha(\Lambda)$ is a fixed point. Bacon and Fine have shown ([4], 13-14) that any subvaluation v^- can be extended to a McKinsey valuation which is also a Med-valuation. Now it is clear that a given subvalution v^- is extended by *at most one* Δ-valuation for any Δ. Hence, it follows that every Med-valuation is a McKinsey valuation, so that Med is McKinsey. So we have:

Proposition 8.2.9. Med is a McKinsey logic.

Corollary 8.2.10. $L_\alpha(\mathsf{Med}) = \mathsf{Med}$.

Now we know that the class of fixed points of L_α is nonempty, it remains to establish some restrictions on this class. We will show first of all that any fixed point of L_α is a normal extension of $\mathsf{S4M}$, admits a certain strong disjunction rule, and is structurally complete. We then use these to show that every fixed point of L_α is a McKinsey logic. We start with:

Proposition 8.2.11. If $L_\alpha(\Lambda) = \Lambda$, then Λ is a normal extension of $\mathsf{S4M}$.

Proof. The characteristic axioms of S4M are K, T, 4, and McKinsey's axiom Mc = $\Box\Diamond p \to \Diamond\Box p$. Each of these is admitted in any fixed point $L_\alpha(\Lambda)$, along with necessitation:

(K) Recall that every instance of K is generally valid, and therefore every instance of K is in $V_\alpha(\Lambda)$. Hence K $\in L_\alpha(\Lambda)$.

(T) Suppose that $m \vDash_{\Lambda,\alpha} \Box\phi$, then $\phi \in \Lambda = L_\alpha(\Lambda) \subseteq V_\alpha(\Lambda)$, meaning that we must also have $m \vDash_{\Lambda,\alpha} \phi$ and hence $m \vDash_{\Lambda,\alpha} \Box\phi \to \phi$. Since every instance of T is in $V_\alpha(\Lambda)$, T $\in L_\alpha(\Lambda)$.

(Nec) Immediate from Proposition 8.2.3.

(4) Suppose that $m \vDash_{\Lambda,\alpha} \Box\phi$, then $\phi \in \Lambda$. Since Λ is normal, $\Box\phi \in \Lambda$ and so $m \vDash_{\Lambda,\alpha} \Box\Box\phi$, meaning we have $m \vDash_{\Lambda,\alpha} \Box\phi \to \Box\Box\phi$. Since every instance of 4 is in $V_\alpha(\Lambda)$, 4 $\in L_\alpha(\Lambda)$.

This shows that any fixed point must be a normal extension of S4. Any such logic must be contained in Triv, and so must be such that $\vdash_\Lambda \Diamond\phi \Rightarrow \nvdash_\Lambda \Diamond\neg\phi$. We use this in establishing McKinsey's axiom.

(Mc) Suppose that $m \vDash_{\Lambda,\alpha} \Box\Diamond\phi$. Then $\Diamond\phi \in \Lambda$, meaning that $\Diamond\neg\phi \notin \Lambda$. So $m \vDash_{\Lambda,\alpha} \neg\Box\Diamond\neg\phi$, and so $m \vDash_{\Lambda,\alpha} \Diamond\Box\phi$. Hence $m \vDash_{\Lambda,\alpha} \Box\Diamond\phi \to \Diamond\Box\phi$. Since every instance of Mc is thus in $V_\alpha(\Lambda)$, Mc $\in L_\alpha(\Lambda)$.

Which completes the proof. $\Box$

Next I will show that every fixed point admits a strong kind of disjunction rule.[15] Borrowing terminology from Bacon and Fine [4], I call this the *extended* disjunction rule.

Definition 8.2.12. A logic Λ admits the extended disjunction rule iff for all $\phi_0, \ldots, \phi_n$ where ϕ_0 is nonmodal: if $\vdash_\Lambda \phi_0 \vee \Box\phi_1 \ldots \vee \Box\phi_n$, then $\vdash_\Lambda \phi_k$ for some k with $0 \leq k \leq n$.

In [4], it is shown that a normal logic admits the extended disjunction property iff it is what modal logicians have sometimes called *coherent*. The usual definition of coherence, originated by Meyer [43], says that a logic Λ is coherent just in case for any $\phi \in \Lambda$ we have $v(s\phi) = 1$ for any Λ-valuation v and any substitution s. A shorter and evidently equivalent definition goes as follows:

[15] Though tangential to the present discussion, it is interesting to note that what we have said up to this point rules out the possibility that the logic of logical truth (understood syntactically, as opposed to model-theoretically) is S5—most importantly, it rules out the possibility that $\Box\Diamond p \vee \Box\neg\Diamond p$ is a theorem of the logic of logical truth. If constancy fails then the logic of logical truth is $L_\alpha(\mathrm{PC})$, but $\Box\Diamond p \vee \Box\neg\Diamond p$ is $L_\alpha(\mathrm{PC})$-inconsistent. If constancy holds then the logic of logical truth is a fixed point of L_α, but no fixed point of L_α contains the formula either. Every fixed point is a normal extension of S4M, but the only normal extension of S4M which contains $\Box\Diamond p \vee \Box\neg\Diamond p$ is Triv, and the reader can quickly verify that Triv is not a fixed point itself since $p \to \Box p \notin L_\alpha(\mathsf{Triv})$. This is of interest since it answers a recent argument by Pruss [48] against the identification of necessity with proof-theoretic logical truth, which makes essential use of $\Box\Diamond p \vee \Box\neg\Diamond p$ as a principle which is supposed to govern syntactic logical truth.

Definition 8.2.13. Λ is coherent iff $\Lambda \subseteq L_\alpha(\Lambda)$.

Proposition 8.2.14. If $L_\alpha(\Lambda) = \Lambda$, then Λ admits the extended disjunction rule.

Proof. Every fixed point of L_α is coherent by definition and normal by Proposition 8.2.11. Hence, by Proposition 37 of [4], which says that a normal logic is coherent iff it admits the extended rule of disjunction, every fixed point of L_α admits the rule. $\square$

Finally, I will show that every fixed point of L_α is structurally complete. This concept is taken from the theory of admissible inference rules as developed for instance in Rybakov [58], and is defined in the following way.[16] First,

Definition 8.2.15. A rule is a pair of formulas $\rho = (\phi, \psi)$.

Definition 8.2.16. A rule (ϕ, ψ) is Λ-admissible iff for all s, if $\vdash_\Lambda s\phi$ then $\vdash_\Lambda s\psi$.

Next we define a relation of Λ-derivability for normal logics Λ as follows.

Definition 8.2.17. Where Λ is normal, a rule ρ is Λ-derivable iff for any substitution s, there exists a finite sequence of formulas $(\chi_0, ..., \chi_n)$ such that χ_0 is $s\phi$, χ_n is $s\psi$, and for every χ_k with $1 \leq k \leq n$, either:

$\chi_k \in \Lambda$,
χ_k is $\Box\chi_j$ for some j with $j < k$, or
There are χ_j, χ_i with $j, i < k$ such that χ_j is $(\chi_i \to \chi_k)$.

In other words, a rule (ϕ, ψ) is derivable in the normal logic Λ just in case for every substitution s, $s\psi$ can be proved from $s\phi$ in the Hilbert-style proof system whose axioms are just the theorems of Λ and whose primitive inference rules are just modus ponens and necessitation. Clearly, where Λ is normal, every Λ-derivable rule is Λ-admissible; the converse of this is structural completeness:

Definition 8.2.18. Λ is structurally complete iff all Λ-admissible rules are Λ-derivable.

We may prove the desired result as follows.

Proposition 8.2.19. If $L_\alpha(\Lambda) = \Lambda$, then Λ is structurally complete.

[16] In Rybakov's terminology, this would specifically be structural completeness *in the finitary sense*, and the 'rules' we mention are finitary rules, strictly speaking.

Proof. Suppose that $\Lambda = L_\alpha(\Lambda)$, and that $\rho = (\phi, \psi)$ is Λ-admissible. We first show that for any substitution s, the formula $\Box s\phi \to \Box s\psi$ is in Λ. Since $\rho = (\phi, \psi)$ is Λ-admissible, for any s, if $\vdash_\Lambda s\phi$ then $\vdash_\Lambda s\psi$. Therefore for any model m, we have $m \vDash_{\Lambda,\alpha} \Box s\phi \to \Box s\psi$ so that $\Box s\phi \to \Box s\psi \in V_\alpha(\Lambda)$ for any substitution s. Additionally, however, wherever t is a substitution, $t(\Box s\phi \to \Box s\psi) = (\Box ts\phi \to \Box ts\psi)$. Since substitutions compose, ts is a substitution, and so by the previous argument, for all t and s, $t(\Box s\phi \to \Box s\psi) = (\Box ts\phi \to \Box ts\psi) \in V_\alpha(\Lambda)$. Hence for any substitution s, $(\Box s\phi \to \Box s\psi) \in L_\alpha(\Lambda) = \Lambda$. Having shown this, for any s we can construct a suitable Hilbert style proof in Λ as follows. Note that Λ contains the T axiom by Proposition 8.2.11, since it is a fixed point of L_α.

1.	$s\phi$	
2.	$\Box s\phi$	1, Necessitation
3.	$\Box s\phi \to \Box s\psi$	Λ-theorem
4.	$\Box s\psi$	2, 3, Modus Ponens
5.	$\Box s\psi \to s\psi$	Λ-theorem
6.	$s\psi$	4, 5, Modus Ponens

Hence ρ is Λ-derivable, which completes the proof. □

So far I have shown that every fixed point of L_α is a normal extension of S4M, structurally complete, and admits the extended disjunction rule. These properties will now be used to show that the McKinsey logics are precicely the fixed points of L_α. To arrive at this we make use of the following result proved by Jeřábek [31].

Proposition 8.2.20 (Jeřábek)**.** A valuation v is McKinsey iff v is a Λ-valuation for Λ a structurally complete, normal extension of S4 which admits the disjunction rule.

Here by the 'disjunction rule' is meant the ordinary disjunction rule familiar from Lemmon and Scott [40]. That is: Λ admits the disunction rule iff for any $\phi_1, ..., \phi_n$, if $\vdash_\Lambda \Box\phi_1 \vee ... \vee \Box\phi_n$ then $\vdash_\Lambda \phi_i$ for some $i \leq n$. We use the following consequence of Jeřábek's result.

Proposition 8.2.21. Λ is a McKinsey logic iff Λ is a structurally complete, normal extension of S4 which admits the disjunction rule.

Proof. Straightforward by Proposition 8.2.20. □

Every consistent logic that admits the extended disjunction rule admits the disjunction rule: for any $\phi_1, ..., \phi_n$, if $\vdash_\Lambda \Box\phi_1 \vee ... \vee \Box\phi_n$ then additionally $\vdash_\Lambda \bot \vee \Box\phi_1 \vee ... \vee \Box\phi_n$—here, $\bot$ serves as the nonmodal formula required by the extended disjunction rule. So by the extended disjunction rule either $\vdash_\Lambda \bot$ or $\vdash_\Lambda \phi_i$ for some $i \leq n$. So if Λ is consistent, it follows from all this that $\vdash_\Lambda \phi_i$ for some $i \leq n$, as required. In addition to the fact that every logic with the extended disjunction rule has the disjunction rule, any normal extension of $\mathsf{S4M}$ is also a normal extension of $\mathsf{S4}$. So by use of Jeřábek's result:

Proposition 8.2.22. Every fixed point of L_α is a McKinsey logic.

And combining this with Proposition 8.2.10 we get:

Proposition 8.2.23. The fixed points of L_α are exactly the McKinsey logics.

We can use this to draw some further corollaries. In the first place, it means there are multiple L_α fixed points iff there are multiple McKinsey logics. But it is quick to show that:

Proposition 8.2.24. The Uniqueness Conjecture holds iff there is one McKinsey logic.

Proof. If there is just one McKinsey logic Λ, then any subvaluation v^- will have its unique Λ-extension as its sole McKinsey extension, so that Uniqueness holds. Conversely, if any subvaluation v^- has two McKinsey extensions v_1 and v_2, then there are at least two McKinsey logics, these being $\{\phi : v_1(\Box\phi) = 1\}$ and $\{\phi : v_2(\Box\phi) = 1\}$, by Prop 8.2.22. □

Corollary 8.2.25. The Uniqueness Conjecture holds iff there is one fixed point of L_α.

This leaves us, and our syntactic logicist, in an interesting situation. Recall that originally, we faced our syntactic logicist with the challenge of specifying a logic Λ such that to be a logical truth according to Λ is to be logically true *tout court* and hence absolutely necessary. If it turns out that there is a unique fixed point of L_α, then the syntactic logicist can meet our challenge, at least in the propositional case. If she denies constancy, then by thesis (1) this logic is PC, and so the logic of absolute necessity / logical truth *tout court* is $L_\alpha(\mathrm{PC}) = \mathsf{TS}$. If she accepts constancy, then by thesis (2) and by the uniqueness of fixed points, this logic is Med and the logic of absolute necessity / logical truth *tout court* is also $L_\alpha(\mathsf{Med}) = \mathsf{Med}$.

But we are not currently in a position to show that there is one fixed point of L_α. To do so we would need to establish the Uniqueness Conjecture, which logicians have so far been unable to do. Once the Uniqueness Conjecture is proven, the syntactic logicist will be able to meet our challenge, but what may she say in the meantime? Or alternatively: suppose against current expectation that the Uniqueness Conjecture is false, meaning that there are at least two fixed points of L_α. Would this situation mean disaster for the syntactic logicist, or would she still have some way of responding to our challenge? I think that she would still have a way to meet our challenge, even if there turned out to be more than one fixed point of L_α. This is because we can show the following, independently of the uniqueness conjecture.

Proposition 8.2.26. If $\Lambda = L_\alpha(\Lambda)$ then $\Lambda \subseteq \mathsf{Med}$.

Proof. If $\Lambda = L_\alpha(\Lambda)$ then Λ is McKinsey. Hence there is some Λ-valuation v which is McKinsey. In [4], Theorem 13 says (in our terms) that if v' is McKinsey then $\{\phi : v'(\Box\phi) = 1\} \subseteq \mathsf{Med}$. So since v is McKinsey we have $\{\phi : v(\Box\phi) = 1\} = \Lambda \subseteq \mathsf{Med}$. □

In other words, even if there are other fixed points of L_α, we have that Med is distinguished by being the greatest fixed point of L_α with respect to inclusion. This property provides the syntactic logicist with a potential tie-breaker. Even if fixed points are not unique, there is only one greatest fixed point. Therefore the syntactic logicist might replace (2) above with something like (2') if constancy holds, the logic of logical truth *tout court* is the maximal fixed point of L_α. And so she might try to answer our challenge in this way.

But this is more like a blueprint for a response than it is a fully satisfactory response. Maximality distinguishes Med from other fixed points, if there are others, but the syntactic logicist would need to provide us a reason why maximality provides greater justification for Med to serve as the logic of logical truth *tout court* over other candidates. In other words, how to justify (2') over and above (2)? Moreover, why not endorse some alternate strengthening of (2)? For example: why doesn't the syntactic logicist insist instead that the logic of logical truth *tout court* be some *minimal* fixed point of L_α? What is special about maximality? I will leave this point open; there may after all be something the syntactic logicist can say here. And in the end, she may not need to bother, since the Uniqueness Conjecture might be established in the meantime anyway.

This is as much as we can say currently about the propositional case. To finish we note another aspect of the connection between our treatment of syntactic logical truth and the McKinsey semantics which is independent of the uniqueness conjecture. This is that McK turns out to be identical to the intersection of all fixed points of L_α. This comes by the following result, and by the fact that the McKinsey logics are just the fixed points of L_α.

Proposition 8.2.27. McK is the intersection of all McKinsey logics.

Proof. If $\phi \in \mathsf{McK}$ then for any McKinsey valuation v and s we have $v(s\phi) = 1$ so that $v(\Box\phi) = 1$. Therefore if Λ is a McKinsey logic we have $\phi \in \Lambda$. So $\mathsf{McK} \subseteq \bigcap\{\Lambda : \Lambda \text{ is McKinsey}\}$. We also see that $\bigcap\{\Lambda : \Lambda \text{ is McKinsey}\} \subseteq \mathsf{McK}$ since if $\phi \notin \mathsf{McK}$, there is a McKinsey v and some s with $v(s\phi) = 0$, hence $v(\Box\phi) = 0$. So $\{\psi : v(\Box\psi) = 1\}$, which is a McKinsey logic by Props. 8.2.20 and 8.2.21, does not contain ϕ. So ϕ is not in the intersection of all McKinsey logics. □

Corollary 8.2.28. $\mathsf{McK} = \bigcap\{\Lambda : L_\alpha(\Lambda) = \Lambda\}$

8.3 Quantifiers and Identity

As with the semantics of the previous chapters, we now consider extensions of the above syntactical semantics to first-order languages with identity. First we consider a semantics for objectual quantifiers, showing that the same problems arise here as in previous chapters. Then we consider a semantics for substitutional quantifiers.

8.3.1 Objectual Quantification

We first briefly consider a semantics which incorporates the syntactic logicist's clause for $\Box$ into a standard semantics for first-order logic with objectual quantifiers. Where M is a Tarskian structure and a is any variable assignment on M we evaluate QML$\forall$ formulas according to the following clauses:

$M, a \nvDash_{\Lambda,\mu} \bot$
$M, a \vDash_{\Lambda,\mu} P^n t_1 ... t_n$ iff $([t_1]_{M,a}, ..., [t_n]_{M,a}) \in [P^n]_M$
$M, a \vDash_{\Lambda,\mu} (\phi \to \psi)$ iff $M, a \nvDash_{\Lambda,\mu} \phi$ or $M, a \vDash_{\Lambda,\mu} \psi$.
$M, a \vDash_{\Lambda,\mu} \forall x \phi$ iff for all $u \in |M|$, $M, a^x_u \vDash_{\Lambda,\mu} \phi$.
$M, a \vDash_{\Lambda,\mu} \Box\phi$ iff $\mu(M) \vdash_\Lambda \phi$.

Here Λ is any QML$\forall$ logic and μ, a modality, is any function which assigns M to a set of QML$\forall$ formulas. Note that Tarskian structures form a proper class, not a set. So modalities need to be characterised with some care here. We cannot say that μ is a function which takes *any* Tarskian structure M to a set of formulas.[17] We must be content to say that μ is a function which takes *some* structures to sets of formulas, and insist only that for the semantic clauses to apply, μ must at least be defined at the structure of evaluation.

As with the propositional semantics, we can potentially define many different kinds of validity and semantic consequence in this semantics. But we will just focus on the most general form of each for now. In the context of the present semantics, let us write $\Gamma \vDash \phi$ iff for any structure M, any assignment a on M, any logic Λ, and any modality μ defined on M: if $M, a \vDash_{\Lambda,\mu} \gamma$ for all $\gamma \in \Gamma$ then $M, a \vDash_{\Lambda,\mu} \phi$. And let $\vDash \phi$ abbreviate $\varnothing \vDash \phi$. Then we note failures of universal instantiation and Leibniz's law on the present semantics.

Proposition 8.3.1. For some ϕ and t, we have $\nvDash \forall x \phi \to \phi[t/x]$.

[17] Unless one allowed modalities to be class functions, though that would introduce alternate complications—e.g. in trying to talk about collections of modalities.

Proof. Let ϕ be $\Diamond x \neq y$ and let t be y. Then take any structure M, assignment a on M, and take any μ defined on M and any logic Λ such that $\mu(M) \nvdash_{\Lambda} x = y$ (for instance, let Λ be $\forall$PC and let $\mu(M) = \varnothing$). For any assignment a' on M we have $M, a' \vDash_{\forall\text{PC},\mu} \Diamond x \neq y$, so in particular for all x-variants of a. So $M, a \vDash_{\forall\text{PC},\mu} \forall x \Diamond x \neq y$, even though $M, a \nvDash_{\forall\text{PC},\mu} \Diamond y \neq y$, since $\mu(M) \vdash_{\Lambda} y = y$ just by the definition of a logic. $\square$

Proposition 8.3.2. For some ϕ, t, t', and $\phi[t'//t]$, we have $\nvDash t = t' \to (\phi \to \phi[t'//t])$.

Proof. Let ϕ be $\Box x = x$, let t be x, let t' be y, and let $\phi[t'//t]$ be $\Box x = y$. Let M be any structure, a any assignment with $a(x) = a(y)$, and let μ and Λ be any function and logic with $\mu(M) \nvdash_{\Lambda} x = y$ (for instance, let Λ be $\forall$PC and let $\mu(M) = \varnothing$). Then we have $M, a \vDash_{\forall\text{PC},\mu} x = y$, and we have $M, a \vDash_{\forall\text{PC},\mu} \Box x = x$ since $\vdash_{\forall\text{PC}} x = x$. But we have $M, a \nvDash_{\forall\text{PC},\mu} \Box x = y$. $\square$

Moreover, we may see that these failures are robust on the semantics, in the sense that the only interpretations in which universal instantiation and Leibniz's law are validated also validate some other unappealing principles. In particular, we will show that any restriction of general validity in the present semantics which would validate either universal instantiation or Leibniz's Law would also validate the sentence $\forall x \forall y \Box x = y$ (informal gloss: everything is necessarily identical).

Proposition 8.3.3. Let UI$\forall$ be the set of all QML$\forall$ formulas of the form $\forall x \phi \to \phi[t/x]$, where ϕ is a formula, x a variable and t a term. Then: UI$\forall \vDash \forall x \forall y \Box x \neq y$.

Proof. Suppose $M, a \vDash_{\Lambda,\mu} \phi$ for all $\phi \in$ UI$\forall$. To avoid the counterexample given in Prop 8.3.1, we require that $\mu(M) \vdash_{\Lambda} x = y$. This implies that $M, a \vDash_{\Lambda,\mu} \forall x \forall y \Box x = y$. $\square$

Proposition 8.3.4. Let LL$\forall$ be the set of all QML$\forall$ formulas of the form $t = t' \to (\phi \to \phi[t'//t])$. Then: LL$\forall \vDash \forall x \forall y \Box x = y$.

Proof. Suppose $M, a \vDash_{\Lambda,\mu} \phi$ for all $\phi \in$ LL$\forall$. To avoid the counterexample given in Prop 8.3.2, we require that $\mu(M) \vdash_{\Lambda} x = y$. This implies that $M, a \vDash_{\Lambda,\mu} \forall x \forall y \Box x = y$. $\square$

Therefore, besides universal instantiation and Leibniz's law not being generally valid, there are also no philosophically viable restrictions of general validity and semantic consequence which would recover these principles in the semantics. So as in previous chapters, we must discard this semantics and switch to a substitutional reading of the quantifiers.

8.3.2 Substitutional Quantification

We now consider a semantics which incorporates the syntactic logicist's clause for $\Box$ into a semantics for first-order logic with substitutional quantifiers. Where S is a state description, Λ is any QMLΠ logic, μ is any function which takes each state description S to a set of QMLΠ formulas $\mu(S)$, and ϕ is a QMLΠ *sentence*, we evaluate ϕ at S as follows:

$S \not\vDash_{\Lambda,\mu} \bot$
$S \vDash_{\Lambda,\mu} \phi$ iff $\phi \in S$ where ϕ is basic.
$S \vDash_{\Lambda,\mu} c = d$ iff c and d are the same constant.
$S \vDash_{\Lambda,\mu} (\phi \to \psi)$ iff $S \not\vDash_{\Lambda,\mu} \phi$ or $S \vDash_{\Lambda,\mu} \psi$.
$S \vDash_{\Lambda,\mu} \Pi v\phi$ iff $S \vDash_{\Lambda,\mu} \phi[c/v]$ for every individual constant c.
$S \vDash_{\Lambda,\mu} \Box\phi$ if and only $\mu(S) \vdash_{\Lambda} \phi$.

We then handle open formulas as described in §A.2.4. Namely, we introduce variable assignments $a :$Vr$\to$ Cn for state descriptions and define satisfaction of a formula ϕ with respect to an assignment a according to the following rule:

$S, a \vDash_{\Lambda,\mu} \phi$ iff $S \vDash_{\Lambda,\mu} \phi[a]$

Then we may additionally stipulate that where ϕ is open, $S \vDash_{\Lambda,\mu} \phi$ iff $S, a \vDash_{\Lambda,\mu} \phi$ for all assignments a. Note that where ψ is a sentence we also have $S \vDash_{\Lambda,\mu} \psi$ iff $S, a \vDash_{\Lambda,\mu} \psi$ for all assignments a, since then $\psi[a] = \psi$. In what follows, for readability, whenever we have some variables $x_1, ..., x_n$ and terms $t_1, ..., t_n$, I will abbreviate $\phi[t_1/x_1, ..., t_n/x_n]$ as $\phi[t_i/x_i]_{i\leq n}$. Or, where we also have additional variables and terms—y and t for instance—I abbreviate $\phi[t_1/x_1, ..., t_n/x_n, t/y]$ as $\phi[(t_i/x_i)_{i\leq n}, t/y]$. Hence for example, if $z_1, ..., z_n$ are precisely the variables that occur free in ϕ, then $\phi[a] = \phi[a(z_1)/z_1...a(z_n)/z_n]$ can be written $\phi[a(z_i)/z_i]_{i\leq n}$.

A quirk of this semantics: owing to the way in which we handle variable assignments, we have that where ϕ is open, the truth of $\Box\phi$ is true at S according to a depends on whether $\phi[a]$ is entailed by $\mu(S)$—rather than whether ϕ itself is entailed. But this quirk is a necessary evil to validate Leibniz's Law. Suppose we had set up the semantics (or our treatment of variables) differently so that in general $S, a \vDash_{\Lambda,\mu} \Box\phi$ iff $\mu(S) \vdash_{\Lambda} \phi$, even when ϕ is open. Then take any case where $\mu(S) \not\vdash_{\Lambda} Fx$ but $\mu(S) \vdash_{\Lambda} Fc$. Then if $a(x) = c$ we would have $S, a \vDash_{\Lambda,\mu} x = c \land (\Box Fc \land \neg\Box Fx)$. Now, since state descriptions form a set, modalities can be understood here in a way that is closer to the modalities of the propositional semantics investigated earlier in this chapter. As with the propositional semantics, we can define many different kinds of validity and semantic consequence in this semantics. Where $\Pi\mathbf{Log}$ is the set of QMLΠ logics and $\Pi\mathbf{Mod}$ is the set of modalities (i.e. functions from state descriptions to sets of QMLΠ formulas):

Definition 8.3.5.

