

Phenomenology of Exotic Hadrons - Hybrid Mesons and Pentaquarks

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This thesis is concerned with the properties of two classes of exotic hadron: hybrid mesons, in which an excitation of the gluonic field causes distinction from the conventional mesons and can give rise to J^{PC} quantum numbers not available to a $q\bar{q}$ state; and pentaquarks, where the usual baryonic qqq structure is supplemented by a $q\bar{q}$ pair, potentially giving the state an exotic flavour structure.

In the hybrid meson sector, we work within the flux-tube model and extend upon an observation made by Isgur that the oscillations of the flux-tube can have dynamical effects upon the quarks living on the ends of the tube. We reverse the logic by allowing oscillations of the quarks caused by interactions with currents to excite modes in the flux-tube and hence hybrid mesons.

Electromagnetic, weak and pionic currents are all applied, allowing us to make predictions about the radiative and hadronic decays of hybrid mesons and their production rates in exclusive hadronic decays of the B -meson. Such predictions are of interest to past, present and future experiments at E852, the B -factories and Jefferson Lab.

In light of the possible observation of a pentaquark state, the Θ^+ , we investigate some phenomenological consequences of certain models proposed to describe this state and attempt to justify one such model within a quark model framework.

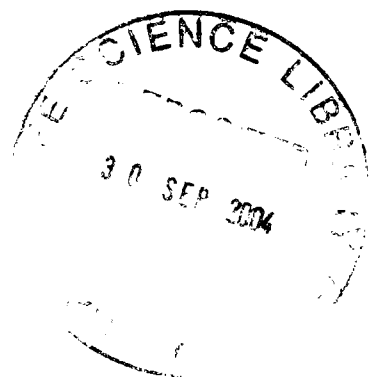
Jaffe and Wilczek's model for pentaquarks predicts in addition to the $\bar{10}$ with $J^P = \frac{1}{2}^+$, a $\bar{10}$ with $J^P = \frac{3}{2}^+$, the multiplets being initially degenerate. Within a dynamical model we are able to calculate the spin-orbit splitting between them and find it to not be large.

We also consider the effect of a near degenerate 8 on the states in a pentaquark $\bar{10}$, outlining a set of characteristic decay systematics that may be used in the search for non-flavour-exotic pentaquarks.

In a constituent quark model with colour-spin interactions it is found to be possible to accommodate a Jaffe-Wilczek-like Θ^+ with a rather low mass, while simultaneously describing the $\Delta - N$ splitting.

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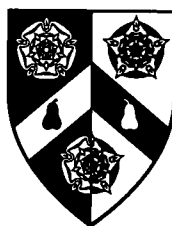
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Introduction

This thesis discusses phenomenological work performed by myself, mostly in collaboration with Prof. Frank Close, on the nature of certain exotic hadronic states.

Part I considers the production and decay of hybrid mesons in the flux-tube model. After discussing the model itself, we outline some of the states which are candidates for hybrid mesons. Building on an insight of Isgur's, we develop a framework in which we are able to make calculations of the radiative and pionic decay rates of light hybrids as well as their production rates in heavy-flavour decays. In the case of the radiative and weak processes, these are the first ever estimates for light hybrids and should be of interest to experimentalists at the upgraded Jefferson Lab GlueX facility and the B -factories, BaBar and Belle.

We propose an alternative strong decay mechanism for hybrids when the decay involves a pion, which allows large rates in the channel $\rho\pi$ which was previously expected to be suppressed. This may help to explain the observation in $\rho\pi$ of an exotic 1^{-+} meson, $\pi_1(1600)$, by E852 at Brookhaven. Extending the model via Vector Meson Dominance, we are able to make radiative transition predictions in the case that the hadronic character of the photon is dominant. With the addition of established theoretical technology it should be possible to use these numbers to make estimates of the photoproduction rates of light hybrids.

•

Part II considers the phenomenological avenues opened up by the possible observation of a narrow, strangeness +1 baryon at 1540 MeV, dubbed the Θ^+ . This state can be interpreted as a pentaquark ($q^4\bar{q}$) and a range of models have been developed to explain its existence. We review a selection of these models and discuss their relative merits.

Correlated quark model approaches to pentaquarks generically have an internal P -wave which couples to a quark spin giving rise to degenerate $J^P = (\frac{1}{2}, \frac{3}{2})^+$ states. We calculate the spin-orbit splitting between these states in a simple quark model, utilising the same interactions

used to model the meson and conventional baryon sectors. With the quark correlation proposed by Jaffe and Wilczek we find a small splitting and suggest the existence of a *nearby* isoscalar $J^P = \frac{3}{2}^+$ partner to the Θ^+ . There is some suggestion of such a state in preliminary data from the CLAS collaboration.

The flavour correlations required to produce an isoscalar, $S = +1$ baryon lead us to expect a $\overline{10}$ of $SU(3)_F$ partnered by an 8 . We consider the decay systematics of the exotic and non-exotic members of the pentaquark $\overline{10} \oplus 8$ under an assumption about the nature of the decay mechanism. We find certain selection rules that may help distinguish between pentaquark states and the more conventional three-quark states and discuss certain experimental state assignments proposed in the literature.

Finally we entertain the possibility of generating the light scalar diquarks required by the Jaffe-Wilczek model for pentaquarks within a constituent quark model with colour-spin interactions. We find that with a sufficiently strong contact interaction we can simultaneously describe the $\Delta - N$ splitting and have a Jaffe-Wilczek Θ of about the right mass. We discuss the effect of a realistic smearing of the contact interaction and consider possible extensions of the work.

Much of the material presented in this thesis has appeared in papers in refereed journals and in preprints on the arXiv:

- **Phys.Rev.Lett.91:142001,2003**, *Electroweak Production of Hybrid Mesons in a Flux Tube Simulation of Lattice QCD* [1]; F.E. Close, J.J. Dudek
and
Phys.Rev.D69:034010,2004, *Hybrid Meson Production by Electromagnetic and Weak Interactions in a Flux Tube Simulation of Lattice QCD* [2]; F.E. Close, J.J. Dudek
contain elements of Chapters 3 & 4;
- **hep-ph/0308099**, *The 'Forbidden' Decays of Hybrid Mesons to $\pi\rho$ Can Be Large* [3];
F.E. Close, J.J. Dudek
contains elements of Chapter 5;
- **Phys.Lett.B583:278-284,2004**, *The $J = 3/2$ Theta* Partner to the Theta(1540) Baryon* [4]; J.J. Dudek, F.E. Close
contains elements of Chapter 7;

- **Phys.Lett.B586:75-82,2004**, *Pentaquark Symmetries, Selection Rules and Another Potentially Narrow State* [5]; F.E. Close, J.J. Dudek
contains elements of Chapter 8;
- **hep-ph/0403235**, *A Model Realisation of the Jaffe-Wilczek Correlation for Pentaquarks* [6]; J.J. Dudek
contains elements of Chapter 9.

Part I

Hybrid Mesons

Chapter 1

Introduction to Hybrids

Experimental physicists have obtained a compendious spectrum of meson states [7] (hereafter labeled “conventional”), spanning a range of masses from the π to the excited Υ , with J^{PC} quantum numbers in the set $\{0^{-+}, 0^{++}, 1^{--}, 1^{+-}, 1^{++}, 2^{--}, 2^{-+}, 2^{++} \dots\}$. This set is explicable in terms of states containing a quark and an anti-quark with relative integral orbital angular momentum, moving in a static potential (transforming as 0^{++}) that can be considered to be due to the ground-state gluonic field.

There is nothing in QCD which prevents a coherent excitation of the gluonic field (which can certainly have $J^{PC} \neq 0^{++}$), in fact in a strongly coupled non-abelian gauge theory we would expect such states to exist. A (local) excitation of the gluonic field without valence quarks is known as a glueball and lattice studies of QCD show a spectrum of massive glueballs with various J^{PC} .

In the presence of valence quarks a gluonic excitation is known as a hybrid. For certain J_g^{PC} in combination with $J_{q\bar{q}}^{PC}$ we can access the quantum numbers absent in the set listed earlier, namely $0^{--}, 0^{+-}, 1^{-+}, 2^{+-} \dots$, we call such states “exotic hybrid mesons”.

The first half of this thesis will focus on the phenomenology of both exotic and non-exotic hybrid meson states. In the remainder of this section I will outline how hybrids are considered in lattice QCD and in the phenomenological bag and flux-tube models. I will then briefly summarise the experimental status of hybrid mesons before considering in more detail the flux-tube model and its predictions.

The bulk of Part I is devoted to an exposition of my work on hybrid meson production and decay in electromagnetic, weak and hadronic processes using the flux-tube model.

date	ref.	mass/MeV
1996	[11]	1880(200)
1996	[12]	1970(90)(300)
1998	[13]	1900(200)
1998	[9]	2110(100)
2002	[14]	2013(26)(71)
2002	[15]	2033(70)
2002	[15]	1854(65)

Table 1.1: Mass estimates for the light quark 1^{-+} hybrid state from lattice QCD studies. Various lattice methods were used. Approximate errors are shown in brackets. (Compilation taken from [16]). The two values quoted from [15] correspond to two different methods for setting the lattice scale.

QCD at the energy scale of hadrons is a non-perturbative gauge field theory for which we have limited calculational techniques. One successful approximate method is to put the theory on a discrete space-time lattice, allowing the path-integral and matrix elements to be computed numerically on a computer, with the true QCD result emerging as the lattice spacing is decreased to zero. The application of this technique to hybrid mesons has principally focused on the mass spectrum of such states, although recently decays [8], wavefunctions [9] and the effective interquark potential [10] have been investigated. For example, in the light quark sector, lattice studies have predicted the mass of the exotic hybrid meson with $J^{PC} = 1^{-+}$ some results are tabulated in Table(1.1).

Before such detailed lattice studies became possible, phenomenological models of hybrids were developed to predict masses and other properties. The bag model was proposed as a way of describing confined quarks of very low mass and worked by considering relativistic solutions to the free Dirac equation with a boundary condition setting the quark wavefunction to zero outside the “bag”. Hybrids are considered by including a gluon in the bag and the possible state quantum numbers set by imposing boundary conditions at the surface of the bag.

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The flux-tube model of hadrons (which I will discuss in much more detail in Chapter 2) exploits an observation made in the strong-coupling limit of lattice QCD, that the gluonic field between separated quark sources forms itself into a tube of flux along the line separating the quarks rather than spreading more evenly over space (as in the electromagnetic case).

The model then considers the tube, discretized into N beads, along with the quarks on the ends as the degrees-of-freedom of the problem. Simply put, the conventional mesons correspond to the string-like tube in its ground state and hybrids to the “plucked” string, i.e. the string

in some higher oscillatory mode.

A rather complete phenomenology has been built out of this model, including predictions for state masses, hadronic decays, charge radii and with the work presented in this thesis, electromagnetic and weak decays [17].

We will return to this model after a brief detour to consider whether any hybrid meson states have been observed in experiment.

1.1 Hybrid Meson Candidates

As motivation for the modelling to follow I will briefly summarise the current experimental status of candidates for hybrid mesons.

Evidence for hybrids has arisen in experiments scattering pions off protons or nuclei where one observes an interesting multi-meson end state along with baryonic debris. The mesons are studied using partial wave analysis (PWA) in which one searches for resonances in separated partial waves using both amplitude and phase information.

The experiment which has contributed most to this study is undoubtedly E852 at Brookhaven who have analysed $\pi^- p \rightarrow X p$ for $X =$

$$\begin{aligned} &\pi^+ \pi^- \pi^- \text{ [18] } (\rho^0 \pi^-, f_2(1270) \pi^-), \\ &\pi^+ \pi^- \pi^- \pi^0 \pi^0 \text{ [19] } (b_1 \pi, \omega \rho), \\ &\eta \pi^+ \pi^- \pi^- \text{ [20] } (a_1 \eta, a_2 \eta, \eta' \pi, f_1 \pi \text{ and } \eta(1295) \pi), \\ &\eta \pi^- \text{ [21]}, \\ &\eta' \pi^- \text{ [22]}. \end{aligned}$$

It has been suggested that in the future hybrids should be made in photon-meson interactions, either using real mesons and virtual photons in the Primakoff effect [23] or using real photons and virtual mesons in photoproduction [24]. The importance of having theoretical estimates for hybrid-meson-photon couplings motivates the work in Chapter 3.

For nomenclature we shall adopt the PDG [7] notation,^{*} where the subscript J denotes the total angular momentum of the meson, and we append the subscript H to denote hybrid (thus for light flavours we have the notation in Table 1.2, with obvious generalisation for flavoured states). This is done for two reasons

- (i) to enable the trivial distinction between hybrid and conventional states to be immediately apparent and reduce confusion in the text;
- (ii) as a reminder that for a conventional and hybrid meson with the same overall J^{PC} , their internal $q\bar{q}$ spin states are inverted. For example, π_H has $S_{q\bar{q}} = 1$ while the conventional

π has $S_{qq} = 0$.

J_H^{PC}	S_{qq}	$I = 1$	$I = 0$
1^{++}	0	a_{1H}	f_{1H}
1^{--}	0	ρ_H	ω_H
$(0, 1, 2)^{+-}$	1	b_{JH}	h_{JH}
$(0, 1, 2)^{-+}$	1	π_{JH}	η_{JH}

Table 1.2: Naming convention for light quark hybrid mesons

Candidates for exotic hybrid mesons with quantum numbers 1^{-+} and 2^{+-} include:

- **$\pi_1(1400)$** , the main evidence for which comes from the $\eta\pi^-$ analysis of E852 [21] and Crystal Barrel [25]. No such resonance was seen in $f_1\pi^-$ [20], $b_1\pi^-$ [26, 27], $\eta'\pi^-$ [22] or $\rho\pi^-$ [18]. A study of $\eta\pi^0$ data from the E852 experiment [28] considered this “state” in some detail. In their PWA they indeed observed an enhancement around 1400 MeV with a large width and some phase motion. They found that the Breit-Wigner fit parameters varied considerably when they took sets of data corresponding to different ranges of $|t|$. Their explanation was that the enhancement was not a resonant state but was due to a combination of soft $\eta\pi$ rescattering and some leakage from the dominant $a_2(1320)$ wave due to incomplete modelling of the angular acceptance of the detector. A phenomenological study by the Indiana group [29] investigated a possible mechanism for the rescattering and found they could quantitatively explain the enhancement if there was also some leakage.
- **$\pi_1(1600)$** , which was seen in $\eta'\pi^-$ [22] and $\rho\pi^-$ [18] at E852. The state width observed in $\rho\pi$ is somewhat narrower than that in $\eta'\pi^-$ but the errors are large. An enhancement around this mass is seen in $b_1\pi$ at VES [26, 27] and at E852 [19, 30], and very recently E852 have reported evidence for this state in $f_1\pi^-$ [20] with a slightly higher mass than in previous observations. Notably this state has not been seen in $\eta\pi$. The soft rescattering mechanism and leakage which “explains away” the $\pi_1(1400)$ does not affect this state in $\eta'\pi$ and it remains a viable hybrid candidate.
- **$\pi_1(2000)$** is the most recently observed 1^{-+} peak with appearances in the E852 analysis of $f_1\pi^-$ [20] and $b_1\pi$ [19, 30]. An earlier Crystal Barrel study [31] had some inconclusive enhancement near 2 GeV in $f_1\pi^-$.
- **$b_2(1650)$** , where this is a broad enhancement in the 2^{+-} wave in $\omega\pi^-$ has been observed by E852 [32].

With more caution owing to the possibility of conventional explanations I list a selection of non-exotic hybrid candidates:

- **$a_1(1640 \rightarrow 1700)$** , an axial enhancement in $f_1\pi$, $f_2\pi$ and $\sigma\pi$, which can probably be assigned the status of radial excitation of the $a_1(1260)$.
- **$a_1(2096)$** was observed in the $f_1\pi$ and $a_1\eta$ channels by E852 [20], with a very large (~ 500 MeV) width. Evidence for hybrid character comes from the ratio of branching ratios $\mathcal{B}(a_1(2096) \rightarrow f_1\pi)/\mathcal{B}(a_1(2096) \rightarrow a_1\eta) = 3.2 \pm 0.6$ where the flux-tube breaking model of [33] predicts 3 for an axial hybrid of this mass. There is no sign of this state in the E852 $\rho\pi^-$ channel [18] where the axial wave contains only the $a_1(1260)$. This non-observation in $\rho\pi$ will be considered in light of decay model predictions in Chapter 5.
- **$\pi_2(2003)$** appears in the E852 analysis of end states $f_1\pi$ and $a_2\eta$ with ~ 300 MeV width. The model of [33] predicts 23 for the ratio $\mathcal{B}(\pi_2(2003) \rightarrow a_2\eta)/\mathcal{B}(\pi_2(2003) \rightarrow f_1\pi)$ which is measured to be 23 ± 8 suggesting a hybrid interpretation. An enhancement in this wave of similar width at 1900 MeV was observed in $a_2\eta$ by Crystal Barrel [34]. This state may have been seen in $\rho\pi$ by E852 [35] but they did not parameterise the enhancement and report only a mass > 2 GeV. The $\omega\rho$ PWA of E852 [19] obtains a better fit with two π_2 states above the conventional 1670, a narrow state at 1880 MeV and a broader one at 1970 MeV. If we interpret the heavier state as being the same as that observed around 2 GeV in other experiments, this observation in a channel with two essentially identical particles casts doubt on the possibility of it being a hybrid, at least within the flux-tube breaking model.
- **$\pi(1750/1800)$** which VES claims are two different states [26]. The lighter is the $3S$ excitation of the pion while the heavier is a pseudoscalar hybrid. Their data is inconclusive and in the E852 study of $\rho\pi$ [18], the 0^{-+} wave is too complex to be simply interpreted.

Chapter 2

The Flux-Tube Model

Description of Hybrid Mesons

The flux-tube is a relativistic object with an infinite number of degrees of freedom. A standard approximation [36–39] has been to fix the longitudinal separation of the $Q\bar{Q} \equiv r$ and to solve the flux-tube dynamics in the limit of a thin relativistic string with purely transverse degrees of freedom. The resulting energies $E(r)$ are then used as adiabatic effective potentials on which the meson spectroscopies are built. Ref. [40] studied the effect of relaxing these strict approximations and found that the spectrum of the conventional and lowest hybrids is robust. The flux-tube ground state is found to give an adiabatic quark potential which is linearly rising. When supplemented with the expected short-distance one-gluon-exchange potential, we get a good description of gross structure of the conventional meson spectrum.

In the simplest formulation of the model, the first excited hybrid is a state with a single phonon of excitation in the string-like flux-tube. This phonon has wavelength $2r$ and hence excitation energy $\frac{\pi}{r}$, which appears in the quark adiabatic potential. An infinite tower of hybrid states exists in principle with various numbers of phonons excited in various modes; our interest, which is phenomenological, is only in the lightest states. In the lowest mode there are in fact two possible flux-tube states corresponding to the two transverse polarisations of the excited phonon. The two polarisations can be superposed to give eigenstates of parity and charge conjugation, such that the J^{PC} of the flux-tube in this excited state is $1^{\pm\mp}$. With the quark and anti-quark in their spatial ground state we can have the following hybrid meson J^{PC} :

Flavour	$I = 0, 1$	$u\bar{s}$	$s\bar{s}$	$c\bar{c}$	$b\bar{b}$
Mass/GeV	1.9	2.0	2.1	4.3	10.8

Table 2.1: Masses, including estimate of non-adiabatic effects, of the lightest hybrid mesons with various flavours. Each flavour has, at this level of approximation, 8 degenerate states of $J^{PC} = \{0^{\mp\pm}, 1^{\pm\pm}, 1^{\pm\mp}, 2^{\pm\mp}\}$, except the kaonic states where C is not a good quantum number.

$S_{q\bar{q}} \setminus J_{\text{fl}}^{PC}$	1^{--}	1^{+-}
0	1^{++}	1^{--}
1	$(0, 1, 2)^{+-}$	$(0, 1, 2)^{-+}$

We see explicitly that there are states with exotic J^{PC} in this model (labelled in bold).

Furthermore, note that the non-exotic hybrid states have inverted quark spin projections with respect to the conventional mesons of the same J^{PC} , for example the conventional axial meson is a $^3P_1$ state with quark spin 1 whereas the hybrid axial has quark spin 0. This will be seen later to produce systematic differences in decays of these states.

In the original flux-tube model paper, Isgur and Paton [36] computed the masses of the lightest hybrids; we quote their estimates in Table 2.1. We see that the light quark hybrid in this model has a mass compatible with the lattice QCD predictions listed in Table 1.1.

2.1 Hadronic decays in the flux-tube model

The flux-tube model has also been extended from this original formulation to consider hadronic decays [37,38,41,42]. The proposal [41] made was that a meson (conventional or hybrid) decays to two lighter mesons by breaking the flux-tube and creating a quark and an anti-quark on the new tube ends. This was shown to be a realization of the phenomenologically successful $^3P_0$ -model for conventional decays and made predictions for the decays of hybrid mesons to conventional mesons. Its most famous prediction was the suppression of light hybrid decays to two identical S -wave mesons. Simply put, in this model there is nowhere for the angular momentum carried by the flux-tube phonon to go in the decay. This observation has lead to the rule, widely used by the hadron community, that the dominant decays of hybrid mesons are to an S -wave meson and a P -wave meson. An alternative flux-tube breaking model [33] which has different quantum numbers for the produced quark pair predicts a broader selection rule that decays to any pair of identical mesons is forbidden. Thus channels like $\rho\pi$ have been considered poor for hybrid-hunting. In fact, since the pion and the rho can be modeled with different spatial wavefunctions the selection rule does not hold exactly and some decay into this channel can occur. We will show in Chapter 5, in an alternative model of decays involving

pions, that this selection rule can be badly broken.

2.2 Dynamical effects of flux-tube degrees-of-freedom on conventional mesons

In 1999, Nathan Isgur [39] demonstrated that the flux-tube could have dynamical effects on conventional hadrons beyond simply providing an adiabatic potential. His observation was that the zero-point oscillation of the tube in its ground-state would give rise to quark motion transverse to the nominal interquark direction. This would be a hitherto overlooked contribution to, for example, the charge radius of a meson. Such an additional effect had for some time been required to fix the quark model under-prediction of charge radii with respect to the experimentally measured values.

His result was that in the flux-tube model, the charge radius of a meson ($Q\bar{d}$) is given by

$$r_Q^2 = \left[\left(\frac{m_d}{m_Q + m_d} \right)^2 + \frac{2b}{\pi^3 m_Q^2} \zeta(3) \right] \langle r^2 \rangle, \quad (2.1)$$

where the first term is the usual quark model result. The additional (positive) term, proportional to the string tension b , can have a large effect of order 50% in light-quark mesons. The work presented in the remainder of Part I is an extension of this observation and as such we will present here, in some detail, Isgur's formalism.

Consider the flux-tube to be discretized into N massless beads with longitudinal separation, a , where eventually we will take $N \rightarrow \infty$ and $a \rightarrow 0$, keeping $r = (N+1)a$ constant. The state of the tube can then be expressed in terms of a complete set of transverse eigenstates

$$|\vec{y}\rangle = |\vec{y}_1 \dots \vec{y}_n \dots \vec{y}_N\rangle.$$

A more useful basis is the set of Fourier mode coefficients,

$$\vec{a}_p = \sqrt{\frac{2}{N+1}} \sum_{n=0}^N \vec{y}_n \sin \frac{\pi p}{N+1} n.$$

The oscillations are in the two dimensional space transverse to the nominal $Q\bar{d}$ axis. Thus there are two polarisations for each Fourier mode $\vec{a}_p \equiv (a_p^1, a_p^2)$ where 1, 2 refer to the two (body-fixed) orthogonal coordinate directions, $\hat{e}_1, \hat{e}_2$. In the small oscillation approximation

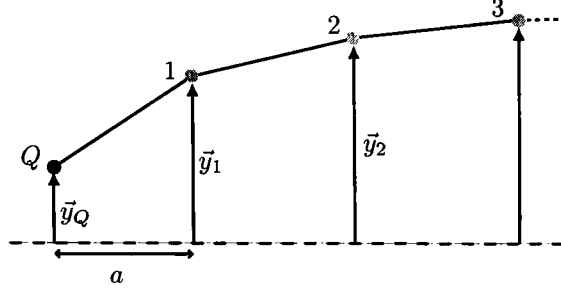


Figure 2.1: Quark - flux-tube system

the system becomes harmonic in $\vec{y}(\vec{a})$, with mode frequency

$$\omega_p = \frac{2}{a} \sin \frac{\pi p}{2(N+1)} \xrightarrow[N \rightarrow \infty]{a \rightarrow 0} \frac{p\pi}{r}.$$

The flux-tube state is defined by a set of phonon-mode occupation numbers. The wavefunctions for the phonon mode variables a_p^1, a_p^2 are the eigenstates of the two-dimensional harmonic oscillator, for example the ground and first excited state wavefunctions for a_p^1 are

$$\chi_0(a_p^1) = (\beta_p^2/\pi)^{1/4} \exp[-\beta_p^2(a_p^1)^2/2] \quad (2.2)$$

$$\chi_1(a_p^1) = \sqrt{2}\beta_p a_p^1 \cdot \chi_0(a_p^1), \quad (2.3)$$

with $\beta_p^2 \xrightarrow[N \rightarrow \infty]{} \frac{b\pi p}{N+1}$.

These eigenfrequencies and eigenstates have been obtained in the approximation that the quarks remain stationary, or at least that they respond over a much longer timescale than that associated with the flux-tube oscillations this is the adiabatic approximation. Isgur considers the validity of this approximation in [39], and investigations into corrections have been performed in [40, 43]. The quarks will only experience the zero-point oscillations of the flux-tube if they have non-infinite mass, hence we must go beyond the pure adiabatic approximation and consider the response of the quarks to the state of the tube. I will go into more detail at this point than is reported in [39]¹.

Consider the force on a quark, Q , due to the string motion in a general state (see Figure 2.1),

$$\vec{F}_Q = -\vec{\nabla}_Q V = -\vec{\nabla}_Q \left(\frac{b}{2a} (\vec{y}_1 - \vec{y}_Q)^2 \right),$$

where the potential is in the small oscillation approximation, which implies an equation of

¹The following derivation is due to Jack Paton

motion for Q ,

$$\ddot{\vec{y}}_Q + \frac{b}{am_Q} \vec{y}_Q = \frac{b}{am_Q} \vec{y}_1.$$

The first departure from the adiabatic approximation will come from the driving force on the RHS, so we neglect the second term on the LHS (which would impact at a higher order in br/m_Q). Then exploiting the harmonic nature of $\vec{a}_p$ ($\ddot{\vec{a}}_p = -\omega_p^2 \vec{a}_p$) we can write

$$\vec{y}_1 = -\sqrt{\frac{2}{N+1}} \sum_p \frac{\ddot{\vec{a}}_p}{\omega_p^2} \sin \frac{p\pi}{N+1} \xrightarrow{N \rightarrow \infty} -\sqrt{\frac{2}{N+1}} \frac{ar}{\pi} \sum_p \frac{\ddot{\vec{a}}_p}{p}.$$

Solving the equation of motion for the quark's transverse departure in terms of the string mode variables we find

$$\vec{y}_Q = -\sqrt{\frac{2}{N+1}} \frac{br}{\pi m_Q} \sum_p \frac{1}{p} \vec{a}_p,$$

which is Isgur's equation (15); Isgur's equation (16) can be derived analogously.

It is the matrix element of this extra term in the quark position in the ground-state of the flux-tube (which is the state of conventional mesons) that gives rise to the new contribution to the meson charge radius. We can draw a more physical conclusion from this - the charge radius is related to the elastic form-factor by $r_Q^2 = -6 \frac{d}{dQ^2} F_{\text{el}}(Q^2)|_{Q^2=0}$, thus an increase in the charge radius is a decrease in the elastic form-factor at a given Q^2 . Loss of elastic probability must be compensated by gain in inelastic channels, and the channel in this case is photo-excitation of a hybrid meson. In the next chapter we will compute photo-couplings to hybrid mesons and see that this holds true.

Chapter 3

Phototransitions of Conventional Mesons and Hybrids

We ended the last section noting that the departure of the elastic form-factor down from 1 indicates loss into inelastic channels. These channels are the excitations of conventional and, with the inclusion of flux-tube degrees-of-freedom, hybrid mesons. The need for calculations of the electromagnetic excitation and decay of hybrid mesons was spelt out in Chapter 1.1. Isgur's formalism as introduced in the previous section can now be applied to such calculations.

The quark positions in a meson $Q\bar{d}$ at lowest non-trivial order in br/m are

$$\vec{r}_{Q,d} = \vec{R} \mp \vec{r} \frac{\mu}{m_{Q,d}} + \frac{br}{\pi m_{Q,d}} \sqrt{\frac{2}{N+1}} \sum_{p=1}^{\infty} \left\{ \frac{(-1)^p}{(-1)^p} \right\} \frac{1}{p} \vec{a}_p. \quad (3.1)$$

where $\mu \equiv m_Q m_d / (m_Q + m_d)$ and where the upper choice corresponds to Q and the lower to $\bar{d}$. Since we will be interested only in conventional states and the $p = 1$ -mode hybrids which are lightest, we need only consider the Fourier mode vector $\vec{a}_1$. All the $\vec{a}_{p \neq 1}$ will be evaluated between Fourier mode ground-states and will only ever give their zero-point contribution, which has already been considered by Isgur in his calculation of the charge radius, or equivalently, the modified form-factor. The quark momenta and orbital angular momenta using the expressions above are derived in Appendix A.

The wavefunctions for mesons must include the state of the flux-tube; for conventional mesons, where the flux-tube is in its ground state, we write

$$\mathcal{C} = \psi_{nlm}^{(0)}(\vec{r}) \chi_0(a^1) \chi_0(a^2),$$

where n, l, m are the usual two-body quantum numbers and the subscript zero indicates the ground state. If either of the transverse modes is excited, one has a hybrid meson. The particular combinations $\frac{1}{\sqrt{2}}[a^1 \pm ia^2] \equiv \frac{1}{\sqrt{2}}a^\pm$ give normalised circularly polarised phonon modes for the flux-tube, which have angular momentum ± 1 about the longitudinal ($Q\bar{d}$) axis. The corresponding wavefunction for such a hybrid may be summarised by

$$\mathcal{H} = \psi_{nlm}^{(\pm)}(\vec{r}) \frac{1}{\sqrt{2}} [\chi_1(a^1)\chi_0(a^2) \pm i\chi_0(a^1)\chi_1(a^2)]$$

or¹

$$\mathcal{H} = \psi_{nlm}^{(\pm)}(\vec{r}) \beta_1 (a^1 \pm ia^2) \chi_0(a^1)\chi_0(a^2).$$

In the above we show only the $\vec{a}$ dependence (where $\vec{a}$ is short for $\vec{a}_1$) as all higher Fourier modes are in their ground-states.

3.1 Conventional $E1$ transition

Consider first, for orientation and later comparison, a conventional electromagnetic $E1$ transition in the flux-tube model which we shall see is equivalent at lowest order to the non-relativistic quark model. The transition operator for a positive helicity photon ($\vec{\epsilon}_+ = -\frac{1}{\sqrt{2}}(1, i, 0)$),

$$\mathcal{O}_{E1} = -i|\vec{q}| \sum_{q=Q, \vec{d}} \vec{\epsilon}_+ \cdot (e_q \vec{r}_q)$$

when acting on a meson $Q\bar{d}$, using equation(3.1) and retaining only $p = 1$ in the sum gives²,

$$\mathcal{O}_{E1} = i|\vec{q}| \left\{ \left(\frac{e_Q}{m_Q} - \frac{e_{\vec{d}}}{m_d} \right) \mu \vec{\epsilon}_+ \cdot \vec{r} + \left(\frac{e_Q}{m_Q} + \frac{e_{\vec{d}}}{m_d} \right) \sqrt{\frac{2b}{\pi^3}} \beta_1 r \vec{\epsilon}_+ \cdot \vec{a} \right\}. \quad (3.2)$$

If l, m denote the orbital angular momentum of the $Q\bar{d}$ system, and its z -projection (on fixed space axes) respectively, then the structure of the matrix element $\mathcal{M}$ becomes

$$\begin{aligned} \mathcal{M} \equiv \langle \mathcal{C}(l', m') | \mathcal{O}_{E1} | \mathcal{C}(l, m) \rangle &= i|\vec{q}| \left(\frac{e_Q}{m_Q} - \frac{e_{\vec{d}}}{m_d} \right) \mu \\ &\times \int d^3\vec{r} \int d^2\vec{a} \psi_{n'l'm'}^{(0)*}(\vec{r}) \chi_0^*(a^1) \chi_0^*(a^2) \vec{\epsilon}_+ \cdot \vec{r} \psi_{nlm}^{(0)}(\vec{r}) \chi_0(a^1) \chi_0(a^2) \end{aligned}$$

¹ $\beta_1 = \sqrt{\frac{b\pi}{N+1}}$

²we justify the use of the “dipole” form for the $E1$ operator in Appendix C.

The integration over $d^2\vec{a} \rightarrow 1$ and the standard integral over $d^3\vec{r}$ gives a transition from l to $l' = l \pm 1$ caused by the presence of $\vec{r}$.

Separating into radial and angular parts, $\psi_{nlm}(\vec{r}) \equiv R_{nl}(r)Y_l^m(\Omega)$ and using

$$\vec{\epsilon}_+ \cdot \vec{r} \equiv \sqrt{\frac{4\pi}{3}} r Y_1^{+1}(\Omega).$$

$\mathcal{M}$ becomes

$$i|\vec{q}|\mu \left(\frac{e_Q}{m_Q} - \frac{e_{\bar{d}}}{m_d} \right) \sqrt{\frac{4\pi}{3}} \langle r \rangle \int d\Omega Y_{l'}^{m'*} Y_1^{+1} Y_l^m$$

where $\langle r \rangle \equiv \int r^2 dr R_{n'l'}^*(r) r R_{nl}(r)$.

This general formula becomes more transparent when applied to the case $l = 0, l' = 1$ for which

$$\mathcal{M} = i|\vec{q}|\mu \left(\frac{e_Q}{m_Q} - \frac{e_{\bar{d}}}{m_d} \right) \langle r \rangle \frac{1}{\sqrt{3}} \delta_{m',+1}$$

We are interested in the specific case:

$$\mathcal{M}(\gamma\pi \leftrightarrow b_1) = i \left(\frac{e_u - e_{\bar{d}}}{m_u} \right) \langle R_b | r | R_\pi \rangle |\vec{q}b| \frac{m_u}{2\sqrt{3}} \delta_{m',+1}. \quad (3.3)$$

In general we can write the radiative width as

$$\Gamma(A \rightarrow B\gamma) = 4 \frac{E_B}{m_A} |\vec{q}| \frac{1}{2J_A + 1} \sum_{m_J^A} |\mathcal{M}(m_J^A, m_J^B = m_J^A + 1)|^2 \quad (3.4)$$

where the sum is over all possible helicities of the initial meson, and the matrix element is understood to be for a positive helicity photon.

This corresponds to the familiar E1 transition formalism of atomic and nuclear physics as traditionally applied to $Q\bar{Q}$ systems. Notice that $\left(\frac{e_Q}{m_Q} - \frac{e_{\bar{d}}}{m_d} \right)$ ensures charge-conjugation conservation; for charge-neutral systems the $Q\bar{Q}$ charges cancel but they are vectorially on opposite sides of the c.m.(which might be called a “longitudinal” electric dipole moment). Hence a non-vanishing E1 amplitude occurs between neutral systems (e.g. $\chi \rightarrow \gamma\psi$).

3.2 Hybrid E1 transition

Transitions to hybrids in the first excited state of the flux-tube arise from the $\vec{a}$ component of the $Q, \bar{d}$ position operators. We consider the general matrix element for transitions between a conventional meson and a first excited mode hybrid with tube oscillation phonon polarisation

$$\begin{array}{c}
m' \backslash m \\
\begin{array}{ccc}
+1 & 0 & -1 \\
+1 & \left(\begin{array}{ccc} \frac{1}{2}(1 + \cos \theta) & -\frac{1}{\sqrt{2}}e^{i\phi} \sin \theta & \frac{1}{2}e^{2i\phi}(1 - \cos \theta) \\ \frac{1}{\sqrt{2}}e^{-i\phi} \sin \theta & \cos \theta & -\frac{1}{\sqrt{2}}e^{i\phi} \sin \theta \\ \frac{1}{2}e^{-2i\phi}(1 - \cos \theta) & \frac{1}{\sqrt{2}}e^{-i\phi} \sin \theta & \frac{1}{2}(1 + \cos \theta) \end{array} \right) \\
0 \\
-1
\end{array}
\end{array}$$

Table 3.1: $\mathcal{D}_{m'm}^{(1)}(\phi, \theta, -\phi)$

± 1 ;

$$\mathcal{M} \equiv \langle \mathcal{H}(\pm, m') | \mathcal{O} | \mathcal{C}(l, m) \rangle = \int d^3\vec{r} \int d^2\vec{a} \mathcal{H}^* \mathcal{O} \mathcal{C} \quad (3.5)$$

where (i) $\mathcal{O}$ = the transition operator, which in this example is

$$\mathcal{O}_{E1} = i|\vec{q}| \left(\frac{e_Q}{m_Q} + \frac{e_d}{m_d} \right) \sqrt{\frac{2b}{\pi^3}} \beta_1 r \vec{\epsilon}_+ \cdot \vec{a} \quad (3.6)$$

(ii) $\mathcal{C}$ = the conventional meson wavefunction,

$$\mathcal{C} = \psi_{nlm}^{(0)}(\vec{r}) \chi_0(a^1) \chi_0(a^2)$$

(iii) $\mathcal{H}^*$ = the hybrid wavefunction (complex conjugate),

$$\mathcal{H}^* = \psi_{nlm}^{(\pm)*}(\vec{r}) \beta_1 (a_1 \mp ia_2) \chi_0^*(a^1) \chi_0^*(a^2)$$

(where the flux tube is excited into a state with polarisation ± 1 along its axis, indicated by the superscript on ψ).

To be specific, we consider the transition between the unexcited tube and a tube that has polarisation $\pm$ along its axis (the “body axis”), that axis in turn being oriented at some angles θ, ϕ in the laboratory (the “fixed axes”). For such a state the hybrid spatial wavefunction can be written [36]

$$\psi_{nlm}^{(\pm)} = R_{nl}(r) \sqrt{\frac{2l+1}{4\pi}} \mathcal{D}_{m\pm}^{(l)}(\phi, \theta, \mp\phi).$$

The rotation matrix $\mathcal{D}$ for $l = 1$ is shown explicitly in Table(3.1), and it is the elements of this matrix that relate the body-fixed unit vectors, $(\hat{e}_1, \hat{e}_2)$ for a tube oriented at angles (θ, ϕ) to the fixed-axis unit vectors $(\hat{x}, \hat{y}, \hat{z})$. Explicitly,

$$\hat{e}_1 \pm i\hat{e}_2 = (\hat{x} \pm i\hat{y})\mathcal{D}_{\mp\mp}^{(1)*} - (\hat{x} \mp i\hat{y})\mathcal{D}_{\mp\pm}^{(1)*} \pm \hat{z}\sqrt{2}\mathcal{D}_{0,\mp}^{(1)*}.$$

I will now demonstrate the calculation of (3.5) for a general operator $\mathcal{O}$ linear in $\vec{a}$, $\mathcal{O} \equiv \vec{a} \cdot \vec{x}_i$.

In this specific example we shall choose $\vec{x}_- \equiv \hat{x} - i\hat{y}$, where these are unit vectors in the fixed axes, and calculate the matrix element,

$$\langle \chi_1^\pm | \vec{a} \cdot \vec{x}_- | \chi_0 \rangle \equiv -\langle \chi_1^\pm | (a^1 + ia^2) \mathcal{D}_{-+}^{(1)*} - (a^1 - ia^2) \mathcal{D}_{--}^{(1)*} | \chi_0 \rangle.$$

Rewrite $a^1 \pm ia^2 \equiv a^\pm$ and the integral becomes

$$-\beta_1 \int d^2 \vec{a} \, a^\mp (a^+ \mathcal{D}_{-+}^{(1)*} - a^- \mathcal{D}_{--}^{(1)*}) | \chi_0(a^1) |^2 | \chi_0(a^2) |^2.$$

Now use $a^+ a^- = (a^1)^2 + (a^2)^2$ and note that $a^\pm a^\pm$ vanishes under integration. This brings us to

$$-\beta_1 (\delta^+ \mathcal{D}_{-+}^{(1)*} - \delta^- \mathcal{D}_{--}^{(1)*}) \int d^2 \vec{a} \, ((a^1)^2 + (a^2)^2) | \chi_0(a^1) |^2 | \chi_0(a^2) |^2$$

Then using the integral

$$\int d^2 \vec{a} \, ((a^1)^2 + (a^2)^2) | \chi_0(a^1) |^2 | \chi_0(a^2) |^2 = \frac{\beta_1^2}{\pi} \int d^2 \vec{a} \, ((a^1)^2 + (a^2)^2) e^{-\beta_1^2 ((a^1)^2 + (a^2)^2)} = \beta_1^{-2}$$

we finally obtain the essential angular decompositions as follows:

$$\langle \chi_1^\pm | \vec{a} \cdot \vec{x}_- | \chi_0 \rangle = -\frac{1}{\beta_1} \left(\delta^+ \mathcal{D}_{-+}^{(1)*} - \delta^- \mathcal{D}_{--}^{(1)*} \right) \quad (3.7)$$

$$\langle \chi_1^\pm | \vec{a} \cdot \vec{x}_+ | \chi_0 \rangle = +\frac{1}{\beta_1} \left(\delta^+ \mathcal{D}_{++}^{(1)*} - \delta^- \mathcal{D}_{+-}^{(1)*} \right) \quad (3.8)$$

$$\langle \chi_1^\pm | \vec{a} \cdot \hat{z} | \chi_0 \rangle = -\frac{1}{\sqrt{2}\beta_1} \left(\delta^+ \mathcal{D}_{0+}^{(1)*} - \delta^- \mathcal{D}_{0-}^{(1)*} \right). \quad (3.9)$$

In the previous equations the factors $\delta^\pm$ refer to the flux tube polarisation transverse to the body vector $\vec{r}$.

We are now in a position to calculate the E1 transition matrix element from a conventional $^1S_0$ meson to a $p = 1$ hybrid,

$$\begin{aligned} \mathcal{M}_{\mathcal{H}\mathcal{C}} &= \langle \mathcal{H}(\pm, m') | e^{-i\vec{q} \cdot \vec{r} \frac{\mu}{m_Q}} \mathcal{O}_{E1} | \mathcal{C}(l=0) \rangle = i|\vec{q}| \left(\frac{e_Q}{m_Q} + \frac{e_{\bar{d}}}{m_d} \right) \sqrt{\frac{2b}{\pi^3}} \beta_1 \frac{-1}{\sqrt{2}\beta_1} \\ &\times \int d^3 \vec{r} \left[R_{\mathcal{H}}(r) \sqrt{\frac{3}{4\pi}} \mathcal{D}_{m', \pm}^{(1)} \right]^* r e^{-i\vec{q} \cdot \vec{r} \frac{\mu}{m_Q}} \left(\delta^+ \mathcal{D}_{-+}^{(1)*} - \delta^- \mathcal{D}_{--}^{(1)*} \right) \left[R_{\mathcal{C}}(r) \sqrt{\frac{1}{4\pi}} \right]. \quad (3.10) \end{aligned}$$

Note that we have included the plane-wave factor here, which gives rise to a $|\vec{q}|$ -dependent form factor; we neglected it in the previous, conventional transition. Also note that we have not included the effect of $\vec{a}$ dependence in the plane-wave, which modifies the form-factor

by changing the effective charge radius. Evaluation of the integrals in equation(3.10) is most easily accomplished by expanding the plane-wave factor in spherical harmonics, $\exp -i\vec{q}\cdot\vec{r}\frac{\mu}{m_Q} \equiv \sum_L (-i)^L (2L+1) \mathcal{D}_{00}^{(L)} j_L(|\vec{q}|r\frac{\mu}{m_Q})$. This leaves an angular integral over the product of three $\mathcal{D}$'s which can be expressed as a product of Clebsch-Gordan coefficients,

$$\left[\int d\Omega \mathcal{D}_{m',\pm}^{(1)*} \mathcal{D}_{00}^{(L)} \left(\delta^+ \mathcal{D}_{-+}^{(1)*} - \delta^- \mathcal{D}_{--}^{(1)*} \right) \right]^* = \frac{4\pi}{2L+1} \langle 1m'; 1-1 | L0 \rangle (\delta^+ \langle 1-1; 1+1 | L0 \rangle - \delta^- \langle 1+; 1-1 | L0 \rangle).$$

Thus for the matrix element we find

$$\mathcal{M}_{\mathcal{HC}} = -i|\vec{q}| \left(\frac{e_Q}{m_Q} + \frac{e_{\bar{d}}}{m_d} \right) \sqrt{\frac{b}{3\pi^3}} \delta_{m',+1} \times \left[\left(\langle rj_0 \rangle - \frac{1}{2} \langle rj_2 \rangle \right) (\delta^+ - \delta^-) + \frac{3i}{2} \langle rj_1 \rangle (\delta^+ + \delta^-) \right].$$

The $\delta_{m',\pm 1}$ refers to the meson's total angular momentum projection in the fixed axes $(\hat{x}, \hat{y}, \hat{z})$. The parity eigenstates in the flux tube are given in [36]. They are linear superpositions of states where the flux-tube has polarisation ± 1 . Following that reference we denote the number of positive or negative helicity phonon modes by $\{n_+, n_-\}$, which for our present purposes will be $\{1, 0\}$ or $\{0, 1\}$. Parity eigenstates are then the linear superpositions

$$|\mathcal{P} = \pm\rangle \equiv \frac{1}{\sqrt{2}} \left(|\{1, 0\}\rangle \mp |\{0, 1\}\rangle \right)$$

Thus we have $(\delta^+ \pm \delta^-) = \sqrt{2} \delta(\mathcal{P} = \mp)$, and in the long-wavelength approximation ($|\vec{q}|r \ll 1$), i.e. where the exponential can be set to unity³,

$$\langle \mathcal{H}(\pm, m') | \mathcal{O}_{E1} | \mathcal{C}(l=0) \rangle = -i|\vec{q}| \left(\frac{e_Q}{m_Q} + \frac{e_{\bar{d}}}{m_d} \right) \sqrt{\frac{2b}{3\pi^3}} \langle r \rangle \delta_{m',+1} \delta(\mathcal{P} = +),$$

where $\langle r \rangle$ is shorthand for $\int r^2 dr R_{\mathcal{H}}^*(r) r R_{\mathcal{C}}(r)$. We see that only the spin-singlet axial (1^{++}) hybrid is accessible from the pseudoscalar meson by an $E1$ transition. Note the charge factor, $(e_Q/m_Q + e_{\bar{d}}/m_d)$, which vanishes for neutral $q\bar{q}$ mesons. This is a non-trivial selection rule that arises because of the spatial symmetry of the $p = 1$ mode flux-tube state. Consider Figure 3.1. Excitation of the conventional state involves oscillations along the interquark axis - since for $p = 1$ in a charge neutral meson, the quarks lie on opposite sides of the c.m. with opposite charges, there is a finite “longitudinal” electric dipole moment and neutral $E1$ transitions can

³This is equivalent to setting the form-factor equal to 1

occur. In contrast, hybrid meson excitation involves oscillations transverse to the interquark axis and since the quarks lie on the *same* side of the c.m. there is no net “transverse” electric dipole moment and no excitation of charge neutral hybrids. Hence in the flux-tube model we would not expect there to be $E1$ transitions between hybrid charmonium and conventional charmonium.

