NON-ZERO COEFFICIENTS OF HALF-INTEGRAL WEIGHT MODULAR FORMS MOD ℓ

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Abstract. We obtain new lower bounds for the number of Fourier coefficients of a weakly holomorphic modular form of half-integral weight not divisible by some prime ℓ. Among the applications of this we show that there are $\gg \sqrt{X}/\log\log X$ integers $n \leq X$ for which the partition function $p(n)$ is not divisible by ℓ, and that there are $\gg \sqrt{X}/\log\log X$ values of $n \leq X$ for which $c(n)$, the nth Fourier coefficient of the j-invariant, is odd.

1. Introduction

Let K be a number field and $\mathcal{O}$ its ring of integers. Let ℓ be a rational prime and let λ be a maximal ideal of $\mathcal{O}$ above ℓ. We denote by $\mathbb{F}$ the residue field $\mathcal{O}/\lambda$, a finite extension of $\mathbb{F}_\ell$. The reader will lose little by supposing that $K = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$, $\lambda = (\ell)$ and $\mathbb{F} = \mathbb{F}_\ell$; our main applications use only this case.

Theorem 1. *Let $k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ be a half-integer, and N a positive integer. Let $f = \sum_{n=n_0}^{\infty} a_n q^n$ be a weakly holomorphic modular form[1] of weight k and level $\Gamma_1(N)$. Suppose that the coefficients a_n lie in the ring $\mathcal{O}$. If $\ell \geq 3$, we assume that $f \not\equiv 0 \pmod{\lambda}$, and for $\ell = 2$ we assume that $f \pmod{\lambda}$ is not a constant. Then*

$$\#\{n \leq X,\ a_n \not\equiv 0 \pmod{\lambda}\} \gg \frac{\sqrt{X}}{\log\log X}.$$

Theorem 1 improves, by a factor of about $\log X$, the earlier work of Ahlgren and Boylan [2]. Here are some sample applications of Theorem 1.

Example 1. Take $f = \eta_1(z)^{-1}$ with $\eta_1(z) = \eta(24z)$ (Dedekind's eta function), so that f is a weakly holomorphic modular form of weight $-1/2$ and level $\Gamma_0(576)$. The Fourier expansion of f is

$$f(q) = q^{-1}\prod_{n=1}^{\infty}\left(1-q^{24n}\right)^{-1} = \sum_{n=0}^{\infty} p(n)q^{24n-1}$$

where $p(n)$ is the *partition function* (cf. [17, Theorem 1.60, Corollary 1.62 and Theorem 5.3] for a proof of these well-known facts). Applying the theorem to f, we conclude that

$$\#\{n \leq X,\ p(n) \not\equiv 0 \pmod{\ell}\} \gg \frac{\sqrt{X}}{\log\log X}. \tag{1}$$

Joël Bellaïche was supported by NSF grant DMS 1405993. Ben Green was supported by a Simons Investigator grant from the Simons Foundation. Kannan Soundararajan was partially supported by NSF grant DMS 1500237, and a Simons Investigator grant from the Simons Foundation. Part of the work was carried out when the second and third authors were in residence at MSRI, Berkeley during the Spring semester of 2017, supported in part by NSF grant DMS 1440140.

[1] *Weakly holomorphic* allows for polar singularities at the cusps; for this and other basic definitions, we refer the reader to [17, Chapter 1].

This improves, by a factor of about $(\log X)^{\frac{3}{4}}$, earlier results of Ahlgren [1], Chen [9] and Dai & Fang [10] for odd ℓ. In the case $\ell = 2$, (1) improves upon previous results (established by somewhat different methods than for ℓ odd) by a factor of about $(\log X)^{\frac{7}{8}}$; see [5, 14, 15, 16].

Example 2. Theorem 1 applies in particular when f is a *holomorphic* cusp form of half-integral weight k (which must then be positive). In this case, it improves on the main theorem of [8] (itself an improvement of [18]) which proves the slightly weaker estimate $\#\{n \leq X, a_n \not\equiv 0 \pmod{\ell}\} \gg \sqrt{X}/\log X$ under the supplementary assumption that the coefficients $a_n \pmod{\lambda}$ are not supported in a finite union of sequences of the form $(cn^2)_{n\in\mathbb{N}}$. We remark that in [8, 18] the Shimura correspondence between holomorphic half-integral weight modular forms and integral weight modular forms plays a crucial role, whereas our proof of Theorem 1 does not involve the Shimura correspondence.