$\Gamma \vDash_{\Lambda,\mu} \phi$ iff for all S and a, if $S, a \vDash_{\Lambda,\mu} \psi$ for all $\psi \in \Gamma$ then $S, a \vDash_{\Lambda,\mu} \phi$.
$\Gamma \vDash_{\mathbf{L},\mathbf{M}} \phi$ iff $\Gamma \vDash_{\Lambda,\mu} \phi$ for all $\Lambda \in \mathbf{L}$ and $\mu \in \mathbf{M}$.
$\Gamma \vDash \phi$ iff $\Gamma \vDash_{\Pi\mathbf{Log},\Pi\mathbf{Mod}} \phi$.

Now we will consider a number of instances of general validities, as well as some restrictions of particular interest to the syntactic logicist. Then we will talk about the peculiar semantic clause given for identity statements, and why—as in Bacon and Fine's proposed extension to the McKinsey semantics—it is essential to validate Leibniz's Law.

8.3.2.1 Classical Logic Recovered

It will be important first to show that both universal instantiation and Leibniz's law are recovered within this semantics. So we will show that all instances of each principle are generally valid. Taking Leibniz's Law first:

Proposition 8.3.6. $\vDash t = t' \to (\phi \to \phi[t'//t])$ for all t, t', ϕ and $\phi[t'//t]$.

Proof. Take any S, a, Λ, μ and suppose that $S, a \vDash_{\Lambda,\mu} t = t'$. Therefore $S \vDash_{\Lambda,\mu} (t = t')[a]$ (or $t[a] = t'[a]$), meaning that $t[a]$ and $t'[a]$ are the same constant. So if $S \vDash_{\Lambda,\mu} \phi[a]$ then $S \vDash_{\Lambda,\mu} \phi[t'//t][a]$, since $\phi[a]$ and $\phi[t'//t][a]$ are the same formula. So if $S, a \vDash_{\Lambda,\mu} \phi$ then $S, a \vDash_{\Lambda,\mu} \phi[t'//t]$. Hence $S, a \vDash_{\Lambda,\mu} t = t' \to (\phi \to \phi[t'//t])$. $\square$

The proof that universal instantiation is also generally valid is a bit more complicated. We recall the fact (Prop A.2.15) that $S, a \vDash_{\Lambda,\mu} \Pi v\phi$ iff for all constants c we have $S, a^v_c \vDash_{\Lambda,\mu} \phi$. We then establish some basic lemmas before arriving at the validity of instantiation.

Proposition 8.3.7. If a and a' agree on free variables in ϕ then $S, a \vDash_{\Lambda,\mu} \phi$ iff $S, a' \vDash_{\Lambda,\mu} \phi$.

Proof. If a and a' agree on the free variables in ϕ then $\phi[a]$ and $\phi[a']$ are the same formula. Hence $S \vDash_{\Lambda,\mu} \phi[a]$ iff $S \vDash_{\Lambda,\mu} \phi[a']$ so that $S, a \vDash_{\Lambda,\mu} \phi$ iff $S, a' \vDash_{\Lambda,\mu} \phi$. $\square$

Proposition 8.3.8. $S, a \vDash_{\Lambda,\mu} \phi[t/v]$ iff $S, a^v_{t[a]} \vDash_{\Lambda,\mu} \phi$.

Proof. It suffices to show that $\phi[t/v][a]$ and $\phi[a^v_{t[a]}]$ are always the very same sentence. If v does not occur free in ϕ then $\phi[t/v]$ just is ϕ, so $\phi[t/v][a]$ just is $\phi[a]$; since a and $a^v_{t[a]}$ agree on all free variables in ϕ we have that $\phi[a]$ is $\phi[a^v_{t[a]}]$ so we are done. Otherwise, let $x_1, \ldots, x_n$ be exactly the variables that occur free in ϕ, with v among the x_i. And let $y_1, \ldots, y_{n-1}$ be exactly the variables which occur free in ϕ besides v. Now we have:

$$\phi[a^v_{t[a]}] = \phi[a^v_{t[a]}(x_i)/x_i]_{i \leq n} = \phi[a^v_{t[a]}(v)/v, (a^v_{t[a]}(y_i)/y_i)_{i<n}]$$

Since $a^v_{t[a]}(v) = t[a]$ and since $a^v_{t[a]}(y_i) = a(y_i)$ for each y_i, we may simplify this to

$$\phi[a^v_{t[a]}] = \phi[t[a]/v, (a(y_i)/y_i)_{i<n}]$$

At this point there are two cases to consider, depending on whether or not t is one of the y_i. (i) if t is y_k for some $k < n$, we can rewrite the above as

$$\phi[a^v_{a(y_k)}] = \phi[a(y_k)/v, (a(y_i)/y_i)_{i<n}]$$

Now it is clear that the result of the substitution $\phi[a(y_k)/v, (a(y_i)/y_i)_{i<n}]$ will be the same as the result of first substituting v out for y_k and then replacing each y_i with $a(y_k)$. But this just means that $\phi[a(y_k)/v, (a(y_i)/y_i)_{i<n}] = \phi[y_k/v][a(y_i)/y_i]_{i<n}$. And since $v, y_1, \ldots, y_{n-1}$ are precisely the variables that occur free in ϕ, $y_1, \ldots, y_{n-1}$ are precisely the variables that occur free in $\phi[y_k/v]$. Hence altogether we get:

$$\phi[a^v_{a(y_k)}] = \phi[a(y_k)/v, (a(y_i)/y_i)_{i<n}] = \phi[y_k/v][a(y_i)/y_i]_{i<n} = \phi[y_k/v][a]$$

Which upon rewriting gets us $\phi[a^v_{t[a]}] = \phi[t/v][a]$ as desired. This deals with the first case. (ii) If t is not among the y_i, then by the fact that v is not free in $\phi[t[a]/v]$, we get

$$\phi[a^v_{t[a]}] = \phi[t[a]/v, (a(y_i)/y_i)_{i<n}] = \phi[t[a]/v][a(y_i)/y_i]_{i<n} = \phi[t[a]/v][a]$$

Now if t is a constant we immediately get $\phi[t[a]/v][a] = \phi[t/v][a]$ as desired. Similarly if t is v then we trivially have $\phi[t/v][a] = \phi = \phi[a^v_{a(v)}] = \phi[a^v_{t[a]}]$ as desired. Otherwise, if t is a variable and not v then since it isn't among the y_i, it is also not among the x_i. In this case t is a variable which does not occur free in ϕ. But this means that $\phi[t[a]/v] = \phi[t/v][t[a]/t]$, so combining this with the previous equations get us:

$$\phi[a^v_{t[a]}] = \phi[t[a]/v][a(y_i)/y_i]_{i<n} = \phi[t/v][t[a]/t][a(y_i)/y_i]_{i<n}$$

But also note that since t is a free variable distinct from v and each y, we have that $t, y_1, \ldots, y_{n-1}$ are precisely the variables that occur free in $\phi[t/v]$. Hence

$$\phi[a^v_{t[a]}] = \phi[t/v][t[a]/t][a(y_i)/y_i]_{i<n} = \phi[t/v][t[a]/t, (a(y_i)/y_i)_{i<n}] = \phi[t/v][a]$$

This then deals with the second case, completing the proof. □

Proposition 8.3.9. $\vDash \Pi v\phi \to \phi[t/v]$

Proof. Take any S, a, Λ, μ and suppose that $S, a \vDash_{\Lambda,\mu} \Pi v\phi$. Then for all constants c we have $S, a^v_c \vDash_{\Lambda,\mu} \phi$, hence in particular we have $S, a^v_{t[a]} \vDash_{\Lambda,\mu} \phi$. So by Prop. 8.3.8 we have $S, a \vDash_{\Lambda,\mu} \phi[t/v]$, meaning that $S, a \vDash_{\Lambda,\mu} \Pi v\phi \to \phi[t/v]$. □

While we are at it, it will also be useful to show that the other basic principles of classical logic are all generally valid on the semantics. Clearly, all classical tautologies will be generally valid just thanks to the semantics provided for $\to$ and $\bot$. The reflexivity schema for identity ($v = v$) is evidently going to be generally valid just by the semantic clauses. Universal generalisation is respected also. To see this we proceed like so.

Proposition 8.3.10. If $S \vDash_{\Lambda,\mu} \phi \to \psi$ then $S \vDash_{\Lambda,\mu} \phi \to \Pi v\psi$ where v is not free in ϕ.

Proof. Suppose that $S \vDash_{\Lambda,\mu} \phi \to \psi$, meaning that $S, a \vDash_{\Lambda,\mu} \phi \to \psi$ for all a. Then for all a we either have $S, a \nvDash_{\Lambda,\mu} \phi$ or $S, a \vDash_{\Lambda,\mu} \psi$. In each case: if $S, a \nvDash_{\Lambda,\mu} \phi$ then $S, a \vDash_{\Lambda,\mu} \phi \to \Pi v\psi$ as required, and if $S, a \vDash_{\Lambda,\mu} \phi$ then by Prop. 8.3.7 we have $S, a^v_c \vDash_{\Lambda,\mu} \phi$ for all c since v is not free in ϕ. Hence, given that $S, a \vDash_{\Lambda,\mu} \phi \to \psi$ for all a, we have $S, a^v_c \vDash_{\Lambda,\mu} \psi$ for all c, meaning that $S, a \vDash_{\Lambda,\mu} \Pi v\psi$ and $S, a \vDash_{\Lambda,\mu} \phi \to \Pi v\psi$ as required. Putting this together we get that $S, a \vDash_{\Lambda,\mu} \phi \to \Pi v\psi$ for all a, meaning $S \vDash_{\Lambda,\mu} \phi \to \Pi v\psi$. □

8.3.2.2 Modal Principles

Now let us move on to consider some properly modal principles. We may show that all instances of the converse Barcan formula are generally valid on this semantics. The Barcan formula and its converse are rendered as:

BFΠ	$\Pi v\Box\phi \to \Box\Pi v\phi$
CBFΠ	$\Box\Pi v\phi \to \Pi v\Box\phi$

The argument is straightforward:

Proposition 8.3.11 (CBFΠ). $\vDash \Box\Pi v\phi \to \Pi v\Box\phi$

Proof. Take any S, a, μ, and Λ, and suppose that $S, a \vDash_{\Lambda,\mu} \Box\Pi v\phi$. By the semantic clause $\mu(S) \vdash_\Lambda (\Pi v\phi)[a]$. note that where $x_1, ..., x_n$ are exactly the free variables occurring in $\Pi v\phi$, $(\Pi v\phi)[a]$ is just $(\Pi v\phi)[a(x_i)/x_i]_i$. Since v is not among these free variables, note that $(\Pi v\phi)[a(x_i)/x_i]_i$ is just $\Pi v(\phi[a(x_i)/x_i]_i)$. In addition, we have for each constant c: $\vdash_\Lambda \Pi v(\phi[a(x_i)/x_i]_i) \to \phi[a(x_i)/x_i]_i[c/v]$. So putting this together we have that $\mu(S) \vdash_\Lambda \phi[a(x_i)/x_i]_i[c/v]$ for each constant c. But since v is not among $x_1, ..., x_n$, the formula $\phi[a(x_i)/x_i]_i[c/v]$ is just $\phi[c/v][a(x_i)/x_i]_i$. And since $x_1, ..., x_n$ are exactly the free variables of $\Pi v\phi$, they are also exactly the free variables of $\phi[c/v]$, meaning that $\phi[c/v][a(x_i)/x_i]_i$ is just $\phi[c/v][a]$. So $\mu(S) \vdash_\Lambda \phi[c/v][a]$ for all c. Meaning that $S, a \vDash_\Lambda \Box(\phi[c/v])$ for all c. But $\Box(\phi[c/v])$ is $(\Box\phi)[c/v]$, so we have $S, a \vDash_\Lambda (\Box\phi)[c/v]$ for all c, meaning $S, a \vDash_\Lambda \Pi v\Box\phi$. $\Box$

Importantly, however, the Barcan formula itself is not generally valid. Here is a counterinstance. If $\mu(S) = \{Fc : c \in \mathrm{Cn}\}$, then $S \vDash_{\Pi\mathrm{PC},\mu} \Pi x\Box Fx$. And yet since $\mu(S) \nvdash_{\Pi\mathrm{PC}} \Pi xFx$, we have $S \vDash_{\Pi\mathrm{PC},\mu} \neg\Box\Pi xFx$. But it is possible to find restrictions of general validity in which BFΠ does feature. As a simpler example, re-appropriating previous notation, let 'α' refer to the unique modality such that $\alpha(S) = \varnothing$ for every state description S. Then:

Proposition 8.3.12 (BFΠ). $\vDash_{\Pi\mathbf{Log},\alpha} \Pi v\Box\phi \to \Box\Pi v\phi$

Proof. Take any S, a, Λ and ϕ, and suppose that $S, a \vDash_{\Lambda,\alpha} \Pi v\Box\phi$. Then $S, a^v_c \vDash_{\Lambda,\alpha} \Box\phi$ for all c, which implies that $S \vDash_{\Lambda,\alpha} (\Box\phi)[a^v_c]$ for all c. Since $(\Box\phi)[a^v_c] = \Box(\phi[a^v_c])$, we then have $S \vDash_{\Lambda,\alpha} \Box(\phi[a^v_c])$ for all c, and from here by the semantic clause we have $\vdash_\Lambda \phi[a^v_c]$ for all c. So in particular we have $\vdash_\Lambda \phi[a^v_{c_0}]$ where c_0 is some constant which does not occur in ϕ and which is distinct from $a(x_i)$ for all x_i in ϕ. Now by closure under uniform substitution we have $\vdash_\Lambda \varsigma_{c_0\to v}(\phi[a^v_{c_0}])$, where $\varsigma_{c_0\to v}$ is the unique substitution that sends c_0 to v and leaves everything else alone. But since c_0 does not occur in ϕ and is distinct from $a(x_i)$ for all x_i in ϕ, we have $\varsigma_{c_0\to v}(\phi[a^v_{c_0}]) = \phi[a(x_i)/x_i]_{i\leq n}$ where $x_1, ..., x_n$ are exactly those variables other than v which occur free in ϕ. So $\vdash_\Lambda \phi[a(x_i)/x_i]_{i\leq n}$. Now by universal generalisation we have $\vdash_\Lambda \Pi v(\phi[a(x_i)/x_i]_{i\leq n})$, and since v is not among the x_i we have:

$$\Pi v(\phi[a(x_i)/x_i]_{i\leq n}) = (\Pi v\phi)[a(x_i)/x_i]_{i\leq n} = (\Pi v\phi)[a]$$

Hence $\vdash_\Lambda (\Pi v\phi)[a]$, so that by the semantic clauses we have $S, a \vDash_{\Lambda,\alpha} \Box\Pi v\phi$. $\Box$

Some additional statements of interest are the necessity of existence, of identity, and of distinctness. These are rendered in substitutional quantifiers as follows.

NEΠ	$\Pi x \Box \Sigma y \, x = y$
NIΠ	$\Pi x \Pi y (x = y \to \Box x = y)$
NDΠ	$\Pi x \Pi y (x \neq y \to \Box x \neq y)$

Proposition 8.3.13 (NEΠ). $\vDash \Pi x \Box \Sigma y \, x = y$

Proof. For all c and any Λ we have $\vdash_\Lambda \Sigma y \, c = y$, so that we always have $S, a \vDash_{\Lambda,\mu} \Box \Sigma y \, c = y$. This holds for all constants meaning we always have $S, a \vDash_{\Lambda,\mu} \Pi x \Box \Sigma y \, x = y$. □

Proposition 8.3.14 (NIΠ). $\vDash \Pi x \Pi y (x = y \to \Box x = y)$

Proof. As an instance of Leibniz's Law (Prop. 8.3.6) we have $\vDash x = y \to (\Box x = x \to \Box x = y)$, which by propositional logic holds iff $\vDash \Box x = x \to (x = y \to \Box x = y)$. But $\vDash \Box x = x$ since $\mu(S) \vdash_\Lambda (x = x)[a]$ for any choice of S, a, Λ, μ. Hence $\vDash (x = y \to \Box x = y)$. So via universal generalisation $\vDash \Pi x \Pi y (x = y \to \Box x = y)$. □

On the other hand, we do not have the necessity of distinctness as generally valid. We can construct counterexamples relatively easily:

Proposition 8.3.15 (NDΠ). $\nvDash \Pi x \Pi y (x \neq y \to \Box x \neq y)$

Proof. Let c and d be distinct constants, and pick any S, Λ, and μ such that $\mu(S) \nvdash_\Lambda c \neq d$. (For example, let $\Lambda = \Pi\text{PC}$ and let $\mu(S) = \varnothing$). Then $S \vDash_{\Lambda,\mu} \neg(c \neq d \to \Box c \neq d)$, so $S \nvDash_{\Lambda,\mu} \Pi x \Pi y (x \neq y \to \Box x \neq y)$. □

Now the necessity of distinctness does not suffer from the kinds of robust failures suffered by universal instantiation and Leibniz's Law on the objectual quantifier semantics. For example:

Definition 8.3.16. Call a modality μ *discerning* iff for all S, $\mu(S)$ contains $c \neq d$ wherever c and d are distinct constants. Call the set of discerning modalities 'Dis'.

Proposition 8.3.17. $\vDash_{\Pi\mathbf{Log},\text{Dis}} \Pi x \Pi y (x \neq y \to \Box x \neq y)$

Proof. Take any S, a, Λ and discerning μ. Suppose that $S, a \vDash_{\Lambda,\mu} x \neq y$. Therefore $x[a]$ and $y[a]$ must be distinct constants, meaning $\mu(S) \vdash_\Lambda x[a] \neq y[a]$. Hence $S, a \vDash_{\Lambda,\mu} \Box x \neq y$. So $S, a \vDash_{\Lambda,\mu} x \neq y \to \Box x \neq y$, which since a was arbitrary also implies that the same holds for all variants $a^{x\,y}_{c\,d}$, meaning that $S, a \vDash_{\Lambda,\mu} \Pi x \Pi y (x \neq y \to \Box x \neq y)$. □

8.3.2.3 α Again

At this point we should turn to consider a class of restrictions to general validity that the syntactic logicist will find particularly interesting. In the present semantics we once again use 'α' to refer to the unique modality such that $\alpha(S) = \varnothing$ for every state description S. By the same considerations as those given at the start of §8.2, the syntactic logicist looks to be committed to saying that there is some special QMLΠ logic Λ such that for a QMLΠ-formula to be a theorem of Λ is for it to be a logical truth *tout court*. So we can pose a fresh version of our challenge to the syntactic logicist in this setting: how can one decide which QMLΠ logic plays this distinguished role? And we can consider, at least in a sketch, a way in which the syntactic logicist might try to answer this challenge in this new setting. For it is possible to define an analogue of the function L_α on PML logics in the case of QMLΠ logics. That can be done as follows.

Definition 8.3.18. Where Λ is a QMLΠ-logic, let $\Pi V_\alpha(\Lambda) = \{\phi : \vDash_{\Lambda,\alpha} \phi\}$.

$\Pi V_\alpha(\Lambda)$ is never a logic, because it is never closed under substitutions. For example, for any distinct constants c and d, we have $c \neq d \in \Pi V_\alpha(\Lambda)$ even though $c \neq c \notin \Pi V_\alpha(\Lambda)$. This is the same problem we had with the function V_α in the propositional case, and we solve it in a similar way by instroducing a derivative function:

Definition 8.3.19. Where Λ is a QMLΠ-logic, let $\Pi L_\alpha(\Lambda) = \{\phi : \varsigma\phi \in \Pi V_\alpha \text{ for all } \varsigma\}$

It is possible to show that $\Pi L_\alpha(\Lambda)$ is always a QMLΠ logic. The proof is straightforward if a little laborious, especially in its first step. One proceeds by first showing that $\Pi L_\alpha(\Lambda)$ contains all tautologies, and all instances of UI, Ref, and LL. Second, one shows that $\Pi L_\alpha(\Lambda)$ is closed under the required inference rules: the proof that $\Pi L_\alpha(\Lambda)$ is closed under modus ponens is straightforward, and closure under substitutions comes by definition. This leaves us to prove the case of universal generalisation.

Proposition 8.3.20. For all Λ, $\Pi L_\alpha(\Lambda)$ is closed under universal generalisation.

Proof. Suppose that $(\phi \to \psi) \in \Pi L_\alpha(\Lambda)$ where v is not free in ϕ. This means that for all S and ς we have $S \vDash_{\Lambda,\alpha} \varsigma(\phi \to \psi)$ $(= (\varsigma\phi \to \varsigma\psi))$. If v is not free in ϕ it is also not free in $\varsigma\phi$. So by Prop. 8.3.10 we have $S \vDash_{\Lambda,\alpha} \varsigma\phi \to \Pi v\varsigma\psi$ $(= \varsigma(\phi \to \Pi\psi))$ for all S and ς. So $(\phi \to \Pi v\psi) \in \Pi L_\alpha(\Lambda)$. $\square$

Corollary 8.3.21. $\Pi L_\alpha : \Pi\mathbf{Log} \to \Pi\mathbf{Log}$.

Now that we have a function ΠL_α to play the analogue of L_α in QMLΠ, the syntactic logicist can formulate quantificational analogues of theses (1) and (2) from §8.2 in an attempt to meet the new version of the challenge posed to them:

(1Π) If constancy fails, the logic of logical truth *tout court* is $\Pi L_\alpha(\Pi\mathrm{PC})$.

(2Π) If constancy holds, the logic of logical truth *tout court* is a ΠL_α fixed point.

The arguments for these theses can be generalised from those for (1) and (2) without substantial revision. (1Π) is justified on the basis that, if $\Box$ is not treated as a logical constant, then ΠPC is the only system which can sensibly be taken to embody logical truth *tout court*. Since it is the only system which contains no principles for $\Box$ which do not just come by instances of propositional logic, quantification theory, or by the logic of identity. (2Π) is justified on the basis that, if constancy holds, the logic of logical truth *tout court* ought to contain the T axiom and the rule of Necessitation. And, once again, we have the following implication:

Proposition 8.3.22. $\phi \in \Lambda$ iff $\Box\phi \in \Pi L_\alpha(\Lambda)$.

Proof. If $\phi \in \Lambda$, then for all S and a, we have $S, a \vDash_{\Lambda,\alpha} \Box\phi$, so that $\Box\phi \in \Pi V_\alpha(\Lambda)$. By the same token, since Λ is closed under substitutions then, if $\phi \in \Lambda$ then for all substitution instances ϕ' of ϕ, $\Box\phi' \in \Pi V_\alpha(\Lambda)$. So: $\Box\phi \in \Pi L_\alpha(\Lambda)$. Conversely, if $\Box\phi \in \Pi L_\alpha(\Lambda) \subseteq \Pi V_\alpha(\Lambda)$ then for all S and a, we have $S, a \vDash_{\Lambda,\alpha} \Box\phi$, which is so iff $\vdash_\Lambda \phi$, iff $\phi \in \Lambda$. $\Box$

Proposition 8.3.23. $\Pi L_\alpha(\Lambda)$ contains T and admits Necessitation iff $\Lambda = \Pi L_\alpha(\Lambda)$.

Proof. (Right to left.) If $\Lambda = \Pi L_\alpha(\Lambda)$ then necessitation comes immediately by Prop 8.3.22. Additionally for any S, a and ϕ one can see that we have $S, a \vDash_{\Lambda,\alpha} \Box\phi \to \phi$ since $\Lambda \subseteq \Pi L_\alpha(\Lambda)$. Hence $\Box p \to p \in \Lambda = \Pi L_\alpha(\Lambda)$. (Left to right.) By Prop 8.3.22, $\phi \in \Lambda$ iff $\Box\phi \in \Pi L_\alpha(\Lambda)$. If $\Pi L_\alpha(\Lambda)$ contains T and admits Necessitation, then for all ϕ, $\phi \in \Pi L_\alpha(\Lambda)$ iff $\Box\phi \in \Pi L_\alpha(\Lambda)$, from which we get $\phi \in \Lambda$ iff $\phi \in \Pi L_\alpha(\Lambda)$. $\Box$

Now, on the existence of fixed points of ΠL_α, there is little that we can say presently. We cannot easily make use of connections to the McKinsey semantics as we did in the propositional case. So we do not even have a ready argument to establish that there *is* a fixed point for ΠL_α. Nevertheless, it seems quite plausible that ΠL_α should have a fixed point, given that L_α does. Moreover, we might make an informed guess as to what logic might provide a witness to this. Since Med was a fixed point for L_α, perhaps some extension of $\Pi\mathsf{Med}$ with the Barcan formula is a fixed point—where, recall, $\Pi\mathsf{Med}$ is the smallest logic in the present language that includes all propositional theorems from Med. Although it is a disappointment not to have a clearer way forward in classifying the fixed points of ΠL_α—felt all the more by the syntactic logicist herself, who must solve this problem in order to fully meet our challenge—there are still some more things we can say before we move on. We know at least that the syntactic logicist who accepts constancy must identify the first-order logic of absolute necessity with some fixed point of ΠL_α, and that the syntactic logicist who denies constancy must identify the first-order logic of absolute necessity with $\Pi L_\alpha(\Pi\mathrm{PC})$. Taking the first of these first: even without an example of a fixed point, we can still put some restrictions on this class. For example, in analogy to Prop. 8.2.11 we have:

Proposition 8.3.24. Every ΠL_α-fixed-point is a normal extension of ΠS4.