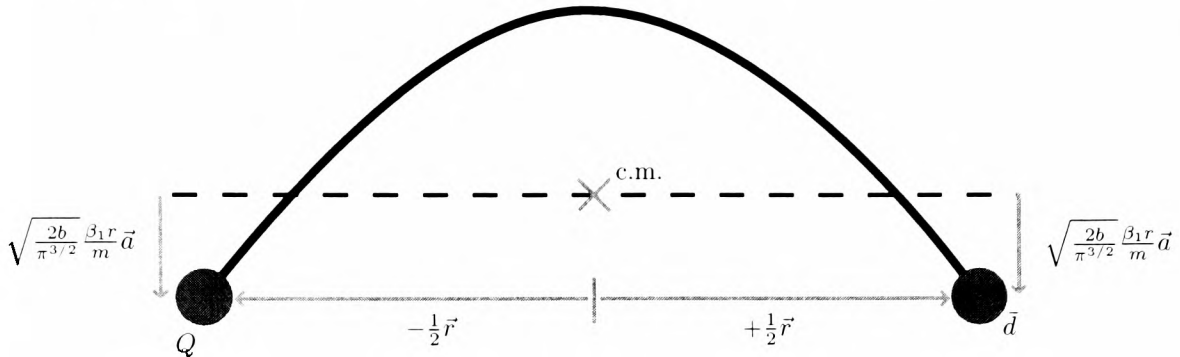


Figure 3.1: Snapshot of a $p = 1$ hybrid state with $m_Q = m_d$, note that the transverse oscillations keep the quarks on the same side of the meson centre-of-mass, while they are on opposite sides longitudinally.

We can, however, have transitions between the electrically charged, light isovector states, e.g. $\gamma\pi^\pm \rightarrow a_{1H}^\pm$. We show below the matrix element for this transition and for comparison, the matrix element for a conventional $E1$ transition without spin-flip, equation(3.3),

$$\begin{aligned}\mathcal{M}(\gamma\pi^+ \leftrightarrow a_{1H}^+) &= \left(\frac{e_u + e_{\bar{d}}}{m_u} \right) \langle R_{\mathcal{H}} | r | R_{\pi} \rangle |\vec{q}_H| \sqrt{\frac{2b}{3\pi^3}} \\ \mathcal{M}(\gamma\pi^+ \leftrightarrow b_1^+) &= \left(\frac{e_u - e_{\bar{d}}}{m_u} \right) \langle R_{\mathcal{C}} | r | R_{\pi} \rangle |\vec{q}_b| \frac{m_u}{2\sqrt{3}}.\end{aligned}\quad (3.11)$$

At this point we can demonstrate the consistency of our calculation with Isgur’s result for the charge radius. Recall equation(2.1) considered in the equal quark mass case,

$$\langle \pi | r_Q^2 | \pi \rangle = \left[\frac{1}{4} + \frac{2b}{\pi^3 m^2} \zeta(3) \right] \langle \pi | r^2 | \pi \rangle. \quad (3.12)$$

Insert a complete set of meson states (which will contain both conventional and hybrid states) and sum over the three polarisations $\vec{\epsilon}_{\pm,0}$,

$$\langle \pi | r_Q^2 | \pi \rangle = \sum_N \langle \pi | r_Q | N \rangle \langle N | r_Q | \pi \rangle = 3 \sum_N |\langle \pi | \vec{\epsilon}_+ \cdot \vec{r}_Q | N \rangle|^2.$$

Using the expression (3.1) we can write this in terms of the $E1$ operators for conventional and

hybrid transitions (3.2),

$$3 \sum_N \left| \left\langle \pi \left| \frac{\mathcal{O}_{E1}^C}{i|\vec{q}|(e_Q - e_d)} \right| N \right\rangle \right|^2 + \left| \left\langle \pi \left| \frac{\mathcal{O}_{E1}^H}{i|\vec{q}|(e_Q + e_d)} \right| N \right\rangle \right|^2.$$

Previously we computed the matrix elements of $\mathcal{O}_{E1}$ from the pion to the lowest states only (b_1 and $a_{1H}^{(p=1)}$). To get a full set of meson states with non-zero matrix element we must consider each mode p which will have a complete set of radial wavefunctions, as will the conventional mesons, hence the sums over radial wavefunctions and over p in the following, where also we set $m_Q = m_d$ for simplicity,

$$= 3 \left(\frac{1}{12} \sum_{\{R_C\}} |\langle R_\pi | r | R_C \rangle|^2 + \frac{2b}{3\pi^3 m^2} \sum_p \frac{1}{p^3} \sum_{\{R_H\}} |\langle R_\pi | r | R_H \rangle|^2 \right),$$

and since the radial wavefunctions form a complete set over $\int r^2 dr$, we obtain equation(3.12). $\zeta(3) = \sum_p p^{-3} \sim 1.2$ is 80% saturated by the $p = 1$ which justifies us only considering excitations to these lightest hybrids - they have the largest matrix element.

3.3 Hybrid $E1$ rates

In order to consider the likely radiative widths of $p = 1$ hybrid states we need estimates for their masses and radial wavefunctions - the flux-tube model gives us these as the eigenvalues and eigenstates of the quark adiabatic Hamiltonian. In Appendix B we give details of the construction and approximate solution of this Hamiltonian.

Using equation(3.4) for the width we can write the ratio of hybrid to conventional radiative width as

$$\frac{\Gamma_{E1}(a_{1H}^+ \rightarrow \pi^+ \gamma)}{\Gamma_{E1}(b_{1Q}^+ \rightarrow \pi^+ \gamma)} = \frac{72}{\pi^3} \frac{b}{m_u^2} \left| \frac{\langle R_H | r | R_\pi \rangle}{\langle R_C | r | R_\pi \rangle} \right|^2 \frac{|\vec{q}(a_{1H})|^3}{|\vec{q}(b_1)|^3}.$$

This is in the long-wavelength approximation where the form-factor is simply 1 in both cases. More realistic, mass-dependent, form-factor estimates are obtained if one retains the plane-wave exponential and the associated spherical Bessel functions.

Using the adiabatic radial wavefunctions of Appendix B we obtain $\left| \frac{\langle R_H | r | R_\pi \rangle}{\langle R_C | r | R_\pi \rangle} \right|^2 \approx 1.0$, so the radial moments do not suppress hybrid production. We follow ref. [36] and use the standard parameters $b = 0.18 \text{ GeV}^2$, $m_u = 0.33 \text{ GeV}$ so that the prefactor $\frac{72}{\pi^3} \frac{b}{m_u^2} \approx 3.8$ and hence there is no hybrid suppression from the flux-tube dynamics.

Within our variational solution $\beta_H = 255 \text{ MeV}$, $\beta_b = 281 \text{ MeV}$, $\beta_\pi = 335 \text{ MeV}$, so we see the

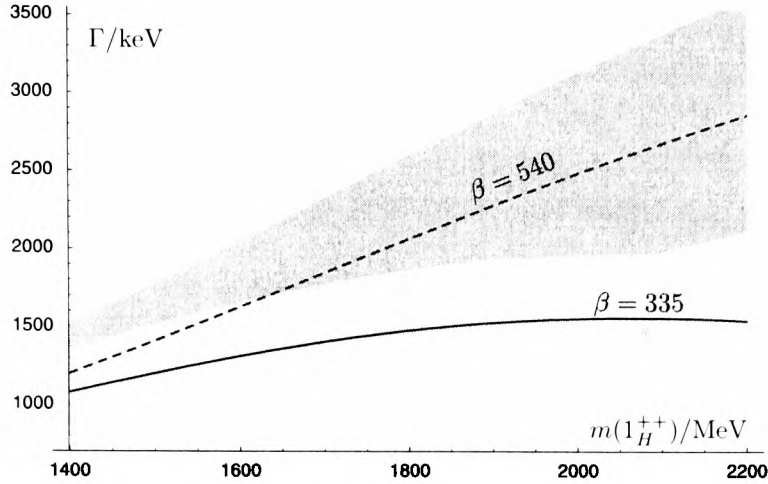


Figure 3.2: $E1$ width as a function of 1^{++} hybrid mass. The solid line is for $\beta_\pi = 335$ MeV. The dashed line is for $\beta_\pi = 540$ MeV [44]. The shaded grey areas are the uncertainties due to the error on the experimental rate used as normalisation.

$p = 1$ hybrid state being only a little larger than the $L = 1$ conventional state. The main uncertainty is the computed size of the π ; one needs an artificially large value of β_π to explain the light mass of the pion (using a spin-spin contact interaction in a quark model). As a guide on this uncertainty we use $\beta_\pi = 540$ MeV [44] as well as the value above. Assuming that this hybrid has mass ~ 1.9 GeV [36, 40], and using the measured width $\Gamma(b_1^+ \rightarrow \pi^+ \gamma) = 230 \pm 60$ keV [7] we predict that

$$\Gamma(a_{1H}^+ \rightarrow \pi^+ \gamma) = 2.1 \pm 0.9 \text{ MeV}. \quad (3.13)$$

where the error allows for the uncertainty in β_π [44]. We present in Figure 3.2 the dependence of the radiative width on the mass of the hybrid assuming a simple Gaussian form-factor⁴ $\exp[-\frac{|\vec{q}|^2}{8\beta_\pi^2}]$.

The equivalent $E1$ process for $S = 1$ is $b_{JH} \rightarrow \rho \gamma$, where the only difference from the $S = 0$ case is the addition of L, S Clebsch-Gordan factors coupling the $Q\bar{Q}$ spin and flux-tube angular momentum to the total J of the hybrid meson in question. As above, the charge conjugation of the initial and final states is the same, thus $\Delta C = 0$, and the amplitude is proportional to $e_1 + e_2$. Consider absorption of a positive helicity photon. The hybrid state is constructed as follows

$$|J^{+-}, m_J\rangle = \sum_{m_S, m'} \langle 1m'; 1m_S | Jm_J \rangle \frac{1}{\sqrt{2}} (|\mathcal{H}(+, m')\rangle - |\mathcal{H}(-, m')\rangle) |S = 1, m_S\rangle$$

⁴which is almost identical to what one obtains by retaining the full plane wave exponential

and the matrix element becomes

$$\mathcal{M}(\gamma\rho^+ \leftrightarrow b_{JH}^+) = \langle 1+1; 1m_\rho | Jm_J \rangle \left(\frac{e_u + e_d}{m_u} \right) \langle R_{\mathcal{H}} | r | R_\rho \rangle |\vec{q}_{\mathcal{H}}| \sqrt{\frac{2b}{3\pi^3}}.$$

We find for $J = 0, 1, 2$ in this $E1$ limit, normalising against measured $\Gamma(f_1 \rightarrow \rho\gamma)$,

$$\Gamma(b_{JH}^+ \rightarrow \rho^+\gamma) = 2.3 \pm 0.8 \text{ MeV}. \quad (3.14)$$

where the error reflects the uncertainties in the conventional $E1$ strength and β_{f_1} and where we have taken $m_{\mathcal{H}} = 1.9 \text{ GeV}$.

We present in Table 3.2, formulae for $E1$ radiative matrix elements between conventional and hybrid states. Of particular interest is the rate of production of the isovector $1^{-+}(\pi_{1H})$ at 1.6 GeV, see Section 1.1. We use the $E1$ decay of this state to a_2 as an explicit example of the use of Table 3.2. With a positive helicity photon there are three helicity amplitudes, corresponding to $m_J = -2, -1, 0$,

$$\begin{aligned} \mathcal{M}_{-2} &= \sqrt{\frac{3}{4}} \langle 1-1; 1-1 | 2-2 \rangle \langle 10; 1-1 | 1-1 \rangle \mathcal{M} = \frac{1}{\sqrt{2}} \sqrt{\frac{3}{4}} \mathcal{M} \\ \mathcal{M}_{-1} &= \sqrt{\frac{3}{4}} (\langle 10; 1-1 | 2-1 \rangle \langle 1+1; 1-1 | 10 \rangle + \langle 1-1; 10 | 2-1 \rangle \langle 10; 10 | 10 \rangle) \mathcal{M} = \frac{1}{2} \sqrt{\frac{3}{4}} \mathcal{M} \\ \mathcal{M}_0 &= \sqrt{\frac{3}{4}} (\langle 1-1; 1+1 | 20 \rangle \langle 10; 1+1 | 1+1 \rangle + \langle 10; 10 | 20 \rangle \langle 1+1; 10 | 1+1 \rangle) \mathcal{M} = \frac{1}{2\sqrt{3}} \sqrt{\frac{3}{4}} \mathcal{M}, \end{aligned}$$

where $\mathcal{M} = |\vec{q}| 87 \times 10^{-3} \text{ GeV}^{-1}$. Using equation (3.4) we obtain

$$\begin{aligned} \Gamma_{E1}(\pi_{1H} \rightarrow a_2\gamma) &= 4 \frac{E_{a_2}}{m_{\pi_{1H}}} |\vec{q}| \frac{1}{3} (|\mathcal{M}_{-2}|^2 + |\mathcal{M}_{-1}|^2 + |\mathcal{M}_0|^2) \\ &= \frac{E_{a_2}}{m_{\pi_{1H}}} |\vec{q}|^3 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{12} \right) (87 \times 10^{-3} \text{ GeV}^{-1})^2, \end{aligned}$$

so that for a π_{1H} at 1.6 GeV the width is $\sim 90 \text{ keV}$. Given that a_2 exchange is suppressed relative to f_2 in photoproduction by a smaller coupling at the $\vec{p}a_2n$ vertex, this is unlikely to be a major production route in $\gamma p \rightarrow \pi_{1H}n$.

A state potentially interesting in heavy-flavour decay (see next chapter) is the axial hybrid kaon, we find that this state has an $E1$ width to $K\gamma$ of $300-1000 \text{ keV}$ (assuming mass $\sim 2 \text{ GeV}$). This state could be seen in photoproduction by looking in the $K\pi\pi\Lambda$ end-state.

Note that these $E1$ transitions are only possible with charge exchange and so cannot occur between flavourless states. In particular they are absent for $c\bar{c}$ and $b\bar{b}$. Thus, for example, the transitions $\psi(3685) \rightarrow \gamma\chi_J$ can receive no contribution from any hybrid component of the

state		$u\bar{d}$	$u\bar{s}$
$^1S_0$	$\times 1$	$\gamma\pi^+ \rightarrow a_{1H}^+$ 56 (23)	$\gamma K^+ \rightarrow K_{1AH}^+$ 43 (23)
$^3S_1$	$\times \langle 11; 1m_i J_H m_H \rangle$	$\gamma\rho^+ \rightarrow b_{JH}^+$ 56	$\gamma K^{*+} \rightarrow K_{JBH}^+$ 43
$^1P_1$	$\times \sqrt{\frac{3}{2}} \langle 11; 1m_i 1m_H \rangle$	$\gamma b_1^+ \rightarrow \rho_H^+$ 87	$\gamma K_{1B}^+ \rightarrow K_H^{*+}$ 68
^3P_J	$\times \sum_{m_L, m_S} \langle 1m_L; 1m_S Jm_J \rangle$ $\cdot \sqrt{\frac{3}{4}} \langle 1m_L + 1; 1m_S J_H m_H \rangle^*$	$\gamma a_J^+ \rightarrow \pi_{JHH}^+$ 87	$\gamma K_{JA}^+ \rightarrow K_{JHH}^+$ 68

Table 3.2: Photon-Meson-Hybrid $E1$ matrix elements: $\mathcal{M} = \left(\frac{e_1}{m_1} + \frac{e_2}{m_2}\right) |\vec{q}| \sqrt{\frac{2b}{3\pi^3}} \langle R_{\mathcal{H}} | r | R_{\mathcal{C}} \rangle$ should be multiplied by the Clebsch-Gordan factor in the second column to give the overall matrix element for a positive helicity photon. The numbers quoted in columns three and four are $\mathcal{M}/|\vec{q}|$ (10^{-3}GeV^{-1}), evaluated using the results of Appendix B, except those in brackets which use the β -values of [44].

$\psi(3685)$ wavefunction (assuming here that the χ_{JH} states are $\gg 4$ GeV in mass and so do not mix measurably into the $c\bar{c}$ states). We outlined earlier the symmetry which gives rise to this selection rule. We will now move on to a higher multipole, $M1_{\text{orb}}$, which does allow radiative decay between neutrals.

3.4 $M1$ transitions

Consider the spin-independent part (convection current) of the interaction Hamiltonian for a quark and a photon [45],

$$H_I = -\frac{eQ}{m_Q} e^{i\vec{q}\cdot\vec{r}_Q} \vec{\epsilon}\cdot\vec{p}_Q + \dots$$

and expand the exponential to the first power of $\vec{q}\cdot\vec{r}$, giving an interaction proportional to $\vec{q}\cdot\vec{r}_Q\vec{\epsilon}\cdot\vec{p}_Q$. This can be written in terms of multipoles $E2$ and $M1$ as

$$\vec{q}\cdot\vec{r}_Q\vec{\epsilon}\cdot\vec{p}_Q = \vec{\epsilon}\cdot\vec{r}_Q\vec{q}\cdot\vec{p}_Q|_{E2} + (\vec{\epsilon} \times \vec{q})\cdot(\vec{r}_Q \times \vec{p}_Q)|_{M1}.$$

The second term is an “orbital” $M1$, transforming with $J^P = 1^+$, that does not contribute in conventional meson radiative transitions as can easily be seen: The $M1$ operator is proportional to $|\vec{q}|\vec{\epsilon}\cdot\vec{L}$ which is an angular momentum raising or lowering operator and hence induces transitions between states of different m but the same L ; such states are degenerate in zero applied field so that $|\vec{q}| = 0$ and there is no transition amplitude.

In the hybrid case there is also the flux-tube angular momentum to consider and so this simple no-transition proof cannot be used. In Appendix A we derive the quark angular mo-

menta, $\vec{L}_Q, \vec{L}_d$. As in the $E1$ case, an operator linear in $\vec{a}$ can induce a transition between a conventional state and a hybrid, and as such we will focus on these terms in $\vec{L}_Q, \vec{L}_d$, which we display below,

$$\begin{aligned}\vec{L}_Q &= \frac{\tilde{m}_d}{m_T} \frac{\sqrt{2(N+1)}}{\pi} \sum_{pq} \frac{1}{p} T_{pq}^{-1} \vec{r} \times \vec{p}_{a_q} + \frac{br\tilde{m}_d}{\pi m_T M} \sqrt{\frac{2}{N+1}} \sum_p \frac{1}{p} \vec{a}_p \times \vec{p}_r \\ \vec{L}_d &= \frac{\tilde{m}_Q}{m_T} \frac{\sqrt{2(N+1)}}{\pi} \sum_{pq} \frac{(-1)^p}{p} T_{pq}^{-1} \vec{r} \times \vec{p}_{a_q} + \frac{br\tilde{m}_Q}{\pi m_T M} \sqrt{\frac{2}{N+1}} \sum_p \frac{(-1)^p}{p} \vec{a}_p \times \vec{p}_r.\end{aligned}$$

The full $M1$ operator is $-i|\vec{q}| \sum_q \frac{e_q}{m_q} \vec{\epsilon} \cdot \vec{L}_q$. We can simplify the operator expression if we assume that it will act on a conventional, $L = 0$ meson as then $\vec{p}_{a_q} \rightarrow i\beta_q^2 \vec{a}_q = i\frac{b\pi q}{N+1} \vec{a}_q$, and using Gaussian wavefunctions (see Appendix B), $\vec{p}_r \rightarrow i\beta_c^2 \vec{r}$ so that

$$\mathcal{O}_{M1} = |\vec{q}| \frac{b}{m_T} \sqrt{\frac{2}{N+1}} \sum_{pq} \left(\frac{e_Q \tilde{m}_d}{m_Q} + (-1)^p \frac{e_d \tilde{m}_Q}{m_d} \right) \left[\frac{q}{p} T_{pq}^{-1} - \frac{\beta_c^2 r}{\pi M} \delta_{pq} \right] \vec{r} \times \vec{a}_q.$$

In order to compute matrix elements of this operator we will need an explicit expression for T_{pq}^{-1} . At lowest order in $\frac{br}{m}$ it is simply δ_{pq} and in the equal quark mass case one can easily show that the first correction is $-\frac{4br}{\pi^2 m} \frac{1}{pq} \delta_{p+q, \text{even}}$. For now we consider only the zeroth order in $\frac{br}{m}$ we will return to the correction when discussing phenomenological estimates.

In the equal quark mass case, for exciting a $p = 1$ hybrid

$$\mathcal{O}_{M1} = |\vec{q}| \frac{b}{2m} \sqrt{\frac{2}{N+1}} (e_Q - e_d) \left(1 - \frac{2\beta_c^2 r}{\pi m} \right) \vec{\epsilon} \cdot \vec{r} \times \vec{a}.$$

Note that the difference of charges appears and so we are able to excite neutral mesons using this operator. Writing the angle dependent piece as

$$\vec{\epsilon}_+ \cdot \vec{r} \times \vec{a} = -ir \sqrt{\frac{4\pi}{3}} \left(Y_1^0(\Omega) \frac{1}{\sqrt{2}} \vec{a} \cdot \hat{x}_+ + Y_1^{+1}(\Omega) \vec{a} \cdot \hat{z} \right),$$

we can use the results derived earlier (equations (3.8,3.7,3.9)) to evaluate the flux-tube integrals.

One obtains

$$\langle \mathcal{H}(\mathcal{P} = -, m') | \mathcal{O}_{M1} | \mathcal{C}(l = 0) \rangle = -i \frac{|\vec{q}|}{2m} \sqrt{\frac{2b}{3\pi}} (e_Q - e_d) \left\langle R_{\mathcal{H}} \left| r \left(1 - \frac{2\beta_c^2 r}{\pi m} \right) \right| R_{\mathcal{C}} \right\rangle. \quad (3.15)$$

Only the negative parity hybrid is obtained in line with the 1^+ of the operator acting on the initial pseudoscalar state. The prefactor $\frac{|\vec{q}|}{2m}$ indicates that this multipole is v/c suppressed relative to $E1$.

Radiative rates will be sensitive to the quantity $f \equiv \left(1 - \frac{2\beta_c^2}{\pi m} \frac{\langle r^2 \rangle}{\langle r \rangle}\right)$, which we can easily compute using our variational wavefunctions (Appendix B). We find a considerable cancellation in light flavours:

$$f_{c\bar{c}} \sim 1 - 0.4 \sim 0.6$$

$$f_{s\bar{s}} \sim 1 - 0.8 \sim 0.2$$

$$f_{u\bar{u}} \sim 1 - 1.04 \sim -0.04.$$

It would appear from this that $M1$ transitions, already suppressed by v/c will be very small, except perhaps in charmonium. This cancellation appears to be accidental, and so we should ask if there are any additional terms that can appear to mollify the zero. We approximated T_{pq}^{-1} by its lowest term in the derivation; including the first correction term introduces an extra $-\frac{4br}{\pi^2 m}$ in the $p = 1$ transition operator. Such an addition modifies f as follows: $\delta f_{c\bar{c}} \sim -0.1$, $\delta f_{s\bar{s}} \sim -0.9$, $\delta f_{u\bar{u}} \sim -1.1$. The correction to charmonium is relatively small but in light flavours the observed cancellation is considerably modified and we must associate a large uncertainty with the $\mathcal{O}(br/m)$ corrections, of which this is just one.

For phenomenological estimates we use the following ranges

$$f_{c\bar{c}} \sim 0.5 \rightarrow 0.7$$

$$f_{s\bar{s}} \sim -0.7 \rightarrow 0.2$$

$$f_{u\bar{u}} \sim -1 \rightarrow 0.$$

The spin-singlet transitions are easily evaluated; for a $c\bar{c}$ vector hybrid at ~ 4.2 GeV the $M1$ radiative width to the η_c is

$$\Gamma(\psi_H \xrightarrow{M1} \eta_c \gamma) \sim 30 \rightarrow 60 \text{keV}, \quad (3.16)$$

which is not small on the usual scale of charmonium radiative widths. The equivalent light quark meson transition is

$$\Gamma(\rho_H \xrightarrow{M1} \pi \gamma) \lesssim 2 \text{MeV},$$

where the upper limit is for $f = -1$. This is potentially rather large and suggests that the photoproduction of this hybrid off a pion exchange could be considerable, however the ρ_H does not have exotic quantum numbers and as such is not an attractive prey for hybrid-hunters. With an $M1$ photon we can get the 1^{-+} exotic using an ω/ρ exchange⁵. As in the $E1$ case we simply need to include the Clebsch-Gordan coupling the quark spin to the flux-tube angular

⁵Which would be zero in the flux-tube breaking model supplemented with VMD in the approximation that the off-shell wavefunctions are identical.

momentum to compute the radiative decay rate. We find

$$\Gamma(\pi_{1H}(1600) \xrightarrow{M1} \rho\gamma) \lesssim \frac{0.7}{6.5} \text{ MeV},$$

which suggests that if the $\pi_{1H}(1600)$ is a light hybrid meson, there should be healthy production of it in $M1$ photoproduction via the ω exchange given that ω is enhanced over ρ at the nucleon vertex by roughly a factor of three [46].

The charmonium 1^{-+} hybrid will decay via $M1$ to the J/ψ with a width very similar to that in equation(3.16).

3.5 Estimating the non-adiabatic corrections

We saw in the last section that in at least one case there are large corrections when considering higher orders in br/m . Isgur's expressions for $\vec{r}_{Q,d}$ which are the foundation of this study are accurate to first non-trivial order in br/m and we expect there to be corrections at the next order which are further step beyond the adiabatic approximation. We have been unable to construct a systematic approach to considering higher orders, but we are able to present here estimates for the effect of higher non-adiabatic corrections using work done previously, in a slightly different context, by Merlin and Paton [43].

In [47] it was demonstrated that many of the non-adiabatic corrections to the mass of a state in the flux-tube model can be accounted for by performing an r -dependent orthogonal transformation of the flux-tube Fourier modes $\vec{a}_p$,

$$\vec{a}_p = \frac{1}{p} \sum_q x_{qp} \mu_q(r) \vec{b}_q, \quad (3.17)$$

where $\mu_q(r)$ are r -dependent effective mode numbers and x_{qp} are the elements of the orthogonal transformation matrix. •

As a rough estimate of non-adiabatic corrections we will see what effect these modified Fourier modes have on the charge radius and on $E1$ rates. The quantity which appears in the transverse part of $\vec{r}_Q$ is $\sum_p \frac{1}{p} \vec{a}_p$, applying the transform in equation(3.17) we find, for the $m_Q = m_d$ case,

$$\sum_p \frac{1}{p} \vec{a}_p = \frac{4br}{\pi^2 m} \sum_{pq} \frac{1}{p^2} \frac{\mu_q(r) \vec{v} \cdot \vec{x}_q}{\mu_q(r)^2 - p^2} \delta_{p+q, \text{even}} \vec{b}_q,$$

where $\vec{v} = (1, 1, 1, 1, \dots)$ and $\vec{x}_q$ is a row of the orthogonal transformation matrix x_{qp} . With

the aid of equation(3.9) in [47] the sums over p can be evaluated and we find

$$\sum_p \frac{1}{p} \vec{a}_p = \sum_q \frac{\vec{v} \cdot \vec{x}_q}{\mu_q(r)} \left(1 + \frac{br}{2m} \delta_{q,\text{odd}} + \frac{br}{6m} \delta_{q,\text{even}} \right) \vec{b}_q.$$

Similarly we find

$$\sum_p \frac{(-1)^p}{p} \vec{a}_p = \sum_q \frac{\vec{v} \cdot \vec{x}_q}{\mu_q(r)} \left((-1)^q - \frac{br}{2m} \delta_{q,\text{odd}} + \frac{br}{6m} \delta_{q,\text{even}} \right) \vec{b}_q.$$

The flux-tube correction to the charge radius $\langle (r_Q^\perp)^2 \rangle$ is easily evaluated, noting that now the flux-tube eigenstates are functions of $\vec{b}_q$ with effective mode numbers $\mu_q(r)$. We find

$$\langle (r_Q^\perp)^2 \rangle = \left\langle \frac{2br^2}{\pi^3 m^2} \sum_q \frac{|\vec{v} \cdot \vec{x}_q|^2}{\mu_q(r)^3} \left(\left[1 + \frac{br}{2m} \right] \delta_{q,\text{odd}} + \left[1 + \frac{br}{6m} \right] \delta_{q,\text{even}} \right) \right\rangle, \quad (3.18)$$

where the remaining expectation is with respect to the radial wavefunctions. In equations(3.13a, 3.13b) of [47], an explicit form for $\vec{v} \cdot \vec{x}_q$ is given, however the effective mode numbers $\mu_q(r)$ are only defined as the solution of transcendental equations, equations(3.10a, 3.10b) in [47]. The dominant change in charge radius with respect to the Isgur solution comes from the first few modes q ; we solved the transcendental equations numerically for $q < 5$, and using the Gaussian wavefunctions of Appendix B computed the value of equation(3.18) which we found to be approximately 30% larger than in the Isgur case. This was for the light quark sector ($m \sim 330$ MeV, $\beta \sim 330$ MeV), in charmonium the effects are much smaller, reflecting the lower value of br/m . Hence we see that higher non-adiabatic effects of the type considered by Merlin do change numerical values but do not overwrite the essential predictions. We will now consider this in the closely related context of hybrid $E1$ rates, where, in light of what we just found, we anticipate there to be $\mathcal{O}(10\%)$ corrections in light quark rates.

The net change to $E1$ conventional-hybrid($p = 1$) matrix elements in this formalism is an additional factor,

$$\frac{\left(1 + \frac{br}{2m} \right) \vec{v} \cdot \vec{x}_1}{\mu_1(r)^{3/2}}.$$

Including this and evaluating the $E1$ matrix element using the Appendix B Gaussian wavefunctions we find a 13% increase in $\mathcal{M}$. From this we propose that to take account of non-adiabatic uncertainties one should associate an $\sim 30\%$ theoretical error on our light quark $E1$ rates.

3.6 Summary

In this chapter we have developed the formalism originally presented in [39], using it to compute $E1$ radiative transition rates between hybrids and conventional mesons. We found that the symmetry of the $p = 1$ hybrids is such that there can be no $E1$ transitions for charge neutral states. Numerical estimates of the light quark hybrid radiative decay rates were performed and found not to be small, which encourages the hybrid photoproduction efforts of, for example, the proposed GlueX experiment at Jefferson Lab.

The v/c suppressed $M1_{\text{orb}}$ multipole was briefly considered, but an accidental cancellation made the numerical results rather sensitive to higher order non-adiabatic corrections. The class of non-adiabatic corrections considered by Merlin and Paton [43] were incorporated approximately into this formalism, changing the $E1$ and charge radii predictions at the $\mathcal{O}(10\%)$ level.

Chapter 4

Heavy Flavour Decays to Conventional and Hybrid Mesons

In Chapter 3 we described the method of calculation used to compute amplitudes for processes involving a hybrid meson, a conventional meson and a current by considering the specific example of hybrid meson radiative decay.

We will now use the same techniques to calculate the rate of production of hybrids in heavy flavour decays. Such a calculation is timely in view of the orders of magnitude increase in statistics on a wide range of exclusive decay channels anticipated at present and upgraded B and charm factories.

Our analysis is based upon an extension of the quark model analysis of semileptonic B -decays performed in [48], hereafter referred to as ISGW. In this paper the authors attempted to explain the *inclusive* semileptonic B -decay spectrum as a sum over quark model meson resonances. Their basic calculational premise was to expand the hadronic matrix elements of weak flavour-changing currents in terms of Lorentz invariant form-factors multiplying all possible kinematic four-vector combinations, and to evaluate the form-factors in the quark model (weak-binding limit). These form-factors were then extrapolated away from the zero-recoil point ($|\vec{q}| \neq 0$) and they are hoped to remain at least qualitatively accurate. For example, in the case of B decay to a vector meson X , with polarisation $\vec{\epsilon}$, the axial current decomposition is

$$\langle X(\vec{p}_X, \vec{\epsilon}) | A_\mu | B(\vec{p}_B) \rangle \equiv f \epsilon_\mu^* + a_+ (\epsilon^* \cdot p_B) (p_B + p_X)_\mu + a_- (\epsilon^* \cdot p_B) (p_B - p_X)_\mu,$$

where $f, a_\pm$ are the Lorentz invariant form-factors, which in the quark model are explicit func-

tions of the parameters of the model (constituent quark masses, wavefunction parameters, β) and of the momentum transfer. We refer the reader to [48] for tabulations of the decompositions, quark model form-factors and a fuller discussion of the model foundations.

Inclusive semileptonic spectra are “clean” in that there can be no strong interactions in the end state, however the bulk of experimental data is in exclusive hadronic decay rates and it is here that there is the best chance of finding new states. There exist theoretical models which aim to describe hadronic decays in terms of hadronic matrix elements of the flavour changing weak currents: these models are usually called “generalised factorisation models” and are reviewed in [49]. We present here one such formulation which we will use to predict decay rates to both conventional and hybrid mesons.

4.1 Factorised Model

The matrix element for decays $B \rightarrow M_1 M_2$ will be written in the generic form (see e.g. [49]),

$$\langle M_1 M_2 | \mathcal{H}_{\text{eff}} | B \rangle = \frac{G_F}{\sqrt{2}} V_{q'_1 b} V_{q'_2 q_2} \mathcal{O} \quad (4.1)$$

with

$$\begin{aligned} \mathcal{O} = & a_1(\mu) \langle M_1 | \bar{q}'_1 (V - A) b | B \rangle \langle M_2 | \bar{q}_2 (V - A) q'_2 | 0 \rangle \\ & + a_2(\mu) \langle M_2 | \bar{q}_2 (V - A) b | B \rangle \langle M_1 | \bar{q}'_1 (V - A) q'_2 | 0 \rangle. \end{aligned} \quad (4.2)$$

The model arises from performing the QCD renormalisation of the weak interaction, supposing that one of the mesons is created from the vacuum by a current with the same quantum numbers. Final state interactions are ignored. a_1 , a_2 are considered as phenomenological parameters (independent of the scale μ) determined by fitting to known conventional decay rates and not as the Wilson coefficients they would be in the strict theory, where their scale dependence cancels with that of the hadronic matrix element. These scale dependences are in the Operator Product Expansion sense, but using quark model hadronic form-factors we lose the scale dependence in the matrix element and have to abandon it in the $a_{1,2}$ -factors - hence the “generalised”.

Decays proceeding by the first term in equation (4.2) only are labelled “Class I”, an example

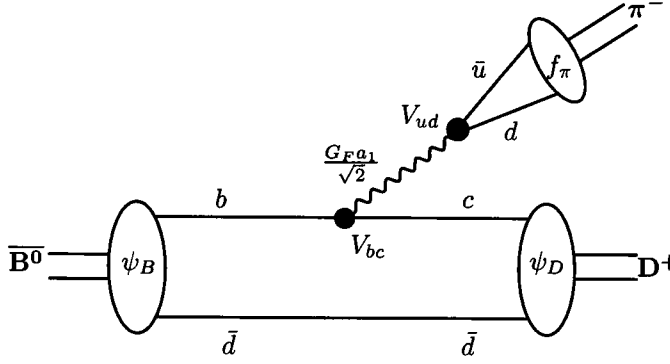


Figure 4.1: A schematic diagram for the Class I decay $\overline{B}^0 \rightarrow \pi^- D^+$.

being $\overline{B}^0 \rightarrow \pi^- D^{(*)+}$ which in this model has amplitude,

$$\frac{G_F}{\sqrt{2}} V_{bc} V_{ud} a_1 \langle \pi^- | \bar{d} \gamma_\mu \gamma^5 u | 0 \rangle \langle D^{(*)+} | \bar{c} (\gamma^\mu - \gamma^\mu \gamma^5) b | \overline{B}^0 \rangle. \quad (4.3)$$

The pion creation current is conventionally parameterised by $\langle \pi^-(q) | \bar{d} \gamma_\mu \gamma^5 u | 0 \rangle \equiv i f_\pi q_\mu$, with $f_\pi \sim 130 \text{ MeV}$ obtained from the π^+ leptonic decay width. The hadronic matrix element must be evaluated using a model; here we will use the ISGW form-factors for conventional mesons. A Class I decay is displayed schematically in Figure 4.1.

Decays proceeding only by the second term in (4.2) are labelled “Class II” or “color-suppressed” and include the well tested channel $B^+ \rightarrow K^{(*)+} J/\psi$ with amplitude

$$\frac{G_F}{\sqrt{2}} V_{bc} V_{cs} a_2 \langle J/\psi | \bar{c} \gamma_\mu c | 0 \rangle \langle K^{(*)+} | \bar{s} (\gamma^\mu - \gamma^\mu \gamma^5) b | B^+ \rangle, \quad (4.4)$$

where the J/ψ current has Lorentz decomposition $\langle J/\psi(q, \epsilon) | \bar{c} \gamma_\mu c | 0 \rangle \equiv \epsilon_\mu^* f_\psi m_\psi$. $f_\psi \sim 400 \text{ MeV}$ is extracted from the leptonic decay width of the J/ψ [49]. A Class II decay is displayed schematically in Figure 4.2.

Generalised factorisation models are more thoroughly discussed in [49] where they are demonstrated to successfully predict weak decay rates of the B -meson to a wide range of conventional exclusive hadronic channels.

We use the ISGW meson state normalisations so that a factor $\sqrt{4m_B m_X}$ appears in all hadronic matrix elements (where X is the meson not created from the vacuum, e.g. $X = D$ in Figure 4.1 and $X = K$ in Figure 4.2), then decay widths are written in the following manner,

$$\Gamma(\overline{B}^0 \rightarrow \rho^- D^{(*)+}) = \frac{G_F^2}{16\pi} \frac{|\vec{q}|}{m_B^2} |V_{bc} V_{ud}|^2 |a_1|^2 \sum_{\text{helicities}} |\mathcal{M}|^2 \quad (4.5)$$

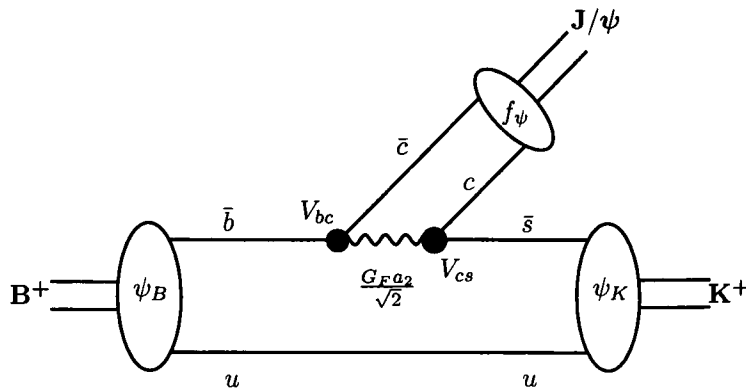


Figure 4.2: A schematic diagram for the Class II decay $B^+ \rightarrow J/\psi K^+$.

where $\mathcal{M} = \langle \rho^- | \bar{d} \gamma_\mu u | 0 \rangle \langle D^{(*)+} | \bar{c} (\gamma^\mu - \gamma^\mu \gamma^5) b | \bar{B}^0 \rangle$ and where for clarity we have shown the case of the Class I decay to a conventional vector. We will use $a_1 = 1.05$, $a_2 = 0.35$ for all B -decays, these numbers are compatible with those in [49] (they vary according to parameterisation of form-factors) and as we shall see in the next section, give a reasonable fit to some conventional decays.

4.2 Exclusive Hadronic B Decays to Conventional Mesons

We will now demonstrate that the combination of the factorized model with ISGW form-factors gives reasonable predictions for some exclusive hadronic B decay rates.

Class I decays are expected to be best described in the factorised formalism [49]. We consider the decays $\bar{B}^0 \rightarrow D^{(*)+} \pi^-$ where experimental branching ratios are available.

Working in the (excellent) approximation $0 \approx m_\pi^2 \ll (m_B^2 - m_D^2)$, the matrix element for $\bar{B}^0 \rightarrow D^+ \pi^-$ is $|\mathcal{M}| = f_\pi (m_B^2 - m_D^2) f_+$, where f_+ is an analytic function of the quark model parameters (equation(B8) in ISGW [48]). Importantly, f_+ is proportional to a quantity denoted F_3 by ISGW:

$$F_3 = \sqrt{\frac{\tilde{m}_D}{\tilde{m}_B}} \frac{\beta_B^{3/2} \beta_D^{3/2}}{\beta_{BD}^3} \exp - \left[\frac{m_d |\vec{q}|}{2\kappa \tilde{m}_D \beta_{BD}} \right]^2,$$

which comes from the overlap of Gaussian radial wavefunctions. The only non-trivial part of this expression is the parameter $\kappa \sim 0.7$ which is an *ad-hoc* addition by ISGW which corrects for the overprediction of the pion elastic-form-factor, i.e. the under-prediction of the pion charge radius. In light of Isgur's findings of the effect of flux-tube degrees-of-freedom in [39] we can interpret this term as dealing with the suppressed flux-tube mode form-factor. ISGW

apply this same $\kappa \sim 0.7$ to all decays; with our new interpretation of its origin, we suggest that it should not have the same value for all hadrons, in particular for a D -meson, where the correction ($\sim 1/m_c^2$) is much smaller, we will use $\kappa \sim 0.8$. Later we will consider the kaon for which we take $\kappa = 0.75$; these values have been scaled using equation (2.1).

For the parameters $m_d, m_c, m_b, \beta_B, \beta_D$ we use the values given in Appendix B. We find

$$\begin{aligned} \mathcal{B}(\overline{B^0} \rightarrow D^+ \pi^-) &= 5.6 \times 10^{-3} [2.8 \pm 0.3 \times 10^{-3}] \\ \mathcal{B}(\overline{B^0} \rightarrow D^{*+} \pi^-) &= 2.2 \times 10^{-3} [2.8 \pm 0.2 \times 10^{-3}] \end{aligned} \quad (4.6)$$

where the values in square brackets are the experimental world averages [7], and where the theoretically predicted widths have been divided by the experimental total widths to give branching ratios. As an example of the kind of theoretical uncertainties present in this model consider the masses $\tilde{m}_{B,X}$ that appear in the ISGW form-factors. Within the quark model with weak binding these are simply the sum of the constituent quark masses, however when we move away from this limit we might equally well use the physical state masses in an attempt to capture some of the higher-order binding physics. Doing so affects the first rate above at the 20% level.

4.2.1 Class II decays $B^+ \rightarrow J/\psi K^+$ and the Mixing Angle in the Axial Kaon Sector

The factorised model is on less firm ground when it comes to Class II decays, but it has been used with some success to predict branching fractions, while longitudinal/transverse ratios remain unexplained [49]. We will consider the channel $B^+ \rightarrow J/\psi K^{(*)+}$ where we have experimental data. This channel contributes to the inclusive decay $B \rightarrow J/\psi X$, which we will see later has some implications for hybrid hunting.

Guided by equation(2.1) we take $\kappa \sim 0.75$ and find

$$\begin{aligned} \mathcal{B}(B^+ \rightarrow K^+ J/\psi) &= 0.4 \times 10^{-3} [1.00 \pm 0.04 \times 10^{-3}] \\ \mathcal{B}(B^+ \rightarrow K^{*+} J/\psi) &= 1.2 \times 10^{-3} [1.4 \pm 0.1 \times 10^{-3}]. \end{aligned} \quad (4.7)$$

The axial kaons $K_1(1270), K_1(1400)$, are not eigenstates of C and in the quark model are linear superpositions of the $^1P_1$ and the $^3P_1$ states. Attempts to determine the mixing angle have considered the state masses, which are rather insensitive to the angle, and their strong

decays [50, 51]. The tools outlined in this section allow us to put limits on the mixing angle using the experimental result [52],

$$\frac{\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1400))}{\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1270))} < 0.30.$$

The ISGW formulae are derived for the decay of the state containing a b -quark which for the charged B would be the decay $B^- \rightarrow J/\psi K^-$. K^- contains a strange quark and, as Barnes points out [50], the mixing angle for this state is opposite to the mixing angle for the state containing an anti-strange anti-quark (i.e. $\theta \rightarrow -\theta$). The compensating minus that renders the B^+ decay rate equal to the B^- decay rate is in the $\vec{\sigma}$ operator in the weak current. This operator acts only for the $^3P_1$ end-state and has opposite phase depending upon whether it acts on the quark or the anti-quark in the meson. We are consistent with Barnes [50] if we write the mixing matrix,

$$\begin{pmatrix} K^-(1270) \\ K^-(1400) \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} \begin{pmatrix} K^-(^1P_1) \\ K^-(^3P_1) \end{pmatrix}. \quad (4.8)$$

Using the formulae from Appendix B of ISGW and the parameters in Appendix B of this thesis, we obtain the ratio $\frac{\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1400))}{\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1270))}$ as a function of the mixing angle θ displayed in Figure 4.3. Systematic errors such as incorrect parameterisation of the form-factor¹ should be mostly cancelled in the ratio since the value of $|\vec{q}|$ differs by less than 10% between the two states. The experimental upper limit is satisfied for $25^\circ < \theta < 80^\circ$, which agrees with the approximate bounds from strong decays [50]. The branching ratio $\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1270))$ as a function of θ is displayed in Figure 4.4. We observe that in the range allowed by the previous, less form-factor dependent analysis, the predicted branching ratio is within a factor of two of the experimentally measured value $\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1270)) = (1.8 \pm 0.34 \pm 0.39) \times 10^{-3}$ [52].

The approximate (to within roughly a factor of 2) agreement we have seen in equations (4.6, 4.7) and for the axial kaons is pleasing, given that there is virtually no parameter tuning (beyond setting a_1, a_2) and that we have extrapolated the form-factors well beyond the region in which we can expect them to be applicable on the grounds of a non-relativistic expansion. There are more sophisticated form-factor models in the literature which do a better job of predicting these numbers, but they do not have the simplicity of ISGW and in particular they are not so obviously connected to the flux-tube model. In the next section we will consider the case in which one hadron in the end-state is a hybrid meson and we shall compute the form-

¹here as earlier we have used $\kappa = 0.75$

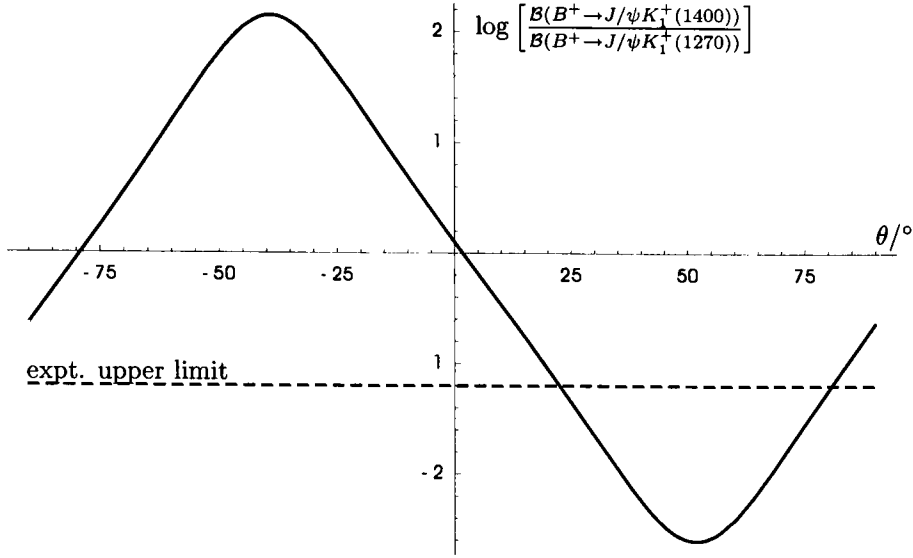


Figure 4.3: $\log \left[\frac{\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1400))}{\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1270))} \right]$ computed in the factorised model with ISGW form-factors as a function of mixing angle θ . The region below the dashed line is allowed by experiment.

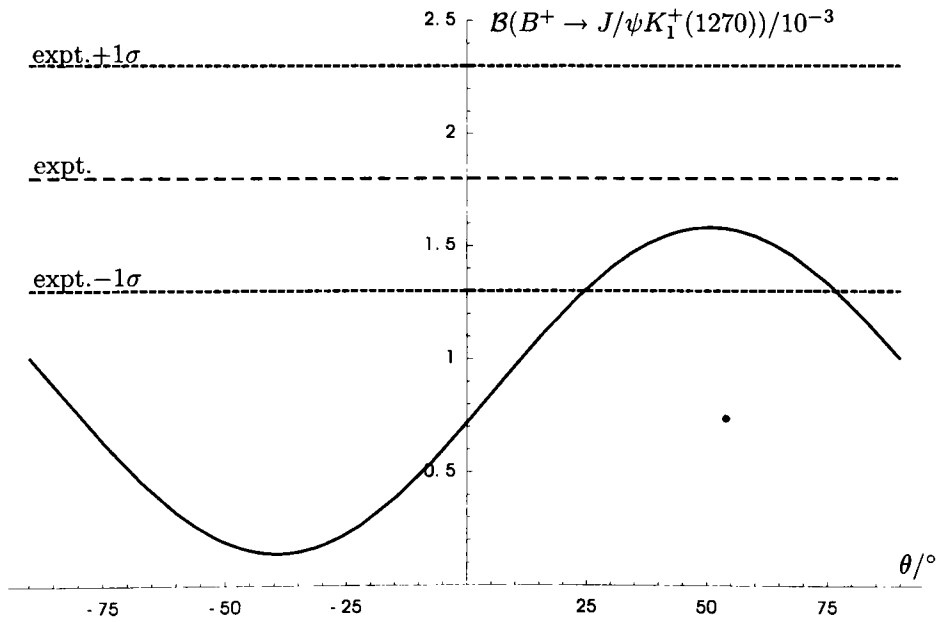


Figure 4.4: $\mathcal{B}(B^+ \rightarrow J/\psi K_1^+(1270))/10^{-3}$ computed in the factorised model with ISGW form-factors as a function of mixing angle θ . The experimental value and the 1σ error bars are also displayed.

factor using the flux-tube model, with strong analogy to the radiative processes considered in Chapter 3. In terms of $|\vec{q}|/m$ being small, such calculations will have greater validity than those for conventional mesons as the hybrids are rather heavier.

