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A COMPREHENSIVE APPROACH TO REVEALED PREFERENCE THEORY

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A Comprehensive Approach to Revealed Preference Theory*

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Abstract

Richter's theorem and Afriat's theorem are two fundamental results underlying modern revealed preference analysis. In this paper, we provide a version of Richter's theorem that characterizes the rationalizability of a choice data set with a continuous utility function (rather than simply a complete preorder as in the original result) and extend Afriat's theorem so it becomes applicable in choice environments other than the classical setting of consumer demand. Furthermore, while standard treatments give very different proofs for these two results, we introduce a framework within which both results can be formulated and established in tandem. We also demonstrate how our generalized versions of these theorems can be used in empirical studies. In particular, we apply our results to devise tests for rationalizability in the context of choice data over lotteries, contingent consumption, intertemporal consumption, and positions in policy space. Some new results on the revealed preference theory of consumer demand (for instance, on the possibility of deriving utility functions from estimated Engel curves) are also reported.

JEL Classification: D11, D81.

Keywords: Revealed Preference, Rational Choice, Afriat's Theorem, Richter's Theorem, Engel curves.

1 INTRODUCTION

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Modern (non-random) revealed preference theory is based on the fundamental theorems of Richter (1966) and Afriat (1967). These theorems are structurally distinct, and have different advantages and disadvantages in terms of providing empirical tests of rationalizability. In a nutshell, the objective of this paper is to develop an approach which (i) allows both of these theorems to be established in tandem, and more importantly, (ii) extends both of them in a way that enlarges their scope for empirical applications.

1.1 Limitations of Richter's and Afriat's theorems. The primitives of revealed preference theory is a nonempty set X of choice alternatives, a collection $\mathcal{A}$ of nonempty subsets of X, and a correspondence c on $\mathcal{A}$ such that $c(A)$ is a nonempty subset of A for each $A \in \mathcal{A}$. The interpretation is that $\mathcal{A}$ contains all feasible sets at which the choices of the agent are observed, and for each feasible set A, the (observed) choices of the agent constitute the set $c(A)$. Richter's theorem applies at this level of generality, and provides a necessary and sufficient condition under which there is a preference relation $\succsim$ (that is, a complete and transitive binary relation) on X which rationalizes c in the sense that

$$c(A) = \{x \in A : x \succsim y \text{ for every } y \in A\}. \tag{1}$$

This theorem has certainly had a wide impact, but it has two features that limit its applicability. First, it implicitly assumes that the alternatives in $c(A)$ are *all* of the agent's optimal choices from A. If, instead, the researcher is not in a position to exclude the optimality of some of the alternatives that she did not see the agent choose (which could be due to, say, potential indifference or indecisiveness), the right rationalizability condition would be

$$c(A) \subseteq \{x \in A : x \succsim y \text{ for every } y \in A\}. \tag{2}$$

Second, Richter's theorem imposes no assumptions at all on X. In particular, this set does not possess a topological, linear and/or order structure, and as a result, the revealed preference relation $\succsim$ that it finds does not have any sort of continuity, convexity and/or monotonicity property. This is problematic in the case when X is an infinite set and $\mathcal{A}$ consists only of *some* of the feasible sets over which the observer is interested in developing a theory of the agent's behavior. The fact that one has found a preference relation the maximization of which explains the agent's behavior on feasible sets in the sample $\mathcal{A}$ does not guarantee that it can also explain the agent's behavior on a set B that does not belong to $\mathcal{A}$. For this, at the very least, we need the preference relation $\succsim$ to be well-behaved enough to guarantee that $\{x \in B : x \succsim y \text{ for every } y \in B\}$ is a nonempty set. Thus, there is a second sense in which Richter's theorem "requires"

the data to be comprehensive – this result assumes implicitly that $\mathcal{A}$ consists of *all* the feasible sets over which the analyst is interested in developing a theory of the agent's choice behavior, thereby ruling out the issue of out-of-sample choices entirely.[1]

Neither of these difficulties is present in Afriat's theorem, which is situated in the context of consumer demand theory. In that case, X is taken as $\mathbb{R}^n_+$, so it has intrinsic topological, linear, and order structures. In turn, $\mathcal{A}$ is a collection of linear budget sets, and the theorem provides a preference relation that can be represented by a strictly increasing, continuous, and concave utility function, and that rationalizes the observations in the sense of (2). The continuity property is important because it allows one to hypothesize that the maximization of this utility function determines the consumer's behavior on *every* budget set (in fact, on any compact feasible set) which may or may not belong to $\mathcal{A}$.

Nonetheless, Afriat's theorem also has some features that limit its applicability. First, and most obviously, it is not clear if, and when, one can utilize this result outside the context of consumer demand. To give an example, suppose we would like to see if we can rationalize the behavior of an individual in the context where the alternatives consist of lotteries over n many certain outcomes (that is, X is the $(n-1)$-dimensional unit simplex), and $\mathcal{A}$ is a collection of finite sets of such lotteries. While this is an entirely standard scenario (which is adopted in numerous experimental studies), there is simply no straightforward way of applying Afriat's theorem to it. Second, notice that rationalization in the sense of (2) is completely trivial if we do not impose some restrictions on the rationalizing preference, because it obviously holds if we suppose the agent is indifferent across all alternatives in X. Afriat's theorem is not trivial, because it provides us with a *strictly* increasing utility function. But we may wish to test for some other restrictions on that utility function; for example, in the case where the elements of $X = \mathbb{R}^n_+$ are interpreted as contingent consumption bundles over n states (with, say, known probabilities) a natural restriction on the utility function is that it be strictly increasing with respect to first order stochastic dominance. As it stands, Afriat's theorem is not flexible enough to provide us with a utility function that satisfies this property. A third limitation of this result is that it assumes that the data set is finite (in the sense that $\mathcal{A}$ is finite and c is finite-valued). While this is natural enough in most applications,

[1] Put another way, suppose the researcher wants to test the hypothesis that the agent's choice in some collection $\mathcal{B}$ of feasible sets is determined by the maximization of a preference relation. Empirically, however, he observes the agent's behavior only in a sample set $\mathcal{A}$ of feasible sets such that $\mathcal{A} \subset \mathcal{B}$. To test his hypothesis, then, the analyst has to find a preference relation with respect to which (i) the agent's behavior in $\mathcal{A}$ is rationalized, and (ii) an optimum exists for all feasible sets in $\mathcal{B}$. Finding a preference relation that merely rationalizes the observations in $\mathcal{A}$ does not suffice in general.

it is sometimes useful to perform a revealed preference analysis using estimates of a consumer's demand function defined on a non-finite domain. Mathematically speaking, checking the rationalizability of these estimates requires us to work with infinite data sets. It would thus be advantageous to have a version of Afriat's theorem where its finiteness assumption is relaxed.

1.2 Our results. This paper provides a unified treatment of Richter and Afriat's theorems. Our objective is to prove one rationalizability theorem which generalizes both of these results in a way that allows us to address the issues recounted above. Formally speaking, our approach is order-theoretic, and hence, it is closer to that of Richter than that of Afriat, except that we also consider here the case where the set X of choice alternatives is endowed with a topology. However, we do not require X to be a linear space and hence, unlike Afriat's theorem, convexity does not play a role in our approach. The following is a short summary of our main results which we organize under four headings.

[**1**] As an immediate consequence of our main rationalizability theorem, we obtain a version of Richter's theorem where rationalization is in the sense of (1) but with a preference relation that is representable by a *continuous* utility function. As we have noted above, continuity of the utility function ensures that we can speak of out-of-sample predictions in a meaningful way within any collection of compact feasible sets.

[**2**] Again as an immediate consequence of our main rationalizability theorem, we deduce a *flexible* version of Afriat's theorem that broadens the scope of the result well beyond consumer choice problems. All we require of the set of alternatives is that it be a locally compact and separable metric space (rather than just $\mathbb{R}^n_+$). Moreover, we consider such a space as ordered by an arbitrarily given dominance relation (rather than just the coordinatewise order, say, in the case of $\mathbb{R}^n_+$), and make sure that the rationalizing utility functions we find in this theorem are strictly increasing with respect to this order.

[**3**] Even in the familiar context of consumer theory with consumers maximizing strictly increasing utility functions on budget sets, our approach has something new to say. One issue we discuss is *recoverability* (Varian (1982)), that is, the extent to which an observer is able to recover information about the agent's preference from her observed choices, without subscribing to a particular utility function that happens to rationalize the choice data at hand. We identify the set of recoverable preference pairs when rationalizing utility functions are allowed to be just continuous and strictly increasing. (Our treatment differs from that of Varian (1982), who also requires them to be concave.)

In empirical revealed preference studies of consumer demand, it is sometimes useful

to consider a situation where the researcher can observe a finite set of Engel curves. As even a single Engel curve corresponds to an infinite set of choice observations, we cannot apply Afriat's theorem in its original form in this case. However, we show using our approach that the theorem *does* extend to this case and it is possible to characterize those families of Engel curves that are rationalizable by continuous and strictly increasing utility functions.

[4] Our generalized versions of Richter's and Afriat's theorems provide ready-to-use rationalizability tests in a variety of contexts of economic interest. We illustrate this with four applications in which the set X of alternatives is a subset of $\mathbb{R}^n_+$ (even though the theory we develop here applies in infinite-dimensional settings as well). Yet, in these applications, either (i) X is a proper subset of $\mathbb{R}^n_+$, and/or (ii) we are interested in rationalizing the choice data with a continuous utility function that is required to be strictly increasing with respect to some preorder other than the usual (coordinatewise) ordering of $\mathbb{R}^n$. These applications are:

(a) *Choice over lotteries.* The set of alternatives is the unit simplex in $\mathbb{R}^n$, and we consider rationalizability by any continuous utility function, or with a continuous utility function that is consistent with first (or second, third, etc.) order stochastic dominance.

(b) *Choice over contingent consumption.* The consumption space is $\mathbb{R}^n_+$ (n being the number of states with known probabilities) and the feasible sets are linear budget sets (induced by finitely many state price vectors), but we require the consumer's utility function to be consistent with first order stochastic dominance.

(c) *Choice over intertemporal consumption.* The consumption space is $\mathbb{R}^n_+$ (n being the number of dates) and the feasible sets are budget sets, but we require the utility function to be strictly increasing with respect to preorders that are stronger than the usual (coordinatewise) ordering. In particular, our results readily yields rationalizability tests to check for positive (or negative) time preference.

(d) *Choice over policy positions.* The set of alternatives is a subset of $\mathbb{R}^n$ which is interpreted as policy positions in a multi-attribute policy space. In this context, it is hardly sensible to use the coordinatewise ordering to rank policies, so the utility function of a "rational" policy maker need not be strictly increasing. Instead, we may either choose not to impose any monotonicity requirement on the policy maker's utility function or we may require that it increases strictly with an incomplete preorder that corresponds to unanimity among those party mem-

bers represented by the policy maker. It is straightforward to obtain tests of such notions of rationalizability from our main theorems.

1.3 Organization of the paper. Section 2 introduces the notation and mathematical concepts used throughout the paper. As we wish to improve upon Richter's and Afriat's theorems simultaneously, we need to allow for multiple notions of rationalizability. This requires us to allow for a fairly general domain of alternatives and arbitrarily fixed dominance relations on those domains. The language that is needed to capture these variations are developed in Section 3, where we introduce the notion of a *revealed preference framework*.

Section 4 begins with a discussion of Richter's and Afriat's theorems in their classic forms. We then prove a general theorem which encompasses both of these results. We also explain in that section that the inner workings of our generalized version of Afriat's theorem is actually not the same as the one obtained by Afriat, when we specialize to the context of consumer demand. Indeed, the rationalizing preference relation we obtain is not convex and, for certain data sets, *cannot* be convex, because it possesses a "tightness" property.

In Section 4, we make no assumptions on the cardinality of the data set nor do we impose any topology on the set X of alternatives. Starting from Section 5, we take X as a metric space, and impose some basic restrictions on the data set (such as the compactness of feasible sets). Within this framework, we show that the rationalizing preference relation we found in Section 4 is actually by a continuous utility function, thereby obtaining a continuous version of Richter's theorem and getting one step closer to the spirit of Afriat's original approach.

The generalizations of Richter's and Afriat's theorems that we achieve here yields a revealed preference theory whose scope for applications is quite substantial. We demonstrate this in the remainder of the paper. In Section 6, we apply our findings to the classic framework of consumer demand and obtain a number of novel results in this classic setting. In Section 7, we show how our rationalizability theorems apply in the contexts of lottery choice, contingent consumption, intertemporal choice, and choice of policy positions. Unless they are particularly short, all proofs of the technical results are contained in the Appendix.

2 PRELIMINARIES

The primary tool of analysis in this paper is order theory. The present section catalogues the definitions of all the order-theoretic notions that we utilize throughout the present

work. As these notions are largely standard, this section is mainly for the reader who may need a clarification about them in the main body of the paper.

2.1 Order-theoretic nomenclature. Let X be a nonempty set, and denote the diagonal of $X \times X$ by $\triangle_X$, that is, $\triangle_X := \{(x, x) : x \in X\}$. By a **binary relation** on X, we mean any nonempty subset of $X \times X$. For any binary relation R on X and $x, y \in X$, we adopt the convention of writing $x\ R\ y$ instead of $(x, y) \in R$, and say that x and y are R**-incomparable** if neither (x, y) nor (y, x) belongs to R. (Thus, $x \triangle_X y$ iff $x = y$ for any $x, y \in X$, so any distinct x and y in X are $\triangle_X$-incomparable.) Similarly, for any $x \in X$ and subset A of X, by $x\ R\ A$ we mean $x\ R\ y$ for every $y \in A$, and interpret the expression $A\ R\ x$ analogously. Moreover, for any binary relations R and S on X, we write $x\ R\ y\ S\ z$ to mean $x\ R\ y$ and $y\ S\ z$, and so on. For any subset A of X, the **decreasing closure** of A with respect to R is defined as

$$A^{\downarrow,R} := \{x \in X : y\ R\ x \text{ for some } y \in A\},$$

but when R is apparent from the context, we may denote this set simply as $A^{\downarrow}$. The **increasing closure** of A is defined dually. By convention, $x^{\downarrow} := \{x\}^{\downarrow}$ and $x^{\uparrow} := \{x\}^{\uparrow}$ for any x in X. If $A \subseteq X$ satisfies $A^{\downarrow,R} = A$ then we say that A is R**-decreasing**; similarly, A is said to be R**-increasing** if $A^{\uparrow,R} = A$.

The **inverse** of a binary relation R on X is itself a binary relation, defined as $R^{-1} := \{(y, x) : x\ R\ y\}$. The **composition** of two binary relations R and R' on X is defined as $R \circ R' := \{(x, y) \in X \times X : x\ R\ z\ R'\ y \text{ for some } z \in X\}$. In turn, we let $R^1 := R$ and $R^n := R \circ R^{n-1}$ for any integer $n > 1$; here R^n is said to be the n**th iterate** of R.

The **asymmetric part** of a binary relation R on X is defined as $P_R := R \backslash R^{-1}$ and the **symmetric part** of R is $I_R := R \cap R^{-1}$. We say that a binary relation R on X is S**-monotonic**, where S is another binary relation, if $S \subseteq R$. R is said to **extend** the binary relation S if $S \subseteq R$ and $P_S \subseteq P_R$. For any nonempty subset A of X, the set of all maximal elements with respect to R is denoted as $\mathrm{MAX}(A, R)$, that is, $\mathrm{MAX}(A, R) := \{x \in A : y\ P_R\ x \text{ for no } y \in A\}$. Similarly, the set of all maximum elements with respect to R is denoted as $\max(A, R)$, that is, $\max(A, R) := \{x \in A : x\ R\ y \text{ for all } y \in A\}$. We also define $\mathrm{MIN}(A, R) := \mathrm{MAX}(A, R^{-1})$ and $\min(A, R) := \max(A, R^{-1})$.