Proof. The distribution formula K can be shown generally valid on this first-order semantics by the same argument as in the propositional semantics. Combining this with Prop 8.3.22 we have that if Λ is a ΠL_α-fixed-point then it is a normal extension of ΠT. From this it suffices to show that Λ must also contain the 4 axiom.

> (4) For any S, a and ϕ, suppose that $S, a \vDash_{\Lambda,\alpha} \Box\phi$. Then $\phi[a] \in \Lambda$, but since Λ is normal we have $\vdash_\Lambda \Box(\phi[a])$. But $\Box(\phi[a])$ is $(\Box\phi)[a]$ so that $\vdash_\Lambda (\Box\phi)[a]$. Hence $S, a \vDash_{\Lambda,\alpha} \Box\Box\phi$ meaning that we have $S, a \vDash_{\Lambda,\alpha} \Box\phi \to \Box\Box\phi$. Since every instance of 4 is in $\Pi V_\alpha(\Lambda)$, $4 \in \Pi L_\alpha(\Lambda)$.

This shows that any fixed point must be a normal extension of ΠS4. □

We can also show that every fixed point contains McKinsey's axiom, though the argument is slightly more complicated in the first-order case than it was for PML. First we show that every fixed point has some consistent extension that proves $\Box p \leftrightarrow p$:

Proposition 8.3.25. If Λ is a consistent normal extension of ΠT, then there is a consistent extension of Λ which contains $\Box p \leftrightarrow p$.

Proof. By reductio. Suppose that Λ is a consistent normal extension of ΠT and has no consistent extension which contains $\Box p \leftrightarrow p$. In this case there must be some formula ϕ such that $\vdash_\Lambda \neg(\Box\phi \leftrightarrow \phi)$, meaning that by PC, $\vdash_\Lambda (\Box\phi \to \phi) \to \neg(\phi \to \Box\phi)$. So since Λ extends ΠT we have $\vdash_\Lambda \neg(\phi \to \Box\phi)$, which again by PC means that $\vdash_\Lambda \phi \wedge \neg\Box\phi$, so that $\vdash_\Lambda \phi$ and $\vdash_\Lambda \neg\Box\phi$. But this means that Λ is not normal, which is absurd. □

From this we get as a corollary:

Proposition 8.3.26. If Λ is a consistent normal extension of $\Pi\mathsf{T}$, then for all ϕ, if $\vdash_\Lambda \neg\Box\phi$ then $\nvdash \neg\Box\neg\phi$.

Proof. If Λ were a consistent normal extension of $\Pi\mathsf{T}$ and there were a formula ϕ such that $\vdash_\Lambda \neg\Box\neg\phi \wedge \neg\Box\phi$, then there would be no consistent extension of Λ that contained $\Box p \leftrightarrow p$. Since this is false by Prop. 8.3.25, we have that there can be no such Λ and ϕ. □

Hence, finally, we can show:

Proposition 8.3.27. Every ΠL_α-fixed-point is a normal extension of $\Pi\mathsf{S4M}$.

Proof. By Prop. 8.3.24, every ΠL_α-fixed-point is a normal extension of $\Pi\mathsf{S4}$. Every such fixed point is also consistent since $\Pi L_\alpha(\text{PML}) \neq \text{PML}$. So every fixed point Λ is a consistent normal extension of $\Pi\mathsf{T}$, meaning that for all for all ϕ, if $\vdash_\Lambda \neg\Box\phi$ then $\nvdash \neg\Box\neg\phi$. We use this to show that where Λ is a fixed point, $\Box\Diamond p \rightarrow \Diamond\Box p$ is a theorem as follows.

> (Mc) For any S, a and ϕ, suppose that $S, a \vDash_{\Lambda,\alpha} \Box\Diamond\phi$. Therefore $S \vDash_{\Lambda,\alpha} (\Box\Diamond\phi)[a]$, which implies $S \vDash_{\Lambda,\alpha} \Box\Diamond(\phi[a])$ since $(\Box\Diamond\phi)[a]$ is $\Box\Diamond(\phi[a])$. By the semantic clause we have $\vdash_\Lambda \neg\Box\neg(\phi[a])$, meaning $\nvdash_\Lambda \neg\Box(\phi[a])$, so that $S, a \vDash_{\Lambda,\alpha} \Diamond\Box(\phi[a])$, which implies $S, a \vDash_{\Lambda,\alpha} (\Diamond\Box\phi)[a]$. Hence $S, a \vDash_{\Lambda,\alpha} \Box\Diamond\phi \rightarrow \Diamond\Box\phi$.

So every instance of Mc is in $\Pi V_\alpha(\Lambda)$, implying that Mc must be in $\Pi L_\alpha(\Lambda) = \Lambda$. □

Now let us consider some further principles. We were already able to show that the converse Barcan formula is generally valid, so it is in any fixed point. And since we are considering α, we have by Prop. 8.3.12 that the Barcan formula is in any fixed point. So too are the necessity of existence and the necessity of identity. By the fact that every fixed point is normal, this also implies that every fixed point must contain the *necessary* necessity of existence just by an instance of necessitation (informal gloss: necessarily everything is necessarily something):[18]

NNEΠ $\quad \Box\Pi x \Box\Sigma y\, x = y$

By contrast, the logic $\Pi L_\alpha(\Pi\text{PC})$, which is not normal, does *not* contain this formula; instead it contains its negation (informal gloss: possibly something is possibly nothing):[19]

[18] This does not mean that the syntactic logicist who accepts constancy is committed to necessitism, at least not in the sense of Williamson [68]. Necessitism is the thesis that necessarily everything is necessarily something, but specifically where 'necessarily' means *metaphysically* necessarily. This semantic logicist is only committed to this where 'necessarily' means *absolutely* necessarily, but our syntactic logicist may not believe that there is such a thing as metaphysical necessity, given her reductionist view of necessity in general.

[19] As in previous footnote, this does not mean that the syntactic logicist who rejects constancy is committed to contingentism, at least not in the sense of Williamson [68]. Contingentism is the thesis that possibly something is possibly nothing, but specifically where 'possibly' means *metaphysically* possibly. But our syntactic logicist may not believe that there is such a thing as metaphysical possibility, given her reductionist view of possibility in general.

Proposition 8.3.28. $\vdash_{\Pi L_\alpha(\Pi\mathrm{PC})} \neg\Box\Pi x\Box\Sigma y\, x = y$

Proof. We have $\nvdash_{\Pi\mathrm{PC}} \Pi x\Box\Sigma y\, x = y$, but since for any ς we have that $\varsigma(\Pi x\Box\Sigma y\, x = y)$ is just $\Pi x\Box\Sigma y\, x = y$, it follows from the semantics that $\vdash_{\Pi L_\alpha(\Pi\mathrm{PC})} \neg\Box\Pi x\Box\Sigma y\, x = y$. $\Box$

Finally we may consider the necessity of distinctness. Recall that this principle is not generally valid, but that there are straightforward and fairly natural restrictions to general validity which do validate it. What becomes of this principle in fixed points and in $\Pi L_\alpha(\Pi\mathrm{PC})$? The answer is that it fails very badly. In fact:

Proposition 8.3.29. For any consistent Λ: $\vDash_{\Lambda,\alpha} \Pi x\Pi y\Diamond x = y$

Proof. For any S, a, and consistent Λ we have $S, a \vDash_{\Lambda,\alpha} \Diamond x = y$ since else we would need $\vdash_\Lambda x[a] \neq y[a]$, which is impossible since then by substitution we would have $\vdash_\Lambda c \neq c$. Hence via the semantic clauses we have $S, a \vDash_{\Lambda,\alpha} \Pi x\Pi y\Diamond x = y$. $\Box$

From this it follows that the only case in which a logic $\Pi L_\alpha(\Lambda)$ contains the necessity of distinctness is where Λ is the inconsistent logic.

8.3.2.4 De Re Elimination

One final property of interest which belongs to all fixed points. We recall the standard distinction made between modal formulas that are *de dicto* and those that are *de re*. In particular, a formula ϕ is *de re* iff it contains some subformula of the form $\Box\psi$ where ψ is an open formula. A formula is *de dicto* iff it is not *de re*. Historically, reservations surrounding use and reflection upon quantified modal logic centered around the legitimacy of *de re* formulas. Such formulas make up the third grade of 'modal involvement' identified by Quine [50] as essentially problematic (because problematically essentialist). For this reason, logicians have studied the following property of some modal systems.

Definition 8.3.30. A logic Λ admits *de re* elimination iff for every formula ϕ, there is some *de dicto* formula ψ such that $\vdash_\Lambda \phi \leftrightarrow \psi$.

See for example the papers of Fine [21] and Cresswell [14]. The term 'elimination' is a misnomer; the device of *de re* modal ascription is not actually eliminated from such a logic Λ, since we remain in the full language QMLΠ which still allows for boxes to be attached to open formulas. Perhaps a better term would be 'reduction'. Fine [21] at least suggests that for a certain moderate form of skeptic about the *de re*, logics that admit *de re* elimination might be considered acceptable, despite the allowance of *de re* ascriptions. For they provide one a way of giving sense to such ascriptions in terms of the purely *de dicto*. Faced with a certain *de re* formula ϕ,

Definition 8.3.31 (Cresswell [14]). Where ϕ is a formula of QMLΠ, let $\overline{\phi}$ be the result of taking each maximal subformula of the form $\Box\psi$ in ϕ and replacing it with $\Box\Pi v_1, ...\Pi v_n\psi$, where $v_1, ..., v_n$ are precisely the variables that occur free in ψ.

Note that for any formula ϕ, the result $\overline{\phi}$ of this operation is *de dicto*. Additionally, if ϕ itself is *de dicto* then note that we have $\overline{\phi} = \phi$. Using this construction we can show that any restriction of general validity based on a restriction of considered modalities to just α will admit *de re* elimination.

Proposition 8.3.32. $\vDash_{\Pi\mathbf{Log},\alpha} \phi \leftrightarrow \overline{\phi}$

Proof. By induction on formula complexity of ϕ we show that for any S, a, Λ, we always have $S, a \vDash_{\Lambda,\alpha} \phi$ iff $S, a \vDash_{\Lambda,\alpha} \overline{\phi}$. The base case is trivial: since all atomic formulas are *de dicto*, we have $\overline{\phi} = \phi$. The induction step for $\to$ is routine since we have $\overline{\psi \to \chi} = \overline{\psi} \to \overline{\chi}$. Therefore the only steps which present special difficulty are those for Π and $\Box$. These proceed as follows:

(Π) Suppose that the proposition holds for all formulas of less complexity than $\Pi v\psi$. Note that $\overline{\Pi v\psi} = \Pi v\overline{\psi}$, and also note that by induction hypothesis we have $\vDash_{\Pi\mathbf{Log},\alpha} \psi \leftrightarrow \overline{\psi}$. Now by Prop. A.2.15, we have that $S, a \vDash_{\Lambda,\alpha} \Pi v\psi$ iff for all c: $S, a^v_c \vDash_{\Lambda,\alpha} \psi$. By assumption, the latter holds iff $S, a^v_c \vDash_{\Lambda,\alpha} \overline{\psi}$, which is so iff $S, a \vDash_{\Lambda,\alpha} \Pi v\overline{\psi}$ ($= \overline{\Pi v\phi}$). Hence $S, a \vDash_{\Lambda,\alpha} \Pi v\psi$ iff $S, a \vDash_{\Lambda,\alpha} \overline{\Pi v\psi}$.

($\Box$) Suppose that the proposition holds for all formulas of less complexity than $\Box\psi$. Now if $\Box\psi$ is *de dicto* we are done since then $\overline{\Box\psi} = \Box\psi$. Else, if $\Box\psi$ is *de re* then we have $\overline{\Box\psi} = \Box\Pi v_1...v_n\psi$ where $v_1, ..., v_n$ are precisely the free variables that occur in ψ. But $\vdash_\Lambda \psi$ iff $\vdash_\Lambda \Pi v_1...v_n\psi$, so that $S, a \vDash_{\Lambda,\alpha} \Box\psi$ iff $S, a \vDash_{\Lambda,\alpha} \overline{\Box\psi}$.

Which completes the proof. □

And from this is follows that any logic of the form $\Pi L_\alpha(\Lambda)$, including any fixed points that ΠL_α might have, admits *de re* elimination.

Corollary 8.3.33. Every logic $\Pi L_\alpha(\Lambda)$ admits *de re* elimination.

Proof. For any ϕ we have $\phi \leftrightarrow \overline{\phi} \in \Pi V_\alpha(\Lambda)$ by Prop. 8.3.32. So in particular for any ϕ and ς we have $\varsigma\phi \leftrightarrow \overline{\varsigma\phi} \in \Pi V_\alpha(\Lambda)$. But note that $\varsigma(\overline{\phi}) = \overline{\varsigma\phi}$. Hence for all ϕ and all ς we have $\varsigma(\phi \leftrightarrow \overline{\phi}) \in \Pi V_\alpha(\Lambda)$. Hence for all ϕ we have $(\phi \leftrightarrow \overline{\phi}) \in \Pi L_\alpha(\Lambda)$. □

8.3.2.5 Syntactic Identity and Leibniz's Law

One final point which will be necessary for the argument of the next chapter. The reader will note that our semantics for the language of substitutional quantification includes the same clause for identity as was present in Carnap's modal semantics and as was considered by Bacon and Fine [4] in their remarks on how to extend the McKinsey semantics. A sentence $c = d$ expressing identity is true in an interpretation iff c and d are the very same constant. As we have said already, there is something a bit off-putting about this clause, but I leave my full criticisms until the following chapter. Here, to round out discussion, I am mainly concerned to show why this clause is once again necessary in order to avoid widespread violations of Leibniz's Law similar to those observed for universal instantiation and Leibniz's Law seen under the semantics for objectual quantification. So, suppose that we had set up the semantics using some alternate clause for identity:

$$S, a \vDash_{\Lambda,\mu} c = d \text{ iff } F(S, c, d).$$

Where the imaginary condition $F(S, c, d)$ does not require that c and d be the very same constant—meaning that it is possible for $F(S, c, d)$ to hold even if c and d are syntactically distinct.[20] In this case we could have violations of Leibniz's law like the following.

Since syntactic identity is not necessary for condition F to hold, there must be some state description S and some distinct constants c and d such that $F(S, c, d)$ is true. Now suppose that we have some Λ and μ such that $\mu(S) \nvdash_\Lambda c = d$. Then we have $S \vDash_{\Lambda,\alpha} c = d$, since $F(S, c, d)$, and we have $S \vDash_{\Lambda,\alpha} \Box c = c$ since $\vdash_\Lambda c = c$, but we have $S \nvDash_{\Lambda,\alpha} \Box c = d$ since $\mu(S) \nvdash_\Lambda c = d$. Therefore we have $S \nvDash_{\Lambda,\mu} c = d \to (\Box c = c \to \Box c = d)$, meaning a failure of Leibniz's Law. For a concrete example, let $\mu = \alpha$ and let $\Lambda = \Pi$PC.

8.4 Conclusion

In this chapter we have developed a semantic framework for propositional and first-order modal logics which fits the needs of the syntactic logicist. We have shown how this logicist can grapple with certain philosophical difficulties stemming from the initially obscure idea of syntactic logical truth *tout court*, and we have drawn out the implications of her view for the logic of absolute necessity. In the quantificational case we saw technical difficulties similar to those of previous chapters in

[20] The condition F might be very complicated, and might build in all sorts of extra resources beyond those present in the machinery of state descriptions. For example, perhaps F associates each S with a domain $F(S)$ and each constant c with a denotation $[c]_F \in F(S)$, so that perhaps $F(S, c, d)$ iff $[c]_F = [d]_F$. Or perhaps F involves some hitherto unfamiliar devices. Note however that we would still need the syntactic identity of c and d to be *sufficient* for $F(S, c, d)$ to hold. For reflexivity to be valid, we need $F(S, c, c)$ to always be true.

the effort to validate principles of classical logic and quantification theory; by now a theme has emerged. In order to avoid widespread failures of classical principles, the semantics of this and previous chapters have needed to make a number of concessions. Two in particular are held in common. First, a confinement to subtitutional readings of the quantifiers. Second, variations on the assumption of 'logical perfection' articulated in Bacon and Fine [4], and rendered in a clause for the identity predicate which says that all identity statements between distinct constants are false. Until now I have not expressed my own philosophical concerns with these concessions. This has been to keep the way clear: so as not to too heavily interrupt the technical development of the semantic systems, and to maintain a general attitude of sympathy to the ways in which each species of logicist might best develop their views within the frameworks provided. Now that this technical development is sufficiently complete, in the next chapter we can consider the problems with logicism from a more philosophical angle, informed by the results of these foregoing chapters.

9

Logicism IV: Critique

Logicism is the view that, in some way or other, absolute necessity just is logical truth. Unlike modalism it is intended as a reductive thesis, reducing the concept of necessity to some prior understanding of logical truth. In the previous three chapters we have seen and developed several forms of modal semantics which correspond to this general outlook, offering readings of modal formulas on which $\Box$ is read variously as expressing semantic, substitutional, or syntactic logical truth. These semantics are technically intriguing, formally rich, and of interest on their own terms. It increases our understanding of the varieties of logical truth to study these systems of semantics. However, as I will argue, each of these systems has shown us, in one way or another, the inadequacy of a logicist understanding of necessity.

We have seen that in order to obtain adequate semantics for quantified modal logic given a logical reading of the $\Box$, one must make two important maneuvers. The first is a rejection of objectual readings of the quantifiers in favour of substitutional readings. The second is to make a requirement that an identity claim can only be true if the very same name occurs on both sides of the identity predicate. In the absence of either of these maneuvers, we get failures of either universal instantiation or Leibniz's Law. Now, if we merely understand these systems of semantics as laying out the peculiar interactions between logical truth operators and other logical operators, then this 'trade-off' that we see between conventional semantics and classical logic for quantification and identity can be seen as ultimately innocuous—a common quirk of some formal interest, which is confined to formal investigations of the modal logics of logical truth. But the logicist claims that we are to understand necessity as logical truth. Therefore, for the logicist, trade-offs between conventional semantics and classical logic are inherent to all interactions of quantification, identity, and modality in general (not merely logical 'modalities'). So in particular these trade-offs will infect natural language as well. Even in English, we will have such trade-offs.

9.1 Substitutional Quantification

To illustrate my point, first in reference to quantification, consider the following inference in natural language.

(1) Everything is possibly distinct from Socrates.

(2) Therefore, Socrates is possibly distinct from Socrates.[1]

The conclusion is uncontroversially false, so either (1) is false or the inference is invalid. Now, if we disregard logicism for a moment, we can see that since the inference is a simple instance of universal instantiation, the correct diagnosis of this inference should be that its premise is false. And indeed, it is false because Socrates provides a counterinstance. However, here is the logicist's problem: for them, it looks like (1) might actually be true. At least, if the quantifier in (1) is read objectually, the logicist will have a hard time denying (1). For if we do read (1) objectually, then it is equivalent to the following semi-formal sentence:

(1′) $\forall x$: possibly, x is distinct from Socrates.

And this looks to be true on any form of logicism. No matter what designation is given to x, the sub-clause 'possibly x is distinct from Socrates' is true since 'x is distinct from Socrates' is a logically consistent statement—even if 'x' happens to designate Socrates. Therefore, analogously to the formal discussions of the previous chapters: if the logicist agrees to read the quantifier in (1) objectually, she will evaluate (1) as true, even though she denies (2) like everyone else, and so she must say that universal instantiation is invalid.

Conversely, it seems the only way that she can provide the correct diagnosis of this inference, by affirming the use of universal instantiation and reading (1) as false, is to adopt a substitutional reading of the quantifier in (1). Rather than reading (1) as equivalent with (1′), she needs to read it as equivalent to some (possibly infinitary) conjunction of all claims of the form 'c is possibly distinct from Socrates' wherever c is a name taken from some stipulated background class of names N. So long as the name 'Socrates' features in N, our logicist will be able to recognise the inference from (1) to (2) as valid and hence recognise (1) as false. This is just one isolated example of this kind

[1]Note the similarity to the formula $\forall x \Diamond x \neq c \rightarrow \Diamond c \neq c$ used in the previous chapters.

of trade-off. But in general, the logicist cannot read English quantifiers objectually if she wants to make use of universal instantiation in natural language reasoning.

9.1.1 Adequacy and Compositionality

Some philosophers have argued that at least some natural language quantifiers are best interpreted substitutional.[2] But the claim that all quantification in natural language is substitutional is highly implausible. General substitutional analyses of natural language quantification deliver clearly incorrect truth-conditions for many quantified statements. On a substitutional analysis of quantification, 'Everything is F' is true iff every claim of the form 'c is F' is true where 'c' is drawn from some background class of names N. What is N in this case? Most plausibly it is the class of all English proper names. This answer fits best with the formal treatment of substitutional quantification seen in previous chapters, where we saw that statements of the form $\Pi v\phi$ were true on an interpretation iff $\phi[c/v]$ was true on that interpretation for every constant c taken from the language QMLΠ itself. But, if N is just the class of all English names, the substitutional analysis of English quantification is extensionally inadequate, in that it predicts some claims to be true which are in fact false. Consider that the sentence 'Everything is named by some English word' is false; there are some things that have never and will never be named by some English word. But for every name 'c' $\in N$, the substitution instance 'c is named by some English word' is true, since in each case c is named by 'c'. So the substitutional analysis of quantification predicts falsely that 'Everything is named by some English word' should come out true. Therefore it is an incorrect account of natural language quantification.

An advocate of the substitutional analysis might try to resist this consideration by insisting that the substitution class N not be identified with the class of all English names, but instead with some much more encompassing class that happens to contain a name for every existing thing. This version of the substitutional analysis correctly allows that 'Everything is named by some English word' might be false, since some things might be named only by non-English names in N. But the view still falsely predicts that 'Everything has a name' is true. Advocates are less likely to budge here however; the new non-English names posited to exist in N might be metaphysically quite unlike English phrases. Perhaps the new names are Lagadonian (cf. [41], 145). A Lagadonian name names an object by being identical with it. Since everything is identical with something and the class of Lagadonian names can be extended to include everything, perhaps 'Everything

[2] Ioup [30], following Fodor [23], suggests a substitutional reading of the quantifier in the sentence "there is a particular dragon that Alberta believes ate her petunias", so that the sentence will be able to come out true even though there are in fact no dragons.

has a name' is no more objectionable (and in some sense, means nothing more) than the claim that everything is identical with something.

There is something strange about this kind of reply. Presumably, one motivation for advocating a substitutional analysis of quantification is deflationary. Objectual analyses of quantification make unavoidable reference to the world, and if one is an anti-metaphysician, one might wish for an alternative analysis of quantification which only makes use of standard linguistic concepts. But here we have seen that the best way, seemingly the only way, to ensure that the substitutional analysis does not make false predictions is to require that that the background substitution class N be far richer and more expansive than an anti-metaphysician would probably want to countenance. This is especially true if one opts for Lagadonian names (e.g. to defend the claim that 'Everything has a name' is actually true), since then we need to say that N itself contains everything in the world. But even without Lagadonian names, to claim that the substitution class contains a name for every object in the world (every set, every concrete individual, etc.) looks very much like a metaphysician's move. Indeed, even in claiming that 'the substitution class N contains a name for everything', the word 'everything' seems to call out for an objectual gloss rather than a substitutional one, if only so that the claim itself will not come out as vacuous. So the advocate of substitutional quantification must be willing to partake in metaphysical discourse in order to resist the errors which plague the view otherwise. But in this case, why not just accept objectual readings of the quantifiers instead? Objectual semantics for quantification seems to provide a much simpler account of English quantification than a theory which posits the existence of a class consisting of a repository of alien names cataloguing everything that exists.

To make the point clearer, here is one way in which objectual quantification provides a simpler account of natural language quantification than the rival view just sketched. A presupposition which guides most contemporary work in natural language semantics is that natural language meaning is compositional: the meanings of the complex expressions of a particular language are a function of the meanings of their constituent expressions. In the case of declarative sentences, whose meanings are propositions: the proposition expressed by a compound sentence is a function of the propositions expressed by its constituent sentences. Whether or not one identifies propositions with truth-conditions, this yields the result that the truth-conditions of a complex declarative sentence will be a function of the truth-conditions of its constituent declarative sentences. I claim that if a version of the substitutional theory of natural language quantification just discussed is true—complete with alien names for every individual—then one form of compositionality fails for natural languages. This is one way in which the proposed substitutional analysis suggests a more complicated picture of natural language quantification—indeed, of natural language in general—than does the objectual account. Even though this is ultimately a point about natural language, in order to argue this point clearly I am going to talk about the truth-conditions of quantified statements made in formal languages. Here as I have done previously, I will use $\forall$ and $\exists$ as objectual quantifiers and Π and Σ as substitutional quantifiers.

Suppose we have a non-modal first-order object language L_0 with both objectual and substitutional quantifiers. We have the usual stock of individual variables, predicates, and importantly, a fixed stock $C_0 = \{c_1, c_2, c_3, ...\}$ of individual constants, besides all the usual logical connectives. We may posit an intended model $M_0 = (D, I_0)$ where D is a non-empty set and where I_0 provides every constant c_i with an interpretation $I_0(c_i) \in D$ and every predicate F^n with an interpretation $I_0(F^n) \subseteq D^n$. We will assume that there is at least one $u \in D$ such that for all constants c_i, $I_0(c_i) \neq u$; in other words, there is at least one thing in the domain which has no name in L_0. Because we have fixed an intended model, we can talk about L_0-formulas as being simply true or false (meaning: true or false in the intended model) with respect to a particular variable assignment a. In this set-up, variable assignments are as in Tarskian semantics: functions which assign variables to items in the domain. In the metalanguage, let '$\textsc{True}^0_a(\phi)$' read as 'the L_0-formula ϕ is true (with respect to a)'. We assume that this truth predicate conforms with standard semantic clauses for the vocabulary of L_0. Now without further ado: in our metalanguage we can articulate the truth-conditions of various classes of formulas. For example we have:

For all $\phi, \psi \in L_0$: $\textsc{True}^0_a(\phi \wedge \psi)$ if and only if $\textsc{True}^0_a(\phi)$ and $\textsc{True}^0_a(\psi)$.