4.3 Exclusive Hadronic B Decays to a Hybrid Meson

We shall now extend the established model described above so that it can be applied to decays with hybrid mesons in the end state, in much the same way as we extended the conventional radiative decay formalism previously: We simply include the additional flux-tube transverse degree of freedom, $\vec{a}$, in the currents and wavefunctions. As in the radiative case this both modifies conventional decay rates (explaining the factor κ in the ISGW formalism) and allows for hybrid excitation.

Previous models of hybrid production in this type of process [53], [54] have assumed that hybrids are created by a colour-octet current with some undetermined strength. The Isgur-Paton flux-tube model has the hybrid meson's gluonic field dominated by a flux-tube configuration which is in a colour singlet and as such we will create hybrids using the same current that creates conventional mesons. This model has the huge advantage over previous attempts that there are no undetermined parameters to be set beyond those in the model describing conventional end states.

We will follow the ISGW philosophy of computing the form-factors with a model truncated at some low power of $|\vec{q}|/m$ (here the flux-tube model with all terms linear in $\vec{a}$) but retaining the full kinematic dependence. Since we will have no further use for the form-factors beyond the computation of exclusive hadronic decays we will not calculate them separately, instead computing the matrix elements directly - the result is not dependent on which method we use.

Firstly we show how the flux-tube degrees-of-freedom enter into a general hadronic matrix element. Following Isgur's equations(17,18,24) in [39] we can represent a meson, X , with centre-of-mass momentum, $\vec{p}_X$ by the state vector,

$$|X(\vec{p}_X)\rangle = \int \frac{d^3\vec{R}}{(2\pi)^{3/2}} \exp i\vec{p}_X \cdot \vec{R} \int d^3\vec{r} \psi(\vec{r}) \int \prod_p d^2\vec{a}_p \chi(a_1^1) \chi(a_1^2) \dots \chi(a_N^1) \chi(a_N^2) \\ \left| Q \left(\vec{R} - \frac{\vec{m}_d}{m_X} \vec{r} - \frac{b\vec{r}}{\pi m_Q} \sqrt{\frac{2}{N+1}} \sum_p \frac{1}{p} \vec{a}_p \right) \right\rangle \otimes \prod_p |\vec{a}_p\rangle \otimes \left| \vec{d} \left(\vec{R} + \frac{\vec{m}_Q}{m_X} \vec{r} + \frac{b\vec{r}}{\pi m_d} \sqrt{\frac{2}{N+1}} \sum_p \frac{(-1)^p}{p} \vec{a}_p \right) \right\rangle, \quad (4.9)$$

where the choice of internal wavefunctions $\psi(\vec{r}), \{\chi(a)\}$ determines whether the meson is a

conventional, $p = 1$ hybrid etc...

Now allow a flavour changing weak current, $J \sim \exp -i\vec{q} \cdot \vec{r}_b$ to act on the b -quark of a B -meson, turning it into a c -quark (we are suppressing for now any Lorentz structure in the current). The transition matrix element $\mathcal{M}(B \rightarrow D)$ is

$$\begin{aligned} \langle D(\vec{p}_D) | J(\vec{q}) | B(\vec{p}_B) \rangle &= \int d^3 \vec{R} \int d^3 \vec{R}' \frac{\exp i(\vec{p}_B \cdot \vec{R} - \vec{p}_D \cdot \vec{R}')}{(2\pi)^3} \\ &\times \int d^3 \vec{r} \int d^3 \vec{r}' \psi_D^*(\vec{r}') \psi_B(\vec{r}) \int \prod_p d^2 \vec{a}_p \int \prod_q d^2 \vec{a}_q \chi_D^*(a_1^1) \dots \chi_B(a_1^1) \dots \\ &\times \left\langle \bar{d} \left(\vec{R}' + \frac{\vec{m}_c}{m_D} \vec{r}' + \frac{br}{\pi m_d} \sqrt{\frac{2}{N+1}} \sum_p \frac{(-1)^p}{p} \vec{a}_p' \right) \right| \bar{d} \left(\vec{R} + \frac{\vec{m}_b}{m_B} \vec{r} + \frac{br}{\pi m_d} \sqrt{\frac{2}{N+1}} \sum_p \frac{(-1)^p}{p} \vec{a}_p \right) \right\rangle \\ &\times \prod_p \langle \{\vec{a}_p'\} | \{\vec{a}_p\} \rangle \\ &\times \left\langle c \left(\vec{R}' - \frac{\vec{m}_d}{m_D} \vec{r}' - \frac{br}{\pi m_c} \sqrt{\frac{2}{N+1}} \sum_p \frac{1}{p} \vec{a}_p' \right) \right| e^{-i\vec{q} \cdot \vec{r}_b} \left| b \left(\vec{R} - \frac{\vec{m}_d}{m_B} \vec{r} - \frac{br}{\pi m_b} \sqrt{\frac{2}{N+1}} \sum_p \frac{1}{p} \vec{a}_p \right) \right\rangle. \end{aligned}$$

The first two *bra-ket* overlaps indicate that the $\bar{d}$ does not instantaneously change its position when the b -quark changes mass and that the flux-tube Fourier modes do not change. The second statement is certainly true in the adiabatic approximation where the flux-tube mode spectrum is independent of the quark state, but in general it will receive non-adiabatic corrections at higher order in br/m . The overlaps supply delta-functions such that we can trivially integrate out the redundant variables $\vec{R}', \vec{r}, \{\vec{a}_p'\}$ and the remaining centre-of-mass variable leaving

$$\begin{aligned} \delta^3(\vec{p}_B - \vec{p}_D - \vec{q}) \\ \times \int d^3 \vec{r} \int \prod_p d^2 \vec{a}_p \chi_D^*(a) \dots \psi_D^*(\vec{r}) \exp -i\vec{q} \cdot \left[-\frac{\vec{m}_d}{m_D} \vec{r} - \frac{br}{\pi m_c} \sqrt{\frac{2}{N+1}} \sum_p \frac{1}{p} \vec{a}_p \right] \\ \times \psi_B \left(\vec{r} + \frac{br}{\pi} \sqrt{\frac{2}{N+1}} \left(\frac{1}{m_c} - \frac{1}{m_b} \right) \sum_p \frac{1}{p} \vec{a}_p \right) \chi_B(a) \dots \end{aligned}$$

The argument of the B -meson spatial wavefunction depends upon the flux-tube mode variables, which will make explicit computation of the matrix element rather difficult. Isgur [39] ensures that such a term is not present by working in the heavy-quark symmetry limit so that $m_c, m_b = m_Q \rightarrow \infty$. In this case $m_c^{-1} - m_b^{-1}$ represents an explicit breaking of this symmetry and is neglected. We find another reason to neglect this term in our calculations that will be valid even for light quarks. We will work only to lowest order in br/m and as such we expand ψ_B

in a power series in br/m ,

$$\begin{aligned}\psi_B(\vec{r} + \vec{x}) &= \psi_B(\vec{r}) + \vec{x} \cdot \vec{\nabla}_r \psi_B(\vec{r}) + \mathcal{O}(x^2) \\ &= \psi_B(\vec{r}) + f(r) \sum_p \hat{r} \cdot \vec{a}_p \psi_B(\vec{r}) + \mathcal{O}(br/m)^2,\end{aligned}$$

where we have used the fact that B is an internal S -wave to set $\vec{\nabla}_r \psi_B(\vec{r}) \sim \hat{r}$. The $\mathcal{O}(br/m)$ term does not contribute because at lowest order in br/m , $\vec{r}$ and $\vec{a}_p$ are orthogonal. We are left with just $\psi_B(\vec{r})$ and $\mathcal{O}(br/m)^2$ corrections, which we neglect.

Exciting a hybrid then will involve extracting the terms linear in $\vec{a}$ in the non-relativistic reduction of the weak current which break the orthogonality of χ_1 and χ_0 . In Appendix C we present the currents responsible for exciting hybrids.

4.3.1 $\overline{B}^0 \rightarrow \pi^- D_H^+$

We considered earlier the decays $\overline{B}^0 \rightarrow \pi^- D^{(*)+}$ as examples of the reasonable success of the generalised factorisation model with quark model form-factors. We can now consider the analogous decays to hybrid D mesons $\overline{B}^0 \rightarrow \pi^- D_H^+$ using the flux-tube model. This mode is actually not particularly interesting phenomenologically as the flavoured D -mesons are not eigenstates of C and hence cannot have manifestly exotic quantum numbers. As such any state seen in this channel could be a conventional meson. Nevertheless we will compute branching ratios to the spin-singlet hybrids as a simple demonstration of the method.

The matrix element for such a decay is the hybrid analogue of equation(4.3). The hybrid currents can be read off from equations (C.3, C.5, C.7, C.9) in Appendix C and give, for the maximally parity violating current,

$$\begin{aligned}q \cdot V_H = e^{i\vec{q} \cdot \vec{r} \frac{m_d}{m_D}} \sqrt{\frac{2b}{\pi^3}} \beta_1 |\vec{q}| \left\{ i\vec{a} \cdot \hat{z} \left[\frac{\pi}{2m_b} \left(1 + \frac{m_b}{m_c} + \frac{|\vec{q}|}{2m_c} \right) + |\vec{q}| \frac{r}{m_c} \left(1 + \frac{|\vec{q}|}{2m_c} \right) \right] \right. \\ \left. + \vec{\sigma} \cdot \hat{z} \times \vec{a} \frac{\pi}{2m_b} \left(-1 + \frac{m_b}{m_c} - \frac{|\vec{q}|}{2m_c} \right) \right\} \quad (4.10)\end{aligned}$$

and for the parity conserving current,

$$\begin{aligned}q \cdot A_H = e^{i\vec{q} \cdot \vec{r} \frac{m_d}{m_D}} \sqrt{\frac{2b}{\pi^3}} \beta_1 |\vec{q}| \left\{ -i\vec{\sigma} \cdot \vec{a} \frac{\pi}{2m_b} \left(1 + \frac{m_b}{m_c} + \frac{|\vec{q}|}{2m_c} \right) \right. \\ \left. - i|\vec{q}| \sigma_z \vec{a} \cdot \hat{z} \frac{r}{m_c} \left(1 + \frac{|\vec{q}|}{2m_c} \right) \right\}\end{aligned}$$

having approximated $q^2 = m_\pi^2 \approx 0$, as appropriate for B decays.

The integrals over $\vec{a}$ and angles θ, ϕ are performed exactly as in the radiative case, Section 3.2. For the spin-singlet hybrids $D_H(1^{\pm(\pm)})$, $\langle \vec{\sigma} \rangle = 0$ and the matrix elements are $\mathcal{M}(\overline{B}^0 \rightarrow \pi^- D_H^+(1^{-(\cdot)})) = 0$ and

$$\begin{aligned} \mathcal{M}(\overline{B}^0 \rightarrow \pi^- D_H^+(1^{+(+)}) &= -i f_\pi \sqrt{4m_B m_D} \sqrt{\frac{2b}{3\pi^3}} |\vec{q}| \\ &\times \left\{ \frac{\pi}{2m_b} \left(1 + \frac{m_b}{m_c} + \frac{|\vec{q}|}{2m_c} \right) (\langle j_0 \rangle + \langle j_2 \rangle) + \frac{3(m_c + m_d)}{m_c m_d} \left(1 + \frac{|\vec{q}|}{2m_c} \right) \langle j_1 \rangle \right\}, \end{aligned}$$

where $\langle j_L \rangle = \langle R_{\mathcal{H}} | j_L(|\vec{q}| r \frac{m_d}{m_d + m_c}) | R_{\mathcal{C}} \rangle$.

That the $1^{-(\cdot)}$ amplitude is zero was to be expected as one cannot maximally violate parity and conserve angular momentum in a process $0^- \rightarrow 1^- 0^-$ in any partial wave, whereas the $1^{+(+)}$ amplitude is non-zero as $0^- \rightarrow 1^+ 0^-$ in a P -wave respects the symmetries.

Using the parameters and wavefunctions of Appendix B we can evaluate the branching ratio for this decay (equation (4.5) gives the width in terms of the matrix element). We find, for an axial hybrid D at 3 GeV,

$$\mathcal{B}(\overline{B}^0 \rightarrow \pi^- D_H^+(1^{+(+)}) \sim 4 \times 10^{-4}. \quad (4.11)$$

Note that this is an order of magnitude smaller than the conventional rates quoted in equation(4.6). Given this suppression and the fact that we would expect a hybrid like this to be rather broad (it can decay strongly by flux-tube breaking into a conventional D -meson and a light meson with several channels likely to be kinematically open), it is rather unlikely that such a state would be easily extracted from experimental data.

The spin-triplet matrix elements can be calculated similarly by constructing the appropriate L, S coupling with Clebsch-Gordan coefficients. Numerically the branching ratios do not exceed that predicted for the spin-singlet axial above.

The potential weakness in these estimates is the form-factor, which in this model is obtained from matrix elements of the spherical Bessel functions with the Gaussian wavefunctions of Appendix B and are polynomials in $|\vec{q}|$ times an exponential. They are of course sensitive to the quality of the radial wavefunctions which we have only estimated approximately - for example, at large $|\vec{q}|$ we are sampling the small r behaviour of the wavefunctions and hence the short-distance potential, which we could only guess was of Coulomb form. In a sequel to the ISGW paper, [55], a power-law form for the form-factor was found to give better agreement with the experimental inclusive semileptonic decay spectra. The quality of the form-factors at large $|\vec{q}|$ is likely to be the limiting factor in any numerical prediction.

We can make qualitative and probably more general statements about the relative sizes of hybrid production rates by examining their internal structure and the spin-structure of the currents. In Appendix C we see that the vector current is principally $\vec{\sigma}$ -independent with $|\vec{q}|/m$ -suppressed $\vec{\sigma}$ terms. The axial current on the other hand is leading in $\vec{\sigma}$. Thus we expect that where the decay is to a spin-singlet hybrid, the vector current dominates whereas to a spin-triplet it is the axial current that contributes most.

A state with $J^P = 1^+$ has Lorentz covariant axial matrix element decomposition $ig\epsilon_{\mu\nu\rho\sigma}\epsilon^{*\nu}(p_B + p_X)^\rho q^\sigma$ and hence with a pion in the decay ($\langle\pi|\gamma^\mu\gamma^5|0\rangle = if_\pi q^\mu$) there can be no axial current. This is simply the statement of conservation of angular momentum; this is a parity conserving term which requires the end state to have $\mathcal{P} = -$ which is not possible in the required P -wave. The vector current can contribute and as we have seen is large for the spin-singlet $1^{+(+)}$. Explicit computation confirms our expectation that the spin-triplet axial $1^{+(-)}$ branching ratio is relatively small ($\mathcal{O}(10^{-6})$).

For a state with $J^P = 1^-$, conservation of angular momentum forbids the vector current, and we get a reversal of the spin-dependence with the spin-triplet $1^{-(+)}$ being larger than the spin-singlet $1^{-(-)}$ (which to the order presented in equations (C.7, C.9) is zero).

Of the remaining spin-triplet states $(0, 2)^{\pm(\mp)}$, the positive parity states have the larger branching ratios as their decays proceed through the leading $\vec{\sigma}$ terms of the axial current. Explicit calculation in the flux-tube model confirms these expectations.

4.3.2 $B^+ \rightarrow J/\psi K_H^+$ and the “Excess” in the Inclusive Decay $B \rightarrow J/\psi X$

The channel $B^+ \rightarrow J/\psi K_H^+$ does not have exotic quantum numbered hybrids but is more phenomenologically interesting than the previous case. The spectrum of the inclusive decay $B \rightarrow J/\psi X$ has been measured [56] at BaBar and the authors claim that at low J/ψ momentum there is an excess of events above what one would expect in models. The momentum range of the excess corresponds to X masses around 2 GeV and it has been previously suggested that a possible contributor could be the production of a kaonic hybrid in the $B \rightarrow J/\psi K_H$ process [57]. We have the calculational tools to estimate the branching ratio for this process.

We will use our general current structure arguments to lead us to the most likely candidate. The J/ψ is created by a vector current, $\langle J/\psi(\vec{q}, \vec{\epsilon}_\psi) | \gamma^\mu | 0 \rangle = m_\psi f_\psi \epsilon_\psi^\mu$, where ϵ_ψ can be longitudinal $\epsilon_\psi^\mu(L) = m_\psi^{-1}(|\vec{q}|, 0, 0, E_\psi)$, or transverse $\epsilon_\psi^\mu(\pm) = \mp \frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$.

The spin-singlet vector hybrid kaon isn't produced in longitudinal polarisation - the vector

current matrix element decomposition $\sim \epsilon_{\mu\nu\rho\sigma}\epsilon^{\nu*}(p_B + p_X)^\rho q^\sigma$ can be evaluated in the B -rest frame so that p_B^μ only has a $\mu = 0$ component and with the J/ψ moving along the z -axis we see that only transverse ϵ can contribute. The axial current is leading in $\vec{\sigma}$ and the sub-leading $\vec{\sigma}$ -independent term requires transverse ϵ . The transverse process proceeds in a P -wave.

The spin-singlet axial hybrid kaon can be produced in both longitudinal and transverse polarisations and for the vector current is in an S -wave. As such we expect the $J/\psi K_H(1^+)$ channel to be considerable in B -decay. We will explicitly compute the branching fraction in the flux-tube model.

For an axial kaon hybrid at 2 GeV, $|\vec{q}| \sim 0.7$ GeV and $E_\psi \sim m_\psi$. We can neglect $\langle j_2 \rangle$ next to $\langle j_0 \rangle$ at this $|\vec{q}|$, and also $|\vec{q}|/2m_s$ against $1 + m_b/m_s$. With these approximations the longitudinal matrix element is

$$\mathcal{M}_L = -if_\psi \sqrt{\frac{2b}{3\pi^3}} \sqrt{4m_B m_K} \left\{ \frac{3|\vec{q}|(m_d + m_s)}{m_d m_s} \left(\frac{m_\psi}{2m_s} + 1 \right) \langle j_1 \rangle + \frac{\pi m_\psi}{2m_b} \left(\frac{m_b}{m_s} + 1 \right) \langle j_0 \rangle \right\},$$

and the transverse matrix element is

$$\mathcal{M}_\pm = if_\psi \sqrt{\frac{2b}{3\pi^3}} \sqrt{4m_B m_K} \frac{\pi m_\psi}{2m_b} \left(\frac{m_b}{m_s} + 1 \right) \langle j_0 \rangle.$$

Summing over all three helicities in the width we obtain, for a spin-singlet axial hybrid kaon at 2 GeV a branching ratio

$$\mathcal{B}(B^+ \rightarrow J/\psi K_H) \sim 2 \times 10^{-4}. \quad (4.12)$$

For the spin-triplet hybrids, the axial current which is leading in $\vec{\sigma}$, produces the largest rates. This current can only act (while conserving angular momentum) for the positive parity hybrids which appear in P -waves, and as such are likely to be somewhat suppressed relative to the positive parity spin-singlet which is in an S -wave. •

From Figure 10 of [56](reproduced here as Figure 4.5), the supposed excess of events corresponds to a branching ratio of a few times 10^{-4} at a mass between 2 and 2.2 GeV. One or more positive parity hybrid kaons in this mass range with branching ratios of $\mathcal{O}(10^{-4})$ would indeed saturate this excess. However we should also consider more conventional explanations, the most obvious being that the NRQCD fit used in [56] is incorrect or not valid over the whole kinematic region. We could instead take the line of ISGW [48] that inclusive spectra are well described by summing over hadronic resonances with negligible non-resonant background. In

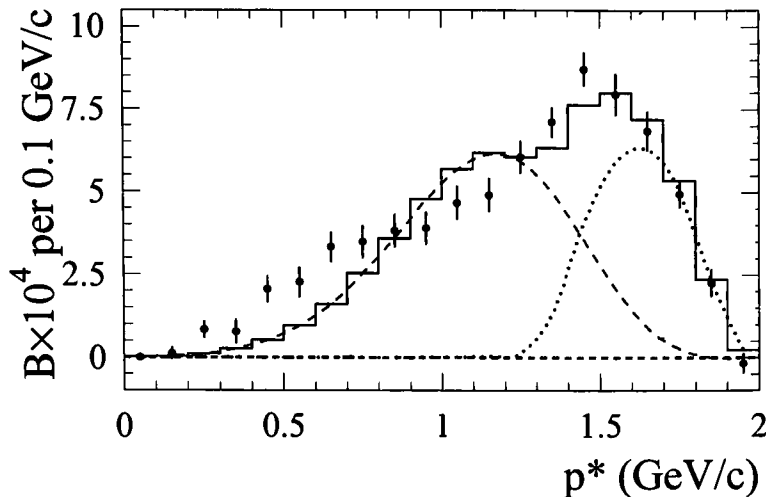


Figure 4.5: p^* of J/ψ mesons produced directly in B decays (points). The histogram is the sum of the color-octet component from a recent NRQCD calculation (dashed line) and the color-singlet $J/\psi K^{(*)}$ component from simulation (dotted line).

this case we would have the K and K^* dominating the spectrum at large p^* , the intermediate region would have contributions from the P -wave kaons, $K_{0,1,2}$ and perhaps the radially excited K, K^* and we would require excited states still to explain the events at very low p^* . We have seen that in the flux-tube model the positive parity hybrids and in particular the spin-singlet axial state can contribute, but there are also expected to be conventional quark model resonances around 2 GeV, for example the radial excitation of the P -wave kaon states [58]. With ISGW-like decompositions for a 2^1P_1 kaon at 2 GeV, we find a branching ratio $\sim 9 \times 10^{-5}$ but this is rather sensitive to the form of the radial wavefunction which we parameterised by the appropriate harmonic oscillator wavefunction. Higher internal angular momentum states might contribute, but are likely to be suppressed by orthogonality constraints at the relatively low $|\vec{q}|$.

4.3.3 Other Hybrid Channels

The most interesting channels for hybrid searches are those in which the produced hybrids are flavour singlets and hence can have exotic J^{PC} . One such channel is $\overline{B}^0 \rightarrow D^{(*)0}(n\bar{n})_{\mathcal{H}}$, which is a Class II decay with the D -meson produced from the vacuum. It has in its favour a rather large phase space for production of even relatively heavy light-quark hybrids and no small CKM matrix elements relative to conventional B decay. However, there is theoretical evidence [59] that factorisation is likely to be a poor approximation in decays of this type and in addition, given the large values of $|\vec{q}|$ we will have for light hybrids, we will not be able to trust our form-factor estimates to any great degree. As such we do not place much

faith in the numerical prediction of such a decay rate in the flux-tube model, but performing the computation nonetheless we find that for the $D^0\pi_1(1600)$ end state we have a branching ratio of $\mathcal{O}(10^{-5})$, which is only slightly less than the measured $D^0(\rho,\omega)$ rates. A search for $\pi_1(1600)$ could be performed in the $D^0\pi^+\pi^-\pi^0$ end state. CLEO II has observed events $\overline{B^0} \rightarrow D^{0*}\pi^+\pi^-\pi^+\pi^-$ [60] which is a possible end state for light-quark hybrids decaying via $a_1^\pm\pi^\mp$ for example. According to the flux-tube breaking model of hybrid hadronic decays [38] isovector $2^{+-}, 1^{+-}, 0^{+-}, 1^{--}$ and isoscalar $1^{-+}, 1^{++}$ hybrids have large branching ratios to $a_1\pi$. Unfortunately the CLEO II statistics are not good enough to clearly resolve any structures in the 4π invariant mass spectrum.

D -meson decays such as the Class I Cabibbo suppressed $D^0 \rightarrow \pi^+(n\bar{n})_{\mathcal{H}}$ and the Cabibbo favoured $D_s \rightarrow \pi^+(s\bar{s})_{\mathcal{H}}$ would be entrees into flavourless hybrids were it not for the limited phase space. Even for the relatively light $\pi_1(1600)$, the need for a P -wave in the end-state renders this channel unfavoured, explicit calculation in the flux-tube model giving a branching fraction 10,000 times smaller than the measured $D \rightarrow \pi\rho$.

4.4 Summary

We have considered hadronic decays of the B -meson within a generalised factorisation model. In the conventional sector, using the form-factor parameterisations of ISGW, we found reasonable agreement with experimental decay rates and extracted a $K_1(1400), K_1(1270)$ mixing angle in line with previous estimates.

In the hybrid sector, using the flux-tube model, we estimated rates for decays of the B -meson and found them to be suppressed with respect to conventional channels, but in favourable cases only by around one order of magnitude. The “excess” reported by BaBar in $B \rightarrow J/\psi X$ was shown to have a possible explanation in terms of production of a hybrid kaon.

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Chapter 5

Hybrid Meson Decays in an Elementary Pion Emission Model

Hybrid mesons, like conventional meson resonances, are expected to decay mostly to quasi-two-body states which are pairs of lighter mesons. For conventional light-quark mesons a common decay is $M \rightarrow \rho \pi$ and in this chapter we will consider whether such a decay channel is favoured in hybrid meson decay.

The flux-tube breaking model [33, 37, 38, 41, 42], in the limit of identical π , ρ spatial wave-functions predicts that this decay mode is zero for any light hybrid, and the absence of a $\rho\pi$ decay is often taken as being a signature of hybrid character. The assumption that the wave-functions are identical is a rather strong one, as we have considerable reason to suspect that the pion has different internal character to the ρ , not least as reflected in its abnormally low mass. While we will not discuss the underlying reasons for this difference here, we will take the opposite, asymmetrical, extreme and propose that in hadronic decays the pion acts as a pointlike current, transforming as a pseudoscalar or the divergence of an axial current (to the order of the non-relativistic expansion we use they are identical, see Appendix D).

Such a model is known as “elementary pion emission” and has a considerable phenomenological pedigree in the conventional sector (see e.g. [61]). In extending it to include hybrid decays will follow the same basic method outlined in the previous two chapters we simply include the flux-tube degrees-of-freedom and since the quark transverse displacement can be excited by the current we can excite the flux-tube.

Firstly we will derive the matrix element for a meson decay $M_i \rightarrow M_f \pi$. Emission of a π^+

by the quark in a meson($q_i\bar{q}$) has matrix element

$$\begin{aligned}\mathcal{M}(q_i \rightarrow q_f + \pi^+) &= \int d^3\vec{p} \int d^3\vec{p}' \phi_f^*(\vec{p}') \phi_i(\vec{p}) \left\langle \bar{q}(\tfrac{1}{2}\vec{P}_f + \vec{p}') \left| \bar{q}(\tfrac{1}{2}\vec{P}_i + \vec{p}) \right. \right\rangle \\ &\quad \times \left\langle q_f(\tfrac{1}{2}\vec{P}_f - \vec{p}') \left| \frac{g}{2m} \int d^3\vec{r} \bar{\psi}(\vec{r}) \Gamma \frac{\tau_-}{\sqrt{2}} \psi(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} \right| q_i(\tfrac{1}{2}\vec{P}_i - \vec{p}) \right\rangle,\end{aligned}$$

where Γ is the Dirac structure of the current, $\frac{\tau_-}{\sqrt{2}}$ is the isospin lowering operator, and the $\phi(\vec{p})$ are the internal momentum wavefunctions. g is the pion-quark-quark coupling constant which we will determine by fitting conventional meson decay rates.

Expanding the Dirac spinors ψ in terms of quark creation and annihilation operators and retaining only those which will annihilate q_i and create q_f gives

$$\begin{aligned}\int d^3\vec{r} \bar{\psi}(\vec{r}) \Gamma \frac{\tau_-}{\sqrt{2}} \psi(\vec{r}) e^{-i\vec{q}\cdot\vec{r}} &= \int d^3\vec{r} \int d^3\vec{k} \int d^3\vec{k}' a^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{r}} \bar{u}(\vec{k}) \Gamma u(\vec{k}') e^{i\vec{k}'\cdot\vec{r}} a(\vec{k}') e^{-i\vec{q}\cdot\vec{r}} \\ &= \delta(\vec{k}' - \vec{k} - \vec{q}) \int d^3\vec{k} \int d^3\vec{k}' a^\dagger(\vec{k}) \bar{u}(\vec{k}) \Gamma u(\vec{k}') a(\vec{k}'),\end{aligned}$$

where we have suppressed spin and flavour labels. Using $a(\vec{k}')|q_i(\tfrac{1}{2}\vec{P}_i - \vec{p})\rangle = \delta(\vec{k}' - (\tfrac{1}{2}\vec{P}_i - \vec{p}))|0\rangle$ and the equivalent for the end state we obtain, after integrating out the redundant $\vec{k}, \vec{k}', \vec{p}'$,

$$\mathcal{M}(q_i \rightarrow q_f + \pi^+) = \delta(\vec{P}_i - \vec{P}_f - \vec{q}) \frac{g}{2m} F(q_i, q_f) \int d^3\vec{p} \phi_f^*(\vec{p} + \vec{q}/2) \phi_i(\vec{p}) [\bar{u}(\vec{P}_f - \vec{P}_i/2 - \vec{p}) \Gamma u(\vec{P}_i/2 - \vec{p})].$$

If we work in the rest frame of the initial meson

$$\mathcal{M}(q_i \rightarrow q_f + \pi^+) = \frac{g}{2m} F(q_i, q_f) \int d^3\vec{p} \phi_f^*(\vec{p} + \vec{q}/2) \phi_i(\vec{p}) [\bar{u}(-\vec{q} - \vec{p}) \Gamma u(-\vec{p})],$$

where $F(q_i, q_f) = \langle q_f | \frac{\tau_-}{\sqrt{2}} | q_i \rangle$ is a flavour factor accounting for isospin conservation at the pion-quark-quark vertex. The object in square brackets is evaluated in Appendix D for both pseudoscalar and divergence of axial currents. For either of these the net result is

$$\mathcal{M}(q_i \rightarrow q_f + \pi^+) = -\frac{g}{2m} F(q_i, q_f) \int d^3\vec{p} \phi_f^*(\vec{p} + \vec{q}/2) \phi_i(\vec{p}) [\vec{\sigma} \cdot \vec{q} (1 + \frac{g^0}{2m}) + \frac{g^0}{m} \vec{\sigma} \cdot \vec{p} + \dots].$$

The equivalent process for emission by the antiquark yields

$$\mathcal{M}(\bar{q}_i \rightarrow \bar{q}_f + \pi^+) = +\frac{g}{2m} F(\bar{q}_i, \bar{q}_f) \int d^3\vec{p} \phi_f^*(\vec{p} - \vec{q}/2) \phi_i(\vec{p}) [\vec{\sigma} \cdot \vec{q} (1 - \frac{g^0}{2m}) - \frac{g^0}{m} \vec{\sigma} \cdot \vec{p} + \dots].$$

These expressions are more use to us in position space; Fourier transforming with $\phi(\vec{p}) =$

$\int d^3\vec{r} e^{-i\vec{p}\cdot\vec{r}} \psi(\vec{r})$ gives

$$\mathcal{M}(q_i^* \rightarrow q_f^* \pi^+) = \mp \frac{g}{2m} F(q_i^*, q_f^*) \int d^3\vec{r} \psi_f^*(\vec{r}) [\vec{\sigma} \cdot \vec{q} \mp \frac{q^0}{m} \vec{\sigma} \cdot \vec{p}'] e^{\pm i\vec{q}\cdot\vec{r}/2} \psi_i(\vec{r}) \quad (5.1)$$

where the operator $\vec{p}'$ is $-i\vec{\nabla}_{\vec{r}}$ acting backwards onto the final state wavefunction.

Compare equation (5.1) with equation (19) in [58], which has the opposite sign definition for $\vec{r}$. Their decay widths are in terms of two independent form-factor parameters, g, h which they fit to data. Our approach determines for example h in terms of g and we hence have less freedom to fit, but more predictive power.

Using the representation $u \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $d \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\bar{u} \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\bar{d} \sim \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and the explicit form $\frac{\tau^-}{\sqrt{2}} \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ we see that the only non-zero flavour factors are $F(u, d) = +1$ and $F(\bar{d}, \bar{u}) = -1$. We will see that equation (5.1) with these charge factors correctly conserves isospin and G -parity.

With our normalisations, the partial width of a meson of spin- J is given by

$$\Gamma(M_J \rightarrow \pi V) = \frac{C}{2J+1} \frac{|\vec{q}|}{2\pi} \sum |\mathcal{M}|^2, \quad (5.2)$$

where the sum is over quark and anti-quark emission and either helicities or partial waves. C is the number of end-state charge possibilities (e.g. $C = 2$ for isovector to $\rho\pi$, $C = 1$ for isovector to $\omega\pi$).

We now have all the tools required to calculate decay widths. We will demonstrate the method with the important channel $b_1 \rightarrow \omega\pi$.

5.1 $b_1 \rightarrow \omega\pi$ in the pion emission model

Considering first the spin and spatial dependence and leaving for now the flavour dependence we can write for the amplitude

$$A_{q,\bar{q}} = \int r^2 dr \int d\Omega R_f^*(r) \frac{1}{\sqrt{4\pi}} \langle S=1, m_s | [\vec{\sigma}_{q,\bar{q}} \cdot \vec{q} \mp \frac{q^0}{m} \vec{\sigma}_{q,\bar{q}} \cdot \vec{p}'] | S=0 \rangle e^{\pm i\vec{q}\cdot\vec{r}/2} Y_1^{m_L}(\Omega) R_i(r).$$

Evaluation of the angular integral is easiest if we expand $e^{\pm i\vec{q}\cdot\vec{r}/2} = \sum_L (\pm i)^L \sqrt{4\pi(2L+1)} Y_L^{0*} j_L(|\vec{q}|r/2)$ and $\vec{\sigma} \cdot \vec{p}' = \frac{1}{2}(\sigma_+ p'_- + \sigma_- p'_+) + \sigma_z p'_z$, where $v_{\pm} = \vec{v} \cdot (\hat{x} \pm i\hat{y})$. The spin matrix elements are

$$\langle S=1, m_s | \sigma_{+, -, z} | S=0 \rangle = \begin{cases} -\sqrt{2}\delta(m_s, +1) \\ \sqrt{2}\delta(m_s, -1) \\ \delta(m_s, 0) \end{cases},$$

and for $S=0 \rightarrow S=1$ transitions $\vec{\sigma}_{\vec{q}} = -\vec{\sigma}_{\vec{q}}$.

$\vec{p}'$ acts back onto the final state wavefunction which is an S -wave, hence we can write

$$\begin{aligned} p'_{\pm} &= \mp i \sqrt{\frac{8\pi}{3}} Y_1^{\pm 1} \overleftarrow{\frac{\partial}{\partial r}} \\ p'_z &= -i \sqrt{\frac{4\pi}{3}} Y_1^0 \overleftarrow{\frac{\partial}{\partial r}}. \end{aligned}$$

Integrating the product of three Y_L^m 's gives Clebsch-Gordan coefficients which can be evaluated giving us the helicity amplitudes

$$\begin{aligned} A_{q,\vec{q}}(0) &= i \frac{g}{2m} \frac{|\vec{q}|}{\sqrt{3}} \left[3\langle j_1 \rangle + \frac{\langle \overleftarrow{\partial} j_0 \rangle}{m} - 2 \frac{\langle \overleftarrow{\partial} j_2 \rangle}{m} \right] \\ A_{q,\vec{q}}(\pm) &= i \frac{g}{2m} \frac{|\vec{q}|}{\sqrt{3}} \left[\frac{\langle \overleftarrow{\partial} j_0 \rangle}{m} + \frac{\langle \overleftarrow{\partial} j_2 \rangle}{m} \right], \end{aligned}$$

where $\langle \overleftarrow{\partial} j_L \rangle$ is shorthand for $\int r^2 dr \frac{dR_f^*}{dr} j_L(|\vec{q}|r/2) R_i$.

Partial wave amplitudes can be constructed according to Table XI of [58]. We find

$$\begin{aligned} A_{q,\vec{q}}(S) &= i \frac{g}{2m} |\vec{q}| \left[\langle j_1 \rangle + \frac{\langle \overleftarrow{\partial} j_0 \rangle}{m} \right] \\ A_{q,\vec{q}}(D) &= -i \sqrt{2} \frac{g}{2m} |\vec{q}| \left[\langle j_1 \rangle - \frac{\langle \overleftarrow{\partial} j_2 \rangle}{m} \right]. \end{aligned}$$

Including the flavour factors for the decay of b_1^+ we obtain a non-zero matrix element for the end state $\omega\pi^+$ only, and not for example, $\rho^+\pi^0$, in line with conservation of isospin and G -parity. The D/S amplitude ratio then for this decay is

$$\frac{D}{S}(b_1 \rightarrow \omega\pi) = -\sqrt{2} \frac{\langle j_1 \rangle - \frac{\langle \overleftarrow{\partial} j_2 \rangle}{m}}{\langle j_1 \rangle + \frac{\langle \overleftarrow{\partial} j_0 \rangle}{m}}, \quad (5.3)$$

<i>S</i> -waves	<i>D</i> -waves	
$\Gamma_S(a_1 \rightarrow \rho\pi) = \frac{8}{3}\Sigma$	$\Gamma_D(a_1 \rightarrow \rho\pi) = \frac{4}{3}\Delta$	$\Sigma \equiv \frac{ \vec{q} ^3}{2\pi} \left(\frac{q}{2m}\right)^2 \left(\langle j_1 \rangle + \frac{\langle \vec{\partial} j_0 \rangle}{m}\right)^2$
$\Gamma_S(b_1 \rightarrow \omega\pi) = \frac{2}{3}\Sigma$	$\Gamma_D(b_1 \rightarrow \omega\pi) = \frac{4}{3}\Delta$	$\Delta \equiv \frac{ \vec{q} ^3}{2\pi} \left(\frac{q}{2m}\right)^2 \left(\langle j_1 \rangle - \frac{\langle \vec{\partial} j_2 \rangle}{m}\right)^2$
	$\Gamma_D(a_2 \rightarrow \rho\pi) = \frac{12}{5}\Delta$	
<i>P</i> -waves	<i>F</i> -waves	
$\Gamma_P(\pi_2 \rightarrow \rho\pi) = \frac{8}{5}\Pi$	$\Gamma_F(\pi_2 \rightarrow \rho\pi) = \frac{12}{5}\Phi$	$\Pi \equiv \frac{ \vec{q} ^3}{2\pi} \left(\frac{q}{2m}\right)^2 \left(\langle j_2 \rangle + \frac{\langle \vec{\partial} j_1 \rangle}{m}\right)^2$
	$\Gamma_F(\omega_3 \rightarrow \rho\pi) = \frac{24}{7}\Phi$	$\Phi \equiv \frac{ \vec{q} ^3}{2\pi} \left(\frac{q}{2m}\right)^2 \left(\langle j_2 \rangle - \frac{\langle \vec{\partial} j_3 \rangle}{m}\right)^2$
	$\Gamma_F(\rho_3 \rightarrow \omega\pi) = \frac{8}{7}\Phi$	

Table 5.1: πV decay widths for $L = 1, 2$ conventional light-quark mesons in the pion emission model.

and the partial widths are

$$\Gamma(b_1 \rightarrow \omega\pi)_S = g^2 \frac{|\vec{q}|^3}{m^2} \frac{1}{12\pi} \left| \langle j_1 \rangle + \frac{\langle \vec{\partial} j_0 \rangle}{m} \right|^2$$

$$\Gamma(b_1 \rightarrow \omega\pi)_D = g^2 \frac{|\vec{q}|^3}{m^2} \frac{1}{6\pi} \left| \langle j_1 \rangle - \frac{\langle \vec{\partial} j_2 \rangle}{m} \right|^2.$$

A sensitive test of hadron decay models comes from comparing the D/S amplitude ratios for the decays $b_1 \rightarrow \omega\pi, a_1 \rightarrow \rho\pi$ with the experimental world averages $+0.277(27), -0.108(16)$ [7]. For the $a_1 \rightarrow \rho\pi$ decay in this model we obtain $\frac{D}{S}(a_1 \rightarrow \rho\pi) = -\frac{1}{2} \frac{D}{S}(b_1 \rightarrow \omega\pi)$, which is also found in the $^3P_0$ model and the “ sKs ” and “ $j^0 K j^0$ ” models discussed in [62].

Using wavefunctions obtained variationally from the Isgur-Paton meson Hamiltonian (“IP”) (see Appendix B) we obtain ratios $+0.45$ and -0.22 for $\frac{D}{S}(b_1 \rightarrow \omega\pi), \frac{D}{S}(a_1 \rightarrow \rho\pi)$, which have the right sign but are roughly a factor of 2 too large in magnitude. A standard approximation [62] is to describe the mesons by harmonic oscillator wavefunctions with a single β value for all states. If we do this and fit to the experimental b_1 ratio we find $\beta = 0.39(3)\text{GeV}$, which is considerably larger than $\bar{\beta} \approx 0.31\text{GeV}$ in “IP” but is in good agreement with $\beta \approx 0.4$ in [62].

That the effective β is larger than our “IP” value may be due to our pointlike pion approximation. Experimentally the pion is not pointlike, it has a charge radius comparable with other light mesons and our radial wavefunction overlap should really take some account of this. Quite possibly we are feeling this in the increased β , which has subsumed the effect of the pion wavefunction¹

5.2 Conventional meson decays to πV

The π_L^0 decays of the spin-triplet $L = 1$ mesons can be computed in this model, the results being displayed in Table 5.1. Another precision test of the decay model is the F/P ratio of the $\pi_2(1670) \rightarrow \rho\pi$ decay, which has recently been measured for the first time by the E852 collaboration who find $F/P = -0.72 \pm 0.07 \pm 0.14$ [35]. In the pion emission model we find

$$\frac{F}{P} = -\sqrt{\frac{3}{2}} \frac{\langle j_2 \rangle - \frac{\langle \vec{\partial} j_3 \rangle}{m}}{\langle j_2 \rangle + \frac{\langle \vec{\partial} j_1 \rangle}{m}}.$$

With “IP”, $\beta = 0.4$ wavefunctions this would equal $+0.57, +0.31$, neither of which is compatible with the experimental value. Since this is an $L = 2 \rightarrow L = 0$ transition we might expect there to be a different effective β , hence we can attempt to fit β using the experimental value. With equal β harmonic oscillator wavefunctions

$$\begin{aligned} \langle j_2 \rangle &= \frac{1}{8\sqrt{15}} \frac{|\vec{q}|^2}{\beta^2} e^{-\frac{|\vec{q}|^2}{16\beta^2}} \\ \langle \vec{\partial} j_1 \rangle &= \frac{1}{32\sqrt{15}} \frac{|\vec{q}|}{\beta^2} (|\vec{q}|^2 - 40\beta^2) e^{-\frac{|\vec{q}|^2}{16\beta^2}} \\ \langle \vec{\partial} j_3 \rangle &= -\frac{1}{32\sqrt{15}} \frac{|\vec{q}|^3}{\beta^2} e^{-\frac{|\vec{q}|^2}{16\beta^2}}, \end{aligned}$$

and hence

$$\frac{F}{P} = -\sqrt{\frac{3}{2}} \frac{1 + \frac{|\vec{q}|}{4m}}{1 + \frac{|\vec{q}|}{4m} - \frac{10\beta^2}{|\vec{q}|m}}, \quad (5.4)$$

which cannot be satisfied by any real β with the experimental numbers. So we see that the pion emission model as formulated here cannot accommodate the E852 value for F/P . The $^3P_0$ decay model [63–69] successfully accommodates the D/S ratios discussed earlier with parameter values which do a good job of describing a wide range of hadron decays. With the same parameters it predicts $F/P \sim +0.4$ [70] which is in the same region as the pion emission predictions and not compatible with experiment. If the E852 result is confirmed it casts some doubt over the validity of these commonly used hadron decay models, at least for high partial waves.

We present in Table 5.2 the partial widths for a number of meson decays computed in the model using $g = 2, 3$ for $\beta = 0.4$ wavefunctions and $g = 3$ for “IP” wavefunctions. Also shown are the predictions of the $^3P_0$ model taken from [62, 71] and experimental values taken from [7].

¹Note that the effective β values obtained by fitting the rms momentum of the wavefunctions of [58] are rather higher than the “IP” values also, so it may be that the simple “IP” Hamiltonian does not include an effect important in determining the size of states (for example the spin-dependence)

Mode	“IP” ($g = 3$)	$\beta = 0.4(g = 2, 3)$	$^3P_0$	Data
$\Gamma_S(a_1 \rightarrow \rho\pi)$	280	(255, 580)	530	150 $\rightarrow$ 360
$\Gamma_D(a_1 \rightarrow \rho\pi)$	14	(5,10)	15	3 $\rightarrow$ 8
$\Gamma_S(b_1 \rightarrow \omega\pi)$	70	(64,145)	132	< 132
$\Gamma_D(b_1 \rightarrow \omega\pi)$	14	(5,10)	11	< 10
$\Gamma_D(a_2 \rightarrow \rho\pi)$	52	(18,40)	54	75 $\pm$ 7
$\Gamma_{F+P}(\pi_2 \rightarrow \rho\pi)$	162	(131, 297)	118	81 $\pm$ 11
$\Gamma_F(\omega_3 \rightarrow \rho\pi)$	77	(16,36)	50	< 74
$\Gamma_F(\rho_3 \rightarrow \omega\pi)$	19	(5,12)	19	26 $\pm$ 13

Table 5.2: Numerical estimates of πV decay widths in MeV for $L = 1, 2$ conventional light-quark mesons in the pion emission model using wavefunction parameterisations as described in the text.

Unfortunately there is little precision data on π_L^ρ decays available, so the best we can say is that the pion emission model does a reasonable job of describing the data. We cannot accurately pin down the value of the coupling using the experimental data. Note also that despite their failure to predict the precision D/S ratios, the “IP” wavefunctions do as good a job overall of describing the data as the $\beta = 0.4$ wavefunctions.

5.3 Hybrid meson decays to πV - including the flux-tube degrees-of-freedom

We can derive the matrix element for decay in the same manner as we did for conventional decays, but now explicitly including the flux-tube degrees-of-freedom. We relegate the full derivation to Appendix D. Starting from equation (D.4) and considering the $p = 1$ hybrid decay to $\rho\pi$ we have, retaining only terms linear in $\vec{a}$,

$$\begin{aligned} \mathcal{M}_{q,\bar{q}} = & \mp ib \frac{g}{2m} F(q_i^i, q_f^f) \sqrt{\frac{2}{N+1}} \int d^3\vec{r} \int d^2\vec{a} \psi_\rho^*(\vec{r}) \chi_0^*(\vec{a}) \\ & \times \left\langle S = 1, m_\rho \left| \left(\vec{\sigma}_{q,\bar{q}} \cdot \vec{q} \mp \frac{q^0}{m} \vec{\sigma}_{q,\bar{q}} \cdot \vec{p}' \right) \frac{\vec{r}}{\pi m} \cdot \vec{q} \cdot \vec{a} - \frac{q^0}{m} \vec{\sigma}_{q,\bar{q}} \cdot \vec{a} \right| S, m'_s \right\rangle e^{\pm i\vec{q} \cdot \vec{r}/2} \psi_{m'}(\vec{r}) \chi_1(\vec{a}). \end{aligned} \quad (5.5)$$

The $\vec{\sigma} \cdot \vec{p}' \vec{q} \cdot \vec{a}$ term is formally suppressed at order $|\vec{q}|/m$ relative to the leading term and as such we will neglect it initially. We will return at the end of the chapter to consider the effect it and other neglected terms might have.

We have already detailed elsewhere in this thesis all the methods required to compute decay rates using this matrix element and as such we will move immediately to the results found, presented in Table 5.3.