A binary relation R on X is said to be **reflexive** if $\triangle_X \subseteq R$, **antisymmetric** if $R \cap R^{-1} \subseteq \triangle_X$, **transitive** if $R \circ R \subseteq R$, and **complete** if $R \cup R^{-1} = X \times X$. The **transitive closure** of R, denoted by $\mathrm{tran}(R)$, is the smallest transitive relation on X that contains R, and is given by $\mathrm{tran}(R) := R \cup R^2 \cup \cdots$. In other words, $x\ \mathrm{tran}(R)\ y$ iff we can find a positive integer k and $x_0, ..., x_k \in X$ such that $x = x_0\ R\ x_1\ R \cdots R\ x_k = y$.

If R is reflexive and transitive, we refer to it as a **preorder** on X. (In particular, $\text{tran}(R)$ is a preorder on X for any reflexive binary relation R on X.) If R is an antisymmetric preorder, we call it a **partial order** on X. The ordered pair (X, R) is a **preordered set** if R is a preorder on X, and a **poset** if R is a partial order on X. In this paper, we often denote preorders by $\succsim$ or $\gtrsim$ (with $\succ$ and $>$ respectively acting as their asymmetric parts). Finally, we say that R is **acyclic** if $\triangle_X \cap P_R^n = \emptyset$ for every positive integer n. It is readily verified that transitivity of a binary relation implies its acyclicity, but not conversely.

For any preorder $\succsim$ on X, a complete preorder on X that extends $\succsim$ is said to be a **completion** of $\succsim$. It is a set-theoretical fact that every preorder on a nonempty set admits a completion. This result, which is based on the Axiom of Choice, is known as *Szpilrajn's theorem*.[2]

Given any preordered set $(X, \succsim)$, a function $f : X \to \mathbb{R}$ is said to be **increasing with respect to** $\succsim$, or simply $\succsim$**-increasing**, if $f(x) \geq f(y)$ holds for every $x, y \in X$ with $x \succsim y$. If, in addition, $f(x) > f(y)$ holds for every $x, y \in X$ with $x \succ y$, we say that f is **strictly increasing with respect to** $\succsim$, or that it is **strictly** $\succsim$**-increasing**.

Finally, given a poset $(X, \succcurlyeq)$ and a subset A of X, we denote by $\bigvee A$ the unique element of $\min(\{x \in X : x \succcurlyeq A\}, \succcurlyeq)$, provided that this set is nonempty (and hence a singleton). Analogously, $\bigwedge A$ is the unique element of $\max(\{x \in X : A \succcurlyeq x\}, \succcurlyeq)$, provided that this set is nonempty. If $\bigvee A$ exists for every nonempty finite $A \subseteq X$, then $(X, \succcurlyeq)$ is said to be a $\vee$**-semilattice**, and if $\bigvee A$ exists for every $A \subseteq X$, then $(X, \succcurlyeq)$ is said to be a **complete** $\vee$**-semilattice**. If $(X, \succcurlyeq^{-1})$ is a $\vee$-semilattice, we say that $(X, \succcurlyeq)$ is a $\wedge$**-semilattice**.

2.2 Topological Nomenclature. Let $(X, \succsim)$ be a preordered set such that X is a topological space. We say that $\succsim$ is a **continuous** preorder on X if it is a closed subset of $X \times X$ (relative to the product topology).[3] We note that the closure of a preorder on X (in $X \times X$) need not be transitive, nor is the transitive closure of a closed binary relation on X in general continuous. One needs additional conditions to ensure such inheritance properties to hold (cf. Ok and Riella (2014)).

Given a continuous preorder $\succsim$ on X, the topological conditions that would ensure the existence of a continuous real map on X that is strictly increasing with respect to $\succsim$

[2] Szpilrajn (1938) has proved this result for partial orders, but the result easily generalizes to the case of preorders; see Corollary 1 in Chapter 1 of Ok (2007).

[3] While there are other notions of continuity for a preorder (for instance, openness of its strict part on X), this terminology is adopted quite widely in the literature. See, for instance, Evren and Ok (2011) and references cited therein.

are well-studied in the mathematical literature. It is known that such a function exists if X is a locally compact and separable metric space. This is *Levin's theorem.*[4]

Notational Convention. Throughout this paper, we write $[k]$ to denote the set $\{1, ..., k\}$ for any positive integer k.

3 CHOICE ENVIRONMENTS AND CHOICE DATA

3.1 Choice environments and choice correspondences. By a **choice environment**, we mean an ordered pair $((X, \gtrsim), \mathcal{A})$, where $(X, \gtrsim)$ is a preordered set and $\mathcal{A}$ is a nonempty collection of nonempty subsets of X. We interpret X as the consumption space, that is, the grand set of all mutually exclusive choice alternatives. We think of $\gtrsim$ as an exogenously given domination relation on X, and view the statement $x \gtrsim y$ as saying that x is an unambiguously better alternative than y for any individual. (If the environment one wishes to study lacks such a dominance relation, we may set $\gtrsim$ as $\triangle_X$ so that $x \gtrsim y$ holds iff $x = y$.) Finally, $\mathcal{A}$ is interpreted as the set of all feasible sets from which a decision maker is observed to make a choice. For instance, if the data at hand is so limited that we have recorded the choice(s) of an agent in the context of a single feasible set $A \subseteq X$, we would set $\mathcal{A} = \{A\}$. At the other extreme, if we have somehow managed to keep track of the choices of the agent from every possible feasible set $A \subseteq X$ (as sometimes is possible in the controlled environments of laboratory experiments), we would set $\mathcal{A} = 2^X \backslash \{\emptyset\}$.

Given a nonempty set X and a nonempty subset $\mathcal{A}$ of $2^X \backslash \{\emptyset\}$, we call a map $c : \mathcal{A} \to 2^X$ such that $c(A)$ is a nonempty subset of A for each $A \in \mathcal{A}$ a **choice correspondence**. We denote the family of all choice correspondences by $\mathfrak{C}_{(X,\mathcal{A})}$. By a **choice correspondence on the choice environment** $((X, \gtrsim), \mathcal{A})$, we simply mean any one member of $\mathfrak{C}_{(X,\mathcal{A})}$. A natural way of ordering the elements of $\mathfrak{C}_{(X,\mathcal{A})}$ is given by the binary relation $\sqsupseteq$ defined as

$$c \sqsupseteq d \quad \text{iff} \quad c(A) \supseteq d(A) \text{for every } A \in \mathcal{A}.$$

Clearly, $(\mathfrak{C}_{(X,\mathcal{A})}, \sqsupseteq)$ is a poset. It is also plain that this poset is a complete $\vee$-semilattice, but it is not an $\wedge$-semilattice unless all members of $\mathcal{A}$ are singleton subsets of X.

3.2 Revealed preference frameworks. By a **revealed preference (RP) framework**, we mean an ordered triplet

$$((X, \gtrsim), \mathcal{A}, \mathbb{C}),$$

[4] See Levin (1983), where this result is proved in the more general case where X is a locally compact, σ-compact and second countable Hausdorff space.

where $((X, \gtrsim), \mathcal{A})$ is a choice environment and $\mathbb{C}$ is a nonempty collection of choice correspondences on this environment. We refer to $\mathbb{C}$ as a **choice data** on $((X, \gtrsim), \mathcal{A})$, and interpret it as summarizing the choices of a given decision maker across all feasible sets in $\mathcal{A}$ in the sense that $\mathbb{C}$ is precisely the set of *all* choice correspondences on $((X, \gtrsim), \mathcal{A})$ that are compatible with the (observed) choices of that agent. This setup is quite general and departs from how revealed preference theory is often formulated in the literature mainly in two ways. First, it features the notion of an unambiguous ordering of the alternatives (in terms of some form of a domination relation). Second, it takes as a primitive not one choice correspondence, but potentially a multiplicity of them.

Let $((X, \gtrsim), \mathcal{A}, \mathbb{C})$ be an RP framework. Suppose we wish to model the situation in which the analyst can observe *all* choices of an individual from any given feasible set. (This is the basic assumption of Richter-type approaches to revealed preference theory.) In other words, we posit that the outside observer is able to identify the "true" choice correspondence, say, c, of the decision-maker in its entirety. Naturally, this situation requires us to set $\mathbb{C} = \{c\}$. Consequently, we refer to $((X, \gtrsim), \mathcal{A}, \mathbb{C})$ as a **Richter-type RP framework** whenever $\mathbb{C}$ is a singleton.

This sort of a model is, however, not suitable for a variety of empirical revealed preference studies in which the researcher observes only some of the choices of an individual from each feasible set. To capture these situations, we should model the choice data as

$$\mathbb{C} := \{C \in \mathfrak{C}_{(X,\mathcal{A})} : C \sqsupseteq c\}$$

or, more succinctly, $\mathbb{C} = c^{\uparrow}$, where $c^{\uparrow}$ is the increasing closure of $\{c\}$ with respect to $\sqsupseteq$. We refer to the corresponding RP framework, that is, $((X, \gtrsim), \mathcal{A}, c^{\uparrow})$ as an **Afriat-type RP framework**. Here the choice correspondence c is interpreted as the observed choices of the individual (that is, $c(A)$ is what the observer sees the individual choose from A for each $A \in \mathcal{A}$). Since we cannot exclude the possibility that the individual was indifferent, or indecisive, between her choice from a feasible set A and some other alternatives in A, the framework allows for the "true" choice correspondence of the individual to be any element of $c^{\uparrow}$.

Virtually all work in revealed preference theory can be thought of as being carried out in either a Richter- or an Afriat-type framework; the following examples are representative of the literature.

Example 1. (*Richter's model*) Let X be a nonempty set, $\mathcal{A}$ a nonempty collection of nonempty subsets of X, and c a choice correspondence on $\mathcal{A}$. Then, $((X, \triangle_X), \mathcal{A}, \{c\})$ is

an RP framework that corresponds to the choice model of Richter (1966). The interpretation of the model is that one is able to observe *all* elements that are deemed choosable by the decision maker from any member of $\mathcal{A}$. No exogenous order (or otherwise) structure on the consumption set X is postulated. □

Many of the revealed preference analyses conducted in the literature on choice theory work with instances of Richter's model. A common specification is one in which X is finite and $\mathcal{A}$ consists of all nonempty subsets of X. (This is precisely the model studied by Arrow (1959) and is the most commonly adopted framework in the recent literature on boundedly rational choice.) However, it is also common to assume that we can observe choice behavior only for certain types of feasible sets. This is the case in the next example.

Example 2. (*Classical consumption choice model*) Let n be a positive integer. Take X as $\mathbb{R}^n_+$, $\geq$ as the coordinatewise ordering of n-vectors, and suppose that $\mathcal{A}$ is a nonempty subset of $\{B(\mathbf{p}, I) : (\mathbf{p}, I) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}\}$, where $B(\mathbf{p}, I)$ is the budget set at prices $\mathbf{p}$ and income I, that is, $B(\mathbf{p}, I) := \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{px} \leq I\}$ for every positive $(n+1)$-vector $(\mathbf{p}, I)$. Then $((\mathbb{R}^n_+, \geq), \mathcal{A}, \{c\})$, where c is a choice correspondence on $\mathcal{A}$ is an instance of a Richter-type framework and corresponds to the classical consumption choice model.[5] □

Example 3. (*Nash bargaining model*) Put $X = \mathbb{R}^n_+$ and take $\mathcal{A}$ as the set of all compact and convex subsets of X that contain the origin $\mathbf{0}$ in their interior. Then the RP framework $((\mathbb{R}^n_+, \geq), \mathcal{A}, \{c\})$ is a Richter-type framework and corresponds to the classical n-person cooperative bargaining model. In this model, elements of X are interpreted as the utility profiles of the individuals involved (of which there are n), while $\mathbf{0}$ is the (normalized) utility profile that corresponds to the disagreement outcome. When c is single-valued this model reduces to the one considered by Nash (1950) and a large fraction of the literature on axiomatic bargaining theory. If we relax the convexity requirement, we obtain the model of non-convex collective choice problems (cf. Ok and Zhou (1999)). □

Example 4. (*Afriat's classical consumption choice model*) In Afriat's (1967) model, the consumption set is $\mathbb{R}^n_+$ and endowed with the usual (coordinatewise) order $\geq$. The model imagines an observer who can record the bundle(s) bought by an agent at a *finite*

[5] Here, and throughout the paper, by the partial order $\geq$ on any subset of $\mathbb{R}^n$, we mean the usual (coordinatewise) ordering of n-vectors, that is, $\mathbf{x} \geq \mathbf{y}$ iff x_i is larger than y_i for each $i \in [n]$.

collection of price vectors. Formally, there is a nonempty finite subset P of $\mathbb{R}^n_{++}$, and a map d that assigns to every $\mathbf{p} \in P$ a nonempty finite subset $d(\mathbf{p})$ of $\mathbb{R}^n_+$ such that $\mathbf{py} = \mathbf{pz}$ for every $\mathbf{y}$ and $\mathbf{z}$ in $d(\mathbf{p})$ and $B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) \neq B(\mathbf{p}', \mathbf{p}'d(\mathbf{p}'))$ for any distinct $\mathbf{p}$, $\mathbf{p}'$ in P. (With a slight abuse of notation, by $\mathbf{p}d(\mathbf{p})$ we mean $\mathbf{py}$ for any $\mathbf{y} \in d(\mathbf{p})$.) We interpret P as a set of price vectors and d as the (observed) **demand correspondence** on P, that is, $d(\mathbf{p})$ is the set of bundles that the individual was observed to choose from the budget set $B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$.[6] This model is captured by the Afriat-type RP framework $((\mathbb{R}^n_+, \geq), \mathcal{A}, c^{\uparrow})$, where $\mathcal{A} = \{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$ and c is the choice correspondence on $\mathcal{A}$ that maps $B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$ to $d(\mathbf{p})$. □

Example 5. (*The Forges-Minelli model of consumption choice*) To deal with nonlinearities arising from price differentiation that depend on quantity thresholds and other considerations, many authors have considered consumption models with nonlinear budget sets (cf. Matzkin (1991) and Chavas and Cox (1993)). In particular, Forges and Minelli (2009) take as a primitive finitely many ordered pairs, say, $(g^1, \mathbf{x}^1), (g^2, \mathbf{x}^2), ..., (g^k, \mathbf{x}^k)$, where g^i is a strictly increasing and continuous real map on $\mathbb{R}^n_+$ with $g^i(\mathbf{x}^i) = 0$ for each $i \in [k]$. They interpret this data as the situation in which we observe a decision maker choosing the bundle $\mathbf{x}^i$ from the generalized budget set $B^i(g^i) := \{\mathbf{x} \in \mathbb{R}^n_+ : g^i(\mathbf{x}) \leq 0\}$ for each $i \in [k]$. This setup is captured by the Afriat-type RP framework $((\mathbb{R}^n_+, \geq), \mathcal{A}, \mathbb{C})$ where $\mathcal{A} := \{B^i(g^i) : i \in [k]\}$ and $\mathbb{C}$ is the set of all choice correspondences C on $\mathcal{A}$ such that $\mathbf{x}^i \in C(B(g^i))$ for each $i \in [k]$. □

3.3 Alternative revealed preference frameworks. As it allows for unobserved optimal choices, an Afriat-type framework is in general more permissive than a Richter-type framework. However, such an RP framework too requires the choice from a feasible set be observed in its entirety. And yet there are instances where an observer receives only *partial* information about the choice from a feasible set. This occurs in situations where the observer sees an agent committing to choosing from some subset of a feasible set, without observing her ultimate choice. For example, suppose that $X = X_1 \times X_2$, where X_1 is the set of alternatives available at date 1 and X_2 the set available at date 2. The agent chooses (x_1, x_2) from a nonempty feasible set $A \subseteq X$. If, for some reason, the observer sees that the choice of the agent at date 1 is, say, a, but he does not see her choice at date 2, she would still be able to surmise that the agent's choice (in its

[6] The restriction that bundles in $d(\mathbf{p})$ incur the same expenditure is without loss of generality. Indeed, by modifying the domain P if necessary, we can assume that $\mathbf{p}d(\mathbf{p}) = 1$ for all $\mathbf{p} \in P$. This is because $B(\mathbf{p}, \mathbf{py}) = B(\lambda\mathbf{p}, \lambda\mathbf{py})$ for any $\lambda > 0$ and $\mathbf{y} \in \mathbb{R}^n_+$, so requiring income to equal 1 imposes no restrictions on a budget set provided the price can be scaled up or down.

entirety) lies in the set $\{(x_1, x_2) \in A : x_1 = a\}$.

