

## GLOBAL SOLUTIONS OF A TWO-DIMENSIONAL RIEMANN PROBLEM FOR THE PRESSURE GRADIENT SYSTEM

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*Dedicated to Professor Shuxing Chen on the occasion of his 80th birthday*

**ABSTRACT.** We are concerned with a two-dimensional Riemann problem for the pressure gradient system that is a hyperbolic system of conservation laws. The Riemann initial data consist of four constant states in four sectorial regions such that two shocks and two vortex sheets are generated between the adjacent states. The solutions keep the four constant states and four planar waves outside the outer sonic circle in the self-similar coordinates, while the two shocks keep planar until meeting the outer sonic circle at two different points and then generate a diffracted shock to connect these points, whose location is *a priori* unknown. Then the problem can be formulated as a free boundary problem, in which the diffracted transonic shock is the one-phase free boundary to connect the two points, while the other part of the sonic circle forms a fixed boundary. We establish the global existence of a solution and the optimal Lipschitz regularity of both the diffracted shock across the two points and the solution across the outer sonic boundary. Then this Riemann problem is solved globally, whose solution contains two vortex sheets and one global shock containing the two originally separated shocks generated by the Riemann data.

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2020 *Mathematics Subject Classification.* Primary: 35L65, 35M10, 35M12, 35R35, 35B36, 35L67; Secondary: 76L05, 76N10, 35D30, 35J67, 76G25.

*Key words and phrases.* Two-dimensional Riemann problems, global solutions, pressure gradient system, Euler equations, hyperbolic conservation laws, mixed type, degenerate elliptic equations, shocks, transonic shock, vortex sheets, free boundary problem.

Gui-Qiang G. Chen's research was supported in part by the UK Engineering and Physical Sciences Research Council under Grant EP/L015811/1 and the Royal Society–Wolfson Research Merit Award WM090014 (UK). Qin Wang's research was supported in part by National Natural Science Foundation of China (11761077), China Scholarship Council (201807035046), and the Key Project of Yunnan Provincial Science and Technology Department and Yunnan University (No.2018FY001-014). Shengguo Zhu's research was supported in part by the Royal Society–Newton International Fellowships NF170015 and the Monash University–Robert Bartnik Visiting Fellowship. Qin Wang would also like to thank the hospitality and support of the Mathematical Institute, University of Oxford, during his visit in 2019–20.

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**1. Introduction.** The two-dimensional (2-D) full Euler equations are of the conservation form:

$$\mathbf{U}_t + \operatorname{div}_{\mathbf{x}} \mathbf{F} = 0 \quad \text{for } t \geq 0 \text{ and } \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \quad (1.1)$$

with

$$\mathbf{U} := (\rho, \rho \mathbf{u}, \rho E), \quad \mathbf{F} := (\rho \mathbf{u}, \rho \mathbf{u} \otimes \mathbf{u} + pI, (\rho E + p)\mathbf{u}),$$

where  $\rho > 0$  is the density,  $\mathbf{u} = (u, v)$  the velocity,  $p$  the pressure, and

$$E = \frac{|\mathbf{u}|^2}{2} + e$$

represents the total energy per unit mass with the internal energy  $e$  given by  $e = \frac{p}{(\gamma-1)\rho}$  for the adiabatic constant  $\gamma > 1$  for polytropic gases.

There are two mechanisms in the fluid motion: inertia and pressure difference. Corresponding to a separation of these two mechanisms, a natural flux-splitting of  $\mathbf{F}$  is to split it into two parts:  $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$  with

$$\mathbf{F}_1 := (\rho \mathbf{u}, \rho \mathbf{u} \otimes \mathbf{u}, \rho E \mathbf{u}), \quad \mathbf{F}_2 := (0, pI_{2 \times 2}, p\mathbf{u}),$$

where  $I_{2 \times 2}$  is the diagonal identity matrix. Correspondingly, the Euler equations (1.1) can be split into two subsystems of conservation laws:

$$\mathbf{U}_t + \operatorname{div} \mathbf{F}_1 = 0, \quad \mathbf{U}_t + \operatorname{div} \mathbf{F}_2 = 0,$$

which are called the pressureless Euler system and the pressure gradient system, respectively; also see [1, 28]. Similar flux-splitting ideas have been widely used in order to design the so-called flux-splitting schemes and their high-order accurate extensions. Many flux-splittings have been derived in the literature for the compressible Euler equations of gas dynamics and are currently used in fluid dynamics codes. See [1, 13, 28] and the references cited therein.

In this paper, we focus on the pressure gradient system that corresponds to flux  $\mathbf{F}_2$ . The explicit form for the pressure gradient system is

$$\begin{cases} \rho_t = 0, \\ (\rho \mathbf{u})_t + \nabla_{\mathbf{x}} p = 0, \\ (\rho E)_t + \operatorname{div}_{\mathbf{x}}(p\mathbf{u}) = 0. \end{cases} \quad (1.2)$$

By a suitable scaling in (1.2) and taking  $\rho \equiv 1$ , the pressure gradient system is of the following form:

$$\begin{cases} u_t + p_{x_1} = 0, \\ v_t + p_{x_2} = 0, \\ E_t + (pu)_{x_1} + (pv)_{x_2} = 0, \end{cases} \quad (1.3)$$

where  $E = \frac{|\mathbf{u}|^2}{2} + p$ . Furthermore, system (1.3) can also be deduced from the physical validity when the velocity is small and the adiabatic gas constant  $\gamma$  is large; see Zheng [37]. An asymptotic derivation of system (1.3) has also been presented by Hunter as described in [39]. We refer the reader to [27, 40] for further background on system (1.3). Besides the pressure gradient system, there are also several other important nonlinear partial differential equations (PDEs) derived from the full Euler equations, such as the potential flow equation that has been widely used in aerodynamics, as well as the nonlinear wave system, the unsteady transonic small disturbance equations, and the pressureless Euler system as mentioned above; see [4, 10, 11, 12, 24, 25] and the references cited therein. The analysis of these

nonlinear PDEs has motivated and inspired the developments of new techniques and ideas to deal with the corresponding problems for the Euler equations.

The Riemann problem was first introduced by B. Riemann in 1860 in his pioneering work [32] to analyze discontinuous solutions of the 1-D Euler equations for gas dynamics. It is an initial value problem with the simplest discontinuous initial data, which are scaling invariant and piecewise constant. The Riemann solutions have played a fundamental role in the mathematical theory of 1-D hyperbolic systems of conservation laws; see [9, 12, 18, 22, 26, 33] and the references cited therein. The 2-D Riemann problem is substantially different and much more complicated than the 1-D case; see [7, 8, 9, 14, 15, 17, 27, 40] and the references cited therein.

One of the prototypical 2-D Riemann problems is that the Riemann initial data consist of four different constant states in the four quadrants so that there is only one wave that is generated between two adjacent states. Each wave between any two adjacent states is of one of at least three types of planar waves: shock wave, rarefaction wave, and vortex sheet. Then the 2-D Riemann problem is to analyze the different combinations/interactions of these four waves in a domain containing the origin. The solutions of such a 2-D Riemann problem for system (1.3) were analyzed via the characteristic method and the corresponding numerical simulations were presented in Zhang-Li-Zhang [36]. It has been observed that the mathematical structure of the pressure gradient system is strikingly in agreement with that of the Euler equations. In [27, 36], it was shown that there are twelve genuinely different cases, besides three trivial cases, for the solutions of the 2-D Riemann problem for the pressure gradient system. To our knowledge, there have been few rigorous mathematical results on the global existence for the non-trivial cases, owing to lack of effective techniques for handling several main difficulties in the analysis of nonlinear PDEs such as equations of mixed elliptic-hyperbolic type, free boundary problems, and corner singularities.

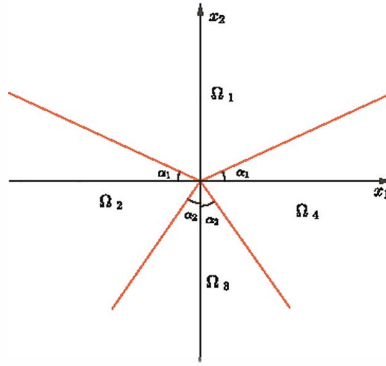


FIGURE 1. The general Riemann initial data

In this paper, we consider a more general Riemann problem for system (1.3), whose initial data consist of four constant states in four sectorial regions  $\Omega_i$  with symmetric sectorial angles (see Fig. 1):

$$(p, u, v)(0, x_1, x_2) = (p_i, u_i, v_i) \quad \text{for } (x_1, x_2) \in \Omega_i, i = 1, 2, 3, 4. \quad (1.4)$$

One of our motivations is to understand the intersections of two shock waves and two vortex sheets. For this purpose, the four initial constant states are required to satisfy the following conditions:

$$\begin{cases} \text{A forward shock } S_{41}^+ \text{ is formed between states (1) and (4);} \\ \text{A backward shock } S_{12}^- \text{ is formed between states (1) and (2);} \\ \text{A vortex sheet } J_{23}^+ \text{ is formed between states (2) and (3);} \\ \text{A vortex sheet } J_{34}^- \text{ is formed between states (3) and (4).} \end{cases} \quad (1.5)$$

These four waves can be obtained by solving four 1-D Riemann problems in the self-similar coordinates  $(\xi, \eta) = (\frac{x_1}{t}, \frac{x_2}{t})$ , which form the configuration as shown in Fig. 2.

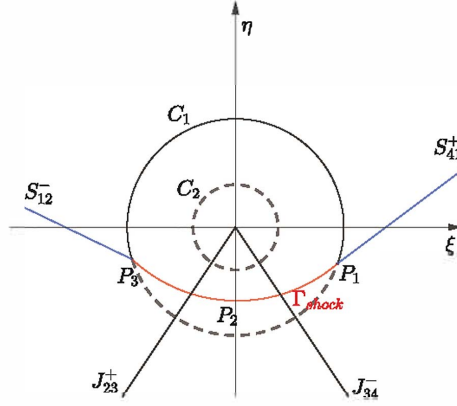


FIGURE 2. The configuration of the four initial waves

When the two shock waves  $S_{12}^-$  and  $S_{41}^+$  meet the outer sonic circle  $C_1$  of state (1), shock diffraction occurs, and then  $S_{12}^-$  and  $S_{41}^+$  are expected to bend and form a diffracted shock, denoted by  $\Gamma_{\text{shock}}$ . One of the main difficulties is that the location of  $\Gamma_{\text{shock}}$  is *a priori* unknown, so that it is *a priori* unclear whether  $\Gamma_{\text{shock}}$  could intersect with the inner sonic circle  $C_2$  of state (2). Zheng [38] studied this Riemann problem initially with the assumption that angle  $\alpha_1 = \alpha_2$  is close to zero. This assumption ensures that the two shocks bend slightly and the diffracted shock  $\Gamma_{\text{shock}}$  does not meet the inner sonic circle  $C_2$ . However, it has been an open problem when the angle between the two shocks is not close to  $\pi$ , since the work of Zheng [38].

The purpose of this paper is to solve the Riemann problem globally for the general case so that an affirmative answer to this open problem is provided. In particular, we establish the global existence of entropy solutions for this Riemann problem allowing all angles  $\alpha_i \in (0, \frac{\pi}{2}), i = 1, 2$ . To solve this problem, we first reformulate the problem as a free boundary problem involving a transonic shock. Then we carefully establish the required appropriate properties and uniform estimates of approximate and exact solutions so that the techniques developed in Chen-Feldman [11, 12] can be employed; also see [2, 3, 16, 35] and the references cited therein. This involves several core difficulties in the theory of the underlying nonlinear PDEs: optimal estimates of solutions of nonlinear degenerate PDEs across the outer sonic circle  $C_1$

and corner singularities (at corners  $P_1$  and  $P_3$  formed between the transonic shock as a free boundary and the sonic circle  $C_1$ ), in addition to the involved nonlinear PDE of mixed elliptic-hyperbolic type and free boundary problem.

The organization of this paper is as follows: In §2, we reformulate the Riemann problem into the free boundary problem and present our main results and strategies. In §3, we give a complete proof of the global existence of solutions of the free boundary problem. In §4, we obtain the optimal  $C^{0,1}$ -regularity of solutions near the degenerate sonic boundary  $C_1$  and at corners  $P_1$  and  $P_3$ . Finally, in §5, we obtain the existence and regularity of global solutions of the 2-D Riemann problem of the pressure gradient system (1.3).

**2. Reformulation of the Riemann problem and main theorem.** Based on the invariance of both the system and the Riemann initial data under the self-similar scaling, we seek self-similar solutions in the self-similar coordinates. For this purpose, in this section, we first reformulate the Riemann problem into a free boundary problem, present the results in the main theorem, Theorem 2.1, and then describe the strategies to achieve them in §2.3–§2.4.