We show in Figure 5.1, $\langle j_L \rangle$ as a function of $m_{\mathcal{H}}$ for “IP” and $\beta = 0.4$ wavefunctions. Note that only $\langle j_0 \rangle$ differs considerably between the two wavefunction choices and as such we

$\mathcal{P}_H = +$		$\Sigma_H \equiv g^2 \frac{ \vec{q} ^3}{m^2} \frac{b}{\pi^2 m^2} \left \frac{2}{\pi} \langle j_1 \rangle - \langle j_0 \rangle \right ^2$ $\Delta_H \equiv g^2 \frac{ \vec{q} ^3}{m^2} \frac{b}{\pi^2 m^2} \left \frac{4}{\pi} \langle j_1 \rangle - \langle j_2 \rangle \right ^2$ $\Pi_H \equiv g^2 \frac{ \vec{q} ^3}{m^2} \frac{b}{\pi^2 m^2} \langle j_1 \rangle ^2$
S-waves	D-waves	
$\Gamma_S(a_{1H} \rightarrow \rho\pi) = \frac{1}{3} \Sigma_H$	$\Gamma_D(a_{1H} \rightarrow \rho\pi) = \frac{1}{6} \Delta_H$	
$\Gamma_S(b_{1H} \rightarrow \omega\pi) = \frac{1}{3} \Sigma_H$	$\Gamma_D(b_{1H} \rightarrow \omega\pi) = \frac{1}{24} \Delta_H$	
	$\Gamma_D(b_{2H} \rightarrow \omega\pi) = \frac{3}{40} \Delta_H$	
$\mathcal{P}_H = -$		
P-waves	F-waves	
$\Gamma_P(\rho_H \rightarrow \omega\pi) = \frac{1}{4} \Pi_H$		
$\Gamma_P(\pi_H \rightarrow \rho\pi) = \Pi_H$		
$\Gamma_P(\pi_{1H} \rightarrow \rho\pi) = \frac{1}{4} \Pi_H$		
$\Gamma_P(\pi_{2H} \rightarrow \rho\pi) = \frac{1}{4} \Pi_H$		

Table 5.3: πV decay widths for $p = 1$ light-quark hybrid mesons in the pion emission/flux-tube model.

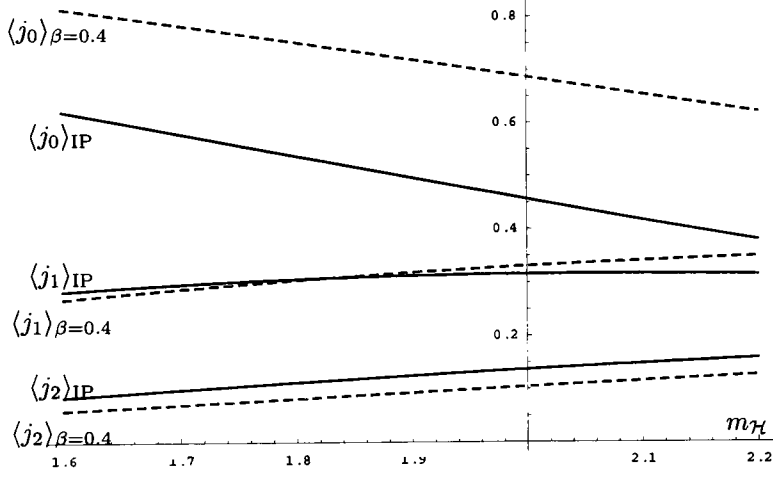


Figure 5.1: Transition matrix elements of the spherical Bessel functions, $\langle R_\rho(r) | j_L(|\vec{q}|r/2) | R_H(r) \rangle$, using wavefunction parameterisations as described in the text.

expect that while P and D -wave predictions will be rather robust with respect to wavefunction parameterisations, S -wave rates will be quite sensitive. We will quote rates predicted with both wavefunction choices where they differ considerably, and we will use $g = 3$ throughout.

5.3.1 Negative Parity Hybrids

1^{--}

The spin-singlet negative parity hybrid, $\rho_H \rightarrow \pi\omega$ and may be compared with the states $\rho(1460)$ and $\rho(1700)$ [7], which have been suggested to have hybrid vector meson content [72–74].

	Γ / MeV	b.r.
$\rho(1460) \rightarrow \omega\pi$	29	9 %
$\rho(1700) \rightarrow \omega\pi$	93	33 %

Although $\omega\pi$ decays are seen for these states, the branching fractions have not been accu-

rately determined. If the states are mixtures of hybrid and conventional, as proposed in [72–74], the $\pi\omega$ widths become rather model dependent and no clear conclusions can be drawn.

1^{-+}

$$\Gamma(\pi_1(1600) \rightarrow \rho\pi) \approx 57\text{MeV} \quad (5.6)$$

This width for the 1^{-+} corresponds to a branching ratio² $\sim 34_{-17}^{+6}\%$ if we identify with the state in [18] with $m = 1600\text{MeV}$ and $\Gamma = 168 \pm 20_{-12}^{+150}\text{MeV}$. This is encouraging; had the branching ratio been $\sim 1\%$ it would be implausible for the state to have been seen in this mode; conversely had the branching ratio been predicted to be $\sim 100\%$ the required absence of other channels would have disagreed with the experimental observation of this state in various channels. This result is consistent within errors with:

1. an experimental limit $b.r.(1^{-+}(1600) \rightarrow \pi\rho) \leq 40\%$ [75];
2. with the relative branching ratios of ref. [76]:
 $br(b_1\pi) : br(\eta'\pi) : br(\rho\pi) = 1 : 1.0 \pm 0.3 : 1.6 \pm 0.4$;
3. with an analysis of the E852 data, assuming purely ρ exchange in the production mechanism which gave a branching ratio of $20 \pm 2\%$ [77].

This is an appropriate point to compare with the flux-tube breaking model, where $\rho\pi$ decays come about only when one allows different radial wavefunctions for the ρ and the π . In [38] the authors consider a particular realisation of this symmetry breaking and find a width for $\pi_1 \rightarrow \rho\pi$ of 8 MeV, where this assumes the hybrid is at 2 GeV; a state at 1600 MeV will have reduced phase-space and a further reduction in width.

0^{-+}

•

$$\Gamma(\pi(1800) \rightarrow \rho\pi) \approx 480\text{MeV}$$

In this model the pseudoscalar hybrid has a width significantly larger than the 1^{-+} state. We will show in Section 5.5 that this is a rather general prediction of the flux-tube model. This numerical prediction, which would signal a very broad state indeed, may be better considered an upper limit. In Section 5.2 we found that with $\beta = 0.4$ wavefunctions there was little

²where the error is just that from experiment.

difference between the quality of fit for $g = 2, 3$. Using $g = 2$ here would reduce the partial width to ~ 210 MeV.

Unfortunately we cannot really use our result to make any statement about the hybrid character or otherwise of the $\pi(1800)$ state. As well as our considerable theoretical uncertainties, we have no experimental measurement of the state's $\rho\pi$ branching ratio. The experiment that we would look to for such data is E852 but they note in [35] that the pseudoscalar partial wave suffers badly from the required $(\pi\pi)_S$ parameterisation uncertainties and as such no reliable data can be extracted.

2^{-+}

The isovector 2^{-+} state in this model has the same partial width as the 1^{-+} state and hence should be as prominent in experiment. There is a very broad ($\Gamma \sim 600$ MeV) candidate π_2 state seen at 2100 MeV in $\rho\pi$ and $f_2\pi$ modes [7], which may correspond to the broad, unparameterised enhancement in $\rho\pi$ above 2 GeV seen by E852 [35].

5.3.2 Positive Parity Hybrids

1^{++}

$$\Gamma_S(a_{1H}(2100) \rightarrow \rho\pi) \approx \frac{160}{660} \text{ MeV}$$

$$\Gamma_D(a_{1H}(2100) \rightarrow \rho\pi) \approx \frac{110}{170} \text{ MeV}$$

The upper/lower values are with “IP”, $\beta = 0.4$ wavefunctions and with $g = 3$. We can probably consider these values to be lower and upper limits on the S -wave width - the $\beta = 0.4$ wavefunctions are the optimum choice for $L = 1 \rightarrow L = 0$ conventional transitions, the “IP” solution shows that hybrids have a slightly smaller β than $L = 1$ states and hence we expect $\langle j_0 \rangle$ to fall faster with $|\vec{q}|$ than in the $\beta = 0.4$ case. The effective β in the “IP” solution is too small in the conventional sector and hence probably too small for hybrids too - hence our upper/lower assignment for $\beta = 0.4$, “IP”. Furthermore it has been noted by Close [78] that the quark-model has a tendency to not describe well decays in which the end state is in an S -wave, while the quarks in the initial meson are in a higher angular momentum eigenstate.

What is clear is that we have a considerable partial width into $\rho\pi$ for the axial hybrid. This

is at odds with the claim that the $a_1(2096)$ state seen by E852 has hybrid character as it is not seen at all in $\rho\pi$ by E852, who see only the dominant $a_1(1260)$ in 1^{++} .

2^{+-}

In the positive parity sector there are exotics $(0, 2)^{+-}$. The spin-0 state has no decay into πV , but we have some hope of seeing the spin-2 state if this model is correct. Normalising against the exotic 1^{-+} we have,

$$\frac{\Gamma(b_{2H} \rightarrow \omega\pi)}{\Gamma(\pi_{1H} \rightarrow \rho\pi)} \approx \frac{72}{5\pi^2} \left(\frac{|\vec{q}_2|}{|\vec{q}_1|} \right)^3 \left(1 - \frac{\pi \langle j_2 \rangle}{4 \langle j_1 \rangle} \right)^2,$$

where we've neglected the slow mass dependence of $\langle j_{1,2} \rangle$. This ratio suggests a similar partial width for the two states.

Normalising against the conventional 2^{++} decay gives

$$\frac{\Gamma(b_{2H} \rightarrow \omega\pi)}{\Gamma(a_2 \rightarrow \rho\pi)} = \left(\frac{|\vec{q}_H|}{|\vec{q}_C|} \right)^3 \frac{4b}{\pi^3 m^2} \left| \frac{\langle j_1 \rangle_H - \frac{\pi}{4} \langle j_2 \rangle_H}{\langle j_1 \rangle_C - \frac{\langle \vec{\partial} j_2 \rangle_C}{m}} \right|^2,$$

where there is some suppression for hybrids from $\frac{4b}{\pi^3 m^2} \approx 0.2$ but which is compensated by the increase in phase space. For a b_{2H} at 1600 MeV we anticipate a partial width around 50% of the $a_2 \rightarrow \rho\pi$ partial width. A heavier b_{2H} at 2100 MeV, with the increase in phase space would be around 150% of a_2 . Thus we expect $b_{2H} \rightarrow \omega\pi$ with a partial width of tens of MeV.

As well as photoproduction of 2^{+-} in pion exchange, there is also the possibility of diffractive photoproduction where the photon fluctuates into a ρ which fuses in a P -wave with the Pomeron. We don't currently have the tools to calculate a rate for this process, but we have no reason to expect it to be small - the ρ is already in the required spin-triplet so the Pomeron current could either excite the flux-tube by oscillating the quark or by interacting directly with the tube.

The possible sighting of such a state around 1650 MeV by E852 in $\pi^- p \rightarrow (\omega\pi^-)p$ [32] is, in light of our estimates, rather interesting and a dedicated study of this observation would be enlightening.

5.4 Higher order effects

The reader will recall that in equation (5.5) we chose to neglect a term transforming as $\vec{\sigma} \cdot \vec{p} \vec{q} \cdot \vec{a}$ on the grounds that it is sub-leading by one power of $|\vec{q}|/m$, or equivalently v/c . Unfortunately,

for the states we are considering, $|\vec{q}|/m$ is not necessarily small and our truncation appears artificial. A common approach in quark model treatments of hadron decays is to truncate the non-relativistic expansion of the operator at the highest order in v/c for which we know all possible terms. Any effects that we are able to calculate occurring at the next order might be negated by other effects at the same order that we are not able to calculate, and as such we do not include them in our estimates.

Explicit calculation of the effect of the “suppressed” $\vec{\sigma} \cdot \vec{p} \vec{q} \cdot \vec{a}$ term shows that it has a considerable effect on our predictions, especially in the negative parity sector. Its net effect is to modify the P and D -wave amplitudes according to

$$\begin{aligned}\Pi_{\mathcal{H}} \rightarrow \Pi'_{\mathcal{H}} &= g^2 \frac{|\vec{q}|^3}{m^2} \frac{b}{\pi^2 m^2} \left| \langle j_1 \rangle + \frac{2}{\pi} \frac{\langle \overleftarrow{\partial} j_1 \rangle}{m} \right|^2 \\ \Delta_{\mathcal{H}} \rightarrow \Delta'_{\mathcal{H}} &= g^2 \frac{|\vec{q}|^3}{m^2} \frac{b}{\pi^2 m^2} \left| \frac{4}{\pi} \langle j_1 \rangle - \langle j_2 \rangle - \frac{6}{\pi} \frac{\langle \overleftarrow{\partial} j_2 \rangle}{m} \right|^2.\end{aligned}$$

Note the similarity of $\Pi'_{\mathcal{H}}$ to the factor f in $M1$ radiative transitions (see Section 3.4) with harmonic oscillator wavefunctions $\overleftarrow{\partial} \rightarrow -\beta_C^2 r$ and $\Pi'_{\mathcal{H}} \sim \left| \left\langle \left(1 - \frac{2\beta_C^2 r}{\pi m} \right) j_1 \right\rangle \right|^2$. Recall that for $M1$ radiative decays in the light-quark sector, $f_{u\bar{u}} \approx 0$ as there is an accidental cancellation between the two terms. We considered possible corrections to this at $\mathcal{O}(br/m)$ and found that the approximate zero was not robust. Similar effects can occur here and we cannot trust that the cancellation is physically relevant. Unfortunately this also means that we must associate a considerable theoretical error with our predictions.

In light of this disturbing sensitivity to the order of truncation used, we should ask if there are any *more general* results to be extracted from this study. We find that there are and we discuss them in the next section.

5.5 General current structure arguments

Making only the assumptions that the pion current should transform as a pseudoscalar, be linear in $\vec{a}$ and at most first order in $\vec{\sigma}$ we can have only the following possible structures,

$$\begin{aligned}j_{\pi}(\mathcal{P}_{\mathcal{H}} = +) &= \alpha \sigma_z \vec{a} \cdot \hat{z} + \beta [\sigma_- \vec{a} \cdot \hat{x}_+ + \sigma_+ \vec{a} \cdot \hat{x}_-] \\ j_{\pi}(\mathcal{P}_{\mathcal{H}} = -) &= \gamma [\sigma_- \vec{a} \cdot \hat{x}_+ + \sigma_+ \vec{a} \cdot \hat{x}_-].\end{aligned}$$

Coupling the internal L, S by the Clebsch-Gordan coefficient, $\langle L = 1m'; Sm_S | Jm_J \rangle$ we find helicity amplitudes³

$\mathcal{P} = -$	$\mathcal{M}_0$	$\mathcal{M}_\pm$	$\mathcal{P} = +$	$\mathcal{M}_0$	$\mathcal{M}_\pm$
1^{--}	0	$\mp\sqrt{2}\gamma$	1^{++}	α	$-\sqrt{2}\beta$
0^{-+}	$\sqrt{\frac{8}{3}}\gamma$	0	0^{+-}	0	0
1^{-+}	0	$\pm\gamma$	1^{+-}	2β	$-\frac{1}{\sqrt{2}}\alpha + \beta$
2^{-+}	$\frac{2}{\sqrt{3}}\gamma$	γ	2^{+-}	0	$\pm(\frac{1}{\sqrt{2}}\alpha + \beta)$

Using the conversion from helicity to partial-wave amplitudes in Table XI of [58], these can be succinctly expressed as follows,

$\mathcal{P} = +$	1^{++}	0^{+-}	1^{+-}	2^{+-}	
$A_S =$	$\sqrt{3}$	0	$\sqrt{6}$	0	$\times S$
$A_D =$	$\sqrt{6}$	0	$\sqrt{3}$	3	$\times D$

$\mathcal{P} = -$	1^{--}	0^{-+}	1^{-+}	2^{-+}	
$A_P =$	$\frac{3}{\sqrt{2}}$	$\sqrt{3}$	$\frac{3}{2}$	$\frac{\sqrt{15}}{2}$	$\times P$

where $S = \frac{1}{3}(\alpha - 2\sqrt{2}\beta)$, $P = \frac{\sqrt{8}}{3}\gamma$ and $D = -\frac{1}{3}(\alpha + \sqrt{2}\beta)$.

These correlate with the relative amplitudes in Table 6 of [38] who computed these decays in the assumption that the flux tube breaks with creation of a new $q\bar{q}$ in ${}^3P_0$ state.

With the assumption only that the partial wave amplitudes $S, D; (P)$ are common to the supermultiplets of hybrids with $\mathcal{P} = +(-)$ respectively, then for equal masses where $\Gamma \sim \frac{C}{2J+1} \sum_L |A_L|^2$ we have the following constraint on the widths for the isoscalar/isovector $\mathcal{P} = +$ states to πV .

$$3\Gamma(1^{+-} \{I=0 \rightarrow \rho\pi\}) + 5\Gamma(2^{+-} \{I=0 \rightarrow \rho\pi\}) = 9\Gamma(1^{++} \{I=1 \rightarrow \rho\pi\})$$

For the $\mathcal{P} = -$ states we have,

$$\left(\Gamma(1^{-+}) = \Gamma(2^{-+}) = \frac{1}{4}\Gamma(0^{-+}) \right) \{I=1 \rightarrow \rho\pi\} = \frac{\mathbf{r}}{3}\Gamma(1^{--}) \{I=0 \rightarrow \rho\pi\}.$$

It is trivial to check that the expressions we have derived explicitly satisfy these rules. As we have already mentioned, the predictions of Close and Page [38] satisfy these rules, as do the predictions of an alternative flux-tube breaking model [33] (which has different quantum numbers at the breaking point). These “sum-rules” appear to be a rather general property depending only upon the spin-orbit coupling structure of the states, consequently we expect

³Unless ρ or ω is specifically denoted, our amplitudes for $A \rightarrow \pi V$ refer to a single charge mode for π and V . Widths for $I = (0; 1) \rightarrow \pi(\rho \text{ or } \omega)$ then require relevant charge counting factors to be included

them to be good if the flux-tube model is a good description of hybrid meson structure.

Their practical use is that once we have a candidate in $(\rho/\omega)\pi$ we can estimate the partial widths of other hybrid states - even the non-exotic ones - in a relatively model-independent way. For example if the $\pi_1(1600)$ is a hybrid, then $\pi_{0H} \rightarrow \pi\rho$ must also be prominent.

5.6 Radiative decays by Vector Meson Dominance

For the conventional hadrons, the widths into πV may be used to give estimates for the widths into $\pi\gamma$ by converting the $V \rightarrow \gamma$ as in vector dominance. The basic premise is that the photon has some hadronic character - it can fluctuate into an off-shell vector meson with some amplitude and interact strongly with the hadron target. Our phenomenological treatment will follow that of Babcock & Rosner [79].

In [79] the radiative helicity amplitude is

$$A_\lambda(M \rightarrow \gamma\pi) = \frac{e}{g_\rho} A_\lambda(M \rightarrow \rho\pi) + \frac{e}{g_\omega} A_\lambda(M \rightarrow \omega\pi) + \dots \quad (5.7)$$

where considering the quark electromagnetic charges between meson wavefunctions gives $g_\omega = 3g_\rho$ and where for on-shell photons $\lambda = \pm 1$ only.

In [79], g_ρ is set using

- $\Gamma(\rho \rightarrow e^+e^-) = \frac{m_\rho}{3} \alpha^2 \left(\frac{g_\rho^2}{4\pi} \right)^{-1}$ which using the PDG [7] value $\Gamma(\rho \rightarrow e^+e^-) = 6.85(11)\text{keV}$ gives $\frac{g_\rho^2}{4\pi} \sim 2.0$;
- $\Gamma(\rho \rightarrow \pi\pi) = \frac{2}{3} \left(\frac{g_\rho^2}{4\pi} \right) \frac{|\vec{q}|^3}{m_\rho^2}$ giving $\frac{g_\rho^2}{4\pi} \sim 2.9$;
- $\frac{\Gamma(\pi^0 \rightarrow \gamma\gamma)}{\Gamma(\omega \rightarrow \pi\gamma)} = \frac{2}{3} \alpha \left(\frac{g_\rho^2}{4\pi} \right)^{-1} \frac{|\vec{q}_\pi|^3}{|\vec{q}_\omega|^3}$ giving $\frac{g_\rho^2}{4\pi} \sim 2.9$.

While the application of VMD to each of these processes might be questioned theoretically, there does seem to be a rough agreement in their predictions for $\frac{g_\rho^2}{4\pi}$. We follow Babcock & Rosner and use an average value $\frac{g_\rho^2}{4\pi} = 2.7$ in what follows - we will see that is reasonably successful in relating conventional radiative and hadronic decays.

5.6.1 Conventional radiative decays

- $a_2 \rightarrow \rho\pi$

$$\Gamma(a_2 \rightarrow \gamma\pi) = \alpha \left(\frac{g_\rho^2}{4\pi} \right)^{-1} \frac{1}{2} \frac{|\vec{q}_\gamma|^3}{|\vec{q}_\rho|^3} \Gamma(a_2 \rightarrow \rho\pi).$$

The factor $\frac{1}{2}$ accounts for the fact that the hadronic width is the sum over two charge modes, one of which has a charged ρ . We have also accounted for the difference in phase space between $\gamma\pi$ and $\rho\pi$ end-states by following equations (9,10,19) of [79] and using a $|\vec{q}|^3$ dependence. Numerically we find

$$\frac{\Gamma(a_2 \rightarrow \gamma\pi)}{\Gamma(a_2 \rightarrow \rho\pi)} \approx 5 \times 10^{-3}$$

in reasonable agreement with the experimental value [7] of $(3.9 \pm 0.4) \times 10^{-3}$.

- $b_1 \rightarrow \omega\pi$

This channel is a little more complicated as we need to remove the $\lambda = 0$ amplitude which can contribute in the hadronic decay but not in the radiative decay, hence

$$\Gamma(b_1 \rightarrow \gamma\pi) = \alpha \left(\frac{g_\omega^2}{4\pi} \right)^{-1} \frac{|\vec{q}_\gamma|^3}{|\vec{q}_\omega|^3} \frac{\Gamma(b_1 \rightarrow \omega\pi)}{1 + \left| \frac{A_0}{\sqrt{2}A_+} \right|^2}.$$

$\frac{A_0}{\sqrt{2}A_+}$ can be expressed in terms of the experimentally measured D/S ratio, $\frac{A_0}{\sqrt{2}A_+} = \frac{1-\sqrt{2}D/S}{\sqrt{2+D/S}} \approx 0.36(4)$. Hence we find

$$\frac{\Gamma(b_1 \rightarrow \gamma\pi)}{\Gamma(b_1 \rightarrow \omega\pi)} \approx 1.4 \times 10^{-3}$$

which does not violate the experimental bound [7] $> (1.6 \pm 0.4) \times 10^{-3}$.

- $a_1 \rightarrow \rho\pi$

$$\Gamma(a_1 \rightarrow \gamma\pi) = \alpha \left(\frac{g_\rho^2}{4\pi} \right)^{-1} \frac{1}{2} \frac{|\vec{q}_\gamma|^3}{|\vec{q}_\rho|^3} \frac{\Gamma(a_1 \rightarrow \rho\pi)}{1 + \left| \frac{A_0}{\sqrt{2}A_+} \right|^2}.$$

So that

$$\frac{\Gamma(a_1 \rightarrow \gamma\pi)}{\Gamma(a_1 \rightarrow \rho\pi)} \approx 4.4 \times 10^{-3}$$

which is within the loose experimental range $(1.1 \rightarrow 5.8) \times 10^{-3}$ [7].

This approximate success encourages us to apply the same method to our hybrid πV predictions with some confidence, modulo the caveats in Section 5.4.

5.6.2 Hybrid radiative decays

For the exotic hybrid candidate $\pi_1(1600)$, using the $\rho\pi$ partial width prediction of 57 MeV we find a $\gamma\pi$ partial width of ~ 170 keV. This is a healthy width, comparable to the conventional $b_1 \rightarrow \gamma\pi$ width of 230 ± 60 keV. The expectation of the flux-tube breaking model supplemented with VMD is of a maximum width of ~ 70 keV with a more realistic prediction of 20% of this [80].

The non-exotic 2^{-+} hybrid, according to the “sum rules” of Section 5.5, will have a partial width equal to this with modification only for the potentially different state mass.

The exotic b_2 , if it has a mass ~ 2100 MeV has $\gamma\pi$ width ~ 50 keV. Much of the suppression relative to the π_1 width is down to the factor of $1/9$ caused by $g_\omega = 3g_\rho$. This does not appear for the isosinglet 2^{+-} hybrid and there we expect $\Gamma(h_2(2100) \rightarrow \gamma\pi) \sim 450$ keV.

The axial hybrid a_{1H} was predicted to have a potentially very large $\rho\pi$ width which was rather sensitive to wavefunction parameterisation. This large width unsurprisingly leads in VMD to a large radiative width $\sim 550 \rightarrow 1600$ keV which is comparable with our prediction in Section 3.3 on the basis of an $E1$ photon current exciting the flux-tube in a pion.

Photoproduction of hybrids through pion exchange At high photon energies and low momentum transfer, t , photoproduction of mesons is believed to be dominated by the pion exchange mechanism. There is very little data available, but what exists is consistent with one pion exchange expectations (see e.g. [81]). If this is really the dominant mechanism then we should expect the hybrids we have identified as having large $\gamma\pi$ partial widths to be produced prominently in photoproduction.

A study similar to that in [81], using the full theoretical formalism of photoproduction modeling could be carried out using the matrix elements found in this model.

5.7 Summary

We have used the formalism in which currents acting on quarks can excite the flux-tube to consider the pionic decays of hybrid mesons. The pion is considered to act as a pointlike current which couples only to the quarks (isospin ensures that it cannot couple directly to the flux-tube) and reasonable success in describing conventional meson decays is observed.

Previous studies of hadronic hybrid meson decays had assumed the need for pair production at some point on the flux-tube and had found that decay to a pair of mesons with identical spatial wavefunctions were forbidden. In the quark model with no spin-dependent effects, the

pion and the rho have identical wavefunctions and this has lead to the expectation that the $\pi\rho$ channel for hybrid decay should be suppressed. The model outlined in this chapter maximally breaks the π/ρ symmetry and finds that $\pi\rho$ rates can be large, which seems to be required by the data showing a 1^{-+} resonance in the $\pi\rho$ channel.

Detailed consideration of the model shows that the v/c expansion of the pionic current is not under control and hence that numerical predictions may not be robust. In light of this, more general results were extracted which link $\pi\rho$ partial widths of different hybrid states.

Radiative decay widths of hybrids were discussed under the assumption of VMD converting $\rho \rightarrow \gamma$ and were found to be not necessarily small. This offers hope to the experiments intending to produce hybrids via photoproduction.

Part II

Pentaquarks

Chapter 6

Introduction to Pentaquarks

Within QCD, with quarks transforming in the fundamental representation of $SU(3)$ and hence having a three-fold colour quantum number, the seemingly complete division of hadrons into mesons and baryons is easily understood. Mesons, with $q\bar{q}$ structure and baryons, with qqq structure have the minimal quark content required to produce an overall colour singlet in line with the experimental datum of colour confinement.

QCD does not, however, disbar states with a greater quark multiplicity - one can always add extra colour singlet $q\bar{q}$ pairs or qqq triplets to any colour singlet configuration, rearrange the internal colour structure and be left with a new allowed colour singlet state. But here lies the immediate problem with such states - we only require a trivial internal colour rearrangement to convert the state into two or more colour singlets which are then free (unless prevented by some other symmetry or kinematics) to separate into asymptotic states. Since internal colour rearrangement is not expected to be suppressed in the strongly coupled low energy regime of hadronic QCD it is believed that these “multiquark” hadrons will decay rapidly to $q\bar{q}$ and qqq states and be sufficiently broad as to be undetectable in experiment.

Actually, we should be a little more careful when we say that, for example, a baryon has qqq content only. Deep Inelastic Scattering experiments examine the proton with far-off-mass-shell photons (high Q^2) and find there to be a much richer quark and antiquark structure. The principle is that one can always add a colour and flavour singlet $q\bar{q}$ pair to a baryon without changing its overall quantum numbers, so that we can consider a baryon to have a schematic quark Fock state expansion,

$$|B\rangle \sim |qqq\rangle(1 + a_1|q\bar{q}\rangle + a_2|q\bar{q}q\bar{q}\rangle + \dots). \quad (6.1)$$

Why then do we usually think of a baryon as being qqq ? Importantly this is the *minimal* quark content required to produce the state, for example the proton might contain $uuds\bar{s}$ or $uud(u\bar{u} + d\bar{d})$ but it must always contain uud .

Furthermore when we examine baryons with near on-shell photons (low Q^2) we see dominance of the three quark state. The “parton distribution functions” for the u and d quarks are roughly peaked near to a momentum fraction of $1/3$ and the peak amplitude for u is close to twice than of d . Antiquark distribution functions (which signal the extra $q\bar{q}$ pairs) are only non-zero for small momentum fractions and have much smaller amplitudes than the quark distributions. See Figure 6.1.

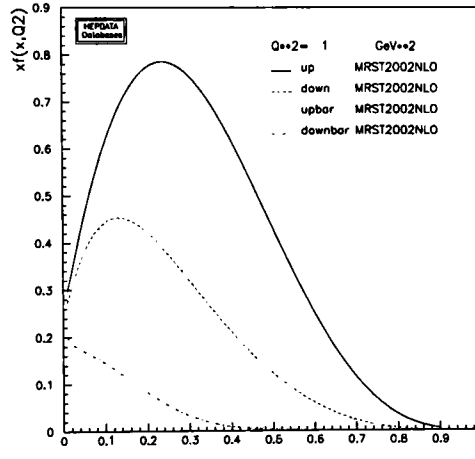


Figure 6.1: “MRST” Parton Distribution Functions at low Q^2 [82]

Constituent quark models have been developed which treat mesons and baryons in terms of only their minimal or “valence” quark content. In such models the valence quarks have large masses (for $u, d \sim 350$ MeV), much larger than the few MeV masses expected in the QCD Lagrangian. The possibility of additional quark-antiquark pairs in this framework is then suppressed by the large quark mass.

Such models have been rather successful in predicting many properties of hadrons that might be described as “on-shell” such as masses, decay rates and magnetic moments. They are clearly not suited to describing the high Q^2 DIS observations.

There will presumably be, somewhere higher in the spectrum, a state orthogonal to (6.1) where the $qqqq\bar{q}$ Fock state dominates; however without low Q^2 DIS measurements (which are not possible for unstable states) or model-dependent analysis of its mass and decays, we cannot be sure it is not simply a conventional excited state of qqq . There is the further possibility that such a state is not bound at all, and is simply part of the meson+baryon continuum.

But, we might ask, what if we add a non-flavour-singlet $q\bar{q}$ pair to qqq ? For example, consider adding $s\bar{u}$ to udd . QCD allows annihilation of the $u\bar{u}$ pair leaving a state sdd and so what we have considered is equivalent to adding a flavour singlet to a Σ^- state. On the other hand, what if we add a $u\bar{s}$ state to udd ? We can no longer annihilate a $q\bar{q}$ pair and we are stuck with a state of *minimal quark content* $udud\bar{s}$. This is a “flavour exotic” pentaquark; because it has $S = +1$ it cannot be a qqq state. The observation of such a state requires no model-dependent analysis of its properties to determine its pentaquark nature as it is manifestly exotic in its flavour quantum numbers.

We still have for this state the general problems with multi-quark hadrons discussed earlier by a simple internal colour rearrangement a $udud\bar{s}$ state can become an nK^+ state. In the naive constituent quark model this pentaquark would weigh around 1800 MeV and hence there would be considerable phase space for the unsuppressed decay to nK^+ . The $udud\bar{s}$ state would be unobservable due to its enormous width.

6.1 Pentaquark Experimental Status

The arguments of the previous section would lead one to believe that the search for narrow resonances in the $I_z = 0, S = +1$ (nK^+) channel would be fruitless, and indeed up until 2003, give or take a few transient candidates, it was. Because they believed that previous searches in KN scattering experiments had missed the low mass region due to too-large kaon momenta and motivated by a prediction of the Chiral Soliton Model [83], the LEPS collaboration at SPring-8 investigated the reaction $\gamma n \rightarrow K^+ K^- n$ with the neutron from a ^{12}C nucleus. In the missing mass spectrum they found a peak at 1.54(1) GeV with a width below 25 MeV at a statistical significance of 4.6σ [84].

While clearly a most exciting observation, it is far from convincing as a display of the data will demonstrate. Figure 6.2 is the kinematically cut data from [84]; the solid histogram is the nK^+ missing mass and the dashed histogram the missing mass with a proton in the end state. This author has added a rough quadratic fit to the proton data which we can treat as an approximate background estimate for the nK^+ data. According to [84] there are 36 events in the peak region, defined as $1.51 \rightarrow 1.57$ GeV from a total of 109. Using our approximate background estimate we would assign $5 + 6 + 7 = 18$ of these events to background, leaving only 18 signal events in the sample.

Since the LEPS announcement several experiments have reanalysed data to search for this enhancement. We give here a brief summary of these experimental searches, in no particular

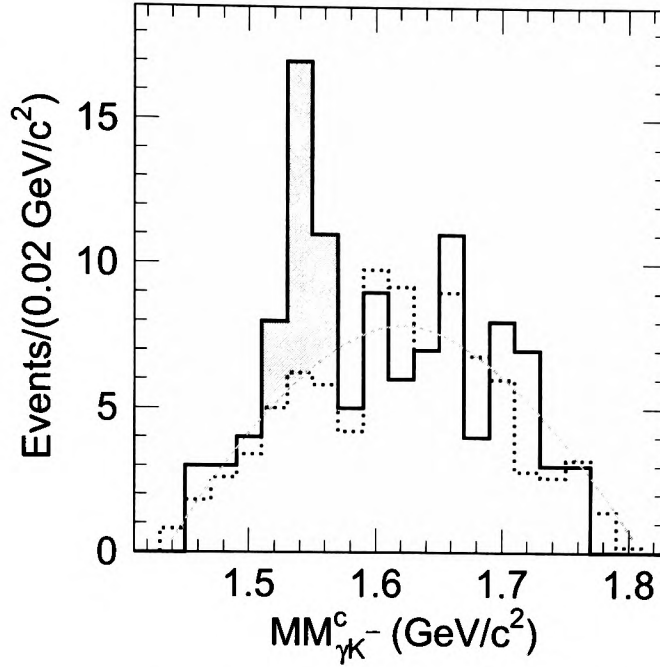


Figure 6.2: LEPS nK^+ missing mass spectrum [84]

order:

- **CLAS** $\gamma d \rightarrow K^+ K^- pn$ [85]

$$nK^+(S=+1) \quad m = 1.542(5) \text{ GeV} \quad \Gamma < 21 \text{ MeV} \quad 5.2\sigma$$

Notable kinematic cuts on the data are: $M(K^+ K^-) > 1.07 \text{ GeV}$ (removing photoproduced $\phi(1020) \rightarrow KK$); rejection of events $1.485 < M(pK^-) < 1.551 \text{ GeV}$ (removing the $\Lambda(1520)$); $|\vec{p}(n)| > 80 \text{ MeV}$ (ensuring the neutron is not a spectator) and $|\vec{p}(K^+)| < 1.0 \text{ GeV}$.

After these cuts, the experiment claims 43 signal events above an estimated background of 54 events in the peak region.

The pK^+ invariant mass was also considered and no sign of an enhancement near 1542 MeV was seen, suggesting that the observed state is an isosinglet.

- **CLAS** $\gamma p \rightarrow \pi^+ K^+ K^- n$ [86]

$$nK^+(S=+1) \quad m = 1.555(10) \text{ GeV} \quad \Gamma < 26 \text{ MeV} \quad 7.8\sigma$$

After an initial kinematic cut $M(K^+ K^-) > 1.06 \text{ GeV}$, removing about 200 ϕ events, the nK^+ mass spectrum showed no signs of resonant structures. A further cut $\cos \theta_{\pi^+}^* > 0.8$, selecting pions traveling in the direction of the photon, caused the clear appearance of a

narrow peak at ~ 1.54 GeV. To remove background events with $\gamma \rightarrow$ meson resonance $\rightarrow K^+K^-$ and baryon resonance $\rightarrow n\pi^+$ or meson resonance $\rightarrow K^+K^-\pi^+$ where the momentum transfer is small and the K^+ moving forward along the photon direction, the cut $\cos\theta_{K^+}^* < 0.6$ was applied. After this a clear peak of around 35 events on a fitted background of about 25 events was observed.

No resonant structures were seen in 130k events in the pK^+ invariant mass spectrum of the reaction $\gamma p \rightarrow K^-K^+p$.

- **HERMES** $\gamma d \rightarrow pK_S^0 + \dots$ [87]

$$pK_S^0(|S|=1) \quad m = 1.528(3) \text{ GeV} \quad \Gamma = 19 \pm 6 \text{ MeV} \quad 4 - 6\sigma$$

This experiment did not use a missing mass method, instead detecting the proton and both pions in $K_S^0 \rightarrow \pi^+\pi^-$. The only cuts made were to remove background in the invariant mass of the two pions and hence give a clean K_S^0 signal. A peak in the pK_S^0 invariant mass spectrum was immediately seen.

Depending upon the background fit used, the experiment found 50-75 signal events on a background of ~ 150 events.

The spectrum of pK^+ shows no structure, once again signaling the likely isoscalar nature of the observed enhancement.

The problem with this and all other experiments looking at the pK_S^0 end-state is that we cannot definitively state that the resonance has strangeness $+1$ and not the more conventional strangeness -1 . The only reason to suspect $S = +1$ is the near co-incidence of the mass of the resonance with that seen in nK^+ . Enhancements in $S = -1$ have been seen before in this general mass region - the PDG [7] list Σ^* states of one or two star status at 1480, 1560 and 1580 MeV.

- **DIANA** $K^+[Xe] \rightarrow K^0p[Xe]'$ [88]

$$pK^0(|S|=1) \quad m = 1.539(2) \text{ GeV} \quad \Gamma < 9 \text{ MeV} \quad 4.4\sigma$$

Although the end-state of this experiment is pK^0 , the production channel must be nK^+ so we can in fact be sure of $S = +1$.

This is a bubble-chamber experiment, where to reduce the chance of nuclear medium rescatterings, topological cuts were applied; $\theta_p < 100^\circ, \theta_K < 100^\circ$ i.e. relatively forward going (with respect to the incident K^+) and $\cos\Phi_{pK} < 0$ i.e. back-to-back proton and

K^0 in the transverse plane. The effect of these cuts was to increase the significance from 2.6σ to 4.4σ .

- **ITEP neutrino scattering on Nuclei** [89]

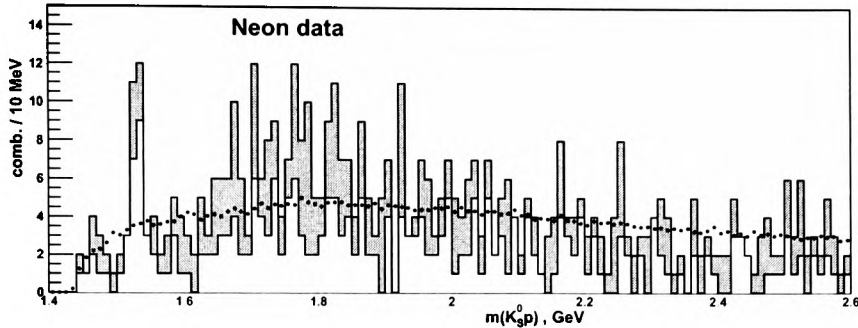


Figure 6.3: ITEP pK_S^0 mass spectrum [89]

$$pK^0(|S| = 1) \quad m = 1.533(5) \text{ GeV} \quad \Gamma < 20 \text{ MeV} \quad 6.7\sigma$$

The main statistics of this experiment come from data taken with a Neon target. Their data is displayed here in Figure 6.3 and shows that 6.7σ would seem to be rather optimistic. The PDG [7] has Σ^* states of four star status at 1660, 1670, 1775 MeV and higher and while it may not be possible to produce all these in the experiment it would add weight to the experiment's claims for an exotic enhancement if it could at least identify established non-exotic resonances.

- **SVD $pA \rightarrow pK_S^0 + X$** [90]

$$pK^0(|S| = 1) \quad m = 1.526(4) \text{ GeV} \quad \Gamma < 24 \text{ MeV} \quad 5.6\sigma$$

The uncut data in this experiment shows no real sign of structure near 1530 MeV. Cuts applied were: $490 < M(\pi^+\pi^-) < 505 \text{ MeV}$ which ensures only $\pi\pi$ from K_S^0 ; $\cos\alpha > 0$ i.e forward going pK_S^0 ; $|\vec{p}(K_S^0)| < |\vec{p}(p)|$ which is supposed to remove high mass Σ^* resonances and hence lower the “background”. With these cuts the quoted 5.6σ significance is obtained.

Note that in their Figure 4(b), reproduced here as Figure 6.4, displaying the $\Lambda\pi^+$ invariant mass spectrum, as well as the $\Sigma(1385)$ which they fit, there is also a possible enhancement at about 1540 MeV. $\Lambda\pi^+$ is a purely $S = -1$ channel and if this enhancement can be shown to be real and at the right mass it would suggest that the pK_S^0 experiments are not seeing an $S = +1$ exotic but simply a non-exotic $S = -1$ state.

- **SAPHIR $\gamma p \rightarrow K_S^0 K^+ n$** [91]

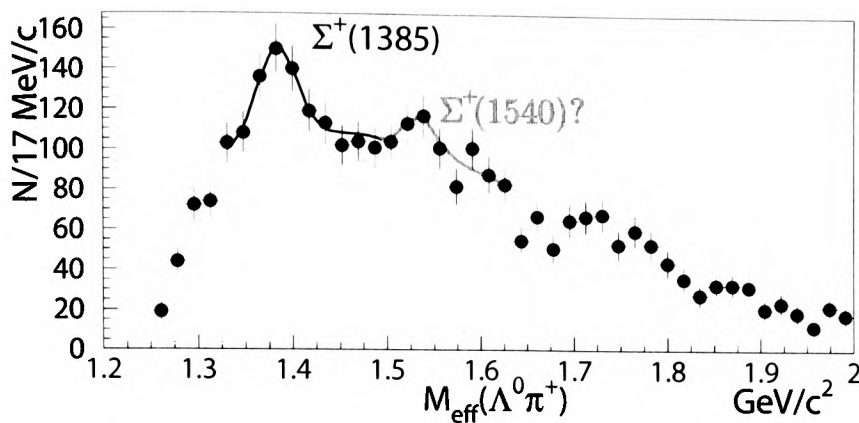


Figure 6.4: $\Lambda\pi^+$ spectrum from SVD [90]. This author has added the grey curve showing a peak near 1540 MeV with low statistical significance.

$$nK^+(S = +1) \quad m = 1.540(5) \text{ GeV} \quad \Gamma < 25 \text{ MeV} \quad 4.8\sigma$$

A cut $480 < M(\pi^+\pi^-) < 518 \text{ MeV}$ selected events with a K_S^0 as did $\cos\theta_{K_S^0} > 0.5$ which also reduced background due to reflection from $\Lambda(1520)K^+$.

A rough estimate of the mean cross-section for production of the Θ^+ with photon energies between 1.74 and 2.6 GeV was made of $\sim 300 \text{ nb}$. Conventional processes like $\gamma p \rightarrow K^+(\Lambda, \Sigma, \Lambda(1520))$ have cross-sections $\sim 800 - 1200 \text{ nb}$ at $E_\gamma = 2 \text{ GeV}$.

An examination of the pK^+ channel showed a tiny excess at 1540 MeV of about 75 events above a rather large background. On this basis of isospin arguments and the improved acceptance of their detector for this channel SAPHIR estimate that they should have ~ 5000 events if there is a Θ^{++} at 1540 MeV. They hence exclude such a possibility.

- **COSY-TOF** $pp \rightarrow \Sigma^+ K_S^0 p$ [92]

$$pK_S^0(|S| = 1) \quad m = 1.530(5) \text{ GeV} \quad \Gamma < 22 \text{ MeV} \quad 4 - 6\sigma$$

Despite being considered in pK_S^0 any observed resonance in this reaction must have $S = +1$ to compensate the $S = -1$ of the Σ .

Invariant mass cuts were applied to ensure the presence of the Σ and the K^0 .

The cross-section for Θ production was estimated to be $0.4(2)\mu\text{b}$.

- **ZEUS** $\gamma^* p \rightarrow K_S^{0p} + X$ [93]

$$pK_S^0(|S| = 1) \quad m = 1.521(3) \text{ GeV} \quad \Gamma < 3 \text{ MeV} \quad 4 - 4.5\sigma$$

This is the first truly high-energy experiment to observe the enhancement and the first to see the anti-particle $\bar{\Theta}^-$. No explanation is offered for the enhancement only arising

at high photon virtuality, $Q^2 > 20\text{GeV}^2$.

No enhancement is observed in pK^+ .

- **HERA-B** $pA \rightarrow K_S^0 p + X$ [94]

$pK_S^0(|S| = 1)$ No Signal

This experiment claims an upper limit on yields of $\Theta(1540)/\Lambda(1520) < 2\%$.

- **BES** $e^+e^- \rightarrow \frac{J/\psi}{\psi(2S)} \rightarrow \frac{K_S^0 p K^- \bar{n}}{K_S^0 \bar{p} K^+ n}$ [95]

$nK^+(S = +1)$ No Signal

Given that the total number of events in nK^+ presented in their Figure 6 appears to be about 20 across a mass range of 200 MeV we should probably not draw too many conclusions from this null result.

As noted by Close and Zhao [96], the observations in the ambiguous strangeness, pK_S^0 channel seem to have a systematically lower mass than those in the $S = +1$, nK^+ channel. In the SVD analysis we noted a possible enhancement around 1540 MeV in the $\Lambda\pi^+$ end state; a higher statistics study of this region would be welcomed as the appearance of an enhancement at this mass in this channel would question the assumed $S = +1$ nature of the pK_S^0 observations.

In [97], the Indiana group suggest that the pentaquark signal in low energy photoproduction experiments could actually have its origin in kinematic reflections from the photoproduction of excited meson states such as ρ_3, f_2, a_2 . From their analysis such an issue is unlikely to affect the hadronic production mechanisms or photoproduction at higher energies. [98] has some experimental criticisms of the DIANA analysis [88].

We will see later when we consider the phenomenology of pentaquarks that we expect there to be further states, some of which have exotic flavour quantum numbers. Searches for these states have also been performed. The NA49 experimental collaboration [99] claim an $S = -2, Q = -2$ state in $\Xi^- \pi^-$ at a mass of 1862(2) MeV with a width below 18 MeV at 4σ significance. The production mechanism is pp collisions at $\sqrt{s} = 17$ GeV. This experiment also considers the end state $\Xi^- \pi^+$ with lower statistics and sees an enhancement roughly degenerate with that in the exotic channel, however given that none of the three star status Ξ^* states listed in the PDG [7] are clearly seen we might question the significance of this observation.

The HERA-B experiment [94] which saw no sign of the Θ also has no $S = -2, Q = -2$ signals between 1.5 and 2.2 GeV. Fischer and Wenig [100] present a set of experimental reasons why the NA49 data is inconsistent with other data and even with itself.

The most recent observation of an exotic baryon comes from the H1 Collaboration who claim a narrow, isoscalar, anti-charmed baryon at a mass of 3099(6) MeV and a width < 15 MeV in $D^{*-}p$ and its anti-partner in $D^{*+}\bar{p}$. Their significance estimate is near 6σ . The minimal quark content of such a state is $udud\bar{c}$ which makes it the charmed partner of the Θ . However, a search by the FOCUS collaboration [101] finds no sign of such a state.

At the Quarks & Nuclear Physics 2004 conference in Bloomington, IN, new results were presented by several experimental groups [102].

- LEPS find a Θ^+ signal in photoproduction on deuterium.
- E690 (Fermilab) find no evidence for a Θ^+ or Ξ^{++} signal in 800 GeV $pp \rightarrow pX$ data with very large statistics.
- CDF find no evidence for $\Theta^+, \Xi^{--}, \Theta_c$ in large statistics $p\bar{p}$ collisions at 2 TeV.
- HyperCP (Fermilab) find no evidence for the Θ^+ in pK_S^0 and note that one can generate a spurious signal near 1540 MeV if “ghost tracks” from the $p\pi$ decay of the Λ are not removed.
- BaBar see no narrow resonances in pK_S^0 except the Λ_c .
- ALEPH, OPAL and DELPHI see no narrow resonances in pK_S^0 .
- PHENIX (RHIC) see no $\bar{\Theta}^-$ signal in $\bar{n}K^-$ in deuterium-gold collisions at 200 GeV.