To accommodate this sort of a situation, we can use the RP framework, $((X, \succsim), \mathcal{A}, \mathbb{C})$, where

$$\mathbb{C} := \{C \in \mathfrak{C}_{(X,\mathcal{A})} : t \sqsupseteq C\} \tag{3}$$

for some choice correspondence $t : \mathcal{A} \to 2^X \backslash \{\emptyset\}$.[7] For any $A \in \mathcal{A}$, we interpret $t(A)$ as a set, declared by the agent, from which her ultimate choice will be made. Somewhat analogous to Richter's framework, this framework implicitly requires *all* optimal choices of the agent from A to be contained in $t(A)$. If we allow the agent to be indifferent between her ultimate choice from $t(A)$ and some alternative in $A \backslash t(A)$ then we would define $\mathbb{C}$ as

$$\mathbb{C} := \{C \in \mathfrak{C}_{(X,\mathcal{A})} : t(A) \cap C(A) \neq \emptyset \text{ for each } A \in \mathcal{A}\}. \tag{4}$$

Such RP frameworks would be suitable in a variety of contexts, such as intertemporal choice problems, where more information about the final choice(s) of the agent is partly revealed in stages, but we have introduced them here only to point to the flexible nature of the abstract notion of an RP framework. Mainly for purposes of brevity, we will not work with such structures in the remainder of this paper.

4 RATIONALIZABILITY OF CHOICE DATA (with arbitrary data sets)

4.1 Rationalizability of choice correspondences. Let X be a nonempty set and $\mathcal{A}$ a nonempty subset of $2^X \backslash \{\emptyset\}$. A choice correspondence c on $\mathcal{A}$ is said to be **rationalizable** if there is a complete preorder $\succsim$ on X such that

$$c(A) = \max(A, \succsim) \quad \text{for every } A \in \mathcal{A}. \tag{5}$$

In his seminal paper, Richter (1966) has provided a characterization of such choice correspondences by means of what he dubbed the "congruence axiom." To state this property, let us define the binary relation $R(c)$ on X by

$$x \; R(c) \; y \quad \text{if and only if} \quad (x, y) \in c(A) \times A \text{ for some } A \in \mathcal{A}.$$

This relation, introduced first by Samuelson (1938) in the special case of consumption problems, is often called the **direct revealed preference relation** induced by c in the literature, while the transitive closure of $R(c)$ is referred to as the **revealed preference**

[7] Such frameworks are scarcely considered in the literature on choice theory. Fishburn (1976) is an exception to this, but no economic motivation for studying such structures is provided in that paper.

relation induced by c. Then, given X and $\mathcal{A}$, a choice correspondence c on $\mathcal{A}$ is said to satisfy the **congruence axiom** if

$$x \text{ tran}(R(c)) \ y \quad \text{and} \quad y \in c(A) \quad \text{imply} \quad x \in c(A)$$

for every $A \in \mathcal{A}$ that contains x. Richter's Theorem says that a choice correspondence c on $\mathcal{A}$ is rationalizable iff it satisfies the congruence axiom.

4.2 Monotonic rationalizability and the monotonic congruence axiom. The notion of rationalizability readily extends to the more general context of RP frameworks. Where $((X, \gtrsim), \mathcal{A}, \mathbb{C})$ is an RP framework, we say that the choice data $\mathbb{C}$ is **rationalizable** if at least one c in $\mathbb{C}$ is a rationalizable choice correspondence on $\mathcal{A}$. Given the interpretation of $\gtrsim$ as a dominance relation, it is natural to require that "rationalizability" takes place by means of preference relations that are consistent with $\gtrsim$. A choice correspondence c on $\mathcal{A}$ is said to be **monotonically rationalizable** if there is a complete preorder $\succsim$ on X such that (5) holds and $\succsim$ extends $\gtrsim$.[8] In turn, the choice data $\mathbb{C}$ is **monotonically rationalizable** if there is $c \in \mathbb{C}$ that is monotonically rationalizable. Obviously, the notions of rationalizability and monotonic rationalizability coincide if $\gtrsim$ equals $\triangle_X$, as in Richter's model (Example 1).

It is plain that we need to strengthen Richter's congruence axiom to deliver a characterization of monotonic rationalizability. A choice correspondence c on a choice environment $((X, \gtrsim), \mathcal{A})$ is said to satisfy the **monotone congruence axiom** if

$$x \text{ tran}(R(c) \cup \gtrsim) \ y \quad \text{and} \quad y \in c(A) \quad \text{imply} \quad x \in c(A) \tag{6}$$

for every $A \in \mathcal{A}$ that contains x, and

$$x \text{ tran}(R(c) \cup \gtrsim) \ y \quad \text{implies} \quad \text{not } y > x. \tag{7}$$

More generally, in an RP framework $((X, \gtrsim), \mathcal{A}, \mathbb{C})$, the choice data $\mathbb{C}$ is said to satisfy the **monotone congruence axiom** if there is a $c \in \mathbb{C}$ that satisfies the axiom. Given Richter's theorem, it is natural to conjecture that a choice correspondence c is monotonically rationalizable iff it satisfies the monotone congruence axiom. We will later prove this conjecture as an immediate consequence of our main rationalizability theorem.

4.3 Generalized cyclical consistency. Consider Afriat's model as outlined in Example 4. We are given a nonempty finite subset P of $\mathbb{R}^n_{++}$, and a demand correspondence d on P such that $d(\mathbf{p})$ is finite for each $\mathbf{p} \in P$. The choice data is $c^{\uparrow}$, where c maps each

[8] *Reminder.* By definition, $\succsim$ extends $\gtrsim$ if $\gtrsim \subseteq \succsim$ and $> \subseteq \succ$.

$B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$ to $d(\mathbf{p})$. Afriat (1967) shows that $c^{\uparrow}$ is monotonically rationalizable iff d obeys the *cyclical consistency axiom*, which may be stated as follows: For every positive integer k, $\mathbf{p}^1, ..., \mathbf{p}^k \in P$ and $\mathbf{x}^1 \in d(\mathbf{p}^1), ..., \mathbf{x}^k \in d(\mathbf{p}^k)$,

$$\mathbf{p}^2\mathbf{x}^1 \leq \mathbf{p}^2\mathbf{x}^2, ..., \mathbf{p}^k\mathbf{x}^{k-1} \leq \mathbf{p}^k\mathbf{x}^k \text{ and } \mathbf{p}^1\mathbf{x}^k \leq \mathbf{p}^1\mathbf{x}^1$$

imply that each of these inequalities hold as equalities. This axiom is also commonly known in its equivalent formulation, due to Varian (1982), as the *generalized axiom of revealed preference* or GARP, for short.

The generalization of Afriat's theorem that we are after requires a generalization of the notion of cyclical consistency. Let $((X, \gtrsim), \mathcal{A})$ be a choice environment. We say that a choice correspondence c on this environment satisfies **generalized cyclical consistency** if $c(A) \subseteq \text{MAX}(A, \gtrsim)$ for each $A \in \mathcal{A}$, and for every $k \in \mathbb{N}$, $A_1, ..., A_k \in \mathcal{A}$, and $x_1 \in c(A_1), ..., x_k \in c(A_k)$,

$$x_1 \in A_2^{\downarrow}, ..., x_{k-1} \in A_k^{\downarrow} \text{ and } x_k \in A_1^{\downarrow} \tag{8}$$

imply

$$x_1 \in \text{MAX}(A_2^{\downarrow}, \gtrsim), ..., x_{k-1} \in \text{MAX}(A_k^{\downarrow}, \gtrsim) \text{ and } x_k \in \text{MAX}(A_1^{\downarrow}, \gtrsim).^9 \tag{9}$$

Note that if $\gtrsim$ is a partial order here (that is, it is antisymmetric), then $\text{MAX}(A_i^{\downarrow}, \gtrsim)$ is the same as $\text{MAX}(A_i, \gtrsim)$ for all $i \in [k]$, so (9) can be simplified accordingly.

Generalized cyclical consistency posits two requirements. The first of these is that, for every feasible set A, any one choice of the agent is not dominated (in terms of $\gtrsim$) within A. Clearly, this requirement is implicit in Afriat's modeling where $\gtrsim$ is the coordinatewise ordering on $\mathbb{R}^n_+$, and every choice problem is of the form $B(\mathbf{p}, I)$ where $\mathbf{p}$ is a price n-vector and $I = \mathbf{py}$ with $\mathbf{y}$ being the consumption bundle that corresponds to the choice of the agent at prices $\mathbf{p}$. On the other hand, the second requirement imposed by generalized cyclical consistency axiom is a straightforward reflection of Afriat's original cyclical consistency axiom.

We show below that the generalized cyclical consistency axiom is precisely what is needed for a natural generalization of Afriat's theorem. This may not be surprising, at least in the sense that, given Afriat's theorem, one may expect this sort of result by analogy. However, Afriat's proof of his theorem (and its extensions by other authors) relies very much on the linear structure of $\mathbb{R}^n$, which makes it inapplicable in our general context. Our proof instead uses an order-theoretic technique which is not common in Afriat-type approaches to revealed preference theory. Besides, the present approach has

[9] The decreasing closures of each A_i are taken here with respect to the preorder $\gtrsim$.

the additional advantage of highlighting the connection between the results of Afriat and Richter and, consequently, the connection between the monotone congruence and generalized cyclical consistency properties. This will become transparent in the next section.

4.4 Characterizations of rationalizability. Generalized cyclical consistency property and the monotone congruence axiom are distinct postulates, even when applied to the same choice correspondence of a choice environment. However, it turns out that there is a close connection between them: In any choice environment, a choice correspondence satisfies generalized cyclical consistency iff the increasing closure of c with respect to $\sqsupseteq$, that is, $c^{\uparrow}$, satisfies the monotone congruence axiom. As the former property yields a rational representation in the sense of Afriat and the latter in the sense of Richter, this fact yields, in turn, a connection that ties these two notions of rationalizability together. This is the content of the first main result of the present paper:

The Rationalizability Theorem I. *Let $((X, \succeq), \mathcal{A})$ be a choice environment and c a choice correspondence on $\mathcal{A}$. Then, the following are equivalent:*

a. *$c^{\uparrow}$ satisfies the monotone congruence axiom*;

b. *$c^{\uparrow}$ is monotonically rationalizable*;

c. *c satisfies generalized cyclical consistency*;

d. *There is a complete preorder $\succsim$ on X that extends both* $\operatorname{tran}(R(c) \cup \succeq)$ *and $\succeq$, and that satisfies*

$$c(A) \subseteq \max(A, \succsim) \quad \text{for each } A \in \mathcal{A}; \tag{10}$$

e. *There is a complete preorder $\succsim$ on X that extends $\succeq$ and that satisfies* (10).

As we will make it precise in the following two sections, this theorem generalizes the Richter- and Afriat-type approaches to revealed preference theory simultaneously. As such, it unifies these two approaches, and demonstrates that, unlike their initial appearance, and how they are treated in the literature, each of these approaches are in fact special cases of a more general viewpoint.

4.5 The monotone version of Richter's theorem. As a corollary of the Rationalizability Theorem I, we obtain a fairly substantial generalization of Richter's Theorem.

Proposition 1. *A choice correspondence c on a choice environment $((X, \succsim), \mathcal{A})$ is monotonically rationalizable if, and only if, it satisfies the monotone congruence axiom.*

Proof. We omit the straightforward proof of the "only if" part of this assertion. Since c obeys the monotone congruence axiom, so does $c^{\uparrow}$. By the Rationalizability Theorem I, there is a complete preorder $\succsim$ on X that extends both $\succsim' := \text{tran}(R(c) \cup \gtrsim)$ and $\gtrsim$, and that satisfies (10). Fix an arbitrary $A \in \mathcal{A}$, and take any y in $\max(A, \succsim)$. Then, for an arbitrarily picked $x \in c(A)$, we have $x \succsim' y$. As $y \succsim x$ and $\succsim$ extends $\succsim'$, however, we cannot have $x \succ' y$. It follows that we also have $y \succsim' x$, and using (6) yields $y \in c(A)$. Conclusion: $c(A) = \max(A, \succsim)$. ■

Note that we recover Richter's theorem as a special case of Proposition 1 by setting $\gtrsim = \triangle_X$. Note also that if c satisfies the monotone congruence axiom then so does $c^{\uparrow}$, and hence by the Rationalizability Theorem I, c obeys generalized cyclical consistency. However, if c satisfies generalized cyclical consistency, then c need not satisfy the monotone congruence axiom.[10]

Example 2. **[Continued]** Consider the classical consumption choice model which corresponds to the RP-framework $((\mathbb{R}^n_+, \geq), \mathcal{A}, \{c\})$, where $\mathcal{A} \subseteq \{B(\mathbf{p}, I) : (\mathbf{p}, I) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}\}$. In this framework, c is said to satisfy the **budget identity** if $\mathbf{x} \in c(B(\mathbf{p}, I))$ implies $\mathbf{px} = I$ for every $B(\mathbf{p}, I) \in \mathcal{A}$. Now, note that $\mathbf{x}\, R(c)\, \mathbf{y}$ means here that there is a budget set $B(\mathbf{p}, I)$ in $\mathcal{A}$ such that $\mathbf{x} \in c(B(\mathbf{p}, I))$ and $\mathbf{py} \leq I$. Consequently, if $\mathbf{x} \in c(B(\mathbf{p}, I))$ for some $B(\mathbf{p}, I) \in \mathcal{A}$, then $\mathbf{x}\, R(c)\, \mathbf{y}$ for all $\mathbf{y} \in \mathbb{R}^n_+$ such that $\mathbf{x} \geq \mathbf{y}$. Given this observation, one can easily check that a choice correspondence c obeys the monotone congruence axiom iff it obeys the congruence axiom and the budget identity. In view of Proposition 1, therefore, we reach the following conclusion in the context of the classical consumption model: *A choice correspondence c is monotonically rationalizable iff it satisfies the congruence axiom and the budget identity.* By contrast, Richter's theorem says that c is rationalizable iff it satisfies the congruence axiom.