More precisely, we seek self-similar solutions with the form:

$$(p, u, v)(t, x_1, x_2) = (p, u, v)(\xi, \eta) \quad \text{with } (\xi, \eta) = \left(\frac{x_1}{t}, \frac{x_2}{t}\right), t > 0.$$

In the  $(\xi, \eta)$ -coordinates, system (1.3) can be rewritten as

$$\begin{cases} (\xi u)_\xi + (\eta u)_\eta - p_\xi - 2u = 0, \\ (\xi v)_\xi + (\eta v)_\eta - p_\eta - 2v = 0, \\ (\xi E)_\xi + (\eta E)_\eta - (pu)_\xi - (pv)_\eta - 2E = 0. \end{cases} \quad (2.1)$$

**2.1. Shock waves and vortex sheets in the self-similar coordinates.** Let  $\eta = \eta(\xi)$  be a  $C^1$ -discontinuity curve of a bounded discontinuous solution of system (2.1). From the Rankine-Hugoniot relation on  $\eta(\xi)$ :

$$\begin{cases} (\xi\sigma - \eta)[u] - \sigma[p] = 0, \\ (\xi\sigma - \eta)[v] + [p] = 0, \\ (\xi\sigma - \eta)[E] - \sigma[pu] + [pv] = 0, \end{cases}$$

we find either the nonlinear discontinuities:

$$\begin{cases} \frac{d\eta}{d\xi} = \sigma_\pm = -\frac{[u]}{[v]} = \frac{\xi\eta \pm \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\xi^2 - \bar{p}}, \\ [p]^2 = \bar{p}([u]^2 + [v]^2), \end{cases} \quad (2.2)$$

or linear discontinuity:

$$\begin{cases} \sigma_0 = \frac{\eta}{\xi} = \frac{[v]}{[u]}, \\ [p] = 0, \end{cases} \quad (2.3)$$

where  $\bar{p}$  is the average of the pressure on the two sides of the discontinuity, and  $[w]$  denotes the jump of  $w$  across the discontinuity.

A discontinuity is called a shock if it satisfies (2.2) and the entropy condition — the pressure  $p$  increases across it in the flow direction; that is, the pressure on the wave front is larger than that on the wave back. The shock is of two types  $S^\pm$ :

- $S = S^+$  if  $\nabla_{(\xi, \eta)} p$  and the flow direction form a right-hand system;
- $S = S^-$  if  $\nabla_{(\xi, \eta)} p$  and the flow direction form a left-hand system.

A discontinuity is called a vortex sheet if it satisfies (2.3). A vortex sheet is of two types according to the sign of the vorticity:

$$J^\pm : \quad \text{curl}(u, v) = \pm\infty.$$

**2.2. Reformulation of the Riemann problem into a free boundary problem.** We first show the following lemma:

**Lemma 2.1.** *For fixed  $(p_1, u_1, v_1)$  and  $p_2 = p_3 = p_4$  satisfying  $p_1 > p_2$ , there exist states  $(u_i, v_i), i = 2, 3, 4$ , such that the conditions in (1.5) for the Riemann initial data hold, and  $(u_i, v_i), i = 2, 3, 4$ , depend on angles  $(\alpha_1, \alpha_2)$  continuously.*

*Proof.* For given  $(p_1, u_1, v_1)$  and  $p_1 > p_2$ , we first consider  $(u_2, v_2)$ . Since the Rankine-Hugoniot conditions on  $S_{12}^-$  hold:

$$\begin{cases} [u \sin \alpha_1 + v \cos \alpha_1] = -\frac{[p]}{\sqrt{p}}, \\ [-u \cos \alpha_1 + v \sin \alpha_1] = 0, \end{cases}$$

a direct computation shows that

$$S_{12}^- := \{(\xi, \eta) : \xi \sin \alpha_1 + \eta \cos \alpha_1 = -\sqrt{p}\}$$

with

$$(u_2, v_2) = (u_1, v_1) - \frac{[p]}{\sqrt{p}}(\sin \alpha_1, \cos \alpha_1).$$

Next, we turn to  $(u_4, v_4)$ . The Rankine-Hugoniot conditions on  $S_{41}^+$  are:

$$\begin{cases} [u \sin \alpha_1 - v \cos \alpha_1] = \frac{[p]}{\sqrt{p}}, \\ [u \cos \alpha_1 + v \sin \alpha_1] = 0, \end{cases}$$

which imply

$$S_{41}^+ := \{(\xi, \eta) : \xi \sin \alpha_1 - \eta \cos \alpha_1 = \sqrt{p}\}$$

with

$$(u_4, v_4) = (u_1, v_1) + \frac{[p]}{\sqrt{p}}(\sin \alpha_1, -\cos \alpha_1).$$

Finally, we consider  $(u_3, v_3)$ . To guarantee the existence of two vortex sheets  $J_{23}^+$  and  $J_{34}^-$ , we have

$$\begin{cases} u_2 \cos \alpha_2 - v_2 \sin \alpha_2 = u_3 \cos \alpha_2 - v_3 \sin \alpha_2, \\ u_4 \cos \alpha_2 + v_4 \sin \alpha_2 = u_3 \cos \alpha_2 + v_3 \sin \alpha_2. \end{cases}$$

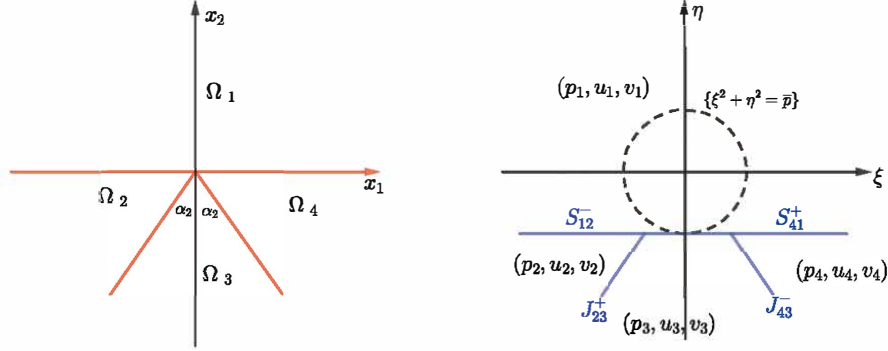
Solving  $(u_3, v_3)$  from the above two equations, we obtain

$$(u_3, v_3) = (u_1, v_1) = \frac{[p]}{\sqrt{p}}(0, \frac{\sin \alpha_1}{\sin \alpha_2} - \cos \alpha_1).$$

It is clear that  $(u_i, v_i), i = 2, 3, 4$ , depend on angles  $(\alpha_1, \alpha_2)$  continuously. □

There is a critical case when  $\alpha_1 = 0$ . Then the Riemann initial data satisfy

$$p_1 > p_2 = p_3 = p_4, \quad u_1 = u_2 = u_3 = u_4, \quad v_1 > v_2 = v_3 = v_4.$$

FIGURE 3. The Riemann data and the global solution when  $\alpha_1 = 0$ 

The global Riemann solution is piecewise constant with two planar shocks:

$$S_{12}^-/S_{41}^+ : \begin{cases} \eta = -\sqrt{\bar{p}} & \text{with } \bar{p} = \frac{p_1 + p_2}{2}, \\ [v] = -\frac{[p]}{\sqrt{\bar{p}}}, & [u] = 0, \\ \xi < 0 \text{ for } S_{12}^-, & \xi > 0 \text{ for } S_{41}^+, \end{cases}$$

and two characteristic lines  $J_{23}^+$  and  $J_{34}^-$  (reduced from the two vortex sheets), as shown in Fig. 3. The two planar shocks  $S_{12}^-$  and  $S_{41}^+$  are both tangential to the circle,  $\{\xi^2 + \eta^2 = \bar{p}\}$ , with the tangent point on the circle as the end-point. From the expression of  $J_{23}^+$  given in (2.3), we know that  $p_2 = p_3$  on both sides of  $J_{23}^+$ . At the point where  $J_{23}^+$  intersects with  $S_{12}^-$ , we deduce that  $J_{23}^+$  does not effect the shock owing to  $p_2 = p_3$ . The intersection between  $J_{34}^-$  and  $S_{41}^+$  can be handled in the same way.

In the following, we focus on the case that  $\alpha_1 \in (0, \frac{\pi}{2})$ , for which we want to solve.

From (2.1), we can derive a second-order nonlinear equation for  $p$ :

$$(p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{(\xi p_\xi + \eta p_\eta)^2}{p} - 2(\xi p_\xi + \eta p_\eta) = 0. \quad (2.4)$$

It is easy to verify that equation (2.4) is of mixed hyperbolic-elliptic type, which is hyperbolic when  $\xi^2 + \eta^2 > p$  and elliptic when  $\xi^2 + \eta^2 < p$ . The sonic circle is:  $\xi^2 + \eta^2 = p$ .

Furthermore, in the polar coordinates:

$$(r, \theta) = (\sqrt{\xi^2 + \eta^2}, \arctan(\frac{\eta}{\xi})),$$

equation (2.4) becomes

$$Qp := (p - r^2)p_{rr} + \frac{p}{r^2}p_{\theta\theta} + \frac{p}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0, \quad (2.5)$$

which is hyperbolic when  $p \in (0, r^2)$  and elliptic when  $p > r^2$ . The sonic circle is given by  $r = r(\theta)$  satisfying that  $r^2(\theta) = p(r(\theta), \theta)$ .

In the  $(\xi, \eta)$ -coordinates, the four elementary waves come from the far-field (at infinity corresponding to  $t = 0$ ) and keep planar waves before the two shocks meet

the outer sonic circle  $C_1$  of state (1):

$$C_1 := \{(\xi, \eta) : \xi^2 + \eta^2 = p_1\}.$$

When the two shocks  $S_{12}^-$  and  $S_{41}^+$  meet the sonic circle  $C_1$  at points  $P_3$  and  $P_1$ , respectively, our main concern is whether they bend and meet to form a diffracted shock, denoted by  $\Gamma_{\text{shock}}$ ; see Fig. 2. Since the whole configuration is symmetric with respect to the  $\eta$ -axis, we infer that  $\Gamma_{\text{shock}}$  must be vertical to  $\xi = 0$  at point  $P_2$  where the two diffracted shocks meet. We should point out here particularly that the two vortex sheets  $J_{23}^+$  and  $J_{34}^-$  and the diffracted shock  $\Gamma_{\text{shock}}$  have no influence each other during the intersection. Therefore, from now on, we *ignore* the two vortex sheets first and focus only on the diffracted shock.

Moreover, we have not excluded yet the case that the diffracted shock may degenerate partially into a portion of the inner sonic circle  $C_2$  of state (2). Once this case occurs,  $p = p_2$  on the sonic circle. It will be seen that  $p = p_2$  satisfies the oblique derivative conditions on the diffracted shock automatically.

On  $\Gamma_{\text{shock}}$ , the Rankine-Hugoniot conditions in the polar coordinates must be satisfied:

$$\begin{cases} r[u] - \cos \theta[p] = \frac{1}{r} \frac{dr}{d\theta} \sin \theta[p], \\ r[v] - \sin \theta[p] = -\frac{1}{r} \frac{dr}{d\theta} \cos \theta[p], \\ r[E] - \cos \theta[pu] - \sin \theta[pv] = \frac{1}{r} \frac{dr}{d\theta} (\sin \theta[pu] - \cos \theta[pv]), \end{cases} \quad (2.6)$$

where  $[w]$  denotes the jump of  $w$  across  $\Gamma_{\text{shock}}$ . Owing to

$$[pu] = \bar{p}[u] + \bar{u}[p],$$

with  $\bar{p}$  as the average of the two neighboring states of  $p$ , we eliminate  $[u]$  and  $[v]$  in the third equation in (2.6) to obtain

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{r^2(r^2 - \bar{p})}{\bar{p}}.$$

The shock diffraction can also be considered to be created from point  $P_2$  in two directions, which implies that  $r'(\theta) > 0$  for  $\theta \in [\frac{3\pi}{2}, \theta_1]$ , and  $r'(\theta) < 0$  for  $\theta \in [\theta_3, \frac{3\pi}{2}]$ , where  $\theta_i$  are denoted as the  $\theta$ -coordinates of points  $P_i$ ,  $i = 1, 3$ , respectively. Thus, we choose

$$\frac{dr}{d\theta} = g(p(r(\theta), \theta), r(\theta)) := \begin{cases} r\sqrt{\frac{r^2 - \bar{p}}{\bar{p}}} & \text{for } \theta \in [\frac{3\pi}{2}, \theta_1], \\ -r\sqrt{\frac{r^2 - \bar{p}}{\bar{p}}} & \text{for } \theta \in [\theta_3, \frac{3\pi}{2}]. \end{cases} \quad (2.7)$$

Moreover, it follows from (2.6) that

$$[p]^2 = \bar{p}([u]^2 + [v]^2). \quad (2.8)$$

From (2.8), taking the derivative  $r'(\theta)\partial_r + \partial_\theta$  along the shock yields the derivative boundary condition on  $\Gamma_{\text{shock}}$ :

$$\mathcal{B}p = \sum_{i=1}^2 \beta_i D_i p := \beta_1 p_r + \beta_2 p_\theta = 0, \quad (2.9)$$

where  $\beta = (\beta_1, \beta_2)$  is a function of  $(p, p_2, r(\theta), r'(\theta))$  with

$$\beta_1 = 2r' \left( \frac{r^2 - \bar{p}}{r^2} - \frac{[p]}{4\bar{p}} + \frac{\bar{p}(r^2 - p)}{r^2 p} \right),$$



$$\beta_2 = \frac{4(r^2 - \bar{p})}{r^2} - \frac{[p]}{2\bar{p}}. \quad (2.10)$$

The obliqueness becomes

$$(\beta_1, \beta_2) \cdot (1, -r'(\theta)) = -2r'(\theta)\left(1 - \frac{\bar{p}}{p}\right) \equiv: \mu.$$

Note that  $\mu$  vanishes at point  $P_2$  where  $r'(\frac{3\pi}{2}) = 0$ . When the obliqueness fails, we have

$$\beta_1 = 0, \quad \beta_2 = -\frac{[p]}{2\bar{p}} < 0,$$

owing to  $p > p_2$ .

Let  $\Gamma_{\text{sonic}}$  be the larger portion  $\widehat{P_1 P_3}$  of the sonic circle  $C_1$  of state (1). On  $\Gamma_{\text{sonic}}$ ,  $p$  satisfies the Dirichlet boundary condition:

$$p = p_1. \quad (2.11)$$

Let  $\Omega$  be the bounded domain enclosed by  $\Gamma_{\text{sonic}}$  and  $\Gamma_{\text{shock}}$ .