6.2 Pentaquark Phenomenology & Models

In the fifteen months since the first report of a Θ candidate by SPring-8, there have been well over one hundred preprints submitted to the arXiv on pentaquarks and related subjects. This thesis cannot possibly give a comprehensive review of all this material, but we will in this chapter attempt to give the reader a flavour of the directions pentaquark phenomenology has taken.

We will begin by discussing the flavour structures open to pentaquark states. Conventional baryons can be grouped into representations (multiplets) of (broken) flavour $SU(3)$,

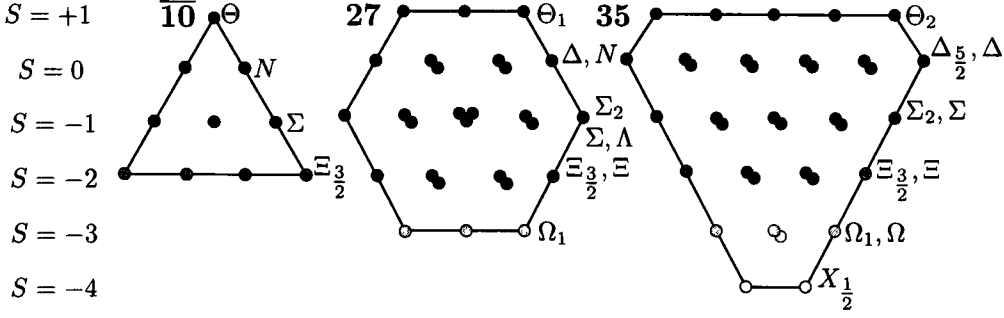


Figure 6.5: Non-trivial pentaquark flavour multiplets. Subscripts, where shown, indicate isospin. The state $X_{\frac{1}{2}}$ is an isodoublet with 4 strange quarks.

examples include: a complete $J^P = \frac{1}{2}^+$ octet containing $\{N(939), \Sigma(1193), \Lambda(1116), \Xi(1315)\}$; a complete $J^P = \frac{3}{2}^-$ octet containing $\{N(1520), \Sigma(1670), \Lambda(1690), \Xi(1820)\}$; a complete $J^P = \frac{3}{2}^+$ decuplet containing $\{\Delta(1232), \Sigma(1385), \Xi(1530), \Omega(1672)\}$ and $J^P = (\frac{1}{2}, \frac{3}{2})^-$ singlets $\Lambda(1405), \Lambda(1520)$.

This empirical observation is explained if baryons have a qqq structure and if quarks transform in the fundamental representation of (broken) $SU(3)_F$. Then $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$ so that the allowed multiplets for baryons are singlets, octets and decuplets, all of which are seen to be present in the experimental spectrum.

In light of this success we can attempt to construct the possible multiplets for pentaquark states ($q^4\bar{q}$) as

$$3 \otimes 3 \otimes 3 \otimes 3 \otimes \bar{3} = 1^3 \oplus 8^8 \oplus 10^4 \oplus \bar{10}^2 \oplus 27^3 \oplus 35$$

The new multiplets are antidecuplets, **27s** and **35s**, all of which contain $S = +1, Q = +1$ states. We display in Figure 6.5 the state content of these multiplets, where we see that the antidecuplet contains an isoscalar Θ , the **27** an isovector Θ_1 and the **35** an isotensor Θ_2 . The possibility that the Θ^+ belongs to an isotensor set was suggested soon after its observation by Capstick, Page and Roberts [103] their aim was to explain the extremely narrow width to nK^+ by making this an isospin-violating and hence heavily suppressed transition. Negative searches for charge partners of the Θ seem to have ruled out the possibility that the Θ is anything but an isoscalar, but see [104] for a counter-argument.

The work which motivated the SPring-8 search was concerned with the properties of an anti-decuplet [83]. In this paper Diakonov, Petrov and Polyakov(DPP) computed within the Chiral Soliton Model (hereafter denoted χ SM) masses and widths of the members of a $\bar{10}$.

The χ SM considers baryons to be topological solitons in the chirally symmetric pion and kaon fields. In the formulation used in [83] it does not contain either quark or gluon degrees-

of-freedom. In the $SU(2)_F$ version (the Skyrme model) there is a mapping between real-space rotations (the $SO(3) \sim SU(2)$ rotations on the circle at infinity) and the $SU(2)$ symmetry of the pion field; this connects spin and isospin, $I = J$. In the $SU(3)_F$ extension, consistent quantisation of the solitons implies the Wess-Zumino constraint which selects the allowed multiplets and their spins. The result of this is that the lightest three multiplets are expected to be a $J^P = \frac{1}{2}^+ \mathbf{8}$, a $J^P = \frac{3}{2}^+ \mathbf{10}$ and a $J^P = \frac{1}{2}^+ \overline{\mathbf{10}}$. The octet is associated with the nucleon multiplet and the decuplet with the Δ multiplet, the $\overline{\mathbf{10}}$ which contains the Θ is predicted to lie $\frac{3}{2I_2}$ above the nucleon octet, where I_2 is a moment-of-inertia of the soliton and is not known except in explicit models.

DPP set the mass of the $\overline{\mathbf{10}}$ by proposing that we have already observed one of its non-exotic members; they associated the $N(1710)$ [7] with the $\overline{\mathbf{10}}$ nucleon doublet. The rows in the $\overline{\mathbf{10}}$, as in the octet and the decuplet are split by the explicit breaking of $SU(3)$ flavour caused by the non-zero m_s . DPP worked to first order in m_s which allows for explicit breaking of the degeneracy and wavefunction mixing with e.g. the nucleon octet, but not mass differences due to mixing (which occurs at second order).

The splitting between rows in their formalism can be expressed in terms of measured quantities by

$$\delta m_{\overline{\mathbf{10}}} = \frac{5}{4} \delta m_{\mathbf{10}} - \frac{1}{6} r \Sigma + \frac{35}{32} (m_\Sigma - m_\Lambda) \approx 180 \text{ MeV}, \quad (6.2)$$

where r is the current quark mass ratio $r = \frac{m_s}{m_u + m_d}$ and Σ is the nucleon sigma term. This value is in line with the expectation of Praszalowicz [105] who states that the splittings should, to a good approximation, be linear in hypercharge with proportionality coefficient of order 150 MeV. Note though that DPP used $\Sigma \sim 45$ MeV while a more recent experimental value $\Sigma \sim 80$ MeV [106] would give $\delta m_{\overline{\mathbf{10}}} \sim 105$ MeV.

Using their value of $\delta m_{\overline{\mathbf{10}}}$, DPP predict a Θ^+ at a mass of 1530 MeV, a Σ^* at 1890 and a $\Xi_{\frac{3}{2}}$ at 2070. This appears to be a very successful prediction of the Θ mass, but look at the splitting from a constituent quark point of view and it looks very peculiar indeed. The Θ has quark content $udud\bar{s}$ and hence contains one strange quark mass; the exotic Ξ^+ , $usus\bar{d}$ has two strange quark masses and hence the mass gap from the top to the bottom of the $\overline{\mathbf{10}}$ should be one strange quark mass excess, $m_s - m_u \sim 150$ MeV. With equally spaced levels we would expect from this a level spacing ~ 50 MeV. Providing higher order breaking effects to do not overwrite the χ SM splittings there is a discriminator of models here a small row splitting supports a quark-like picture while a larger splitting supports the χ SM.

With the χ SM, with parameters set using measured decuplet decays and explicit model

results, the partial widths of the $\overline{10}$ states were predicted. In particular the $\Theta \rightarrow nK^+$ width was found to be 15 MeV, but as Jaffe [107] points out, this value is based upon a mistake in arithmetic and should in fact be 30 MeV. Compared with experiment this would seem to be an overprediction. See also the refutation by DPP [108] who argue that the width can go to zero with a suitable choice of parameters.

Since the observation of the Θ , the χ SM has been the subject of further study and the DPP numerical predictions the target of some criticism. For example in [106], Ellis, Karliner and Praszalowicz consider the DPP Θ mass prediction to be “fortuitous”, and in their own model dependent analysis can only conclude that the Θ should lie in the region 1450-1640 MeV. There seems to be a good deal of uncertainty in how to describe the $SU(3)$ breaking in the χ SM and unless more definitive statements can be made we should not treat χ SM mass estimates as gospel.

In particular considering the $N(1710)$ to be a member of a $\overline{10}$ is not consistent with the experimental data. For example it has a considerable decay rate to $\Delta\pi$, which would violate the group theory result that $\overline{10} \not\rightarrow 10 \otimes 8$ in $SU(3)$. Similarly this state has been seen in photoproduction off the proton, but since the photon is a U -spin singlet we cannot get the required $U = \frac{1}{2} \rightarrow U = \frac{3}{2}$ transition to take us to a $\overline{10}$.

The Wess-Zumino constraint allows larger representations than the $\overline{10}$, for example:

- with $J^P = \frac{1}{2}^+$ we can have **27, $\overline{35}$, 64**...
- with $J^P = \frac{3}{2}^+$ we can have **27, 35, $\overline{35}$, 64**...
- with $J^P = \frac{5}{2}^+$ we can have **35, 64**...

Note that to get $\overline{35}, 64$ in a quark model one must introduce a further $q\bar{q}$ pair to the $q^4\bar{q}$ making a “heptoquark”. Later we will see that the absence in the χ SM of a $\overline{10}$ with $J^P = \frac{3}{2}^+$ and the fact that we do not expect an octet to be naturally degenerate with the $\overline{10}$ in the χ SM admits clear model discriminators with respect to quark pictures.

6.2.1 Karliner & Lipkin - Colour-Magnetic Interaction

One of the first models to attempt to describe the Θ^+ is due to Karliner and Lipkin(KL) [109]. They applied the colour-magnetic interaction model, which has some success describing the ground state mesons and baryons, to the $q^4\bar{q}$ system. They proposed that the structure which maximises the colour-magnetic binding energy is one consisting of a well separated diquark, triquark pair. The diquark has flavour ud , is a $\bar{3}$ of colour and has spin-0, while the triquark has

flavour $ud\bar{s}$ (in a flavour $\bar{6}$), colour **3** and spin-1/2 with its ud component in a colour **6**, spin-1 state. A P -wave is included between the diquark and triquark which prevents colour-magnetic interactions between quarks in different clusters, the effect being to keep apart the identical pairs uu, dd which have a repulsive interaction. This P -wave also has the effect of rendering the Θ parity positive.

KL evaluate the colour-magnetic interaction energy (in the $SU(3)_F$ limit) and express the diquark-triquark mass as

$$m_N + m_K - \frac{m_\Delta - m_N}{6}$$

where the $u\bar{s}$ interaction energy has been eliminated by using the mass of the kaon. As we now show this introduces a large error, as this model as presented is not able to simultaneously fit the meson and baryon spectrum. With colour-magnetic interaction Hamiltonian

$$\delta H = - \sum_{i>j} \vec{\lambda}_i \cdot \vec{\lambda}_j \vec{S}_i \cdot \vec{S}_j \frac{v_{ij}}{m_i m_j},$$

the Σ^*, Σ mass difference is $\frac{v_{us}}{m_u m_s}$ and the K^*, K mass difference is $\frac{4}{3} \frac{v_{us}}{m_u m_s}$. If we set $\frac{v_{us}}{m_u m_s}$ using the experimental Σ masses and use the experimental K^* mass, this model predicts a kaon mass of ~ 620 MeV, over 100 MeV greater than the experimental value. Thus by replacing $m_u + m_s - \frac{v_{us}}{m_u m_s}$ with m_K in their equations, KL have lowered the pentaquark mass by over 100 MeV relative to the value it would have been if we set $\frac{v_{us}}{m_u m_s}$ using baryon masses.

The P -wave excitation energy for the Θ is determined by KL by noting that the diquark-triquark reduced mass is rather similar to that in the D_s meson. They use experimental masses in the D_s sector to estimate $\omega_P = 207$ MeV. This “small” value is critical for them to obtain a low mass Θ . They outline a few mild caveats in their paper, but we have some more serious objections to this estimate, which we now detail.

KL used the mass of the recently observed $D_s(2317)$ meson - proposing that it is a P -wave $c\bar{s}$ excitation (presumably the 0^+). The structure of this state is the subject of some discussion, with many authors (including Lipkin himself [110]) suggesting that it is a four-quark state, an idea related to the fact that it is very close to the DK threshold and much lighter than the anticipated mass of a $c\bar{s}$ 0^+ state.

Furthermore, even if the $D_s(2317)$ is the $^3P_0$ $c\bar{s}$ state, the mass difference KL use is not the P -wave excitation energy. Figure 6.6 shows the D_s experimental spectrum where the gap used by KL is indicated. With colour-magnetic interactions (exactly what KL are exploiting in this paper), the $0^-, 1^-$ system is split from its zeroth-order value which we can find from the

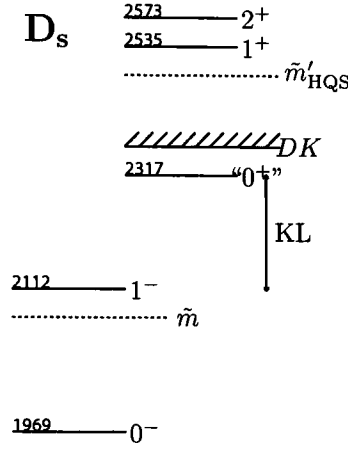


Figure 6.6: The experimental D_s spectrum. The lower dashed line is the spin-averaged ground state mass. The upper dashed line is the average P -wave mass in the limit of heavy-quark symmetry. The hatched line denotes the DK kinematic threshold. KL denotes the gap used by Karliner and Lipkin as the P -wave excitation energy.

spin-averaged mass, $\tilde{m} = \frac{3}{4}m(1^-) + \frac{1}{4}m(0^-) \sim 2080$ MeV and which is shown in Figure 6.6 by the lower dashed line. This mass should have been the lower value in the KL estimate, but the difference here is relatively small. A much larger difference appears in the upper value. The experimental states are clearly split by a large spin-orbit force (if we accept the $^3P_0$ proposal for the $D_s(2317)$) and we should use the pre-splitting mass to compute the P -wave excitation energy if we wish to be consistent. In this case KL's estimate can only be correct if the 0^+ state has not been moved by the spin-orbit splitting, while the $2^+, 1^+$ states have been pushed up. We can examine this possibility:

Eichten and Quigg [111] parameterise the splittings in their equations(2.16, 2.20). We display below the 2^+ and 0^+ shifts,

$$\begin{aligned}\delta m(2^+) &= \frac{1}{4} \left(\frac{1}{m_c^2} + \frac{1}{m_s^2} \right) \tilde{T}_1 + \frac{1}{m_s m_c} \tilde{T}_2 - \frac{2}{5} \frac{1}{m_s m_c} \tilde{T}_4 \\ \delta m(0^+) &= -\frac{1}{2} \left(\frac{1}{m_c^2} + \frac{1}{m_s^2} \right) \tilde{T}_1 - \frac{2}{m_s m_c} \tilde{T}_2 - 4 \frac{1}{m_s m_c} \tilde{T}_4.\end{aligned}\quad (6.3)$$

Setting $\delta m(0^+) = 0$ (as KL require) and using the resulting equality to eliminate the first two terms from $\delta m(2^+)$ we obtain

$$\delta m(2^+) = -\frac{12}{5} \frac{1}{m_c m_s} \tilde{T}_4,$$

so that to agree with experiment we require $\tilde{T}_4 < 0$. In terms of one-gluon-exchange, Eichten and Quigg have $\tilde{T}_4 = \frac{\alpha_s}{3} \langle r^{-3} \rangle$ which is clearly positive. If $\tilde{T}_4$ generalises to $\sim \langle \frac{1}{r} \frac{dV}{dr} \rangle$ as expected from the Breit-Fermi Hamiltonian, then $\tilde{T}_4$ negative would require the potential to

have negative slope over the range of r where the wavefunction is peaked - this is highly unlikely for an attractive, confining potential.

We conclude that the use by KL of the 0^+ state mass as the upper value is badly flawed. We can make a still model-dependent, but we believe more reliable, estimate of ω_P by considering the splitting in the heavy-quark limit where

$$\begin{aligned}\delta m(2^+, 1^+) &= +x \\ \delta m(1^+, 0^+) &= -2x.\end{aligned}$$

If the observed 1^+ is the lighter of the two then $\omega_P \approx 470$ MeV, if it is the heavier then we require further information and we will use the “ 0^+ ” mass (considering it to be a lower limit on the true 0^+ mass) which gives $\omega_P \approx 370$ MeV which is still much larger than the KL estimate of 207 MeV.

We are forced to conclude that the KL estimate for the mass of the Θ in their model is a considerable under-estimate and that their prediction should have been at least 200 MeV larger. This undermines the original aim of trying to explain the lightness of a positive parity Θ . We will give this model further consideration when we look at spin-orbit splittings in Chapter 7.

6.2.2 “Flavour-Spin” models

Constituent Quark Models usually consider the binding dynamics in baryons to be due to the exchange of coloured objects between quarks, however some successful results arise if one instead considers exchanges to be flavoured rather than coloured. These models have many names, one of which is “Goldstone Boson Exchange” which suggests a possible origin for these exchanges - they are the flavoured Goldstone boson fields of Chiral Symmetry Breaking (pion, kaon...). There is much discussion of which model is more successful in the baryon sector [112,113].

Such models have been applied to the pentaquark sector. While some authors have used full, spatially dependent forms for these exchanges and the wavefunctions (e.g. [114]), others have employed the “schematic approximation” [115–119] where spatial dependence is neglected. In Chapter 9 we will discuss whether this approximation is good for a system in which there is a P -wave.

6.2.3 Jaffe & Wilczek - Light Scalar Diquarks

In [120] Jaffe and Wilczek proposed a phenomenological model for pentaquarks in terms of light scalar diquark degrees-of-freedom. While the dynamics causing the formation of diquarks were not specified, this model has the advantage of also possibly explaining the nature of the light scalar mesons ($a_0(980)$, $f_0(980)$...) as diquark-antidiquark states. A P -wave giving the pentaquarks positive parity is incorporated in a natural way. As much of what follows in Chapters 7, 8, 9 is based upon this model, we will now go into some detail in describing it.

The essential feature of the Jaffe-Wilczek model (hereafter JW) is the scalar diquark - a qq system strongly correlated into a spin-0, flavour and colour $\bar{\mathbf{3}}$, i.e. transforming like a scalar antiquark. Pentaquark states consist of two scalar diquarks and an antiquark, a concrete example being the Θ which has structure $[ud][ud]\bar{s}$. To obtain an overall colour singlet we must couple the diquark colours antisymmetrically to a $\mathbf{3}_c$, while to obtain a $\bar{\mathbf{10}}_F$ the diquark flavours must be coupled symmetrically to a $\bar{\mathbf{6}}_F$. Since the diquarks are identical bosons we demand that they satisfy Bose symmetry (except when they are very close together and can resolve each others fermion substructure) and hence the spatial wavefunction must be antisymmetric which can be achieved by invoking a P -wave between the diquarks.

In Table 6.1 we display the possible internal couplings of two diquarks and an antiquark.

Colour	$[qq][qq]$ Flavour	Space	Flavour	J^P	$[qq][qq]\bar{q}$ e.g.
$\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$	$\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$				
$\mathbf{3}$ (AS)	$\bar{\mathbf{6}}$ (S)	P -wave (AS)	$\bar{\mathbf{10}} \oplus \mathbf{8}$	$(\frac{1}{2}, \frac{3}{2})^+$	$[ud][ud]\bar{s}, \frac{1}{\sqrt{2}}([ud][us] + [us][ud])\bar{s}$
$\mathbf{3}$ (AS)	$\mathbf{3}$ (AS)	S -wave (S)	$\mathbf{8} \oplus \mathbf{1}$	$\frac{1}{2}^-$	$\frac{1}{\sqrt{2}}([ud][us] - [us][ud])\bar{s}$

Table 6.1: Possible couplings of two scalar diquarks and an anti-quark into colour-singlet baryons.

The octet and singlet of negative parity pentaquarks are discarded by JW - with an S -wave between the diquarks they are not spatially separated and can overlap resulting in a repulsive “Pauli blocking” effect due to the identical flavour pairs (e.g. uu in $[ud][us]$). The masses of these states are expected to be considerably higher than the positive parity states - we will consider this possibility in an explicit model for diquarks in Chapter 9.

JW thus predict a degenerate set of multiplets, $\bar{\mathbf{10}}(\frac{1}{2}^+)$, $\bar{\mathbf{10}}(\frac{3}{2}^+)$, $\mathbf{8}(\frac{1}{2}^+)$, $\mathbf{8}(\frac{3}{2}^+)$, with splittings within the multiplets expected from the breaking of $SU(3)_F$ by the strange quark mass difference. The spin- $\frac{3}{2}$ states are discarded by assuming them to be pushed up in mass by some unspecified spin-orbit effect; it is assumed that they become broad and do not produce observable resonances. We consider this in a dynamical model in Chapter 7.

We are thus left with a degenerate $(\overline{10} \oplus 8)(\frac{1}{2}^+)$ for which JW propose a phenomenological Hamiltonian incorporating $SU(3)_F$ breaking,

$$H_{JW} = M_0 + (n_s + n_{\bar{s}})\delta m_s + n_s \alpha. \quad (6.4)$$

The first term sets the scale of the $\overline{10}$ multiplet mass, the second term counts the number of strange quark masses and the third term allows $[ud]$ diquarks to have a different binding energy to $[us]$ diquarks. This Hamiltonian is diagonalised by “ideally mixed” states, those with or without hidden strangeness (like the $\phi = s\bar{s}$ and the $\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$).

The three unknowns in H_{JW} are set using:

- the Σ, Λ mass difference ($\alpha \approx 60$ MeV);
- the $N(1440)$ mass - where the Roper is interpreted as the $[ud][ud]\bar{d}$ state¹ ($M_0 \approx 1440$ MeV);
- the $\Theta(1540)$ mass ($\delta m_s \approx 100$ MeV).

Then the JW predictions for the masses of the members of the $(\overline{10} \oplus 8)(\frac{1}{2}^+)$ are as listed in Table 6.2.

State	Flavour	Mass/MeV	Possible State Assignment
Θ^+	$[ud][ud]\bar{s}$	<i>1540</i>	
N	$[ud][ud]\bar{d}$	<i>1440</i>	$N(1440)$ “Roper”
N_s	$[ud][us]\bar{s}$	1700	$N(1710)$
Λ	$[ud][ds]\bar{d} _{I=0}$	1600	$\Lambda(1600)$
Σ	$[ud][ds]\bar{d} _{I=1}$	1600	$\Sigma(1560), \Sigma(1660)?$
Σ_s	$[us][us]\bar{s}$	1850	?
Ξ	$[us][us]\bar{u}$	1750	?
$\Xi_{\frac{3}{2}}$	$[us][us]\bar{d}$	1750	?

Table 6.2: Jaffe-Wilczek ideally mixed $\overline{10} \oplus 8$. Masses in *italics* are used as inputs

The $N(1710)$ in this framework is a $\overline{10}$ -8 mixture and its observation in γp , which was a problem for a pure $\overline{10}$ can now be explained by the 8 component.

JW have addressed the problem of a low mass for the Θ by proposing a diquark structure and equating these diquarks to the possible constituents of the light scalar mesons; the effect is to render diquarks, and hence the Θ relatively light compared to naive quark counting expectations. What then, within this framework, is the reason for the extremely narrow width of the Θ state?

¹we will question the validity of this assignment in Chapter 8

The Θ (if it has $J^P = \frac{1}{2}^+$) can “fall-apart” into K^+n/K^0p in a P -wave, that is it can decay without pair-production. As an estimate of the width for this process JW calculate the width for a kaon to tunnel through a P -wave barrier from a 1 fm square well potential, 100 MeV above threshold. Their result² is above 175 MeV and to get down to 10 MeV they require a range of 0.05 fm, which does not look much like QCD.

A solution to this conundrum is to recognise that we have not computed all factors in the decay rate we have begun with a kaon formed inside the Θ which can then tunnel out - we should have also accounted for the probability for a kaon state to be formed. The Θ internal wavefunction is likely to differ in colour, spin, flavour and spatial terms from the product wavefunction of asymptotic NK states and the overlap < 1 will lead to some suppression relative to the 175 MeV potential tunneling estimate. We will discuss an explicit realisation of this in Chapters 7, 8.

Θ -like pentaquarks with heavy antiquarks, denoted $\Theta_{c,b}$ can be considered in the JW framework quite simply as the antiquark almost decouples from the scalar diquark system. As estimate for the mass of the Θ_c is

$$M(\Theta_c) \approx M(\Theta) + (M(\Lambda_c) - M(\Lambda)) \approx 2710 \text{ MeV} \quad (6.5)$$

where hyperfine effects for the s in Λ are assumed similar to those for $\bar{s}$ in Θ (the validity of this assumption will be considered in Chapter 9). 2710 MeV is below threshold for strong decay to ND and if this state exists at this mass it will be exceedingly narrow and must be searched for in charmless (weak-decay) end-states.

²But note that in a similar calculation by Jennings and Maltman [115] a width of 280 MeV is found with a range of 0.8 fm.

Chapter 7

Pentaquark Spin-Orbit Splittings

At the end of the previous chapter we gave a brief introduction to the pentaquark model of Jaffe and Wilczek. Their proposed internal structure for the Θ gave them, in addition to the desired $J^P = \frac{1}{2}^+$ state, a degenerate $J^P = \frac{3}{2}^+$ state, with otherwise identical quantum numbers. In [120] they discarded such states, assuming that some spin-orbit dynamics would act to push them up in mass and make them broad. In this chapter we will investigate the spin-orbit splitting within a model analogous to that used with some success in the conventional sector.

Empirically conventional mesons and baryons have small spin-orbit splittings. In the case of the mesons, this has been explained in the quark model by proposing that there are both short-range Lorentz vector and a long-range Lorentz scalar interactions between quarks. These each produce a large spin-orbit effect, but with opposite signs, such that the net spin-orbit splitting is small. Early studies(e.g. [121]) seemed to suggest that such a cancellation was not possible in the baryon sector owing to the presence of so-called “three-body terms”, but recent work [122] seems to suggest that, at least for a certain class of baryons, a cancellation can occur.

7.1 Isgur model for baryon spin-orbit splittings

We intend to treat the JW pentaquark as an effective three-body system bound by QCD-inspired quark model forces. This picture is a close analogue of the conventional baryons. Work by Isgur [122] showed that certain baryon spin-orbit splittings are well described by the quark model providing one correctly deals with the so-called “three-body terms” It will be worthwhile to repeat Isgur’s calculation here to emphasize the important features and clarify some ambiguities in his text.

The Λ_c^* baryons, $\Lambda_c(1/2^-)(2593)$, $\Lambda_c(3/2^-)(2625)$ are $L = 1$ excited udc states. The ud are in an isosinglet, colour **3**, and Isgur proposes that they have $S = 0, L = 0$. The state with $S_{ud} = 1, L_{ud} = 1$ is assumed to be much higher in the spectrum and to have negligible mixing with the spin-singlet¹. Hence the unit of angular momentum is between the c -quark and the centre-of-mass of the ud pair, i.e. with respect to $\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_u + \vec{r}_d - 2\vec{r}_c)$ (see Figure 7.1).

Isgur uses the standard quark model binding forces; Lorentz vector, Coulomb-like one gluon exchange and a Lorentz scalar linear confining potential. Such an assignment of forces has met with much success in describing the meson spectrum, where in particular there is a strong cancellation between large vector and scalar components leading to the empirically observed small spin-orbit splittings in the meson spectrum.

The non-relativistic reduction of vector and scalar exchanges between spin-1/2 quarks in a general frame yields the following spin-orbit Hamiltonians (for $\vec{r} = \vec{r}_1 - \vec{r}_2$);

$$\begin{aligned} H_V^{SO} &= \left(\frac{\vec{\sigma}_1 \cdot \vec{r} \times \vec{p}_1}{4m_1^2} - \frac{\vec{\sigma}_2 \cdot \vec{r} \times \vec{p}_2}{4m_2^2} + \frac{\vec{\sigma}_2 \cdot \vec{r} \times \vec{p}_1}{2m_1m_2} - \frac{\vec{\sigma}_1 \cdot \vec{r} \times \vec{p}_2}{2m_1m_2} \right) \frac{1}{r} \frac{dV_V}{dr} \\ H_S^{SO} &= - \left(\frac{\vec{\sigma}_1 \cdot \vec{r} \times \vec{p}_1}{4m_1^2} - \frac{\vec{\sigma}_2 \cdot \vec{r} \times \vec{p}_2}{4m_2^2} \right) \frac{1}{r} \frac{dV_S}{dr}. \end{aligned} \quad (7.1)$$

This structure includes the Thomas precession terms which occur for purely kinematic reasons - hence their presence even for a scalar potential, where the exchange particle carries no spin information.

The full spin-orbit Hamiltonian for a baryon consists of the sum, pairwise, of exchanges between the three quarks. For the Λ_c^* baryons with $S_{ud} = 0$ this reduces to the relatively simple form

$$\begin{aligned} H_{S+V}^{SO} &= - \frac{1}{4m_3^2} \vec{\sigma}_3 \cdot [\vec{r}_{13}(v_{13} - s_{13}) + \vec{r}_{23}(v_{23} - s_{23})] \times \vec{p}_3 \\ &\quad + \frac{1}{2mm_3} \vec{\sigma}_3 \cdot [\vec{r}_{13}v_{13} \times \vec{p}_1 + \vec{r}_{23}v_{23} \times \vec{p}_2], \end{aligned}$$

where e.g. $v_{13} = \frac{1}{r_{13}} \frac{dV_V}{dr_{13}}$.

A convenient set of variables is $\vec{\rho}, \vec{\lambda}$ defined by;

$$\begin{aligned} \vec{r}_{1,2} &= \vec{R} + \sqrt{\frac{3}{2}} \frac{m}{2m_0 + m} \vec{\lambda} \pm \frac{1}{\sqrt{2}} \vec{\rho} \\ \vec{r}_3 &= \vec{R} - \sqrt{\frac{3}{2}} \frac{2m_0}{2m_0 + m} \vec{\lambda}, \end{aligned}$$

¹The ud system looks rather diquark-like and Isgur notes as much, describing this as a “meson-like baryon”

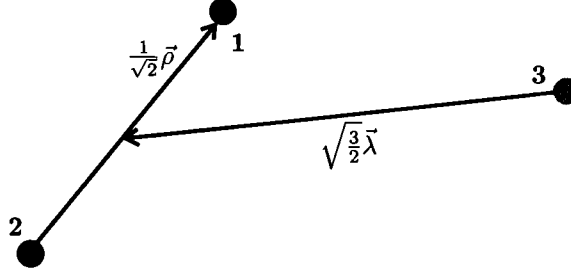


Figure 7.1: 3-body variables

where $m_1 = m_2 = m_0$ and $m_3 = m$. See Figure 7.1 for a visualisation. In the baryon rest frame the internal momenta are

$$\begin{aligned}\vec{p}_{1,2} &= \frac{1}{\sqrt{6}}\vec{p}_\lambda \pm \frac{1}{\sqrt{2}}\vec{p}_\rho \\ \vec{p}_3 &= -\sqrt{\frac{2}{3}}\vec{p}_\lambda.\end{aligned}$$

With these variables H_{S+V}^{SO} contains terms proportional to

$$\vec{\sigma}_3 \cdot \vec{\lambda} \times \vec{p}_\lambda \quad \vec{\sigma}_3 \cdot \vec{\rho} \times \vec{p}_\rho \quad \vec{\sigma}_3 \cdot \vec{\rho} \times \vec{p}_\lambda \quad \vec{\sigma}_3 \cdot \vec{\lambda} \times \vec{p}_\rho.$$

The first of these is clearly what we want for a spin-orbit splitting term $\sim \vec{S}_3 \cdot \vec{L}_\lambda$; the second is trivially zero since $L_\rho = 0$; the third and fourth are the “three-body terms”. The fourth term does not in fact contribute - $\vec{p}_\rho$ acting on the $L_\rho = 0$ wavefunctions gives something in the direction of $\hat{\rho}$ - when we perform the Ω_ρ integration we average over directions of $\hat{\rho}$ and obtain something in the direction of $\hat{\lambda}$ (equation (7.2)) and hence this term $\sim \hat{\lambda} \times \hat{\lambda} = 0$. The $\vec{\sigma}_3 \cdot \vec{\rho} \times \vec{p}_\lambda$ term is often considered to be problematical but is in fact vital to obtain a consistent answer as we shall see.

We thus have the remaining terms,

$$\begin{aligned}& \frac{1}{4m_3^2} \vec{\sigma}_3 \cdot [\vec{r}_{13}(v_{13} - s_{13}) + \vec{r}_{23}(v_{23} - s_{23})] \times \sqrt{\frac{2}{3}}\vec{p}_\lambda \\ & + \frac{1}{4mm_3} \vec{\sigma}_3 \cdot [\vec{r}_{13}v_{13} + \vec{r}_{23}v_{23}] \times \sqrt{\frac{2}{3}}\vec{p}_\lambda\end{aligned}$$

where we recall that $\vec{r}_{13}$ contains both $\vec{\rho}$ and $\vec{\lambda}$ components. $-\vec{r}_{13}v_{13}$ for example has a simple physical interpretation as the force on the c -quark due to the Coulomb potential from the u -quark,

$$-\vec{r}_{13}v_{13} = -\frac{\vec{r}_{13}}{r_{13}} \frac{dV_V}{dr_{13}} = -\vec{\nabla}_{13}V_V = \vec{F}_{13}^V.$$

In computing the matrix element of the Hamiltonian and hence the spin-orbit splitting one

comes across integrals of the form $\int d\Omega_\rho |Y_0^0(\Omega_\rho)|^2 \vec{F}_{(1,2)3}^{(V,S)}$ which are somewhat non-trivial. I will demonstrate here how one can compute them. Very similar integrals will be required in the pentaquark case.

Consider for example the Coulomb potential², $V_V = -\frac{2\alpha_S}{3} \frac{1}{r}$. The force is given by $\vec{F}_{13}^V = -\frac{2\alpha_S}{3} (\sqrt{3/2} \vec{\lambda} + \sqrt{1/2} \vec{\rho}) / r_{13}^3$. Using the standard multipole expansion for $1/r$ (which differs inside and outside the spherical shell defined by $\rho = \sqrt{3}\lambda$) we find

$$\begin{aligned} \frac{1}{r_{(1,2)3}} &= \left(\frac{1}{2} \rho^2 \pm \sqrt{3} \lambda \rho \cos \tilde{\theta} + \frac{3}{2} \lambda^2 \right)^{-1/2} \\ &= \sum_L P_L(\cos \tilde{\theta}) \left[\frac{1}{\frac{1}{\sqrt{2}} \rho} \left(\mp \frac{\sqrt{3} \lambda}{\rho} \right)^L \Theta(\rho - \sqrt{3} \lambda) + \frac{1}{\sqrt{\frac{3}{2}} \lambda} \left(\mp \frac{\rho}{\sqrt{3} \lambda} \right)^L \Theta(\sqrt{3} \lambda - \rho) \right]. \end{aligned}$$

We obtain r^{-3} by differentiation:

$$\begin{aligned} \frac{1}{r_{(1,2)3}^3} &= \mp \frac{2}{\sqrt{3} \lambda \rho} \frac{d}{d \cos \tilde{\theta}} \left(\frac{1}{r_{(1,2)3}} \right) \\ &= 4\pi \sum_{L,m} Y_L^m(\Omega_\rho) Y_L^{m*}(\Omega_\lambda) \left[\frac{2\sqrt{2}}{\rho^3(1-3\lambda^2/\rho^2)} \left(\mp \frac{\sqrt{3} \lambda}{\rho} \right)^L \Theta(\rho - \sqrt{3} \lambda) \right. \\ &\quad \left. + \frac{2\sqrt{2}/3\sqrt{3}}{\lambda^3(1-\rho^2/3\lambda^2)} \left(\mp \frac{\rho}{\sqrt{3} \lambda} \right)^L \Theta(\sqrt{3} \lambda - \rho) \right], \end{aligned}$$

where use has been made of recursion relations for the P_L and the expression $P_L(\cos \tilde{\theta}) = 4\pi/(2L+1) \sum_m Y_L^m(\theta_\rho, \phi_\rho) Y_L^{m*}(\theta_\lambda, \phi_\lambda)$ [123]. Written in this form, the required angular integrals are immediate³:

$$\begin{aligned} \int d\Omega_\rho |Y_0^0(\Omega_\rho)|^2 \frac{\vec{\lambda}}{r_{(1,2)3}^3} &= \vec{\lambda} \frac{2\sqrt{2}}{3\sqrt{3}} \left[\frac{3\sqrt{3}}{\rho^3(1-3\lambda^2/\rho^2)} \Theta(\rho - \sqrt{3} \lambda) + \frac{1}{\lambda^3(1-\rho^2/3\lambda^2)} \Theta(\sqrt{3} \lambda - \rho) \right] \\ \int d\Omega_\rho |Y_0^0(\Omega_\rho)|^2 \frac{\vec{\rho}}{r_{(1,2)3}^3} &= \mp \vec{\lambda} \frac{2\sqrt{2}}{3} \left[\frac{3\sqrt{3}}{\rho^3(1-3\lambda^2/\rho^2)} \Theta(\rho - \sqrt{3} \lambda) + \frac{\rho^2/3\lambda^2}{\lambda^3(1-\rho^2/3\lambda^2)} \Theta(\sqrt{3} \lambda - \rho) \right]. \end{aligned} \quad (7.2)$$

Hence we find for the forces,

$$\int \frac{d\Omega_\rho}{4\pi} \vec{F}_{23}^V = \int \frac{d\Omega_\rho}{4\pi} \vec{F}_{13}^V = -\frac{2\alpha_S}{3} \frac{\sqrt{\frac{3}{2}} \vec{\lambda}}{\left(\sqrt{\frac{3}{2}} \lambda\right)^3} \Theta(\sqrt{3} \lambda - \rho).$$

²the numerical factor is the expectation of $\vec{\lambda}_1 \cdot \vec{\lambda}_2$ in a colour state $\mathbf{3}_1 \otimes \mathbf{3}_2 \rightarrow \bar{\mathbf{3}}_{12}$.

³but note that equation (7.2) does not agree with the statement made in [122] that this quantity is zero. Isgur's results only hold with what I present here, his statement in the text that this is zero is inconsistent with his calculation and is presumably a mistake

This is obvious when one remembers Gauss's Law - averaging over Ω_ρ defines a spherical shell $\sqrt{3}\lambda = \rho$ which acts like a uniform spherical shell of charge - when the c -quark is inside the shell it feels no force and when it is outside it feels a Coulomb force from the centre of the sphere.

From this analysis it is clear that inclusion of the $\vec{\sigma}_3 \cdot \vec{\rho} \times \vec{p}_\lambda$ "three-body" term leads to something of the form $\vec{S}_3 \cdot \vec{L}_\lambda$ and is vital to ensure consistency.

An analogous procedure can be followed for the scalar potential⁴, $V_S = \frac{1}{2}br$, leading to the following results for the spin-orbit splitting:

$$\begin{aligned} H_{\text{Thom}}^V &= - \left(\frac{2}{3} \right)^{5/2} \frac{\alpha_S}{m_3^2} \frac{\vec{S}_3 \cdot \vec{L}_\lambda}{\lambda^3} \Theta(\sqrt{3}\lambda - \rho) \\ H_{\text{dyn}}^V &= - \left(2 + \frac{m_3}{m} \right) H_{\text{Thom}}^V \\ H_{\text{Thom}}^S &= - \frac{1}{\sqrt{6}} \frac{b}{m_3^2} \vec{S}_3 \cdot \vec{L}_\lambda \left[\frac{1}{\lambda} (1 - \rho^2/9\lambda^2) \Theta(\sqrt{3}\lambda - \rho) + \frac{2}{\sqrt{3}\rho} \Theta(\rho - \sqrt{3}\lambda) \right]. \end{aligned}$$

Parameterising the Λ_c^* wavefunctions by Gaussians

$$Y_0(\Omega_\rho) \frac{2}{\pi^{1/4}} \alpha_\rho^{3/2} \exp -\alpha_\rho^2 \rho^2 / 2 \cdot Y_1^m(\Omega_\lambda) \frac{2}{\pi^{1/4}} \sqrt{\frac{2}{3}} \alpha_\lambda^{5/2} \lambda \exp -\alpha_\lambda^2 \lambda^2 / 2$$

we obtain

$$\begin{aligned} H_{\text{Thom}}^V &= - \frac{16\sqrt{2}}{9\sqrt{\pi}} \frac{\alpha_S}{m_3^2} \vec{S}_3 \cdot \vec{L}_\lambda \left(\frac{\alpha_\lambda}{\sqrt{3+k^2}} \right)^3 \\ H_{\text{Thom}}^S &= - \frac{2\sqrt{2}}{3\sqrt{\pi}} \frac{b}{m_3^2} \vec{S}_3 \cdot \vec{L}_\lambda \frac{\alpha_\lambda}{\sqrt{3+k^2}}, \end{aligned}$$

where $\alpha_\lambda = k\alpha_\rho$. An estimate for α_ρ comes from fitting the experimental Σ_c, Λ_c hyperfine splitting,

$$M(\Sigma_c) - M(\Lambda_c) = \frac{16\pi}{9} \frac{\alpha_S}{m} \left(\frac{1}{m} - \frac{1}{m_c} \right) \left(\frac{\alpha_\rho}{\sqrt{2\pi}} \right)^3,$$

whence $\alpha_\rho \approx 470\text{MeV}$. An estimate for α_λ comes from the P -wave mass gap in the Λ_c sector: $\omega_P = \left[\frac{2}{3}m(\Lambda_c^*(\frac{3}{2}^-)) + \frac{1}{3}m(\Lambda_c^*(\frac{1}{2}^-)) \right] - m(\Lambda_c(\frac{1}{2}^+)) \approx 330\text{MeV}$. With effective harmonic oscillator wavefunctions, $\omega_P = \frac{\alpha_\lambda^2}{m_\lambda}$ where $m_\lambda = \frac{3m_c m}{2m+m_c} \approx 730\text{MeV}$ and hence $\alpha_\lambda \approx 490\text{MeV}$.

With these values $k \approx 1.04$, which is only about a 20% deviation from the value we would expect on the basis of identical harmonic oscillator forces between each quark, $\tilde{k} = \left(\frac{3m_c}{2m+m_c} \right)^{1/4} \approx 1.22$. With other standard quark model parameters taken from [122],

⁴note that this formulation of the scalar potential differs at short distances from the form used by Isgur. This choice is consistent with that used to describe mesons and we will see leads to results very similar to Isgur's

we find (in MeV):

$$\begin{array}{ccc|c|c} \delta E_{\text{dyn}}^V & \delta E_{\text{Thom}}^V & \delta E_{\text{Thom}}^S & \delta E_{\text{tot}} & \delta E_{\text{expt}} \\ +41 & -6 & -11 & +24 & 33 \pm 1 \end{array}$$

Isgur [122] has a more complete model in which $\alpha_\rho, \alpha_\lambda$ are determined by the assumed dynamics and in which he finds

$$\begin{array}{ccc|c|c} \delta E_{\text{dyn}}^V & \delta E_{\text{Thom}}^V & \delta E_{\text{Thom}}^S & \delta E_{\text{tot}} & \delta E_{\text{expt}} \\ +75 & -10 & -13 & +52 & 33 \pm 1 \end{array}$$

That the prediction using the “experimentally” obtained α ’s is 50% lower is consistent with the expectation voiced in [122] that these predictions are only likely to be good to around 50% due to e.g. matrix element uncertainties. Given this caveat we have reasonable agreement with the data. It would appear that the consistent inclusion of the “three-body” terms in this “meson-like baryon” can give us sensible predictions for spin-orbit splittings. Inspired by this we can apply similar methods to the JW pentaquark.

7.2 Spin-orbit splittings in Jaffe-Wilczek pentaquark picture

One possible version⁵ of the Jaffe-Wilczek model has the scalar diquarks as relatively compact⁶ objects kept apart by a P -wave. This is self-consistent since provided the P -wave keeps the diquarks sufficiently far apart they will not resolve their spin-1/2 substructure and the diquark-diquark wavefunction is forced to be Bose symmetric and hence in a P -wave.

In this picture the Θ is therefore an effective 3-body system of two spin-0 diquarks and an antiquark with each of the objects in a color $\bar{3}$ coupled anti-symmetrically to a color singlet. For compact diquarks this is rather like an (anti-)baryon with two spinless antiquarks. We will assume that since the color representations are the same, the lowest order potentials are the same as in a conventional baryon. The higher order effects e.g. the spin-orbit terms will come from non-relativistically reducing the standard quark model⁷ exchanges but now between a spin-0, $\bar{3}_c$ and a spin-1/2 $\bar{3}_c$ ⁷.

The non-relativistic reduction of vector and scalar exchanges between a scalar and an antiquark can be performed in a manner analogous to the reduction for two fermions (see [124]).

⁵JW do not specify detailed dynamics for their model

⁶relative to the typical distance between them

⁷exchanges between the scalar diquarks will not contribute to the spin-orbit splitting

Vector Exchange, $\Gamma = \gamma^\mu$ The scattering matrix element can be written in terms of the fermion and scalar currents,

$$\mathcal{M}^V = j_f^\mu(p', p) D_{\mu\nu}^V(q) j_0^\nu(k', k),$$

where $j_f^\mu(p', p) = \bar{u}(p')\gamma^\mu u(p)$ has non-relativistic reduction,

$$\begin{aligned} j_f^0 &\rightarrow 2m\chi'^\dagger \left(1 - \frac{|\vec{q}|^2}{8m^2} + i \frac{\vec{\sigma} \cdot \vec{q} \times \vec{p}}{4m^2} + \dots \right) \chi \\ \vec{j}_f &\rightarrow \chi'^\dagger (\vec{p}' + \vec{p} + i\vec{\sigma} \times \vec{q} + \dots) \chi, \end{aligned}$$

and where

$$\begin{aligned} j_0^0 &\rightarrow 2m_0 + \dots \\ \vec{j}_0 &\rightarrow \vec{k}' + \vec{k} + \dots \end{aligned}$$

The vector propagator can be written in Feynman gauge as $D_{\mu\nu} = g_{\mu\nu}D$ so that the matrix element, retaining only the lowest order spin-independent and dependent terms is

$$\mathcal{M}^V = 4mm_0\chi'^\dagger \left(1 + i \frac{\vec{\sigma} \cdot \vec{q} \times \vec{p}}{4m^2} - i \frac{\vec{\sigma} \cdot \vec{q} \times \vec{k}}{2mm_0} \right) D^V(q)\chi.$$

From this we can extract the particle interaction operator (which appears in the Hamiltonian) in the momentum representation,

$$U^V(\vec{p}, \vec{k}, \vec{q}) = -D^V(q) \left(1 + i \frac{\vec{\sigma} \cdot \vec{q} \times \vec{p}}{4m^2} - i \frac{\vec{\sigma} \cdot \vec{q} \times \vec{k}}{2mm_0} \right).$$

Fourier transforming,

$$U^V(\vec{p}, \vec{k}; \vec{r}) = \int \frac{d^3\vec{q}}{(2\pi)^{3/2}} e^{i\vec{q} \cdot \vec{r}} [-D^V(q)] \left(1 + i \frac{\vec{\sigma} \cdot \vec{q} \times \vec{p}}{4m^2} - i \frac{\vec{\sigma} \cdot \vec{q} \times \vec{k}}{2mm_0} \right).$$

Using $\vec{q} = -i\vec{\nabla}$ under the integral and the definition of the static potential as the Fourier transform of the propagator,

$$V(r) = \int \frac{d^3\vec{q}}{(2\pi)^{3/2}} e^{i\vec{q} \cdot \vec{r}} [-D(q)]$$

we obtain for the spin-orbit interaction

$$U_{SO}^V(\vec{p}, \vec{k}; \vec{r}) = \left(\frac{\vec{\sigma} \cdot \vec{r} \times \vec{p}}{4m^2} - \frac{\vec{\sigma} \cdot \vec{r} \times \vec{k}}{2mm_0} \right) \frac{1}{r} \frac{dV_V}{dr},$$

which has both dynamic and Thomas-type terms.