Example 3. When $\ell = 2$, our theorem applies as well to weakly holomorphic modular forms of *integral* weight, since those forms are congruent modulo λ to forms of *half-integral* weight (see Lemma 3 below). In particular, for the modular invariant $j(q) = \sum_{n=-1}^{\infty} c(n)q^n$, which is of weight 0 and level $\mathrm{SL}_2(\mathbb{Z})$, we obtain that

$$\#\{n \leq X, c(n) \text{ is odd}\} \gg \frac{\sqrt{X}}{\log\log X}. \tag{2}$$

This improves upon recent results in [3, 20] obtained by different methods.

Theorem 1 unfortunately does not give any improvement for the number of class numbers of imaginary quadratic fields not divisible by a prime ℓ. Although these class numbers arise as the Fourier coefficients of θ^3, in order to produce distinct fields we need a variant of Theorem 1 that produces square-free n with $a_n \not\equiv 0 \pmod{\lambda}$, which is not something that our methods allow.

For completeness, we remark that [6] establishes, for $\ell \geq 3$, an asymptotic for the number of non-zero coefficients (mod ℓ) of holomorphic modular forms, and [5] establishes such an asymptotic for $\ell = 2$ and holomorphic forms of level 1. The situation for *weakly* holomorphic forms of integral weight modulo a prime $\ell > 2$ remains mysterious, and (for example) we do not have lower bounds for the number of $c(n) \neq 0 \pmod{\ell}$ for primes $3 \leq \ell \leq 11$. (It is known [19, Théorème 5.2(a)] that $\sum_{n\geq 0} c(\ell n)q^n$ is an holomorphic modular form mod ℓ of positive integral weight, and this form is constant if and only if $\ell \leq 11$ [19, Exercise (6.16)]. Thus in the case $\ell \geq 13$, from [6] it follows that there are $\gg X/(\log X)^{3/4}$ values of $n \leq X$ with $c(n)$ not divisible by ℓ.)

The proof of Theorem 1 uses the standard idea of multiplying f by a suitable lacunary holomorphic cusp form g of half-integral weight, to obtain an holomorphic cusp form $h = fg$ of integral weight. The panoply of results stemming from the existence of Galois representations associated to integral weight holomorphic eigenforms may then be used to study the coefficients of h. Finding a suitable form g is easy when $\ell > 2$, and somewhat less so in the case $\ell = 2$. We describe this deduction in Section 2 below for the sake of completeness, but we should note that closely related arguments appear already in the work of Ahlgren and Boylan [2]. It will follow from the work of Section 2 (and this is also implicit in [2]) that the sumset of the set of squares and the set of non-zero Fourier coefficients of f

(mod ℓ) contains all numbers of the form up for a fixed integer u and p lying in a positive density subset of the primes. Our improvement over previous work comes from a more careful analysis of this problem in analytic number theory/additive combinatorics.

Theorem 2. *Let $u \geq 1$ be a fixed natural number, and let X be large. For any subset $\mathcal{A} \subset \{1\ldots, X\}$ the number of primes p such that $pu \leq X$ and pu may be written as $a+m^2$ for some $a \in \mathcal{A}$ and some integer m is*

$$\ll \frac{\sqrt{X}}{\log X}\Big(|\mathcal{A}|\log\log X + |\mathcal{A}|^{\frac{1}{2}}X^{\frac{1}{4}}\Big).$$

Our interest in Theorem 2 is in the situation where a positive proportion of the primes p are known to be of the form $a+m^2$, when it follows that $|\mathcal{A}|$ must have $\gg \sqrt{X}/\log\log X$ elements. This statement is optimal, as we shall show in Section 4 by constructing an example of a set $\mathcal{A}$ with $|\mathcal{A}| \asymp \sqrt{X}/\log\log X$, and with a positive proportion of primes below X being of the form $a+m^2$.

Theorem 1, on the other hand, is almost certainly not optimal. For any weakly holomorphic form $f(q) = \sum a_n q^n$ of half-integral weight, one might expect

$$\#\{n \leq X,\ a_n \not\equiv 0 \pmod{\lambda}\} \gg \sqrt{X},$$

and this bound is attained for $\eta_1(q)$ (see (4) below). Theorem 1 comes close to this estimate. For most forms f of half-integral weight however (specifically for $f(q) = \eta_1^{-1}(q) = \sum_n p(n)q^{24n-1}$, and perhaps for all forms that are not congruent mod λ to a one-variable theta series), it is expected that $f \pmod{\lambda}$ is not *lacunary*, which is to say that $\#\{n \leq X,\ a_n \not\equiv 0 \pmod{\lambda}\} \gg X$.