4.6 Relation to Afriat's theorem and its variants. The equivalence of statements (c) and (e) in the Rationalizability Theorem I shows that *a choice correspondence c on any choice environment $((X, \gtrsim), \mathcal{A})$ satisfies generalized cyclical consistency iff $c^{\uparrow}$ is monotonically rationalizable.* As we discuss in some detail below, this very much captures the choice-theoretic gist of Afriat's classical theorem. Unlike that theorem, however, this result allows for arbitrary choice environments, and hence it requires neither that $\mathcal{A}$ be finite nor c be finite-valued. On the other hand, just like in Afriat's theorem, monotonicity property is what makes this result non-trivial. Indeed, in (e), we find a

[10] For instance, consider any RP framework $((X, \triangle_X), \mathcal{A}, \{c\})$, where c is not rationalizable. Then, c fails the monotone congruence axiom, but c trivially satisfies generalized cyclical consistency.

complete preference relation $\succsim$ that extends the dominance preorder $\gtrsim$ (which implies that if an alternative x *strictly* dominates another alternative y, then the agent *strictly* prefers x to y), thereby excluding the preference relation where the decision maker is indifferent across all elements. Similarly, and unlike the equivalence of the statements (a) and (b), the equivalence of (c) and (e) is a triviality when $\gtrsim = \triangle_X$. (For, *any* choice correspondence on any choice enviroment of the form $((X, \triangle_X), \mathcal{A})$ obeys generalized cyclical consistency and is rationalizable in the sense of (10) by the preference relation that deems the decision maker indifferent across all elements of X.)

Let us now compare Afriat's original theorem, and some of its more recent variants, with what we found in Rationalizability Theorem I more concretely.

Example 4. **[Continued]** Let P and d be exactly as in Example 4, and recall that Afriat's classical consumption model is captured by the RP framework $((\mathbb{R}^n_+, \geq), \mathcal{A}, c^{\uparrow})$, where $\mathcal{A} = \{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$ and c is the choice correspondence on $\mathcal{A}$ that maps $B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$ to $d(\mathbf{p})$. Given that P is finite and d is finite-valued, Afriat's theorem says that d is cyclically consistent iff $c^{\uparrow}$ is monotonically rationalizable; the latter means that there is a complete preorder $\succsim$ which extends $\geq$ such that

$$d(\mathbf{p}) \subseteq \max(B(\mathbf{p}, \mathbf{p}d(\mathbf{p}, \succsim))) \quad \text{for each } \mathbf{p} \in P. \tag{11}$$

Afriat also shows that $\succsim$ is representable by a strictly increasing, continuous, and concave utility function.

The equivalence of statements (c) and (e) in Rationalizability Theorem I gives us an infinite version of Afriat's theorem. Applying that fact, and dispensing with Afriat's finiteness assumptions on P and $d(\mathbf{p})$, we again obtain that the cyclical consistency of d is necessary and sufficient for the existence of a complete preorder that extends $\geq$ and satisfies (11). Yet, unlike Afriat's theorem, we cannot conclude here that $\succsim$ can be represented by a utility function with those nice properties. In fact, Reny (2015) has shown recently that it is *not* possible to obtain a continuous preference relation in the setting of Afriat's theorem without the finiteness of P and finite-valuedness of $d(\mathbf{p})$. Having said that, we should note that Reny (2015) proves that one can choose $\succsim$ to be convex even without these assumptions. For reasons which we shall discuss in the next section, our rationalization will, in general, preclude such a property. □

Example 5. **[Continued]** Consider an RP framework of the form $((\mathbb{R}^n_+, \geq), \mathcal{A}, c^{\uparrow})$, where each set in $\mathcal{A}$ is $\geq$-decreasing. Then, by the Rationalizability Theorem I, c satisfies

(e) iff $c(A) \subseteq \text{MAX}(A, \geq)$ for any $A \in \mathcal{A}$, and

$$x_1 \in \text{MAX}(A_2, \geq), ..., x_{k-1} \in \text{MAX}(A_k, \geq) \text{ and } x_k \in \text{MAX}(A_1, \geq)$$

for every $k \in \mathbb{N}$, $A_1, ..., A_k \in \mathcal{A}$, and $x_1 \in c(A_1), ..., x_k \in c(A_k)$ such that $x_1 \in \text{MAX}(A_2, \geq), ..., x_{k-1} \in \text{MAX}(A_k, \geq)$ and $x_k \in \text{MAX}(A_1, \geq)$. This fact captures the choice-theoretic gist of Forges and Minelli's (2009) extension of Afriat's theorem. Once again, we dispense with the finiteness assumptions made in that theorem, at the cost of not getting a continuous utility representation for the rationalizing preference relation.[11] □

4.7 On the structure of rationalizability. The Rationalizability Theorem I provides a necessary and sufficient condition under which the choice data $c^{\uparrow}$ is monotonically rationalizable but, with the exception of some trivial situations, there are a multitude of complete preorders that can rationalize such choice data. Our next result, whose proof is relegated to the Appendix, shows that there is a complete preorder which is, in some precise sense, "minimal" among all possible rationalizations of $c^{\uparrow}$.

Proposition 2. *Let $((X, \gtrsim), \mathcal{A})$ be a choice environment and c a choice correspondence on $\mathcal{A}$. Let $\succsim$ be a complete preorder satisfying the properties in part* (d) *of the Rationalizability Theorem I. Then,*

$$\max(A, \succsim) = A \cap c(A)^{\uparrow, \text{tran}(R(c) \cup \gtrsim)} \subseteq \max(A, \trianglerighteq) \tag{12}$$

for every $A \in \mathcal{A}$ and every $\gtrsim$-monotonic rationalization $\trianglerighteq$ of $c^{\uparrow}$.

Given any feasible set A in the choice environment, the second part of (12) says that any element in $c(A)$, or any element in A that is revealed preferred to at least one chosen alternative in A, has to be declared optimal with respect to *every* rationalization of $c^{\uparrow}$ that is $\gtrsim$-monotonic. Furthermore, the first part of (12) says that *exactly* these elements are declared optimal by the rationalization identified in part (d) of the Rationalizability Theorem I. (In particular, an optimal point with respect to this rationalization must be optimal for *any* $\gtrsim$-monotonic rationalization of c.) It is in this sense that we have found a rationalization that is "tight" among all possible rationalizations for c. The elements that are declared optimal by this preference are the only ones that an observer can robustly conclude to be optimal by the "true" preference relation of the decision maker (which can, in general, only be partially identified).

[11] The Forges-Minelli theorem does not guarantee the convexity of the rationalizing preference, and indeed, it is not possible to guarantee this in general. The presence of this property in Afriat's theorem arises from the linearity of the budget sets whereas the constraint sets in the Forges-Minelli's more general framework need not be linear.

At this point it is worth returning again to the Afriat's consumption choice model in order to highlight the distinction between the rationalization obtained here and those obtained in Afriat's analysis and Reny's (2015) extension of that analysis to the case of infinite choice data. Both of those analyses yield a rationalizing preference relation that can be represented by a quasiconcave utility function on $\mathbb{R}^n_+$. This implies that the set of optimal bundles in a given budget set can be strictly larger than that according to the rationalization found in the Rationalizability Theorem I. (This is the content of (12).) To wit, suppose we have choice data about an individual at the same prices $\mathbf{p}$ at two different times, say, $\mathbf{x}^1$ and $\mathbf{x}^2$, with $\mathbf{p}\mathbf{x}^1 = \mathbf{p}\mathbf{x}^2$. Suppose also that $\mathbf{x}^1 \neq \mathbf{x}^2$, so, what we observe is precisely two distinct elements in $d(\mathbf{p})$. Rationalizing such demands by a convex preference would then entail that every bundle on the line segment between $\mathbf{x}^1$ and $\mathbf{x}^2$ is also optimal for the individual at prices $\mathbf{p}$. Such a rationalization may seem unduly coarse, especially in situations where there is no a priori reason to suppose that convexity holds. By contrast, Proposition 2 says that there is a preference that declares *only* $\mathbf{x}^1$ and $\mathbf{x}^2$ as optimal at prices $\mathbf{p}$. (Clearly, such a preference *cannot* be convex.)

5 RATIONALIZABILITY OF CHOICE DATA (with continuous utility)

As we have already mentioned, the preference relation that rationalizes the choice data can be chosen to have a continuous utility representation in Afriat's model. Obviously, this fact owes to the particular choice domain of the classical consumption model, which possesses a well-behaved topological structure. It also depends crucially on Afriat's assumption that one faces *finitely* many choice problems. (Without this assumption, Reny (2015) has demonstrated that continuity cannot be guaranteed.) The situation is similar in our more general setting. That is, it is possible to obtain a rationalization admitting a continuous utility representation if we endow the alternative space X with a well-behaved topological structure and restrict our attention to a choice environment with finitely many choice problems. This is actually quite pleasant because the "finiteness" assumption is unexceptionable from an empirical point of view. This section and the next is devoted to this key result and its many applications.

5.1 Rationalizability by a continuous utility function. Where $((X, \gtrsim), \mathcal{A}, \mathbb{C})$ is an RP framework, we say that the choice data $\mathbb{C}$ is **rationalizable by a utility function** if there is at least one c in $\mathbb{C}$ and a (utility) function $u : X \to \mathbb{R}$ such that

$$c(A) = \arg\max_{x \in A} u(x) \quad \text{for each } A \in \mathcal{A}.$$

It is natural to ask for u to be strictly increasing with respect to $\gtrsim$ and, when X is a topological space, we would also like u to be continuous.

5.2 Characterizations of continuous rationalizability. We now show that in finite environments, that is, when the set $\mathcal{A}$ of all choice problems is finite, mild topological conditions allow restating the Rationalizability Theorem I in terms of continuous utility functions.

The Rationalizability Theorem II. *Let $((X, \gtrsim), \mathcal{A})$ be a choice environment such that X is a locally compact and separable metric space, $\gtrsim$ a continuous preorder on X, and $\mathcal{A}$ a nonempty finite collection of nonempty compact subsets of X. Let c be a closed-valued choice correspondence on this environment. Then, the following are equivalent:*

a. *$c^{\uparrow}$ satisfies the monotone congruence axiom*;

b. *$c^{\uparrow}$ is rationalizable by a continuous and strictly $\gtrsim$-increasing function $u : X \to \mathbb{R}$*;

c. *c is satisfies generalized cyclical consistency*;

d. *There is a continuous function $u : X \to \mathbb{R}$ which is strictly increasing with respect to both $\operatorname{tran}(R(c) \cup \gtrsim)$ and $\gtrsim$, and which satisfies*

$$c(A) \subseteq \arg\max_{x \in A} u(x) \quad \text{for each } A \in \mathcal{A}; \tag{13}$$

e. *There is a continuous and strictly $\gtrsim$-increasing function $u : X \to \mathbb{R}$ that satisfies* (13).

An examination of the proof of Rationalizability Theorem II in the Appendix shows that it consists of two parts. The first part shows that, under the assumptions of the theorem and, in particular, the finiteness of $\mathcal{A}$, the relation $\operatorname{tran}(R(c) \cup \geq)$ is a continuous preorder on X. The second part shows that the continuity of $\operatorname{tran}(R(c) \cup \geq)$ is sufficient to guarantee the equivalence of the statements (a) to (e). This is worth bearing in mind because there are interesting cases where $\operatorname{tran}(R(c) \cup \geq)$ is continuous, and thus the conclusion of this theorem holds, even when $\mathcal{A}$ is not finite. We will investigate one such case in Section 6.3.

5.3 Continuous versions of Richter's theorem and Afriat's theorems. It is not a priori obvious how one may obtain a utility representation in the context of Richter's theorem, for the arbitrariness of $\mathcal{A}$ makes it difficult to ensure the continuity of the rationalizing preference relations. However, at least when $\mathcal{A}$ is finite, this sort of difficulty does not arise. Indeed, the argument we used to deduce Proposition 1 from the

Rationalizability Theorem I can also be used to derive the following characterization from the Rationalizability Theorem II.

Proposition 3. *Let $((X, \gtrsim), \mathcal{A})$ and c be as in the Rationalizability Theorem II. Then, c satisfies the (monotone) congruence axiom if, and only if, it is rationalizable by a continuous (and strictly $\gtrsim$-increasing) utility function on X.*

This proposition consists of two parts. Its first part characterizes rationalizability with a continuous utility function. It makes no reference to $\gtrsim$, and as such it is a continuous version of Richter's classical theorem. The second part of Proposition 3 characterizes rationalizability with a continuous *and* strictly $\gtrsim$-increasing utility function.

There is, of course, an analog to the second part of Proposition 3 in an Afriat-type framework. Indeed, this is provided by the equivalence of (c) and (e) in the Rationalizability Theorem II. This equivalence extends the results of both Afriat (1967) and Forges and Minelli (2009) to a general context, in which the set of alternatives need not be the positive orthant and the dominance relation is arbitrary.

5.4 On empirical testing of generalized cyclical consistency. To clarify the scope of the Rationalizability Theorem II for empirical applications, we propose to segment the procedure for checking whether or not a choice correspondence c on $\mathcal{A}$ satisfies generalized cyclical consistency into four steps.

(**Step I**) Check that $c(A) \subseteq \text{MAX}(A, \gtrsim)$ for each $A \in \mathcal{A}$. This involves finitely many checks if $\mathcal{A}$ is finite and c is finite-valued.

(**Step II**) Let $c(\mathcal{A})$ stand for the set of all choices across all choice problems, that is, $c(\mathcal{A}) := \bigcup\{c(A) : A \in \mathcal{A}\}$. Construct the binary relation $\mathcal{S}$ on $c(\mathcal{A})$ in the following manner: $x \,\mathcal{S}\, y$ if there is $A \in \mathcal{A}$ such that $x \in c(A)$ and $y \in A^{\downarrow}$.

(If $c(\mathcal{A})$ has m elements, then there are at most $m(m-1)$ nontrivial checks that one has to perform in the construction of $\mathcal{S}$. Note that x_i for all $i \in [k]$ obey (8) iff they form a cycle in $\mathcal{S}$ in the sense that $x_1 \,\mathcal{S}\, x_k$, $x_k \,\mathcal{S}\, x_{k-1}$,...,$x_2 \,\mathcal{S}\, x_1$.)

(**Step III**) Identify all the cycles in $\mathcal{S}$. This is computationally straightforward when $c(\mathcal{A})$ is a finite set; for example, one could use Warshall's algorithm (see Varian, 1982).

(**Step IV**) Check whether (9) holds for every cycle found in step (III).

This description sets out a procedure for checking for generalized cyclical consistency that is analogous to the one for checking cyclical consistency. It is well known that

verifying cyclical consistency is computationally undemanding, so long as $c(\mathcal{A})$ is a finite set. Whether this is also true of generalized cyclical property will depend on the complexity of $\gtrsim$ and the structure of the sets in $\mathcal{A}$, but it seems likely that it will also be relatively straightforward in many applications.

5.5 On empirical testing of monotone congruence. It is not immediately obvious how one could test the validity of this axiom. The following proposition reformulates the axiom in a way that highlights its similarity with generalized cyclical consistency. It has the advantage of reducing the problem of checking for monotone congruence to one of checking whether the binary relation $\mathcal{S}$ defined (as above) on the domain $c(\mathcal{A})$ (which is finite if $\mathcal{A}$ is finite and c is finite-valued) has certain properties.