Scalar Exchange, $\Gamma = 1$ In this case we have

$$\mathcal{M}^S = \bar{u}(p')u(p)D^S(q)2m_0,$$

which yields a pure Thomas-type spin-orbit interaction

$$U_{SO}^S(\vec{p}, \vec{k}; \vec{r}) = -\frac{\vec{\sigma} \cdot \vec{r} \times \vec{p}}{4m^2} \frac{1}{r} \frac{dV_S}{dr}.$$

The total spin-orbit Hamiltonian for vector and scalar exchanges between an anti-quark and a pointlike scalar diquark is,

$$H_{SO} = \frac{\vec{\sigma} \cdot \vec{r} \times \vec{k}}{4m^2}(v - s) - \frac{\vec{\sigma} \cdot \vec{r} \times \vec{p}}{2mm_0}v. \quad (7.3)$$

Note that this agrees with equation(7.1) if one sets $\vec{\sigma}_2 = 0$. The full spin-orbit Hamiltonian for this system is obtained by summing the pairwise exchanges, leading to $H_{SO} =$

$$\begin{aligned} & \left\{ \left(1 + \frac{m}{m_0} \right) (v_{13} + v_{23}) - (s_{13} + s_{23}) \right\} \frac{\vec{\sigma} \cdot \vec{\lambda} \times \vec{p}_\lambda}{4m^2} + (v_{13} + v_{23}) \frac{\vec{\sigma} \cdot \vec{\rho} \times \vec{p}_\rho}{4mm_0} \\ & + \frac{1}{\sqrt{3}} \left\{ \left(1 + \frac{m}{m_0} \right) (v_{13} - v_{23}) - (s_{13} - s_{23}) \right\} \frac{\vec{\sigma} \cdot \vec{\rho} \times \vec{p}_\lambda}{4m^2} + \sqrt{3}(v_{13} - v_{23}) \frac{\vec{\sigma} \cdot \vec{\lambda} \times \vec{p}_\rho}{4mm_0}, \end{aligned} \quad (7.4)$$

where 1, 2 are the diquarks and 3 is the anti-quark. •

The principal difference between this system and the Λ_c^* baryon considered previously is that now the $L = 1$ is in the ρ variable. This leads to a considerable simplification in the spin-orbit dynamics of the Θ in that the terms proportional to $\vec{p}_\lambda$ in eqn(7.4) are zero; the $\vec{\lambda} \times \vec{p}_\lambda$ term trivially so since $L_\lambda = 0$, and the $\vec{\rho} \times \vec{p}_\lambda$ term $\sim \vec{\rho} \times \hat{\rho} = 0$ after integration over Ω_λ . These zeros remove all spin-orbit effects of the scalar confining potential. The remaining spin-orbit Hamiltonian is simply proportional to the difference of Coulomb forces:

$$H_{SO} = \frac{\sqrt{2}}{4mm_0} \vec{\sigma} \cdot [\vec{r}_{13}v_{13} - \vec{r}_{23}v_{23}] \times \vec{p}_\rho.$$

One can perform the angular integral over Ω_λ using expressions analogous to equations(7.2) to find

$$H_{SO} = \frac{4\sqrt{2}}{3} \frac{\alpha_S}{mm_0} \frac{\vec{S} \cdot \vec{L}_\rho}{\rho^3} \Theta(\rho - \sqrt{3}\lambda),$$

which is exactly what we would expect from Gauss's law(recall that the vector potential is Coulomb-like). If again we parameterise the baryon wavefunction with Gaussians (but with $L_\rho = 1$ and $L_\lambda = 0$) we find for the $3/2^+ - 1/2^+$ splitting,

$$\frac{8\sqrt{2}}{3\sqrt{\pi}} \frac{\alpha_S}{mm_0} \alpha_\rho^3 \frac{k^3}{(3 + k^2)^{3/2}}, \quad (7.5)$$

where the diquark mass m_0 and the wavefunction parameter α_ρ need to be determined and $k = \frac{\alpha_\lambda}{\alpha_\rho} \approx \tilde{k} = \left(\frac{3m_s}{2m_0 + m_s} \right)^{1/4}$ if the diquarks feel the same forces as (anti)quarks.

Jaffe and Wilczek do not explicitly state the masses of their diquarks. The standard colour-spin interaction in first-order of perturbation theory would give $m_0 \sim 500\text{MeV}$, but this is hard to confront with a $\Theta(1540)$ containing two diquarks, together with m_s and a P -wave excitation energy (ω_P) as well. Further, one should ensure that $[ud]_0[ud]_0[ud]_0$ with two P -waves, is not more stable than the deuteron. This requires that $\omega_P \gtrsim 450\text{MeV}$. Such a result is at least consistent with the (anti)baryon spectrum, which has the same internal colour arrangement and by assumption similar binding dynamics for which the energy gap between $(m_N + m_\Delta)/2$ and the negative parity $N^*(1520 - 1750)$ is $\sim 500\text{MeV}$. This leaves $\sim 1 - 1.1\text{GeV}$ to be shared between two diquarks and the $\bar{s}$. With $m_s \sim 450\text{MeV}$ we thus assign a mass of $\sim 350\text{MeV}$ to each diquark. This is the minimum we can tolerate, and even with this we shall find that the $\Theta^* - \Theta$ mass gap is only tens of MeV; any larger mass for the diquarks would reduce it even further. This is line with the very rough estimate we would obtain from considering the light scalar mesons to be diquark-antidiquark states. The $\sigma = f_0(600)$ is interpreted as $[ud]_0[\overline{u\bar{d}}]_0$ and hence taking the central mass of the broad Breit-Wigner we would guess $m([ud]) \sim 300\text{MeV}$ for the effective mass of a diquark. •

We can make an estimate of the $\Theta^{(*)}$ spatial wavefunction from baryon measurements. Assuming that the diquark environment in the Θ is rather like the quark environment in the N , which is reasonable given approximate $SU(3)$ flavour symmetry and the fact that the colour structures are identical, we can express the harmonic oscillator parameter, α_ρ in terms of the $\Delta - N$ splitting,

$$M(\Delta) - M(N) \approx 300\text{MeV} = \sqrt{\frac{8}{9\pi}} \frac{\alpha_S}{m_u^2} \alpha_\rho^3.$$

Eliminating α_ρ from equation(7.5) we obtain for the Θ^*, Θ splitting

$$M(\Theta^*) - M(\Theta) = \frac{m_u}{m_s} \frac{m_u}{m_0} \frac{4k^3}{(3 + k^2)^{3/2}} 300\text{MeV}. \quad (7.6)$$

Using a diquark mass range $380 \rightarrow 550$ MeV, we would estimate from equation (7.6) a Θ^* 60-100 MeV above the Θ . Alternatively we can attempt to set $\alpha_\lambda, \alpha_\rho$ using a dynamical model for the Jaffe-Wilczek correlation. One possible model is outlined in Chapter 9, where a JW correlation is found to be possible with about the right mass to match with the experimental Θ . The diquarks are not pointlike, but they are small on the scale of their separation and we find harmonic binding parameters which correspond to $\alpha_\rho \approx 480\text{MeV}, \alpha_\lambda \approx 460\text{MeV}$ and hence $k \approx 0.96$. With 350 MeV diquarks we would have $\tilde{k} \approx 1.04$ which differs by less than 10%. Using these values in equation 7.5 we find a splitting of ~ 80 MeV, which lies within the range predicted using the Δ, N splitting as input.

Other members of the pentaquark $\overline{10}$ and 8 will have similar splittings modified slightly by the different diquark and anti-quark masses. The possible admixture of 3-quark components into the non-exotic pentaquark states might change considerably their splittings which offers a way out of the quandary that there is no known P_{13} partner to the $P_{11}(1440)$ Roper around 1500 MeV. Admixture of a 3-quark component is also suggested by the decay systematics of this state (see [5]). For the $[(ud)_0][(us)_0]\bar{s}$ state, the mass gap to the $3/2^+$ partner is predicted to be several tens of MeV. This is somewhat too large to describe the $P_{11}(1710) - P_{13}(1720)$ pair.

The lack of obvious success in predicting the splittings for the non-exotic states here may indicate that it is only to the flavour exotic states, where no qqq component can be admixed, that we can apply our method. The dynamics of $\vec{L}, \vec{S}$ may therefore be probed also by the exotic Ξ^+, Ξ^{--} and the heavy flavour analogues $[ud][ud]\bar{Q}$, with $Q \equiv c, b$. We estimate $\Delta m(\Xi^* - \Xi) \sim 60 - 90\text{MeV}$. The reason that this is similar to $\Delta m(\Theta^* - \Theta)$ is due to the effect of the heavier s mass being “diluted” within clusters, and the $\bar{s}$ being replaced by the lighter $\bar{d}(\bar{u})$. The splitting scales as the inverse constituent mass for large m_Q (equation(7.5)) and hence the splittings for $\Theta_c^* - \Theta_c$ are likely to be as low as a few MeV.

How can we explain the small splitting for the $\Theta - \Theta^*$ in light of the absence of scalar exchanges, which for the Λ_c cancelled much of the effect of the vector exchange? The explanation lies in Gauss’s law which here has meant that there is only a splitting effect when the diquarks lie outside the spherical shell defined by averaging over directions of the $\bar{q}$. Were this not the case the force on the diquark would go like ρ^{-3} , the inclusion of Gauss’s law introduces the

“effective charge” contained within the sphere of radius $\rho/\sqrt{2}$ so that the force $\sim q_{\text{eff}}(\rho)/\rho^3$. $q_{\text{eff}}(\rho)$ interpolates between 0 at $\rho = 0$ and 1 as $\rho \rightarrow \infty$ (see Figure 7.2). The diquark radial wavefunction is peaked around $\rho \sim 1/\alpha_\rho$ where $q_{\text{eff}} < 0.5$, so there is significant suppression.

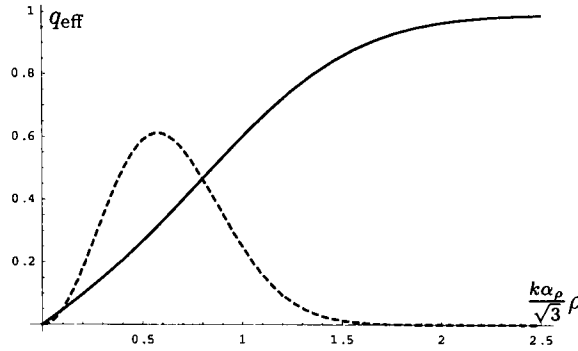


Figure 7.2: Effective charge q_{eff} as a function of ρ , solid curve. Dashed curve is $|R(\rho)|^2$ with arbitrary normalisation.

7.2.1 Width of Θ^*

In Section 6.2.3 we entertained the possibility that the decay width of the Θ can be explained in terms of a kaon tunneling through a P -wave barrier with suppression from the limited overlap between the Θ wavefunction and the KN product wavefunction. We outline here a possible realisation of this suggestion and indicate how it relates the Θ^* and Θ widths.

Colour Overlap While this could be computed using $SU(3)$ recoupling coefficients, we can avoid introducing the technology by using explicit representations. In the JW framework the Θ has two diquarks (each $\mathbf{3} \otimes \mathbf{3} \rightarrow \bar{\mathbf{3}}$) coupled antisymmetrically to a $\mathbf{3}$. The Θ colour state can be written,

$$|\Theta\rangle_c = \frac{1}{\sqrt{3}}(R''\bar{R} + G''\bar{G} + B''\bar{B}),$$

where the double prime indicates that this is the representation $\mathbf{6f}$ of $\mathbf{3}$ made from $\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$; explicitly,

$$R'' = \frac{(\bar{G}'\bar{B}' - \bar{B}'\bar{G}')}{\sqrt{2}}.$$

The single primes indicate $\mathbf{3} \otimes \mathbf{3} \rightarrow \bar{\mathbf{3}}$ so that in full,

$$R'' = \frac{1}{\sqrt{2}} \left(\frac{BR - RB}{\sqrt{2}} \frac{RG - GR}{\sqrt{2}} - \frac{RG - GR}{\sqrt{2}} \frac{BR - RB}{\sqrt{2}} \right),$$

with similar expressions for G'' , B'' . The kaon and nucleon colour states are the unique colour singlets,

$$|K\rangle_c = \frac{1}{\sqrt{3}}(R\bar{R} + G\bar{G} + B\bar{B}),$$

$$|N\rangle_c = \frac{1}{\sqrt{3}} \left(\frac{RB - BR}{\sqrt{2}}G - \frac{RG - GR}{\sqrt{2}}B - \frac{GB - BG}{\sqrt{2}}R \right).$$

Hence we find an overlap

$$\langle \Theta | NK \rangle_c = 3 \times \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \left[(-1)(-\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}} \right] = \frac{1}{\sqrt{3}}.$$

Flavour-Spin Overlap

$$|\Theta\rangle_F = \frac{ud - du}{\sqrt{2}} \frac{ud - du}{\sqrt{2}} \bar{s} = \frac{1}{\sqrt{2}} \left(\left[\frac{ud - du}{\sqrt{2}} u \right] [d\bar{s}] - \left[\frac{ud - du}{\sqrt{2}} d \right] [u\bar{s}] \right)$$

Which overlaps with the mixed-antisymmetric part of the nucleon flavour wavefunction with value $\frac{1}{\sqrt{2}}$. The $[qq][qq]\bar{q}$ spin wavefunction with the antiquark spin-up is

$$|\Theta\rangle_\sigma = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \uparrow = \frac{1}{\sqrt{2}} \left(\left[\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \uparrow \right] \frac{1}{\sqrt{2}} [V_0 - P] - \left[\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \downarrow \right] V_+ \right),$$

where $V_{0,+}$ is a vector meson with spin-component 0, +1 and P is a pseudoscalar meson. The flavour-spin overlap with nK^+ is then $-1/4$. This is a factor $\frac{1}{\sqrt{2}}$ different from this result as computed in [115], which is presumably because they have summed over nK^+ , pK^0 modes.

Spatial Overlap In [4] the approximation was made that the only spatial suppression for the Θ decay comes from the spin-orbit Clebsch-Gordan coefficient $\langle 1, 0; \frac{1}{2}, \frac{1}{2} | \frac{3}{2}, \frac{1}{2} \rangle$. This is clearly very crude and to make a believable statement about the absolute spatial suppression we would require a model for the internal dynamics of the diquarks which allows us to compute the cost of “cleaving” a diquark - steps in this direction have recently been taken [125].

However, it is the case that the relative spatial suppression of Θ^* versus Θ is simply the ratio of the differing spin-orbit Clebsch-Gordan’s,

$$\frac{\langle 1, 0; \frac{1}{2}, \frac{1}{2} | \frac{3}{2}, \frac{3}{2} \rangle}{\langle 1, 0; \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle} = \sqrt{2}.$$

This is, in fact, the only difference between Θ^* , Θ decay suppression in terms of wavefunction overlaps and on this basis we would expect a Θ^* width roughly twice the Θ width, with a

small correction for the increased phase space due to the larger Θ^* mass. If, however, the decay proceeds as suggested by tunneling through a P -wave barrier, the increase in width can be far more dramatic as the tunneling rate is exponentially dependent on the distance from threshold.

The suppression from colour-flavour-spin alone corresponds to a total width suppression of $1/24$ for the Θ ; with the spin-orbit Clebsch-Gordan and possibly further spatial suppression from cleaving the diquark we can see that it might be possible to cope with a very small Θ width despite the naturally large width due to tunneling. We have found that the Θ^* should have a width at least twice that of the Θ and possibly a lot larger if the state is massive enough for the tunneling to be less suppressed.

7.2.2 Radiative decay $\Theta^* \rightarrow \Theta\gamma$

The only open decay channels for Θ^* if our estimate of the spin-orbit splitting is correct, are KN , $KN\pi$ (with small phase space and probable suppression from the three-body non-resonant nature of the decay) and $\Theta\gamma$. We can easily calculate the rate for the last of these in this model.

This transition can occur via $M1$ and $E2$ multipoles as is also the case for the conventional $\Delta \rightarrow N\gamma$. The experimental suppression of $E2$ in the latter is attributed in the quark model to dominance of $L = 0$ in the Δ wavefunction. The Θ has a richer internal structure involving diquarks and a unit of orbital angular momentum which allows a wider range of internal transitions.

The Hamiltonian for photon emission by the antiquark has an $M1$ piece,

$$H(M1_\sigma) = i \frac{e_{\bar{q}}}{2m} \vec{\sigma} \cdot \vec{q} \times \vec{\epsilon},$$

where the plane wave $e^{i\vec{q} \cdot \vec{r}_{\bar{q}}}$ has been approximated by 1, which is quite reasonable considering the small $\Theta^* - \Theta$ mass gap.

The photon-diquark Hamiltonian has both $M1$ and $E2$ pieces,

$$H(M1_L) = -i \frac{e_0}{m_0} \vec{L} \cdot \vec{q} \times \vec{\epsilon}$$

$$H(E2) = -e_0 |\vec{q}| (\vec{\epsilon} \cdot \vec{r})(\vec{q} \cdot \vec{r}).$$

We are assuming that the photon couples to the diquark and not to its quark constituents. If

the spin-orbit splitting is small on the scale of the diquark excitation energy we would expect this to hold. Given that we expect the mass gap to the vector diquark to be at least 300 MeV this should be a realistic approximation.

Summing over interactions with all three constituents and using the ρ, λ variables we obtain, for a positive helicity photon,

$$\begin{aligned} H(M1_\sigma) &= +\frac{e_{\bar{q}}}{m} |\vec{q}| \vec{\epsilon}_+ \cdot \vec{S}_{\bar{q}} \\ H(M1_L) &= -\frac{e_0}{m_0} |\vec{q}| \vec{\epsilon}_+ \cdot \vec{L}_\rho \\ H(E2) &= -e_0 |\vec{q}| (\vec{\epsilon}_+ \cdot \vec{\rho}) (\vec{q} \cdot \vec{\rho}). \end{aligned}$$

Computing the $M1$ matrix elements is trivial as one recognises the operators as the angular momentum raising operators which act to raise m by one unit, keeping J constant. We find, for the two helicity amplitudes:

$$\mathcal{M}_{M1} \begin{Bmatrix} 3/2 \\ 1/2 \end{Bmatrix} = \frac{|\vec{q}|}{3} \left(\frac{e_{\bar{q}}}{m} + \frac{e_0}{m_0} \right) \begin{Bmatrix} \sqrt{3} \\ 1 \end{Bmatrix}$$

The $E2$ helicity amplitudes are found to be:

$$\mathcal{M}_{E2} \begin{Bmatrix} 3/2 \\ 1/2 \end{Bmatrix} = e_0 \frac{|\vec{q}|^2 \langle \rho^2 \rangle}{5\sqrt{3}} \begin{Bmatrix} 1 \\ -\sqrt{3} \end{Bmatrix}$$

In this model $E2$ amplitudes are heavily suppressed relative to $M1$:

$$\frac{\mathcal{M}_{E2}}{\mathcal{M}_{M1}} = \frac{|\vec{q}|}{2\alpha_\rho^2} \frac{m_0}{1 + (e_{\bar{q}} m_0 / e_0 m)} \sim \frac{1}{30}.$$

The $M1$ radiative width (for a $\Theta^*(1600)$) is then

$$\Gamma_{M1}(\Theta^* \rightarrow \Theta \gamma) \approx \frac{4}{9} \alpha_{em} |\vec{q}|^3 \left(\frac{e_{\bar{q}}}{m} + \frac{e_0}{m_0} \right)^2 \approx 8 \text{keV},$$

which will correspond to a branching ratio well below 1% and most likely be unobservable.

7.3 Spin-orbit splitting in the Karliner-Lipkin model

In Section 6.2.1 we discussed the Karliner-Lipkin (KL) model for pentaquarks and raised some serious doubts about its numerical prediction of the Θ mass. In this section we give brief

consideration to the magnitude of the spin-orbit splitting within this model.

The KL model [109] has the Θ as an effective 2-body system consisting of a scalar diquark and a spin-1/2 triquark in a relative P -wave. In their paper the masses are quoted as 720 MeV and 1260 MeV but this does not include any subtraction of the spin-spin interaction energy that binds them. The Θ and the D_s composite systems then have roughly the same reduced mass and are assumed [109] to be bound by the same (QCD) dynamics. We exploit this analogy and set the diquark-triquark binding potential to be $V_V(r) = -\frac{4}{3}\frac{\alpha_S}{r}$, $V_S(r) = br$ with $b = 0.18\text{GeV}^2$ and $\alpha_S \sim 0.5$ (which gives a good fit to the hyperfine shifts of $q\bar{q}$ and qqq) [126]. In this potential the P -wave D_s mesons can be approximately described by a variational harmonic oscillator wavefunction, $R(r) \sim r \exp -\beta^2 r^2/2$ with $\beta \sim 0.4\text{GeV}$, which reproduces the results of Godfrey and Isgur [58] to $\sim 10\%$.

With $\vec{r} = \vec{r}_{\text{tri}} - \vec{r}_{\text{di}}$ the internal momenta are $\vec{p} = \vec{p}_r$, $\vec{k} = -\vec{p}_r$ and the orbital angular momentum, $\vec{L} = \vec{r} \times \vec{p}_r$. The spin-orbit splitting term, treating the triquark as a Dirac fermion and both diquark and triquark as pointlike is then

$$H_{SO} = \frac{\vec{S} \cdot \vec{L}}{2m_{\text{tri}}^2} \left(\frac{4\alpha_S}{3} \left\langle \frac{1}{r^3} \right\rangle \left[1 + 2\frac{m_{\text{tri}}}{m_{\text{di}}} \right] - b \left\langle \frac{1}{r} \right\rangle \right)$$

and using $\langle L = 1 | r^{-3(-1)} | L = 1 \rangle = \frac{4}{3\sqrt{\pi}}\beta^{3(1)}$ this gives a splitting of

$$\Delta E(\Theta^* - \Theta) = \frac{1}{\sqrt{\pi}m_{\text{tri}}^2} \left(\frac{4\alpha_S}{3} \beta^3 \left[1 + 2\frac{m_{\text{tri}}}{m_{\text{di}}} \right] - b\beta \right). \quad (7.7)$$

Using $m_{\text{di}} = 720\text{MeV}$ and $m_{\text{tri}} = 1260\text{MeV}$ [109]

$$\Delta E(3/2 - 1/2) = (63\text{MeV})_V + (-25\text{MeV})_S = 38\text{MeV},$$

where a cancellation between vector and scalar terms is observed much as in the conventional $q\bar{q}$ and qqq states. Each term is individually small due to the large m_{di} and m_{tri} .

But this is an overly simple treatment of the system the triquark has internal colour structure and is hence quite likely to have an anomalous chromo-magnetic moment. If we simply add together the quark moments inside the triquark (as is done in the quark model for the magnetic moments of the ground state baryons) we would obtain a triquark g -factor

$$g_{\text{tri}} \approx 3 + \frac{4m_s}{3m_u} + \frac{2m_u}{3m_s} \approx 5.6, \quad (7.8)$$

which is much larger than the $g = 1$ for a Dirac fermion. The dynamic term in the spin-

orbit calculation is modified by a factor g_{tri} destroying the dynamic-Thomas cancellation and yielding $\Delta E(3/2 - 1/2) \sim 270 \text{ MeV}$, far in excess of our previous estimate. If we replace m_s by m_c and thus describe the Θ_c we observe that the second term in g_{tri} cancels the $\sim \frac{1}{m_c}$ dependence of the dynamic term again allowing a large splitting $\sim 300 \text{ MeV}$.

However, we find it hard to justify this model (as we found it hard to believe the KL model at all in Section 6.2.1) as the diquark and triquark are certainly not pointlike and the simple addition of quark chromomagnetic moments is likely to be a crude approximation.

7.4 Comparisons with the Chiral Soliton Model

We emphasize here that there is a clear difference between the $3/2^+$ states in the JW model (and indeed any quark model containing $L = 1$) and those in Chiral Soliton Models. In the JW model $3/2^+, 1/2^+$ states come from the two ways of coupling $L = 1$ to a quark spin, this has no impact on the flavour structure so that the $3/2^+$ states should also sit in $\overline{10} \oplus 8$.

In the χSM this is not the case. Allowed multiplets in such models are determined by the Wess-Zumino constraint which allows only those representations which contain states of hypercharge $Y = +1$. Furthermore the states of $Y = +1$ must have isospin equal to their spin $I = J$, so that for baryon spin $1/2$, states sit in $8, \overline{10}, 27, \dots$. For baryon spin $3/2$ the smallest allowed representations are $10, 27, 35, \dots$ - specifically no $\overline{10}$ or 8 .

Thus the minimum isospin allowed for a narrow $\Theta(3/2^+)$ in the χSM is $I = 1$; observation of a Θ^* with $I = 0$ would place it in a $\overline{10}$ and rule out the χSM as a description of the Θ . Larger flavour representations ($27, 35$) and hence higher $\Theta(3/2^+)$ isospin can be admitted in quark models, for example within the diquark paradigm by having flavour 6 diquarks, so observation of a non-isosinglet $\Theta(3/2^+)$ would not rule out such models. Even with such an observation we would still demand there be the isosinglet Θ^* somewhere: non-observation of this state would be problematical for quark models with $L = 1$ and without further dynamics explaining its absence we would be required to abandon such models.

In Chapter 6.1 we listed several experiments which have looked at the pK^+ spectrum. None have reported any sign of a Θ^{++} resonance.

7.5 Experimental Status

While no published experiment yet has any claim for a second state in nK^+, pK^0 which could correspond to the Θ^* , there is preliminary data from CLAS [127] in the reaction $\gamma p \rightarrow \Theta K^0$

that indicates the possibility of two narrow peaks separated by around 50 MeV near 1550 MeV.

In [128], the anticharmed pentaquark state observed is significantly heavier than the mass prediction of the JW model. The authors of [128] suggest that the observed state could be the $J^P = \frac{3}{2}^+$ spin-orbit partner of the as yet unseen Θ_c . If our study of spin-orbit splittings as presented in this chapter is correct, this is not a possibility as we anticipate a splitting of order a few tens of MeV.

7.6 Summary

We have developed, by analogy to a model used with some success in the conventional meson and baryon sectors, an effective model of the JW correlation in which we have been able to compute the spin-orbit splitting of the $J^P = (\frac{1}{2}, \frac{3}{2})^+$ states. Setting parameters using experimental baryon measurements and as a cross-check, a dynamical model, we find a splitting below 100 MeV.

Consideration of the width of the $\frac{3}{2}^+$ state was made under the assumption of “fall-apart” decay dynamics and it was found to be at twice as large as the Θ width and possibly much wider if the mass sensitivity of tunneling is realised.

The radiative $\Theta^* \rightarrow \Theta\gamma$ decay rate was found to be rather small.

Comparison has been made with the χ SM, where no $S = +1$ isoscalar $\frac{3}{2}^+$ narrow state is expected.

Chapter 8

Symmetries and Selection Rules in Pentaquark Decay

In the Jaffe-Wilczek model, as well as the $\overline{\mathbf{10}} (\frac{1}{2}^+)$ multiplet we required to accommodate the Θ , we also generated degenerate $\overline{\mathbf{10}} (\frac{3}{2}^+)$ and $\mathbf{8} (\frac{1}{2}^+, \frac{3}{2}^+)$ multiplets. In the previous chapter we considered the splitting between the $\frac{1}{2}^+$ and the $\frac{3}{2}^+$ states in a simple dynamical model. In this chapter we will investigate the properties of the non-exotic states in the $\overline{\mathbf{10}} \oplus \mathbf{8}$ on the basis of their flavour and spin structure. Some consideration of the $\overline{\mathbf{10}}$ flavour symmetries have already appeared in the literature [129, 130].

Our first task is to construct the flavour wavefunctions of the states in the $\overline{\mathbf{10}} \oplus \mathbf{8}$. Starting with the Θ^+ wavefunction,

$$|\Theta^+\rangle = \frac{ud - du}{\sqrt{2}} \frac{ud - du}{\sqrt{2}} \bar{s} \equiv (ud)(ud)\bar{s},$$

we can construct the wavefunctions of the other members of the $\overline{\mathbf{10}}$ by repeated application of the I, U, V -spin raising and lowering operators. A reminder of their properties can be found in Appendix E.

In terms of U -spin, the Θ is a $U = \frac{3}{2}, U_3 = +\frac{3}{2}$ state so that

$$\begin{aligned} |\overline{\mathbf{10}}; p\rangle &= |U = \frac{3}{2}, U_3 = +\frac{1}{2}\rangle = \sqrt{\frac{1}{3}} U_- |\Theta\rangle \\ &= -\sqrt{\frac{1}{3}} [(su)(ud)\bar{s} + (ud)(su)\bar{s} + (ud)^2\bar{d}]. \end{aligned}$$

This is the proton-like state in the $\overline{\mathbf{10}}$ constructed from $\bar{\mathbf{6}} \otimes \bar{\mathbf{3}}$. This combination also gives

rise to an $\mathbf{8}$ which contains a proton with orthogonal flavour wavefunction,

$$|\mathbf{8}; p\rangle = -\sqrt{\frac{1}{6}} [(su)(ud)\bar{s} + (ud)(su)\bar{s} - 2(ud)^2\bar{d}].$$

In general these two degenerate states will mix to produce mass eigenstates,

$$\begin{pmatrix} |p_H\rangle \\ |p_L\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |\bar{\mathbf{10}}; p\rangle \\ |\mathbf{8}; p\rangle \end{pmatrix}.$$

With the JW phenomenological Hamiltonian (equation(6.4)) which gives states mass according to their strangeness content, the mass eigenstates are the ideal mixtures ($\tan\theta = \sqrt{\frac{1}{2}}$),

$$\begin{aligned} |p_H\rangle &= -\sqrt{\frac{1}{2}} [(su)(ud) + (ud)(su)] \bar{s} \\ |p_L\rangle &= -(ud)^2\bar{d}, \end{aligned}$$

where the heavier state has hidden strangeness. The Jaffe-Wilczek assignment of the experimental states $N(1710), N(1440)$ to these mass eigenstates will be discussed later when we consider the decay systematics of the $\bar{\mathbf{10}} \oplus \mathbf{8}$.

By multiple applications of $I_{\pm}, U_{\pm}, V_{\pm}$ we can construct the flavour wavefunctions for all the states in the $\bar{\mathbf{10}}$ and its partner $\mathbf{8}_5$, the results being presented in Table 8.1. The $\bar{\mathbf{3}} = (\mathbf{3} \otimes \mathbf{3})_{\text{AS}}$ basis states are written in shorthand,

$$\begin{aligned} A &\equiv (ud) \equiv (ud - du)/\sqrt{2} \sim \bar{s} \\ B &\equiv (ds) \equiv (ds - sd)/\sqrt{2} \sim \bar{u} \\ C &\equiv (su) \equiv (su - us)/\sqrt{2} \sim \bar{d} \end{aligned}$$

for which $U_-A = -C; V_-B = -A; I_-C = -B$, and see also Appendix E. Note that an identical flavour construction occurs in the KL model so that the wavefunctions presented in Table 8.1 are applicable to this model too, provided one understands that the latter pair of labels represent the vector diquark and antiquark in the triquark.

The $\mathbf{8}_5$ wavefunctions we have constructed above are of mixed-symmetric ($\mathbf{8}^{M_S}$) type; in the JW model the scalar diquarks, separated by a P -wave, are coupled symmetrically in flavour giving rise to this symmetry, the mixed antisymmetric ($\mathbf{8}^{M_A}$) analogues (e.g. $p = \frac{1}{\sqrt{2}}(AC - CA)A$) appear in the negative parity states with an S -wave between the diquarks

Θ^+	$\overline{10}$ AAA	8_5
p	$-\frac{1}{\sqrt{3}}(ACA + CAA + AAC)$	$-\frac{1}{\sqrt{6}}(ACA + CAA - 2AAC)$
n	$\frac{1}{\sqrt{3}}(ABA + BAA + AAB)$	$\frac{1}{\sqrt{6}}(ABA + BAA - 2AAB)$
Σ^+	$\frac{1}{\sqrt{3}}(CAC + ACC + CCA)$	$\frac{1}{\sqrt{6}}(CAC + ACC - 2CCA)$
Σ^0	$-\frac{1}{\sqrt{6}}(ABC + BAC + ACB + CAB + BCA + CBA)$	$-\frac{1}{\sqrt{12}}(ABC + BAC + ACB + CAB - 2BCA - 2CBA)$
Λ^0		$-\frac{1}{2}(ABC - ACB + BAC - CAB)$
Σ^-	$\frac{1}{\sqrt{3}}(BAB + ABB + BBA)$	$\frac{1}{\sqrt{3}}(BAB + ABB - 2BBA)$
Ξ^+	$-\overline{CCC}$	
Ξ^0	$\frac{1}{\sqrt{3}}(CBC + BCC + CCB)$	$\frac{1}{\sqrt{6}}(CBC + BCC - 2CBB)$
Ξ^-	$-\frac{1}{\sqrt{3}}(CBB + BCB + BBC)$	$-\frac{1}{\sqrt{6}}(CBB + BCB - 2BBC)$
Ξ^{--}	BBB	

Table 8.1: Pentaquark wavefunctions where ABC are defined in the text. Note that consistency requires the meson octet to be defined with each $q\bar{q}$ positive except for $\pi^+ = -u\bar{d}$; $\bar{K}^0 = -s\bar{d}$ and then $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ (see Appendix E). In this convention $\eta_8 = \frac{1}{\sqrt{6}}(2s\bar{s} - u\bar{u} - d\bar{d})$. For JW [120] A_1, A_2 refer to diquarks and A_3 to the antiquark; for KL [109] A_1 is diquark and $[A_2 A_3]$ is the triquark in $\bar{\mathbf{6}}_F$. The $\bar{\mathbf{6}} \otimes \mathbf{3}$ gives the $\overline{10}$ and 8 as listed above.

which are assumed to be at much higher masses due to the repulsive Pauli blocking.

We gave in Section 7.2.1 a demonstration that the limited overlap between Θ and NK wavefunctions can suppress the otherwise large Θ width. This is an example of the use of the “fall-apart” mechanism for decay where no operator (e.g. pair-production $^3P_0$, gluon exchange etc...) is required to trigger the decay. Such dynamics have been assumed in [78, 115, 117] among others. We will continue to use this method as our dominant decay mechanism in this chapter, taking advantage of the fact that with this picture the relative rates to different flavoured end-states are determined solely by the flavour wavefunction overlaps. These flavour arguments are likely to be a good approximation in quark models with colour (not flavour) dependent forces. While spin and colour overlaps might be modified by, say, single gluon exchange between diquarks, the flavour overlaps cannot and should be robust.

We will use the flavour wavefunctions of Table 8.1 to account for the relative strengths of various baryon plus meson final states under the assumption of “fall-apart” decay dynamics, but first we will take a brief detour to consider the spin wavefunctions of the pentaquark states in the JW and KL models, noting that these predictions⁸ may be more sensitive to the assumption of “fall-apart” dynamics.

8.1 Spin wavefunctions

In the JW model the diquarks are spin-0 combinations of qq and hence the spin-wavefunction for the pentaquark is trivial:

$$|q^4 \bar{q}; \uparrow\rangle_{\text{JW}} = \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \uparrow. \quad (8.1)$$

We have already (in Section 7.2.1) considered how this maps on to baryon plus meson spin states and found

$$|q^4 \bar{q}; \uparrow\rangle_{\text{JW}} = \sqrt{\frac{1}{2}} \left(\chi_1^{M_A} \sqrt{\frac{1}{2}} [V_0 - P] - \chi_1^{M_A} V_+ \right),$$

so that, for example, the ratio of decay rates to vector and pseudoscalar mesons (neglecting phase space difference) is

$$\frac{\Gamma(q^4 \bar{q} \rightarrow BV)}{\Gamma(q^4 \bar{q} \rightarrow BP)} \Big|_{\text{JW}} = \frac{|\frac{1}{2}|^2 + |\frac{1}{\sqrt{2}}|^2}{|\frac{1}{2}|^2} = 3.$$

In the KL model one diquark is scalar while the other is a vector which couples to the antiquark to give a spin-1/2 triquark:

$$\begin{aligned} |q^4 \bar{q}; \uparrow\rangle_{\text{KL}} &= \frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}} \sqrt{\frac{1}{3}} \left(-\frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}} \uparrow + \sqrt{2}(\uparrow\uparrow) \downarrow \right) \\ &= \sqrt{\frac{1}{3}} \left(\chi_1^{M_A} \frac{1}{2} [V_0 + 3P] - \chi_1^{M_A} \frac{1}{\sqrt{2}} V_+ \right), \end{aligned}$$

and hence

$$\frac{\Gamma(q^4 \bar{q} \rightarrow BV)}{\Gamma(q^4 \bar{q} \rightarrow BP)} \Big|_{\text{KL}} = \frac{|\frac{1}{2}|^2 + |\frac{1}{\sqrt{2}}|^2}{|\frac{3}{2}|^2} = \frac{1}{3}.$$

Thus we see that in the JW model, decays to vector mesons are preferred over pseudoscalars while the converse is true for KL. This is a moot point for decays as the Θ is below kinematic threshold for decay to NK^* , but it might have some bearing on the production mechanism, for example in photoproduction, $\gamma p \rightarrow K^0 \Theta^+$ where a K^* exchange might produce the Θ [131].

If the charmed version of the Θ lies above D^*p threshold we might expect an enhanced D^*p decay relative to Dp (modulo the phase space differences) if the JW model with “fall-apart” decay dynamics describes this decay and the opposite relative rate for KL. The H1 candidate Θ_c [128] is above D^*p threshold and indeed the observation is in that channel, however since no report of the Dp spectrum has appeared we cannot yet draw any conclusions about the relative branching ratios.

8.2 Decays of Ξ_5 states

Starting with the wavefunction (see Table 8.1)

$$|\overline{10}; \Xi^-\rangle = -\frac{1}{2\sqrt{3}} \left([(ds - sd)(su - us) + (su - us)(ds - sd)]\bar{u} + (ds - sd)^2\bar{d} \right)$$

we can rewrite this in flavour space in the form $(qqq)(q\bar{q})$.

$$|\overline{10}; \Xi^-\rangle = -\frac{1}{2\sqrt{3}} \left([(ds - sd)s](u\bar{u} - d\bar{d}) + [(su - us)d + (sd - ds)u](s\bar{u}) - \right. \\ \left. [(su - us)s](d\bar{u}) + [(ds - sd)d](s\bar{d}) \right)$$

which maps onto the following ground state hadrons

$$\Xi^-(\overline{10}) \rightarrow -\frac{1}{\sqrt{6}} \left(\sqrt{2}\Xi^-\pi^0 + \Xi^0\pi^- - \sqrt{2}K^-\Sigma^0 - \bar{K}^0\Sigma^- \right)$$

These agree in relative magnitudes and phases with the standard de Swart results [7, 132]; they agree in relative magnitudes with Oh et al. [130] but their phases differ from ours. Refs [129, 130] do not discuss the **8** decays as these depend in general on an undetermined F/D . However with the pentaquark wavefunctions as specified as in Table 8.1, the octet from $\bar{\mathbf{6}}_F \otimes \bar{\mathbf{3}}_F$ that is orthogonal to the $\overline{10}$ is

$$|\mathbf{8}; \Xi^-\rangle = -\frac{1}{2\sqrt{3}} \left([(ds - sd)(su - us) + (su - us)(ds - sd)]\bar{u} - \sqrt{2}(ds - sd)^2\bar{d} \right)$$

and for “fall-apart” dynamics, the particle decomposition is

$$\Xi_5^-(\mathbf{8}) \rightarrow -\frac{1}{\sqrt{24}} \left(\sqrt{6}\Xi^-\eta_1 - \sqrt{3}\Xi^-\eta_8 - \Xi^-\pi^0 + \sqrt{2}\Xi^0\pi^- + 2\sqrt{2}\Sigma^-\bar{K}^0 - 2\Sigma^0K^- + 0\Lambda K^- \right)$$

which corresponds to $\mathbf{8} \rightarrow \mathbf{8} \otimes \mathbf{8}$ with $F/D=1/3$ (or $g_1 = \sqrt{5}g_2$ in the de Swart convention [7, 132]). With this one can therefore deduce the branching ratios for N, Σ, Λ states in $\mathbf{8}_5$ immediately from existing tables [7, 132] and we do not discuss them further here.

For the Ξ_5 we see immediately distinctions between the two $(\mathbf{8}, \overline{10})$ states.

- Isospin ($I = 3/2$ versus $I = 1/2$) is responsible for the distinctive ratios

$$\frac{\Gamma(\Xi_5^- \rightarrow \Xi^-\pi^0)}{\Gamma(\Xi_5^- \rightarrow \Xi^0\pi^-)} = \begin{cases} 1/2 & \mathbf{8} \\ 2 & \overline{10} \end{cases}$$

and analogous for the ΣK modes.

- There is a selection rule that ΛK^- modes vanish. For the $\overline{10}$ this is a trivial consequence of isospin; for the 8_5 it is a result of the pentaquark wavefunction, in particular that the $qqqq$ flavour wavefunction of the pair of diquarks is symmetric in flavour, (i.e. $\bar{6}_F = \bar{3}_F \otimes \bar{3}_F$) leading to $F/D=1/3$.

A pedagogic explanation of the selection rule is as follows. The Ξ_5 state wavefunctions contain two pieces of generic structure $(dssu)\bar{u}$ and $(dssd)\bar{d}$. The $I = 3/2$ and $I = 1/2$ states differ in the relative proportions of these two. However, only the first component $(dssu)\bar{u}$ contains the $\bar{u}$ required for the K^- and this is common to both the $\Xi(I = 3/2)$ and $\Xi(I = 1/2)$. Thus as the $\Xi(I = 3/2) \rightarrow K\Lambda$ is trivially forbidden by isospin, the $\Xi(I = 1/2) \rightarrow K\Lambda$ must be also unless there is cross-talk between the two components in the wavefunction. This would happen if annihilation $(dssu)\bar{u} \rightarrow (dss) \rightarrow (dssd)\bar{d}$ occurs. Thus observation of ΛK^- could arise if there are admixtures of 8_3 in the wavefunction.

Rescattering from kinematically forbidden channels, such as $\Xi\eta$ for a Ξ_5 mass below 2.2 GeV, can feed both $K\Sigma$ and $K\Lambda$, though this is not expected to be a large effect if experience with light hadrons is relevant (such as the small width of the $f_1(1285)$ not being affected by rescattering from the kinematically closed KK^* channel, and the predicted $\pi_2 \rightarrow b_1\pi \sim 0$ [133] not being affected by rescattering from the allowed channels $\pi f_2; \pi\rho$). Whether this carries over to pentaquarks may be tested qualitatively in models by comparing the relative suppression of Θ , Ξ^{--} and Ξ^- states; if there is no rescattering and the $\Xi\eta$ channels are closed in the initial pentaquark wavefunction, its width will be further suppressed by a factor 5/8 and $K\Lambda \sim 0$. In this case the width of Ξ^- (after phase space effects have been removed) will be less than that of Ξ^{--} . A dominance of $K\Lambda > K\Sigma$ can arise if there are pentaquark configurations having $F = D$. In this latter case the $\Sigma^0 K^-$ would be forbidden but ΛK^- allowed. The $\Lambda K : \Sigma K$ ratio in general can be used to constrain the F/D ratio and begin to discriminate between various dynamical schemes.

- Decays to $\Xi^*\pi$ and Σ^*K for Σ^*, Ξ^* in the 10 are forbidden (even if allowed by phase space). For the $\overline{10}$ this is a result of $\overline{10} \not\rightarrow 8 \otimes 10$ as noted in [129] who also discuss $SU(3)_F$ breaking as a potential source of violation of this zero. However, this selection rule may be stronger in the pentaquark models of refs. [109,120] due to the diquarks having antisymmetric flavour ($\bar{3}$) and spin zero, both of which prevent simple overlap of flavour-spin with the $10, S = 3/2$ baryon decuplet resonances. Thus although $SU(3)_F$

allows $\mathbf{8} \rightarrow \mathbf{10} \otimes \mathbf{8}$ to occur, for the $\mathbf{8}_5$ states of Table 8.1 it is again forbidden as a consequence of the antisymmetric flavour content of the wavefunction, at least within the models of suppressed decay widths considered here.

If our assumptions about the “fall-apart” nature of the decay process are right, then the narrow width for the Θ implies that the $I = 3/2$ states will all be narrow. They are degenerate up to electromagnetic mass shifts and other isospin violating effects [5, 134].

Violation of these relations would imply either mixing with excited $\mathbf{8}_3$ states, be due to pentaquark components in the wavefunction beyond those above, or because the width suppression is realised by some dynamics other than implicit in [78, 115, 117]. In the former case one would expect the $\mathbf{8}_3$ components to decay without suppression and dominate the systematics of the widths. In this case there will be narrow $\Xi^{-,0}$ with $I=3/2$ partnering the exotic $\Xi^{+,-}$ and broad $I=1/2$ states that are akin to normal excited Ξ states. Clearly to extract any meaningful conclusions we need to observe both exotic and non-exotic Ξ states.

8.3 N_5 states

The proton-like states in $\overline{\mathbf{10}} \oplus \mathbf{8}$ have baryon-meson decomposition,

$$\begin{aligned} p(\overline{\mathbf{10}}) &= \frac{1}{\sqrt{12}} \left(p\pi^0 - \sqrt{2}n\pi^+ - \sqrt{3}p\eta_8 - \Sigma^0 K^+ + \sqrt{2}\Sigma^+ K^0 - \sqrt{3}\Lambda K^+ \right) \\ p(\mathbf{8}) &= -\frac{1}{\sqrt{12}} \left(\sqrt{2}p\pi^0 - 2n\pi^+ - \sqrt{3}p\eta_1 - \frac{1}{\sqrt{2}}\Sigma^0 K^+ + \Sigma^+ K^0 - \sqrt{\frac{3}{2}}\Lambda K^+ \right). \end{aligned}$$

With the ideal mixing proposed by JW the mass eigenstates are

$$\begin{aligned} p_H &= \frac{1}{\sqrt{8}} \left(\sqrt{2}p\eta_8 - \Sigma^0 K^+ + \sqrt{2}\Sigma^+ K^0 - \sqrt{3}\Lambda K^+ \right) \\ p_L &= -\frac{1}{2} \left(p\pi^0 - \sqrt{2}n\pi^+ - p\eta_8 \right). \end{aligned}$$

JW assign these states to the experimental $N(1710), N(1440)$ on the basis of an approximate mass agreement. In Table 8.2 we reproduce a selection of branching ratio measurements for the states from the PDG [7].

While the $N(1440)$ decays dominantly into $N\pi$ it does so with a very large width, far larger than the few MeV suspected to be the Θ width. We do not see dominance of strange end states for the $N(1710)$, and for both states we observe considerable $\Delta\pi$ modes. This mode is not possible for the p_5 states in $\overline{\mathbf{10}}$ nor in $\mathbf{8}_5$ unless overwritten by rescattering or mixing with $\mathbf{8}_3$.

$N(1440)$		$N(1710)$	
$\Gamma \sim 350\text{MeV}$		$\Gamma \sim 100\text{MeV}$	
$N\pi$	60 – 70%	$N\pi$	10 – 20%
$\Delta\pi$	20 – 30%	$\Delta\pi$	15 – 40%
$N\rho$	< 8%	$N\rho$	5 – 25%
		$N\eta$	$6 \pm 1\%$
		ΛK	5 – 25%
		ΣK	seen

Table 8.2: $N(1440)$ (“Roper”) and $N(1710)$ decay branching ratios from the PDG.

There is no obvious sign of pentaquarks here. A possibility, which would make this system very model-dependent, is that the states are linear combinations of p_3 and p_5 ; the p_5 could even dominate the wavefunction but its $O(1\text{MeV})$ width is swamped by the $O(100\text{MeV})$ width of the unsuppressed p_3 component. The p_5 decays listed above would then show up as rare decays at the $O(1\%)$ level.

Λ_5 state with $J^P = 3/2^+$ - a further narrow pentaquark?

There is one further potentially narrow state in pentaquark models, which has little opportunity for mixing with qqq states. This is the Λ_5 state that is the $J^P = 3/2^+$ spin-orbit partner of Λ_5 in $\mathbf{8}_5$.

First note that $\overline{\mathbf{10}}$ contains Σ_5 but has no Λ_5 . The $\mathbf{8}_5$ contains a Λ_5 , and there will be no mixing with $\overline{\mathbf{10}}$ so long as isospin is good. If there were no mixing with $\Lambda(qqq)$ excited states, on the basis of our assumptions about “fall-apart” decays, this Λ_5 would be narrow, with width identical to that of the Θ apart from phase space factors.