Next, applying the usual Tarskian clause for $\forall$ we have:

For all $\phi \in L_0$ and v: $\textsc{True}^0_a(\forall v\phi)$ iff $\textsc{True}^0_{a^v_u}(\phi)$ for every object u

Finally, applying the usual clause for substitutional quantification, we have:

For all $\phi \in L_0$ and v: $\textsc{True}^0_a(\Pi v\phi)$ iff $\textsc{True}^0_a(\phi[c/v])$ for every c in C_0.

Before going further we note three things. First, these three clauses provide compositional truth conditions for L_0-formulas of the form $(\phi \wedge \psi)$, $\forall v\phi$, and $\Pi v\phi$. In each case, the conditions under which the more complex L_0-formula is true are given solely in terms of when less complex L_0-formulas are true (with respect to certain variable assignments). Second, there is a disanalogy between the way in which statements involving substitutional and objectual quantification depend for their truth-conditions on less complex formulas. The truth-conditions of $\forall v\phi$ are given in terms of the truth of the subformula ϕ according to given variable assignments, whereas the truth-conditions of $\Pi v\phi$ are given in terms of the range of formulas $\phi[c/v]$ which result by substitutions.

Third and finally, we note that because of the way that the intended interpretation was set up, in general $\forall v\phi$ and $\Pi v\phi$ do not have the same truth-conditions. Recall that some members of the domain are not named by any constant in C_0. So for example if we assume that $I_0(F) \subseteq D$ is just the set of those items in D that are named by some constant in C_0, we have $\textsc{True}^0_a(\neg\forall xFx)$ and yet $\textsc{True}^0_a(\Pi xFx)$. This provides a formal rendering of the objection raised at the start of the section: if not everything has a name, substitutional readings of the quantifier diverge from objectual readings and deliver incorrect truth-conditions for quantified statements. In this case, because some item in the domain has no name in the class of constants C_0, the statement ΠxFx diverges in truth-condition (indeed, in truth value) from $\forall xFx$. In this case, the result of that divergence is that the objectually quantified statement is a more accurate rendering of the informal claim that everything satisfies F.

Here is how to start to render the reply we have been just been considering against this objection. We will posit the existence of a class of names C_1 which extends our original class C_0. We will extend our intended interpretation to a new interpretation $M_1 = (D, I_1)$ where $I_1(\varepsilon) = I_0(\varepsilon)$ for all expressions ε from L_0. We stipulate that every object in D *is* named by some name in C_1, so that for all $u \in D$, there is some $d \in C_1$ such that $I_1(d) = u$. Now, let L_1 be the result of expanding L_0 with all of the new names from C_1 according to a standard grammar. And where ϕ is an L_1-formula, read '$\textsc{True}^1_a(\phi)$' as 'the L_1-formula ϕ is true (according to assignment a)', and we will assume again that this truth predicate conforms with standard semantic clauses. Intuitively, C_1 represents the extended universal substitution class of names N, which was posited earlier in defense of the substitutional account, and which contains names for every individual in the world (or specifically, in the intended domain). And L_1 represents an ideal language which is extended by this new class of names, while our original language L_0 represents something like ordinary natural language (say English) and C_0 the class of ordinary names in that natural language. This process of extension as a few interesting result. The first is that, in L_1, substitutional and objectual quantification now possess the same truth-conditions (confounding factors seen in Chapter 7 are absent since L_0 and L_1 have no modal vocabulary). That is, by an easy argument one can show that, for all $\phi \in L_1$ and v: $\textsc{True}^1_a(\phi[d/v])$ iff $\textsc{True}^1_{a^v_u}(\phi)$ where $I_1(d) = u$. Hence we have that for all $\phi \in L_1$ and v: $\textsc{True}^1_a(\forall v\phi)$ iff $\textsc{True}^1_a(\Pi v\phi)$. This means that in the ideal language L_1, the substitutional analysis of quantification delivers the right truth-conditions. However, what of our original, mundane language L_0? Well, the success of the substitutional account in L_1 actually implies its success, in one way, in L_0. For any $\phi \in L_0$, we clearly have $\textsc{True}^0_a(\forall v\phi)$ if and only if $\textsc{True}^0_a(\forall v\phi)$. Putting together what we have just seen with the semantic clause for substitutional quantification yields the following fact:

For all $\phi \in L_0$ and v: $\textsc{True}^0_a(\forall v\phi)$ iff $\textsc{True}^1_a(\phi[c/v])$ for every c in C_1.

Intuitively this renders the idea—considered above as a reply to our first objection—that a substitutional analysis of quantification in our original language can be made to deliver the right truth-conditions for quantified statements *if* we allow reference to an ideal substitution class C_1 which has a name for every object, rather than only making use of the stock C_0 of names from the original language. This brings us back to where we started; now we may properly state our follow-up objection to the substitutional account.

The offset biconditional just given is supposed to explicate the truth-conditions of L_0-formulas of the form $\forall v\phi$ in substitutional terms—much like the previous explications given above for formulas of the form $\phi \wedge \psi$, $\forall v\phi$, and $\Pi v\phi$. But this new explication is importantly unlike those previous explications of truth-conditions. It fails to be compositional in the following sense: rather than explicating the truth-conditions of the L_0-formula in terms of the truth in L_0 of other L_0-formulas, it makes essential use of L_1's truth predicate, and reference to the truth of expressions in L_1. Such reference is required because it is only by considering the extended language L_1, allowing substitutions of new names from C_1, that the substitutional truth-conditions can be made adequate. But because of this, it means that on a substitutional account of quantification, the truth-conditions of L_0 formulas are not generally a function of the truth-conditions of simpler L_0-formulas. The meaning of the L_0 formula $\forall v\phi$ is not composed from the meanings of other L_0 formulas, but must be composed in part from the meanings of L_1-formulas. In this way, if the substitutional account is right and we are to dispense with the objectual account of quantification, then we should not be able to give a fully adequate compositional semantics for L_0.[3]

Moving back to the case of English, the point may be put as follows. Suppose N is some posited substitution class which extends the set of English names and contains a name for each object in the world. If this class is appealed to to make the substitutional analysis of English quantification work, we get the following situation. Let English$_N$ be that extension of English which admits all of the new names in N, extending English via ordinary grammatical rules. Then the truth-conditions of English sentence like 'everything is wonderful' can only be explicated by appealing to the truth of new English$_N$ sentences of the form 'c is wonderful' for non-English names c. In this case, the meanings of English sentences are not built up compositionally from other English sentences; an important kind of compositionality then fails for English as a whole. This is undesirable; it means that a substitutional reading of the quantifiers is incompatible with a guiding assumption of modern linguistics, and that logicism is too because it requires such a reading.

[3]Even if one can give such a semantics for the extended language L_1.

To sum up the argument of this section: logicism requires us to adopt a substitutional analysis of natural language (for instance, English) quantification. But such an analysis must proceed in one of two ways, both unsatisfactory. The analysis might only appeal to substitutions of English names, in which case it will be extensionally inadequate because it will give false truth-conditions to statements like 'everything is named by an English word'. Instead, the analysis might appeal to some ideal extended class N of names that includes some name for every existing thing. But in this case, we will lose compositional truth-conditions for quantified statements in ordinary English, since to explicate the truth conditions of such statements, essential reference must be made to the truth of non-English sentences. We should reject substitutional analyses of substitutional quantification.

9.2 Co-Reference and Leibniz's Law

In the case of identity, we also find a trade-off between conventional semantics and classical logic. Suppose our logicist, accepting the assumptions made in Chatepr 1, wants to accept the general validity of Leibniz's Law in natural language. In this case she will have a problem with the following inference.

(1) Hesperus = Phosphorus.
(2) Necessarily, Hesperus = Hesperus.
(3) Therefore, necessarily, Hesperus = Phosphorus.

For the logicist, (3) is clearly false because the sentence 'Hesperus = Phosphorus' is perfectly logically consistent. Moreover, for her, (2) is clearly true since 'Hesperus = Hesperus' is just an instance of reflexivity, hence logically true. So if the logicist wants to uphold Leibniz's law, she must say that (1) is false. But here the names 'Hesperus' and 'Phosphorus' were just an arbitrary example of two names which attach to the same object. Therefore, in order to assert the general validity of Leibniz's Law in natural langauge, the logicist must assert the non-existence of such counter-instances, by insisting that no two English names attach to the same object. But this claim is false. 'Hesparus' and 'Phosphorus' easy counterexamples, as are 'Tully' and 'Cicero' and '$2 + 2$' and '4'. Because of this, it is perfectly possible to have truth identity claims in which different names flank the identity predicate, meaning that this consequence of logicism is false, so that logicism should be rejected.

How might a logicist resist this type of argument? In the first place, perhaps the logicist will propose to restrict attention to some augmented version of English or logic in which there are never two names for the same object (similarly to Bacon and Fine's [4] suggestion to make use of a logically perfect language). But this is just to concede the objection. We use Leibniz's Law when reasoning in

English, not just in purpose-built languages. Unless our logicist wants to deny that Leibniz's Law should be used in English, which would be undesirable in itself, she must explain why it is okay for English speakers to use Leibniz's Law. So she really does need to say that no two *English* names co-denote. She will need to say, for instance, that Hesperus is not numerically identical to Phosphorus; both are planets, and these planets are presumably spaciotemorally coincident, etc. But really, 'Hesparus' and 'Phosphorus' *do* name distinct objects. This line of thought must be repeated in any case of apparently co-denoting names. Another example: the terms '$2+2$' and '4' must not actually denote the same number. $2+2$ and 4 are equivalent in many ways. For instance, any set with $2+2$ elements also has 4 elements and vice versa.[4] But in fact, $2+2$ and 4 are two numbers. In general, apparently co-denoting names really denote 'logically coincident' objects: objects that are exactly alike, except for their being identical to themselves and distinct from one another.

This is not yet a reply to our objection; it is just what the logicist needs to say if she is being consistent. But on this basis, a response might now be formulated as follows. Even though the logicist's claim, that identity statements are only true if the same name appears twice, seems obviously false, we can make sense of it as long as we posit the existence of logically coincident objects. Metaphysicians are no strangers to coincident objects: they are often willing to posit objects that overlap with respect to location, or with respect to all sorts of other properties. So why should the existence of such logically coincident objects, which share all properties save for identity relations, be anything to balk at? And if these are nothing to balk at, why should the logicist's original claim about identity statements be anything to balk at either? To answer this reply we will consider three further points. The first concerns ordinary linguistic practices of stipulative synonymy. The second concerns some of the surprising and unattractive consequences this view has with respect to mathematics. The third, which is in some respects the most basic, concerns the unsatisfactory way in which Leibniz's Law ends up being preserved by the logicist through this strategy.

9.2.1 Synonymy by Stipulation

Names can be introduced by stipulations that fix their reference. This implies that names can co-refer. For example, suppose I announce 'henceforth I will refer to Cicero as 'Tully''? Now by my stipulation, when I use 'Tully' I am referring to the same object as when I use 'Cicero'. Since we can introduce synonymous names in this way, the claim that no English names co-refer is clearly false.

[4] Further objection. If *the* number of elements in a set A is $2+2$ and also 4, doesn't it follow that $2+2$ and 4 are the same number? Possible reply: sets do not have unique cardinalities, only unique cardinalities *upto* some form of equivalence such as that possessed by $2+2$ and 4.

I see two lines of resistance to this argument, of which neither looks appealing. First, perhaps Leibniz's Law fails in any extension or augmentation of English which is arrived at by stipulation in this way. So my stipulation succeeds: I now use 'Tully' and 'Cicero' as co-denoting names. But now by that very fact I am working in a version of English in which Leibniz's Law fails. This is unappealing because it is clear that English itself is a language which has often been extended by stipulative means. For example, in cases of purposeful *re-naming*, not only in theoretical discourse but when a group may decide to give a new name to a place or object, or when a person changes their name legally. If stipulative augmentations or additions to a language do not preserve Leibniz's Law—meaning that they may take us from a language in which the law holds to one where it doesn't—then there is no guarantee that our present iteration of English continues to validate Leibniz's Law, given that some such validity-destroying augmentation have already happened. So now a second line of response which avoids this: perhaps stipulation never succeeds. I say that I intend to use 'Tully' to refer to Cicero, but in fact I fail to carry this out, and I continue to merely use 'Tully' to refer to something logically coincident with Cicero. This is obviously unappealing for other reasons: it makes little sense why stipulation should always fail; if stipulation always fails to yield a co-denoting term, this undermines our ordinary reasoning practices, and so on. The response has the benefit of implying that individual acts of stipulation cannot destroy the validity of Leibniz's Law, but at the cost of implying that acts of stipulation also cannot do what they are intended to.

One more alternative: why not say that, since the stipulation fails, the name 'Tully' fails to signify anything? This makes a certain amount of sense: given that in this scenario I have introduced the name by stipulation, why should the name refer to anything if that stipulation misfires? But this alternative undermines the use of stipulative synonymy in ordinary reasoning even more completely. Suppose that I want to show that a certain property holds of some objects $a_1, \ldots, a_n$. I reason in a general way: 'let b be one of $a_1, \ldots, a_n$, then...' proceeding to show that b must have the property in question. But the stipulation 'let b be one of $a_1, \ldots a_n$' misfires because it essentially means 'suppose $b = a_1$ or ... or $b = a_n$', which will be false since 'b' is a different name from each 'a_i'. If, as above, this means that b is some logically coincident object with one of the a_is, then they following argument is unlikely to be much affected. But if it means that 'b' does not refer to anything, the following argument will be nonsensical.

All of this shows how cases of stipulative synonymy make further trouble for the logicist, and that she is committed to undesirable conclusions about the success of such stipulations if she wants to maintain her view. Now we will consider how ordinary mathematics also makes trouble, by providing further instances of co-referring names.

9.2.2 Mathematics

Pure mathematics provides us with many instances of coarse-grained individuals. Sets obey a principle of extensionality: if A and B share all their elements then they are identical. This should give us easy examples of co-denoting names. We have '$\{a, b\}$' and '$\{b, a\}$', for instance. But perhaps these are not really names; perhaps terms constructed using set-builder notation are really just contracted definite descriptions (so '$\{a, b\}$' abbreviates 'the set whose elements are just a and b'). But then consider another example. Georg is a mathematician. On Monday he is thinking about the set of all even primes and calls it 'E'. On Tuesday he is thinking about the set of all natural numbers $n > 1$ such that the diophantine equation $x^n + y^n = z^n$ has infinitely many solutions, and calls it 'F'. On Wednesday he realises that E and F have exactly the same elements; hence by extensionality, 'E' and 'F' are co-denoting names for the set $\{2\}$—clearly in this case they are not descriptions, even if Georg has certain descriptions in mind when he provides these names.

The logicist might try to resist this case as follows. It would be bad for her to deny the principle of extensionality for sets outright; this would be to adopt revisionism about mathematics and invite an entirely fresh set of objections. What she can do instead is draw a hard distinction between discussions of sets in natural languages and discussions of sets in formal (most importantly, extensional) languages. The reason why logicists are compelled to adopt the view that no English names co-refer is because English contains modal vocabulary, hence co-referring names generate counterexamples to Leibniz's Law in English. She would not be compelled to say that English names never co-refer if English did not contain modal vocabulary, since then Leibniz's Law would not be in jeopardy. Hence in extensional formal languages like the language of ZFC or of Peano Arithmetic, which have no modal vocabulary, it should be possible for her to allow co-referring names and so allow for principles like extensionality to be true without coming up against Leibniz's Law.

But there is something odd about this maneuver. The logicist is still compelled to say that the principle of extensionality fails when discussing sets in English: even if non-modal languages can accommodate the use of extensionality, in discussing collections and sets in English, where modality lurks, extensionality is ruled-out by Leibniz's Law. So there must be a fundamental difference between set theory conducted in a formalism and set theory conduced in English (and similarly for arithmetic). This is implausible; in practice, much mathematical reasoning is formulated in natural language, not formal languages. The principle of extensionality itself is often expressed in the English claim 'sets with the same members are identical', and not exclusively in the formal statement '$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$'. If extensionality must fail in any natural language discussion of sets, this undermines the common use of natural language in formulating set theoretic arguments, and so results in mathematical revisionism of a different kind.

Moreover, on this approach, we seem to preclude the possibility of *adding* modal expressions to a preexisting theory of arithmetic or set theory. We must not add an operator $\Box$ expressive of absolute modality (i.e. logical truth) to Peano Arithmetic, for $1 + 1 = 2$ is a theorem of Peano Arithmetic, and $\Box(2 = 2)$ ought to be a theorem of the extended theory since $2 = 2$ is a logical truth. But then by Leibniz's Law we would need to have $\Box(1 + 1 = 2)$ as a theorem, even though $1 + 1 = 2$ is not a logical truth. Conversely, if $\Box(1 + 1 = 2)$ is omitted as a theorem as required, we would end up with a failure of Leibniz's Law again. So modalisation of existing theories is prohibited. Yet it is hard to see how such a prohibition on modalising existing theories could be independently motivated.

9.2.3 Leibniz's Law as Vacuous

Our final objection to the logicist's strategy of retaining Leibniz's Law is probably the simplest, which is that on the logicist's view, Leibniz's Law reduces to a redundancy. In logic and in natural language, to save Leibniz's Law the logicist must say there are no co-denoting names. But if there are no co-denoting names, then every instance of Leibniz's Law is vacuous. If t and t' are the same term then $t = t' \rightarrow (\phi \rightarrow \phi[t'//t])$ is vacuously true because $\phi[t'//t]$ and ϕ are the same formula, so the instance is really just $t = t' \rightarrow (\phi \rightarrow \phi)$. If t and t' are distinct names, then $t = t' \rightarrow (\phi \rightarrow \phi[t'//t])$ is vacuously true because the antecedent is false. We never have non-vacuously true instances of Leibniz's law. This means that Leibniz's Law is only saved in the form of a vacuity, not useful for any purpose.

Here too the logicist might have something to say in reply. First of all, even though this objection shows that every *sentential* instance of Leibniz's Law is a vacuity on the logicist's view, instances of Leibniz's Law involving variables are also validated and are not similarly vacuous. Suppose we consider a standard non-modal first order language without modal operators, and build a semantics for the language in terms of state descriptions and variable assignments a for state descriptions as detailed in §A.2.4. Then where $\vDash \phi$ means that $S, a \vDash \phi$ for every S and a, we have:

$$\vDash x = y \rightarrow (Fx \rightarrow Fy)$$

Even though 'x' and 'y' are distinct variables. This response to our objection would, however, turn on the particular way in which we have decided to deal with free variables in state-description semantics. Recall that the definition of satisfaction of a formula ϕ with respect to a state-description S and a variable assignment a and a state description S. Recall that $S, a \vDash \phi$ means by definition just that $S \vDash \phi[a]$, where $\phi[a]$ is the sentence obtained by replacing all free variables x in ϕ with the constants $a(x)$ to which a assigns them. In this case, all it means to say that $x = y \to (Fx \to Fy)$ is valid is to note that for all assignments a, $(x = y \to (Fx \to Fy))[a]$ is true in all state descriptions. But this is precicely because, for any assignment a, the sentence $(x = y \to (Fx \to Fy))[a]$ is always vacuous. If $a(x) = a(y) = c$ then $(x = y \to (Fx \to Fy))[a]$ is just the vacuity $c = c \to (Fc \to Fc)$. But if $a(x) \neq a(y)$, then where $a(x) = c$ and $a(y) = d$, the sentence $(x = y \to (Fx \to Fy))[a]$ is $c = d \to (Fc \to Fd)$, which is vacuously valid because $c = d$ is false at every state description. So this argument does not stand up to scrutiny; as soon as we ask why open instances of Leibniz's Law are valid, we see that their validity is based on the vacuity of all sentential instances of Leibniz's Law.

The logicist is not the first to make the logic of identity in some sense redundant by banning co-referential terms. In one sense, this is what Wittgenstein ([70], 5.53ff.) did with his own attempt at implementing a logically perfect language—though he went even further by also preventing distinct variables from taking the same value. So perhaps our logicist can respectably bite the bullet here and acknowledge the sense in which, for them, identity is redundant. Yet there is a key difference between the position of the Tractatus and the position of the logicist here. Wittgenstein only bans co-reference in his logic, not from ordinary language. He does not claim that, in ordinary language, identity of object is signified by identity of the sign, but only that this is will be his convention in logic. But the logicist is not only committed to the claim that an identity sign can be made redundant in certain formal languages via a convention of banning co-reference; they are committed to the claim that reasoning with identity in English is always redundant because co-reference is banned in English. This claim is far less defensible; as we have seen, there are many easy counterexamples that are costly to resist.

9.3 Conclusion

Logicism about necessity requires English to be embeddable in a logically perfect language: a language with a name for every object (§9.1), so that an adequate substitutional reading of quantification can be given, and a language with *no more* than one name for every object so that identity statements will be true iff the same name flanks the identity sign (§9.2). But English is not embeddable in a logically perfect language, because it has many instances of co-referential names. Additionally, even if English can be embedded in some language with a name for every object, a substitutional semantics for English quantification which made use of such an extended language would fail to be compositional, and so not be fully adequate according to prevailing standards in natural language semantics.

To sum up our argument more generally: in order for a logicist to uphold classical logic in any context, including in English, she must take up nonstandard views on quantification and identity. In particular, she must adopt a substitutional reading of the quantifiers and the assumption that nothing is named twice. These views are false and theoretically costly, as we have seen in this chapter. If one wants to avoid such costs—to retain *both* classical logic and defensible views on the semantics of identity and quantification in logic or in natural language—then logicism must be rejected.

10
Conclusion

Logical truth and necessity are altogether different things. Necessity is metaphysical—a feature of the world. Logical truth is a property (or a family of properties) of certain declarative sentences and formulas—a feature of language. There is an elementary distinction which must be made between the two in order to avoid confusion. In this essay I have tried to develop and survey the problems which arise when this distinction is ignored. Due to the centrality of logical truth and necessity as concepts in theoretical philosophy, as well as some logical features which seem to be shared by the two, it is tempting to try to give an account of one in terms of the other. As I have defined the positions here, a modalist is someone who tries to understand logical truth in terms of necessity, and a logicist is someone who tries to understand necessity in terms of logical truth.

Modalism leads to a peculiar kind of revisionism about the practice of logic. If modalism is right, then most logicians are mistaken about the object of their study, and a decent amount of enquiry currently undertaken under the heading of 'logic' is not really logic. Despite the fact that there have been substantial periods in the history of philosophy during which modalism enjoyed predominance among traditional logicians (the scholastic period, for instance), it is not true to the way in which modern logic has been conceived since its emergence. Moreover, modalism makes questions on the validity of arguments implausibly dependent on empirical and metaphysical questions.

Logicism comes in many flavours. We developed three versions of logicism in parallel—semantic, substitutional, and syntactic—and found that each gave rise to a trade-off between classical logic and plausible semantics for quantification and identity. As argued in the previous chapter, if one wants to retain *both* classical logic and conventional views on the semantics of identity and quantification, either in logic or in natural langauge, then logicism is a no go.

In the course of this essay we have also had the opportunity to discuss non-reductive theories of necessity (Ch 4-5), and to investigate the propositional logic of various forms of logical truth under constancy and inconstancy. The following table collects together our results on these logics.

	Semantic	Substitutional	Syntactic
Constancy	S5	McK(= Med?)	Some $\Lambda = L_\alpha(\Lambda)$
Inconstancy	$\times$	TS	TS

For a given type of logicist (e.g. a substitutional logicist), once constancy or inconstancy is assumed, the logic of absolute necessity is to be identified with some logic in the corresponding cell. As also discussed in Chapters 7 and 8, if the Uniqueness Conjecture (Conj. 7.1.9) turns out to be correct, then this table can be rewritten as follows.

	Semantic	Substitutional	Syntactic
Constancy	S5	Med	Med
Inconstancy	$\times$	TS	TS

Therefore the Uniqueness Conjecture also ensures that substitutional and syntactic logical truth a alike with respect to their propositional logics.

So to conclude. Logical truth is different from necessity, and necessity cannot be reduced to logical truth. Therefore metaphysics, in its inclusion of necessity as a proper object of study, cannot be reduced to logical theory. This conclusion is in line with the basic conclusions of Kripke [34], and so with those broadly post-Kripkean currents in analytic philosophy. Metaphysics is an independent discipline and is about the world. Despite any recurrent temptations to reduce it and its core subject matters to logic, or to linguistic analysis more generally, indulging this line of thought leads quickly to problems and confusions. In this essay I hope to have effectively reiterated this basic point, and to have offered some new ways of elaborating it in detail.

Bibliography

[1] ARISTOTLE. *Prior Analytics.* Robin Smith, Ed. Hackett Publishing Company, Cambridge, 1989.

[2] BACON, A. The Broadest Necessity. *Journal of Philosophical Logic 47* (2018), 733–783.

[3] BACON, A. Logical Combinatorialism. *The Philosophical Review 129*, 4 (2020), 537–589.

[4] BACON, A., AND FINE, K. The Logic of Logical Necessity. *MS.* (March 12, 2021).

[5] BERTO, F., AND JAGO, M. *Impossible Worlds*. Oxford University Press, Oxford, 2019.

[6] BOLZANO, B. *Wissenschaftslehre. 4 Volumes*. J. E. v. Seidel, Sulzbach, 1837.