We thank Scott Ahlgren for kindly drawing our attention to [2], and the referees for a careful reading.

2. Deduction of Theorem 1 from Theorem 2

2.1. **A preliminary lemma.** Let $M_k(\Gamma_1(N), \mathcal{O})$ be the $\mathcal{O}$-module of holomorphic modular forms of integral weight $k \geq 0$, level $\Gamma_1(N)$, and coefficients in $\mathcal{O}$. Let $\mathcal{O}_\lambda$ be the completion of $\mathcal{O}$ at the place defined by the ideal λ, and set $M_k(\Gamma_1(N), \mathcal{O}_\lambda) = M_k(\Gamma_1(N), \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}_\lambda$. Let A be the closure of the $\mathcal{O}_\lambda$-subalgebra of $\mathrm{End}_{\mathcal{O}_\lambda}(M_k(\Gamma_1(N), \mathcal{O}_\lambda))$ generated by the Hecke operators T_n for n running among integers relatively primes to $N\ell$.

Throughout, we denote the n-th coefficient of a modular form h by $a_n(h)$. We recall that if $h = \sum_{n=0}^{\infty} a_n q^n$ is a holomorphic modular form of integral weight k for $\Gamma_1(N)$, and if p is a prime not dividing N, the m-th coefficient of the form $T_p h$ is

$$a_m(T_p h) = a_{mp}(h) + p^{k-1}a_{m/p}(\langle p\rangle h), \tag{3}$$

where $\langle p\rangle$ is the diamond operator, and with the convention that $a_{m/p}(-) = 0$ when $p \nmid m$; see, for example, Chapter 5 of [11].

Denote by $G_{\mathbb{Q},N\ell}$ the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside $N\ell$, and by Frob_p, for p a prime not dividing $N\ell$, the Frobenius element of p in $G_{\mathbb{Q},N\ell}$, well-defined up to conjugation.

Lemma 1. *There exists a unique continuous map $t : G_{\mathbb{Q},N\ell} \to A$ which is central and satisfies $t(\mathrm{Frob}_p) = T_p$ for every prime p not dividing $N\ell$. This maps also satisfies $t(1) = 2$.*

Proof. This follows from a well-known argument of Wiles based on the existence of Galois representations attached to eigenforms due to Deligne; see, for example, [4, Thm 1.8.5] for a detailed proof. □

2.2. **The case $\ell > 2$.** We begin with a lemma.

Lemma 2. *Assume that ℓ is odd. Let $k \in \mathbb{N}$ and $h(q) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N), \mathcal{O})$. Assume that $u \geq 1$ is an integer such that $a_u \not\equiv 0 \pmod{\lambda}$. Then there is a positive density set of primes $\mathcal{P}$ such that $a_{up} \not\equiv 0 \pmod{\lambda}$ for every $p \in \mathcal{P}$.*

Proof. With t as in Lemma 1, the map from $G_{\mathbb{Q},N\ell}$ to $\mathbb{F}$ sending g to $a_u(t(g)h) \pmod{\lambda}$ is a continuous map. Thus there exists an open neighborhood U of 1 in $G_{\mathbb{Q},N\ell}$ such that $g \mapsto a_u(t(g)h) \pmod{\lambda}$ is constant on U. Let $\mathcal{P}$ be the set of primes not dividing $N\ell u$ such that $\mathrm{Frob}_p \in U$. By Chebotarev, $\mathcal{P}$ has positive density. Further for $p \in \mathcal{P}$ we have using (3) and Lemma 1:

$$a_{up}(h) = a_u(T_p h) = a_u(t(\mathrm{Frob}_p)h) = a_u(t(1)h) = 2a_u(h) \not\equiv 0 \pmod{\lambda},$$

since $\ell \neq 2$ and $a_u(h) \not\equiv 0 \pmod{\lambda}$. □

We can now deduce Theorem 1 from Theorem 2. Let $f = \sum_{n \geq n_0} a_n q^n$ be a weakly holomorphic modular form of half-integral weight, level $\Gamma_1(N)$, and coefficients in $\mathcal{O}_\lambda$. Let $\eta(z)$ be the usual Dedekind's eta function and set $\eta_1(z) = \eta(24z)$ so that (see [17])

$$\eta_1(q) = q\prod_n (1-q^{24n}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2} \tag{4}$$

is a holomorphic cuspidal modular form of weight $1/2$. Let m be an even integer such that ℓ^m is larger than the order of any pole of f. Since f has half-integer weight k, the holomorphic cuspidal modular form $h = f\eta_1^{\ell^m}$ has integral weight $k + \ell^m/2$. Since $f \pmod{\lambda}$ and $\eta_1 \pmod{\lambda}$ are non-zero, the power series $h \pmod{\lambda} \in \mathbb{F}[[q]]$ is also non-zero, and indeed $h \pmod{\lambda}$ is not a constant (because h is cuspidal, and a cuspidal constant form must be 0).