**Problem 2.1** (Free boundary problem). The Riemann problem for the pressure gradient system (1.3) with the Riemann initial data satisfying (1.5) can be reformulated into the following free boundary problem:

$$\begin{cases} \text{Equation (2.5)} & \text{in } \Omega, \\ \text{The derivative boundary conditions (2.9)–(2.10)} & \text{on } \Gamma_{\text{shock}}, \\ \text{The Dirichlet boundary condition (2.11)} & \text{on } \Gamma_{\text{sonic}}, \end{cases}$$

where  $\Gamma_{\text{shock}}$  is a free boundary to be determined as given by (2.7).

**2.3. Main theorem.** We now state our main theorem of this paper.

**Theorem 2.2.** *There exists a global solution  $p(r, \theta)$  of Problem 2.1 in domain  $\Omega$  with the free boundary  $r = r(\theta)$ ,  $\theta \in [\theta_3, \theta_1]$ , such that*

$$p \in C^{2,\alpha}(\Omega) \cap C^\alpha(\bar{\Omega}), \quad r \in C^{2,\alpha}((\theta_3, \theta_1)) \cap C^{1,1}([\theta_3, \theta_1]),$$

where  $\alpha \in (0, 1)$  depends only on the Riemann initial data. Moreover, the global solution  $(p(r, \theta), r(\theta))$  satisfies the following properties:

- (i)  $p > p_2$  on  $\Gamma_{\text{shock}}$ , that is, the diffracted shock  $\Gamma_{\text{shock}}$  does not meet the sonic circle  $C_2$  of state (2);
- (ii) The shock curve  $\Gamma_{\text{shock}}$  is convex in the self-similar coordinates;
- (iii) The global solution  $p(r, \theta)$  is  $C^\alpha$  up to the sonic boundary  $\Gamma_{\text{sonic}}$  and Lipschitz continuous across  $\Gamma_{\text{sonic}}$ ;
- (iv) The Lipschitz regularity of the solution across  $\Gamma_{\text{sonic}}$  from the inside of the subsonic region is optimal.

**2.4. Main strategies.** There are three main difficulties in establishing the existence of solutions of Problem 2.1:

- (i) On the sonic boundary  $\Gamma_{\text{sonic}}$ , owing to  $p_1 = r^2$ , the ellipticity of equation (2.5) degenerates;
- (ii) At point  $P_2$  where the diffracted shock  $\Gamma_{\text{shock}}$  meets the  $\eta$ -axis  $\xi = 0$ , the obliqueness of derivative boundary conditions fails, since

$$(\beta_1, \beta_2) \cdot (1, -r'(\theta)) = 0;$$

- (iii) The diffracted shock  $\Gamma_{\text{shock}}$  is a free boundary, which is not clear yet whether it would coincide with the inner sonic circle  $C_2$  of state (2).

In the proof of the existence result, we first assume that  $p \geq p_2 + \delta$  holds on  $\Gamma_{\text{shock}}$  for some  $\delta > 0$ , which means that  $\Gamma_{\text{shock}}$  cannot coincide with the sonic circle  $C_2$  of state (2). This fact is eventually true and will be proved in §4. For the second difficulty, we may express this as a one-point Dirichlet condition  $p(P_2) = \hat{p}$  by solving

$$2r(\theta_2) = p(r(\theta_2), \theta_2) + p_2.$$

We now illustrate a sketch of the proof for the existence of solutions in the  $(r, \theta)$ -coordinates established in §3. We divide the existence proof into four steps:

Step 1. Since equation (2.5) degenerates on the sonic boundary, we consider the regularized operator:

$$Q^\varepsilon = Q + \varepsilon \Delta_{(\xi, \eta)}.$$

We first fix a diffracted shock boundary  $\Gamma_{\text{shock}}$ , and then linearize the equation and the derivative boundary condition. We employ the techniques developed in a series of works in [10, 25, 29, 38] to establish the existence result for the linear fixed mixed-type boundary problem for the regularized equation in the polar coordinates.

Step 2. Based on the estimates of solutions to the linear fixed boundary problem obtained in Step 1, we prove the existence of a solution of the nonlinear fixed boundary problem via the Schauder fixed point theorem.

Step 3. We apply the Schauder fixed point theorem again to obtain the existence of a solution of the free boundary problem with the oblique derivative boundary condition for the regularized elliptic equation. We conclude that the diffracted shock never meets the sonic circle  $C_2$  of state  $p_2$ .

Step 4. Finally, we study the limiting solution as the elliptic regularization parameter  $\varepsilon$  tends to 0 and complete the proof of the existence of solutions of Problem 2.1.

In §4, we introduce the new coordinates  $(x, y) = (r_1 - r, \theta - \theta_1)$ , which can flatten the sonic boundary. It is shown that the optimal regularity of solutions across the sonic boundary is of  $C^{0,1}$ -regularity. The most interesting point is the position of the diffracted shock, which is a free boundary. This kind of free boundary problems occurs in many applications, such as the shock reflection-diffraction problem [4, 5, 10, 11, 12, 25, 38], the Prandtl-Meyer shock configuration problem [3, 20], among others. In §5, we establish a corresponding theorem for the existence and regularity of solutions of the 2-D Riemann problem for the pressure gradient system (1.3).

**3. Global existence of solutions of the free boundary problem.** In this section, we follow the strategies introduced in §2.4 to obtain the global existence of a solution of the free boundary problem, Problem 2.1. We first introduce the weighted norm used in this paper.

Let  $\Omega' := \overline{\Omega} \setminus \overline{\Gamma_{\text{shock}}}$ . For  $P \in \{P_1, P_3\}$ , we introduce the corner regions:

$$\Omega_P(\sigma) := \{x \in \Omega : \text{dist}(x, P) \leq \sigma\}, \quad \Omega_V(\sigma) := \Omega_{p_1}(\sigma) \cup \Omega_{p_3}(\sigma).$$

Define

$$\begin{aligned} \Gamma'(\sigma) &:= \{P \in \Gamma_{\text{shock}} : \text{dist}(P, P_1) > \sigma, \text{dist}(P, P_3) > \sigma\}, \\ \Gamma(\sigma) &:= \Omega \cap \left( \bigcup_{P \in \Gamma'(\sigma)} B_\sigma(P) \right), \end{aligned}$$

where  $B_\sigma(P)$  is a ball of radius  $\sigma$  centered at  $P$ . Hence,  $\Gamma(\sigma)$  is a region that is close to  $\Gamma_{\text{shock}}$ , but does not contain corners  $P_1$  and  $P_3$ .

We introduce the weighted norm

$$\|u\|_a^b := \sup_{\sigma > 0} \{ \sigma^{a+b} \|u\|_{a, \bar{\Omega} \setminus (\Gamma(\sigma) \cup \Omega_V(\sigma))} \} \quad \text{for any } a > 0 \text{ and } a + b \geq 0. \quad (3.1)$$

The set of functions with finite norm  $\|\cdot\|_a^b$  is denoted by  $C_b^a(\Omega)$ .

We now prove the existence of a solution of Problem 2.1 in the following four subsections.

**3.1. Regularized linear fixed boundary value problem.** For a fixed  $\varepsilon \in (0, 1)$ , we consider the regularized operator  $Q^\varepsilon := Q + \varepsilon \Delta_{(\xi, \eta)}$ . The equation for  $p$  in the subsonic region is

$$Q^\varepsilon p := (p - r^2 + \varepsilon)p_{rr} + \frac{p + \varepsilon}{r^2}p_{\theta\theta} + \frac{p + \varepsilon}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0. \quad (3.2)$$

Since the position of the free boundary is not known *a priori*, we impose a cut-off function in (3.2). Let  $\zeta(\cdot) \in C^\infty(\mathbb{R})$  such that

$$\zeta(s) = \begin{cases} s & \text{if } s \geq 0, \\ -\frac{\varepsilon}{2} & \text{if } s < -\varepsilon, \end{cases} \quad (3.3)$$

and  $|\zeta'(s)| \leq 1$ . We then consider the following modified governing equation:

$$Q^{\varepsilon,+} p := (\zeta(p - r^2) + \varepsilon)p_{rr} + \frac{p + \varepsilon}{r^2}p_{\theta\theta} + \frac{p + \varepsilon}{r}p_r + \frac{1}{p}(rp_r)^2 - 2rp_r = 0. \quad (3.4)$$

We define the iteration set  $\mathcal{R}^\varepsilon$  for shock  $r(\theta)$ , which is a closed, convex subset of the Hölder space  $C^{1,\alpha_1}([\theta_3, \theta_1])$  as follows:

**Definition 3.1.** Let  $r_i = \sqrt{p_i}$  be the radius of the sonic circle  $C_i$  of state  $p_i$ ,  $i = 1, 2$ . The iteration set  $\mathcal{R}^\varepsilon \subset C^{1,\alpha_1}([\theta_3, \theta_1])$  consists of elements  $r(\theta)$  such that

- (R1)  $r(\theta_1) = r(\theta_3) = r_1$ ;
- (R2)  $r_2 + \delta \leq r(\theta) \leq r_1$  for all  $\theta \in [\theta_3, \theta_1]$ ;
- (R3)  $0 < r'(\theta) \leq r'(\theta_1)$  for  $\theta \in (\frac{3\pi}{2}, \theta_1]$ ,  $r'(\theta_3) \leq r'(\theta) < 0$  for  $\theta \in [\theta_3, \frac{3\pi}{2})$ ,  
and  $r'(\frac{3\pi}{2}) = 0$ .

In order to linearize the equation and the boundary conditions, we define a function space  $\mathcal{W}$ .

**Definition 3.2.** The function space  $\mathcal{W} \subset C_{(-\gamma_1)}^2$  consists of elements  $\omega$  such that

- (W1)  $p_2 < \hat{p}^\varepsilon \leq \omega \leq p_1$ ,  $\omega(P_2) = \hat{p}^\varepsilon$  and  $\omega = p_1$  on  $\Gamma_{\text{sonic}}$ ;
- (W2)  $\max\{\|\omega\|_{\alpha_0}, \|\omega\|_{1+\mu, \Gamma(d_0)}\} \leq K_0$ ;
- (W3)  $\|\omega\|_2^{(-\gamma_1)} \leq K_1$ .

The values of  $\gamma_1, \alpha_0, \mu \in (0, 1)$ , and constants  $d_0, K_0$ , and  $K_1$  will be specified later. The function set  $\mathcal{W}$  is clearly closed, bounded, and convex.

For a given  $r(\theta) \in \mathcal{R}^\varepsilon$ , let  $\Gamma_{\text{shock}}^\varepsilon$  be the fixed shock defined by

$$\Gamma_{\text{shock}}^\varepsilon := \{(r(\theta), \theta) : \theta_3 \leq \theta \leq \theta_1\}.$$

The nonlinear equation (3.2) and the boundary conditions (2.9)–(2.10) are now replaced by the linearized equation:

$$L^{\varepsilon,+} p := (\zeta(\omega - r^2) + \varepsilon)p_{rr} + \frac{\omega + \varepsilon}{r^2}p_{\theta\theta} + \frac{\omega + \varepsilon}{r}p_r + \frac{r^2\omega_r}{\omega}p_r - 2rp_r = 0, \quad (3.5)$$

and the linearized oblique derivative boundary condition on  $\Gamma_{\text{shock}}^\varepsilon$ :

$$\mathcal{B}p := \beta_1(\omega)p_r + \beta_2(\omega)p_\theta = 0, \quad (3.6)$$

with  $\omega \in \mathcal{W}$ , and  $\beta_i, i = 1, 2$ , given in (2.10). Because of the bound of (W1), equation (3.5) is uniformly elliptic in  $\Omega$  with ellipticity ratio depending on the Riemann initial data and  $\varepsilon$ .

The other boundary condition is

$$p = p_1 \quad \text{on } \Gamma_{\text{sonic}}. \quad (3.7)$$

Now we consider the following mixed-type boundary value problem for the linear elliptic equation (3.5).

**Problem 3.1** (Linear fixed boundary value problem). *Seek a solution  $p$  of the linear elliptic equation (3.5) subject to the derivative boundary condition (3.6) on given  $\Gamma_{\text{shock}}^\varepsilon$  and the Dirichlet boundary condition (3.7) on  $\Gamma_{\text{sonic}}$ .*

There have been several papers on the tangential oblique derivative problems for linear equations; see [19, 23, 31, 34] and the references cited therein. However, we can not apply them directly because the obliqueness of the derivative boundary condition fails at point  $P_2$ . The main point is to find a way to remove the degeneracy. We have the following result.

**Lemma 3.3** (Existence of solutions of Problem 3.1). *Assume that  $\Gamma_{\text{shock}}^\varepsilon$  is given by  $r(\theta) \in \mathcal{R}^\varepsilon$  for some  $\alpha_1 \in (0, 1)$ , and  $\omega \in \mathcal{W}$  for given  $\alpha_0, \gamma_1, d_0, K_0$ , and  $K_1$ . Then there exist  $\gamma_V, \alpha_\Omega \in (0, 1)$  depending on  $\varepsilon$ , but independent of  $\alpha_1$  and  $\gamma_1$ , such that there is a solution*

$$p^\varepsilon \in C^1(\bar{\Omega}) \cap C^{2,\alpha}(\Omega') \cap C^\gamma(\Omega_V(d)) \quad (3.8)$$

of Problem 3.1 for any  $\alpha \leq \alpha_\Omega$ ,  $\gamma \leq \gamma_V$ , and  $d \leq d_0$ . Furthermore, the solution,  $p^\varepsilon$ , satisfies the following estimates:

$$\begin{aligned} \|p^\varepsilon\|_{\gamma, \Omega_V(d)} &\leq M_1 \|p^\varepsilon\|_0 && \text{for any } \gamma \leq \gamma_V, \\ \|p^\varepsilon\|_{2,\alpha; \Omega'_{\text{loc}}} &\leq M_2 \|p^\varepsilon\|_0 && \text{for any } \alpha \leq \alpha_\Omega, \end{aligned}$$

where  $M_1$  is independent of  $K_0$  and  $K_1$ , and  $M_2$  is independent of  $K_1$  but depends on  $K_0$ .