[7] BURGESS, J. P. Which Modal Logic is the Right One? *Notre Dame Journal of Formal Logic 40*, 1 (1999), 81–93.

[8] CALARDO, E., AND ROTOLO, A. Variants of multi-relational semantics for propositional non-normal modal logics. *Journal of Applied Non-Classical Logics 24*, 4 (2014), 293–320.

[9] CARNAP, R. Modalities and quantification. *The Journal of Symbolic Logic 11* (1946), 33–64.

[10] CARNAP, R. *Meaning and Necessity*. University of Chicago Press, Chicago, 1947.

[11] CARNAP, R. Carnap's Intellectual Biography. In *The Philosophy of Rudolph Carnap*, P. A. Schilp, Ed. Open Court, Illinois, 1963, pp. 3–84.

[12] CHALMERS, D. Does Conceivability Entail Possibility? In *Conceivability and Possibility*, Gendler and Hawthorne, Eds. Oxford University Press, 2002.

[13] CHIHARA, C. S. *Introduction to Boolean Algebras*. Springer, New York, 2009.

[14] CRESSWELL, M. The Elimination of De Re Modalities. *Journal of Symbolic Logic 34*, 3 (1969), 329 – 330.

[15] Cresswell, M. Revisiting Mckinsey's 'Syntactical' Construction of Modality. *The Australasian Journal of Logic 17*, 2 (2020).

[16] Cresswell, M. Carnap: Modal logic. *Internet Encyclopaedia of Philosophy* (2022).

[17] Cresswell, M. J. Note on the interpretation of s0.5. *Logique et Analyse 13* (1970), 376–378.

[18] Dorr, C. To be F is to be G. *Philosophical Perspectives, Metaphysics 30*, 1 (2016), 39–134.

[19] Dunn, M., and Nuel D. Belnap, J. The Substitution Interpretation of the Quantifiers. *Noûs 2*, 2 (1968), 177–185.

[20] Field, H. Metalogic and modality. *Philosophical Studies 62*, 1 (1991), 1–22.

[21] Fine, K. Model Theory for Modal Logic II: The Elimination of de re Modality. *Journal of Philosophical Logic 7* (1978), 277–306.

[22] Fine, K. *Modality and Tense: Philosophical Papers*. Clarendon Press, London, 2005.

[23] Fodor, J. D. *The Linguistic Description of Opaque Contexts*. Unpublished PhD Thesis, Massachusetts Institute of Technology, 1970.

[24] Friedman, H. One hundred and two problems in mathematical logic. *Journal of Symbolic Logic 40*, 2 (1975), 113–129.

[25] Gould, J. B. *The Philosophy of Chrysippus*. Brill, Leiden, 1970.

[26] Halbach, V. The Substitutional Analysis of Logical Truth. *Noûs 54*, 2 (March 12, 2021), 431–450.

[27] Hale, B., and Leech, J. Relative necessity reformulated. *Journal of Philosophical Logic 46*, 1 (2016), 1–26.

[28] Hanson, W. H. Actuality, Necessity, and Logical Truth. *Philosophical Studies 130* (2006), 437–459.

[29] Huntington, E. V. Sets of Independent Postulates for the Algebra of Logic. *Transactions of the American Mathematical Society 5*, 3 (1904), 288–309.

[30] Ioup, G. Specificity and the Interpretation of Quantifiers. *Linguistics and Philosophy 1*, 2 (1977), 233–245.

[31] Jeřábek, E. Interpretations of modal logic where $\Box$ means "valid". MathOverflow. URL: mathoverflow.net/q/368010 (version: 2020-07-31).

[32] King, P. Consequence as inference: Mediaeval proof theory 1300-1350. In *Medieval Formal Logic*, Yrjönsuuri, Ed. Springer, 2001.

[33] Kratzer, A. What 'must' and 'can' must and can mean. *Linguistics and Philosophy 1*, 3 (1977), 337–355.

[34] Kripke, S. *Naming and Necessity*. Harvard University Press, Cambridge, MA., 1972.

[35] Leech, J. The varieties of (relative) modality. *Pacific Philosophical Quarterly 97* (2015), 158–180.

[36] Leibniz, G. W. *Discourse on Metaphysics*. Oxford University Press, Oxford, 2020. Translated and Edited by Gonzalo Rodriguez-Pereyra.

[37] Lemmon, E. J. New foundations for the lewis modal systems. *The Journal of Symbolic Logic 22*, 2 (1957), 176–186.

[38] Lemmon, E. J. Algebraic Semantics for Modal Logics i. *The Journal of Symbolic Logic 31*, 1 (1966), 46–65.

[39] Lemmon, E. J. Algebraic Semantics for Modal Logics ii. *The Journal of Symbolic Logic 31*, 2 (1966), 191–218.

[40] Lemmon, E. J., and Scott, D. *An Introduction to Modal Logic*. Basil Blackwell, Oxford, 1977.

[41] Lewis, D. *On the Plurality of Worlds*. Blackwell, Oxford, 1986.

[42] McKinsey, J. C. C. On the syntactical construction of systems of modal logic. *Journal of Symbolic Logic 10*, 3 (1945), 83–94.

[43] Meyer, R. On coherence in modal logics. *Logique et Analyse 14* (1971), 658–668.

[44] Novaes, C. D. Reassessing logical hylomorphism and the demarcation of logical constants. *Synthese 185*, 3 (2012), 387–410.

[45] Paseau, A. C., and Griffiths, O. Propositional logics of truth by logical form. *MS.* (2019).

[46] Plantinga, A. *The Nature of Necessity*. Oxford University Press, Oxford, 1978.

[47] Portner, P. *Modality*. Oxford University Press, Oxford, 2009.

[48] Pruss, A. Possibility is not Consistency. *Philosophical Studies 172*, 9 (2015), 2341–2348.

[49] Quine, W. V. On What There Is. *The Review of Metaphysics 2*, 5 (1948), 21–38.

[50] Quine, W. v. O. Three grades of modal involvement. In *The Ways of Paradox and Other Essays*. Random House, New York, 2005, pp. 156–174.

[51] Rayo, A. On the open-endedness of logical space. *MS.* (September 29th 2018), `http://web.mit.edu/arayo/www/oels.pdf`.

[52] Read, S. Formal and material consequence. *Journal of Philosophical Logic 23*, 3 (1994), 247–265.

[53] Read, S. The medieval theory of consequence. *Synthese 187* (2012), 899–912.

[54] Roberts, A. Relative necessity and propositional quantification. *Journal of Philosophical Logic 49* (2020), 703–726.

[55] Routley, R. The decidability and semantical incompleteness of so.5. *Logique et Analyse 11* (1968), 413–421.

[56] Rumfitt, I. *The Boundary Stones of Thought: An Essay in the Philosophy of Logic*. Oxford University Press, Oxford, 2015.

[57] Russell, B. *The Philosophy of Logical Atomism*. Routledge, 1940.

[58] Rybakov, V. V. *Admissibility of Logical Inference Rules*. Elsevier, 1978.

[59] Scroggs, S. J. Extensions of the Lewis system S5. *Journal of Symbolic Logic 16*, 2 (1951), 112 – 120.

[60] Shieh, S. *Necessity Lost*. Oxford University Press, Oxford, 2019.

[61] Sider, T. *Logic for Philosophy*. Oxford University Press, Oxford, 2010.

[62] Sider, T. *Writing the Book of the World*. Oxford University Press, Oxford, 2011.

[63] Soames, S. *Understanding Truth*. Oxford University Press, Oxford, 1999.

[64] Spinoza, B. D. Treatise on the Emendation of the Intellect. In *Complete Works of Spinoza*, M. L. Morgan, Ed. Indianapolis: Hackett., 2002. Transl. Samuel Shirley.

[65] Strohminger, M., and Yli-Vakkuri, J. Knowledge of objective modality. *Philosophical Studies 176*, 5 (2019), 1155–1175.

[66] Tarski, A. On the concept of logical consequence. In *Logic, semantics, metamathematics: papers from 1923 to 1938*, J. Corcoran, Ed. Indianapolis: Hackett., 1983.

[67] Urquhart, A. Anderson and Belnap's Invitation to Sin. *Journal of Philosophical Logic 39*, 4 (2010), 453–472.

[68] Williamson, T. *Modal Logic as Metaphysics*. Oxford University Press, Oxford, 2013.

[69] Williamson, T. Modal Science. *Canadian Journal of Philosophy 46*, (4-5) (2016), 453–492.

[70] Wittgenstein, L. *Tractatus Logico-Philosophicus*. Routledge, Oxford, 2001.

[71] Zalta, E. N. Logical and Analytic Truths That are Not Necessary. *Journal of Philosophy 85*, 2 (1988), 57–74.

Part III

Appendices

A
Logical Background

In this appendix I fill in the technical background assumed in the main discussion. §A.1 discusses propositional modal logic; §A.2 discusses quantified modal logic.

A.1 Propositional Modal Logic

First I introduce the language of Propositional Modal Logic (PML), and the idea of a PML logic. Firstly then, the language PML is built up from a denumerable supply of sentence letters At $= \{p, q, r, ...\}$ via the connectives $\rightarrow$, $\bot$, and $\Box$ according to this grammar:

> Every sentence letter is a PML formula.
> $\bot$ is a PML formula.
> If ϕ, ψ are PML formulas then $\Box\phi$ and $(\phi \rightarrow \psi)$ are PML formulas.
> Nothing else is a PML formula.

PML is identified with the set of PML formulas. In this language the other standard connectives of $\wedge$, $\vee$, $\leftrightarrow$ and $\Diamond$ are given standard definitions. Standard conventions are also adopted for omitting brackets, where doing so does not generate ambiguity.

Next we introduce PML logics according to a standard definition. We understand a PML logic to be a set Λ of PML formulas that fulfills three basic conditions, which we will first consider individually. The first condition is that a logic must be closed under modus ponens:

Definition A.1.1. A set Γ is closed under modus ponens (MP) iff whenever $\phi, (\phi \to \psi) \in \Gamma$, it is also the case that $\psi \in \Gamma$

The second condition is that a logic must include the classical propositional calculus. In particular, every logic must include the following set:

Definition A.1.2. PC is the smallest set of PML formulas which contains all instances of PL1-PL3, and which is closed under modus ponens.

(PL1) $\phi \to (\psi \to \phi)$

(PL2) $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$

(PL3) $((\phi \to \psi) \to \phi) \to \phi$

Note that PC, which is characterised via a standard axiomatisation of the non-modal propositional calculus, still contains formulas with instances of $\Box$. For example, $\Box p \to (\Diamond q \to \Box p)$ is in PC. We can also understand PC alternatively as the set of all classical tautologies expressible in the language PML. Now, to introduce the third and final condition, we will first need to introduce substitution functions.

Definition A.1.3. A PML-substitution (in context just a substitution or uniform substitution) is a function s: PML $\to$ PML such that for all ϕ and ψ:

$s\bot = \bot$

$s(\phi \to \psi) = (s\phi \to s\psi)$

$s\Box\phi = \Box s\phi$.

The third condition for logics is that they be closed under such substitution functions:

Definition A.1.4. A set Γ of PML formulas is closed under uniform substitution iff for all substitutions s and formulas ϕ, if $\phi \in \Gamma$ then $s\phi \in \Gamma$.

To sum up these three conditions, the definition of of a PML logic is as follows:

Definition A.1.5. A PML logic is a set Λ of PML formulas such that

PC $\subseteq \Lambda$,

Λ is closed under modus ponens, and

Λ is closed under uniform substitution.

As I have already been doing, I will often use the letter Λ, sometimes with dashes or subscripts, to denote arbitrary logics. Now, according to this definition of a logic, it turns out that PC is itself a logic. Trivially it meets the first and second conditions, and it is known to meet the third condition. So we have:

Proposition A.1.6. PC is the smallest PML logic with respect to inclusion.

New logics can be defined by reference to additional formulas added as axioms, and rules of closure to supplement modus ponens and uniform substitution. For example, the PML logic K is defined as the smallest logic containing the formula $\Box(p \to q) \to (\Box p \to \Box q)$, which is called K, and closed under the rule of necessitation (Nec), which says that if ϕ is a theorem then so is $\Box\phi$. In general:

Definition A.1.7. Where each X_i is either a set of formulas, a formula or a rule we write $(X_1 + ... + X_n)$ to denote the smallest logic which includes/contains/admits each X_i.

So, using the previous example, $\mathsf{K} =$ (K + Nec). As I have been doing already, I will often use sans serif fonts to name logics and theories, and ordinary serif font to name formulas and rules. This is especially helpful for differentiating between logics and formulas denoted by the same letter, like K and K for example.

Finally, there are many cases where a set of formulas Γ is not a logic, but includes logics within it. Under certain conditions we can show that there is a largest logic included in Γ, which we name $\ell\Gamma$ according to the following convention.

Definition A.1.8. Where Γ is a set of formulas, $\ell\Gamma = \{\phi : \text{ for all substitutions } s, s\phi \in \Gamma\}$.

In particular we have:

Proposition A.1.9. If $\Gamma \supseteq$ PC is closed under modus ponens, $\ell\Gamma$ is the largest logic in Γ by inclusion.

Proof. Straightforward. $\square$

A.1.1 Named Propositional Modal Logics, Axioms and Rules

AXIOMS

K	$\Box(p \to q) \to (\Box p \to \Box q)$
D	$\Box p \to \Diamond p$
T	$\Box p \to p$
B	$p \to \Box\Diamond p$
4	$\Box p \to \Box\Box p$
5	$\Diamond p \to \Box\Diamond p$
Mc	$\Box\Diamond p \to \Diamond\Box p$
Triv	$p \leftrightarrow \Box p$
Ver	$\Box p$
Fal	$\neg\Box p$
N	$\Box\top$
C	$(\Box p \wedge \Box q) \to \Box(p \wedge q)$
M	$\Box(p \wedge q) \to (\Box p \wedge \Box q)$

RULES

Nec	If ϕ is a theorem, $\Box\phi$ is a theorem.
Mon	If $\phi \to \psi$ is a theorem, $\Box\phi \to \Box\psi$ is a theorem.

OTHER SETS OF FORMULAS

$\Box(\Gamma)$	$\{\Box\psi : \psi \in \Gamma\}$
$\Box^-(\Gamma)$	$\{\psi : \Box\psi \in \Gamma\}$.
Mon(Γ)	$\{\Box\phi \to \Box\psi : \phi \to \psi \in \Gamma\}$

LOGICS (NORMAL)

K	(K + Nec)
D	(K + D + Nec)
T	(K + T + Nec)
B	(K + T + B + Nec)
S4	(K + T + 4 + Nec)
S5	(K + T + 5 + Nec)
S4M	(K + T + 4 + Mc + Nec)
Triv	(PC + Triv)
Ver	(PC + Ver)
Med	In Kripke semantics, the logic of Medvedev frames (cf. §X.YY).
McK	In McKinsey semantics, the logic of the full substitution class (cf. §X.YY).
PML	The inconsistent logic, (PC $+ \bot$).

LOGICS (NON-NORMAL)

$\mathsf{S0.5}^0$	(K + $\Box$PC)
S0.5	(K + T + $\Box$PC)
TS	The logic of tautological validity (cf. §X.XX).
Fal	(PC + Fal)
$\mathsf{S0.2}^0$	(PC + Mon(PC))
S0.2	(PC + T + Mon(PC))

A.2 First-Order Modal Logic

In chapter 8 we will consider two languages of first-order modal logic. The first contains the objectual quantifier $\forall$, while the second contains the substitutional quantifier Π. This is the only difference between the languages: both languages contain a logical identity predicate, and share the same stock of individual variables and constants, as well as predicate symbols, connectives, and sentence letters. The grammar for both languages is essentially the same, but the definition of a logic differs more substantially between the two. I will first introduce the language containing $\forall$ along with the definition of a logic in this language. Then I will say how all of this is altered in the case of the language containing Π. I will also make some general remarks about providing semantics for these languages.

A.2.1 QML$\forall$

The language QML$\forall$ is constructed from a denumerable supply of individual variables $\mathrm{Vr} = \{x, y, z, ...\}$, a denumerable supply of individual constants $\mathrm{Cn} = \{c, d, e, ...\}$, and a denumerable supply of predicate constants $\mathrm{Pr}_n = \{P^n, Q^n, R^n, ...\}$ for every arity $n \geq 0$. Predicate constants of arity 0 are identified with sentence letters and are written $p, q, r, ...$ as in PML. Predicate constants and individual constants are *non-logical symbols*, with every other kind of symbol being logical. Alongside the connectives $\to$, $\bot$ and $\Box$, we have the additional symbols $\forall$ and $=$ (a predicate of arity 2, also counted as a logical symbol). The grammar is as follows. A term is either an individual constant or an individual variable. A predicate is either a predicate constant P^n or $=$.

> All sentence letters are formulas, and $\bot$ is a formula.
> If F^n is an n-adic predicate and $t_1, ..., t_n$ are terms, $F^n t_1 ... t_n$ is a formula.
> If ϕ and ψ are formulas then so are $\Box\phi$ and $(\phi \to \psi)$.
> If ϕ is a formula and v an individual variable, $\forall v \phi$ is a formula.
> Nothing else is a formula.

An atomic formula is either a sentence letter, $\bot$, or an n-adic predicate followed by t terms. Now for some notational conventions. The other connectives like conjunction, disjunction, possibility, and negation are given the same definitions as before. Additionally we define $\exists$ by the abbreviation $\exists x\phi := \neg\forall x\neg\phi$. The identity predicate is written infix, meaning we write $t = t'$ rather than $= tt'$. We also write $t \neq t'$ to abbreviate $\neg t = t'$.

The following notions are all standard. The scope of an occurrence of some connective or quantifier $\forall x$ in a formula ϕ is the smallest subformula of ϕ in which the occurrence occurs. An occurrence of a variable x in a formula ϕ is bound iff it occurs within the scope of an occurrence of '$\forall x$', otherwise the occurrence is free.[1] For the sake of stating some definitions, any occurrence of an individual constant is also 'free'. A formula ϕ is called closed iff it contains no free occurrences of any variables, else it is called open. Closed formulas are also called *sentences*. We say that ϕ and ψ are alphabetic variants iff they are identical except for replacements of their bound variables, which preserve the same pattern of identity and distinctness. For example, $\forall xPx$ and $\forall yPy$ are alphabetic variants, but $\forall x\forall z(Px \vee Qy)$ and $\forall y\forall y(Py \vee Qy)$ are not alphabetic variants. Since in moving from the first to the second two variables are substituted for the same variable, the pattern of distinctness is not preserved. We write $\phi[t'/t]$ to denote one of two things: as long as ϕ contains no free occurrences of t within the scope of some occurrence of '$\forall t'$', then $\phi[t'/t]$ denotes the result of replacing all free occurrences of t with an occurrence of t' which is then also free. Otherwise, $\phi[t'/t]$ denotes the result of carrying out such a replacement on some alphabetic variant ϕ' of ϕ in which t does *not* occur free within the scope of some $\forall t'$. When multiple terms are indicated in one set of square brackets as in $\phi[t'_1/t_1, ..., t'_n/t_n]$, this indicates a *simultaneous* replacement of $t_1, ..., t_n$ with $t'_1, ..., t'_n$, following similar caveats as in the single-term case. We write $\phi[t'//t]$ to symbolise any result of replacing zero or more (but not necessarily all) free occurrences of t with t' in a similar way. With this out of the way, we may introduce the definition of a logic in this language. As in PML, QML$\forall$ logics are defined as sets of formulas which satisfy certain special conditions. The conditions themselves are also analogues to the defining conditions of PML logics, and we will go through them each in turn now. In the first place, QML$\forall$ logics are closed under modus ponens. They are additionally closed under the following conditions. First a rule of universal generalisation.

Definition A.2.1. A set of QML$\forall$ formulas is closed under universal generalisation iff whenever $(\phi \to \psi) \in \Gamma$ and v is not free in ϕ then $(\phi \to \forall v\psi) \in \Gamma$.

And additionally two rules of substitution. One for variables:

Definition A.2.2. A set of QML$\forall$ formulas is closed under free variable substitution iff whenever $\phi \in \Gamma$, v is a variable, and t is a term, then also $\phi[t/v] \in \Gamma$.

And another rule of substitution for non-logical symbols.

Definition A.2.3. A QML$\forall$-substitution is a mapping ς such that for all ϕ, ψ:

$\varsigma(\phi \to \psi) = (\varsigma\phi \to \varsigma\psi)$

$\varsigma\Box\phi = \Box\varsigma\phi$

[1] This means that the variable x which features in the quantifier '$\forall x$' is bound.

$\varsigma(\forall v\phi) = \forall v\varsigma(\phi)$
$\varsigma(P^n t_1 ... t_n) = \varsigma(P^n)\varsigma(t_1)...\varsigma(t_n)$
$\varsigma(P^n)$ is an n-ary predicate for all predicate constants P^n ($n \geq 1$).
$\varsigma(p)$ is a formula for all sentence letters p.
$\varsigma(c)$ is a term for all individual constants c.
$\varsigma(\varepsilon) = \varepsilon$ for all logical symbols ε.

Definition A.2.4. Where c is a constant and t is some term, let $\varsigma_{c\to t}$ be the unique QML∀ substitution such that $\varsigma_{c\to t}(c) = t$ and $\varsigma_{c\to t}(\varepsilon) = \varepsilon$ for all nonlogical symbols ε besides c.

Definition A.2.5. A set of QML∀ formulas is closed under QML∀-substitutions iff whenever $\phi \in \Gamma$, and ς is a QML∀-substitution, $\varsigma\phi \in \Gamma$.

Finally, logics are taken to include all truths of the classical predicate calculus with identity. With all of this said, the definition of a logic in QML∀ can be summed up as follows.

Definition A.2.6. A QML∀ logic is a set of QML∀ formulas which is closed under modus ponens, universal generalisation, free variable substitution, uniform QML∀-substitutions, and which contains all PML tautologies and all instances of the following schemas[2]

(UI) $\forall v\phi \to \phi[t/v]$
(Rf) $v = v$
(LL) $v = v' \to (\phi \to \phi[v'//v])$

I have set up the definition here to mirror the definition of a PML logic as closely as possible. Now, I adopt a rough naming convention for QML∀ logics. Namely, where Λ is a PML logic, let ∀Λ denote the smallest QML∀ logic that includes Λ (note that by the above grammar, and by the identification of sentence letters with 0-ary predicate constants, that every PML formula is a QML∀ formula). Hence ∀PC would be the smallest QML∀ logic, since this is also the smallest logic in the new language which includes the PML logic PC.

A.2.2 QMLΠ

The language QMLΠ is exactly like QML∀ except for the presence of the substitutional quantifier Π in place of the objectual quantifier ∀. Therefore the language is built from the same stock of predicates, constants and variables (etc.) from the same connectives according to a grammar which is identical to that of QML∀ but for the presence of 'Π' in place of '∀'. The distinction between logical and non-logical symbols is also as before, with Π counted as logical. Analogous definitions of

[2] From this one can show that QML∀ logics also admit the distribution schema $\forall v(\phi \to \psi) \to (\forall v\phi \to \forall v\psi)$.

scope, free and bound variables, open formulas and sentences, and alphabetic variance are adopted. Where ϕ is a QMLΠ formula, and t and t' are terms, $\phi[t'/t]$ and $\phi[t'//t]$ have basically the same meaning as before. Also retained are the standard definitions of the truth-functional connectives, and $\Diamond$. In place of $\exists$, the dual of Π is written Σ, defined $\Sigma x\phi := \neg\Pi x\neg\phi$.

The definition of a QMLΠ logic proceeds exactly as for QML∀ logics, with similar definitions for closure under universal generalisation, QMLΠ-substitutions, and so on, with 'Π' now being used everywhere rather than '∀'. I adopt the same naming convention for QMLΠ logics as I do for QML∀ logics. Namely, where Λ is a PML logic, ΠΛ is the smallest QMLΠ logic that includes Λ. Hence ΠPC is the smallest QMLΠ logic.

A.2.3 Formula Complexity

For each language PML, QML∀ and QMLΠ, we may define a natural notion of formula complexity as follows.

Definition A.2.7. The complexity of a PML formula, $comp(\phi)$ is defined as follows.

$comp(\phi) = 0$ if ϕ is atomic

$comp_\forall(\Box\phi) = comp(\phi) + 1$

$comp_\forall(\phi \to \psi) = max\{comp_\forall(\phi), comp_\forall(\psi)\} + 1$

Definition A.2.8. The complexity of a QML∀ formula, $comp_\forall(\phi)$ is defined as follows.

$comp_\forall(\phi) = 0$ if ϕ is atomic

$comp_\forall(\forall x\phi) = comp_\forall(\Box\phi) = comp_\forall(\phi) + 1$

$comp_\forall(\phi \to \psi) = max\{comp_\forall(\phi), comp_\forall(\psi)\} + 1$

Definition A.2.9. The complexity of a QMLΠ formula, $comp_\Pi(\phi)$ is defined as follows.

$comp_\Pi(\phi) = 0$ if ϕ is atomic

$comp_\Pi(\Pi x\phi) = comp_\Pi(\Box\phi) = comp_\Pi(\phi) + 1$

$comp_\Pi(\phi \to \psi) = max\{comp_\Pi(\phi), comp_\Pi(\psi)\} + 1$

A.2.4 Semantics of Objectual and Substitutional Quantification

In this section I will give some of the general background on the ways in which objectual and substitutional quantification are evaluated in various semantic frameworks. I will not be exhaustive, since I cover the same topic in greater detail and specificity in chapter 8 with respect to the two languages just introduced.