Proposition 4. *Let c be a choice correspondence on a choice environment $((X, \gtrsim), \mathcal{A})$. Then, c satisfies the monotone congruence axiom if, and only if, $c(A) \subseteq \text{MAX}(A, \gtrsim)$ for each $A \in \mathcal{A}$, and for every positive integer k, $A_1, ..., A_k \in \mathcal{A}$, and $x_1 \in c(A_1), ..., x_k \in c(A_k)$,*

$$x_1 \in A_2^{\downarrow}, ..., x_{k-1} \in A_k^{\downarrow} \text{ and } x_k \in A_1^{\downarrow} \tag{14}$$

imply (9) *as well as*

$$x_1^{\uparrow} \cap A_2 \subseteq c(A_2), ..., x_{k-1}^{\uparrow} \cap A_k \subseteq c(A_k) \text{ and } x_k^{\uparrow} \cap A_1 \subseteq c(A_1). \tag{15}$$

This proposition formulates the monotone congruence axiom in a way that is analogous to generalized cyclical consistency except that now (15) is added to (9). Given this, we can simply use the above four-step procedure set out for checking generalized cyclical consistency to check for monotone congruence, provided we modify step (IV) to check that condition (9) *and* (15) holds on every cycle in $\mathcal{S}$.[12]

5.6 Extrapolation to choice behavior outside $\mathcal{A}$. The key feature of Rationalizability Theorem II is that it delivers a *continuous* utility function. This property is crucial when we are interested in understanding the agent's behavior, not in $\mathcal{A}$ as such, but in a larger (and possibly infinite) collection $\mathcal{B}$ of nonempty compact subsets of X. In that case, $\mathcal{A}$ should be interpreted as a random sample of feasible sets drawn from $\mathcal{B}$ and the observer is interested in testing the hypothesis that the agent chooses by maximizing a utility function for every set in $\mathcal{B}$. The existence of a continuous utility function that rationalizes the choice data in $\mathcal{A}$ then furnishes us with a utility function that could

[12] This proposition also confirms directly our earlier observation that the monotone congruence of a choice correspondence implies its generalized cyclical consistency.

potentially be the one used by the agent when making her choices more generally; if the rationalizing utility function is not continuous, it would be difficult to view that function as the agent's "true" utility function, since there is then no guarantee that an optimum exists for the compact feasible sets in $\mathcal{B}$.[13]

More formally, let $((X, \gtrsim), \mathcal{A})$ be a choice environment, $\mathcal{B}$ a collection of feasible subsets containing $\mathcal{A}$, and $C : \mathcal{B} \to X$ a choice correspondence. We say that C is a **rational prediction on $\mathcal{B}$ consistent with choice data** $\mathbb{C}$ if there is $c \in \mathbb{C}$ and a continuous and strictly $\gtrsim$-increasing utility function $u : X \to \mathbb{R}$ such that $C|_{\mathcal{A}} = c$ and

$$C(B) = \arg\max_{x \in B} u(x) \quad \text{for each } B \in \mathcal{B}.$$

The following result guarantees the existence of rational predictions, under both the Richter and Ariat-type frameworks. It is an immediate consequence of Propositions 3, the equivalence of (c) and (e) in the Rationalizability Theorem II, and the fact that a continuous function real map on a compact set achieves its maximum.

Proposition 5. (Existence of Rational Predictions) *Let $((X, \gtrsim), \mathcal{A})$ and c be as in the Rationalizability Theorem II. Let $\mathcal{B}$ be any collection of nonempty compact subsets of X with $\mathcal{A} \subseteq \mathcal{B}$. Then, there exists a rational prediction on $\mathcal{B}$ consistent with choice data $\{c\}$ (resp., $c^{\uparrow}$) if, and only if, c satisfies the monotone congruence axiom (resp., generalized cyclical consistency).*

The next result uses our continuous and monotone version of Richter's Theorem to show how one may determine whether a choice correspondence on $\mathcal{B}$ is a rational prediction induced by a given choice correspondence on $\mathcal{A}$.

Proposition 6. (Characterization of Rational Predictions) *Let $((X, \gtrsim), \mathcal{A})$ and c be as in Proposition 3, and let $\mathcal{B}$ be a finite collection of nonempty compact subsets of X with $\mathcal{A} \subseteq \mathcal{B}$. Then, a closed-valued choice correspondence C on $\mathcal{B}$ is a rational prediction on $\mathcal{B}$ induced by c if, and only if, $c = C|_{\mathcal{A}}$ and C satisfies the monotone congruence axiom.*

Proof. The "only if" part of the assertion is again straightforward. On the other hand, its "if" part follows by applying Proposition 3 to C. ■

6 APPLICATIONS TO THE THEORY OF CONSUMER DEMAND

[13] As a matter of fact, if the agent's utility function is not continuous, then it is not even clear why, in the first place, we should hypothesize that the constraint sets in the random sample $\mathcal{A}$ should each contain an optimal choice.

The rest of this paper focuses on applications of Rationalizability Theorem II. In the present section we show how it can be used to obtain new and interesting results, even in the classical setting of consumer demand.

6.1 Rationalizability. Let us call an ordered pair (P, d) an **Afriat choice model** if P is a nonempty finite set of prices in $\mathbb{R}^n_{++}$ and $d : P \rightrightarrows \mathbb{R}^n_+$ is the (observed) demand correspondence of a consumer such that for any $\mathbf{p} \in P$, the set $d(\mathbf{p})$ is nonempty and compact, and $\mathbf{pz} = \mathbf{py}$ for every $\mathbf{z}$ and $\mathbf{y}$ in $d(\mathbf{p})$. As we have discussed in Example 4, an Afriat choice model induces the RP framework $((\mathbb{R}^n_+, \geq), \mathcal{A}, c^{\uparrow})$, where $\mathcal{A} := \{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$, and the (observed) choice correspondence c on $\mathcal{A}$ is defined by $c(B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))) := d(\mathbf{p})$. With a minor abuse of terminology, let us agree to say that d satisfies generalized cyclical consistency, congruence or monotone congruence whenever c satisfies the corresponding property. Clearly, in this context, d satisfies generalized cyclical consistency iff it satisfies cyclical consistency (the latter in the original sense defined by Afriat). Using Proposition 4, it is also straightforward to check that d satisfies the congruence axiom iff it has the following property: For every positive integer k, $\mathbf{p}^1, ..., \mathbf{p}^k \in P$ and $\mathbf{x}^1 \in d(\mathbf{p}^1), ..., \mathbf{x}^k \in d(\mathbf{p}^k)$,

$$\mathbf{p}^2\mathbf{x}^1 \leq \mathbf{p}^2\mathbf{x}^2, ..., \mathbf{p}^k\mathbf{x}^{k-1} \leq \mathbf{p}^k\mathbf{x}^k \text{ and } \mathbf{p}^1\mathbf{x}^k \leq \mathbf{p}^1\mathbf{x}^1 \tag{16}$$

imply

$$\mathbf{x}^1 \in d(\mathbf{p}^2), ..., \mathbf{x}^{k-1} \in d(\mathbf{p}^k) \text{ and } \mathbf{x}^k \in d(\mathbf{p}^1).$$

Comparing this with the definition of cyclical consistency given in Section 4.2 shows that the congruence axiom is a stronger property.

Proposition 7. *Let (P, d) be an Afriat choice model. Then,*

(i) *d satisfies the congruence axiom if, and only if, there is a continuous and strictly $\geq$-increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that*

$$d(\mathbf{p}) = \underset{\mathbf{x} \in B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))}{\arg\max}\ u(\mathbf{x}) \quad \textit{for every } \mathbf{p} \in P; \tag{17}$$

(ii) *d satisfies cyclical consistency if, and only if, there is a continuous and strictly $\geq$-increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that*

$$d(\mathbf{p}) \subseteq \underset{\mathbf{x} \in B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))}{\arg\max}\ u(\mathbf{x}) \quad \textit{for every } \mathbf{p} \in P. \tag{18}$$

Proof. Let $((\mathbb{R}^n_+, \geq), \mathcal{A}, c^{\uparrow})$ be the RP framework induced by (P, d). We have seen in Section 4.6 that c satisfies the monotone congruence axiom iff it obeys the budget

identity and the congruence axiom. Here c satisfies the budget identity by construction, while it obeys the congruence axiom iff d obeys the congruence axiom. Consequently, part (i) follows readily from Proposition 3. Part (ii) is a special case of the equivalence between statements (c) and (e) of the Rationalizability Theorem II. ■

We emphasize that part (ii) of Proposition 7 is none other than a version of Afriat's theorem, but it is stronger than the standard version of that result since d need not be finite-valued here. Moreover, for the reasons outlined in Section 4.7, the utility function we identify here is not the same as the concave utility function constructed from the classical Afriat inequalities.

Part (i) of this Proposition 7 is simply the continuous and monotone version of Richter's theorem specialized to the context of consumer demand. The literature on the rationalizability of consumer demand is quite large but, to the best of our knowledge, there is no result characterizing a representation of the form (17) with a *continuous* utility function.[14] This lacuna may be due to the reliance on convex analysis techniques, which typically lead to concave utility functions, whereas a representation of the form (17) will, for some data sets, *require* using a non-concave (but still continuous) utility function.

In passing, we note that the gap between the two conditions on the demand correspondence d in parts (i) and (ii) above is negligible from a practical point of view. While empirical work on revealed preference analysis has invariably focused on testing cyclical consistency, data sets that obey cyclical consistency will almost always do so because there are simply no observations obeying (16). In these cases, both cyclical consistency and the congruence axiom are satisfied, and therefore, one can invoke part (i) of Proposition 7 to obtain a utility function obeying (17) rather than just (18).

6.2 Recoverability. Let (P, d) be an Afriat choice model such that the demand correspondence d satisfies cyclical consistency. Let $\mathcal{U}(d)$ stand for the collection of all continuous and strictly $\geq$-increasing $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that (18) holds. By part (ii) of Proposition 7, $\mathcal{U}(d) \neq \emptyset$. However, different members of $\mathcal{U}(d)$ would entail different preference rankings (which are required to be in agreement only in the case of the observed choices of the agent). A natural question is to what extent we may identify the actual preference relation of the agent without subscribing to any one utility function

[14] Chiappori and Rochet (1987) characterize finite data sets that are rationalizable (as in (17)) by a strictly quasiconcave, strictly $\geq$-increasing, and smooth utility function. However, the choice data in their case excludes distinct bundles chosen at the same budget and the same bundle chosen at distinct budgets; neither phenonmenon is excluded by the congruence axiom.

that rationalizes d as in (18). Formally, we would like to characterize the subsets $S(d)$ and $S'(d)$ of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ such that

$$(\mathbf{x}, \mathbf{y}) \in S(d) \quad \text{iff} \quad u(\mathbf{x}) \geq u(\mathbf{y}) \text{ for each } u \in \mathcal{U}(d)$$

and

$$(\mathbf{x}, \mathbf{y}) \in S'(d) \quad \text{iff} \quad u(\mathbf{x}) > u(\mathbf{y}) \text{ for each } u \in \mathcal{U}(d).$$

This formulation of the problem is the same as what Varian (1982) calls the *Recoverability Problem*, except that Varian considers the case where $\mathcal{U}(d)$ consists of strictly $\geq$-increasing, continuous, *and concave* utility functions with (18). Given the weaker assumptions on the utility functions we impose here, the relations $S(d)$ and $S'(d)$ are bound to be smaller than those studied by Varian (1982). Moreover, in certain contexts, it is sensible not to impose the concavity, or even the quasiconcavity, requirement on the utility functions (even when it is possible to rationalize d by such a utility function). To wit, consider the case where the consumer chooses a contingent consumption subject to a linear budget set, where a bundle $\mathbf{x} \in \mathbb{R}^n_+$ specifies the levels of the representative good in different states of the world (of which there are n), and $\mathbf{p} \in \mathbb{R}^n_{++}$ are the state prices.[15] In such a context, we may well wish to draw inferences of the agent's preferences (based on her observed choices) without assuming that she has a quasiconcave utility function, since that assumption would exclude risk-seeking and/or elation-seeking preferences.[16,17]

For bundles $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$, we say that $\mathbf{x}$ **is revealed preferred to** $\mathbf{y}$ if $\mathbf{x}\, \text{tran}(R(c) \cup \geq)\, \mathbf{y}$, where, as usual, $c(B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))) = d(\mathbf{p})$ for each $\mathbf{p} \in P$. This is equivalent to saying that there are finitely many $\mathbf{p}^1, ..., \mathbf{p}^k \in P$ and $\mathbf{x}^1 \in d(\mathbf{p}^1), ..., \mathbf{x}^k \in d(\mathbf{p}^k)$ such that

$$\mathbf{x} \geq \mathbf{x}^1,\ \mathbf{p}^1\mathbf{x}^1 \geq \mathbf{p}^1\mathbf{x}^2,\ ..., \mathbf{p}^{k-1}\mathbf{x}^{k-1} \geq \mathbf{p}^{k-1}\mathbf{x}^k \text{ and } \mathbf{p}^k\mathbf{x}^k \geq \mathbf{p}^k\mathbf{y}. \tag{19}$$

We say $\mathbf{x}$ **is revealed strictly preferred to** $\mathbf{y}$ if any of the inequalities in (19) is strict. It is clear that if $\mathbf{x}$ is revealed preferred (revealed strictly preferred) to $\mathbf{y}$, then $(\mathbf{x}, \mathbf{y})$ belongs to $S(d)$ (respectively $S'(d)$). The next result characterizes $S(d)$ and $S'(d)$ by showing that the converse is also true. When P is finite and d is finite-valued, working out

[15] Data from laboratory experiments of this type have been collected and tested for cyclical consistency, for instance, by Choi et al. (2007), among others.

[16] Halevy et al. (2014) make a similar point.

[17] For example, suppose that $n = 2$ and the consumer's true utility function is $u(x_1, x_2) = \pi_1 v(x_1) + \pi_2 v(x_2)$, where $\pi_i > 0$ for $i = 1, 2$, and v is strictly increasing but not concave (so the agent is not everywhere risk averse). Then u will not be quasiconcave and predicting the consumer's preference from d while assuming quasiconcavity can lead to false conclusions. On the other hand, the predictions captured by $S(d)$ and $S'(d)$ will be correct (even though they will be coarser than what would have been if the observer assumed, correctly in this case, that u is additive across states).

the pairs of bundles that are related by revealed preference (or revealed strict preference) is computationally straightforward. It follows that it is easy to compute $S(d)$ and $S'(d)$ in practice.

Proposition 8. *Let (P, d) be an Afriat choice model such that d satisfies cyclical consistency. Then, for any $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n_+$,*

(i) *$(\mathbf{x}, \mathbf{y}) \in S(d)$ if, and only if, $\mathbf{x}$ is revealed preferred to $\mathbf{y}$; and*

(ii) *$(\mathbf{x}, \mathbf{y}) \in S'(d)$ if, and only if, $\mathbf{x}$ is revealed strictly preferred to $\mathbf{y}$.*

6.3 Rationalizability of Engel curves. We pointed out in Section 5.2 that continuous rationalizability relies on the continuity of $\text{tran}(R(c) \cup \geq)$ rather than the finiteness of the set of observations. This can be an important distinction, as we shall illustrate with the following application.

Consider a situation in which we happen to know the entirety of finitely many Engel curves of an individual. When is it the case that these curves correspond to those of a continuous and strictly increasing utility maximizing individual? This rationalizability problem cannot be attacked by Afriat's approach, for even a single Engel curve presumes uncountably many choice situations.[18] We can, however, provide an answer fairly easily by using the (proof of) Rationalizability Theorem II.

Put precisely, the choice environment we consider is $((\mathbb{R}^n_+, \geq), \mathcal{A})$, where

$$\mathcal{A} := \{B(\mathbf{p}, I) : \mathbf{p} \in P \text{ and } I > 0\},$$

with P being a nonempty finite set of prices. In this context, by a **demand correspondence on $\mathcal{A}$** we mean a correspondence $d : P \times \mathbb{R}_{++} \rightrightarrows \mathbb{R}^n_+$ such that $\emptyset \neq d(\mathbf{p}, I) \subseteq B(\mathbf{p}, I)$ for any $\mathbf{p} \in P$ and $I > 0$. (Therefore, the map $I \mapsto d(B(\mathbf{p}, I))$ is the **Engel curve** of the agent in consideration.) Abusing the terminology again, we say that d, or the Engel curve induced by d, satisfies monotone congruence (generalized cyclical consistency) if the choice correspondence $c : B(\mathbf{p}, I) \mapsto d(\mathbf{p}, I)$ on $\mathcal{A}$ satisfies the corresponding property.