*Proof.* It suffices to prove the local existence at point  $P_2$ , where the obliqueness of the derivative boundary condition fails. Let  $B$  be a sufficiently small neighborhood of  $P_2$  with smooth boundary. Let  $L_\sigma$  be the line with  $\sigma$ -distance from point  $P_2$  upward. Let  $\Omega_\sigma$  be the domain enclosed by  $\partial B$ ,  $\Gamma_{\text{shock}}^\varepsilon$ , and  $L_\sigma$ . Now we consider the following boundary value problem:

$$\begin{cases} Q^{\varepsilon,+} p = 0 & \text{in } \Omega_\sigma, \\ Bp = 0 & \text{on } \partial\Omega_\sigma \cap \Gamma_{\text{shock}}^\varepsilon, \\ p = h & \text{on } \partial B \cap \Omega, \\ p = \hat{p}^\varepsilon & \text{on } L_\sigma, \end{cases} \quad (3.9)$$

where  $h$  is a smooth function satisfying that  $\hat{p}^\varepsilon < h \leq p_1$ . Following [29], there exists a solution

$$p_\sigma \in C(\bar{\Omega} \cap \bar{\hat{B}}) \cap C^{2,\alpha}(\Omega_\sigma \cap \hat{B})$$

for a smaller neighborhood  $\hat{B}$  of point  $P_2$ . By the maximum principle,  $p_\sigma$  converges locally in  $C^2(\Omega \cap \hat{B})$  to a solution  $p \in C^{2,\alpha}(\Omega \cap \hat{B})$  as  $\sigma \rightarrow 0+$ .

Next we construct a barrier function to prove the continuity of  $p$  at  $P_2$ . Define

$$u = \hat{p}^\varepsilon + C(1 - e^{-l(\theta - \frac{3\pi}{2})}),$$

where  $C > 0$  and  $l > 0$  are specified later. For the equation, we have

$$Q^{\varepsilon,+}u = -\frac{Cl^2(u+\varepsilon)}{r^2}e^{-l(\theta-\frac{3\pi}{2})} < 0.$$

It is direct to see that  $u \geq \hat{p}^\varepsilon$  on  $L_\varepsilon$ . Choose  $C$  large enough such that

$$u \geq \sup |h| \quad \text{on } \partial\hat{B} \cap \Omega.$$

For the oblique derivative boundary condition along  $\partial\Omega_\sigma \cap \Gamma_{\text{shock}}^\varepsilon$ , we find that

- $(\beta_1, \beta_2) \cdot \nu > 0$  and  $\mathcal{B}u > 0$  for  $\theta < \frac{3\pi}{2}$ ,
- $(\beta_1, \beta_2) \cdot \nu < 0$  and  $\mathcal{B}u < 0$  for  $\theta > \frac{3\pi}{2}$ ,

where  $\nu$  denotes the outward normal to  $\partial\Omega_\sigma \cap \Gamma_{\text{shock}}^\varepsilon$ . Thus, by the comparison principle, we have

$$\hat{p}^\varepsilon \leq p \leq u,$$

which implies that  $p$  is continuous at  $P_2$ .

Then we can follow the arguments in [21, 30] to prove the existence of a solution away from point  $P_2$  and to obtain the estimates near both points  $P_1$  and  $P_3$ , and both boundaries  $\Gamma_{\text{sonic}}$  and  $\Gamma_{\text{shock}}^\varepsilon \setminus \Omega_{P_2}(d)$ . This completes the proof.  $\square$

Since the interior Schauder estimates can be further applied, any solution in  $C_{\text{loc}}^{2,\alpha}(\Omega')$  is actually in  $C_{\text{loc}}^3(\Omega)$ . We next establish the Hölder gradient estimate of the solution on  $\Gamma_{\text{shock}}^\varepsilon$ .

**Lemma 3.4.** *Assume that  $\Gamma_{\text{shock}}^\varepsilon$  is given by  $\{(r(\theta), \theta)\}$  with  $r(\theta) \in \mathcal{R}^\varepsilon$  for some  $\alpha_1 \in (0, 1)$ . Then there exists a positive constant  $d_0$  such that, for every  $d \leq d_0$ , any solution  $p^\varepsilon \in C_{\text{loc}}^1(\Omega \cup \Gamma_{\text{shock}}) \cap C_{\text{loc}}^3(\Omega)$  of Problem 3.1 satisfies*

$$\|p^\varepsilon\|_{1+\mu, \Gamma(d) \setminus \Omega_V(d_0)} \leq C(\varepsilon, \alpha_1, \mu, \gamma_1, K_0, K_1) \|p^\varepsilon\|_0 \quad (3.10)$$

for any  $\mu < \min\{\alpha_1, \gamma_1\}$ .

*Proof.* Away from a neighborhood  $B_{d_0}(P_2)$  of  $P_2$ , we can employ Theorem 6.30 in [21] to obtain (3.10) in  $\Gamma(d) \setminus (\cup_{i=1,2,3} B_{d_0}(P_i))$ . For the estimates near  $P_2$ , we follow the technique used in [6]. The main idea is that, for a given solution  $p$  of the linear problem (3.5)–(3.7), we define

$$u = \frac{p}{1 + \|Dp\|_0}, \quad z = \mathcal{B}u := \sum_{i=1}^2 \beta_i(P) D_i u.$$

Taking  $f(\zeta) = f_0 \zeta^\mu$  for any  $\mu < \gamma_2 := \min\{\alpha_1, \gamma_1\}$ , we can prove that  $\pm f(\zeta)$  are barrier functions for  $z$  on  $B_{d_0}(P_2) \cap \bar{\Omega}$  such that  $|z| \leq f$ . The barrier functions lead to

$$|D(z + f)| \leq \|(z + f)\|_{1+\gamma_2}^{1-\mu} d^{\mu-1} \leq C d^{\mu-1} \quad \text{for } d < d_0,$$

which implies that  $\|u\|_{1+\mu} \leq C$ .

It follows from the interpolation inequality that

$$\|p\|_{1+\mu} \leq C(1 + \|Dp\|_0) \leq C(1 + \alpha \|p\|_{1,\mu} + C_\alpha \|p\|_0)$$

with small  $\alpha > 0$ , so that (3.10) holds. Also see [6] for more details.  $\square$

**3.2. Regularized nonlinear fixed boundary value problem.** This subsection is devoted to the proof of the existence of solutions of the nonlinear equation (3.2) with a fixed boundary  $r(\theta) \in \mathcal{R}^\varepsilon$ . We have the following lemma.

**Lemma 3.5.** *For each  $\varepsilon \in (0, 1)$ , given  $r^\varepsilon(\theta) \in \mathcal{R}^\varepsilon$ , there exists a solution  $p^\varepsilon \in C_{-\gamma}^{2,\alpha}(\Omega)$  of equation (3.2) with conditions (2.9)–(2.10) for  $r^\varepsilon(\theta)$  and (3.7) such that*

$$p_2 < \bar{p}^\varepsilon \leq p^\varepsilon < p_1, \quad p^\varepsilon > r^2 \quad \text{in } \bar{\Omega}. \quad (3.11)$$

Moreover, for some  $d_0 > 0$ ,  $p^\varepsilon(r, \theta)$  satisfies

$$\|p^\varepsilon\|_{\gamma, \Gamma(d_0) \cup \Omega_V(d_0)} \leq K_1, \quad (3.12)$$

where  $\gamma$  and  $K_1$  depend on  $\varepsilon$ ,  $\gamma_V$ , and  $K$ , but independent of  $\alpha_1$ . In addition,  $p^\varepsilon$  is monotonically increasing on  $r^\varepsilon(\theta)$  from  $P_2$  to  $P_1$ , and decreasing from  $P_3$  to  $P_2$ .

*Proof.* For simplicity, we suppress the  $\varepsilon$ -dependence in the proof. Using the Hölder gradient bounds for the linear problem, we establish the existence results for the nonlinear fixed boundary problem via the Schauder fixed point theorem.

For any function  $w \in \mathcal{W}$ , we define a mapping

$$T : \mathcal{W} \subset C_{(-\gamma_1)}^2 \rightarrow C_{(-\gamma_1)}^2 \quad (3.13)$$

by  $Tw = p$ , where  $p$  is the solution of the linear regularized fixed boundary value problem (3.5)–(3.7) solved in Lemma 3.3. It is direct to see that  $T$  maps  $\mathcal{W}$  into a bounded set in  $C_{(-\gamma_v)}^2$ , where  $\gamma_v$  is determined in Lemma 3.3. Since  $\gamma_v$  is independent of  $\gamma_1$ , we may take  $\gamma_1 = \frac{\gamma_v}{2}$  so that  $T(\mathcal{W})$  is precompact in  $C_{(-\gamma_1)}^2$ .

Next, we show that  $T$  maps  $\mathcal{W}$  into itself. First, (W1) is satisfied by the boundary conditions and the maximum principle. (W2) is satisfied by the standard interior and boundary Hölder estimates for elliptic equations. In order to prove that  $p$  satisfies (W3), it suffices to prove that there exists  $K > 0$  such that

$$\sup_{\sigma > 0} (\sigma^{2-\gamma_1} \|p\|_{2, \bar{\Omega} \setminus (\Gamma(\sigma) \cup \Omega_V(\sigma))}) < K, \quad (3.14)$$

under the condition that  $\|w\|_2^{(-\gamma_1)} \leq K$ . Lemma 3.4 implies that

$$d^{2-\gamma_1} \|p\|_2 \leq d^{1-\gamma_1+\mu} C \quad \text{for all } d \leq d_0,$$

where  $C$  depends on  $K$ ,  $\alpha_1$ , and  $\gamma_1$ . Moreover, by the interpolation inequality, we can obtain

$$d^{2-\gamma_1} \|p\|_2 \leq K_V \quad \text{for all } d < d_V,$$

where  $\gamma_1 = \frac{\gamma_V}{2}$ , and  $K_V$  is independent of  $K$ . Therefore, we can choose sufficiently small  $\hat{d} \leq \frac{\min\{d_0, d_V\}}{2}$  such that

$$\hat{d}^{1-\gamma_1+\mu} C \leq K.$$

For domain  $\bar{\Omega} \setminus (\Gamma(\sigma) \cup \Omega_V(\sigma))$  with  $\sigma > \hat{d}$ , the solution is smooth, and its  $C^2$ -norm bound is independent of  $K$  by the uniform Hölder estimate. Therefore, (3.14) is satisfied, and parameters  $K$ ,  $K_0$ , and  $\alpha_0$  defining  $\mathcal{W}$  have been chosen such that  $T$  maps  $\mathcal{W}$  into itself.

Finally, by the Schauder fixed point theorem, there exists a fixed point  $p$  such that

$$Tp = p \in C_{(-\gamma_1)}^2.$$

Then  $p$  is a solution as required.

We now prove the monotonicity of  $p$  along  $r = r(\theta)$  from  $P_2$  to  $P_1$  by contradiction. We may label the points on  $r = r(\theta)$  from  $P_2$  to  $P_1$  by their respective

$\xi$ -coordinates and refer to the intervals by the labels, without ambiguity. The lack of increasing monotonicity of  $p$  along  $P_2P_1$  implies that there exist two points  $A$  and  $B$  with  $P_2 < A < B < P_1$  such that  $p(A) > p(B)$ . Then we deduce that

- (a) In  $(A, P_1)$ , there exists  $\hat{C}$  with  $p(\hat{C}) = \max_{[A, P_1]} p$ ;
- (b) In  $(P_2, \hat{C})$ , there exists  $D$  with  $p(D) = \min_{[P_2, \hat{C}]} p$ . Thus, we can find two

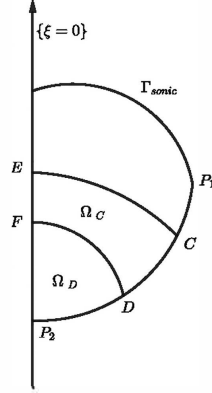


FIGURE 4. Hypothetical curves

points  $C$  and  $D$  on  $\Gamma_{\text{shock}}$  with  $D < C$  such that

- (i)  $p(D) \leq p \leq p(P_2)$  on  $[P_2, D]$ ;
- (ii)  $p(D) \leq p \leq p(C)$  on  $[D, C]$ ;
- (iii)  $p(P_1) \leq p \leq p(C)$  on  $[C, P_1]$ .

Here property (ii) may not hold with  $C = \hat{C}$  because  $p(\hat{C})$  is the maximum value only on interval  $[A, P_1]$ . If  $D < A$  and there is a point in  $(D, A)$  at which  $p > p(\hat{C})$ , then we let  $C$  be the point at which  $p$  achieves its maximum in this interval. Otherwise, we let  $C = \hat{C}$ . Then all the three properties hold.