Let $\mathcal{A} = \{n,\ a_n = a_n(f) \not\equiv 0 \pmod{\lambda}\}$. Note that from (4)

$$\eta_1(q)^{\ell^m} = \Big(\sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2}\Big)^{\ell^m} \equiv \sum_{n=-\infty}^{\infty} (-1)^n q^{\ell^m(6n+1)^2} \pmod{\lambda},$$

so that the Fourier coefficients of $\eta_1^{\ell^m}$ are non-zero $\pmod{\lambda}$ only on squares. Thus if n is such that $a_n(h) \not\equiv 0 \pmod{\lambda}$, then n must be of the form $a + v^2$ for some $a \in \mathcal{A}$ and some integer v. Now, by Lemma 2, the set of n such that $a_n(h) \not\equiv 0 \pmod{\lambda}$ contains a set of the form $u\mathcal{P}$, for some fixed natural number u and a set of primes $\mathcal{P}$ of positive density. For large X, it follows from Theorem 2 that the number of primes p with $up \leq X$ and up of the form $a + v^2$ with $a \in \mathcal{A}$ is $\ll |\mathcal{A} \cap [1, X]|\sqrt{X}(\log\log X)/\log X + |\mathcal{A} \cap [1, X]|^{\frac{1}{2}} X^{\frac{3}{4}}/\log X$. It follows that $|\mathcal{A} \cap [1, X]| \gg \sqrt{X}/\log\log X$, proving Theorem 1.

2.3. **The case $\ell = 2$.** This case needs a little more care, and we begin by recalling a well-known result that (for $\ell = 2$) modular forms of integer weights are congruent to modular forms of half-integer weight.

Lemma 3. *Assume that $\ell = 2$. For every weakly holomorphic modular form f of weight k and level $\Gamma_1(N)$ with coefficients in $\mathcal{O}$ there exists a weakly holomorphic modular form f' of weight $k + 1/2$, some level $\Gamma_1(N')$, with coefficients in $\mathcal{O}$, such that $f \equiv f' \pmod{\lambda}$.*

Proof. Recall that (see, for example, [17, Prop 1.4]) the theta series $\theta_0(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + \sum_{n=1}^{\infty} 2q^{n^2}$ is a holomorphic modular form of weight $1/2$, level $\Gamma_0(4)$, coefficients in $\mathbb{Z}$. Now take $f' = f\theta_0$. □

Lemma 4. *Let n_0 be a non-zero integer, and let N be a positive integer. There are only finitely many natural numbers s such that $2^s + n_0$ equals uy^2 for some square-free divisor u of $2N$ and some integer y.*

Proof. Write $s = 3s_0 + r$ with $r = 0$, 1, or 2, and set $x = 2^{s_0}$. The equation $2^s + n_0 = uy^2$ becomes $2^r x^3 + n_0 = uy^2$, which for a given r and u may be viewed as an elliptic curve (since $n_0 \neq 0$). By Siegel's Theorem there are only finitely many integer points (x, y) on this elliptic curve. Since there are only three possible values for r, and finitely many possibilities for u (being a square-free divisor of $2N$), the lemma follows. □

Lemma 5. *Let $k \in \mathbb{N}$ and $h(q) = \sum_{n=0}^{\infty} a_n q^n \in M_k(\Gamma_1(N), \mathcal{O})$. Assume that there exists an integer $n \geq 1$ and a prime p_0 not dividing $2N$ such that $ord_{p_0} n$ is odd and $a_n \not\equiv 0 \pmod{\lambda}$. Then there exists an integer $u \geq 1$ and a set of primes $\mathcal{P}$ of positive density such that $a_{up} \not\equiv 0 \pmod{\lambda}$ for every $p \in \mathcal{P}$.*