This is essentially the setup of Blundell, et al. (2003), who estimated Engel curves of consumers by using a finite set of prices, and then tested if these curves satisfied cyclical consistency. Now, it is plain that the Engel curves of a utility maximizing consumer must obey cyclical consistency. But empirical welfare analysis based on data that passes

[18] While the rationalizability theorem of Reny (2015) applies to non-finite data sets that satisfy generalized cyclical consistency (as in Proposition 8.(ii)), the utility function it constructs is in general neither strictly $\geq$-increasing nor continuous.

cyclical consistency, such as the one carried out by Blundell, et al. (2003), presumes that cyclical consistency is also *sufficient* for rationalizability. This presumption is by no means self-evident, but the following result, which provides two characterizations of the rationalizability of d, proves that it is nevertheless correct.

Proposition 9. *Let $((\mathbb{R}^n_+, \geq), \mathcal{A})$ be the choice environment defined above and d an upper hemicontinuous demand correspondence on $\mathcal{A}$. Then,*

(i) *d obeys the monotone congruence axiom if, and only if, there is a continuous and strictly $\geq$-increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that*

$$d(\mathbf{p}, I) = \underset{\mathbf{x} \in B(\mathbf{p}, I)}{\arg\max}\, u(\mathbf{x}) \quad \text{for every } \mathbf{p} \in P \text{ and } I > 0;$$

(ii) *d obeys the generalized cyclical consistency if, and only if, there is a continuous and strictly $\geq$-increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that*

$$d(\mathbf{p}, I) \subseteq \underset{\mathbf{x} \in B(\mathbf{p}, I)}{\arg\max}\, u(\mathbf{x}) \quad \text{for every } \mathbf{p} \in P \text{ and } I > 0.$$

7 FURTHER APPLICATIONS

In all of the applications we have considered so far, the alternative space was a subset of the n-dimensional Euclidean space, and the domination relation was the usual (coordinatewise) ordering of n-vectors. In a variety of individual choice frameworks, however, other (more stringent) types of dominance relations are also of interest. As the generalized revealed preference theory developed in this paper is able to deal with such relations, it readily applies to such contexts, thereby yielding novel, and empirically implementable, tests of rationalizability. In this section, where n is an arbitrarily fixed integer, we consider four such applications.

7.1 Choice over lotteries. Let Δ^{n-1} denote the $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$, that is, $\Delta^{n-1} := \{\mathbf{p} \in \mathbb{R}^n_+ : p_1 + \cdots + p_n = 1\}$. As usual, we view Δ^{n-1} as the space of all lotteries whose supports are contained within a given set of n many (certain) prizes. (We regard Δ^{n-1} as a metric subspace of $\mathbb{R}^n_+$.) Let us consider an experiment in which subjects choose among lotteries from various (feasible) sets in Δ^{n-1}. Put more precisely, the set $\mathcal{A}$ of all feasible sets at hand is a nonempty finite collection of nonempty compact subsets of Δ^{n-1}, and we are interested in the choice correspondences of the subjects on $\mathcal{A}$. This is, of course, a common setting for a large number of experiments designed to check whether subjects are, say, expected utility maximizers. Famously, the Allais paradox is framed in such a setting.

There are revealed preference tests that characterize the structure of observations consistent with expected utility maximization in this setting (cf. Fishburn (1975)). However, we may wish to take a step back and begin with a more fundamental question: Does a subject make her choices on the basis of maximizing *some* continuous utility function on the set Δ^{n-1}? At this level of generality (where there are no monotonicity considerations with respect to a dominance relation), the appropriate (indeed, only) result that one could appeal to must be couched in the context of Richter's model where the alternative space is Δ^{n-1} and the set of all feasible sets is $\mathcal{A}$. So, we may use Proposition 3 to conclude that *a choice correspondence on $\mathcal{A}$ is rationalizable by a continuous utility function on Δ^{n-1} iff it satisfies the congruence axiom.*

If a subject passes this basic test, we may then ask if her behavior is consistent with more stringent rationality requirements. Suppose the n outcomes are ranked, from worst to best, totally and strictly. (This could naturally be done, for instance, if the outcomes were monetary.) This, in turn, induces a partial order on Δ^{n-1} which is based on first-order stochastic dominance; we denote this (continuous) partial order by $\geq_{\text{FSD}}$.[19] As usual, we say that a real map on Δ^{n-1} is **stochastically monotonic** if this map is strictly $\geq_{\text{FSD}}$-increasing. We are then led naturally to the following question: Does a subject make her choices on the basis of maximizing some continuous and stochastically monotonic utility function on the set Δ^{n-1}? The following result provides the tests that allow us to answer this question. Its proof obtains readily by applying Proposition 3 and the equivalence of the statements (c) and (d) in the Rationalizability Theorem II (where we set $\gtrsim = \geq_{\text{FSD}}$).

Proposition 10. *Let c be a closed-valued choice correspondence on the choice environment $((\Delta^{n-1}, \geq_{\text{FSD}}), \mathcal{A})$ where $\mathcal{A}$ is a nonempty finite collection of nonempty compact subsets of Δ^{n-1}. Then, c (resp., $c^{\uparrow}$) is rationalizable by a continuous and stochastically monotonic utility function on X if, and only if, c satisfies the monotone congruence axiom (resp., generalized cyclical consistency).*

The tests provided in Proposition 10 are empirically implementable since checking for the monotone congruence axiom and/or generalized cyclical consistency in this context (with the procedure spelled out in Section 5.4) is unlikely to pose any computational difficulties. Indeed, verifying these properties is completely straightforward if each set A in $\mathcal{A}$ has only finitely many elements, and this case is common in experiments. It is also

[19] Given that we order the prizes from worst to best (that is, prize 1 is the worst one, prize 2 is the second worst, and so on), we have $\mathbf{p} \geq_{\text{FSD}} \mathbf{q}$ iff $p_1 + \cdots + p_k \leq q_1 + \cdots + q_k$ for each $k \in [n]$.

worth emphasizing that there are other dominance relations that could be interesting to impose in the context of choice over lotteries. One obvious, and more stringent, alternative is second order stochastic dominance, which is also a continuous partial order on Δ^{n-1}. It is straightforward to formulate tests (similar to those in Proposition 10) for this property.

Proposition 10 provides also some guidance regarding the design of experiments. For instance, suppose the experiment is so conducted that any two lotteries that a subject faces (within and across choice problems) are not comparable with respect to $\geq_{\text{FSD}}$. Then, it is readily checked that *any* choice correspondence c on $\mathcal{A}$ satisfies the generalized cyclical consistency property automatically, so the experiment is ill-designed for the purposes of checking for rationalizability by stochastically monotone preferences. More generally, suppose any two distinct lotteries that are maximal (with respect to $\geq_{\text{FSD}}$) in the feasible sets that they belong to are incomparable with respect to $\geq_{\text{FSD}}$. Then Afriat-type rationalizability has no predictive power besides requiring that choices are $\geq_{\text{FSD}}$-maximal within each constraint set, while this requirement, together with the standard congruence axiom, fully captures Richter-type rationalizability. This is the content of the following result, which is a straightforward corollary of Proposition 10.

Corollary. *Let $((\Delta^{n-1}, \geq_{\text{FSD}}), \mathcal{A})$ be a choice environment as in Proposition 10, and c a closed-valued choice correspondence on $\mathcal{A}$ such that $c(A) \subseteq \text{MAX}(A, \geq_{\text{FSD}})$. Suppose that any lottery in $\text{MAX}(A, \geq_{\text{FSD}})$ is $\geq_{\text{FSD}}$-incomparable to any lottery in $\text{MAX}(B, \geq_{\text{FSD}})$ for any distinct feasible sets A and B in $\mathcal{A}$. Then, there is a continuous and stochastically monotonic utility function U on X such that*

$$c(A) \subseteq \arg\max_{\mathbf{p} \in A} U(\mathbf{p}) \quad \textit{for each } A \in \mathcal{A}.$$

Moreover, c satisfies the congruence axiom if, and only if, there is a continuous and stochastically monotonic utility function U on X such that

$$c(A) = \arg\max_{\mathbf{p} \in A} U(\mathbf{p}) \quad \textit{for each } A \in \mathcal{A}.$$

7.2 Contingent consumption. Suppose there are n states of the world, where the probability of each state i, which we denote by π_i, is known to both the consumer and the observer. A consumer chooses a bundle of contingent consumption subject to a linear budget set, where a bundle $\mathbf{x} \in \mathbb{R}^n_+$ specifies the level of consumption in each state of the world and $\mathbf{p} \in \mathbb{R}^n_{++}$ is the vector of state prices. In such a context, Varian (1983) and Green and Srivastava (1986), among others, have developed characterizations of data

sets consistent with the hypothesis that the consumer maximizes an expected utility function.

Our objective here is to develop a test for a more permissive model of rational behavior. Let us first ask if the agent's behavior is consistent with the maximization of a strictly $\geq$-increasing utility function. Then, depending on the type of rationalizability we require, either parts (i) or (ii) of Proposition 7 will provide us with such a test. If the consumer passes such a test, we could go further and ask if her utility is strictly increasing with respect to first order stochastic dominance. Formally, the choice environment is then $((\mathbb{R}^n_+, \gtrsim_{\text{FSD},\pi}), \mathcal{A})$, where $\mathcal{A} := \{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$ for some nonempty finite set P in $\mathbb{R}^n_{++}$, and $\gtrsim_{\text{FSD},\pi}$ is the first order stochastic dominance on $\mathbb{R}^n_+$ induced by the probability vector $\pi := (\pi_1, ..., \pi_n)$.[20] Our next result sets out necessary and sufficient conditions for rationalizability within this environment.[21] It follows readily from Proposition 5 and and the equivalence of (c) and (e) in the Rationalizability Theorem II, respectively.

Proposition 11. *Let $((\mathbb{R}^n_+, \gtrsim_{\text{FSD},\pi}), \mathcal{A})$ be the choice environment defined above and d a compact-valued demand correspondence on $\mathcal{A}$. Then,*

(i) *d obeys the monotone congruence axiom with respect to $\gtrsim_{\text{FSD},\pi}$ if, and only if, there is a continuous and strictly $\gtrsim_{\text{FSD},\pi}$-increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that* (17) *holds.*

(ii) *d obeys generalized cyclical consistency with respect to $\gtrsim_{\text{FSD},\pi}$ if, and only if, there is a continuous and strictly $\gtrsim_{\text{FSD},\pi}$-increasing utility function $u : \mathbb{R}^n_+ \to \mathbb{R}$ such that* (18) *holds.*

A few comments on the practical issues involved in implementing the test given by Proposition 11 are in order. As we explain in Section 5.4, the procedure for checking generalized cyclical consistency or the monotone congruence axiom can be divided into

[20] For any n-vector $\mathbf{z}$ and real number $a \geq 0$, put $I(\mathbf{z}, a) := \{i \in [n] : z_i \leq a\}$. Then, for any two nonnegative n-vectors $\mathbf{x}$ and $\mathbf{y}$, we have $\mathbf{x} \gtrsim_{\text{FSD},\pi} \mathbf{y}$ iff $\sum_{i \in I(\mathbf{x},a)} \pi_i \leq \sum_{I(\mathbf{y},a)} \pi_i$ for each $a \geq 0$. This is a continuous preorder on $\mathbb{R}^n_+$, but note that, unlike the first order dominance relation we worked with in the previous section, it is not a partial order unless $n = 1$. (For instance, $(1, 2)$ and $(2, 1)$ are equivalent with respect to $\gtrsim_{\text{FSD},(\frac{1}{2},\frac{1}{2})}$.)

[21] It is possible for a preference to be representable by utility function that is strictly $\gtrsim_{\text{FSD},\pi}$-increasing, without it being representable by an expected utility function. For example, it is easy to check that the utility function $U(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2$ is strictly $\gtrsim_{\text{FSD},(\frac{1}{2},\frac{1}{2})}$-increasing, but there is no strictly increasing transformation of U that leads to a function of the form $V(x_1, x_2) = 1/2[v(x_1) + v(x_2)]$. Any preference with such a representation will have the following ordinal property: the ratio of the marginal rate of substitution at (x_1^*, x_2) and at (x_1^{**}, x_2) is independent of x_2. A quick calculation shows that this property does not hold for U.

four stages. Carrying out such a check is straightforward when there are just two states, and experiments of this type are not uncommon (cf. Choi, et al. (2007)). However, when the number of states is larger, the computational demands need more careful examination.

The tricky part of the procedure is in determining whether some given bundle $\mathbf{x}$ is in the decreasing closure of $B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$ with respect to $\gtrsim_{\text{FSD},\pi}$, where $\mathbf{p} \in P$. This holds iff we can find some $\mathbf{y} \in B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$ such that $\mathbf{y} \gtrsim_{\text{FSD},\pi} \mathbf{x}$. Suppose we impose the additional condition that $y_i \leq y_{i+1}$ for each $i \in [n-1]$. Then, it is straightforward to write a *linear* program that checks whether there is a $\mathbf{y}$, with this additional condition, such that $\mathbf{y} \gtrsim_{\text{FSD},\pi} \mathbf{x}$ and $\mathbf{p}\mathbf{y} \leq \mathbf{p}d(\mathbf{p})$. Running such a program is also computationally undemanding. Of course, there is no particular reason why $y_i \leq y_{i+1}$ for each $i \in [n-1]$; this is just one of $n!$ possible orderings of the entries in $\mathbf{y}$, and, in the worst case, we would have to run the linear program $n!$ times. This gives us a rough-and-ready method for checking whether or not $\mathbf{x}$ belongs to the decreasing closure of $B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))$ with respect to $\gtrsim_{\text{FSD},\pi}$. There could be computationally superior methods, but even this method is implementable so long as n is a small number.

7.3. Intertemporal consumption. Let us consider a researcher who observes a consumer choosing consumption of a single good across n different future dates, subject to a given income and prices at different dates. A consumption stream in this framework is thus an element $\mathbf{x}$ of $\mathbb{R}^n_+$ and the price vector is an element $\mathbf{p}$ of $\mathbb{R}^n_{++}$. (For an experiment where subjects are observed to make choices in such a setting, see Andreoni and Sprenger (2012).) To avoid issues that may arise due to dynamic inconsistency, we assume in what follows the anlysist observes the commitment choices of the agent, that is, the consumer cannot deviate from the consumption stream chosen initially.

The canonical model of intertemporal consumption choice is the model of geometric discounted utility. In formal terms, this model is closely related to the expected utility model and there are revealed preference tests that could determine if a data set is consistent with such a model (cf. Echenique, et al. (2015)). We could, of course, also test if the observations are consistent with the maximization of a strictly $\geq$-increasing continuous utility function by testing for cyclical consistency. What we would like to consider here are utility functions with properties that lie between these two extremes.

In the study of intertemporal consumption, a question that has received quite a bit of attention is whether agents display *impatience* or *positive time preference*. This property is an obvious feature of geometric discounting, even though there is considerable evidence challenging its empirical prevalence. (See Frederick et al. (2002) for a survey.) It would

be interesting to test whether the consumer's behavior is consistent with a utility function with this specific property, whether or not it also satisfies geometric discounting. The upshot is that this can be done by checking generalized cyclical consistency, or the monotone congruence axiom, with respect to an appropriately chosen preorder on the consumption space.