Now we look at  $p$  in domain  $\Omega' := \Omega \cap \{\xi > 0\}$ . The idea is to partition  $\Omega'$  into three subdomains by two curves  $\Gamma_C$  and  $\Gamma_D$  from  $C$  and  $D$  to points  $E$  and  $F$  on  $\{\xi = 0\}$ , respectively, such that  $p(E) > p(F)$ . We want to deduce that there is a point  $X_0$  on  $\{\xi = 0\}$  at which  $p$  attains a minimum on either subdomain  $\Omega_C$  or domain  $\Omega_D$ , which violates the Hopf maximum principle, since  $\partial_\xi p = 0$  at  $\{\xi = 0\}$  according to the symmetry.

Let

$$\mu = \frac{1}{4} \min\{p(C) - p(P_1), p(C) - p(D), p(P_2) - p(D)\}.$$

We want to construct the Lipschitz curves on which  $p$  satisfies

$$\begin{aligned} p(E) &\geq p \geq p(C) - \mu \quad \text{on } \Gamma_C, & p(E) &> p(C), \\ p(F) &\leq p \leq p(D) + \mu \quad \text{on } \Gamma_D, & p(F) &< p(D). \end{aligned} \quad (3.15)$$

Since  $p \in C^\alpha(\bar{\Omega})$ , we have

$$p(X_1) - p(X_2) \leq M|X_1 - X_2|^\alpha \quad \text{for any } X_1, X_2 \in \bar{\Omega},$$

for some  $M > 0$ . Thus, on any ball with radius  $r > 0$ , we have

$$\text{osc}(p) \leq 2Mr^\alpha.$$

Let  $R = (\frac{\mu}{2M})^{1/\alpha}$ . Then  $\text{osc}_{B_R \cap \Omega}(p) \leq \mu$ .

We construct  $\Gamma_C$  as follows: In  $B_R(C) \cap \Omega'$ , let  $Y_1$  be a point at which  $p$  attains its maximum value in  $\overline{B_R(C)}$ . Then the first segment of  $\Gamma_C$  is the straight line from  $C$  to  $Y_1$ . On the segment, we have

$$p(Y) \geq p(C) - \mu, \quad p(Y) \leq p(Y_1).$$

Now we continue inductively, forming a sequence of line segments with corners at  $\{Y_i\}$  (take  $Y_0 = C$ ), along which  $p \geq p(C) - \mu$  and  $p(Y_1) \leq p(Y_2) < \dots$ . Since the domain is finite, the process must terminate after finite steps when we reach a point  $E \in \partial\Omega'$ . By construction,  $\Gamma_C$  has the properties indicated in (3.15). Similarly, we construct  $\Gamma_D$  with termination point  $F \in \partial\Omega'$ .

We now show that points  $E$  and  $F$  must lie on  $\{\xi = 0\}$ . First, the two curves can not cross each other, because  $p(X) \geq p(C) - \mu > p(D) + \mu$  at every point  $X \in \Gamma_C$ , while  $p(X) \leq p(D) + \mu$  at every point  $X \in \Gamma_D$ . Also,  $\Gamma_C$  can not terminate at  $\Gamma_{\text{sonic}}$  where  $p > p(P_1) - \mu > p(D) - \mu$ . Furthermore,  $\Gamma_C$  can not come back to  $\Gamma_{\text{shock}}$  in  $[D, C]$  or  $[C, P_1]$  where  $p \leq p(C)$ . Finally,  $E$  can not lie in segment  $[P_2, D]$  because this would trap  $\Gamma_D$  in a region where  $p \leq p(D)$ , which contradicts the fact that  $D$  is a local minimum in  $\Omega$ . Therefore,  $E \in \{\xi = 0\}$ . Similarly,  $F$  must lie on  $\{\xi = 0\}$  between  $P_2$  and  $E$ .

Now we reach the final contradiction. Since  $p(F)$  is smaller than  $p(P_2)$  and  $p(E)$ , then there must be a point  $X_0$  along boundary  $P_2E$  at which  $p$  attains its minimum. It is obvious that  $X_0$  can not be a local minimum for domain  $\Omega'$  by the Hopf maximum principle. However, along the entire boundary of domain  $P_2DCEFP_2$ ,  $p \geq p(X_0)$  which implies that it is a minimum. This is a contradiction, which implies that  $p$  is monotonically increasing, pointing from  $P_2$  to  $P_1$ . Similarly, the monotonicity of  $p$  from  $P_3$  to  $P_2$  can also be proved.  $\square$

**3.3. Regularized nonlinear free boundary problem.** We now prove the existence of a solution of the regularized free boundary problem (3.2), (2.9)–(2.10), and (3.7). For each  $r(\theta) \in \mathcal{R}^\varepsilon$ , using solution  $p$  of the nonlinear fixed boundary value problem given by Lemma 3.3, we define the map,  $J$ , on  $\mathcal{R}^\varepsilon$ :

$$\tilde{r} = Jr := \begin{cases} r_1 + \int_{\theta_1}^{\theta} g(r(s), s, p(s, r(s))) ds & \text{for } \theta \in [\frac{3\pi}{2}, \theta_1], \\ r_1 + \int_{\theta}^{\theta_3} g(r(s), s, p(s, r(s))) ds & \text{for } \theta \in (\theta_3, \frac{3\pi}{2}]. \end{cases} \quad (3.16)$$

First, we check that  $J$  maps  $\mathcal{R}^\varepsilon$  into itself. Property (R1) follows from (3.16). By the definition of  $g$  and  $r^2(\frac{3\pi}{2}) = \bar{p}$ , property (R3) holds. In order to prove property (R2), we need to make clear the position of the diffracted shock as a free boundary.

There are three possibilities for the position of the diffracted shock  $\Gamma_{\text{shock}}$ :

- (i)  $r_2 < r(\theta) \leq r_1$  for all  $\theta \in [\theta_3, \theta_1]$ ;
- (ii) There exists  $\theta^* > 0$  such that  $r(\theta^*) = r_2$  for all  $\theta \in [\frac{3\pi}{2} - \theta^*, \frac{3\pi}{2} + \theta^*]$ , and  $r_2 < r(\theta)$  for all  $\theta \in [\theta_3, \frac{3\pi}{2} - \theta^*) \cup (\frac{3\pi}{2} + \theta^*, \theta_1]$ ;
- (iii)  $r(\frac{3\pi}{2}) = r_2$ , and  $r_2 < r(\theta) \leq r_1$  for all  $\theta \in [\theta_3, \theta_1] \setminus \{\frac{3\pi}{2}\}$ , where  $r_i = \sqrt{p_i}$ , and  $\theta_i$  are the  $\theta$ -coordinates of point  $P_i$ ,  $i = 1, 2, 3$ .

Let

$$d = \text{dist}\{C_2, \Gamma_{\text{shock}}\} = |OP_2| - r_2.$$

**Proposition 3.1.** *Let  $(p, r)$  be the solution of the regularized free boundary problem (3.2), (2.9)–(2.10), and (3.7). Then  $d \geq \delta > 0$  for some constant  $\delta > 0$  depending on the Riemann initial data, when  $\varepsilon > 0$  is sufficiently small. This means that*



cases (ii)–(iii) do not occur, so that the diffracted shock does not meet the sonic circle  $\{r = r_2\}$ .

*Proof.* We first prove (ii) via the method of contradiction. For  $\theta_0 \in [\frac{3\pi}{2} - \theta^*, \frac{3\pi}{2} + \theta^*]$ , let  $\mathcal{N}$  be a small interior neighborhood of point  $(r_2, \theta_0) \in \Gamma_{\text{shock}}$ . We now prove that the optimal regularity of  $p$  in  $\mathcal{N}$  near  $\Gamma_{\text{shock}}$  is  $C^{1/2}$ .

1. We introduce the barrier function

$$w(r, \theta) = p_2 + A_1(r_2 - r)^{\frac{1}{2}} - B_1(r_2 - r)^\beta + D_1(\theta - \theta_0)^2,$$

where  $A_1, B_1, D_1 > 0$ , and  $\beta \in (\frac{1}{2}, 1)$  will be specified later. Let

$$Qw := (p - r^2 + \varepsilon)w_{rr} + \frac{p + \varepsilon}{r^2}w_{\theta\theta} + \frac{p + \varepsilon}{r}w_r + \frac{1}{p}(rw_r)^2 - 2rw_r, \quad (3.17)$$

and let  $\hat{Q}w$  be obtained via replacing the coefficient of  $w_{rr}$  in  $Qw$  by  $w - r^2$ :

$$\hat{Q}w := Qw + (w - p - \varepsilon)w_{rr}.$$

A direct calculation yields

$$\begin{aligned} \hat{Q}w = & \left( \left( \frac{1}{4} - \beta^2 \right) A_1 B_1 (r_2 - r)^{\beta - \frac{3}{2}} + O_1 \right) - \frac{A_1^2}{4p} (p - r^2) (r_2 - r)^{-1} \\ & + \left( B_1^2 \beta (2\beta - 1) (r_2 - r)^{2\beta - 2} + O_2 \right) + \left( -\frac{A_1 D_1}{4} (r_2 - r)^{-\frac{3}{2}} (\theta - \theta_0)^2 + O_3 \right), \end{aligned}$$

where

$$\begin{aligned} O_1 = & \frac{p - r^2}{p} A_1 B_1 \beta (r_2 - r)^{\beta - \frac{3}{2}} + \frac{A_1 (2r^2 - p)}{2r} (r_2 - r)^{-\frac{1}{2}} + \frac{p - 2r^2}{r} B_1 \beta (r_2 - r)^{\beta - 1} \\ & - (p_2 - r^2) \left( \frac{A_1}{4} (r_2 - r)^{-\frac{3}{2}} + B_1 \beta (\beta - 1) (r_2 - r)^{\beta - 2} \right) + \frac{2p D_1}{r^2}, \\ O_2 = & -\frac{B_1^2 \beta^2}{p} (p - r^2) (r_2 - r)^{2\beta - 2}, \\ O_3 = & B_1 D_1 \beta (1 - \beta) (r_2 - r)^{\beta - 2} (\theta - \theta_0)^2. \end{aligned}$$

Notice that there exists  $\alpha \in (0, \frac{1}{2})$  such that  $p - r^2 \leq (r_2 - r)^\alpha$ . Then

$$\left| -\frac{A_1^2}{4p} (p - r^2) (r_2 - r)^{-1} \right| \leq C(p_1, p_2) A_1^2 (r_2 - r)^{\alpha - 1}.$$

Choose  $\beta$  such that  $\beta - \frac{3}{2} < \alpha - 1$ , i.e.,  $\alpha > \beta - \frac{1}{2}$ , so that

$$\left( \beta^2 - \frac{1}{4} \right) A_1 B_1 (r_2 - r)^{\beta - \frac{3}{2}} > 3C(p_1, p_2) A_1^2 (r_2 - r)^{\alpha - 1},$$

which implies

$$B_1 > A_1 C(p_1, p_2, \beta) (r_2 - r)^{\alpha - \beta + \frac{1}{2}}. \quad (3.18)$$

On the other hand, if  $r_2 - r$  is small enough,

$$3B_1^2 \beta (2\beta - 1) (r_2 - r)^{2\beta - 2} < \left( \beta^2 - \frac{1}{4} \right) A_1 B_1 (r_2 - r)^{\beta - \frac{3}{2}},$$

which implies

$$A_1 > \frac{3\beta(2\beta - 1)B_1}{\beta^2 - \frac{1}{4}} (r_2 - r)^{\beta - \frac{1}{2}}. \quad (3.19)$$

We can choose  $A_1$  and  $D_1$  such that

$$w(r, \theta) > p_2 + \frac{A_1}{2} (r_2 - r)^{\frac{1}{2}} + D_1 (\theta - \theta_0)^2 > p \quad \text{on } \partial\mathcal{N} \setminus \{r = r_2\}. \quad (3.20)$$

Moreover, we see that

$$w(r, \theta) = p_2 + D_1(\theta - \theta_0)^2 \geq p \quad \text{on } \partial\mathcal{N} \cap \{r = r_2\}, \quad (3.21)$$

which implies that

$$w(r, \theta) \geq p \quad \text{on } \partial\mathcal{N}. \quad (3.22)$$

On the other hand, we take  $B_1$  sufficiently small such that (3.18)–(3.19) hold. Finally, we have

$$\hat{Q}w < 0 \quad \text{in } \mathcal{N}.$$

Moreover, for sufficiently small  $r_2 - r$ ,

$$\partial_{rr}w(r, \theta) = -\frac{1}{4}A_1(r_2 - r)^{-\frac{3}{2}} + B_1\beta(1 - \beta)(r_2 - r)^{\beta-2} < 0.$$

Assume now that there exists a non-empty open subset  $\mathcal{N}_1 = \{(r, \theta) \in \mathcal{N}, p > w\} \subset \mathcal{N}$ . By the continuity of function  $p - w$ , there exist a maximum point  $P_* \in \mathcal{N}_1$  and a small neighborhood  $\hat{\mathcal{N}}_1 \subset \mathcal{N}_1$  such that

$$p - w > \epsilon \quad \text{on } \hat{\mathcal{N}}_1,$$

when  $\epsilon > 0$  is sufficiently small. Then

$$Q(p - w) = Qp - \hat{Q}w + (\hat{Q}w - Qw) = -\hat{Q}w + (w - p - \epsilon)w_{rr} \geq -\hat{Q}w > 0 \quad \text{in } \hat{\mathcal{N}}_1; \quad (3.23)$$

in particular,

$$Q(p - w)(P_*) > 0 \quad \text{in } \hat{\mathcal{N}}_1. \quad (3.24)$$

On the other hand, at the maximum point  $P_* \in \hat{\mathcal{N}}_1$ , the structure of operator  $Q$  leads to

$$Q(p - w)(P_*) \leq 0,$$

which is a contradiction to (3.24), so that the open subset  $\mathcal{N}_1 \subset \mathcal{N}$  must be empty. Thus,  $p \leq w$  in  $\mathcal{N}$ .