The Λ_5 wavefunction shows that it has only one strange mass quark and hence is similar to the Θ in this regard. [120] estimate ~ 1600 MeV for such a state (the excess ~ 60 MeV relative to the Θ arising because the mass of a $(us)\bar{d}$ set is larger than $(ud)\bar{s}$ due to the relatively smaller downward mass shift in the (us) diquark). Scaling the spin-orbit splitting from [4] and allowing for the relative masses of the $\bar{s}/\bar{d}$ and $m(us)/m(ud)$ gives 60-100 MeV for the Λ_5 mass gap of $3/2^+ - 1/2^+$ and hence 1600-1700 as a conservative estimate for the mass range for the partner $\Lambda_5(3/2^+)$.

Perusal of the data [7] shows that, for the $1/2^+$, mixing with qqq states is likely (given the existence of a candidate $\mathbf{56}, 0^+$ multiplet containing $P_{11}(1440), \Lambda(1600), \Sigma(1660), \Xi(?)$). However there is no $3/2^+$ multiplet with a $\Lambda(1600 - 1700)$ seen, nor is one expected in standard qqq models. The first such is the set containing $P_{13}(1720), \Lambda(1890) \dots$. Thus there is a significant

gap between $\Lambda(1890)$ and our predicted $\Lambda_5(3/2^+)$, which should suppress any mixing.

The branching ratios for either the spin 1/2 or 3/2 states can be determined from the breakdown

$$\Lambda_5 \rightarrow -\frac{1}{2\sqrt{2}} \left(pK^- - n\bar{K}^0 + \Sigma^- \pi^+ - \Sigma^+ \pi^- - \Sigma^0 \pi^0 - \sqrt{3} \Lambda \eta_{n\bar{n}} \right)$$

Decays to $\Sigma^* \pi$ should be suppressed, even if they are kinematically accessible. The production rate of the spin 1/2 state in $\gamma p \rightarrow K^+ \Lambda_5$ should be similar to that of $\gamma n \rightarrow K^- \Theta$ (perhaps a factor of four smaller if K exchange drives the production and $g(KN\Theta) = 2g(KN\Lambda_5)$). If the arguments about $L \otimes S$ coupling and fall-apart dynamics are correct, then we can expect the spin 3/2 state to be enhanced by a factor of two relative to the spin 1/2 counterpart (see Section 7.2.1). That we have had no sign of such a state in KN scattering may be due to the supposed difficulty of seeing narrow resonances in standard partial wave analyses [135]. Re-analysis of the KN (isoscalar, $S = -1$) database allowing for narrow resonances or a search in $\gamma p \rightarrow K^+ \Lambda_5$ therefore seems appropriate.

8.4 Summary

We have constructed the flavour wavefunctions of the states in $\overline{10} \oplus 8$ constructed from $\bar{6} \otimes \bar{3}$. These states appear in the pentaquark models of JW and KL. Assuming that they are able to decay by simple internal rearrangement, the so-called “fall-apart” mechanism, we have decomposed the flavour wavefunctions in the space of baryon-meson flavour wavefunctions and found the relative decay ratios. Certain selection rules arose, such as the absence of ΛK^- for pentaquark Ξ^- which would be expected for a three-quark state.

The decays of the nucleon states in $\overline{10} \oplus 8$ were considered and compared to the measured $N(1440), N(1710)$. The assignment of pentaquark to these states was found to be incompatible with their decays unless there is admixture of three-quark component which is dominating the decays.

The difference in internal spin coupling between the JW and KL models was presented and gives rise to a model discriminator in the relative decay rate to vector and pseudoscalar mesons, where these channels are open.

A state with Λ quantum numbers and $J^P = \frac{3}{2}^+$ is identified as a further possible narrow pentaquark state as despite not having manifestly exotic flavour there appears to be no nearby three-quark states with which it could mix.

Chapter 9

A Constituent Quark Model Realisation of the Jaffe-Wilczek Correlation

In the previous two chapters we have discussed the consequences of using the Jaffe-Wilczek model for pentaquarks. This model is based upon the supposed existence of light scalar diquarks and so we would like to know whether such objects can be formed by the strong force. Ideally one would perform a calculation in QCD to determine this but so far analytic calculations in strong QCD are an unsolved problem, and we are forced to resort to models and effective theories utilising known symmetries.

In this chapter we will consider the possibility that the diquarks in the JW correlation can be formed from a pair of constituent quarks using a colour-spin interaction. We will investigate the mass of the JW state, the naive quark model ground state and the nucleon within the model.

The possibility of diquarks as bound states of two constituent quarks would be a non-trivial result as it suggests that such objects can arise “post chiral symmetry breaking”, when the effective degrees-of-freedom inside the hadrons are the heavy constituent quarks and not the light current quarks of the QCD Lagrangian. Studies in which diquarks in pentaquarks arise in models with chiral symmetry have recently appeared [136, 137].

The model discussed in this chapter takes into account the internal spatial structure of the pentaquark and as such is an improvement over the “schematic approximation” studies already

in the literature - we begin by examining this approximation.

9.1 Schematic Approximation, Quark Potential Model and a Jaffe-Wilczek-like state

Previous investigations into pentaquark structure have often made use of the ‘‘Schematic’’ approximation to colour-spin forces [114–119]. This approximation discards spatial dependence, having an interaction potential $V_{\text{SCH}}^\sigma = -C \sum_{i,j} \frac{\vec{\lambda}_i}{2} \cdot \frac{\vec{\lambda}_j}{2} \vec{S}_i \cdot \vec{S}_j$, where C is a constant. For example in the total spin-0 channel this would give,

$$\langle \vec{3}_c, \vec{3}_c | V_{\text{SCH}}^\sigma | \vec{3}_c, \vec{3}_c \rangle = -C \begin{cases} 0 \otimes 0 \rightarrow 0 \\ 1 \otimes 1 \rightarrow 0 \end{cases}, \quad (9.1)$$

where e.g. $1 \otimes 1 \rightarrow 0$ indicates that 1 & 2 and 3 & 4 are each coupled to spin-1 and then the two pairs coupled to total spin-0.

We will go beyond this approximation by allowing there to be a non-trivial spatial dependence. In particular we shall see that the schematic approximation, *while capturing all the physics for spatially symmetric states, is not sufficiently versatile to accurately describe states containing a P-wave*. This will be demonstrated by computing the equivalent to (9.1) in a model with non-trivial spatial dependence.

We now introduce the model which, in essence, is a standard quark potential model suitable for describing the light baryon and meson spectrum.

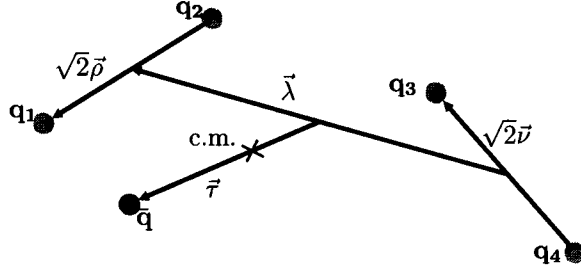
As a binding potential between quarks we take the coloured harmonic oscillator,

$$V(q_i q_j) = -a \frac{\vec{\lambda}_i}{2} \cdot \frac{\vec{\lambda}_j}{2} (\vec{r}_i - \vec{r}_j)^2. \quad (9.2)$$

This is chosen mainly for ease of calculation but has been used in the past as an approximation to the more phenomenologically justified coloured Coulomb + linear potential. This is a purely two-body form which takes no account of the possibility of flux-tube like configurations where three tubes meet in a junction or any other three-body forces between the quarks, except where such forces can be approximated by two-body forces.

In addition to this we introduce a colour-spin contact interaction,

$$V^\sigma(q_i q_j) = -h \frac{\vec{\lambda}_i}{2} \cdot \frac{\vec{\lambda}_j}{2} \vec{S}_i \cdot \vec{S}_j \delta(\vec{r}_{ij}). \quad (9.3)$$

Figure 9.1: Five-body co-ordinate set in the $q^4 \bar{q}$ rest frame.

It is this potential that will act to bind two quarks tightly into an effective light diquark. The motivation we have for a delta-function interaction is that of [126], the Breit-Fermi reduction of one-gluon exchange between quarks, but delta-function interactions have also been used to model instanton effects in light hadrons (see for example [138] and references therein). We will consider in Section 9.3 the validity of such a potential.

Working in the $q^4 \bar{q}$ rest frame, we define the internal variables, $\vec{r}, \vec{\lambda}, \vec{\rho}, \vec{\nu}$ in terms of the quark positions, by

$$\begin{aligned}
 \vec{r}_1 &= -\frac{\mu}{4m} \vec{r} + \frac{1}{2} \vec{\lambda} + \frac{1}{\sqrt{2}} \vec{\rho} \\
 \vec{r}_2 &= -\frac{\mu}{4m} \vec{r} + \frac{1}{2} \vec{\lambda} - \frac{1}{\sqrt{2}} \vec{\rho} \\
 \vec{r}_3 &= -\frac{\mu}{4m} \vec{r} - \frac{1}{2} \vec{\lambda} + \frac{1}{\sqrt{2}} \vec{\nu} \\
 \vec{r}_4 &= -\frac{\mu}{4m} \vec{r} - \frac{1}{2} \vec{\lambda} - \frac{1}{\sqrt{2}} \vec{\nu} \\
 \vec{r}_{\bar{q}} &= \frac{\mu}{m_{\bar{q}}} \vec{r},
 \end{aligned} \tag{9.4}$$

which are clearly well suited to describing a Jaffe-Wilczek-like configuration if the diquarks are spatially correlated (and see figure 9.1). With these variables the kinetic energy is¹

$$T(q^4 \bar{q}) = \frac{\vec{\nabla}_{\vec{r}}^2}{2\mu} + \frac{\vec{\nabla}_{\vec{\lambda}}^2}{2m} + \frac{\vec{\nabla}_{\vec{\rho}}^2}{2m} + \frac{\vec{\nabla}_{\vec{\nu}}^2}{2m}, \tag{9.5}$$

and we can write the Hamiltonian $H(q^4 \bar{q}) = T(q^4 \bar{q}) + V(q^4) + V(\bar{q}) + V^\sigma(q^4) + V^\sigma(\bar{q})$. $V(q^4), V^\sigma(q^4)$ are the spin-independent and dependent potentials, (9.2,9.3) summed pairwise over the four quarks. $V(\bar{q}), V^\sigma(\bar{q})$ are the summed potentials between the anti-quark and each of the four quarks.

¹ μ is the $q^4, \bar{q}$ reduced mass, $\frac{m_{\bar{q}} 4m}{4m + m_{\bar{q}}}$.

9.1.1 Variational trial wavefunction

We can approximately solve for an eigenstate of this Hamiltonian using the variational method. First introduce a trial wavefunction containing a number of variational parameters and then minimize the expectation value of the Hamiltonian with respect to them. Our intention is less to accurately solve to Hamiltonian and more to demonstrate the action of the strong colour-spin force on the wavefunction and the energy. The trial wavefunction is chosen to have spatial form

$$\psi_m = N[\beta\lambda Y_{1m}(\hat{\lambda})e^{-\beta^2\lambda^2/2}][Y_{00}(\hat{\rho})e^{-\gamma^2\rho^2/2}][Y_{00}(\hat{\nu})e^{-\gamma^2\nu^2/2}][Y_{00}(\hat{\tau})e^{-\alpha^2\tau^2/2}], \quad (9.6)$$

which is a P -wave in the $\vec{\lambda}$ variable and an S -wave in the others and is an exact eigenstate in the pure harmonic oscillator approximation (with suitable α, β, γ). The flavour-spin-colour structure is chosen to be $|\bar{\mathbf{3}}_0, \bar{\mathbf{3}}_0\rangle_{\bar{\mathbf{6}},0} \otimes |\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c\rangle_{\mathbf{3}_c}$, where this is a shorthand for quarks 1 & 2 coupled to a $\bar{\mathbf{3}}$ of flavour, spin 0 and colour $\bar{\mathbf{3}}$; quarks 3 & 4 are coupled identically to 1 & 2 and then the pairs $\{12\}, \{34\}$ coupled to a $\bar{\mathbf{6}}$ of flavour, spin 0 and a $\mathbf{3}$ of colour. This is the flavour-spin-colour structure of the Jaffe-Wilczek correlation. The antiquark couples trivially to give an overall colour singlet.

As presented, this wavefunction (flavour-spin-colour-spatial) is only antisymmetric under exchange of labels $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$, but not for example $1 \leftrightarrow 3$. This would seem to violate the generalized Pauli principle, and indeed it does, but this is not necessarily a problem. We need not go to the trouble of antisymmetrising two particles if their wavepackets have very limited overlap - we don't have to worry about electrons on the moon when we solve the Schrödinger equation for a hydrogen atom on Earth nor, more pertinently, do we usually antisymmetrise the quarks in different nucleons when we study the deuteron. Here we are proposing that the dynamics in the Hamiltonian given earlier will be such that the wavefunctions of quarks in different diquarks have very limited overlap such that we don't have to antisymmetrise them. We will show that this assumption is consistent when we variationally solve the Hamiltonian with this trial wavefunction.

With the $|\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c\rangle_{\mathbf{3}_c}$ colour state the expectation values of the potentials are

$$\begin{aligned}
 \langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c | V(q^4) | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c \rangle &= \frac{5}{3}a(\rho^2 + \nu^2) + \frac{2}{3}a\lambda^2 \\
 \langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c | V(\bar{q}) | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c \rangle &= \frac{1}{3}a(\rho^2 + \nu^2) + \frac{1}{3}a\lambda^2 + \frac{4}{3}a\tau^2 \\
 \langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c | V^\sigma(q^4) | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c \rangle &= \frac{2}{3}h(\vec{S}_1 \cdot \vec{S}_2 \delta(\vec{r}_{12}) + \vec{S}_3 \cdot \vec{S}_4 \delta(\vec{r}_{34})) \\
 &\quad + \frac{1}{6}h(\vec{S}_1 \cdot \vec{S}_3 \delta(\vec{r}_{13}) + \vec{S}_1 \cdot \vec{S}_4 \delta(\vec{r}_{14}) + \vec{S}_2 \cdot \vec{S}_3 \delta(\vec{r}_{23}) + \vec{S}_2 \cdot \vec{S}_4 \delta(\vec{r}_{24})) \\
 \langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c | V^\sigma(\bar{q}) | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c \rangle &= \frac{1}{3}h\vec{S}(q^4) \cdot \vec{S}(\bar{q}) \delta(\vec{r}_{1\bar{q}}),
 \end{aligned} \tag{9.7}$$

where the colour overlaps used are tabulated in Appendix F. With the spatial wavefunction (9.6) we have

$$\begin{aligned}
 \langle \delta(\vec{r}_{12}) \rangle &= \langle \delta(\vec{r}_{34}) \rangle = \left(\frac{\gamma}{\sqrt{2\pi}} \right)^3 \\
 \langle \delta(\vec{r}_{13}) \rangle &= \langle \delta(\vec{r}_{14}) \rangle = \langle \delta(\vec{r}_{23}) \rangle = \langle \delta(\vec{r}_{24}) \rangle = \frac{1}{2} \left(\frac{\gamma}{\sqrt{2\pi}} \right)^3 \frac{\beta^5}{((\gamma^2 + \beta^2)/2)^{5/2}} \\
 \langle \rho^2 \rangle &= \langle \nu^2 \rangle = \frac{3}{2\gamma^2}; \quad \langle \lambda^2 \rangle = \frac{5}{2\beta^2} \\
 \langle \vec{\nabla}_\rho^2 \rangle &= \langle \vec{\nabla}_\nu^2 \rangle = \frac{3\gamma^2}{2}; \quad \langle \vec{\nabla}_\lambda^2 \rangle = \frac{5\beta^2}{2}.
 \end{aligned} \tag{9.8}$$

The integrals involved in computing the delta-function expectations are rather non-trivial and we present their evaluation in Appendix F.

9.1.2 Spin-dependent potential and the “schematic” approximation

The spin structure of $\langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c | V^\sigma(q^4) | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c \rangle$ guarantees an attractive potential only for the spin-0, spin-0 correlation,

$$\langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c | V^\sigma(q^4) | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c \rangle = \frac{h}{6} \left(\frac{\gamma}{\sqrt{2\pi}} \right)^3 \begin{pmatrix} -6 \\ 2 - \frac{\beta^5}{((\beta^2 + \gamma^2)/2)^{5/2}} \\ 2 + \frac{1}{2} \frac{\beta^5}{((\beta^2 + \gamma^2)/2)^{5/2}} \end{pmatrix} \begin{cases} 0 \otimes 0 \rightarrow 0 \\ 1 \otimes 1 \rightarrow 0 \\ 1 \otimes 1 \rightarrow 2 \end{cases} \tag{9.9}$$

Compare this with the result of the schematic approximation, equation (9.1). We see that by suitable choice of C , the schematic approximation can duplicate the $0 \otimes 0$ result, but that unless there is accidental cancellation it would be unable to describe the $1 \otimes 1 \rightarrow 0$ result. The origin of the non-zero value for $1 \otimes 1 \rightarrow 0$ is the different expectation values of the delta function with a P -wave and without (see equation(9.8)). We propose that this is a general problem with the schematic approximation when applied to states with non-trivial spatial dependence

and that by its use one can miss significant physics.

9.1.3 Pentaquark and ground-state baryons in the model

Returning to our study of the quark potential model we see that $\langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c | V^\sigma(\bar{q}) | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c \rangle$ is particularly simple. The spatial and colour dependence is identical for all quarks and factors out leaving a sum of the quark spins, which in the $0 \otimes 0 \rightarrow 0$ channel is zero, hence the anti-quark does not change the hyperfine energy of the system in the Jaffe-Wilczek correlation.

The expectation value of the Hamiltonian is thus,

$$\langle JW | H | JW \rangle = \left[\frac{3\alpha^2}{4\mu} + \frac{2a}{\alpha^2} \right] + \left[\frac{5\beta^2}{4m} + \frac{5a}{2\beta^2} \right] + 2 \left[\frac{3\gamma^2}{4m} + \frac{3a}{\gamma^2} \right] - h \left(\frac{\gamma}{\sqrt{2\pi}} \right)^3. \quad (9.10)$$

The minimum is found to be at $\alpha = (8\mu a/3)^{1/4}$, $\beta = (2ma)^{1/4}$ and γ satisfying $\frac{3h}{(2\pi)^{3/2}}\gamma^5 - \frac{3}{m}\gamma^4 + 12a = 0$. We can set the parameters m, a, h using conventional baryon spectroscopy and along the way demonstrate that this model can perfectly well describe the $\Delta - N$ splitting.

a can be set approximately using the S -wave P -wave splitting (ω_P) of roughly 500MeV for the non-strange baryons, since in the harmonic oscillator $\omega_P = \sqrt{4a/m}$. This gives $a \sim \frac{(405\text{MeV})^4}{4 \times 330\text{MeV}}$, where the reason for this unusual presentation will become clear later. If we consider one-gluon-exchange to be the origin of the contact term [121, 126], then $h = \frac{8\pi}{3} \frac{\alpha_S}{m^2}$, where $\alpha_S \sim 0.75$ is usual. The light quark mass takes its conventional value $m = 330\text{MeV}$.

The expectation value of the Hamiltonian for the nucleon between S -wave Gaussian trial wavefunctions (with oscillator parameter α_ρ) is then

$$2 \left[\frac{3\alpha_\rho^2}{4m} + \frac{3a}{\alpha_\rho^2} \right] - \frac{1}{3} \sqrt{\frac{2}{\pi}} \frac{\alpha_S}{m^2} \alpha_\rho^3, \quad (9.11)$$

which is minimised by $\alpha_\rho = 440\text{MeV}$, with colour-spin hyperfine energy -150MeV . This corresponds to a $\Delta - N$ splitting of 300MeV , in good agreement with experiment [7]². The approach presented here differs from that usually taken which considers the hyperfine term only in first order of perturbation theory. By using a variational ansatz we are allowing the hyperfine term to modify both the energy and the wavefunction (α_ρ would have been 405MeV had the hyperfine term been neglected - hence the rather unusual form of a presented earlier which makes this obvious).

Returning to the Jaffe-Wilczek state, using these parameter values we find $\alpha = 370\text{MeV}$, $\beta = 340\text{MeV}$ and $\gamma = 520\text{MeV}$. Had we neglected the hyperfine term, γ would have been

²of course, the parameter α_S has been chosen to give this good agreement

405MeV. The hyperfine term is reducing the mean distance between quarks 1 & 2 and quarks 3 & 4 relative to the $\{12\}\{34\}$ distance, which is exactly what one demands of a spatial diquark-diquark state and what we need for the unsymmetrised ansatz to be justified. Specifically with these numbers the diquarks have mean radius $\sim 1/3$ fm and are separated by an average distance ~ 1 fm.

With $\gamma = 520\text{MeV}$, the hyperfine energy is -510MeV , much larger than the cost of exciting the P -wave ($\frac{p^2}{m} \sim 350\text{MeV}$), so that the Jaffe-Wilczek state is considerably lighter than one would have naively expected. Simply adding together quark masses and including a P -wave energy of ~ 350 MeV would suggest a mass over 2 GeV. Consider the difference $m(\Theta) - m(N)$ which in this model (where the Hamiltonian does not include the quark rest masses) will be

$$\langle JW|H|JW\rangle + m + m_s - \langle N|H|N\rangle.$$

This is found to be (with $m_s = 450\text{MeV}$), $\sim 590\text{MeV}$ and hence $m(\Theta) \sim 1530\text{MeV}$, in surprising agreement with experiment. This agreement is fortuitous we have, for example, neglected tensor interactions which are non-zero due to the P -wave character and which will change this prediction.

We have observed something rather interesting - A Jaffe-Wilczek-like Θ state of low mass with compact diquarks has emerged from a model which also gives an excellent description of the $\Delta - N$ splitting. From conventional quark model analysis one would not have expected this - the usual colour-spin arguments, based upon first-order perturbation theory as applied to the nucleon, suggest a spin-zero “diquark” of mass ~ 600 MeV which if used naively in the Jaffe-Wilczek scheme would hugely overpredict the Θ mass. What we have demonstrated here is that it is possible that the diquarks in the Θ are not like the “diquark” in the nucleon. The quarks in the nucleon all overlap spatially and correct antisymmetrisation must be carried out. This does not allow for the kind of spatially distinct diquark found for the Θ , instead the correlation can only be in spin and flavour.

9.2 Flavour-Spin 210 “ground state”

The Jaffe-Wilczek correlation would not be our first guess for the ground state of the $q^4\bar{q}$ system. Without colour-spin interactions a state with S -waves between all quarks would be expected to have lower energy. Jaffe & Wilczek suggest that the colour-spin interactions would force such a state to a higher energy than their correlation and it is this to which we now turn

in this simple model.

The totally antisymmetric q^4 state with symmetric spatial part is in a **210** of flavour-spin and is coupled to total spin $S(q^4) = 1$ [139, 140]. It has explicit form

$$\frac{1}{\sqrt{3}} \left\{ |6_1, 6_1\rangle \otimes |\bar{3}_c, \bar{3}_c\rangle + \left(\frac{\sqrt{3}}{2} |\bar{3}_1, \bar{3}_0\rangle - \frac{1}{2} |6_0, 6_1\rangle \right) \otimes |6_c, \bar{3}_c\rangle \right. \\ \left. + \left(\frac{\sqrt{3}}{2} |\bar{3}_0, \bar{3}_1\rangle - \frac{1}{2} |6_1, 6_0\rangle \right) \otimes |\bar{3}_c, 6_c\rangle \right\}_{\bar{6}, 1 \otimes \bar{3}_c}, \quad (9.12)$$

where we use the “diquark” notation without implying that diquarks are dynamically generated. The symmetry properties of this state are discussed in Appendix F. The variational spatial form (for q^4) is

$$\psi = N[Y_{00}(\hat{\lambda})e^{-\bar{\beta}^2\lambda^2/2}][Y_{00}(\hat{\rho})e^{-\bar{\beta}^2\rho^2/2}][Y_{00}(\hat{\nu})e^{-\bar{\beta}^2\nu^2/2}], \quad (9.13)$$

whose overall symmetry is exposed by expressing the exponent in terms of the $\vec{r}_i$ as $-2\bar{\beta}^2 \sum_{i>j} (\vec{r}_i - \vec{r}_j)^2$.

In this flavour-spin-colour state the potential $V(q^4)$ is $\frac{4a}{3}(\lambda^2 + \rho^2 + \nu^2)$. A straightforward but somewhat lengthy computation (presented in Appendix F) yields the expectation of $V^\sigma(q^4)$ in this state. This is simplified slightly by the expectation of the delta function, $\langle \delta(\vec{r}_{ij}) \rangle = \left(\frac{\bar{\beta}}{\sqrt{2\pi}} \right)^3$, being i, j independent. We find

$$\langle V^\sigma(q^4) \rangle = \frac{8\pi}{9} \frac{\alpha_S}{m^2} \left(\frac{\bar{\beta}}{\sqrt{2\pi}} \right)^3, \quad (9.14)$$

which is repulsive, as anticipated for such a “Pauli-blocked” configuration. Comparing with the q^4 part of the Hamiltonian in the Jaffe-Wilczek case we find that the **210**_{FS} is around 60MeV heavier. We have not considered the colour-spin interaction between q^4 and the anti-quark which may be non-zero due to the $S(q^4) = 1$ character of the **210**_{FS} state. This could raise the mass of the **210**_{FS} even further.

Thus in this simple model with colour-spin contact interactions, a Jaffe-Wilczek-like state can be consistently defined which is lighter than the naive ground state. The diquark size is small on the scale of their separation, which was assumed in the spin-orbit analysis of Chapter 7.

9.3 Smearing the delta function

One should worry about the validity of using a three-dimensional delta function interaction in a Hamiltonian. Such an object is too singular at the origin to use the normal boundary condition $u(0) = 0$ to quantise the energy levels; we have simply ignored this problem by using a variational ansatz which satisfies the usual boundary condition. We have a further problem if we consider the origin of this term - it came at order $(v/c)^2$ in a non-relativistic reduction of the one-gluon-exchange process, hence in using this term we are implicitly assuming that the momenta of the quarks are much lower than their mass. At very small interquark separations the corresponding momentum scale is rather large and the non-relativistic approximation breaks down. As such we should “smear out” the delta function on the scale of the quark mass. A suitable modification is

$$\delta(\vec{r}) \rightarrow \frac{1}{(r_0\sqrt{\pi})^3} \exp[-r^2/r_0^2] \quad (9.15)$$

with $r_0 \sim m_q^{-1}$. This type of smearing has been used previously in the literature when modeling conventional hadron spectra [58, 138]. The following smearing scales were found to be phenomenologically satisfactory:

$$[138] \quad r_0 \sim 1/(1280\text{MeV}) \sim 1/(4m_u);$$

$$[58] \quad r_0 \sim 1/(1870\text{MeV}) \sim 1/(6m_u).$$

We will consider here the effect of smearing on the results we reported in the previous sections. The change is $\langle \beta | \delta(\vec{r}) | \beta \rangle \equiv (\beta/\sqrt{\pi})^3 \rightarrow (\beta/\sqrt{\pi})^3 (r_0^2 \beta^2 + 1)^{-3/2}$. With $r_0 = 1/(nm_u)$, the following is obtained for the nucleon, Jaffe-Wilczek state and the $\mathbf{210}_{FS}$

n	nucleon		JW		$\mathbf{210}_{FS}$	
	α_ρ	E_{hyp}	γ	E_{hyp}	$\bar{\beta}$	E_{hyp}
1	408	-31	410	-62	365	19
4	428	-124	464	-307	358	54
6	433	-139	483	-379	358	57
10	436	-147	501	-444	358	59

The delta-function is recovered in the limit $n \rightarrow \infty$. So while the nucleon and $\mathbf{210}_{FS}$ state are not strongly affected by the smearing for the phenomenological values $n \sim 4 \rightarrow 6$, the large effects felt by the Jaffe-Wilczek state are diluted considerably. Since the P -wave excitation energy is independent of n , the dilution is such that the Jaffe-Wilczek state is heavier than the $\mathbf{210}_{FS}$.

That the conclusions arrived at in the previous sections are cut-off dependent is disappointing, but at least the trends remain: *The Jaffe-Wilczek-like state still undergoes a considerable*

downward shift and spatially localised diquarks still seem to be formed.

9.4 Extensions

As mentioned previously, an attractive feature of diquark correlations is their supposed ability to describe the light scalar mesons. A straightforward extension to this work would consider $qq\bar{q}\bar{q}$ states; however, without a P -wave to separate the diquark from the anti-diquark it seems unlikely that spatially distinct diquarks of the type found earlier will emerge. Of course such a study would share many similarities with the classic analysis (in the bag model) of Jaffe [141].

Another interesting state to consider is the $ududud$ state with two P -waves. This has the same quantum numbers as the deuteron. If three diquarks are formed with P -waves between them we would hope that it has mass greater than the measured mass of the deuteron which we know to be well described as a bound-state of a proton and a neutron with small overlap. Consideration of such a state has been advocated in [4] as a test on any model proposed to give a light Θ .

This model, extended to allow unequal quark masses, could predict the masses of the other members of the pentaquark $\bar{\mathbf{10}} \oplus \mathbf{8}$, being an explicit realisation of $SU(3)_F$ breaking. In particular this would test the phenomenological Jaffe-Wilczek Hamiltonian $H_s = M_0 + (n_s + n_{\bar{s}})m_s + n_s\alpha$ and its predictions for the other pentaquark states.

If confirmed, the anti-charmed pentaquark observed by H1 [128] with a mass of 3.1 GeV would have its mass significantly underpredicted by this model, as it is in the original Jaffe-Wilczek paper [120]. Setting the charm quark mass using the experimental $\Lambda_c - \Lambda$ mass difference and computing the new Hamiltonian expectation, we find a mass of around 2.8 GeV for the equivalent anti-charmed Jaffe-Wilczek-like state, which is within 100 MeV of the Jaffe-Wilczek prediction and very close to the DN threshold. If the magnitude of spin-orbit splitting calculated in [4] is correct, the possibility that the observed state is the $3/2^+$ state is unlikely. One possibility would be that the lightest anti-charmed pentaquark pair ($1/2^+, 3/2^+$) is still to be found and that the H1 observation is an excited state, such as the vector diquark excitation. The state $1 \otimes 1 \rightarrow 0$ in the model presented in this paper is about 400MeV heavier than the Jaffe-Wilczek-like state. This would be a little heavy for the H1 candidate but probably within model errors; alternatively the state with one vector diquark and one scalar diquark will be somewhat lighter and might be a possibility, but only if the H1 state is the $I_z = 0$ part of an isovector.

The spin-orbit study of Chapter 7 considered the diquarks as pointlike degrees-of-freedom

whereas within this model they have a size ~ 0.3 fm. This model could be extended to include the Breit-Fermi spin-orbit terms and a full 5-body spin-orbit study performed although we should anticipate considerable technical challenges in computing the various integrals that will arise.

Under the assumption of “fall-apart” dynamics we can compute decay widths; the matrix elements involve just the colour-flavour-spin Clebsch-Gordan coefficients we have already considered multiplied by the spatial wavefunction overlap which within this model we have explicit forms for.

9.5 Summary

We have discussed a realisation of the pentaquark structure proposed by Jaffe and Wilczek within a simple quark model with colour-spin contact interactions and coloured harmonic confinement, which accurately describes the $\Delta - N$ splitting. In this model spatially compact diquarks are formed in the pentaquark but no such compact object exists in the nucleon. The colour-spin attraction brings the Jaffe-Wilczek-like state down to a low mass, compatible with the experimental observation and below that of the naive ground state with all S -waves. We found, however, that although these trends are maintained, the extreme effects observed do not survive the required “smearing” of the delta function contact interaction. We also demonstrated the weakness of the “schematic” approximation when applied to a system containing a P -wave, which undermines many of the “group-theoretic” treatments already in the literature. An estimate of the anti-charmed pentaquark mass was made which is in line with the Jaffe-Wilczek prediction and significantly less than the value reported by the H1 collaboration.

Part III

Appendices

Appendix A

Flux-Tube Model Momentum and Angular Momentum

Isgur's equations for the quark positions in the flux-tube model and the position of the n th element of flux-tube are

$$\begin{aligned}\vec{r}_Q &= \vec{R} - \frac{\tilde{m}_d}{m_T} \vec{r} - \frac{br}{\pi m_Q} \sqrt{\frac{2}{N+1}} \sum_{p=1}^{\infty} \frac{1}{p} \vec{a}_p \\ \vec{r}_d &= \vec{R} + \frac{\tilde{m}_Q}{m_T} \vec{r} + \frac{br}{\pi m_d} \sqrt{\frac{2}{N+1}} \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \vec{a}_p \\ \vec{r}_n &= \vec{R} + \left(\frac{n}{N+1} - \frac{\tilde{m}_d}{m_T} \right) \vec{r} + \vec{y}_n,\end{aligned}\tag{A.1}$$

where $\vec{y}_n = \sqrt{\frac{2}{N+1}} \sum_p \vec{a}_p \sin \frac{n\pi p}{N+1}$, $\tilde{m} = m + \frac{br}{2}$ and $m_T = m_Q + m_d + br$. There are certain summations that will be useful to us which we tabulate below

$$\begin{aligned}\sum_{n=1}^N \frac{n}{N+1} &= \frac{N}{2} \\ \sum_{n=1}^N \left(\frac{n}{N+1} \right)^2 &= \frac{N(2N+1)}{6(N+1)} \\ \sum_{n=1}^N \sin \frac{n\pi p}{N+1} &= \frac{2(N+1)}{\pi p} \delta_{p,\text{odd}} \\ \sum_{n=1}^N \frac{n}{N+1} \sin \frac{n\pi p}{N+1} &= -\frac{(N+1)}{\pi p} (-1)^p \\ \sum_{n=1}^N \sin \frac{n\pi p}{N+1} \sin \frac{n\pi q}{N+1} &= \frac{N+1}{2} \delta_{p,q}\end{aligned}$$

It is trivial to check using these formulae that $m_Q \vec{r}_Q + m_d \vec{r}_d + ba \sum_n \vec{r}_n = m_T \vec{R}$, such that $\vec{R}$ should be interpreted as the meson centre-of-mass. The system's kinetic energy can be computed, $T = \frac{m_Q}{2} (\dot{\vec{r}}_Q)^2 + \frac{m_d}{2} (\dot{\vec{r}}_d)^2 + \frac{ba}{2} \sum_n (\dot{\vec{r}}_n)^2$.

$$\begin{aligned} &= \frac{m_Q}{2} \left[\frac{\tilde{m}_d^2}{m_T^2} (\dot{\vec{r}})^2 + 2 \frac{\tilde{m}_d}{m_T} \frac{br}{\pi m_Q} \sqrt{\frac{2}{N+1}} \sum_p \frac{1}{p} \dot{\vec{r}} \cdot \dot{\vec{a}}_p + \left(\frac{br}{\pi m_Q} \right)^2 \frac{2}{N+1} \sum_{p,q} \frac{1}{pq} \dot{\vec{a}}_p \cdot \dot{\vec{a}}_q \right] \\ &+ \frac{m_d}{2} \left[\frac{\tilde{m}_Q^2}{m_T^2} (\dot{\vec{r}})^2 + 2 \frac{\tilde{m}_Q}{m_T} \frac{br}{\pi m_d} \sqrt{\frac{2}{N+1}} \sum_p \frac{(-1)^p}{p} \dot{\vec{r}} \cdot \dot{\vec{a}}_p + \left(\frac{br}{\pi m_Q} \right)^2 \frac{2}{N+1} \sum_{p,q} \frac{(-1)^{p+q}}{pq} \dot{\vec{a}}_p \cdot \dot{\vec{a}}_q \right] \\ &+ \frac{ba}{2} \sum_n \left[\left(\frac{n}{N+1} - \frac{\tilde{m}_d}{m_T} \right)^2 (\dot{\vec{r}})^2 + 2 \left(\frac{n}{N+1} - \frac{\tilde{m}_d}{m_T} \right) \sqrt{\frac{2}{N+1}} \sum_p \sin \frac{n\pi p}{N+1} \dot{\vec{r}} \cdot \dot{\vec{a}}_p \right. \\ &\quad \left. + \frac{2}{N+1} \sum_{p,q} \sin \frac{n\pi p}{N+1} \sin \frac{n\pi q}{N+1} \dot{\vec{a}}_p \cdot \dot{\vec{a}}_q \right].\end{aligned}$$

Evaluating the sums over n one finds that the coefficient of $\dot{\vec{r}} \cdot \dot{\vec{a}}_p$ vanishes and we are left with

$$T = \frac{M}{2} (\dot{\vec{r}})^2 + \frac{ba}{2} \sum_p (\dot{\vec{a}}_p)^2 + ba \frac{br}{\pi^2} \sum_{p,q} \frac{1}{pq} \dot{\vec{a}}_p \cdot \dot{\vec{a}}_q \left(\frac{1}{m_Q} + \frac{(-1)^{p+q}}{m_d} \right),$$

where $M = \frac{m_Q \tilde{m}_d^2}{m_T^2} + \frac{m_d \tilde{m}_Q^2}{m_T^2} - \frac{br \tilde{m}_d \tilde{m}_Q}{m_T^2} + \frac{br}{3} \xrightarrow{br \rightarrow 0} \frac{m_Q m_d}{m_Q + m_d}$. Note that the kinetic energy is not $\{p, q\}$ diagonal at $\mathcal{O}(br/m)$, which is the level of non-adiabaticity we considered in our derivation of $\vec{r}_{Q,d}$.

Since the potential is velocity independent, the conjugate momenta are derivatives of T , $\vec{p}_x = \frac{\partial T}{\partial \dot{\vec{x}}}$ so that

$$\begin{aligned} \vec{p}_r &= M \dot{\vec{r}} \\ \vec{p}_{a_p} &= ba \sum_q \left\{ \delta_{p,q} + 2 \frac{br}{\pi^2} \frac{1}{pq} \left(\frac{1}{m_Q} + \frac{(-1)^{p+q}}{m_d} \right) \right\} \dot{\vec{a}}_q \equiv ba \sum_q T_{pq} \dot{\vec{a}}_q. \end{aligned} \quad (\text{A.2})$$

Eliminating $\dot{\vec{r}}, \dot{\vec{a}}_p$ from the Hamiltonian requires inversion of these expressions,

$$\begin{aligned} \dot{\vec{r}} &= \vec{p}_r / M \\ \dot{\vec{a}}_p &= \sum_q T_{pq}^{-1} \vec{p}_{a_q} / ba. \end{aligned}$$

Inverting the matrix T_{pq} for general masses m_Q, m_d is a non-trivial problem but we can see that $T_{pq}^{-1} = \delta_{pq} + \mathcal{O}(\frac{br}{m})$. For now we will leave T_{pq}^{-1} undetermined and go on to express the quark and flux-tube element momenta as a function of the conjugate momenta in the meson centre-of-mass frame¹,

$$\begin{aligned} \vec{p}_Q &= -\frac{m_Q \tilde{m}_d}{m_T M} \vec{p}_r - \frac{\sqrt{2(N+1)}}{\pi} \sum_{pq} \frac{1}{p} T_{pq}^{-1} \vec{p}_{a_q} \\ \vec{p}_d &= \frac{m_d \tilde{m}_Q}{m_T M} \vec{p}_r + \frac{\sqrt{2(N+1)}}{\pi} \sum_{pq} \frac{(-1)^p}{p} T_{pq}^{-1} \vec{p}_{a_q} \\ \vec{p}_n &= \frac{ba}{M} \left(\frac{n}{N+1} - \frac{\tilde{m}_d}{m_T} \right) \vec{p}_r + \sqrt{\frac{2}{N+1}} \sum_{pq} \sin \frac{n\pi p}{N+1} T_{pq}^{-1} \vec{p}_{a_q}. \end{aligned} \quad (\text{A.3})$$

An immediate check on the consistency of these expressions is to compute the total momentum $\vec{p}_Q + \vec{p}_d + \sum_n \vec{p}_n$, which is found to be zero for a general matrix T_{pq}^{-1} . To compute $M1$ transitions it will be necessary to have expressions for the orbital angular momenta of the quarks and of the whole meson system. Using equations (A.1, A.3) we find

$$\begin{aligned} \vec{L}_Q &= \frac{m_Q \tilde{m}_d^2}{M m_T^2} \vec{r} \times \vec{p}_r + \frac{\tilde{m}_d}{m_T} \frac{\sqrt{2(N+1)}}{\pi} \sum_{pq} \frac{1}{p} T_{pq}^{-1} \vec{r} \times \vec{p}_{a_q} \\ &\quad + \frac{br \tilde{m}_d}{\pi m_T M} \sqrt{\frac{2}{N+1}} \sum_p \frac{1}{p} \vec{a}_p \times \vec{p}_r + \frac{2br}{\pi^2 m_Q} \sum_{spq} \frac{1}{sp} T_{pq}^{-1} \vec{a}_s \times \vec{p}_{a_q} \end{aligned}$$

¹for simplicity, all results follow for general $\vec{R}, \vec{p}_R$

$$\begin{aligned}\vec{L}_d = & \frac{m_d \tilde{m}_Q^2}{M m_T^2} \vec{r} \times \vec{p}_r + \frac{\tilde{m}_Q}{m_T} \frac{\sqrt{2(N+1)}}{\pi} \sum_{pq} \frac{(-1)^p}{p} T_{pq}^{-1} \vec{r} \times \vec{p}_{a_q} \\ & + \frac{br \tilde{m}_Q}{\pi m_T M} \sqrt{\frac{2}{N+1}} \sum_p \frac{(-1)^p}{p} \vec{a}_p \times \vec{p}_r + \frac{2br}{\pi^2 m_d} \sum_{spq} \frac{(-1)^{s+p}}{sp} T_{pq}^{-1} \vec{a}_s \times \vec{p}_{a_q}\end{aligned}$$

$$\begin{aligned}\vec{L}_{\text{tube}} = & \sum_n \vec{L}_n \\ = & \frac{1}{M} \left(\frac{\tilde{m}_d^2}{m_T^2} - \frac{\tilde{m}_d br}{m_T} + \frac{br}{3} \right) \vec{r} \times \vec{p}_r - \frac{\sqrt{2(N+1)}}{\pi} \sum_{pq} \frac{1}{p} \left((-1)^p + 2 \frac{\tilde{m}_d}{m_T} \delta_{p,\text{odd}} \right) T_{pq}^{-1} \vec{r} \times \vec{p}_{a_q} \\ & - \frac{ba}{M} \frac{\sqrt{2(N+1)}}{\pi} \frac{1}{p} \left((-1)^p + 2 \frac{\tilde{m}_d}{m_T} \delta_{p,\text{odd}} \right) \vec{a}_p \times \vec{p}_r + \sum_{pq} T_{pq}^{-1} \vec{a}_p \times \vec{p}_{a_q}.\end{aligned}$$

Adding these together to produce the meson orbital angular momentum we obtain

$$\vec{L}_T = \vec{r} \times \vec{p}_r + \sum_{spq} T_{pq}^{-1} \left\{ \delta_{ps} + 2 \frac{br}{\pi^2} \frac{1}{sp} \left(\frac{1}{m_Q} + \frac{(-1)^{s+p}}{m_d} \right) \right\} \vec{a}_s \times \vec{p}_{a_q},$$

and recognising the object in braces as T_{ps} we finally get the diagonal result

$$\vec{L}_T = \vec{r} \times \vec{p}_r + \sum_p \vec{a}_p \times \vec{p}_{a_p} \quad (\text{A.4})$$

Appendix B

Quark Adiabatic Hamiltonian

We follow the formulation of the flux-tube model described in [36], [43], [47]. In the appendix of [36] the authors show explicitly that the spatial wavefunction of a hybrid state with phonon occupation $\{n_{p+}, n_{p-}\}$ can be written

$$R_{nLN\Lambda}(r) \sqrt{\frac{2L+1}{4\pi}} \mathcal{D}_{m_L\Lambda}^{(L)}(\phi, \theta, -\phi)$$

where the quantum numbers are; n , radial; (L, m_L) , angular momentum; $N = \sum_p (n_{p+} + n_{p-})$; $\Lambda = \sum_p (n_{p+} - n_{p-})$. We are only interested in the lightest hybrids which have one phonon excited in the $p = 1$ mode and hence $N = 1$ and $\Lambda = \pm 1$. In the adiabatic approximation we get a radial Hamiltonian (acting on $rR(r)$) for these hybrids,

$$H^1 = \frac{1}{2\mu} \frac{\partial^2}{\partial r^2} + \frac{L(L+1) - \Lambda^2}{2\mu r^2} + br + \frac{\pi}{r} (1 - e^{-f\sqrt{b}r}) - \frac{\kappa}{r} + c, \quad (\text{B.1})$$

whereas for conventional mesons (no phonons) we would get,

$$H^0 = \frac{1}{2\mu} \frac{\partial^2}{\partial r^2} + \frac{L(L+1)}{2\mu r^2} + br - \frac{\kappa}{r} + c.$$

The modified angular momentum barrier in the hybrid case has its origin in the $\Lambda = \pm 1$ carried by the phonon in the tube. br is the mass energy of the string. $\frac{\pi}{r}$ is the excitation energy of the string in the $p = 1$ mode, the additional factor multiplying this is put in by hand and is designed to model the fact that at short distances we do not expect “stringy” configurations to dominate in QCD. The remaining potential terms $-\frac{\kappa}{r} + c$ are introduced by hand, the first of which represents one-gluon-exchange dominance at short distances and the second is required to describe the observed meson spectrum when the constituent quark masses are added to the energy eigenstates.

The parameters, m_Q, b, κ, c are chosen to reproduce approximately the observed conventional meson spectrum (up to spin-dependent splittings). We use the following set of values;

$b = 0.18\text{GeV}^2$, $f = 1$, $c = -0.7\text{GeV}$, $m_{u,d} = 0.33\text{GeV}$, $m_s = 0.55\text{GeV}$, $m_c = 1.77\text{GeV}$, $m_b = 5.17\text{GeV}$. κ is allowed to run in a reasonable way so that for light mesons ($n\bar{n}, n\bar{s}, s\bar{s}$) $\kappa = 1.07$, for heavy-light mesons ($n\bar{c}, s\bar{c}, n\bar{b}, s\bar{b}$) $\kappa = 0.67$ and for heavy-heavy mesons ($c\bar{c}, c\bar{b}, b\bar{b}$) $\kappa = 0.52$.

We solve the Schrödinger equation variationally using a Harmonic Oscillator basis

$$R_{n,L'}^{HO}(r) = \sqrt{\frac{2\Gamma(n)}{\Gamma(n+L'+1/2)}} \beta^{L'+3/2} r^{L'} \mathcal{L}_{n-1}^{L'+1/2}(\beta^2 r^2) e^{-\beta^2 r^2/2}.$$

For conventional states L' here is just the angular momentum quantum number L . For the hybrid Hamiltonian the modified angular momentum barrier is cancelled if L' is chosen so it

satisfies $L'(L' + 1) = L(L + 1) - \Lambda^2$, for $L = 1, \Lambda = \pm 1$ this means $L' \equiv \delta \approx 0.62$.

	$L = 0$		$L = 1$		Hybrid	
	$\frac{2}{\pi^{1/4}} \beta_{1S}^{3/2} e^{-\beta_{1S}^2 r^2/2}$		$\frac{2}{\pi^{1/4}} \sqrt{\frac{2}{3}} \beta_{1P}^{5/2} r e^{-\beta_{1P}^2 r^2/2}$		$\sqrt{\frac{2}{\Gamma(3/2+\delta)}} \beta_H^{\frac{3}{2}+\delta} r^\delta e^{-\beta_H^2 r^2/2}$	
	β_{1S}	M_{1S}	β_{1P}	M_{1P}	β_H	M_H
$n\bar{n}$	0.334	0.672	0.280	1.296	0.260	1.825
$n\bar{s}$	0.370	0.780	0.306	1.386	0.275	1.961
$s\bar{s}$	0.426	0.857	0.342	1.448	0.311	2.024
$n\bar{c}$	0.389	2.036	0.331	2.544	0.291	3.128
$s\bar{c}$	0.469	2.092	0.387	2.571	0.335	3.195
$c\bar{c}$	0.639	3.129	0.509	3.539	0.426	4.235
$n\bar{b}$	0.408	5.392			0.302	6.449
$s\bar{b}$	0.510	5.455			0.355	6.563
$c\bar{b}$	0.798	6.418			0.486	7.575
$b\bar{b}$	1.239	9.522				

Table B.1: β values/ state masses in GeV

The Hamiltonians H^0, H^1 are found to be diagonal in this basis to a very good approximation if the β values listed in Table(B.1) are used.