Proof. We claim that *if p is a prime not dividing $2N$ and if $T_p h \pmod{\lambda}$ is a constant, then $a_n(h) \equiv 0 \pmod{\lambda}$ if $ord_p(n)$ is odd.* We prove that claim, for all h such that $T_p h \pmod{\lambda}$ is constant, by induction over the odd number $\mathrm{ord}_p(n)$. If $\mathrm{ord}_p(n) = 1$, applying (3) to the form h and the integer $m = n/p$ and reducing mod λ gives (using that $p \equiv -1 \equiv 1 \pmod{\lambda}$):

$$a_n(h) \equiv a_{n/p^2}(\langle p \rangle h) \equiv 0 \pmod{\lambda}.$$

For a general n with $\mathrm{ord}_p(n)$ odd, we get similarly

$$a_n(h) \equiv a_{n/p^2}(\langle p \rangle h) \pmod{\lambda}.$$

By the induction hypothesis applied to the form $\langle p \rangle h$ (which also satisfies $T_p(\langle p \rangle h) \pmod{\lambda}$ constant since the diamond operator $\langle p \rangle$ commutes with T_p and stabilizes the subspace of constants), we get $a_n(h) \equiv 0 \pmod{\lambda}$ which completes the induction step.

By the hypothesis of the Lemma, it follows that $T_{p_0} h \pmod{\lambda}$ is not a constant, that is to say there exists $u \geq 1$ such that $a_u(T_{p_0} h) \not\equiv 0 \pmod{\lambda}$, or equivalently, with t as in Lemma 1, $a_u(t(\mathrm{Frob}_{p_0}) h) \not\equiv 0 \pmod{\lambda}$. By continuity of t, there exists an open set U in $G_{\mathbb{Q},2N}$ such that for p a prime not dividing $2Nu$, if $\mathrm{Frob}_p \in U$, then

$$a_{up}(h) = a_u(T_p h) = a_u(t(\mathrm{Frob}_p) h) \equiv a_u(t(\mathrm{Frob}_{p_0}) h) \not\equiv 0 \pmod{\lambda}.$$

The set $\mathcal{P}$ of such primes p is a set of primes of positive density by Chebotarev. □

We are now ready to prove Theorem 1 in the case $l = 2$ using Theorem 2. Let f be a weakly holomorphic modular form of half-integral weight with coefficients in $\mathcal{O}_\lambda$. By Lemma 3 we may assume that f has integral weight instead.

We consider the form $h := f\eta_1^{2^m}$ for a suitable m that will be specified below. We observe that if $m \geq 1$, $\eta_1^{2^m}$ has integral weight, and so does h. Moreover, since η_1 is cuspidal, h is also cuspidal holomorphic for m large enough.

Write $f \equiv \sum_{n=n_0}^{\infty} a_n q^n \pmod{\lambda}$ with $a_{n_0} \not\equiv 0$. The first term of $h \pmod{\lambda}$ is $a_{n_0} q^{2^m+n_0}$. When $n_0 \neq 0$, Lemma 4 ensures that we can choose m even, large enough in the sense of the preceding paragraph, and such that there is a prime p_0 not dividing $2N$ such that $\mathrm{ord}_{p_0}(2^m + n_0)$ is odd. When $n_0 = 0$, let $a_{n_1} q^{n_1}$ with $n_1 > 0$, $a_{n_1} \not\equiv 0$ the term of smallest positive degree in $f \pmod{\lambda}$ (such an n_1 exists because we assume that $f \pmod{\lambda}$ is not a constant). If m is such that $2^m > n_1$, then the form h has a term $a_{n_1} q^{2^m+n_1}$. Again by Lemma 4 we can find m large enough and such that there is a prime p_0 not dividing $2N$ such that $\mathrm{ord}_{p_0}(2^m + n_1)$ is odd.

So in both cases ($n_0 \neq 0$ and $n_0 = 0$) we have shown the existence of an integer m such that $h = f\eta_1^{2^m}$ is a cuspidal holomorphic modular form of integral weight and such that, by Lemma 5, there is $u \geq 1$ and a set of primes $\mathcal{P}$ of positive density, with $a_{up}(h) \not\equiv 0 \pmod{\lambda}$ for every $p \in \mathcal{P}$. The rest of the proof is now exactly as in the case $\ell > 2$.

3. Proof of Theorem 2

Given a positive integer a, we let $\chi_{-4a} = (\frac{-4a}{\cdot})$ denote the Kronecker symbol, which is a Dirichlet character $\pmod{4a}$. Note that $-4a$ is a discriminant, but it need not be a fundamental discriminant. We denote the associated (negative) fundamental discriminant by $\widetilde{a}$, so that $-4a = \widetilde{a}a_2^2$ for a suitable natural number a_2.