2. Next, we take

$$v(r, \theta) = p_2 + A_2(r_2 - r)^{\frac{1}{2}} + B_2(r_2 - r)^\beta - D_2(\theta - \theta_0)^2,$$

where  $A_2, B_2, D_2 > 0$  and  $\frac{1}{2} < \beta < 1$ . Then

$$\begin{aligned} \hat{Q}v = & \left( (\beta^2 - \frac{1}{4})A_2B_2(r_2 - r)^{\beta-\frac{3}{2}} + \overline{O}_1 \right) - \frac{A_2^2}{4p}(p - r^2)(r_2 - r)^{-1} \\ & + (B_2^2\beta(2\beta - 1)(r_2 - r)^{2\beta-2} + \overline{O}_2) + \left( \frac{A_1D_1}{4}(r_2 - r)^{-\frac{3}{2}}(\theta - \theta_0)^2 + \overline{O}_3 \right), \end{aligned}$$

where

$$\begin{aligned} \overline{O}_1 = & -\frac{p-r^2}{p}A_2B_2\beta(r_2-r)^{\beta-\frac{3}{2}} + \frac{A_2(2r^2-p)}{2r}(r_2-r)^{-\frac{1}{2}} + \frac{2r^2-p}{r}B_2\beta(r_2-r)^{\beta-1} \\ & - (p-r^2)\left(\frac{A_2}{4}(r_2-r)^{-\frac{3}{2}} + B_2\beta(1-\beta)(r_2-r)^{\beta-2}\right) - \frac{2pD_2}{r^2}, \\ \overline{O}_2 = & -\frac{B_2^2\beta^2}{p}(p-r^2)(r_2-r)^{2\beta-2}, \\ \overline{O}_3 = & -B_2D_2\beta(1-\beta)(r_2-r)^{\beta-2}(\theta-\theta_0)^2. \end{aligned}$$

Similarly, there exists  $\alpha \in (0, \frac{1}{2})$  such that  $p - r^2 \leq (r_2 - r)^\alpha$ . Then

$$\left| -\frac{A_2^2}{4p}(p - r^2)(r_2 - r)^{-1} \right| \leq C(p_1, p_2)A_2^2(r_2 - r)^{\alpha-1}.$$

Now we choose  $\beta$  such that  $\beta - \frac{3}{2} < \alpha - 1$ , i.e.,  $\alpha > \beta - \frac{1}{2}$ . Then

$$(\beta^2 - \frac{1}{4})A_2B_2(r_2 - r)^{\beta - \frac{3}{2}} > 3C(p_1, p_2)A_2^2(r_2 - r)^{\alpha - 1},$$

which implies that

$$B_2 > A_2C(p_1, p_2, \beta)(r_2 - r)^{\alpha - \beta + \frac{1}{2}}. \quad (3.25)$$

On the other hand, if  $r_2 - r$  is small enough,

$$3B_2^2\beta(2\beta - 1)(r_2 - r)^{2\beta - 2} < (\beta^2 - \frac{1}{4})A_2B_2(r_2 - r)^{\beta - \frac{3}{2}},$$

which implies that

$$A_2 > \frac{3\beta(2\beta - 1)B_2}{\beta^2 - \frac{1}{4}}(r_2 - r)^{\beta - \frac{1}{2}}. \quad (3.26)$$

Let  $D_2$  be large enough such that  $p > v$  for some  $\theta = \theta_a, \theta_b$ , so that  $\theta_0 \in (\theta_a, \theta_b)$ . We choose  $\tilde{r}_2 < r_2$  such that

$$\begin{aligned} p &> p_2 + 2A_2(r_2 - \tilde{r}_2)^{\frac{1}{2}} - D_2(\theta - \theta_0)^2 \\ &\geq p_2 + A_2(r_2 - \tilde{r}_2)^{\frac{1}{2}} + B_2(r_2 - \tilde{r}_2)^\beta - D_2(\theta - \theta_0)^2 = v(\tilde{r}_2, \theta), \end{aligned}$$

where the second inequality holds, provided that  $\frac{B_2}{A_2} \leq (r_2 - \tilde{r}_2)^{\frac{1}{2} - \beta}$ . Moreover,

$$(p - v)|_{r=r_2} = D_2(\theta - \theta_0)^2 \geq 0 \quad \text{for } \theta \in (\theta_a, \theta_b),$$

which implies that

$$(p - v)|_{\partial\tilde{\mathcal{N}}} \geq 0,$$

where  $\tilde{\mathcal{N}} := \{r \in (\tilde{r}_2, r_2), \theta \in (\theta_a, \theta_b)\}$ .

On the other hand, we take  $B_2$  sufficiently small such that (3.25)–(3.26) hold. Then we derive

$$\hat{Q}v > 0 \quad \text{in } \tilde{\mathcal{N}}.$$

Moreover, in  $\tilde{\mathcal{N}}$ ,

$$\partial_{rr}v(r, \theta) = -\frac{1}{4}A_1(r_2 - r)^{-\frac{3}{2}} + B_1\beta(1 - \beta)(r_2 - r)^{\beta - 2} < 0.$$

Now assume that there exists a non-empty open subset  $\mathcal{N}_2 = \{(r, \theta) \in \tilde{\mathcal{N}}, p < v\} \subset \tilde{\mathcal{N}}$ . By the continuity of function  $p - v$ , there exist a minimum point  $P_{**} \in \mathcal{N}_2$  and a small neighborhood  $\hat{\mathcal{N}}_2 \subset \mathcal{N}_2$  such that

$$p - v < -\epsilon \quad \text{on } \hat{\mathcal{N}}_2$$

when  $\epsilon > 0$  is sufficiently small. Then

$$Q(p - v) = Qp - \hat{Q}v + (\hat{Q}v - Qv) = -\hat{Q}v + (v - p - \epsilon)v_{rr} \leq -\hat{Q}v < 0 \quad \text{in } \hat{\mathcal{N}}_2; \quad (3.27)$$

in particular,

$$Q(p - v)(P_{**}) < 0 \quad \text{in } \hat{\mathcal{N}}_2. \quad (3.28)$$

On the other hand, at the minimum point  $P_{**} \in \hat{\mathcal{N}}_2$ , the structure of operator  $Q$  leads to

$$Q(p - v)(P_{**}) \geq 0,$$

which is a contradiction to (3.28), so that the open subset  $\mathcal{N}_2 \subset \tilde{\mathcal{N}}$  must be empty. Thus,  $p \geq v$  in  $\tilde{\mathcal{N}}$ .

3. Combining Steps 1–2, we conclude that

$$a(r_2 - r)^{\frac{1}{2}} < p - p_2 < A(r_2 - r)^{\frac{1}{2}} \quad \text{in } \mathcal{N}$$

for some constants  $a, A > 0$ , so that the optimal regularity of  $p$  near the sonic circle  $C_2$  is  $C^{\frac{1}{2}}$ .

4. Next, we introduce the coordinates  $(x, y) = (r_2 - r, \theta)$  and denote  $\varphi := p - p_2$ . We scale  $\varphi$  in  $\mathcal{N}$  such that

$$u(S, T) = \frac{1}{S^{\frac{1}{5}}} \varphi(S^{-\frac{12}{5}}, y_0 + S^{-\frac{14}{5}} T) \quad \text{for } (S^{-\frac{12}{5}}, y_0 + S^{-\frac{14}{5}} T) \in \mathcal{N}.$$

From the optimal regularity, it follows that

$$0 < a \leq S^{\frac{7}{5}} u \leq A$$

for some constants  $a, A > 0$ . From (2.5), we obtain the governing equation for  $u$  in the  $(S, T)$ -coordinates:

$$Qu = a_{11}u_{SS} + a_{12}u_{ST} + a_{22}u_{TT} + b_1u_S + b_2u_T + c_1u + \text{h.o.t} = 0, \quad (3.29)$$

where

$$\begin{aligned} a_{11} &= S^{\frac{7}{5}}u + 2r_2S^{-\frac{6}{5}} - S^{-\frac{18}{5}}, \\ a_{12} &= \frac{28T}{5S}(S^{\frac{7}{5}}u + 2r_2S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\ a_{22} &= \frac{144}{25} + \frac{196T^2}{S^2}(S^{\frac{7}{5}}u + 2r_2S^{-\frac{6}{5}} - S^{-\frac{18}{5}}), \\ b_1 &= \frac{19}{5S}(S^{\frac{7}{5}}u + 2r_2S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) + \frac{12M}{5S^{\frac{11}{5}}}\left(\frac{S^{\frac{1}{5}}u + p_2}{M^2} - 2\right) =: \hat{b}_1S^{-1}, \\ b_2 &= \frac{294T}{25S^2}(S^{\frac{7}{5}}u + 2r_2S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) + \frac{168MT}{25S^{\frac{16}{5}}}\left(\frac{S^{\frac{1}{5}}u + p_2}{M^2} - 2\right) =: \hat{b}_2S^{-2}T, \\ c_1 &= \frac{13}{25S^2}(S^{\frac{7}{5}}u + 2r_2S^{-\frac{6}{5}} - S^{-\frac{18}{5}}) + \frac{12M}{25S^{\frac{16}{5}}}\left(\frac{S^{\frac{1}{5}}u + p_2}{M^2} - 2\right) =: \hat{c}_1S^{-2}, \\ \text{h.o.t} &= \frac{M^2S^{\frac{2}{5}}}{S^{\frac{1}{5}}u + p_0}\left(S(u_S)^2 + \frac{196}{25}\frac{T^2}{S}(u_T)^2 + \frac{1}{25}\frac{u^2}{S} + \frac{28}{5}Tu_Su_T + \frac{2}{5}uu_S + \frac{28}{25}\frac{T}{S}uu_T\right), \end{aligned}$$

with  $M := r_0 - S^{-\frac{12}{5}}$ . According to the optimal regularity, we have

$$0 < C^{-1} \leq a_{11}, a_{12}, a_{22}, \hat{b}_1, \hat{b}_2, \hat{c}_1 \leq C, \quad \text{h.o.t.} \rightarrow 0,$$

if  $S^{-1}$  and  $T$  are sufficiently small. Moreover, the eigenvalues of (3.29) are positive and bounded, so that (3.29) is uniformly elliptic for  $u$  in the  $(S, T)$ -coordinates.

Let  $x_0^{-1} < S < x_0^{-2}$  with  $x_0$  small enough. Then, using Theorem 8.20 in [21], we conclude

$$ax_0^{\frac{7}{5}} \leq u(x_0^{-1}, 0) \leq \sup_{x_0^{-1} \leq S \leq x_0^{-2}} u(S, T) \leq C \inf_{x_0^{-1} \leq S \leq x_0^{-2}} u(S, T) \leq Cu(x_0^{-2}, 0) \leq CAx_0^{\frac{14}{5}},$$

where  $C$  is independent of  $x_0$ . This implies that  $x_0^{-\frac{7}{5}} \leq K$  for a bounded constant  $K > 0$ , which is a contradiction if  $x_0$  is sufficiently small. Therefore, case (ii) can not occur.

The impossibility of case (iii) can be proved similarly, so we omit the proof here. This completes the proof.  $\square$

In order to use the Schauder fixed point theorem, we need further to prove that map  $J$  is compact and continuous on  $\mathcal{R}^\varepsilon$ . Evaluating  $g(r, \theta, p)$ , we obtain the bound:

$$\|g\|_{\gamma_V/2} \leq C(K_1),$$

so that

$$\|\tilde{r}\|_{1+\gamma_V/2} \leq C(K_1),$$

where  $\gamma_V$  is independent of  $\alpha_1$ , the Hölder exponent of space  $\mathcal{R}^\varepsilon$ . Thus,  $J(\mathcal{R}^\varepsilon) \subset C^{1+\frac{\gamma_V}{2}}$ , and  $J(\mathcal{R}^\varepsilon) \subset \mathcal{R}^\varepsilon$  if  $\alpha_1 \leq \frac{\gamma_V}{2}$ . We take  $\alpha_1 = \frac{\gamma_V}{3}$  to guarantee that  $J$  is compact. Furthermore, assume that  $r_m, r \in \mathcal{R}^\varepsilon$  and  $r_m \rightarrow r$  as  $m \rightarrow \infty$ . Assume that  $p_m$  is the solution of the nonlinear fixed boundary problem with shock  $\Gamma_{\text{shock}}$  defined by  $r_m$  for each  $m$ . By the standard argument (cf. [6]), we obtain that  $p_m \rightarrow p$ , which solves the problem for  $r$ . Then

$$g(r_m(\theta), \theta, p(r_m(\theta), \theta)) \rightarrow g(r(\theta), \theta, p(r(\theta), \theta)) \quad \text{as } m \rightarrow \infty,$$

which implies that  $Jr_m \rightarrow Jr$  as  $m \rightarrow \infty$ . Therefore, the Schauder fixed point theorem implies that  $J$  has a fixed point  $r^\varepsilon \in C^{1+\frac{\gamma_V}{3}}([\theta_3, \theta_1])$ .

Together with the corresponding solution  $p^\varepsilon$ , we conclude the existence of a solution  $(p^\varepsilon, r^\varepsilon) \in C_{(-\gamma)}^{2,\alpha}(\Omega^\varepsilon) \times C^{1,\alpha_1}([\theta_3, \theta_1])$  of the regularized free boundary problem (3.2), (2.9)–(2.10), and (3.7).