The standard way of providing semantics for languages with objectual quantification is the familiar Tarskian one. That is, formulas of such languages are evaluated at models equipped with a domain and an interpretation function with respect to various variable assignments on the domain. More precisely:

Definition A.2.10. A (Tarskian) structure is a pair $M = (|M|, [\cdot]_M)$ where $|M|$ is a non-empty set and $[\cdot]_M$ is a function on the predicates and constants of the above languages such that $[c]_M \in |M|$ and $[F^n]_M \subseteq |M|^n$ for every constant c and n-adic predicate F^n—in particular, such that $[=]_M$ is always the true identity relation $\{(u, u) : u \in |M|\}$ over the domain.

Definition A.2.11. A variable assignment on M is a function $a : \mathrm{Vr} \to |M|$.

Statements involving $\forall$ are evaluated at such models with respect to variable assignments according to the following general clause:

$M, a \vDash_{(-)} \forall x\phi$ if and only iff for all $u \in |M|$, $M, a^x_u \vDash_{(-)} \phi$.

Here, a^x_u represents the unique variable assignment such that $a^x_u(x) = u$ and such that $a^x_u(y) = a(y)$ for all variables y besides x, and $(-)$ stands for whatever additional parameters are needed by the semantics in question. For example, in the semantics of chapter 8, $(-)$ is replaced by a pair (Λ, μ), where Λ is a logic and μ is a function which assigns formulas to models. Now, it is also possible to provide a semantics for languages of substitutional quantification in terms of Tarskian models as we do for languages of objectual quantification. Here, the appeal to varible assignments is not required, and *sentences* containing Π can be evaluated according to the following general clause.

$M \vDash_{(-)} \Pi x\phi$ if and only iff for all $c \in \mathrm{Cn}$, $M \vDash_{(-)} \phi[c/x]$.

As one might have guessed from comparing this to the previous clause, the apparatus of variable assignments and a domain of quantification is rendered redundant if substitutional quantification is the only mode of quantification in one's language. The chief other function of a domain besides evaluating Π-formulas is to provide interpretations to predicates and constants for the evaluation of atomic sentences. But the latter can be achieved through alternate means via the Carnapian apparatus of *state descriptions*. For our purposes, a state description is simply a set of atomic sentences—in particular, those which only involve non-logical symbols:

Definition A.2.12. A basic sentence is an n-adic predicate constant followed by n individual constants. (Equivalently, a basic sentence is an atomic sentence not containing $=$ or $\bot$.

Definition A.2.13. A state description is a set S of basic sentences.[3]

Tarksi-style semantics for quantificational languages are familiar; semantics in terms of such state descriptions are less so. So we can finish this section by setting out the general idea. In general the handling of open formulas is much more nuanced in this setting. So to start we we define a notion of satisfaction which applies only to sentences of QMLΠ. Ignoring modal formulas for the moment, sentences ϕ are evaluated in state descriptions S with respect to assignments a, and perhaps also with respect to certain additional parameters $(-)$, according to the following clauses:

$S \not\models_{(-)} \bot$
$S \models_{(-)} \phi$ iff $\phi \in S$ where ϕ is basic.
$S \models_{(-)} c = d$ iff c and d are the very same constant.
$S \models_{(-)} \phi \to \psi$ iff $S \not\models_{(-)} \phi$ or $S \models_{(-)} \psi$.
$S \models_{(-)} \Pi x\phi$ iff for all individual constants c: $S \models_{(-)} \phi[c/x]$.

The provision that $c = c'$ be false at a state-description whenever c and c' are different constants might seem oddly strong. It is the semantic clause used in Carnap [9], and advocated in Bacon and Fine [4]. In the main discussion we will see that it is sometimes necessary to avoid widespread failures of Leibniz's Law. We will use this point in Chapter 9 to criticise the logicist semantics of previous chapters. Now, to return to the matter of open formulas: how can one generalise the definition of satisfaction in order to handle open formulas? I have found the following to be the simplest, being a simplification of the strategy of Carnap [10].[4] We first re-appropriate the notion of a 'variable assignment' as follows.

Definition A.2.14. A variable assignment (for st. descriptions) is a function $a : \mathrm{Vr} \to \mathrm{Cn}$.

Then for any assignment a, we also define a notation $[a]$ as follows. Where t is a constant let $t[a]$ just be t itself; where t is a variable let $t[a]$ be $a(t)$. Where ϕ is a formula, let $\phi[a] = \phi[a(x_1)/x_1, ..., a(x_n)/x_n]$ where $x_1, ..., x_n$ are precisely the variables that occur free in ϕ—clearly

[3]Strictly, Carnap's notion of a state-description involves a state-description containing exactly one member of the pair $\{\phi, \neg\phi\}$ for every atomic sentence ϕ. But the inclusion of negations is redundant, and we can mimic the inclusion of $\neg\phi$ by omitting ϕ for any atomic ϕ.

[4]In Carnap's 1947 semantics, assignments are functions from pairs (S, x) of state-descriptions and variables to individual constants. So one and the same assignment might assign x to different constants at different descriptions. This is much in line with Carnap's method of 'extension and intension' taken up in the 1947 book, but is unacceptable from our point of view. As noted in Cresswell [16], Carnap's 1947 semantics allows for failures of Leibniz's Law where modal operators are involved, which we have disallowed by fiat (§1.2). Our discussion of Carnap in Chapter 6 follows more closely the 1946 semantics, in which Leibniz's Law holds in full generality.

$\phi[a] = \phi$ wherever ϕ is a sentence. Finally, as with variable assignments in Tarskian semantics, where a is an assignment, v is a variable and c is a constant, we may write a^v_c to denote the unique assignment such that $a^v_c(v) = c$ and $a^v_c(v') = a(v')$ for all variables v' besides v. With this in hand we may define a notion of satisfaction for any formula ϕ with respect to a state description S and assignment a as follows:

$S, a \vDash_{(-)} \phi$ iff $S \vDash_{(-)} \phi[a]$

An interesting feature of this definition is that we can use it to recover a substitutional counterpart to the Tarskian clause for quantification:

Proposition A.2.15. $S, a \vDash_{(-)} \Pi v\phi$ iff for all constants c: $S, a^v_c \vDash_{(-)} \phi$.

Proof. $S, a \vDash_{(-)} \Pi v\phi$ iff $S \vDash_{(-)} (\Pi v\phi)[a] = (\Pi v\phi)[a(v_1)/v_1, ..., a(v_n)/v_n]$ where $v_1, ..., v_n$ are precisely the variables that occur free in $\Pi v\phi$. Clearly v is not among $v_1, ..., v_n$, so

$$(\Pi v\phi)[a(v_1)/v_1, ..., a(v_n)/v_n] = \Pi v(\phi[a(v_1)/v_1, ..., a(v_n)/v_n])$$

Therefore $S \vDash_{(-)} (\Pi v\phi)[a]$ iff for all constants c we have $S \vDash_{(-)} \phi[a(v_1)/v_1, ..., a(v_n)/v_n][c/v]$. Again, since v is not among $v_1, ..., v_n$ we can see that we have $\phi[a(v_1)/v_1, ..., a(v_n)/v_n][c/v] = \phi[a(v_1)/v_1, ..., a(v_n)/v_n, c/v]$. Then there are two cases to consider. If v does not occur free in ϕ, then

$$\phi[a(v_1)/v_1, ..., a(v_n)/v_n, c/v] = \phi[a(v_1)/v_1, ..., a(v_n)/v_n][c/v] = \phi[a(v_1)/v_1, ..., a(v_n)/v_n]$$

but since $v_1, ..., v_n$ are precisely the variables occurring free in ϕ, $\phi[a(v_1)/v_1, ..., a(v_n)/v_n] = \phi[a^v_c] = \phi[a]$ for all constants c. So putting this together we have $S \vDash_{(-)} (\Pi v\phi)[a]$ iff $S \vDash_{(-)} \phi[a]$ iff $S \vDash_{(-)} \phi[a^v_c]$. Hence $S, a \vDash_{(-)} (\Pi v\phi)$ iff $S, a^v_c \vDash_{(-)} \phi$ as required. Alternately, if v does occur free in ϕ then $v_1, ..., v_n, v$ are exactly the variables that occur free in ϕ, so we have that $\phi[a(v_1)/v_1, ..., a(v_n)/v_n, c/v] = \phi[a^v_c]$. But then putting this together we have that $S \vDash_{(-)} (\Pi v\phi)[a]$ iff $S \vDash_{(-)} \phi[a^v_c]$ for all c, meaning $S, a \vDash_{(-)} (\Pi v\phi)$ iff $S, a^v_c \vDash_{(-)} \phi$ as required. $\square$

Now in fact, we can then use variable assignments to extend the idea of satisfaction of an open formula ϕ at a state description S by itself $S \vDash_{(-)} \phi$. Simply say that in this case, $S \vDash_{(-)} \phi$ iff $S, a \vDash_{(-)} \phi$ for all assignments a. This is a natural extension of the original satisfaction relation defined for sentences, since if ψ is a sentence we also see that $S \vDash_{(-)} \psi$ iff $S, a \vDash_{(-)} \psi$ for all ψ, since $\psi[a] = \psi$. Before moving on, the following proposition is also sometimes useful.

Proposition A.2.16. $S, a \vDash_{(-)} \phi \to \psi$ iff $S, a \nvDash_{(-)} \phi$ or $S, a \vDash_{(-)} \psi$

Proof. $S, a \vDash_{(-)} \phi \to \psi$ iff $S \vDash_{(-)} \phi \to \psi$, but $(\phi \to \psi)[a] = (\phi[a] \to \psi[a])$ so this holds iff $S \nvDash_{(-)} \phi[a]$ or $S \vDash_{(-)} \psi[a]$. But this holds iff $S, a \nvDash_{(-)} \phi$ or $S, a \vDash_{(-)} \psi$. □

Now, the previous semantic clauses are disambiguated by the particular systems of semantics studied in the main discussion. For instance, in the case of Carnap's semantics, these clauses are disambiguated by omitting $(-)$ and adding this clause for modal sentences:

$S \vDash \Box\phi$ iff for all state descriptions S': $S' \vDash \phi$.

By contrast, in our semantics presented in Chapter 8, the first three clauses are disambiguated by replacing $(-)$ with a pair 'Λ, μ' where Λ is a QMLΠ logic and μ assigns every state description a set $\mu(S)$ of QMLΠ sentences, and using the following clause for modal sentences:

$S \vDash_{\Lambda,\mu} \Box\phi$ iff $\mu(S) \vdash_{\Lambda} \phi$.

Now that we have shown how substitutional quantifiers are generally given semantics, I can make one final point, which is important for our discussion but often not remarked upon.[5]

There is a difficulty in defining semantic consequence relations for QMLΠ. If we proceed in the most familiar way, saying that ϕ is a consequence of Γ iff ϕ is true in every state description which validates all of Γ, we will end up with a relation which is both non-compact and not closed under substitutions. This is because of the general form of the clause for Π. Basically, suppose that $\vDash$ is some consequence relation defined over QMLΠ sentences in terms of truth at state descriptions (or something similar so long as the definition of truth incorporates the above clause for Π).[6] Then we will have

$\{\phi[c/v] : c$ is a constant$\} \vDash \Pi v\phi$, where each $\phi[c/v]$ and $\Pi v\phi$ is a sentence.

This is simply because if every $\phi[c/v]$ is true at some model/state description/etc. (according to certain additional parameters, etc.), then by the clause for Π, $\Pi v\phi$ must also be true in the same respect. Now note that since the set of constants is infinite, the set on the left hand side of this sequent is infinite. Therefore if $\vDash$ were compact, there would need to be some finite collection of constants $c_1, ..., c_n$ such that $\{\phi[c_1/x], ..., \phi[c_n/x]\} \vDash \Pi x\phi$. But note in this case that by taking the conjunction of the finitely many formulas on the left we get $(\phi[c_1/x] \wedge ... \wedge \phi[c_n/x]) \vDash \Pi x\phi$. But already by ΠPC and the relevant admissible instances of universal instantitation and

[5]Dunn and Belnap [19]

[6]That is, we will have some version of the following: $\Gamma \vDash \phi$ iff any model (or etc.) at which every sentence in Γ is true is also a model (or etc.) at which ϕ is true.

PC, we also have $\Pi x\phi \vDash (\phi[c_1/x] \wedge ... \wedge \phi[c_n/x])$. Therefore if $\vDash$ is compact, a universal statement is always logically equivalent with a finite conjunction of its instances. Since this is generally not acceptable, we have the result that most workable semantically defined consequence relations for QMLΠ are non-compact.

This also implies that closure under substitution fails for $\vDash$. Suppose we enumerate all the individual constants $c_1, c_2, ...$ and let ς be the unique substitution which sends each c_n to $\varsigma(c_n) = c_{n+1}$, while leaving everything else intact. Then where P is unary:

$$\{Pc_n : n \in \omega\} \vDash \Pi v P v$$

And yet if we apply the the substitution we are lead to

$$\{\varsigma(Pc_n) : n \in \omega\} \vDash \varsigma \Pi v P v$$

which is false. For carrying through the substitution, we see that this is the same as:

$$\{Pc_{n+1} : n \in \omega\} \vDash \Pi v P v$$

And this is clearly false since we might have a state description S where $Pc_1 \notin S$ (so that $S, a \nvDash_{(-)} \Pi x P v$) even though $Pc_k \in S$ for all $k > 1$. Now as a matter of fact, these problems will not make a difference for our own discussion. But it is worth flagging them for the sake of caution.

B
Multiplex Modalities

Here we look at an interesting way of generalising the syntactical semantics for PML from chapter 7. This mode of generalisation is analogous to the multi-relational Kripke semantics studied in for example [8]. To fix some terminology: a monotonic logic is a logic closed under the rule of monotonicity (Mon): that if $\phi \to \psi$ is a theorem then so is $\Box\phi \to \Box\psi$. The normal logics form a proper subset of the monotonic logics, and the smallest monotonic logic is called M. A *regular* logic is a monotonic logic which contains the axiom C = $(\Box p \wedge \Box q) \to \Box(p \wedge q)$. The normal logics also form a proper subset of the regular logics, and the smallest regular logic is called C.[1] Adding the N axiom $(\Box\top)$ to M yields the logic MN and adding N to C yields the smallest normal logic K. Multiplex extensions to our semantics allow for completeness results to be given to logics which do not contain C or N.

B.1 Basic Notions

B.1.1 Adapting Definitions

The basic point of generalisation in this extension to the previous semantics is with the new notion of a *multiplex* modality:

Definition B.1.1. A multiplex modality is a function $\overline{\mu} : \mathcal{P}(\mathrm{At}) \to \mathcal{PP}(\mathrm{PML})$ which assigns each model m a set of sets of formulas $\overline{\mu}(M) \subseteq \mathcal{P}(\mathrm{PML})$.

[1] Sometimes the smallest regular logic is called 'R'.

The set of multiplex modalities is called $\overline{\mathbf{Mod}}$. The definition of a model and of a logic is unchanged, and the semantic clauses for our multiplex semantics are as follows.

> Where m is a model, Λ a logic and $\overline{\mu}$ is a multiplex modality, truth at m according to Λ and $\overline{\mu}$ ($m \vDash_{\Lambda,\overline{\mu}} \phi$) is defined inductively as follows.
>
> $m \nvDash_{\Lambda,\overline{\mu}} \bot$
> $m \vDash_{\Lambda,\overline{\mu}} p$ iff $p \in m$ (for $p \in \mathrm{At}$)
> $m \vDash_{\Lambda,\overline{\mu}} (\phi \to \psi)$ iff $m \nvDash_{\Lambda,\overline{\mu}} \phi$ or $m \vDash_{\Lambda,\overline{\mu}} \psi$
> $m \vDash_{\Lambda,\overline{\mu}} \Box\phi$ iff $\exists\Gamma \in \overline{\mu}(m)$ such that $\Gamma \vdash_{\Lambda} \phi$

As in chapter 7, we can define different varieties of semantic consequence and validity on this semantics in the following general way.

Definition B.1.2. Where Λ is a logic and $\overline{\mu}$ is a multiplex modality, $\Gamma \vDash_{\Lambda,\overline{\mu}} \phi$ iff for all models m: if $m \vDash_{\Lambda,\overline{\mu}} \gamma$ for all $\gamma \in \Gamma$, then $m \vDash_{\Lambda,\overline{\mu}} \phi$.

Definition B.1.3. Where $\mathbf{L} \subseteq \mathbf{Log}$ and $\overline{\mathbf{M}} \subseteq \overline{\mathbf{Mod}}$,
$\Gamma \vDash_{\mathbf{L},\overline{\mathbf{M}}} \phi$ iff $\Gamma \vDash_{\Lambda,\overline{\mu}} \phi$ for all $\Lambda \in \mathbf{L}$ and $\overline{\mu} \in \overline{\mathbf{M}}$.

Notions of soundness and (strong/weak) completeness of a PML logic with respect to a pair $(\mathbf{L}, \mathbf{M})$ are exactly analogous to those given in Chapter 8. In addition, the idea of a 'condition' on modalities from Chapter 8 can be straightforwardly adapted to a notion of a condition on multiplex modalities. V-functions can be adapted accordingly:

Definition B.1.4. A condition on multiplex modalities is a function $\overline{C} : \mathbf{Log} \to \mathcal{P}(\overline{\mathbf{Mod}})$.

Definition B.1.5. Where $\overline{C} : \mathbf{Log} \to \mathcal{P}(\overline{\mathbf{Mod}})$, we define $V_{\overline{C}} : \Lambda \mapsto \mathrm{Va}(\{\Lambda\}, \overline{C}(\Lambda))$.

With these generalised concepts in hand, we can begin to investigate the correspondence theory of the generalised semantics along a similar plan to the discussion in chapter 7.

B.1.2 Multiplex General Validity

The most general variety of consequence in this extended semantics is $\vDash_{\mathbf{Log},\overline{\mathbf{Mod}}}$, and so the most general variety of validity is $\mathrm{Va}(\mathbf{Log}, \overline{\mathbf{Mod}})$. These can be called general semantic consequence and general validity respectively. It is easy to verify that the axiom C is not generally valid on the multiplex semantics. For suppose we have a model m and a multiplex modality $\overline{\mu}$ such that $\overline{\mu}(m) = \{\{p\}, \{q\}\}$. Then $m \vDash_{\mathrm{PC},\mu} (\Box p \wedge \Box q)$ but $m \nvDash_{\mathrm{PC},\mu} \Box(p \wedge q)$. Interestingly, the N

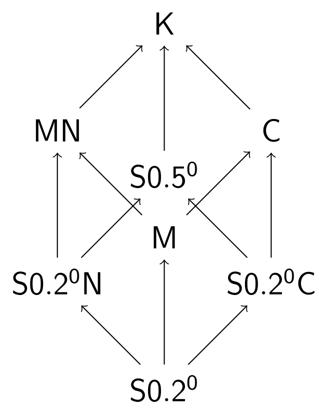

Figure B.1: Inclusion diagram featuring the $\mathsf{S0.2^0}$ logics. Northwest arrows add N, northeast add C, and up-arrows add the full rule Mon.

axiom ($\Box\top$) also fails to be generally valid. This is because if $\overline{\mu}(m) = \varnothing$, then we have $m \not\models_{\Lambda,\overline{\mu}} \Box\top$ by the generalised semantic clause above.

We want a strong completeness result for general semantic consequence similar to that given in the original semantics. However, we cannot simply re-apply the same technique used in the proof of that result here. This is because the general validity of the regularity axiom C (or rather K, which implies C) was relied on in a key way throughout the proofs of chapter 7. Therefore we need a new method of establishing completeness for logics without C. We also need names for the relevant subsystems of $\mathsf{S0.5^0}$ lacking C and N. I propose to give the name '$\mathsf{S0.2^0}$' to the smallest logic which includes the set $\mathrm{Mon(PC)} = \{(\Box\phi \to \Box\psi) : \vdash_{\mathrm{PC}} \phi \to \psi\}$.[2] The systems '$\mathsf{S0.2^0C}$' and '$\mathsf{S0.2^0N}$' are then the logics which add only C or N to $\mathsf{S0.2^0}$ respectively. Adding both N and C to $\mathsf{S0.2^0}$ yields $\mathsf{S0.5^0}$. Note also that adding the full rule Mon to any of the logics $\mathsf{S0.2^0}$, $\mathsf{S0.2^0C}$, $\mathsf{S0.2^0N}$ or $\mathsf{S0.5^0}$ yields M, C, MN and K respectively (see Fig. 1).

B.2 Correspondence Theory

B.2.1 A Satisfaction Lemma

As for our new method of establishing completeness, we will work by proving a modified version of Prop. 8.1.16. In particular we shall use a certain kind of partition over maximal consistent sets of sentences.

[2]Note that this logic contains the axiom M. Since $\vdash_{\mathrm{PC}} (p \wedge q) \to p$ and $\vdash_{\mathrm{PC}} (p \wedge q) \to q$ we have $\vdash_{\mathsf{S0.2^0}} \Box(p \wedge q) \to \Box p$ and $\vdash_{\mathsf{S0.2^0}} \Box(p \wedge q) \to \Box q$. Therefore $\vdash_{\mathsf{S0.2^0}} \Box(p \wedge q) \to (\Box p \wedge \Box q)$.

Definition B.2.1. The Λ-partition $\pi_\Lambda(\Delta)$ on a set Δ of formulas is the partition on Δ generated by the equivalence relation $\{(\phi, \psi) | \vdash_\Lambda \phi \leftrightarrow \psi\}$.

In other words, if $[\phi]_\Lambda$ is the set of all formulas such that $\vdash_\Lambda \phi \leftrightarrow \psi$, then $\pi_\Lambda(\Delta) = \{[\phi]_\Lambda \cap \Delta | \phi \in \Delta\}$. Now for a useful fact about such partitions.

Proposition B.2.2. Suppose that Λ' includes Mon(Λ). Then where Δ is some maximally Λ'-consistent set, every cell of $\pi_\Lambda(\Box^-(\Delta))$ is closed under conjunction formation.

Proof. Suppose that ϕ and ψ inhabit the same cell of the partition, so that both are in $\Box^-(\Delta)$. Then $\vdash_\Lambda \phi \leftrightarrow \psi$, so by PC, $\vdash_\Lambda \phi \leftrightarrow (\phi \wedge \psi)$, and by Mon$(\Lambda)$, $\vdash_{\Lambda'} \Box\phi \leftrightarrow \Box(\phi \wedge \psi)$. Thus $(\phi \wedge \psi) \in \Box^-(\Delta)$ as well. Since $\vdash_\Lambda \phi \leftrightarrow (\phi \wedge \psi)$, $(\phi \wedge \psi)$ is in the same cell of $\pi_\Lambda(\Box^-(\Delta))$ as ϕ and ψ. $\Box$

We now formulate a version of the satisfaction lemma for the multiplex semantics. First we generalise our previous notion of a Λ-modality.

Definition B.2.3. A (Λ_1, Λ_2)-modality is a multiplex modality $\overline{\mu}$ such that for all m, there is some maximally Λ_2-consistent set Δ such that $\overline{\mu}(m) = \pi_{\Lambda_1}(\Box^-(\Delta))$ and $\Delta \cap \mathrm{At} = m$. A $(\Gamma, \Lambda_1, \Lambda_2)$-modality is a (Λ_1, Λ_2)-modality $\overline{\mu}$ such that $\overline{\mu}(m_\Gamma) = \pi_{\Lambda_1}(\Box^-(\Gamma))$.

An existence theorem analogous to Prop. 8.1.5 can be shown along lines to the proof of that proposition. From this we can then prove a multiplex counterpart of Prop. 8.1.16:

Proposition B.2.4. Suppose Λ_1, Λ_2 are consistent and that Γ is a maximally Λ_2-consistent set of formulas, then there is a $(\Gamma, \Lambda_1, \Lambda_2)$-modality.

Proof. Argument proceeds in analogy to Prop. 8.1.5 without special difficulty. $\Box$

Proposition B.2.5 (Multiplex Satisfaction Lemma)**.** If $\overline{\mu}$ is a $(\Gamma, \Lambda_1, \Lambda_2)$-modality where Λ_2 includes Mon(Λ_1), then for all ϕ we have $m_\Gamma \vDash_{\Lambda_1, \overline{\mu}} \phi$ iff $\phi \in \Gamma$.

Proof. By induction on formula complexity. The base and $\rightarrow$-induction steps are trivial; the $\Box$-induction step is as follows. Assume that the theorem holds for ϕ, then:

> ($\Box$-Ind). If $\Box\phi \in \Gamma$, then $\phi \in \Box^-(\Gamma)$. Since $\overline{\mu}(m_\Gamma)$ is a partition on $\Box^-(\Gamma)$, $\exists\Delta \in \overline{\mu}(m_\Gamma)$ such that $\phi \in \Delta$, so that $\Delta \vdash_{\Lambda_1} \phi$ and $m_\Gamma \vDash_{\Lambda_1,\mu} \Box\phi$. Conversely: if $m_\Gamma \vDash_{\Lambda_1,\mu} \Box\phi$, then $\exists\Delta \in \overline{\mu}(m_\Gamma)$ such that $\Delta \vdash_{\Lambda_1} \phi$, which means that there are $\delta_1, ..., \delta_n \in \Delta$ such that $\vdash_{\Lambda_1} (\delta_1 \wedge ... \wedge \delta_n) \supset \phi$. Thus by Mon$(\Lambda_1)$ included in Λ_2, we have $\vdash_{\Lambda_2} \Box(\delta_1 \wedge ... \wedge \delta_n) \supset \Box\phi$, and so since Γ is maximally Λ_2-consistent,

> $\Box(\delta_1 \wedge ... \wedge \delta_n) \supset \Box\phi \in \Gamma$. Finally, since $\overline{\mu}(m_\Gamma)$ is the Λ_1-partition on $\Box^-(\Gamma)$ where Γ is maximally Λ_2-consistent, by Proposition B.2.2. the cell Δ is closed under conjunction introduction: since $\delta_1, ..., \delta_n \in \Delta$, we have $(\delta_1 \wedge ... \wedge \delta_n) \in \Delta$ too. Therefore $\Box(\delta_1 \wedge ... \wedge \delta_n) \in \Gamma$, so that $\Box\phi \in \Gamma$.