Lemma 6. *Let u be a fixed natural number, and let X be large. For every integer $1 \leq a \leq X$,*

$$\#\{p : \ up \leq X, \ p = a + m^2 \text{ for some } m \in \mathbb{Z}\} \ll \frac{\sqrt{X}}{\log X} \prod_{p \leq X^{\frac{1}{4}}} \Big(1 - \frac{\chi_{-4a}(p)}{p}\Big).$$

Proof. Consider the equivalent problem of estimating the number of m below $\sqrt{X}$ such that $a + m^2$ is of the form ur for a prime number r. We may restrict attention to $r > X^{\frac{1}{4}}$, since the smaller primes r contribute negligibly to the number of m. For each prime $p \nmid 2u$ and $p \leq X^{\frac{1}{4}}$ we see that m^2 cannot be $\equiv -a \pmod{p}$, which means that $1 + \chi_{-4a}(p)$ residue classes $\pmod{p}$ are forbidden for m. Any standard upper bound sieve (for example, Brun's sieve or Selberg's sieve; or see Theorem 2.2 of [13]) then shows that the number of possible $m \leq \sqrt{X}$ is

$$\ll \sqrt{X} \prod_{\substack{p \leq X^{\frac{1}{4}} \\ p \nmid 2u}} \Big(1 - \frac{1 + \chi_{-4a}(p)}{p}\Big) \ll \frac{\sqrt{X}}{\log X} \prod_{p \leq X^{\frac{1}{4}}} \Big(1 - \frac{\chi_{-4a}(p)}{p}\Big),$$

and the lemma follows. □

Call a fundamental discriminant d *good* if the corresponding Dirichlet L-function $L(s, \chi_d)$ has no zeros in the region $\{\sigma > 99/100, \ |t| \leq |d|\}$, and call the discriminant d *bad* otherwise.

Lemma 7. *Suppose $1 \le a \le X$ is an integer, and that the fundamental discriminant $\widetilde{a}$ corresponding to $-4a$ is good. Then*

$$\prod_{p \le X^{\frac{1}{4}}} \Big(1 - \frac{\chi_{-4a}(p)}{p}\Big) \ll \log\log X.$$

Proof. By [12, Lemma 2.1], for a good fundamental discriminant $\widetilde{a}$ one has

$$L\Big(1 + \frac{1}{\log X}, \chi_{\widetilde{a}}\Big) \asymp \prod_{p < (\log|\widetilde{a}|)^{100}} \Big(1 - \frac{\chi_{\widetilde{a}}(p)}{p}\Big)^{-1}.$$

Further

$$\begin{aligned}\log L\Big(1 + \frac{1}{\log X}, \chi_{\widetilde{a}}\Big) &= \sum_p \frac{\chi_{\widetilde{a}}(p)}{p^{1+1/\log X}} + O(1) \\ &= \sum_{p \le X^{\frac{1}{4}}} \frac{\chi_{\widetilde{a}}(p)}{p} + O\Big(\sum_{p \le X^{\frac{1}{4}}} \frac{1 - p^{-1/\log X}}{p} + \sum_{p > X^{\frac{1}{4}}} \frac{1}{p^{1+1/\log X}} + 1\Big).\end{aligned}$$

Using $1 - p^{-1/\log X} = O(\frac{\log p}{\log X})$ for $p \le X^{\frac{1}{4}}$, the first error term above is seen to be $O(1)$, and partial summation shows that the second term is also $O(1)$. Therefore

$$\prod_{p \le X^{\frac{1}{4}}} \Big(1 - \frac{\chi_{\widetilde{a}}(p)}{p}\Big) \asymp L(1 + 1/\log X, \chi_{\widetilde{a}})^{-1} \asymp \prod_{p \le (\log|\widetilde{a}|)^{100}} \Big(1 - \frac{\chi_{\widetilde{a}}(p)}{p}\Big). \tag{5}$$