**3.4. Existence of solutions of Problem 2.1.** In this section, we prove that the limit of  $(p^\varepsilon, r^\varepsilon)$  as  $\varepsilon \rightarrow 0$  is a solution of Problem 2.1.

**Lemma 3.6.** *There exists a positive function  $\phi$ , independent of  $\varepsilon$ , such that*

$$p^\varepsilon - (\xi^2 + \eta^2) \geq \phi \quad \text{in } \overline{\Omega} \setminus \Gamma_{\text{sonic}}, \quad (3.30)$$

and  $\phi \rightarrow 0$  as  $\text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) \rightarrow 0$ .

*Proof.* For  $X_0 = (\xi_0, \eta_0) \in \Omega$  and  $0 < R < 1$ , denote

$$\zeta(X) = 1 - \frac{(\xi - \xi_0)^2 + (\eta - \eta_0)^2}{R^2} \quad \text{for } B_R(X_0) \cap \Gamma_{\text{sonic}} = \emptyset.$$

We define

$$\phi = \delta_0(\zeta(X))^\tau,$$

where  $\delta_0$  and  $\tau$  are two positive constants. Then, piecing together these  $B_{\frac{3}{4}R_{X_0}}(X_0)$  for  $X_0 \in \overline{\Omega} \setminus \Gamma_{\text{sonic}}$ , we can obtain a local uniform lower-barrier

$$p^\varepsilon - (\xi^2 + \eta^2) \geq \phi = \delta_0(\zeta(X))^\tau \quad \text{in } B_{\frac{3}{4}R_{X_0}}(X_0) \cap \overline{\Omega},$$

where  $\delta_0$  and  $\tau$  are independent of  $\varepsilon$ . Moreover,  $\delta_0 \rightarrow 0$  as  $\text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) \rightarrow 0$ , so does  $\phi$ . See [6, 10] for more details.  $\square$

The uniform lower bound of  $p^\varepsilon - (\xi^2 + \eta^2)$ , independent of  $\varepsilon$ , implies that the governing equation is locally uniform elliptic, independent of  $\varepsilon$ . Thus, we can apply the standard local compactness arguments to obtain the limit,  $p$ , locally in the interior of the domain.

**Theorem 3.7.** *There exist functions  $r(\theta) \in C^1([\theta_3, \theta_1])$  and  $p \in C_{\text{loc}}^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$  such that*

$$r^\varepsilon \rightarrow r \quad \text{in } C([\theta_3, \theta_1]), \quad p^\varepsilon \rightarrow p \quad \text{in } C_{\text{loc}}^{2,\alpha}(\Omega),$$

and  $(p, r)$  is a solution of the free boundary problem, Problem 2.1.

*Proof.* We have obtained the estimate:

$$\|r^\varepsilon(\theta)\|_{C^1([\theta_3, \theta_1])} \leq C,$$

where  $C$  is independent of  $\varepsilon$ . Then, by the Arzela-Ascoli theorem, there exists a subsequence converging uniformly to a function  $r(\theta)$  in  $C^\alpha([\theta_3, \theta_1])$  as  $\varepsilon \rightarrow 0$ , for any  $\alpha \in (0, 1)$ . By the local ellipticity and the standard interior Schauder estimate,

there exists a function  $p \in C_{\text{loc}}^{2,\alpha}(\Omega)$  such that  $p^\varepsilon \rightarrow p$  in any compact subset, contained by  $\bar{\Omega} \setminus (\Gamma_{\text{sonic}} \cup \Gamma_{\text{shock}})$ , satisfying that  $Qp = 0$ .

Since the shock does not meet the sonic circle of state  $p_2$ ,  $p^\varepsilon > p_2$ . Thus, we have the uniform ellipticity, and the uniform negativity of  $\beta \cdot \nu$  locally. Thus, we can pass the limit to obtain that  $p \in C^{1,\alpha}$ ,

$$\mathcal{B}p = 0 \quad \text{on } \Gamma_{\text{shock}},$$

and  $r'(\theta) = g(r(\theta), \theta)$ . Therefore, the limiting vector function  $(p, r)$  is a global solution of Problem 2.1.  $\square$

The property of  $\Gamma_{\text{shock}}$  is stated as follows:

**Proposition 3.2.** *For the free boundary  $\Gamma_{\text{shock}} = \{(\xi, \eta(\xi)) : \xi_3 < \xi < \xi_1\}$  with  $\xi_3$  and  $\xi_1$  as the  $\xi$ -coordinates of points  $P_3$  and  $P_1$  respectively,*

$$\eta(\xi) \in C^2(\xi_3, \xi_1),$$

*and  $\eta(\xi)$  is convex for  $\xi \in (\xi_3, \xi_1)$ .*

*Proof.* Define

$$F(\xi, \eta) = \xi^2 + \eta^2 - r^2(\theta(\xi, \eta)) = 0 \quad \text{on } \Gamma_{\text{shock}}.$$

Then

$$F_\eta|_{\xi=0} = (2\eta - 2rr'\theta_\eta)|_{\xi=0} = 2\eta(0) \neq 0. \quad (3.31)$$

By the implicit function theorem, there exists  $\eta = \eta(\xi)$  such that (3.31) holds locally on  $\Gamma_{\text{shock}}$  near  $\xi = 0$ . Hence, there exists  $\bar{\xi} > 0$  such that  $(\xi, \eta(\xi)) \in \Gamma_{\text{shock}}$  for  $0 < |\xi| \leq \bar{\xi}$ .

Recall that, for  $\xi \in [0, \xi_1)$ ,

$$\eta'(\xi) = f(\xi, \eta(\xi), \bar{p}) = \frac{\xi\eta + \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}{\xi^2 - \bar{p}}.$$

Then

$$\eta''(\xi) = f_\xi + \eta' f_\eta + f_{\bar{p}} \bar{p}'.$$

Notice that

$$\begin{aligned} f_\xi &= \frac{\eta}{\xi^2 - \bar{p}} + \frac{\xi \bar{p}}{(\xi^2 - \bar{p})\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}} - \frac{2\xi(\xi\eta + \sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})})}{(\xi^2 - \bar{p})^2}, \\ f_\eta &= \frac{\xi}{\xi^2 - \bar{p}} + \frac{\eta \bar{p}}{(\xi^2 - \bar{p})\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}}. \end{aligned}$$

Then  $f_\xi + \eta' f_\eta = 0$ . Thus, the sign of  $\eta''$  is determined by the signs of  $f_{\bar{p}}$  and  $\bar{p}'$ . Since

$$\frac{\partial f}{\partial \bar{p}} = \frac{(\eta\sqrt{\bar{p}} + \xi\sqrt{\xi^2 + \eta^2 - \bar{p}})^2}{2(\xi^2 - \bar{p})^2\sqrt{\bar{p}(\xi^2 + \eta^2 - \bar{p})}},$$

and  $\bar{p}$  is monotonically increasing for  $\xi \in [0, \xi_1)$ , we obtain that, for each  $\xi \in [0, \xi_1)$ ,  $\bar{p}' \geq 0$  and  $\frac{\partial f}{\partial \bar{p}} > 0$ . Therefore,

$$\eta''(\xi) \geq 0 \quad \text{for } \xi \in [0, \xi_1).$$

Similarly, it can be proved that

$$\eta''(\xi) \geq 0 \quad \text{for } \xi \in (\xi_3, 0].$$

This implies that the shock curve  $\eta(\xi)$  is convex in  $(\xi_3, \xi_1)$  in the self-similar coordinates.  $\square$

**4. Optimal regularity near the sonic boundary.** In this section, we first establish the Lipschitz continuity for the solution near the degenerate sonic boundary  $\Gamma_{\text{sonic}}$ .

**Lemma 4.1.** *The solution,  $p(\xi, \eta)$ , of Problem 2.1 is Lipschitz continuous up to the sonic boundary  $\Gamma_{\text{sonic}}$ .*

*Proof.* Since  $p \leq p_1$  in  $\Omega$ , it follows that

$$p - \xi^2 - \eta^2 < p_1 - \xi^2 - \eta^2.$$

On the other hand,  $p - \xi^2 - \eta^2 > \xi^2 + \eta^2 - p_1$  in  $\Omega$ . Then

$$|p - p_1| \leq |p - \xi^2 - \eta^2| + |p_1 - \xi^2 - \eta^2| \leq 2|p_1 - \xi^2 - \eta^2| \leq 4\sqrt{p_1} |\sqrt{p_1} - \sqrt{\xi^2 + \eta^2}|,$$

which implies that  $p$  is Lipschitz continuous up to the degenerate boundary  $\Gamma_{\text{sonic}}$ .  $\square$

Next, we want to show that the Lipschitz continuity is the optimal regularity for  $p$  both across the sonic boundary  $\Gamma_{\text{sonic}}$  and at the intersection points  $P_1$  and  $P_3$ . Since the problem is symmetric, we consider only the right-half sonic circle for convenience.

For  $\epsilon \in (0, \frac{r_1}{4})$ , we denote the  $\epsilon$ -neighborhood of the sonic boundary  $\Gamma_{\text{sonic}}$  within  $\Omega$  by

$$\Omega_\epsilon := \Omega \cap \{(r, \theta) : 0 < r_1 - r < \epsilon, \theta_1 < \theta < \frac{\pi}{2}\},$$

where we take  $\theta_1 \in (-\frac{\pi}{2}, 0)$  if  $P_1$  is below the  $\xi$ -axis.

In  $\Omega_\epsilon$ , we introduce the new coordinates:

$$(x, y) := (r_1 - r, \theta - \theta_1). \quad (4.1)$$

Then

$$\Gamma_{\text{sonic}} := \{(0, y) : 0 < y < \frac{\pi}{2} - \theta_1\}, \quad P_1 = (0, 0).$$

We can take  $P_1$  as an interior point of  $\Gamma_{\text{sonic}}^{\text{ext}}$ , which is obtained by reflecting  $\Gamma_{\text{sonic}}$  with respect to  $y = 0$ . Let

$$Q_{r,R}^+ := \{(x, y) : x \in (0, r), |y| < R\} \quad \text{with } R = \frac{\pi}{2} - \theta_1.$$

Let  $\varphi = p_1 - p$ . Then

$$\varphi > 0 \text{ in } Q_{r,R}^+, \quad \varphi = 0 \text{ on } \partial Q_{r,R}^+ \cap \{x = 0\}. \quad (4.2)$$

It follows from (2.5) that  $\varphi$  satisfies

$$\mathcal{L}\varphi := (2r_1x - \varphi + O_1)\varphi_{xx} + (1 + O_2)\varphi_{yy} + (r_1 + O_3)\varphi_x - (1 + O_4)(\varphi_x)^2 = 0 \quad (4.3)$$

in  $Q_{r,R}^+$ , where

$$\begin{aligned} O_1(x, \varphi) &= -x^2, \quad O_2(x, \varphi) = \frac{x(2r_1 - x) - \varphi}{(r_1 - x)^2}, \\ O_3(x, \varphi) &= \frac{2x^2 - 3r_1x + \varphi}{r_1 - x}, \quad O_4(x, \varphi) = \frac{x^2 - 2r_1x + \varphi}{r_1^2 - \varphi}. \end{aligned} \quad (4.4)$$

**Lemma 4.2.** *There exist  $\varepsilon > 0$  and  $k > 0$  depending on the Riemann initial data such that, for any solution  $p$ ,*

$$0 \leq \varphi \leq (2r_1 - k)x \quad \text{for } x \in (0, \varepsilon). \quad (4.5)$$

*Proof.* The proof is similar to that for Lemma 9.6.4 in Chen-Feldman [12], so we only give the outline. We first define a smooth approximation to  $(\xi, \eta) \rightarrow \text{dist}((\xi, \eta), \Gamma_{\text{sonic}})$ , denoted by  $g(\xi, \eta)$ , and then consider the function:  $p - r^2 - \lambda g(\xi, \eta)$ , where  $\lambda > 0$  will be specified later. According to the ellipticity principle, it can be proved that  $p - r^2 - \lambda g(\xi, \eta)$  cannot attain its minimum in the interior of  $\Omega$ .

Next, we turn to the shock boundary. Suppose that  $p - r^2 - \lambda g(\xi, \eta)$  achieves its minimum at  $P_{\min} \in \Gamma_{\text{shock}}$ . Then

$$(p - r^2)(P_{\min}) \leq \lambda g(P_{\min}) \leq \lambda C \quad \text{for some } C > 0.$$

By the boundary condition (2.9)–(2.10), we can choose  $\lambda$  sufficiently small such that

$$((\beta_1, \beta_2) \cdot \nabla(p - r^2 - \lambda g))(P_{\min}) < 0,$$

which contradicts the Hopf maximum principle. Thus,  $p - r^2 - \lambda g(\xi, \eta)$  must attain its minimum on  $\Gamma_{\text{sonic}}$ , which implies that

$$p - r^2 \geq \lambda \text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) \quad \text{in } \bar{\Omega}. \quad (4.6)$$

It is clear that  $\varphi \geq 0$ . Combining this with (4.6), we can derive

$$\lambda x \leq \lambda \text{dist}((\xi, \eta), \Gamma_{\text{sonic}}) \leq p - (r_1 - x)^2 = -\varphi + 2r_1x - x^2.$$

This completes the proof by taking  $k = \frac{\lambda}{2}$ .  $\square$

According to (4.5), we derive that  $O_i(x, \varphi)$ ,  $i = 1, \dots, 4$ , are continuously differentiable and satisfy

$$\frac{|O_1(x, y)|}{x^2} + \frac{|O_k(x, y)|}{x} + \frac{|DO_1(x, y)|}{x} + |DO_k(x, y)| \leq N \quad \text{for } k = 2, 3, 4, \quad (4.7)$$

for some  $N > 0$  depending on the Riemann initial data. Then the leading terms of equation (4.3) form the following equation:

$$(2r_1x - \varphi)\varphi_{xx} + \varphi_{yy} + r_1\varphi_x - \varphi_x^2 = 0, \quad (4.8)$$

which is uniformly elliptic in every subdomain  $\{x > \delta\}$  with  $\delta > 0$ .