Which completes the proof. □

This then gives us the generalised analogue of Prop. 8.1.17:

Corollary B.2.6. If $\mathrm{Mon}(\Lambda_1) \subseteq \Lambda_2$ and every (Λ_1, Λ_2)-modality is in $\overline{\mathbf{M}}$, then Λ_2 is strongly complete for $(\{\Lambda_1\}, \overline{\mathbf{M}})$.

B.3 General Validity and General Completeness

We can apply this first to the case of general semantic consequence and validity:

Proposition B.3.1. $\mathsf{S0.2}^0$ is sound and strongly complete for $(\mathbf{Log}, \overline{\mathbf{Mod}})$.

Proof. (Soundness) We need only note that everything in Mon(PC) is generally valid: if $\vdash_{\mathrm{PC}} \phi \to \psi$, then supposing that $m \vDash_{\Lambda,\overline{\mu}} \Box\phi$ we can infer than there is some $\Gamma \in \overline{\mu}(m)$ such that $\Gamma \vdash_\Lambda \phi$. Since $\mathrm{PC} \subseteq \Lambda$ we also have $\Gamma \vdash_\Lambda \phi \to \psi$, meaning that we must have $\Gamma \vdash_\Lambda \psi$ entailing that $m \vDash_{\Lambda,\overline{\mu}} \Box\psi$. So $m \vDash_{\Lambda,\overline{\mu}} (\Box\phi \to \Box\psi)$. (Completeness) If $\Gamma \nvdash_{\mathsf{S0.2}^0} \phi$, then $\Gamma \cup \{\neg\phi\}$ is $\mathsf{S0.2}^0$-consistent. Hence by Lindenbaum's lemma there is a maximal $\mathsf{S0.2}^0$-consistent set Γ^* such that $\Gamma \cup \{\neg\phi\} \subseteq \Gamma$. By the multiplex satisfaction lemma, Γ is satisfied at m_Γ according to some $(\mathrm{PC}, \mathsf{S0.2}^0, \Gamma)$-modality with PC set as the logic of evaluation. So ϕ is not in $\mathrm{Va}(\mathrm{PC}, \overline{\mathbf{Mod}})$, hence not in $\mathrm{Va}(\mathbf{Log}, \overline{\mathbf{Mod}})$ either. □

Thus general validity on the multiplex semantics is captured by the particularly weak logic $\mathsf{S0.2}^0$. As previously, stronger logic are determined in various ways by placing conditions on modalities, as we shall see in a moment.

B.4 Some Standard Multiplex Conditions

We shall alter the definition of a *standard* condition for the multiplex setting. This is because necessitation rules are no longer so prominent—$\mathsf{S0.2}^0$ admits no such rules, for example). Instead, monotonicity rules abound:

Proposition B.4.1. For all $\overline{C}$ and Λ, $V_{\overline{C}}(\Lambda)$ admits $\mathrm{Mon}(\Lambda)$.

So we shall say:

Definition B.4.2. A multiplex condition $\overline{C}$ is *standard* iff $\exists\Sigma\,\forall\Lambda : V_{\overline{C}}(\Lambda) = \Sigma + \mathrm{Mon}(\Lambda)$

I will now list some multiplex conditions and show that each of them is standard.

Definition B.4.3. $\overline{\mu}$ is called Λ-*cohesive* iff for all m, ϕ and ψ, if there are $\Gamma, \Gamma' \in \overline{\mu}(m)$ such that $\Gamma \vdash_\Lambda \phi$ and $\Gamma' \vdash_\Lambda \psi$, then there is some $\Gamma'' \in \overline{\mu}(m)$ such that $\Gamma'' \vdash_\Lambda (\phi \wedge \psi)$.

Cohesion
$\overline{Ch}(\Lambda) = \{\overline{\mu} | \overline{\mu} \text{ is } \Lambda\text{-cohesive}\}$
Non-Empty
$\overline{Ne}(\Lambda) = \{\overline{\mu} | \forall m : \overline{\mu}(m) \neq \varnothing\}$
Non-Empty Cohesive
$\overline{NC} = \overline{Ne} \oplus \overline{Ch}$

Here '$\oplus$' carries the same definition as before, such that $\overline{NC}(\Lambda)$ is the intersection of $\overline{Ch}(\Lambda)$ and $\overline{Ne}(\Lambda)$. Recall that there is also a trivial condition $\Lambda \mapsto$ MMod. We shall also show that this condition is standard.

Proposition B.4.4.

(i)	$V_{\overline{\mathbf{Mod}}}(\Lambda)$	$=$	$\mathsf{S0.2}^0 + \mathrm{Mon}(\Lambda)$
(ii)	$V_{\overline{Ch}}(\Lambda)$	$=$	$\mathsf{S0.2}^0\mathsf{C} + \mathrm{Mon}(\Lambda)$
(iii)	$V_{\overline{Ne}}(\Lambda)$	$=$	$\mathsf{S0.2}^0\mathsf{N} + \mathrm{Mon}(\Lambda)$
(iv)	$V_{\overline{NC}}(\Lambda)$	$=$	$\mathsf{S0.5}^0 + \mathrm{Mon}(\Lambda)$[3]

Proof.

(i) ($\supseteq$) $V_{\text{MMod}}(\Lambda)$ contains $\mathsf{S0.2^0}$ since all of$\mathsf{S0.2^0}$s theorems are generally valid. $V_{\text{MMod}}(\Lambda)$ also admits $\text{Mon}(\Lambda)$ by Prop. B.4.1 ($\subseteq$) Immediate by Corollary. B.2.6.
(ii) ($\supseteq$) $V_{\overline{Ch}}(\Lambda)$ must contain $\mathsf{S0.2^0}$ since all of the latter's theorems are generally valid, and admit $\text{Mon}(\Lambda)$ by Proposition B.4.1. Every instance of C must be in $V_{\overline{Ch}}(\Lambda)$. For if $m \vDash_{\Lambda,\overline{\mu}} \Box\phi$ and $m \vDash_{\Lambda,\overline{\mu}} \Box\psi$ then there are $\Gamma, \Gamma' \in \overline{\mu}(m)$ such that $\Gamma \vdash_\Lambda \phi$ and $\Gamma' \vdash_\Lambda \phi$. So by integration there is some $\Gamma'' \in \overline{\mu}$ such that $\Gamma'' \vdash_\Lambda (\phi \wedge \psi)$, meaning $m \vDash_{\Lambda,\overline{\mu}} \Box\phi \wedge \Box\psi$. ($\subseteq$) By Prop. B.2.5, it suffices to show that every $(\Lambda, \mathsf{S0.2^0C} + \text{Mon}(\Lambda))$-modality is in $\overline{Ch}(\Lambda)$. Suppose we have a $(\Lambda, \mathsf{S0.2^0C} + \text{Mon}(\Lambda))$-modality $\overline{\mu}$. Then for any m, suppose there are $\Gamma, \Gamma' \in \overline{\mu}(m)$ such that $\Gamma \vdash_\Lambda \phi$ and $\Gamma' \vdash_\Lambda \psi$. So there are $\gamma_1, ..., \gamma_n \in \Gamma$ and $\gamma'_1, ..., \gamma'_m \in \Gamma'$ such that $\vdash_\Lambda (\gamma_1 \wedge ... \wedge \gamma_n) \to \phi$ and $\vdash_\Lambda (\gamma'_1 \wedge ... \wedge \gamma'_m) \to \psi$. Therefore $\vdash_\Lambda (\gamma_1 \wedge ... \wedge \gamma_n \wedge \gamma'_1 \wedge ... \wedge \gamma'_m) \to (\phi \wedge \psi)$. Now recall that by definition Γ and Γ' are cells in the Λ-partition of a maximally consistent set Δ^μ_m. Therefore since all of the γs and γ's are in this set, their conjunction is too, and must be in some other cell Γ'' of said partition. Therefore there is a $\Gamma'' \in \overline{\mu}(m)$ such that $\Gamma'' \vdash_\Lambda (\phi \wedge \psi)$.
(iii) ($\supseteq$) $V_{\overline{Ne}}(\Lambda)$ must contain $\mathsf{S0.2^0}$ since all of the latter's theorems are generally valid, and admit $\text{Mon}(\Lambda)$ by Prop. B.4.1. Additionally, N must be in $V_{\overline{Ne}}(\Lambda)$ since for all m if $\overline{\mu}$ is nonempty then there is some $\Gamma \in \overline{\mu}$ such that $\Gamma \vdash_\Lambda \top$, so that $m \vDash_{\Lambda,\overline{\mu}} \Box\top$. ($\subseteq$) By Prop. B.2.5. it suffices to note that every $(\Lambda, \mathsf{S0.2^0N} + \text{Mon}(\Lambda))$-modality is in $\overline{Ne}(\Lambda)$. This is trivial since where $\overline{\mu}$ is $(\mathsf{S0.2^0} + \text{Mon}(\Lambda))$-modality, every $\overline{\mu}(m)$ is a partition on a non-empty set, and so non-empty.
(iv) ($\supseteq$) $V_{\overline{NC}}(\Lambda)$ must contain $\mathsf{S0.2^0}$ since all of the latter's theorems are generally valid, and admit $\text{Mon}(\Lambda)$ by Prop. B.4.1. It must also contain N and every instance of C since these are in the more general notions of validity above. ($\subseteq$) By Prop. B.2.5 it suffices to show that every $(\Lambda, \mathsf{S0.5^0} + \text{Mon}(\Lambda))$-modality is in $\overline{NC}(\Lambda)$. Here the arguments from (ii) and (iii) may be repeated without alteration (save relabelling of $\mathsf{S0.2^0C/N}$ to $\mathsf{S0.5^0}$) to show that all such modalities are in $\overline{Ch}(\Lambda)$ and $\overline{Ne}(\Lambda)$.

Which completes the proof. □

B.5 Generation Again

I will make free use of the conventions on transfinite application of V-functions introduced in the main discussion. I will also retain the same definition of generation and simple generation, such that a multiplex condition $\overline{C}$ is said to simply generate Λ iff there is some ordinal β such that $V^\beta_{\overline{C}}(\text{PC}) = \Lambda$. The results relating to transfinite application and generation from the ordinary semantics can be generalised in the multiplex setting without much trouble. First some lemmas.

Proposition B.5.1. Suppose that Λ' is a logic and that $\{\Lambda_n\}_{n\in\omega}$ a set of logics such that $\Lambda_n \subseteq \Lambda_m$ whenever $n \leq m$. Then we have:

$$\bigcup_n(\Lambda' + \text{Mon}(\Lambda_n)) = \Lambda' + \text{Mon}(\bigcup_n \Lambda_n)$$

Proof. ($\supseteq$) By the fact that $\{\Lambda_n\}_{n\in\omega}$ a set of logics such that $\Lambda_n \subseteq \Lambda_m$ whenever $n \leq m$, it follows that $\{\Lambda' + \text{Mon}(\Lambda_n)\}_{n\in\omega}$ is a logic. Additionally, it is clear that $\bigcup_n(\Lambda' + \text{Mon}(\Lambda_n))$ includes Λ', and it can be shown to also include $\text{Mon}(\bigcup_n \Lambda_n)$. For if $\phi \to \psi \in \bigcup_n \Lambda_n$, then for some $k \in \omega$, $\phi \to \psi \in \Lambda_k$, which means that $\Box\phi \to \Box\psi \in \text{Mon}(\Lambda_k) \subseteq (\Lambda' + \text{Mon}(\Lambda_k)) \subseteq \bigcup_n(\Lambda' + \text{Mon}(\Lambda_n))$. ($\supseteq$) It is clear that for each n, $(\Lambda' + \text{Mon}(\Lambda_n))$ will be included in $\Lambda' + \text{Mon}(\bigcup_n \Lambda_n)$, since the latter includes both Λ' and $\text{Mon}(\Lambda_n)$ for each n. Therefore $\bigcup_n(\Lambda' + \Box(\Lambda_n)) \subseteq \Lambda' + \text{Mon}(\bigcup_n \Lambda_n)$. □

Proposition B.5.2. For standard $\overline{C}$ and all Λ, $V_{\overline{C}}(\Lambda) = V_{\overline{C}}(\text{PC}) + \text{Mon}(\Lambda)$.

Proof. For some Λ_0, $V_C(\Lambda) = \Lambda_0 + \Box(\Lambda)$ and $V_C(\text{PC}) = \Lambda_0 + \Box(\text{PC})$. So since $\Box(\text{PC}) \subseteq \Box(\Lambda)$, we have $V_C(\text{PC}) + \Box(\Lambda) = \Lambda_0 + \Box(\text{PC}) + \Box(\Lambda) = \Lambda_0 + \Box(\Lambda) = V_C(\Lambda)$. □

Then with these in hand we may prove our desired result as follows.

Proposition B.5.3. For standard C and all Λ, $V_{\overline{C}}^{\omega}(\Lambda)$ is a V_C-fixed-point if defined.

Proof. Assuming $V_{\overline{C}}^{\omega}(\Lambda)$ is defined we proceed as follows. Where n is a variable ranging over the finite ordinals, then

$$V_{\overline{C}}^{\omega}(\Lambda) = \bigcup_n V_{\overline{C}}^{n}(\Lambda) = V_{\overline{C}}^{0}(\Lambda) \cup \bigcup_{n\geq 1} V_{\overline{C}}^{n}(\Lambda) = \bigcup_{n\geq 1} V_{\overline{C}}^{n}(\Lambda)$$

The final part of this manipulation comes by the fact that, by ω-cumulativity, $V_{\overline{C}}^{0}(\Lambda) = \Lambda \subseteq \bigcup_{n\geq 1} V_{\overline{C}}^{n}(\Lambda)$. Next we have, by simple manipulations and an application of Prop B.5.2,

$$\bigcup_{n\geq 1} V_{\overline{C}}^{n}(\Lambda) = \bigcup_n V_{\overline{C}}^{n+1}(\Lambda) = \bigcup_n V_{\overline{C}}(V_{\overline{C}}^{n}(\Lambda)) = \bigcup_n (V_{\overline{C}}(\text{PC}) + \text{Mon}(V_{\overline{C}}^{n}(\Lambda)))$$

Then by applying Prop B.5.1, we have

$$\bigcup_n (V_{\overline{C}}(\text{PC}) + \text{Mon}(V_{\overline{C}}^{n}(\Lambda))) = V_{\overline{C}}(\text{PC}) + \text{Mon}(\bigcup_n V_{\overline{C}}^{n}(\Lambda))$$

But since $\bigcup_n V_{\overline{C}}^{n}(\Lambda)$ just is $V_{\overline{C}}^{\omega}(\Lambda)$, this means that:

$$V_{\overline{C}}(\text{PC}) + \text{Mon}(\bigcup_n V_{\overline{C}}^{n}(\Lambda)) = V_{\overline{C}}(\text{PC}) + \text{Mon}(V_{\overline{C}}^{\omega}(\Lambda)) = V_{\overline{C}}^{\omega+1}(\Lambda)$$

So that, putting this all together, we get

$$V_{\overline{C}}^{\omega}(\Lambda) = V_{\overline{C}}^{\omega+1}(\Lambda)$$

Which means that $V_{\overline{C}}^{\omega}(\Lambda)$ is a $V_{\overline{C}}$-fixed-point. □

Thus, if $\overline{C}$ is standard, then it simply generates at most countably many logics, starting with $V_{\overline{C}}(\mathrm{PC})$ and ending with $V_{\overline{C}}^{\omega}(\mathrm{PC})$. Now the identity of the limit $V_{\overline{C}}^{\omega}(\mathrm{PC})$ can be determined by a general result. In this case, that result is:

Proposition B.5.4. If $\overline{C}$ is standard, then $V_{\overline{C}}^{\omega}(\Lambda) = V_{\overline{C}}(\Lambda) + \mathrm{Mon}$ where defined.

Proof. ($\subseteq$) $V_{\overline{C}}(\Lambda) + \mathrm{Mon}$ includes every $V_{\overline{C}}^{n}(\Lambda)$ for $n \in \omega$. For it includes $V_{\overline{C}}^{1}(\Lambda) = V_{\overline{C}}(\Lambda) = V_{\overline{C}}(\mathrm{PC}) + \mathrm{Mon}(\Lambda)$, and if we assume it includes $V_{\overline{C}}^{k}(\Lambda)$, it must also include $V_{\overline{C}}^{k+1}(\Lambda) = V_{\overline{C}}(\mathrm{PC}) + \mathrm{Mon}(V_{\overline{C}}^{k}(\Lambda))$ thanks to it admitting full Mon. ($\supseteq$) $V_{\overline{C}}^{\omega}(\Lambda)$ includes $V_{\overline{C}}(\Lambda)$ by definition and admits full Mon. For if $\phi \to \psi \in V_{\overline{C}}^{\omega}(\Lambda)$, then there is some finite n such that $\phi \to \psi \in V_{\overline{C}}^{n}(\Lambda)$, so that therefore $\Box\phi \to \Box\psi \in V_{\overline{C}}^{n+1}(\Lambda) \subseteq V_{\overline{C}}^{\omega}(\Lambda)$. $\square$

From this, finally, we get our analogue of Prop. 8.1.39 by combining Props B.4.4., B.5.3, and B.5.4,.

Corollary B.5.5.

(i) $V_{\overline{\mathbf{Mod}}}^{\omega}(\mathrm{PC}) = \mathsf{M}$

(ii) $V_{\overline{Ch}}^{\omega}(\mathrm{PC}) = \mathsf{C}$

(iii) $V_{\overline{Ne}}^{\omega}(\mathrm{PC}) = \mathsf{MN}$

(iv) $V_{\overline{NC}}^{\omega}(\mathrm{PC}) = \mathsf{K}$

Also by Prop 5.5.3, $V_{\overline{C}}^{\beta}(PC) = V_{\overline{C}}^{\omega}(PC)$ for all $\beta > \omega$ for each of these conditions C.

C
Second-Order Modal Logic

In this appendix I extend the discussion of the syntactical semantics from chapter 8 to languages with second-order quantification. (§C.1) In the case of higher-order objectual quantifiers, the same problems arise as did for first-order objectual quantification—namely, a robust failure of universal instantiation. In addition to this, with the addition of the abstraction operator λ come failures of extensional β-conversion in the standard Tarksian semantics. The nonmodal comprehension schema is retained as generally valid, but the modal comprehension schema exhibits robust failures. (§C.2) Once again, a retreat to substitutional quantification and state-descriptions as opposed to Tarskian models would appear to solve the problem in the case of quantification, and additionally a new primitive semantic clause for the evaluation of complex predications is able to recover β-conversion. Additionally, in the substitutional semantics, both comprehension and modal comprehension would be rendered generally valid.[1] However, higher-order substitutional quantification in the presence of complex predication raises worries with respect to impredicativity which must be overcome before such a semantic strategy can be at all endorsed.

[1] A terminological point: talk of 'second-order objectual quantification' will sound like a contradiction in terms to some readers, since usually the term 'object' is reserved for the values of individual (i.e. first-order) terms, and indeed is often used synonymously with 'individual'. Perhaps a better and more neutral term would be 'metaphysical quantification', but given that this term has less currency in the literature, it is preferable to retain the substitutional/objectual terminology even when moving to higher orders.

C.1 Higher-Order Objectual Quantifiers

In §C.1.1 I introduce the first second-order langauge and its semantics; in §§C.1.2-C.1.4 I demonstrate failures analogous to those seen in chapter 8 for the objectual semantics.

C.1.1 The Language

We extend the language of QML$\forall$ to a second-order language QML$\forall 2$ by adjoining a denumerable stock of propositional variables X^n, Y^n, Z^n... for each arity n, as well as the new symbol λ. The formation rules for this language are as follows.

> $\bot$ is a formula.
>
> Every sentence letter is a formula.
>
> If F^n is an n-adic predicate and $t_1, ..., t_n$ are terms, $F^n t_1 ... t_n$ is a formula.
>
> If ϕ and ψ are formulas, then so are $\Box\phi$ and $(\phi \rightarrow \psi)$.
>
> If ϕ is a formula and x an individual variable, $\forall x \phi$ is a formula.
>
> If ϕ is a formula and X is a predicate variable, $\forall X \phi$ is a formula.
>
> If ϕ is a formula and $x_1 ... x_n$ are individual variables, then $\lambda x_1 ... x_n \phi$ is an n-adic predicate.
>
> There are no formulas besides those obtained via these formation rules.

A predicate formed via λ is called a λ-abstract, or else a complex predicate; every other predicate is called primitive. An atomic formula is defined as any string $F^n t_1 ... t_n$ where F is a primitive predicate and $t_1 ... t_n$ are terms. The distinction between free and bound variables is extended to include predicate variables, and the distinction between open and closed formulas (or sentences) is extended accordingly. Now, in addition, an individual variable occurrence x is also called bound if it occurs within the scope of some occurrence of $\lambda x_1 ... x_n$ where x is x_i for some i with $1 \leq i \leq n$. Here, the 'scope' of an occurrence of $\lambda x_1 ... x_n$ is the smallest predicate in which it occurs. For example, the scope of λy in the formula $(\lambda y(p \rightarrow q)x \rightarrow \bot)$ is the monadic predicate $\lambda y(p \rightarrow q)$. The notation $\phi[F/X]$ and $\phi[F//X]$ extends substitution notation to predicates. Here it is assumed that the predicate F is of equal arity to the variable X. A predicate is called open (else closed) iff it contains an occurrence of a free variable.

The definition of a logic in this second-order language is essentially the same as for first-order languages, with an attendant definition of a second-order or QML$\forall$2-substitution as was sketched in Chapter 7 (§7.3.1). Now that our language contains complex predication we need an additional condition for substitutions ς that will govern how they interact with λ-terms. For that we lay down the rule:

$$\varsigma(\lambda x_1...x_n\phi) = \lambda x_1...x_n\varsigma\phi$$

We include principles adequate for a classical second-order quantification theory—universal generalisation, free variable substitution, plus an axiom schema of universal instantiation:

$\mathrm{UI}_2 \quad \forall X^n\phi \to \phi[F^n/X^n]$

In addition to this we add two further axiom schemas to the definition of a logic: β-conversion and unrestricted Comprehension. These are given as follows:

β-conversion $\quad \lambda x_1...x_n\phi t_1...t_n \leftrightarrow \phi[t_1/x_1, ..., t_n/x_n]$.

Comp $\quad \exists X^n\forall x_1...\forall x_n(Xx_1...x_n \leftrightarrow \phi)$ where X^n is not free in ϕ.

The smallest QML$\forall$2 logic I call $\forall$2PC, and in general where Λ is a PML logic, $\forall 2\Lambda$ is the smallest QML$\forall$2 logic which contains Λ. Where Λ is a QML∀2 logic, the syntactic consequence relation $\vdash_\Lambda$ is defined in the usual way.

C.1.2 Semantics

Semantics for QML$\forall$2 are given in terms of some Tarksian model M, a modality μ which now assigns M a set $\mu(M)$ of QML$\forall$2 formulas, a QML$\forall$2 logic Λ, and a variable assignment. Variable assignments on a model M are now functions A which assign every individual variable x an object $A(x) \in |M|$, and every n-adic predicate X^n a set of n-tuples $A(X^n) \subseteq |M|^n$. A formula ϕ is evaluated with respect to these ($M, A \vDash_{\Lambda,\mu} \phi$) according to the following semantic clauses. In these clauses the assignment A^x_u is defined as the assignment which differs at most from A in assigning x to u, and similarly for A^X_U. Moreover, the notation $[F]_{M,A,\Lambda,\mu}$ is either $A(F)$ if F is a predicate variable, $[F]_M$ if F is a primitive predicate constant, or $[F]_{M,A,\Lambda,\mu}$ if F is a λ-abstract (see below.)

$M, A \nvDash_{\Lambda,\mu} \bot$

$M, A \vDash_{\Lambda,\mu} F^n t_1 ... t_n$ iff $([t_1]_{M,A}, ..., [t_n]_{M,A}) \in [F^n]_{M,A,\Lambda,\mu}$

$M, A \vDash_{\Lambda,\mu} t = t'$ iff $[t]_{M,A} = [t']_{M,A}$

$M, A \vDash_{\Lambda,\mu} (\phi \to \psi)$ iff $M, A \nvDash_{\Lambda,\mu} \phi$ or $M, A \vDash_{\Lambda,\mu} \psi$

$M, A \vDash_{\Lambda,\mu} \forall x \phi$ iff for all $u \in |M|$, $M, A^x_u \vDash_{\Lambda,\mu} \phi$

$M, A \vDash_{\Lambda,\mu} \forall X^n \phi$ iff for all $U \subseteq |M|^n$, $M, A^{X^n}_U \vDash_{\Lambda,\mu} \phi$

$M, A \vDash_{\Lambda,\mu} \Box \phi$ iff $\mu(M) \vdash_\Lambda \phi$

In evaluating formulas containing complex predicates, we need to provide interpretations for λ-abstracts. We do this in a way which adopts a standard approach to interpreting λ-abstracts (cf. Sider [61], 127):

For any ϕ and x, $[\lambda x \phi]_{M,A,\Lambda,\mu} = \{u \in |M| : M, A^x_u \vDash_{\Lambda,\mu} \phi\}$

For any ϕ and $x_1 ... x_n$ ($n \geq 2$),

$$[\lambda x_1 ... x_n \phi]_{M,A,\Lambda,\mu} = \{(u_1, ..., u_n) \in |M|^n : M, A^{x_1 ... x_n}_{u_1 ... u_n} \vDash_{\Lambda,\mu} \phi\}$$

As before, I call a formula ϕ generally valid in the present semantics iff $M, A \vDash_{\Lambda,\mu} \phi$ for all suitable M, A, Λ and μ.