Now write $-4a = \widetilde{a}a_2^2$ for some positive integer $a_2 \le \sqrt{X}$. Then

$$\prod_{p \le X^{\frac{1}{4}}} \Big(1 - \frac{\chi_{-4a}(p)}{p}\Big) = \prod_{p \le X^{\frac{1}{4}}} \Big(1 - \frac{\chi_{\widetilde{a}}(p)}{p}\Big) \prod_{\substack{p \le X^{\frac{1}{4}} \\ p \mid a_2}} \Big(1 - \frac{\chi_{\widetilde{a}}(p)}{p}\Big)^{-1},$$

and using (5) this is

$$\asymp \prod_{\substack{p \le (\log|\widetilde{a}|)^{100} \\ p \nmid a_2}} \Big(1 - \frac{\chi_{\widetilde{a}}(p)}{p}\Big) \prod_{\substack{X^{\frac{1}{4}} \ge p \ge (\log|\widetilde{a}|)^{100} \\ p \mid a_2}} \Big(1 - \frac{\chi_{\widetilde{a}}(p)}{p}\Big)^{-1}. \tag{6}$$

The first product in (6) is clearly at most

$$\prod_{p \le (\log|\widetilde{a}|)^{100}} \Big(1 + \frac{1}{p}\Big) \ll \log\log|\widetilde{a}|.$$

As for the second product in (6), this is

$$\le \prod_{\substack{X^{\frac{1}{4}} \ge p \ge (\log|\widetilde{a}|)^{100} \\ p \mid a_2}} \Big(1 - \frac{1}{p}\Big)^{-1} \le \prod_{(\log\widetilde{a})^{100} \le p \le (\log\widetilde{a})^{100} + (\log X)^2} \Big(1 - \frac{1}{p}\Big)^{-1},$$

since a_2 has at most $\log X$ prime factors, and the product is largest if these prime factors are the first $\le \log X$ primes all larger than $(\log\widetilde{a})^{100}$. This quantity is easily seen to be $\ll \max(1, \frac{\log\log X}{\log\log|\widetilde{a}|})$, proving the lemma. □

Applying Lemmas 6 and 7 we see that the number of primes p with $up \le X$ and p of the form $a + m^2$ with $a \in \mathcal{A}$ coming from a *good* associated fundamental discriminant $\widetilde{a}$ is bounded by

$$\sum_{\substack{a \in \mathcal{A} \\ \widetilde{a} \text{ good}}} \frac{\sqrt{X}}{\log X} \prod_{p \le X^{\frac{1}{4}}} \Big(1 - \frac{\chi_{-4a}(p)}{p}\Big) \ll |\mathcal{A}| \frac{\sqrt{X}}{\log X} \log\log X.$$

It remains to bound the number of primes arising from *bad* fundamental discriminants $\widetilde{a}$. Note that, with $-4a = \widetilde{a}a_2^2$,

$$\prod_{p\le X^{\frac14}}\Big(1-\frac{\chi_{-4a}(p)}{p}\Big) \ll \prod_{p\le X^{\frac14}}\Big(1-\frac{\chi_{\widetilde{a}}(p)}{p}\Big)\prod_{p\mid a_2}\Big(1-\frac{1}{p}\Big)^{-1}$$
$$\ll \frac{a_2}{\phi(a_2)}L(1+1/\log X,\chi_{\widetilde{a}})^{-1} \ll |\widetilde{a}|^{\epsilon}\frac{a_2}{\phi(a_2)},$$

where in the penultimate estimate ϕ denotes the Euler ϕ-function, and the final estimate follows by an obvious modification to Siegel's Theorem which gives $L(1+1/\log X,\chi_{\widetilde{a}}) \gg |\widetilde{a}|^{-\epsilon}$. Thus the number of primes arising from bad fundamental discriminants is

$$\ll \frac{\sqrt{X}}{\log X}\sum_{\substack{a\le X\\ \widetilde{a}\ \text{bad}}}\prod_{p\le X^{\frac14}}\Big(1-\frac{\chi_{-4a}(p)}{p}\Big) \ll \frac{\sqrt{X}}{\log X}\sum_{\substack{|\widetilde{a}|\le X\\ \widetilde{a}\ \text{bad}}}|\widetilde{a}|^{\epsilon}\sum_{\substack{-4a=\widetilde{a}a_2^2\\ a\in\mathcal{A}}}\frac{a_2}{\phi(a_2)}.$$

For a given $\widetilde{a}$ we may bound the sum over a_2 above using Cauchy-Schwarz; thus

$$\sum_{\substack{-4a=\widetilde{a}a_2^2\\ a\in\mathcal{A}}}\frac{a_2}{\phi(a_2)} \le \Big(\sum_{\substack{-4a=\widetilde{a}a_2^2\\ a\in\mathcal{A}}}1\Big)^{\frac12}\Big(\sum_{a_2\le\sqrt{X/|\widetilde{a}|}}\Big(\frac{a_2}{\phi(a_2)}\Big)^2\Big)^{\frac12} \ll \sqrt{|\mathcal{A}|}\frac{X^{\frac14}}{|\widetilde{a}|^{\frac14}}.$$