Let us first take the case of *time indifference*. We say that an intertemporal utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ is a **neutral time preference** if it is strictly increasing and symmetric. (By *symmetry* of U here, we mean that $U(\mathbf{x}) = U(\mathbf{x}_\sigma)$ for any $\mathbf{x} \in \mathbb{R}^n_+$ and any permutation σ on $[n]$, where $\mathbf{x}_\sigma$ stands for the n-vector $(z_{\sigma(1)}, ..., z_{\sigma(n)})$.) In turn, we define the binary relation $\gtrsim_{\text{sym}}$ on $\mathbb{R}^n_+$ by $\mathbf{x} \gtrsim_{\text{sym}} \mathbf{y}$ iff $\mathbf{x} \geq \mathbf{y}_\sigma$ for some permutation σ on $[n]$. Obviously, a person who is indifferent over the timing of consumption would be indifferent between $\mathbf{y}$ and $\mathbf{y}_\sigma$ (for any σ), so, assuming that she prefers more to less, she would prefer $\mathbf{x}$ over $\mathbf{y}$ whenever $\mathbf{x} \gtrsim_{\text{sym}} \mathbf{y}$. It is readily verified that $\gtrsim_{\text{sym}}$ is a continuous preorder on $\mathbb{R}^n_+$ (but it is not a partial order unless $n = 1$). Moreover, an intertemporal utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ is a neutral time preference iff it is strictly $\gtrsim_{\text{sym}}$-increasing. Therefore, as in the case of Proposition 11, we may apply Proposition 5 and the Rationalizability Theorem II to test whether a data set is consistent with the maximization of a continuous neutral time preference, by choosing $\gtrsim_{\text{sym}}$ as the dominance relation of the model.

To test for impatience, we shall require the consumer's utility function to be strictly increasing with respect to a different preorder. Given a bundle $\mathbf{x} \in \mathbb{R}^n_+$, we call a bundle $\mathbf{y}$ an **improving swap** of $\mathbf{x}$ if $\mathbf{y} = \mathbf{x}_\sigma$ for some permutation σ on $[n]$ such that there exist k and l in $[n]$ with $k < l$, $x_k \leq x_l$, $\sigma(k) = l$, $\sigma(l) = k$ and $\sigma(j) = j$ for all $j \in [n] \backslash \{k, l\}$. (For example, $(4, 3, 6, 5)$ is an improving swap of $(4, 3, 5, 6)$.) We say that an intertemporal utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$ is a **positive time preference** if it is strictly increasing and $U(\mathbf{y}) > U(\mathbf{x})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ such that $\mathbf{y}$ an improving swap of $\mathbf{x}$.[22] Now, let us say that $\mathbf{y}$ is an **improving permutation** of $\mathbf{x}$ if $\mathbf{y}$ is obtained from $\mathbf{x}$ by means of a sequence of improving swaps. In turn, we define the binary relation $\gtrsim_+$ on $\mathbb{R}^n_+$ by setting $\mathbf{x} \gtrsim_+ \mathbf{y}$ whenever there is a vector $\mathbf{z}$, which is an improving permutation of $\mathbf{y}$, such that $\mathbf{x} \geq \mathbf{z}$. (Clearly, $\gtrsim_+$ is a continuous preorder on $\mathbb{R}^n_+$.) And, once again, we may apply Proposition 5 and the Rationalizability Theorem II to test whether a data set is rationalizable by a continuous positive time preference, by choosing $\gtrsim_+$ as

[22] For instance, suppose $U(\mathbf{x}) = \sum_{i=1}^n v(i, \mathbf{x}_i)$ where $v(i, y)$ is strictly decreasing in i and strictly increasing in y. Then, U is a positive time preference. Of course, these properties on v hold in the case of geometric discounting, where $v(i, y) = \delta^i h(y)$ for some strictly increasing function h and discount factor $\delta \in (0, 1)$.

our dominance relation. An agent whose choices pass such a test can be deemed as displaying time preferences which is consistent with delay aversion.

7.4 Choice over policy space. In formal political science, it is common to model political positions as points in a Euclidean space $\mathbb{R}^n_+$, where n is interpreted as the number of issues over which a policy maker may form a preference. However, unlike its interpretation as a consumption space, it is no longer sensible to require a preference on this space to be strictly $\geq$-increasing; indeed the policy maker may well have an ideal position in $\mathbb{R}^n_+$ that is preferred to all alternatives. In formal terms, the choice environment is $((\mathbb{R}^n, \Delta_{\mathbb{R}^n_+}), \mathcal{A})$, where $\mathcal{A}$ is a nonempty finite collection of compact sets in $\mathbb{R}^n_+$. The policy maker's observed choices is given by some closed-valued correspondence c on $\mathcal{A}$. Our continuous version of Richter's Theorem (Proposition 3) tells us that c is rationalizable by a continuous utility function on $\mathbb{R}^n_+$ iff c satisfies the congruence axiom.[23]

To exploit our results a bit further, suppose we know that the policy maker's preference over policy positions must take into account the preferences of a given group of agents $\mathcal{I}$. We assume that each agent $i \in \mathcal{I}$ has a preference $\succsim^i$ that is known to the observer and it is a continuous (but not necessarily complete) preorder in $\mathbb{R}^n_+$. Given this, we can define a new relation $\succsim^{\mathcal{I}}$, where $\mathbf{x} \succsim^{\mathcal{I}} \mathbf{y}$ if $\mathbf{x} \succsim^i \mathbf{y}$ for all $i \in \mathcal{I}$, and require the policy maker's utility function to be strictly $\succsim^{\mathcal{I}}$-increasing. In other words, the policy maker's preference relation is a Pareto aggregation of those of the agents in $\mathcal{I}$: If all agents in $\mathcal{I}$ prefer $\mathbf{x}$ to $\mathbf{y}$, then so does the policy maker. (It is plain that $\succsim^{\mathcal{I}}$ is a continuous preorder in $\mathbb{R}^n_+$.) Formally, the choice environment is then $((X, \succsim^{\mathcal{I}}), \mathcal{A})$, and given a choice correspondence c on $\mathcal{A}$, we can apply our continuous version of Richter's (resp., Afriat's) theorem to obtain necessary and sufficient conditions for the rationalizability of c (resp., $c^{\uparrow}$) with a utility function that is strictly $\succsim^{\mathcal{I}}$-increasing.

APPENDIX

Proof of the Rationalizability Theorem I. (a)$\Rightarrow$(b) Assume that (a) is valid. Then, there is a choice correspondence d on $\mathcal{A}$ such that (i) $d \sqsupseteq c$ and (ii) d satisfies the two requirements

[23] See Kalandrakis (2010) for a different revealed preference analysis in a similar setting. Using our terminology, Kalandrakis consider the case where each set $A \in \mathcal{A}$ has just two elements, and c is single-valued; these restrictions on c and $\mathcal{A}$ are natural if they are interpreted as modeling a legislator's voting record in the legislature. He characterizes those functions c that can be rationalized by a *concave* (hence continuous) utility function. Our rationalization imposes weaker assumptions on $\mathcal{A}$ and c, but the rationalizing utility function we obtain, while continuous, need not be concave. Non-concavity can arise naturally in policy space; see, for example, Stiglitz (1974).

of the monotone congruence axiom. Put $\mathcal{B} := \mathcal{A} \cup \{\{x, y\} \in 2^X : x \gtrsim y\}$, and define $e : \mathcal{B} \to 2^X$ as:

$$e(B) := \begin{cases} d(B), & \text{if } B \in \mathcal{A}, \\ \max(B, \gtrsim), & \text{if } B \in \mathcal{B} \backslash \mathcal{A}. \end{cases}$$

Obviously, e is a choice correspondence on $\mathcal{B}$. Moreover, e satisfies the congruence axiom. (To see this, take any $x, y \in X$ such that x tran$(R(e))$ y and $y \in e(B)$ for some $B \in \mathcal{B}$ with $x \in B$. But it is readily checked that $R(e) = R(d) \cup \gtrsim$. Consequently, if $B \in \mathcal{A}$, the monotone congruence axiom yields $x \in d(B) = e(B)$. If $B \in \mathcal{B} \backslash \mathcal{A}$, then $B = \{x, y\}$ and $y \gtrsim x$ (by definition of e), so again by the monotone congruence axiom, we find $x \sim y$ and so $x \in e(B)$.)

We now use Szpilrajn's Theorem to find a complete preorder $\succsim$ on X that extends tran$(R(e))$. Given an arbitrarily fixed B in $\mathcal{B}$, notice that if $x \in e(B)$ and $y \in B$, then x $R(e)$ y, and hence $x \succsim y$, which shows that $e(B) \subseteq \max(B, \succsim)$. Conversely, suppose there is an x in $\max(B, \succsim) \backslash e(B)$. Then, pick any $y \in e(B)$ so that y $R(e)$ x, and hence, y tran$(R(e))$ x. The reverse of this relation cannot hold, because, otherwise, we would get $x \in e(B)$ by the congruence axiom (on e). Thus, y tran$(R(e))$ x holds strictly, that is, y $P_{\text{tran}(R(e))}$ x. As $\succsim$ extends tran$(R(e))$, therefore, we find $y \succ x$, contradicting x being a $\succsim$-maximum in B. Conclusion:

$$e(B) = \max(B, \succsim) \quad \text{for every } B \in \mathcal{B}.$$

Obviously, this implies that $d(A) = \max(A, \succsim)$ for each $A \in \mathcal{A}$. It remains to show that $\succsim$ extends $\gtrsim$. To this end, take any $x, y \in X$ with $x > y$. If $\{x, y\} \in \mathcal{A}$, then $y \in d(\{x, y\})$ cannot hold due to the monotone congruence axiom, and hence $\{x\} = d(\{x, y\}) = e(\{x, y\})$, while if $\{x, y\} \notin \mathcal{A}$, we trivially have $\{x\} = e(\{x, y\})$. Consequently, $\{x\} = \max(\{x, y\}, \succsim)$, that is, $x \succ y$, as we sought.

(b)⇒(c) Assume that (b) is valid. Then, there is a complete preorder $\succsim$ on X and a d in $c^{\uparrow}$ such that $\succsim$ extends $\gtrsim$ and $d(A) = \max(A, \succsim)$ for each $A \in \mathcal{A}$. It follows that

$$c(A) \subseteq \max(A, \succsim) \subseteq \text{ MAX}(A, \gtrsim) \quad \text{for every } A \in \mathcal{A}. \tag{21}$$

Now take any $k \in \mathbb{N}$, $A_1, ..., A_k \in \mathcal{A}$, and $(x_1, ..., x_k) \in c(A_1) \times \cdots \times c(A_k)$ such that $x_1 \in A_2^{\downarrow}, ..., x_{k-1} \in A_k^{\downarrow}$ and $x_k \in A_1^{\downarrow}$. Then, there exists a $(y_1, ..., y_k) \in A_1 \times \cdots \times A_k$ such that $x_2 \succsim y_2 \geq x_1, ..., x_k \succsim y_k \geq x_{k-1}$ and $x_1 \succsim y_1 \geq x_k$. As $\succsim$ extends $\gtrsim$, we obtain, $x_1 \succsim x_k \succsim x_{k-1} \succsim \cdots \succsim x_1$. This implies that $x_{i-1} \in \text{MAX}(A_i^{\downarrow}, \succsim)$ for each $i \in [k]$. If not, there is $z \in A_i$ such that $z > x_{i-1}$ and so $x_i \succsim z \succ x_{i-1}$, which is not possible since $x_{i-1} \succsim x_{i-1}$.

(c)⇒(d) Assume that (c) is valid. Define

$$\succsim' := \text{ tran}\left(R\left(c\right) \cup \gtrsim\right),$$

where R is the direct revealed preference relation induced by c (Section 4.1). Clearly, $\succsim'$ is a preorder on X. We use Szpilrajn's Theorem to find a complete preorder $\succsim$ on X that extends

$\succsim'$. As $R(c) \subseteq \succsim$, we have $x \succsim y$ if there is an $A \in \mathcal{A}$ with $(x, y) \in c(A) \times A$. It follows that $c(A) \subseteq \max(A, \succsim)$ for every $A \in \mathcal{A}$. It remains to show that $\succsim$ extends $\gtrsim$, and for this, it is enough to show that $> \subseteq \succ'$. To this end, take any two elements x and y of X such that $x > y$. By definition of $\succsim'$, we have $x \succsim' y$. To derive a contradiction, suppose $y \succsim' x$ holds as well. Then, there exist an integer $k \geq 4$ and $x_1, ..., x_k$ in X such that

$$y = x_1 \ (R(c) \cup \gtrsim) \cdots \ (R(c) \cup \gtrsim) \ x_k = x. \tag{22}$$

But notice that, for any z and w in X, if $z \ (R(c) \cup \gtrsim) \ w$ holds, then there is a $v \in \{z, w\}$ with $z \ R(c) \ v \gtrsim w$. Consequently, we may conclude that there exist $y_1, ..., y_{k-1}$ in X such that

$$y = x_1 \ R(c) \ y_1 \gtrsim x_2 \ R(c) \ y_2 \gtrsim \cdots \gtrsim x_{k-1} \ R(c) \ y_{k-1} \gtrsim x_k = x. \tag{23}$$

By definition of $R(c)$, then, for each $i \in [k-1]$ there is an $A_i \in \mathcal{A}$ such that $(x_i, y_i) \in c(A_i) \times A_i$. In turn, also by (23), $x_{i+1} \in A_i^{\downarrow}$ for each $i \in [k-1]$. On the other hand, since $x_k = x > y = x_1$ and $x_k \in A_{k-1}^{\downarrow}$ we also have $x_1 \in A_{k-1}^{\downarrow}$. Therefore,

$$x_1 \in c(A_1), \ ..., \ x_{k-1} \in c(A_{k-1}) \quad \text{while} \quad x_2 \in A_1^{\downarrow}, \ ..., \ x_{k-1} \in A_{k-2}^{\downarrow}, \ x_1 \in A_{k-1}^{\downarrow}.$$

It then follows from generalized cyclical consistency that $x_1 \in \text{MAX}(A_{k-1}^{\downarrow}, \gtrsim)$. But this is impossible, since $x_k \in A_{k-1}^{\downarrow}$ and $x_k > x_1$.

(d)⇒(e) This is obvious.

(e)⇒(a) Assume that (e) is valid. Where $\succsim$ is as given in the statement of (e), define $d : \mathcal{A} \to 2^X$ as $d(A) := \max(A, \succsim)$. As $\max(A, \succsim)$ contains $c(A)$, it is nonempty for any A in $\mathcal{A}$, so d is a choice correspondence on $\mathcal{A}$ such that $d \sqsupseteq c$, that is, $d \in c^{\uparrow}$. Take any x and y in X with $x \ \text{tran}(R(d) \cup \geq) \ y$. Then, there is a positive integer k, elements $A_0, ..., A_k$ of $\mathcal{A}$, and $(x_0, ..., x_k) \in A_0 \times \cdots \times A_k$ such that $x = x_0$, $(x_{i-1}, x_i) \in d(A_i) \times A_i$ for each $i \in [k]$, and $y = x_k$. It follows from the definition of d that $x = x_0 \succsim \cdots \succsim x_k = y$, so, by transitivity of $\succsim$, we find $x \succsim y$. As $\succsim$ extends $\geq$, therefore, we cannot have $y > x$. Furthermore, if $y \in d(A)$ for some $A \in \mathcal{A}$ with $x \in A$, then $x \succsim y \succsim z$ for all $z \in A$, and hence, $x \in d(A)$. Thus: $\mathbb{C}$ satisfies the monotone congruence axiom. ■

Proof of Proposition 2. Fix an arbitrary $\gtrsim$-monotonic rationalization $\trianglerighteq$ for $c^{\uparrow}$ and put $R := R(c) \cup \gtrsim$. Let us first prove that

$$\text{tran}(R) \subseteq \trianglerighteq \tag{24}$$

(but note that $\trianglerighteq$ need not be an extension of $\text{tran}(R)$). To this end, take any distinct $x, y \in X$ with $x \ \text{tran}(R) \ y$. Then, there is a positive integer k and $x_0, ..., x_k \in X$ such that $x = x_0 \ R \ x_1 \ R \cdots R \ x_k = y$. If $x_{i-1} \ R(c) \ x_i$ for any $i \in [k]$, then $(x_{i-1}, x_i) \in c(A) \times A$ for some $A \in \mathcal{A}$, and

hence $x_{i-1} \trianglerighteq x_i$ because $c(A) \subseteq \max(A, \trianglerighteq)$. If, on the other hand, $x_{i-1} \gtrsim x_i$ for any $i \in [k]$, then $x_{i-1} \trianglerighteq x_i$ because $\trianglerighteq$ is $\gtrsim$-monotonic. Therefore, $x = x_0 \trianglerighteq x_1 \trianglerighteq \cdots \trianglerighteq x_k = y$, so, by transitivity of $\trianglerighteq$, we find $x \trianglerighteq y$, as we sought.