Merlin [47] and Merlin & Paton [43] consider non-adiabatic corrections to this Hamiltonian, the values quoted in Table(B.1) are actually obtained using this modified Hamiltonian, although the differences in β with respect to using H^1 are usually small.

Appendix C

Hybrid transition operator for V_μ and A_μ

We perform the non-relativistic reduction of the vector and axial currents allowing for flavour changing. We present the particular case of $B(b\bar{d}) \rightarrow D(c\bar{d})$ by quark level $b \rightarrow c$. In the following $m_D = m_d + m_c$ and $m_B = m_d + m_b$, which are indeed the meson masses in the extreme non-relativistic limit.

V^0

For the zeroth (time) component of the vector current we have

$$V^0(p_c, p_b; x) = \bar{c}(x) \gamma^0 b(x) \xrightarrow{\text{N.R.}} e^{-i(p_c - p_b) \cdot x} \left\{ 1 + \frac{\vec{p}_c \cdot \vec{p}_b}{4m_b m_c} + \frac{i}{4} \vec{\sigma} \cdot \left(\frac{\vec{p}_c}{m_c} \times \frac{\vec{p}_b}{m_b} \right) + \dots \right\} \quad (\text{C.1})$$

Considering only the terms linear in $\vec{a}$ obtained from utilising the effect of the momentum operator on flux-tube ground state wavefunctions,

$$\vec{p}_b |\chi_0\rangle \Big|_{\vec{a}=\text{cmpt}} = -i \sqrt{\frac{2b}{\pi}} \beta_1 \vec{a} |\chi_0\rangle \quad (\text{C.2})$$

and expanding the plane wave to leading order in $\vec{q} \cdot \vec{a}$ we have the effective transition operator to the first hybrid excitation

$$V_H^0 = e^{i\vec{q} \cdot \vec{r} \frac{m_d}{m_D}} \sqrt{\frac{2b}{\pi^3}} \beta_1 \left\{ i\vec{q} \cdot \vec{a} \left(\frac{r}{m_c} + \frac{\pi}{4m_b m_c} \right) - \frac{\pi}{4m_b m_c} \vec{\sigma} \cdot \vec{q} \times \vec{a} \right\} \quad (\text{C.3})$$

$\vec{V}$

For the spatial vector current we have

$$\vec{V}(p_c, p_b; x) = \bar{c}(x) \vec{\gamma} b(x) \xrightarrow{\text{N.R.}} e^{-i(p_c - p_b) \cdot x} \left\{ \left(\frac{\vec{p}_b}{2m_b} + \frac{\vec{p}_c}{2m_c} \right) - i\vec{\sigma} \times \left(\frac{\vec{p}_b}{2m_b} - \frac{\vec{p}_c}{2m_c} \right) + \dots \right\} \quad (\text{C.4})$$

and the relevant transition to first excited hybrid becomes

$$\begin{aligned} \vec{V}_H = e^{i\vec{q} \cdot \vec{r} \frac{m_d}{m_D}} \sqrt{\frac{2b}{\pi^3}} \beta_1 \left\{ -i \frac{\pi}{2m_b} \left(\frac{m_b}{m_c} + 1 \right) \vec{a} - i \frac{r}{2m_c^2} (\vec{q} \cdot \vec{a}) \vec{q} \right. \\ \left. + \frac{\pi}{2m_b} \left(\frac{m_b}{m_c} - 1 \right) \vec{\sigma} \times \vec{a} + \frac{r}{2m_c^2} (\vec{q} \cdot \vec{a}) \vec{\sigma} \times \vec{q} \right\} \quad (\text{C.5}) \end{aligned}$$

A^0

$$A^0(p_c, p_b; x) = \bar{c}(x) \gamma^0 \gamma^5 b(x) \xrightarrow{\text{N.R.}} e^{-i(p_c - p_b) \cdot x} \left\{ \vec{\sigma} \cdot \left(\frac{\vec{p}_b}{2m_b} + \frac{\vec{p}_c}{2m_c} \right) + \dots \right\} \quad (\text{C.6})$$

and the relevant hybrid transition becomes

$$A_H^0 = e^{i\vec{q} \cdot \vec{r} \frac{m_d}{m_D}} \sqrt{\frac{2b}{\pi^3}} \beta_1 \left\{ -i \frac{\pi}{2m_b} \left(\frac{m_b}{m_c} + 1 \right) \vec{\sigma} \cdot \vec{a} - i \frac{r}{2m_c^2} (\vec{q} \cdot \vec{a}) \vec{\sigma} \cdot \vec{q} \right\} \quad (\text{C.7})$$

 $\vec{A}$

$$\begin{aligned} \vec{A}(p_c, p_b; x) = \bar{c}(x) \vec{\gamma} \gamma^5 b(x) \xrightarrow{\text{N.R.}} e^{-i(p_c - p_b) \cdot x} \left\{ \vec{\sigma} \left(1 - \frac{\vec{p}_c \cdot \vec{p}_b}{4m_c m_b} \right) \right. \\ \left. + \frac{\vec{p}_b (\vec{\sigma} \cdot \vec{p}_c) + \vec{p}_c (\vec{\sigma} \cdot \vec{p}_b)}{4m_c m_b} - i \frac{\vec{p}_c \times \vec{p}_b}{4m_c m_b} \right\} \end{aligned} \quad (\text{C.8})$$

and the relevant hybrid transition becomes

$$\begin{aligned} \vec{A}_H = e^{i\vec{q} \cdot \vec{r} \frac{m_d}{m_D}} \sqrt{\frac{2b}{\pi^3}} \beta_1 \left\{ i(\vec{q} \cdot \vec{a}) \vec{\sigma} \left(\frac{r}{m_c} - \frac{\pi}{4m_b m_c} \right) + i \frac{\pi}{4m_b m_c} [(\vec{\sigma} \cdot \vec{a}) \vec{q} + (\vec{\sigma} \cdot \vec{q}) \vec{a}] \right. \\ \left. + \frac{\pi}{4m_b m_c} \vec{q} \times \vec{a} \right\} \end{aligned} \quad (\text{C.9})$$

“Dipole” form in the adiabatic flux-tube model

In the usual non-relativistic quark model the matrix element of the lowest order electric dipole operator $\sim \frac{\vec{\epsilon} \cdot \vec{p}}{m}$ can be transformed using the commutator $\vec{p} = im[H, \vec{r}]$ into an explicit dipole form $\sim |\vec{q}| \vec{\epsilon} \cdot \vec{r}$. This commutation relation is valid provided the system Hamiltonian can be written in the form $H = \frac{\vec{p}^2}{2m} + V(r)$, any other dependence of $\vec{p}$ causing deviations from this behaviour. We find that this transformation is justified in the adiabatic approximation to the flux-tube model also:

Minimal coupling of the photon field to a quark at $\vec{r}_Q$ moving with $\vec{p}_Q$ leads to a convection current operator $\sim \vec{p}_Q \cdot \vec{A}(\vec{r}_Q)$. For a plane-wave photon field at lowest order in $\vec{q} \cdot \vec{r}_Q$ we get an E1 operator $\sim \frac{\vec{\epsilon} \cdot \vec{p}_Q}{m_Q}$.

We construct $\vec{p}_Q$ as follows,

$$\vec{p}_Q \equiv m \dot{\vec{r}}_Q = -\frac{m_Q(m_d + br/2)}{m_Q + m_d + br} \dot{\vec{r}} - \frac{b}{\pi} \sqrt{\frac{2}{N+1}} \frac{d}{dt} \left(r \sum_p \vec{a}_p / p \right).$$

The second term is linear in $\vec{a}$ and can hence excite the flux-tube. Using the identity $(d/dt)A = i[H, A]$ we can write the flux tube term as

$$-\frac{b}{\pi} \sqrt{\frac{2}{N+1}} i \left[H, r \sum_p \vec{a}_p / p \right].$$

Then

$$\begin{aligned}
 \langle \mathcal{H} | \vec{p}_Q | \mathcal{C} \rangle &= -\frac{ib}{\pi} \sqrt{\frac{2}{N+1}} (E_{\mathcal{H}} - E_{\mathcal{C}}) \langle \mathcal{H} | r \sum_p \vec{a}_p / p | \mathcal{C} \rangle \\
 &= -\frac{ib}{\pi} \sqrt{\frac{2}{N+1}} |\vec{q}| \langle \mathcal{H} | r \sum_p \vec{a}_p / p | \mathcal{C} \rangle,
 \end{aligned}$$

and hence

$$\langle \mathcal{H} | \frac{\vec{\epsilon} \cdot \vec{p}_Q}{m_Q} | \mathcal{C} \rangle = i |\vec{q}| \langle \mathcal{H} | \vec{\epsilon} \cdot \vec{r}_Q | \mathcal{C} \rangle,$$

so that using the dipole form is justified in the flux-tube case.

Appendix D

Pion Emission Currents and Decay Matrix Elements

Two possible Dirac structures for the pion emission are the pseudoscalar ($\bar{\psi}\gamma^5\psi$) or the divergence of the axial current ($q_\mu\bar{\psi}\gamma^\mu\gamma^5\psi$). They are trivially related when the ψ are eigenstates of the free Dirac equation,

$$\bar{\psi}_f q \gamma^5 \psi_i = \bar{\psi}_f (\not{p}_f - \not{p}_i) \gamma^5 \psi_i = -\bar{\psi}_f (\not{p}_f \gamma^5 + \gamma^5 \not{p}_i) \psi_i = -(m_f + m_i) \bar{\psi}_f \gamma^5 \psi_i, \quad (\text{D.1})$$

where we have allowed for the fermion mass to change between initial and final states.

Performing the non-relativistic reductions of these currents we find

$$\begin{aligned} \bar{u}(\vec{p} - \vec{q}) q \gamma^5 u(\vec{p}) &\rightarrow \left(1, -\frac{\vec{\sigma} \cdot (\vec{p} - \vec{q})}{E(\vec{p} - \vec{q}) + m_f}\right) \begin{pmatrix} -\vec{\sigma} \cdot \vec{q} & q^0 \\ -q^0 & \vec{\sigma} \cdot \vec{q} \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E(\vec{p}) + m_i} \end{pmatrix} \\ &\rightarrow -\vec{\sigma} \cdot \vec{q} \left(1 + \frac{q^0}{2m_f}\right) + \vec{\sigma} \cdot \vec{p} q^0 \frac{m_f + m_i}{2m_f m_i} + \dots \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned} \bar{u}(\vec{p} - \vec{q}) \gamma^5 u(\vec{p}) &\rightarrow \left(1, -\frac{\vec{\sigma} \cdot (\vec{p} - \vec{q})}{E(\vec{p} - \vec{q}) + m_f}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{\vec{\sigma} \cdot \vec{p}}{E(\vec{p}) + m_i} \end{pmatrix} \\ &\rightarrow \frac{\vec{\sigma} \cdot \vec{q}}{m_i + m_f} \left(1 + \frac{q^0}{m_i + m_f}\right) + \vec{\sigma} \cdot \vec{p} \frac{m_f - m_i + q^0}{4m_f m_i} \end{aligned} \quad (\text{D.3})$$

Note that there is a slight subtlety in the $m_f = m_i$ case. If we immediately take the equal mass limit in the equations(D.2,D.3), we obtain

$$\begin{aligned} q \gamma^5 &\sim -\vec{\sigma} \cdot \vec{q} \left(1 + \frac{q^0}{2m}\right) + \frac{q^0}{m} \vec{\sigma} \cdot \vec{p} \\ -2m \gamma^5 &\sim -\vec{\sigma} \cdot \vec{q} \left(1 + \frac{q^0}{2m}\right) + \frac{q^0}{2m} \vec{\sigma} \cdot \vec{p}, \end{aligned}$$

which does not satisfy equation(D.1) in the second term. We must be more careful to include energy conservation in the non-relativistic limit - $m_i + \frac{\vec{p}^2}{2m_i} = m_f + \frac{(\vec{p} - \vec{q})^2}{2m_f} + q^0$ or $m_f - m_i \approx -q^0$. Replacing $m_f - m_i$ by $-q^0$ in the pseudoscalar case we obtain parity between the two results.

Including flux-tube degrees-of-freedom, the matrix element for emission of a pion by the

quark in a meson is

$$\begin{aligned} \mathcal{M}(q_i \rightarrow q_f + \pi^+) &= \int d^3 \vec{p} \phi_i(\vec{p}) \int d^3 \vec{p}' \phi_f^*(\vec{p}') \int \prod_q d^2 \vec{p}_{a_q} \prod_p \chi_i(\vec{p}_{a_p}) \int \prod_q d^2 \vec{p}'_{a_q} \prod_p \chi_f^*(\vec{p}'_{a_p}) \\ &\times \left\langle \bar{q} \left(\frac{m}{m_T} \vec{P}_f + \vec{p}' + \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{(-1)^p}{p} \vec{p}'_{a_p} \right) \right| \bar{q} \left(\frac{m}{m_T} \vec{P}_i + \vec{p} + \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{(-1)^p}{p} \vec{p}_{a_p} \right) \right\rangle \\ &\times \left\langle \text{f.t.} \left(\frac{br}{m_T} \vec{P}_f + 2 \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{\delta_{p,\text{odd}}}{p} \vec{p}'_{a_p} \right) \right| \text{f.t.} \left(\frac{br}{m_T} \vec{P}_i + 2 \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{\delta_{p,\text{odd}}}{p} \vec{p}_{a_p} \right) \right\rangle \\ &\times \left\langle q_f \left(\frac{m}{m_T} \vec{P}_f - \vec{p}' - \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{1}{p} \vec{p}'_{a_p} \right) \right| \frac{g}{2m} \int d^3 \vec{r} \bar{\psi}(\vec{r}) \Gamma_{\frac{\tau}{2}} \psi(\vec{r}) e^{-i\vec{q} \cdot \vec{r}} \left| q_i \left(\frac{m}{m_T} \vec{P}_i - \vec{p} - \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{1}{p} \vec{p}_{a_p} \right) \right\rangle, \end{aligned}$$

where we have conserved total flux-tube momentum $\vec{p}_{\text{f.t.}} = \sum_n \vec{p}_n$. Next we integrate out $\vec{p}' = \vec{p} - \frac{m}{m_T}(\vec{P}_f - \vec{P}_i) - \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{(-1)^p}{p} (\vec{p}'_{a_p} - \vec{p}_{a_p})$ leaving delta-functions

$$\begin{aligned} &\delta \left[\frac{2m}{m_T} (\vec{P}_f - \vec{P}_i) + \vec{q} - 2 \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{\delta_{p,\text{odd}}}{p} (\vec{p}'_{a_p} - \vec{p}_{a_p}) \right] \\ &\delta \left[\frac{br}{m_T} (\vec{P}_f - \vec{P}_i) + \vec{q} + 2 \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{\delta_{p,\text{odd}}}{p} (\vec{p}'_{a_p} - \vec{p}_{a_p}) \right]. \end{aligned}$$

Integrating out $\vec{p}'_{a_p}$ leaves an overall momentum conserving delta-function $\delta[\vec{P}_f - \vec{P}_i - \vec{q}]$. In the initial meson rest frame we have $\vec{p}'_{a_p} = \vec{p}_{a_p} + \vec{q} \sqrt{\frac{2}{N+1}} \frac{br}{\pi m} \frac{1}{p}$ and hence at lowest non-trivial order in br/m ,

$$\begin{aligned} \mathcal{M}(q_i \rightarrow q_f + \pi^+) &= \frac{g}{2m} F(q_i, q_f) \int d^3 \vec{p} \int \prod_q d^2 \vec{p}_{a_q} \phi_f^*(\vec{p} + \vec{q}/2) \phi_i(\vec{p}) \prod_p \chi_f^*(\vec{p}_{a_p} + \vec{q} \sqrt{\frac{2}{N+1}} \frac{br}{\pi m} \frac{1}{p}) \chi_i(\vec{p}_{a_p}) \\ &\times \bar{u} \left(-\vec{p} - \vec{q} - \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{1}{p} \vec{p}_{a_p} \right) \Gamma u \left(-\vec{p} - \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{1}{p} \vec{p}_{a_p} \right). \end{aligned}$$

In position space, for the currents reduced in equations (D.2,D.3) we find, for emission by the quark and the antiquark,

$$\begin{aligned} \mathcal{M}_q &= -\frac{g}{2m} F(q_i, q_f) \int d^3 \vec{r} \int \prod_q d^2 \vec{a}_q \psi_f^*(\vec{r}) \chi_f^*(\vec{a}_p) \left[\vec{\sigma}_q \cdot \vec{q} - \frac{q^0}{m} \vec{\sigma}_q \cdot \left(\vec{p}' + \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{1}{p} \vec{p}'_{a_p} \right) \right] \\ &\exp -i\vec{q} \cdot \left(-\frac{1}{2} \vec{r} - \sqrt{\frac{2}{N+1}} \frac{br}{\pi m} \sum_p \frac{1}{p} \vec{a}_p \right) \psi_i(\vec{r}) \chi_i(\vec{a}_p) \\ \mathcal{M}_{\bar{q}} &= +\frac{g}{2m} F(\bar{q}_i, \bar{q}_f) \int d^3 \vec{r} \int \prod_q d^2 \vec{a}_q \psi_f^*(\vec{r}) \chi_f^*(\vec{a}_p) \left[\vec{\sigma}_{\bar{q}} \cdot \vec{q} + \frac{q^0}{m} \vec{\sigma}_{\bar{q}} \cdot \left(\vec{p}' + \frac{\sqrt{2(N+1)}}{\pi} \sum_p \frac{(-1)^p}{p} \vec{p}'_{a_p} \right) \right] \\ &\exp -i\vec{q} \cdot \left(\frac{1}{2} \vec{r} + \sqrt{\frac{2}{N+1}} \frac{br}{\pi m} \sum_p \frac{(-1)^p}{p} \vec{a}_p \right) \psi_i(\vec{r}) \chi_i(\vec{a}_p). \quad (\text{D.4}) \end{aligned}$$

The primed momentum operators, as in the no flux-tube case, act backwards onto the final state wavefunction.

Appendix E

$SU(3)$ -Flavour Multiplets and I, U, V -spin

We can move between states in an $SU(3)_F$ multiplet using the I, U, V -spin raising and lowering operators defined by the $SU(3)$ sub-algebras,

$$\begin{aligned} [I_+, I_-] &= 2I_3 & [I_+, U_+] &= V_+ & [I_-, V_+] &= 0 \\ [U_+, U_-] &= \frac{1}{2}(1 + 3S) - I_3 & [I_+, V_+] &= U_+ & [I_-, U_+] &= 0 \\ [V_+, V_-] &= \frac{1}{2}(1 + 3S) + I_3 & [V_+, U_-] &= I_+. \end{aligned} \quad (\text{E.1})$$

The simplest $SU(3)_F$ multiplet is the fundamental containing an zero-strangeness isodoublet u, d , and a strangeness minus one isosinglet s . The following transforms satisfy (E.1):

$$I_- u = d \qquad U_- d = s.$$

The anti-fundamental containing the conjugates $\bar{u}, \bar{d}, \bar{s}$ has consistent transforms:

$$I_- \bar{d} = -\bar{u} \qquad U_- \bar{s} = -\bar{d}.$$

Hence for the meson octet ($\mathbf{3}_F \otimes \bar{\mathbf{3}}_F$), starting with $K^+ \equiv u\bar{s}$ we can construct the remaining flavour wavefunctions:

$$\begin{aligned} K^0 &= I_- K^+ = d\bar{s} \\ \pi^+ &= U_- K^+ = -u\bar{d} \\ \pi^0 &= \frac{1}{\sqrt{2}} I_- \pi^+ = \frac{1}{\sqrt{2}} (u\bar{u} - d\bar{d}) \\ &\vdots \end{aligned}$$

The complete set of flavour wavefunctions are shown in Table E.1.

K^0	$d\bar{s}$
K^+	$u\bar{s}$
π^-	$d\bar{u}$
π^0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$
π^+	$-u\bar{d}$
η_8	$\frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} - 2s\bar{s})$
K^-	$s\bar{u}$
$\bar{K}^0$	$-s\bar{d}$

Table E.1: Meson octet flavour wavefunctions

The nucleon octet is formed from $\mathbf{3}_F^{(1)} \otimes \mathbf{3}_F^{(2)} \otimes \mathbf{3}_F^{(3)}$ with (1), (2) coupled either symmetrically to a $\mathbf{6}_F$ (M_S) or antisymmetrically to a $\mathbf{\bar{3}}_F$ (M_A). Hence for the proton,

$$p(M_A) = \frac{ud - du}{\sqrt{2}}u,$$

while the M_S part of the proton is a linear superposition of components $\frac{ud+du}{\sqrt{2}}u$ and uud with isospin Clebsch-Gordan coefficients, $\langle \frac{1}{2}, +\frac{1}{2}; 1, 0 | \frac{1}{2}, +\frac{1}{2} \rangle, \langle \frac{1}{2}, -\frac{1}{2}; 1, +1 | \frac{1}{2}, +\frac{1}{2} \rangle$:

$$p(M_S) = \frac{1}{\sqrt{3}} \left(\frac{ud + du}{\sqrt{2}}u - \sqrt{2}uud \right).$$

The complete set of flavour wavefunctions are shown in Table E.2. The ground state baryon octet is in a $\mathbf{56}$ of $SU(6)$ flavour-spin with flavour-spin wavefunction,

$$B_{\uparrow} = \frac{1}{\sqrt{2}} \left(\phi^{M_S} \chi_{\uparrow}^{M_S} + \phi^{M_A} \chi_{\uparrow}^{M_A} \right),$$

where

$$\begin{aligned} \chi_{\uparrow}^{M_S} &= \frac{1}{\sqrt{3}} \left(\frac{\uparrow\uparrow+\downarrow\downarrow}{\sqrt{2}} \uparrow - \sqrt{2}(\uparrow\uparrow) \downarrow \right) \\ \chi_{\uparrow}^{M_A} &= \frac{\uparrow\downarrow-\downarrow\uparrow}{\sqrt{2}} \end{aligned}$$

	ϕ^{M_S}	ϕ^{M_A}
p	$\frac{1}{\sqrt{3}} \left(\frac{ud+du}{\sqrt{2}}u - \sqrt{2}uud \right)$	$\frac{ud-du}{\sqrt{2}}u$
n	$-\frac{1}{\sqrt{3}} \left(\frac{ud+du}{\sqrt{2}}d - \sqrt{2}ddu \right)$	$\frac{ud-du}{\sqrt{2}}d$
Σ^+	$\frac{1}{\sqrt{3}} \left(\frac{us+su}{\sqrt{2}}u - \sqrt{2}uus \right)$	$\frac{us-su}{\sqrt{2}}u$
Σ^0	$\frac{1}{\sqrt{6}} \left(\frac{ds+sd}{\sqrt{2}}u + \frac{us+su}{\sqrt{2}}d - 2\frac{ud+du}{\sqrt{2}}s \right)$	$\frac{1}{\sqrt{2}} \left(\frac{ds-sd}{\sqrt{2}}u + \frac{us-su}{\sqrt{2}}d \right)$
Λ^0	$\frac{1}{\sqrt{2}} \left(\frac{ds+sd}{\sqrt{2}}u - \frac{us+su}{\sqrt{2}}d \right)$	$\frac{1}{\sqrt{6}} \left(-\frac{ds-sd}{\sqrt{2}}u + \frac{us-su}{\sqrt{2}}d + 2\frac{ud-du}{\sqrt{2}}s \right)$
Σ^-	$\frac{1}{\sqrt{3}} \left(\frac{ds+sd}{\sqrt{2}}d - \sqrt{2}dds \right)$	$\frac{ds-sd}{\sqrt{2}}d$
Ξ^-	$-\frac{1}{\sqrt{3}} \left(\frac{ds+sd}{\sqrt{2}}s - \sqrt{2}ssd \right)$	$\frac{ds-sd}{\sqrt{2}}s$
Ξ^0	$-\frac{1}{\sqrt{3}} \left(\frac{us+su}{\sqrt{2}}s - \sqrt{2}ssu \right)$	$\frac{us-su}{\sqrt{2}}s$

Table E.2: Baryon octet flavour wavefunctions

Appendix F

Matrix Elements for Chapter 9

F.1 Colour Expectations, $\langle \frac{\vec{\lambda}_i}{2} \cdot \frac{\vec{\lambda}_j}{2} \rangle$

The q^4 colour basis states required for the JW correlation and the “ground-state” $\mathbf{210}_{FS}$ are

$$|\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c\rangle; |\mathbf{6}_c, \bar{\mathbf{3}}_c\rangle; |\bar{\mathbf{3}}_c, \mathbf{6}_c\rangle, \quad (\text{F.1})$$

where each state is coupled to an overall $\mathbf{3}_c$. Explicit representations of these are

$$\begin{aligned} |\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c; \mathbf{3}_c, R\rangle &= -\sqrt{\frac{1}{2}} \left(\frac{RG-GR}{\sqrt{2}} \frac{BR-RB}{\sqrt{2}} - \frac{BR-RB}{\sqrt{2}} \frac{RG-GR}{\sqrt{2}} \right) \\ |\mathbf{6}_c, \bar{\mathbf{3}}_c; \mathbf{3}_c, R\rangle &= \frac{1}{2} \left(\sqrt{2} RR \frac{GB-BG}{\sqrt{2}} + \frac{RG+GR}{\sqrt{2}} \frac{BR-RB}{\sqrt{2}} + \frac{RB+BR}{\sqrt{2}} \frac{RG-GR}{\sqrt{2}} \right) \\ |\bar{\mathbf{3}}_c, \mathbf{6}_c; \mathbf{3}_c, R\rangle &= \frac{1}{2} \left(\sqrt{2} \frac{GB-BG}{\sqrt{2}} RR + \frac{BR-RB}{\sqrt{2}} \frac{RG+GR}{\sqrt{2}} + \frac{RG-GR}{\sqrt{2}} \frac{RB+BR}{\sqrt{2}} \right), \end{aligned}$$

with similar expressions for the G, B states.

$\vec{\lambda}_i \cdot \vec{\lambda}_j$ can be expressed as a sum of diagonal and off-diagonal (colour changing) terms,

$$\begin{aligned} \vec{\lambda}_i \cdot \vec{\lambda}_j &= \lambda_i^{(3)} \lambda_j^{(3)} + \lambda_i^{(8)} \lambda_j^{(8)} + 2 \left[\lambda_i^{(B \rightarrow R)} \lambda_i^{(R \rightarrow B)} + \lambda_i^{(R \rightarrow B)} \lambda_i^{(B \rightarrow R)} \right. \\ &\quad \left. \lambda_i^{(G \rightarrow R)} \lambda_i^{(R \rightarrow G)} + \lambda_i^{(R \rightarrow G)} \lambda_i^{(G \rightarrow R)} \right. \\ &\quad \left. \lambda_i^{(B \rightarrow G)} \lambda_i^{(G \rightarrow B)} + \lambda_i^{(G \rightarrow B)} \lambda_i^{(B \rightarrow G)} \right], \quad (\text{F.2}) \end{aligned}$$

where the diagonal terms transform the colours as

$$\lambda_i^{(3)} \begin{Bmatrix} R \\ G \\ B \end{Bmatrix} = \begin{Bmatrix} R \\ -G \\ 0 \end{Bmatrix}; \quad \lambda_i^{(8)} \begin{Bmatrix} R \\ G \\ B \end{Bmatrix} = \sqrt{\frac{1}{3}} \begin{Bmatrix} R \\ G \\ -2B \end{Bmatrix},$$

and an example of the off-diagonal terms is

$$\lambda_i^{(B \rightarrow R)} \begin{Bmatrix} R \\ G \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ R \end{Bmatrix}.$$

A check on this decomposition comes from considering a state where i, j are in a definite colour state since then the result can be expressed in terms of the squared Casimirs of $SU(3)$,

$$\langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c; \mathbf{3}_c, R | \frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_2}{2} | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c; \mathbf{3}_c, R \rangle = \frac{1}{2} (C(\bar{\mathbf{3}}) - 2C(\mathbf{3})) = -\frac{1}{2}C(\mathbf{3}) = -\frac{2}{3},$$

whereas with the decomposition (F.2),

$$\begin{aligned} \vec{\lambda}_1 \cdot \vec{\lambda}_2 |\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c; \mathbf{3}_c, R\rangle = & -\sqrt{\frac{1}{2}} \left(\left[-\frac{RG-GR}{\sqrt{2}} \frac{BR-RB}{\sqrt{2}} - 0 \right] \right. \\ & + \frac{1}{3} \left[\frac{RG-GR}{\sqrt{2}} \frac{BR-RB}{\sqrt{2}} - (-2) \frac{BR-RB}{\sqrt{2}} \frac{RG-GR}{\sqrt{2}} \right] \\ & \left. + 2 \left[-\frac{RG-GR}{\sqrt{2}} \frac{BR-RB}{\sqrt{2}} - (-1) \frac{BR-RB}{\sqrt{2}} \frac{RG-GR}{\sqrt{2}} \right] \right), \end{aligned}$$

so that

$$\langle \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c; \mathbf{3}_c, R | \frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_2}{2} | \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c; \mathbf{3}_c, R \rangle = \frac{1}{4} \frac{1}{2} (-1 + \frac{1}{3} - \frac{2}{3} - 2 - 2) = -\frac{2}{3}.$$

This straightforward approach can be used to obtain the matrix element of $\frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_2}{2}$ for the states (F.1) which we present in Table F.1

$\langle \frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_2}{2} \rangle = \begin{matrix} & \mathbf{6}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \mathbf{6}_c \\ \begin{matrix} \mathbf{6}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \mathbf{6}_c \end{matrix} & \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \end{matrix}$	$\langle \frac{\vec{\lambda}_3}{2} \cdot \frac{\vec{\lambda}_4}{2} \rangle = \begin{matrix} & \mathbf{6}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \mathbf{6}_c \\ \begin{matrix} \mathbf{6}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \mathbf{6}_c \end{matrix} & \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \end{matrix}$
$\langle \frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_3}{2} \rangle = \begin{matrix} & \mathbf{6}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \mathbf{6}_c \\ \begin{matrix} \mathbf{6}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \mathbf{6}_c \end{matrix} & \begin{pmatrix} -\frac{5}{12} & -\sqrt{\frac{1}{8}} & -\frac{1}{4} \\ -\sqrt{\frac{1}{8}} & -\frac{1}{6} & \sqrt{\frac{1}{8}} \\ -\frac{1}{4} & \sqrt{\frac{1}{8}} & -\frac{5}{12} \end{pmatrix} \end{matrix}$	$\langle \frac{\vec{\lambda}_2}{2} \cdot \frac{\vec{\lambda}_3}{2} \rangle = \begin{matrix} & \mathbf{6}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \mathbf{6}_c \\ \begin{matrix} \mathbf{6}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \mathbf{6}_c \end{matrix} & \begin{pmatrix} -\frac{5}{12} & \sqrt{\frac{1}{8}} & \frac{1}{4} \\ \sqrt{\frac{1}{8}} & -\frac{1}{6} & \sqrt{\frac{1}{8}} \\ \frac{1}{4} & \sqrt{\frac{1}{8}} & -\frac{5}{12} \end{pmatrix} \end{matrix}$
$\langle \frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_4}{2} \rangle = \begin{matrix} & \mathbf{6}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \mathbf{6}_c \\ \begin{matrix} \mathbf{6}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \mathbf{6}_c \end{matrix} & \begin{pmatrix} -\frac{5}{12} & -\sqrt{\frac{1}{8}} & \frac{1}{4} \\ -\sqrt{\frac{1}{8}} & -\frac{1}{6} & -\sqrt{\frac{1}{8}} \\ \frac{1}{4} & -\sqrt{\frac{1}{8}} & -\frac{5}{12} \end{pmatrix} \end{matrix}$	$\langle \frac{\vec{\lambda}_2}{2} \cdot \frac{\vec{\lambda}_4}{2} \rangle = \begin{matrix} & \mathbf{6}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c & \bar{\mathbf{3}}_c \mathbf{6}_c \\ \begin{matrix} \mathbf{6}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \bar{\mathbf{3}}_c \\ \bar{\mathbf{3}}_c \mathbf{6}_c \end{matrix} & \begin{pmatrix} -\frac{5}{12} & \sqrt{\frac{1}{8}} & -\frac{1}{4} \\ \sqrt{\frac{1}{8}} & -\frac{1}{6} & -\sqrt{\frac{1}{8}} \\ -\frac{1}{4} & -\sqrt{\frac{1}{8}} & -\frac{5}{12} \end{pmatrix} \end{matrix}$

Table F.1: Colour matrix elements

The same techniques can be used to obtain expectations of $\frac{\vec{\lambda}_1}{2} \cdot \frac{(-\vec{\lambda}_i^*)}{2}$. With the JW colour state, for all $i = 1 \dots 4$:

$$\langle (\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c)_{\mathbf{3}_c} \otimes \bar{\mathbf{3}}_c; \mathbf{1}_c | \frac{\vec{\lambda}_1}{2} \cdot \frac{(-\vec{\lambda}_i^*)}{2} | (\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c)_{\mathbf{3}_c} \otimes \bar{\mathbf{3}}_c; \mathbf{1}_c \rangle = -\frac{1}{3}.$$

F.2 Delta-function matrix elements

The three-dimensional delta-function has the following representation in terms of spherical harmonics:

$$\delta(\vec{r} - \vec{r}') = \frac{\delta(r - r')}{r^2} \sum_{L,m} Y_{Lm}(\hat{r}) Y_{Lm}^*(\hat{r}').$$

For the state (9.6) we have

$$\langle \delta(\vec{r}_{12}) \rangle = \langle \sqrt{2} \hat{\rho} \rangle = |N_\rho|^2 \int \rho^2 d\rho d\Omega_\rho e^{-\gamma^2 \rho^2} |Y_{00}(\hat{\rho})|^2 \frac{1}{(\sqrt{2})^3} \frac{\delta(\rho)}{\rho^2} \sum_{L,m} Y_{Lm}(\hat{\rho}) Y_{Lm}^*(\hat{0}),$$

and since the second spherical harmonic has no argument we can set $Y_{Lm}(\hat{0}) = \frac{1}{\sqrt{4\pi}}\delta_{L,0}\delta_{m,0}$ and hence

$$\langle \delta(\vec{r}_{12}) \rangle = \left(\frac{\gamma}{\sqrt{2\pi}} \right)^3.$$

An identical result is obtained for $\langle \delta(\vec{r}_{34}) \rangle$ as is obvious from the symmetry of the wavefunction.

The expectation of $\delta(\vec{r}_{13})$ is far from trivial as we shall now demonstrate:

$$\begin{aligned} \langle \delta(\vec{r}_{13}) \rangle &= \left\langle \delta \left(\frac{\vec{\rho}}{\sqrt{2}} - \left(\frac{\vec{\nu}}{\sqrt{2}} - \vec{\lambda} \right) \right) \right\rangle \\ &= (\sqrt{2})^3 \left\langle \frac{\delta(\rho - |\vec{\nu} - \sqrt{2}\vec{\lambda}|)}{\rho^2} \sum_{L,m} Y_{Lm}(\hat{\rho}) Y_{Lm}^*(\widehat{\vec{\nu} - \sqrt{2}\vec{\lambda}}) \right\rangle. \end{aligned}$$

Since the final and initial states are S -waves, the integral over Ω_ρ is non-zero only for $L = 0, m = 0$,

$$\langle \delta(\vec{r}_{13}) \rangle = \frac{(\sqrt{2})^3}{4\pi} \left\langle \frac{\delta(\rho - |\vec{\nu} - \sqrt{2}\vec{\lambda}|)}{\rho^2} \right\rangle,$$

integrating out $\vec{\rho}$ and showing the remaining integrals explicitly,

$$\langle \delta(\vec{r}_{13}) \rangle = \frac{N^2 \beta^2}{\sqrt{2\pi}} \int d^3 \vec{\nu} d^3 \vec{\lambda} |\lambda Y_{1m}(\hat{\lambda})|^2 |Y_{00}(\hat{\nu})|^2 \exp[-\lambda^2(\beta^2 + 2\gamma^2) - \nu^2(2\gamma^2) + \vec{\nu} \cdot \vec{\lambda}(2\sqrt{2}\gamma^2)].$$

The difficulty here is the angular dependence of $\vec{\nu} \cdot \vec{\lambda}$ in the exponent. A simple change of variables ($\vec{\lambda}, \vec{\nu} \rightarrow \vec{x}, \vec{y}$) will make this integral tractable

$$\begin{aligned} \vec{\lambda} &= \cos \theta \vec{x} + \sin \theta \vec{y} \\ \vec{\nu} &= -\sin \theta \vec{x} + \cos \theta \vec{y}. \end{aligned}$$

The argument of the exponential becomes

$$\begin{aligned} &-2\gamma^2 \left(x^2 \left[\left(1 + \frac{\beta^2}{2\gamma^2} \right) \cos^2 \theta + \sin^2 \theta + \sqrt{2} \cos \theta \sin \theta \right] \right. \\ &\quad + y^2 \left[\left(1 + \frac{\beta^2}{2\gamma^2} \right) \sin^2 \theta + \cos^2 \theta - \sqrt{2} \cos \theta \sin \theta \right] \\ &\quad \left. + \vec{x} \cdot \vec{y} \left[\frac{\beta^2}{\gamma^2} \cos \theta \sin \theta - \sqrt{2}(\cos^2 \theta - \sin^2 \theta) \right] \right) \end{aligned}$$

and hence we can remove the tricky angular dependence by choosing θ to satisfy

$$\frac{\beta^2}{\gamma^2} \cos \theta \sin \theta - \sqrt{2}(\cos^2 \theta - \sin^2 \theta) = 0$$

$$\tan 2\theta = 2\sqrt{2} \frac{\gamma^2}{\beta^2}.$$

This leaves an exponent $-2\gamma^2(a^2 x^2 + b^2 y^2)$ with

$$\begin{aligned} a^2 &= \left(1 + \frac{\beta^2}{2\gamma^2} + \frac{2\gamma^2}{\beta^2} \right) \cos^2 \theta + \left(1 + \frac{\beta^2}{2\gamma^2} - \frac{2\gamma^2}{\beta^2} \right) \sin^2 \theta \\ b^2 &= \left(1 + \frac{\beta^2}{2\gamma^2} - \frac{2\gamma^2}{\beta^2} \right) \sin^2 \theta + \left(1 + \frac{\beta^2}{2\gamma^2} + \frac{2\gamma^2}{\beta^2} \right) \cos^2 \theta. \end{aligned}$$

The factor $|\lambda Y_{1m}(\hat{\lambda})|^2$ is easily dealt with by noting that the $\lambda Y_{1m}(\hat{\lambda})$ are simply spherical components of the vector $\vec{\lambda}$, $Y_{1m}(\hat{\lambda}) = \vec{\epsilon}_m \cdot \vec{\lambda}$. The integral will have the same value for all m and we choose $m = 0$ for simplicity, hence

$$|\lambda Y_{10}(\hat{\lambda})|^2 = \cos^2 \theta x^2 |Y_{10}(\hat{x})|^2 + \sin^2 \theta y^2 |Y_{10}(\hat{y})|^2 + 2 \cos \theta \sin \theta xy Y_{10}(\hat{x}) Y_{10}^*(\hat{y}).$$

The Jacobean for this transform is 1 as it is simply a rotation in the space of vectors $(\vec{\lambda}, \vec{\nu})$, and hence

$$\langle \delta(\vec{r}_{13}) \rangle = \frac{N^2 \beta^2}{\sqrt{2\pi}} \int d^3 \vec{x} d^3 \vec{y} \frac{1}{4\pi} [\cos^2 \theta x^2 |Y_{10}(\hat{x})|^2 + \sin^2 \theta y^2 |Y_{10}(\hat{y})|^2 + 2 \cos \theta \sin \theta xy Y_{10}(\hat{x}) Y_{10}^*(\hat{y})] \\ \times \exp[-2\gamma^2(a^2 x^2 + b^2 y^2)],$$

which contains only trivial angular and Gaussian integrals which can be performed to give

$$\langle \delta(\vec{r}_{13}) \rangle = \frac{3N^2 \beta^2}{2^9 \sqrt{2} \gamma^8} \frac{1}{(ab)^5} (b^2 \cos^2 \theta + a^2 \sin^2 \theta).$$

With a little trigonometric manipulation one finds that $b^2 \cos^2 \theta + a^2 \sin^2 \theta = 1$ so that finally

$$\langle \delta(\vec{r}_{13}) \rangle = \left(\frac{\beta}{\sqrt{\pi}} \right)^3 \frac{\gamma^3 \beta^2}{(\gamma^2 + \beta^2)^{5/2}}.$$

F.3 210_{FS} “ground-state”

The spatial q^4 wavefunction for the ground-state will be totally symmetric which directs us to antisymmetric colour-flavour-spin q^4 wavefunctions. The colour state is specified by the need to couple with the antiquark $\bar{\mathbf{3}}_c$ to give a colour singlet; the $\mathbf{3}_c(q^4)$ state is of mixed symmetry and the required (to give a $\bar{\mathbf{10}}$) mixed symmetry flavour-spin representation is the 210_{FS} which has $S(q^4) = 1$.

The basis states written in our diquark-diquark notation are

	1, 2	3, 4
$210_{FS} \left\{ \begin{array}{l} \mathbf{6}_1, \mathbf{6}_1\rangle_{\bar{\mathbf{6}},1} \\ (\frac{\sqrt{3}}{2} \bar{\mathbf{3}}_1, \bar{\mathbf{3}}_0\rangle - \frac{1}{2} \mathbf{6}_0, \mathbf{6}_1\rangle)_{\bar{\mathbf{6}},1} \\ (\frac{\sqrt{3}}{2} \bar{\mathbf{3}}_0, \bar{\mathbf{3}}_1\rangle - \frac{1}{2} \mathbf{6}_1, \mathbf{6}_0\rangle)_{\bar{\mathbf{6}},1} \end{array} \right.$	S	S
	AS	S
	S	AS
$\mathbf{3}_c \left\{ \begin{array}{l} \bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c\rangle \\ \mathbf{6}_c, \bar{\mathbf{3}}_c\rangle \\ \bar{\mathbf{3}}_c, \mathbf{6}_c\rangle \end{array} \right.$	AS	AS
	S	AS
	AS	S

where we've indicated the transformation symmetry under $1 \rightarrow 2, 3 \rightarrow 4$. The appropriate overall antisymmetric combination is

$$|210_{FS}, \mathbf{3}_c\rangle = \frac{1}{\sqrt{3}} \left[|\mathbf{6}_1, \mathbf{6}_1\rangle |\bar{\mathbf{3}}_c, \bar{\mathbf{3}}_c\rangle + \left(\frac{\sqrt{3}}{2} |\bar{\mathbf{3}}_1, \bar{\mathbf{3}}_0\rangle - \frac{1}{2} |\mathbf{6}_0, \mathbf{6}_1\rangle \right) |\mathbf{6}_c, \bar{\mathbf{3}}_c\rangle \right. \\ \left. + \left(\frac{\sqrt{3}}{2} |\bar{\mathbf{3}}_0, \bar{\mathbf{3}}_1\rangle - \frac{1}{2} |\mathbf{6}_1, \mathbf{6}_0\rangle \right) |\bar{\mathbf{3}}_c, \mathbf{6}_c\rangle \right].$$

The orthogonal flavour-spin basis makes the evaluation of $\langle 210_{FS}, \mathbf{3}_c | \frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_2}{2} | 210_{FS}, \mathbf{3}_c \rangle$ involve only colour-diagonal terms such as $\langle \bar{\mathbf{3}}_c, \mathbf{6}_c | \frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_2}{2} | \bar{\mathbf{3}}_c, \mathbf{6}_c \rangle$. These can be read off from the matrices in Table F.1.

The spin-dependent factors $\frac{\vec{\lambda}_1}{2} \cdot \frac{\vec{\lambda}_2}{2} \vec{S}_i \cdot \vec{S}_j$ are a little trickier we require the spin matrix elements $\langle \vec{S}_i \cdot \vec{S}_j \rangle$, where the spin basis in (9.12) is $|S_{12} = 1 \otimes S_{34} = 1 \rightarrow S_{q^4} = 1\rangle, |0 \otimes 1 \rightarrow 1\rangle, |1 \otimes 0 \rightarrow 1\rangle$. These matrix elements can be computed using $SU(2)$ raising and lowering operators and are presented in Table F.2.

The evaluation of the colour-spin term is simplified slightly by the orthogonality of the flavour basis and the symmetry of the spatial wavefunction, leaving a sum over products of entries from Table F.1 and Table F.2. It is the lack of definite symmetry in the colour and spin spaces which has made his computation difficult and this difficulty is felt even more acutely if

$$\begin{array}{cc}
\begin{array}{c}
1 \otimes 1 \quad 1 \otimes 0 \quad 0 \otimes 1 \\
\langle \vec{S}_1 \cdot \vec{S}_2 \rangle = \begin{array}{c} 1 \otimes 1 \\ 1 \otimes 0 \\ 0 \otimes 1 \end{array} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & -\frac{3}{4} \end{pmatrix}
\end{array}
\quad
\begin{array}{c}
1 \otimes 1 \quad 1 \otimes 0 \quad 0 \otimes 1 \\
\langle \vec{S}_3 \cdot \vec{S}_4 \rangle = \begin{array}{c} 1 \otimes 1 \\ 1 \otimes 0 \\ 0 \otimes 1 \end{array} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}
\end{array} \\
\\
\begin{array}{c}
1 \otimes 1 \quad 1 \otimes 0 \quad 0 \otimes 1 \\
\langle \vec{S}_1 \cdot \vec{S}_3 \rangle = \begin{array}{c} 1 \otimes 1 \\ 1 \otimes 0 \\ 0 \otimes 1 \end{array} \begin{pmatrix} -\frac{1}{4} & \sqrt{\frac{1}{8}} & -\sqrt{\frac{1}{8}} \\ \sqrt{\frac{1}{8}} & 0 & \frac{1}{4} \\ -\sqrt{\frac{1}{8}} & \frac{1}{4} & 0 \end{pmatrix}
\end{array}
\quad
\begin{array}{c}
1 \otimes 1 \quad 1 \otimes 0 \quad 0 \otimes 1 \\
\langle \vec{S}_2 \cdot \vec{S}_3 \rangle = \begin{array}{c} 1 \otimes 1 \\ 1 \otimes 0 \\ 0 \otimes 1 \end{array} \begin{pmatrix} -\frac{1}{4} & \sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} \\ \sqrt{\frac{1}{8}} & 0 & -\frac{1}{4} \\ \sqrt{\frac{1}{8}} & -\frac{1}{4} & 0 \end{pmatrix}
\end{array} \\
\\
\begin{array}{c}
1 \otimes 1 \quad 1 \otimes 0 \quad 0 \otimes 1 \\
\langle \vec{S}_1 \cdot \vec{S}_4 \rangle = \begin{array}{c} 1 \otimes 1 \\ 1 \otimes 0 \\ 0 \otimes 1 \end{array} \begin{pmatrix} -\frac{1}{4} & -\sqrt{\frac{1}{8}} & -\sqrt{\frac{1}{8}} \\ -\sqrt{\frac{1}{8}} & 0 & -\frac{1}{4} \\ -\sqrt{\frac{1}{8}} & -\frac{1}{4} & 0 \end{pmatrix}
\end{array}
\quad
\begin{array}{c}
1 \otimes 1 \quad 1 \otimes 0 \quad 0 \otimes 1 \\
\langle \vec{S}_2 \cdot \vec{S}_4 \rangle = \begin{array}{c} 1 \otimes 1 \\ 1 \otimes 0 \\ 0 \otimes 1 \end{array} \begin{pmatrix} -\frac{1}{4} & -\sqrt{\frac{1}{8}} & \sqrt{\frac{1}{8}} \\ -\sqrt{\frac{1}{8}} & 0 & \frac{1}{4} \\ \sqrt{\frac{1}{8}} & \frac{1}{4} & 0 \end{pmatrix}
\end{array}
\end{array}$$

Table F.2: $S(q^4) = 1$ matrix elements

we try to compute the interaction with the antiquark. For this reason we have compared only the q^4 parts of the $\mathbf{210}_{FS}$ and the JW state in Chapter 9.

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