We conclude that the number of primes p arising from bad fundamental discriminants is

$$\ll \frac{\sqrt{X}}{\log X}|\mathcal{A}|^{\frac12}X^{\frac14}\sum_{\substack{|\widetilde{a}|\le X\\ \widetilde{a}\ \text{bad}}}\frac{|\widetilde{a}|^{\epsilon}}{|\widetilde{a}|^{\frac14}} \ll \frac{|\mathcal{A}|^{\frac12}X^{\frac34}}{\log X}\sum_{\substack{|\widetilde{a}|\le X\\ \widetilde{a}\ \text{bad}}}|\widetilde{a}|^{-\frac16}, \tag{7}$$

upon choosing $\epsilon = 1/12$. At this stage we note that bad fundamental discriminants are rare by a standard zero density result (see for example [7, Theorem 20]): thus there are at most $\ll Y^{1/10}$ bad fundamental discriminants d with $Y \le |d| \le 2Y$. Therefore the sum over bad $\widetilde{a}$ in (7) converges, and we conclude that the quantity in (7) is $\ll |\mathcal{A}|^{\frac12}X^{\frac34}/\log X$. This completes the proof of Theorem 2.

4. Optimality of Theorem 2

In this section, we show the existence of a subset $\mathcal{A}$ of $[1,X]$ with $|\mathcal{A}| \asymp \sqrt{X}/\log\log X$, and such that a positive proportion of the primes below X may be written as $a+m^2$ with $a\in\mathcal{A}$ and $m\in\mathbb{Z}$. Since this is only an example to show the optimality of Theorem 2, we shall be content with sketching the proof.

Set $Z = \exp((\log X)^{\frac{1}{10}})$. Note that $\log\log Z \asymp \log\log X$. Let $\mathcal{D}$ be a set of about $\sqrt{Z}/\log\log X$ odd square-free numbers d with $Z \le d \le 2Z$ and such that $L(1,\chi_{-4d}) \asymp 1/\log\log X$. Then our set $\mathcal{A}$ will consist of all numbers of the form dk^2 with $d\in\mathcal{D}$ and $k \le \sqrt{X/2Z}$. By construction the set $\mathcal{A}$ has $\asymp \sqrt{X}/\log\log X$ elements.

Arguing using a classical zero-free region for class group L-functions, we may see that for any $d\in\mathcal{D}$ the number of primes up to $X/2$ of the form dk^2+b^2 with $b,k\in\mathbb{N}$ is

$$\gg \frac{\pi(X)}{h(-4d)} \asymp \frac{X}{\sqrt{Z}\log X}\log\log X,$$

upon using the class number formula. Thus if $r_{\mathcal{A}}(p)$ denotes the number of ways of writing p as $a+b^2$ with $a\in\mathcal{A}$ and $b\in\mathbb{N}$, it follows that

$$\sum_{p\le X/2} r_{\mathcal{A}}(p) \gg \frac{X}{\log X}.$$

By similar methods, we may show that for $d_1 \neq d_2 \in \mathcal{D}$, the number of primes up to $X/2$ that may be represented as $d_1k^2 + b^2$ and also as $d_2r^2 + s^2$ is at most

$$\ll \frac{\pi(X)}{h(-4d_1)h(-4d_2)} \asymp \frac{X}{Z \log X}(\log\log X)^2.$$

It follows that

$$\sum_{p \leq X/2} r_{\mathcal{A}}(p)^2 \ll \frac{X}{\log X}.$$

By Cauchy-Schwarz it follows that the number of $p \leq X/2$ with $r_{\mathcal{A}}(p) > 0$ is $\gg X/\log X$, as claimed.

In Theorem 2 we were interested in lower bounds for the size of a set $\mathcal{A} \subset \{1, \dots, X\}$ such that $\mathcal{A} + \mathcal{B} \supset \mathcal{C}$, where $\mathcal{B}, \mathcal{C} \subset \{1, \dots, X\}$ are given sets (in this case, $\mathcal{B}$ is the set of squares, and $\mathcal{C}$ is a set consisting of a positive proportion of the primes). One might say that $\mathcal{A}$ is an *additive complement of $\mathcal{B}$ relative to $\mathcal{C}$*. In the case $\mathcal{C} = \{1, \dots, X\}$ one recovers the usual notion of additive complement. To our knowledge the relative case has not been studied in any generality. A large number of questions suggest themselves.

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