C.1.3 Instantiation Again

Universal instantiation for second-order quantification exhibits the same widespread failures as seen in chapter 8. The same kind of failure occurs for second-order quantification. For instance, take any M, A, Λ and μ such that $\mu(M) \nvdash_\Lambda Xx \leftrightarrow Yx$. Then for all assignments A' on M have

$M, A' \vDash_{\Lambda,\mu} \neg\Box(Xx \leftrightarrow Yx)$, so that in particular we have for all $U \subseteq |M|$,
$M, A^X_U \vDash_{\Lambda,\mu} \neg\Box(Xx \leftrightarrow Yx)$, so that by the semantic clause we have
$M, A \vDash_{\Lambda,\mu} \forall X \neg\Box(Xx \leftrightarrow Yx)$. But since $\vdash_\Lambda Yx \leftrightarrow Yx$, we have
$M, A \nvDash_{\Lambda,\mu} \neg\Box(Yx \leftrightarrow Yx)$.

We therefore have a failure of universal instantiation since $\neg\Box(Yx \leftrightarrow Yx)$ is $\neg\Box(Xx \leftrightarrow Yx)[Y/X]$. The failures of universal instantiation are robust in the same way as for first-order objectual quantification. To see this, let UI_2 be the set of instances of second-order universal instantiation. Then:

Proposition C.1.1. For any M, A, Λ, μ: if $M, A \vDash_{\Lambda,\mu} \phi$ wherever $\phi \in \mathrm{UI}_2$, it is also the case that $M, A \vDash_{\Lambda,\mu} \forall X \forall Y \forall x \Box (Xx \leftrightarrow Yx)$.

Proof. If $\mu(M) \nvdash_\Lambda Xx \leftrightarrow Yx$, then we know from above that some instance of UI_2 fails at M, A, Λ, μ. So if $M, A \vDash_{\Lambda,\mu} \phi$ wherever $\phi \in \mathrm{UI}_2$, it follows that $\mu(M) \vdash_\Lambda Xx \leftrightarrow Yx$. But then we will have $M, A \vDash_{\Lambda,\mu} \forall X \forall Y \forall x \Box (Xx \leftrightarrow Yx)$ via the semantic clauses. $\Box$

It turns out the failure is even more drastic among closed modalities:

Proposition C.1.2. For any M, A, Λ, μ: if $\mu(M)$ consists entirely of sentences and if $M, A \vDash_{\Lambda,\mu} \phi$ wherever $\phi \in \mathrm{UI}_2$, it is also the case that $M, A \vDash_{\Lambda,\mu} \Box \bot$.

Proof. If $\mu(M) \nvdash_\Lambda Xx \leftrightarrow Yx$, then we know from above that some instance of UI_2 fails at M, A, Λ, μ. So if $M, A \vDash_{\Lambda,\mu} \phi$ wherever $\phi \in \mathrm{UI}_2$, it follows that $\mu(M) \vdash_\Lambda Xx \leftrightarrow Yx$. But then via the rule of universal generalisation, since X, Y, x do not occur free in any member of $\mu(M)$, we have $\mu(M) \vdash_\Lambda \forall X \forall Y \forall x (Xx \leftrightarrow Yx)$. But $\vdash_\Lambda \forall X \forall Y \forall x (Xx \leftrightarrow Yx) \rightarrow (P^1 c \leftrightarrow \lambda x P^1 xc)$, thanks to universal instantiation in Λ. Therefore we have $\mu(M) \vdash_\Lambda (P^1 c \leftrightarrow \lambda x \neg P^1 xc)$, but since $\vdash_\Lambda (\lambda x \neg P^1 xc \leftrightarrow \neg P^1 c)$ as an instance of β-conversion, we have $\mu(M) \vdash_\Lambda P^1 c \leftrightarrow \neg P^1 c$. Since $\vdash_\Lambda (P^1 c \leftrightarrow \neg P^1 c) \rightarrow \bot$ by propositional logic, we have $\mu(M) \vdash_\Lambda \bot$ and hence $M, A \vDash_{\Lambda,\mu} \Box \bot$. $\Box$

Therefore the failures of UI_2 are robust in the semantics in the sense that the only way to ensure their validity also ensures the validity of implausible theses like $\forall X \forall Y \forall x \Box (Xx \leftrightarrow Yx)$ and $\Box \bot$.

C.1.4 β-Conversion

The schema of β-conversion, $\lambda x_1 ... x_n \phi t_1 ... t_n \leftrightarrow \phi[t_1/x_1, ..., t_n/x_n]$ also robusly fails in the semantics. For instance, take any M, A, Λ, μ such that $\mu(M) \nvdash_\Lambda x = y$. Then $\mu(M) \nvdash_\Lambda \neg x \neq y$, so that for any assignment A' we have

> $M, A' \vDash_{\Lambda,\mu} \Diamond x \neq y$. In particular for every $u \in |M|$ we have
> $M, A^x_u \vDash_{\Lambda,\mu} \Diamond x \neq y$. This means that $[\lambda z \Diamond z \neq y]_{M,A,\Lambda,\mu} = |M|$, so that
> $M, A \vDash_{\Lambda,\mu} (\lambda z \Diamond z \neq y) y$, and yet of course, $M, A \nvDash_{\Lambda,\mu} \Diamond y \neq y$ since $\mu(M) \vdash_\Lambda \neg y \neq y$. Therefore we have a failure of β-conversion, since
> $M, A \nvDash_{\Lambda,\mu} (\lambda z \Diamond z \neq y) y \leftrightarrow \Diamond y \neq y$.

From such examples we may observe the following.

Proposition C.1.3. For any M, A, Λ, μ: $M, A \vDash_{\Lambda,\mu} \phi$ wherever ϕ is an instance of β-conversion, then $M, A \vDash_{\Lambda,\mu} \forall x \forall y \Box x = y$.

Proof. If $M, A \nvDash_{\Lambda,\mu} \forall x \forall y \Box x = y$, then there are $u, v \in |M|$ such that $M, A^{xy}_{uv} \nvDash_{\Lambda,\mu} \Box x = y$, so that $\mu(M) \nvDash_{\Lambda} x = y$. But we know from the above that in this case, $M, A \nvDash_{\Lambda,\mu} (\lambda z \Diamond z \neq y)y \leftrightarrow \Diamond y \neq y$. So then we arrive at the proposition by contraposition. □

Therefore the failures of β-conversion are robust in the semantics in the sense that the only way to ensure their validity also ensures the validity of implausible theses like $\forall x \forall y \Box x = y$.

C.1.5 Comprehension Schemas

I will next consider two comprehension schemas. The first is the nonmodal comprehension schema, Comp, which we included above as an axiom schema for QML$\forall$2 logics. This is:

Comp $\quad \exists X^n \forall x_1 ... \forall x_n (X^n x_1 ... x_n \leftrightarrow \phi)$ where X^n is not free in ϕ.

As it turns out, non-modal comprehension does not exhibit the same robust failures as the other classical principles considered thus far. In fact, we will see that every instance of Comp is generally valid even on the present semantics. The second comprehension schema we will consider is the modal comprehension schema, which is:

Comp$_\Box$ $\quad \exists X^n \Box \forall x_1 ... \forall x_n (X^n x_1 ... x_n \leftrightarrow \phi)$ where X^n is not free in ϕ.

We will see that the modal comprehension schema exhibits the same kind of robust failures as did universal instantiation and β-conversion. But first, the non-modal schema. To show that every instance of the latter is generally valid, we will need the following lemma.

Proposition C.1.4. If assignments A and A' agree on all of the free variables in ϕ, then $M, A \vDash_{\Lambda,\mu} \phi$ iff $M, A' \vDash_{\Lambda,\mu} \phi$.

Proof. Straightforward induction on formula complexity. □

Proposition C.1.5. All instances of Comp are generally valid.

Proof Sketch. I will just give the case where X is monadic; it is then easy to see how to generalise. To show this we will introduce some special notation. Firstly, for any M, Λ and μ, we will write $|\phi|_{M,\Lambda,\mu}$ to denote the set of M-assignments A such that ϕ is satisfied at M with respect to A, Λ, and μ. That is,

$$|\phi|_{M,\Lambda,\mu} := \{A : M, A \vDash_{\Lambda,\mu} \phi\}$$

Additionally, for an M-assignment A and variables x, we will write $var_{M,x}(A)$ to denote the set of M-assignments which differ from A in at most the assignments of x:

$$var_{M,x}(A) := \{A^x_u : u \in |M|\}$$

With this in place we may proceed. Let ϕ be any formula without free occurrences of X. Then consider any M, A, Λ, and μ. Let

$$T = \{A'(x) : A' \in (var_{M,x}(A) \cap |\phi|_{M,\Lambda,\mu})\}.$$

Then we will show that $M, A^X_T \vDash_{\Lambda,\mu} \forall x(Xx \leftrightarrow \phi)$. For note that for any $u \in |M|$, since $A^{X\,x}_{T\,u}$ and A^x_u only disagree on the assignment of X, they agree on all variables that are free in ϕ. Hence by Proposition C.2.1, we have $M, A^{X\,x}_{T\,u} \vDash_{\Lambda,\mu} \phi$ iff $M, A^x_u \vDash_{\Lambda,\mu} \phi$ iff $A^x_u(x) \in T$.
But since $A^x_u(x) = A^{X\,x}_{T\,u}(x) = u$ and $T = [X]_{M,A^X_T}$, we have $A^x_u(x) \in T$ iff $A^{X\,x}_{T\,u} \in [X]_{M,A^X_T}$, which is so iff $M, A^{X\,x}_{T\,u} \vDash_{\Lambda,\mu} Xx$. Hence, $M, A^{X\,x}_{T\,u} \vDash_{\Lambda,\mu} Xx \leftrightarrow \phi$. Since u was arbitrary, this is so for all $u \in |M|$ and so $M, A^X_T \vDash_{\Lambda,\mu} \forall x(Xx \leftrightarrow \phi)$. Hence, finally, $M, A \vDash_{\Lambda,\mu} \exists X \forall x(Xx \leftrightarrow \phi)$. Then since M, A, Λ, and μ were arbitrary, $\exists X \forall x(Xx \leftrightarrow \phi)$ is generally valid. Since ϕ was arbitrary besides the condition that X is not free in ϕ, the same holds for all (monadic) instances of Comp. $\square$

Next, we show that modal comprehension, Comp$_\Box$ exhibits robust failures. First, a counterexample: take any M, A, Λ and μ. Suppose that $\mu(M) \nvdash_\Lambda \forall x(Xx \leftrightarrow Yx)$. Then for all assignments A' we have

$M, A' \nvDash_{\Lambda,\mu} \Box \forall x(Xx \leftrightarrow Yx)$, so in particular for all $U \subseteq |M|$,
$M, A^X_U \nvDash_{\Lambda,\mu} \Box \forall x(Xx \leftrightarrow Yx)$, so that via the semantic clause,
$M, A \nvDash_{\Lambda,\mu} \exists X \Box \forall x(Xx \leftrightarrow Yx)$, which is an instance of Comp$_\Box$

We can now adapt this into an argument that Comp$_\Box$ fails robustly on the semantics:

Proposition C.1.6. For any M, A, Λ, μ: if $M, A \vDash_{\Lambda,\mu} \psi$ wherever ψ is an instance of Comp$_\Box$, then $M, A \vDash_{\Lambda,\mu} \forall X \forall Y \Box \forall x(Xx \leftrightarrow Yx)$.

Proof. Take any M, A, Λ, μ. From the above, if $\mu(M) \nvdash_\Lambda \forall x(Xx \leftrightarrow Yx)$, then at least one instance of Comp$_\Box$ fails in M, A, Λ, μ. So $\mu(M) \vdash_\Lambda \forall x(Xx \leftrightarrow Yx)$, meaning that for all assignments A', $M, A' \vDash_{\Lambda,\mu} \Box \forall x(Xx \leftrightarrow Yx)$. Therefore, for all $U, U' \subseteq |M|$, we have $M, A^{X\,Y}_{U\,U'} \vDash \Box \forall x(Xx \leftrightarrow Yx)$, so that $M, A \vDash \forall X \forall Y \Box \forall x(Xx \leftrightarrow Yx)$. $\square$

It turns out the failure is even more drastic among closed modalities:

Proposition C.1.7. For any M, A, Λ, μ: if $\mu(M)$ consists entirely of sentences and $M, A \vDash_{\Lambda,\mu} \psi$ wherever ψ is an instance of $\text{Comp}_\Box$, then $M, A \vDash_{\Lambda,\mu} \Box\bot$.

Proof. As seen above, if $M, A \vDash_{\Lambda,\mu} \psi$ wherever ψ is an instance of $\text{Comp}_\Box$, then it must be the case that $\mu(M) \vdash_\Lambda \forall x(Xx \leftrightarrow Yx)$. Since every formula in $\mu(M)$ is closed, by universal generalisation for Λ we have $\mu(M) \vdash_\Lambda \forall Y \forall x(Xx \leftrightarrow Yx)$. But $\vdash_\Lambda \forall Y \forall x(Xx \leftrightarrow Yx) \rightarrow \forall x(Xx \leftrightarrow \lambda y \neg Xyx)$ by UI_2, and $\vdash_\Lambda \forall x(Xx \leftrightarrow \lambda y \neg Xyx) \rightarrow (Xc \leftrightarrow \lambda y \neg Xyc)$ by UI. Finally, via β-conversion, we have $\vdash_\Lambda (Xc \leftrightarrow \lambda y \neg Xyc) \rightarrow (Xc \leftrightarrow \neg Xc)$, so that summing up we get $\mu(M) \vdash_\Lambda Xc \leftrightarrow \neg Xc$. But $\vdash_\Lambda (Xc \leftrightarrow \neg Xc) \rightarrow \bot$, so we therefore have $\mu(M) \vdash_\Lambda \bot$, meaning that $M, A \vDash_{\Lambda,\mu} \Box\bot$. □

Therefore the failures of $\text{Comp}_\Box$ are robust, in the sense that the only way to ensure its validity also ensures the validity of implausible theses like $\forall X \forall Y \forall x \Box(Xx \leftrightarrow Yx)$ and $\Box\bot$.

C.2 Higher-Order Substitutional Quantifiers

We now consider the language QMLΠ2, constructed from QML∀2 in the same way that QMLΠ was constructed from QML∀. Namely, the formulas of QMLΠ2 are exactly those formulas which result from taking a QML∀2 formula and replacing every occurrence of the objectual quantifier ∀ with Π. We can define QMLΠ2 logics likewise.

Our semantics for this language will be like the semantics for QMLΠ given in chapter 8, but for the addition of two new semantic clauses. The definition of a state description S is unchanged from there, since the definition of an atomic sentence is itself unchanged. The first of the new clauses deals with the case of second-order quantifiers:

$S \vDash_{\Lambda,\mu} \Pi X\phi$ iff for all closed F of equal arity with X, $S \vDash_{\Lambda,\mu} \phi[F/X]$.

Here the predicate F can be either a primitive predicate constant or a λ-abstract (we require F to be closed since $\phi[F/X]$ must be a sentence). The second new clause deals with formulas which contain λ-abstracts. The notation $\phi[t_1/x_1, ..., t_n/x_n]$ extends the usual substitution notation and represents the simultaneous substituting-in of the terms $t_1...t_n$ for the variables $x_1...x_n$ respectively, observing the usual caveat that if one of the variables x_i occurs within the scope of an occurance of 'Πt_i,' we consider instead some appropriate alphabetic variant of ϕ instead. Though note this caveat may be ignored where the t_i are all constants, which is the case here:

$$S \vDash_{\Lambda,\mu} \lambda x_1 ... x_n \phi c_1 ... c_n \text{ iff } S \vDash_{\Lambda,\mu} \phi[c_1/x_1, ..., c_n/x_n].$$

For the remainder of this appendix, I will write $\vDash_{\Lambda,\mu} \phi$ to say that $S \vDash_{\Lambda,\mu} \phi$ for all state descriptions S, and I will write $\vDash \phi$ to say that ϕ is generally valid, meaning that $\vDash_{\Lambda,\mu} \phi$ for all QMLΠ2 logics Λ and all functions $\mu : \mathcal{P}(\text{AS}) \to \mathcal{P}(\text{QML}\Pi 2)$.

The way that we have set down our semantic clauses is the most natural extension of the semantics for substitutional quantifiers given in chapter 8. However, these particular clauses hint at problems of impredicatvivity which would threaten the coherence of the semantics. Suppose that we wish to evaluate a sentence $\Pi X \phi$ at some S with respect to Λ and μ. By our clause, $\Pi X \phi$ is true in this case iff every formula $\phi[F/X]$ is true, for all predicates F of equal arity to X. Suppose that X is of arity n. Then among these predicates F are the complex predicates $\lambda x_1, ..., x_n \psi$. Now whether or not a formula including $\lambda x_1, ..., x_n \psi$ is true in this case depends on whether certain substitution instances of ψ are true, and the formula ψ may be of much greater complexity than ϕ itself. Therefore it is not clearly the case that our clauses succeed in determining a semantics. The problem here is essentially similar to that faced in miniature in the McKinsey semantics for propositional modal logic; there, it is a highly non-trivial result that there exist valuations v of the propositional language which obey McKinsey's clause, and it remains an open question whether or not a particular valuation of atomic formulas fully determines valuations of compound formulas (i.e. whether every subvaluation is uniquely extended by a McKinsey valuation). Therefore when dealing with the McKinsey sematics, one cannot talk of a single satisfaction relation '$\vDash$' which holds between models m and formulas ϕ which obeys McKinsey's clause (i.e. such that $m \vDash \Box\phi$ iff $m \vDash s\phi$ for all substitutions s), but one is required to speak more generally about valuation functions v which obey the clause

In the case of our clauses for QMLΠ2, the case is even worse. Given a state description S, logic Λ and modality μ, we do not even have an existence theorem to ensure that there is some valuation function $V : Sen(\text{QML}) \to \{0, 1\}$ which accords with our clauses—i.e. a function such that:

$$V(\bot) = 0$$
$$V(\phi) = 1 \text{ iff } \phi \in S \text{ for } \phi \in \text{AS}.$$
$$V(\phi \to \psi) = 1 \text{ iff } V(\phi) = 0 \text{ or } V(\psi) = 1$$
$$V(\Box\phi) = 1 \text{ iff } \mu(S) \vdash_\Lambda \phi$$
$$V(\Pi x \phi) = 1 \text{ iff } V(\phi[c/x]) = 1 \text{ for every constant } c.$$
$$V(\Pi X \phi) = 1 \text{ iff } V(\phi[F/X]) = 1 \text{ for every closed } F \text{ of equal arity with } X.$$

This is not even touching on the question of whether there might be a *unique* such function for each S, Λ and μ, which is what would be needed to show that our clauses succeed in defining a unique determinate satisfaction relation. I am not in a position to provide a proof of existence or uniqueness of such valuation functions, or in other words, that our clauses for QMLΠ2 formulas succeed in picking out a determinate satisfaction relation between models and formulas. Instead, in the remaining Propositions this section (C.2.1-C.2.4), marked $(*)$ I will assume that they do, and show what such a semantics *would* do to resolve the robust failures of the QML∀2 semantics if shown to be coherent. I will then discuss the nature of this assumption in the conclusion.

For sake of brevity we will not consider how to deal with free variables and open formulas here as we did previously for first-order languages.

C.2.1 Instantiation, β-Conversion, and Modal Comprehension

We can infer immediately by the semantic clauses just given that all sentential instances of universal instantiation for second-order quantification and β-conversion are generally validated on this semantics:

Proposition C.2.1 $(*)$**.** $\vDash \Pi X\phi \rightarrow \phi[F/X]$ where F is closed.

Proposition C.2.2 $(*)$**.** $\vDash \Pi y_1...\Pi y_n(\lambda x_1...x_n\phi y_1...y_n \leftrightarrow \phi[y_1/x_1...y_n/x_n])$.

Proofs. Immediate. □

In addition, we may show that the substitutional counterparts of Comp and Comp$_\Box$ are both generally valid in all of their instances. These substitutional comprehension schemas are as follows:

Comp$^\Pi$ $\quad \Sigma X^n \Pi x_1...\Pi x_n(X^n x_1...x_n \leftrightarrow \phi)$ where X^n is not free in ϕ.

Comp$^\Pi_\Box$ $\quad \Sigma X^n \Box \Pi x_1...\Pi x_n(X^n x_1...x_n \leftrightarrow \phi)$ where X^n is not free in ϕ.

Note that the following arguments make use of (i) the fact that β-conversion is generally validated on the semantics and (ii) that β-conversion is also included in every QMLΠ2 logic as a result of the definition of QMLΠ2 and QML∀2 logics that we have adopted.

Proposition C.2.3 $(*)$**.** $\vDash \Sigma X^n \Pi x_1...\Pi x_n(X^n x_1...x_n \leftrightarrow \phi)$ for X^n not free in ϕ.

Proof. Take any ϕ where X^n does not occur free. Then consider the λ-abstract $\lambda x_1...x_n\phi$. By the semantic clauses, $\vDash \Pi x_1...\Pi x_n(\lambda x_1...x_n\phi x_1...x_n \leftrightarrow \phi[x_1/x_n, ..., x_n/x_n])$, but given that $\phi[x_1/x_n, ..., x_n/x_n]$ is ϕ, we have $\vDash \Pi x_1...\Pi x_n(\lambda x_1...x_n\phi x_1...x_n \leftrightarrow \phi)$. Then via the derived semantic clause for Σ, we have $\vDash \Sigma X^n \Pi x_1...\Pi x_n(X^n x_1...x_n \leftrightarrow \phi)$, with $\lambda x_1...x_n\phi$ acting as witness. □

Proposition C.2.4 ($*$). $\vDash \Sigma X^n \Box \Pi x_1 ... \Pi_n (X x_1 ... x_n \leftrightarrow \phi)$ for X^n not free in ϕ.

Proof. Take any S, Λ, and μ, and any ϕ such that X^n is not free in ϕ. Consider $\lambda x_1 ... x_n \phi$, which is of equal arity with X^n. By β-conversion, $\vdash_\Lambda \Pi x_1 ... \Pi x_n (\lambda x_1 ... x_n \phi . x_1 ... x_n \leftrightarrow \phi[x_1/x_1]...[x_n/x_n])$, or, in other words, $\vdash_\Lambda \Pi x_1 ... \Pi x_n (\lambda x_1 ... x_n \phi . x_1 ... x_n \leftrightarrow \phi)$. By the semantic clause we can infer that $S \vDash_{\Lambda,\mu} \Box \Pi x_1 ... \Pi x_n (\lambda x_1 ... x_n \phi . x_1 ... x_n \leftrightarrow \phi)$ so that, by the derived clause for Σ, we have $S \vDash_{\Lambda,\mu} \Sigma X^n \Box \Pi x_1 ... \Pi x_n (X x_1 ... x_n \leftrightarrow \phi)$. □

Once again, in the above propositions marked with ($*$), it is assumed that the semantic clauses given above for QMLΠ2 succeed in determining a definite satisfaction relation.

C.3 Conclusion

Many of the technical results discussed regarding the syntactical semantics for first-order modal languages in Chapter 8 can be extended to the second-order case. Robust failures of basic principles beset the language with objectual quantifiers; such failures now include β-conversion and the comprehension schemas. The retreat to substitutional quantification once again fixes such failures, but now with an additional caveat. The impredicativity of the semantics leaves us in a state of ignorance as to whether the resulting clauses are consistent.

It might be thought that the threat of inconsistency via impredicativity could be overcome by omitting λ, the device for forming complex predicates, from the second-order language. On the one hand, this indeed removes the impredicativity, and ensures that the semantic clauses given above (omitting the clause for λ-abstracts) succeeds in determining a definite satisfaction relation. On the other hand, going this route means sacrificing the validity of both the non-modal and modal comprehension schemas. To see why this is the case, let S be the unique state description which contains every atomic formula except $\bot$ and $c = c'$ where c and c' are differing constants. Then for every monadic predicate F (which, in the absence of λ, means just the primitive monadic predicate constants) and every constant c, we have $S \nvDash_{\Lambda,\mu} (Fc \leftrightarrow \bot)$ since $Fc \in S$ (here Λ and μ can be arbitrary). But in this case we have $S \vDash_{\Lambda,\mu} \Sigma x \neg (Fx \leftrightarrow \bot)$ for every predicate F and hence $S \vDash_{\Lambda,\mu} \Pi X \Sigma x \neg (Xx \leftrightarrow \bot)$, which means that $S \vDash_{\Lambda,\mu} \neg \Sigma X \Pi x (Xx \leftrightarrow \bot)$, showing that non-modal comprehension fails to be valid. In the modal case: in the absence of λ-abstracts, and hence in the absence of β-conversion in QMLΠ2 logics, for many logics Λ there will be no monadic predicate F in the language such that $\vdash_\Lambda Fc \leftrightarrow \bot$. But in this case we have $S \vDash_{\Lambda,\alpha} \neg \Sigma X \Box \Pi x (Xx \leftrightarrow \bot)$ for any state description S. Now of course, a strategy which attempts to avoid inconsistency in the semantics by restricting the expressive power of the object language already seems desperate.

But it is also important to note, as we just have, that such a strategy has further knock-on effects with respect to the task of validating ordinary principles of higher-order modal logic.

Philosophically, all this seems to compound the case against syntactic logicism about modality. The point is less central than those already made in the main discussion, but the fact that syntactic logicism may be unable to provide a natural semantics for higher-order modal logics as a result of impredicativity is certainly no point in its favour. If taking up syntactic logicism requires one, in retreating to the use of substitutional quantifiers, to refrain from higher-order languages, then that is a cost of deducibility logicism. But suppose that the problems surrounding impredicativity can be ultimately solved—there is reason to think this is possible, given that it is possible to provide a valuation existence theorem for the McKinsey semantics in PML, which suffers from similar impredicativity. Even in this case, one is still faced with the higher-order analogues of the problems raised in the main discussion for the first-order semantics.