**Lemma 4.3.** *Let  $\varphi \in C^2(Q_{r,R}^+) \cap C(\overline{Q_{r,R}^+})$  be the solution of equation (4.3) with condition (4.2). Then, for any  $\alpha \in (0, 1)$  and  $|y| < \frac{R}{2}$ ,*

$$\varphi \in C^{1,\alpha}(\overline{Q_{r/2,R/2}^+}), \quad \varphi_x(0, y) = r_1, \quad \varphi_y(0, y) = 0. \quad (4.9)$$

*Proof.* The proof is similar to that in Bae-Chen-Feldman [2], and we only list the major procedure and the points of difference here.

1. By constructing barrier functions and the maximum principle for strictly elliptic equations, we can prove that  $\varphi = p_1 - p$  has a positive lower bound, *i.e.*, there exist  $\hat{r} > 0$  and  $\mu > 0$ , depending on the Riemann initial data and  $\inf_{Q_{\hat{r},R}^+ \cap \{x > \frac{\hat{r}}{2}\}} \varphi$ , such that, for all  $r \in (0, \frac{\hat{r}}{2}]$ ,

$$\varphi \geq \mu r_1 x \quad \text{in } Q_{r, \frac{15R}{16}}^+.$$

2. We can now obtain more precise estimates for  $\varphi$  near the sonic boundary:

$$|\varphi(x, y) - r_1 x| \leq Cx^{1+\alpha} \quad \text{in } Q_{\hat{r}, \frac{7R}{8}}^+$$

for any  $\alpha \in (0, 1)$  and some constant  $C$  depending on  $\hat{r}, R, \alpha$ , and the Riemann initial data.



To achieve this, we denote  $W(x, y) := r_1 x - \varphi(x, y)$  and introduce a cutoff function  $\zeta(s) \in C^\infty$  such that

$$\zeta(s) = \begin{cases} s & \text{for } s \in (-r_1, r_1), \\ 0 & \text{for } s \in \mathbb{R} \setminus (-r_1 - 1, r_1 + 1). \end{cases} \quad (4.10)$$

Then  $W(x, y)$  satisfies the following equation:

$$\begin{aligned} x(r_1 + \zeta(\frac{W}{x}) + \frac{O_1}{x})W_{xx} + (1 + O_2)W_{yy} - (r_1 - O_3 + 2r_1 O_4)W_x + (1 + O_4)W_x^2 \\ = r_1 O_3 - r_1^2 O_4, \end{aligned} \quad (4.11)$$

where  $O_i$ ,  $i = 1, \dots, 4$ , are given in (4.4).

Next, for fixed  $z_0 = (x_0, y_0) \in Q_{r/2, R/2}^+$ , we define

$$W^{(z_0)}(S, T) = \frac{1}{x_0^{1+\alpha}} W(x_0 + \frac{x_0}{8}S, y_0 + \frac{\sqrt{x_0}}{8}T) \quad \text{for } (S, T) \in Q_1,$$

where  $Q_1 = (-1, 1)^2$ . By estimating the coefficients carefully, we can show that equation (4.11) is uniformly elliptic with ellipticity constants independent of  $z_0$ . Then, by Theorem A.1 in Chen-Feldman [11], we can derive

$$\|W^{(z_0)}\|_{C^{2,\alpha}(\overline{Q_{1/2}})} \leq C(r_1 r^{-\alpha} + r^{1-\alpha}) =: \widehat{C},$$

where  $C$  depends only on the data and  $\alpha$ . Thus, we have

$$|D_x^i D_y^j W(x_0, y_0)| \leq C x_0^{2+\alpha-i-\frac{j}{2}} \quad \text{for all } (x_0, y_0) \in Q_{r/2, R/2}^+, 0 \leq i+j \leq 1,$$

which implies that  $DW(0, y) = 0$ . This completes the proof.  $\square$

The following lemma states the regularity of solutions near the interior of the sonic boundary.

**Lemma 4.4.** *Let  $p \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$  be a solution of Problem 2.1 and satisfy that*

$$p_2 < p < p_1, \quad r^2 < p \quad \text{in } \Omega.$$

*Then  $p$  cannot be  $C^1$  across the degenerate sonic boundary  $\Gamma_{\text{sonic}}$ .*

*Proof.* Suppose that  $p$  is  $C^1$  across  $\Gamma_{\text{sonic}}$ , so is  $\varphi = p_1 - p$ . Since  $\varphi \equiv 0$  in the supersonic domain  $(\{\xi \leq \xi_1\} \setminus \Omega) \cap \Gamma_{\text{sonic}}$ , it follows that

$$D\varphi(0, y) = 0 \quad \text{for any } (0, y) \in \Gamma_{\text{sonic}}.$$

On the other hand, for  $(0, y_0) \in \Gamma_{\text{sonic}}$  and small  $r, R > 0$ ,

$$\hat{Q}_{r,R}^+ := \{(x, y) : x \in (0, r), |y - y_0| < R\} \subset \Omega_\varepsilon.$$

Since (4.3) is invariant under the transformation:  $(x, y) \rightarrow (x, y - y_0)$ , we can let  $(0, y_0) = (0, 0)$  so that  $\hat{Q}_{r,R}^+ \subset \Omega_\varepsilon$ . From the proof of Lemma 4.3, there exist  $r, \mu > 0$  such that

$$\varphi \geq \mu r_1 x \quad \text{in } Q_{r, 15R/16}^+,$$

which contradicts to  $D\varphi(0, y) = 0$ .  $\square$

For the regularity near the interaction points  $P_1$  and  $P_3$ , we have the following result.

**Lemma 4.5.** *Let  $p$  be the solution of Problem 2.1 and satisfy the properties that there exists a neighborhood  $\mathcal{N}(\Gamma_{\text{sonic}})$  such that, for  $\varphi = p_1 - p$ ,*

- (i)  $\varphi$  is  $C^{0,1}$  across the degenerate boundary  $\Gamma_{\text{sonic}}$ ;
- (ii) There exists  $\mu_0 > 0$  such that, in the  $(x, y)$ -coordinates,

$$0 \leq \varphi \leq (2r_1 - \mu_0)x \quad \text{in } \Omega \cap \mathcal{N}(\Gamma_{\text{sonic}});$$

- (iii) There exist  $\epsilon_0 > 0$ ,  $\omega > 0$ , and  $y = f(x) \in C^{1,1}([0, \epsilon_0])$  such that

$$\Gamma_{\text{shock}} \cap \partial\Omega_{\epsilon_0} = \{(x, y) : x \in (0, \epsilon_0), y = f(x)\},$$

$$\Omega_{\epsilon_0} = \{(x, y) : x \in (0, \epsilon_0), f(x) < y < \frac{\pi}{2} - \theta_1\},$$

$$\frac{\partial f}{\partial x} \geq \omega > 0 \quad \text{for } x \in (0, \epsilon_0).$$

Then both limits,  $\lim_{\substack{(\xi, \eta) \rightarrow P_1 \\ (\xi, \eta) \in \Omega}} D_r \varphi$  and  $\lim_{\substack{(\xi, \eta) \rightarrow P_3 \\ (\xi, \eta) \in \Omega}} D_r \varphi$ , do not exist.

*Proof.* We prove this assertion by contradiction as in Bae-Chen-Feldman [2]. We choose two different sequences of points converging to  $P_1$  and show that the two limits  $\varphi_x$  along the two sequences are different, which leads to a contradiction.

We first take a sequence close to  $\Gamma_{\text{sonic}}$ . Let  $\{y_m\}_{m=1}^\infty$  be a sequence such that  $y_m \in (0, \frac{\pi}{2} - \theta_1)$  and  $\lim_{m \rightarrow \infty} y_m = 0$ . By (4.9), there exists  $x_m \in (0, \frac{1}{m})$  such that

$$|\varphi_x(x_m, y_m) - r_1| < \frac{1}{m}.$$

Then we see that  $(x_m, y_m) \in \Omega$ ,  $\lim_{m \rightarrow \infty} (x_m, y_m) = 0$ ,

$$\lim_{m \rightarrow \infty} \varphi_x(x_m, y_m) = r_1, \quad \lim_{m \rightarrow \infty} \varphi_y(x_m, y_m) = 0. \quad (4.12)$$

We now construct the second sequence close to  $\Gamma_{\text{shock}}$ . Suppose that the limit,  $\lim_{\substack{(\xi, \eta) \rightarrow P_1 \\ (\xi, \eta) \in \Omega}} D_r \varphi$ , exists. Then

$$\lim_{x \rightarrow 0} \varphi(x, f(x)) = \varphi(0, 0) = 0.$$

From Lemma 4.3, it follows that

$$\lim_{x \rightarrow 0} \varphi_y(x, f(x)) = 0.$$

We rewrite the boundary conditions (2.9)–(2.10) for  $\varphi$  in the  $(x, y)$ -coordinates as

$$\widehat{\beta}_1 \varphi_x + \widehat{\beta}_2 \varphi_y = 0.$$

It is direct to see that there exists  $\lambda > 0$  such that  $\widehat{\beta}_1 > \lambda$  and  $|\widehat{\beta}_2| \leq \frac{1}{\lambda}$  on  $\Gamma_{\text{shock}} \cap \partial\Omega_\epsilon$ . Then

$$|\varphi_x(x, f(x))| \leq K |\varphi_y(x, f(x))| \quad \text{for some } K > 0,$$

which implies that  $\lim_{x \rightarrow 0} \varphi_x(x, f(x)) = 0$ .

Denote  $H(x) := \varphi(x, f(x) + \frac{\omega}{2}x)$ . Then there exists  $\{x_k\}_{k=1}^\infty$  with  $x_k \in (0, \epsilon_0)$  such that

$$\lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} H'(x_k) = 0.$$

Moreover, since

$$H'(x) = \varphi_x(x, g(x)) + \varphi_y(x, g(x))g'(x),$$

with  $g(x) = f(x) + \frac{\omega}{2}x$  and  $|g'| \leq K$ , it follows that, for  $(x_k, g(x_k)) \in \Omega_\epsilon$ ,

$$\lim_{k \rightarrow \infty} \varphi_x(x_k, g(x_k)) = 0. \quad (4.13)$$

It yields that, along sequence  $(x_k, g(x_k))$ , the limit of  $\varphi_x(x, y)$  is 0. Combining with (4.12), we conclude that  $\varphi_x(x, y)$  does not have a limit at  $P_1$  from  $\Omega$ . Similarly, we can prove that, as a sequence  $\{(\xi_i, \eta_i)\}_{i=1}^\infty \subset \Omega$  tends to  $P_3$ , the limit of  $D_r\varphi$  does not exist. This completes the proof.  $\square$

**5. Existence and regularity of global solutions of the pressure gradient system.** In Theorem 2.2, we have constructed a global solution  $p$  of the second order equation (2.4) in  $\Omega$ , which is piecewise constant in the supersonic region. Moreover, we have proved that  $p$  is Lipschitz continuous across the degenerate sonic boundary  $\Gamma_{\text{sonic}}$  from  $\Omega$  to the supersonic region.

To recover the velocity components  $u$  and  $v$ , we consider the first two equations in (2.1). We can write these equations in the radial variable  $r$  as

$$\frac{\partial u}{\partial r} = \frac{1}{r}p_\xi, \quad \frac{\partial v}{\partial r} = \frac{1}{r}p_\eta,$$

and integrate from the boundary of the subsonic region toward the origin. It is direct to see that  $(u, v)$  are at least Lipschitz continuous across  $\Gamma_{\text{sonic}}$ . Furthermore,  $(u, v)$  have the same regularity as  $p$  inside  $\Omega$  except origin  $r = 0$ . However,  $(u, v)$  may be multi-valued at origin  $r = 0$ .

In conclusion, we have

**Theorem 5.1.** *Let the Riemann initial data satisfy (1.5). Then there exists a global solution  $(u, v, p)(r, \theta)$  with the free boundary  $r = r(\theta)$ ,  $\theta \in [\theta_3, \theta_1]$ , such that*

$$(u, v, p) \in C^{2,\alpha}(\Omega), \quad p \in C^\alpha(\overline{\Omega}), \quad r \in C^{2,\alpha}((\theta_3, \theta_1)) \cap C^{1,1}([\theta_3, \theta_1]),$$

*and  $(u, v, p)$  are piecewise constant in the supersonic region. Moreover, the global solution  $(u, v, p)$  with the free boundary  $r = r(\theta)$  satisfies the following properties:*

- (i)  $p > p_2$  on  $\Gamma_{\text{shock}}$ , that is, shock  $\Gamma_{\text{shock}}$  does not meet the sonic circle of state  $p_2$ ;
- (ii) The shock,  $\Gamma_{\text{shock}}$ , is convex in the self-similar coordinates;
- (iii) The solution,  $(u, v, p)$ , is  $C^\alpha$  up to the sonic boundary  $\Gamma_{\text{sonic}}$  and Lipschitz continuous across  $\Gamma_{\text{sonic}}$ ;
- (iv) The Lipschitz regularity of both solution  $(u, v, p)$  across  $\Gamma_{\text{sonic}}$  from the subsonic region  $\Omega$  and shock  $\Gamma_{\text{shock}}$  across points  $\{P_1, P_3\}$  is optimal.

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