We now move to prove (12). Fix an arbitrary A in $\mathcal{A}$, and take any $x \in A$ with x tran(R) y for some $y \in c(A)$. Then, by (24), $x \trianglerighteq y$ while $y \in \max(A, \trianglerighteq)$ because $c(A) \subseteq \max(A, \trianglerighteq)$. It follows that $x \in \max(A, \trianglerighteq)$, establishing the second part of (12). Next, notice that $\succsim$ is obviously a rationalization for c, so the second part of (12) entails the $\supseteq$ part of the asserted equality in (12). To complete our proof, then, take any x in $\max(A, \succsim)$. Now pick any y in $c(A)$ and notice that, by (10), we must have $x \sim y$. On the other hand, as $(y, x) \in c(A) \times A$, we have y $R(c)$ x, and hence, y tran(R) x. As $\succsim$ extends tran(R) by hypothesis, and $x \sim y$, therefore, y tran(R) x cannot hold strictly, that is, we have x tran(R) y, which means $x \in c(A)^{\uparrow, \text{tran}(R)}$, as we sought. ■

Proof of Proposition 4. Suppose that c satisfies the monotone congruence axiom, and take any $k \in \mathbb{N}$, $A_1, ..., A_k \in \mathcal{A}$, and $x_1 \in c(A_1), ..., x_k \in c(A_k)$ such that (14) holds. Since $x_k \in A_1^{\downarrow}$, there is a $y \in A_1$ such that $y \gtrsim x_k$. Combining this with (14), therefore, we have $y \gtrsim x_k$ tran$(R(c) \cup \gtrsim)$ x_1, and hence $y \in c(A_1)$. This also means that x_k tran$(R(c) \cup \gtrsim)$ y, and hence, the monotone congruence axiom implies that $y > x_k$ is false. Thus: $x_k \sim y$. We conclude that $x_k \in \text{MAX}(A_1^{\downarrow}, \gtrsim)$ and $x_k^{\uparrow} \cap A_1 \subseteq c(A_1)$. The other cases are established analogously.

To prove the converse assertion, let x and y be any alternatives in X with x tran$(R(c) \cup \gtrsim)$ y. Then, there is an integer $k \geq 4$ and $x_1, ..., x_k \in X$ such that

$$x = x_k \ (R(c) \cup \gtrsim) \cdots \ (R(c) \cup \gtrsim) \ x_1 = y.$$

But recall that, for any z and w in X, if z $(R(c) \cup \gtrsim)$ w holds, then there is a $v \in \{z, w\}$ with z $R(c)$ $v \gtrsim w$. Consequently, there exist $y_1, ..., y_k \in X$ such that

$$x = x_k \ R(c) \ y_k \gtrsim x_{k-1} \ R(c) \ y_{k-1} \gtrsim \cdots \gtrsim x_2 \ R(c) \ y_2 \gtrsim x_1 = y.$$

Now, take any $A \in \mathcal{A}$ with $y \in c(A)$ and $x \in A$. (We wish to show that $x \in c(A)$.) By definition of $R(c)$, for each $i \in \{2, ..., k\}$ there is an $A_i \in \mathcal{A}$ such that $(x_i, y_i) \in c(A_i) \times A_i$. Putting $A_1 := A$, therefore, we have

$$x_i \in c(A_i) \quad \text{and} \quad x_i \in A_{i+1 \ (\text{mod } k)}^{\downarrow}$$

for each $i \in [k]$. By the converse hypothesis of the proposition, therefore, $x_k^{\uparrow} \cap A_1 \subseteq c(A_1)$, and hence $x \in c(A)$, as we sought. To conclude that c satisfies the monotonic congruence axiom, it remains to show that $y > x$ cannot hold. To derive a contradiction, then, suppose we have

$y > x$. In that case, $y_2 \gtrsim y > x = x_k$, so $x_k \in A_2^{\downarrow}$. Thus,

$$x_2 \in c(A_2),\ ...,\ x_k \in c(A_k) \quad \text{while} \quad x_2 \in A_3^{\downarrow},\ ...,\ x_{k-1} \in A_k^{\downarrow},\ x_k \in A_2^{\downarrow}.$$

By the converse hypothesis of the proposition, therefore, $x_k \in \text{MAX}(A_2^{\downarrow}, \gtrsim)$, contradicting the fact that $y_2 > x_k$. ■

Proof of the Rationalizability Theorem II. It is plain that (b) and (e) are equivalent, and (d) implies (e). From Rationalizability Theorem I we know that (a) implies (c) and that (e) implies (a). We will complete the proof of the theorem by showing that (c) implies (d). Let us denote the direct revealed preference induced by c as R, that is, we put $R := R(c)$. We first show that $\succsim' := \text{tran}(R \cup \gtrsim)$ is a continuous preorder on X. This requires the following lemma.

Lemma. *Let R, S and T be binary relations on X such that R and T are closed, and S is compact, in $X \times X$. Then $R \circ S$, $S \circ R$ and $R \circ S \circ T$ are closed in $X \times X$.*

Proof. We shall confine ourselves to proving that $R \circ S \circ T$ is closed; the other cases have similar proofs. Suppose (x_m) and (y_m) are two sequences that converge to x and y and we have $x_m \, R \circ S \circ T \, y_m$ for each m. Then, there are sequences (z_m) and (w_m) such that $x_m \, R \, z_m \, S \, w_m \, T \, y_m$. Since S is compact, we may assume, after passing to subsequences if necessary, that (z_m) and (w_m) are convergent. Since R, S and T are closed in $X \times X$, we thus have $x \, R \lim z_m \, S \lim w_m \, T \, y$, which means $x \, R \circ S \circ T \, y$. □

Note that $R(c)$ is a compact relation because $\mathcal{A}$ is a finite collection of compact sets and c is compact-valued. Furthermore, $\gtrsim$ is closed in $X \times X$ by assumption. Therefore, the Lemma above guarantees that the relation $E^1 := \gtrsim \circ R(c) \circ \gtrsim$ is closed in $X \times X$. More generally,

$$E^k := \gtrsim \circ R(c) \circ \gtrsim \circ R(c) \circ .... \circ R(c) \circ \gtrsim$$

where $R(c)$ appears k times on the right hand side is also closed relation in $X \times X$ (by repeated application of the Lemma). Suppose $x \, \text{tran}(R \cup \gtrsim) \, y$. Then either $x \gtrsim y$ or there is an integer $k \geq 4$ and $x_1, ..., x_k \in X$ such that

$$x \gtrsim x_1 \, R(c) \, x_2 \gtrsim x_3 \, R(c) \ \cdots \ R(c) \, x_k \gtrsim y.$$

In other words, $x \, E^k \, y$. Crucially, we can always ensure that $x \, E^k \, y$ for $k \leq |\mathcal{A}|$. Indeed, if $x \, E^k \, y$ and $k > |\mathcal{A}|$, then there must be $A \in \mathcal{A}$, and i and $j > i$ such that $(x_i, x_{i+1}) \in c(A) \times A$ and $(x_j, x_{j+1}) \in c(A) \times A$. In that case we can write

$$x \gtrsim x_1 \, R(c) \, x_2 \gtrsim \cdots \gtrsim x_i \, R(c) \, x_{j+1} \gtrsim \cdots x_k \gtrsim y,$$

which means that $x\, R^n\, y$ for some $n < k$. We conclude that

$$\text{tran}(R \cup \gtrsim) = \gtrsim \cup E^1 \cup E^2 \cup \cdots \cup E^{|\mathcal{A}|}.$$

Therefore, $\text{tran}(R \cup \gtrsim)$ is closed in $X \times X$, being the union of finitely many closed relations.

We are now ready to complete the proof that (c) implies (d). Combining what is established in Steps 4 and 5, we see that $\succsim' := \text{tran}(R \cup \gtrsim)$ is a continuous preorder on X. We may thus apply Levin's Theorem to find a continuous real map u on X such that u is strictly increasing with respect to $\succsim'$. From the proof of Rationalizability Theorem I, we know that $\succsim'$ is an extension of $\gtrsim$. Therefore, u is also strictly increasing with respect to $\gtrsim$. Lastly, since $R \subseteq \succsim'$, for any $A \in \mathcal{A}$ with $x \in c(A)$ we have $x \succsim' y$, and hence, $u(x) \geq u(y)$, for all $y \in A$. Our proof is complete. ■

Proof of Proposition 8. (i) Suppose $\mathbf{x}$ is not revealed preferred to $\mathbf{y}$. In that case, put $\mathcal{A}^* := \mathcal{A} \cup \{\mathbf{x}, \mathbf{y}\}$, where $\mathcal{A} = \{B(\mathbf{p}, \mathbf{p}d(\mathbf{p})) : \mathbf{p} \in P\}$, and define the correspondence $c^* : \mathcal{A}^* \rightrightarrows \mathbb{R}^n_+$ by $c^*(B(\mathbf{p}, \mathbf{p}d(\mathbf{p}))) := d(\mathbf{p})$ for each $p \in P$, and $c^*(\{\mathbf{x}, \mathbf{y}\}) := \mathbf{y}$. Since $\mathbf{x}$ is not revealed preferred to $\mathbf{y}$, that is, $(\mathbf{x}, \mathbf{y})$ is not in $\text{tran}(R(c) \cup \geq)$, and c obeys generalized cyclical consistency, c^* also obeys generalized cyclical consistency. By construction, $(\mathbf{y}, \mathbf{x}) \in R(c^*)$ but $(\mathbf{x}, \mathbf{y})$ does not belong to $\text{tran}(R(c^*) \cup \gtrsim)$, and thus, by the equivalence of (c) and (d) in Rationalizability Theorem II, there exists u^* with (13), and hence $u^* \in \mathcal{U}(d)$ such that $u^*(\mathbf{y}) > u^*(\mathbf{x})$. Thus, $(\mathbf{x}, \mathbf{y})$ does not belong to $S(d)$.

(ii) We define $\mathcal{A}^*$ and c^* as in the proof of (i). If $\mathbf{x}$ is not revealed preferred to $\mathbf{y}$, then we know from the proof of (i) that $(\mathbf{x}, \mathbf{y})$ is not in $S(d)$ and hence not in $S'(d)$. Now suppose $\mathbf{x}$ is revealed preferred to $\mathbf{y}$, but not strictly so. In that case, one could check that c^* still satisfies cyclical consistency and hence Rationalizability Theorem II tells us that there is a strictly increasing and continuous function u^{**} with (13). Since $\mathbf{x}\ \text{tran}(R(c^*) \cup \gtrsim)\, \mathbf{y}$ and, by construction, $\mathbf{y}\, R(c^*)\, \mathbf{x}$, we obtain $u^{**}(\mathbf{x}) = u^{**}(\mathbf{y})$. Finally, note that $u^{**} \in \mathcal{U}(d)$, and thus $(\mathbf{x}, \mathbf{y})$ does not belong to $S'(d)$. ■

Proof of Proposition 9. We only need to show that $\text{tran}(R(c) \cup \geq)$ is continuous for the choice correspondence $c : B(\mathbf{p}, I) \mapsto d(\mathbf{p}, I)$. Note that, by generalized cyclical consistency, $\mathbf{x} \in d(\mathbf{p}, I)$ implies $\mathbf{p}\mathbf{x} = I$ for any $\mathbf{p} \in P$ and $I > 0$. Also, note that if $\mathbf{y}\, \text{tran}(R(c) \cup \geq)\, \mathbf{z}$, then there are $(\mathbf{p}^1, I^1), \ldots, (\mathbf{p}^k, I^k) \in P \times I$ and $\mathbf{x}^1 \in d(\mathbf{p}^1, I^1), \ldots, \mathbf{x}^k \in d(\mathbf{p}^k, I^k)$ such that

$$\mathbf{y} \geq \mathbf{x}^1, \mathbf{p}^1\mathbf{x}^1 \geq \mathbf{p}^1\mathbf{x}^2, ..., \mathbf{p}^{k-1}\mathbf{x}^{k-1} \geq \mathbf{p}^{k-1}\mathbf{x}^k \text{ and } \mathbf{p}^k\mathbf{x}^k \geq \mathbf{p}^k\mathbf{z}. \tag{25}$$

Crucially, we may choose $\mathbf{p}^1, \ldots, \mathbf{p}^k$ to be distinct. Indeed, generalized cyclical consistency requires that if $\mathbf{p}^r = \mathbf{p}^s$ for some $r < s$, then $I^r \geq I^s$; thus $B(\mathbf{p}^s, I^s) \subseteq B(\mathbf{p}^r, I^r)$ and it follows

that $\mathbf{p}^r\mathbf{x}^r \geq \mathbf{p}^r\mathbf{x}^{s+1}$, that is, we may 'snip off' the part of the sequence in (25) between $r+1$ and s.

Suppose there are sequences $\mathbf{y}_m \to \bar{\mathbf{y}}$ and $\mathbf{z}_m \to \bar{\mathbf{z}}$ such that $\mathbf{y}_m \operatorname{tran}(R(c) \cup \geq) \mathbf{z}_m$ for each m. We claim that $\bar{\mathbf{y}} \operatorname{tran}(R(c) \cup \geq) \bar{\mathbf{z}}$. Indeed, each pair of $\mathbf{y}_m$ and $\mathbf{z}_m$ is linked by a sequence of inequalities like (25), where the price vectors in that sequence are distinct. This property, together with the finiteness of P, guarantees that we can find subsequences $\mathbf{y}_{m_t}$ and $\mathbf{z}_{m_t}$ and distinct vectors $\mathbf{p}^1, \ldots, \mathbf{p}^k$ in P such that

$$\mathbf{y}_{m_t} \geq \mathbf{x}^1_{m_t}, \mathbf{p}^1\mathbf{x}^1_{m_t} \geq \mathbf{p}^1\mathbf{x}^2_{m_t}, ..., \mathbf{p}^{k-1}\mathbf{x}^{k-1}_{m_t} \geq \mathbf{p}^{k-1}\mathbf{x}^k_{m_t} \text{ and } \mathbf{p}^k\mathbf{x}^k_{m_t} \geq \mathbf{p}^k\mathbf{z}_{m_t}.$$

By the upper hemicontinuity of d, and after taking further subsequences if necessary, we obtain $I^l_{m_t} \to \bar{I}^l > 0$ and $\mathbf{x}^l_{m_t} \to \bar{\mathbf{x}}^l \in d(\mathbf{p}^l, \bar{I}^l)$. Thus,

$$\bar{\mathbf{y}} \geq \bar{\mathbf{x}}^1, \mathbf{p}^1\bar{\mathbf{x}}^1 \geq \mathbf{p}^1\bar{\mathbf{x}}^2, ..., \mathbf{p}^{k-1}\bar{\mathbf{x}}^{k-1} \geq \mathbf{p}^{k-1}\bar{\mathbf{x}}^k \text{ and } \mathbf{p}^k\bar{\mathbf{x}}^k \geq \mathbf{p}^k\bar{\mathbf{z}},$$

which means that $\bar{\mathbf{y}} \operatorname{tran}(R(c) \cup \geq) \bar{\mathbf{z}}$, as we sought. ■

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