

EXCHANGEABLE ARRAYS AND INTEGRABLE SYSTEMS FOR CHARACTERISTIC POLYNOMIALS OF RANDOM MATRICES

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Abstract

The joint moments of the derivatives of the characteristic polynomial of a random unitary matrix, and also a variant of the characteristic polynomial that is real on the unit circle, in the large matrix size limit, have been studied intensively in the past twenty five years, partly in relation to conjectural connections to the Riemann zeta-function and Hardy's function. We completely settle the most general version of the problem of convergence of these joint moments, after they are suitably rescaled, for an arbitrary number of derivatives and with arbitrary positive real exponents. Our approach relies on a hidden, higher-order exchangeable structure, that of an exchangeable array, which, as far as we know, had never been used before in the study of characteristic polynomials of random matrices. We then develop a systematic method, based on a class of Hankel determinants shifted by partitions, that allows us for the first time to give an exact representation of all these joint moments, for finite matrix size, in terms of derivatives of σ -Painlevé V transcendents. As an application, we can also represent all the joint moments of power sum linear statistics of a certain determinantal point process behind this problem in terms of derivatives of σ -Painlevé III' transcendents. This gives an efficient way to compute all these quantities explicitly. Our methods can be used to obtain analogous results for a number of other models sharing the same features.

Keywords: Joint moments, Random unitary matrices, Exchangeable arrays, Painlevé equations.

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1 INTRODUCTION AND MAIN RESULTS

1.1 RANDOM MATRIX THEORY AND RIEMANN ζ -FUNCTION

Over the past fifty years, there has been significant interest in the surprising duality between problems in analytic number theory and analogous ones in random matrix theory. This connection dates back to the work of Montgomery [76], who conjectured that the eigenvalues of $N \times N$ unitary matrices, as the matrix size tends to infinity, can be used to model the behaviour of the zeros of the Riemann ζ -function, $\zeta(z)$, high up the critical line, $\Re(z) = 1/2$; see [20, 22, 21, 85, 43] for subsequent extensions of Montgomery's work. It is natural to ask whether this connection between the zeros of the Riemann ζ -function and those of the characteristic polynomials of random unitary matrices (i.e. the eigenvalues) extends to the values of these two functions. Keating and Snaith [70] suggested that the value distribution of the Riemann ζ -function high on the critical line can be modeled by the value of the characteristic polynomial of a random unitary matrix at 1, when the matrix size is large enough. As an application, they conjectured the leading-order asymptotics of the moments of the ζ -function on the critical line by considering the analogous moments for the characteristic polynomials of unitary random matrices.

To be more precise, let $\mathbf{A} \in \mathbb{U}(N)$, where, here and throughout this paper, $\mathbb{U}(N)$ denotes the group of $N \times N$ unitary matrices, and let $e^{i\theta_1}, \dots, e^{i\theta_N}$ denote the eigenvalues of \mathbf{A} . Consider the characteristic polynomial $V_{\mathbf{A}}(\theta)$ of \mathbf{A} on the unit circle $\theta \in [0, 2\pi)$, given by,

$$V_{\mathbf{A}}(\theta) = \det(\mathbf{I} - e^{-i\theta} \mathbf{A}) = \prod_{j=1}^N (1 - e^{i(\theta_j - \theta)}).$$

It was conjectured in [70] (see also [35, 36] for $s = 3, 4$), that the $T \rightarrow \infty$ asymptotics of

$$\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2s} dt$$

can be obtained from the $N \rightarrow \infty$ asymptotics of

$$\int_{\mathbb{U}(N)} |V_{\mathbf{A}}(0)|^{2s} d\mu_{\text{Haar}}(\mathbf{A}),$$

where μ_{Haar} is the Haar measure on $\mathbb{U}(N)$. This conjecture was extended to later terms in the asymptotic expansion of the moments in [32, 33], and has been verified heuristically using correlations of divisor functions in [37, 38, 39, 40, 41].

Inspired by this, Hughes [60] considered a related quantity, aiming to obtain conjectural values for joint moments of the Riemann ζ -function and its derivative. For a unitary matrix \mathbf{A} , instead of the characteristic polynomial itself, he considered the associated

function

$$Z_{\mathbf{A}}(\theta) = \exp\left(iN\frac{\theta + \pi}{2} - i\sum_{j=1}^N \frac{\theta_j}{2}\right) V_{\mathbf{A}}(\theta)$$

which has the property that for $\theta \in [0, 2\pi)$, $Z_{\mathbf{A}}(\theta) \in \mathbb{R}$ and $|Z_{\mathbf{A}}(\theta)| = |V_{\mathbf{A}}(\theta)|$. This relation between $V_{\mathbf{A}}$ and $Z_{\mathbf{A}}$ is analogous to the relation between the ζ function and Hardy's \mathcal{Z} -function, defined as follows,

$$\mathcal{Z}(t) = \pi^{-it/2} \frac{\Gamma(1/4 + it/2)}{|\Gamma(1/4 + it/2)|} \zeta(1/2 + it),$$

which also satisfies $|\mathcal{Z}(t)| = \left|\zeta\left(\frac{1}{2} + it\right)\right|$, and $\mathcal{Z}(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. Hughes conjectured in [60] that, for arbitrary real parameters s, h such that $s + 1/2 > h > -1/2$, one should have asymptotic convergence of the joint moments as follows,

$$\lim_{N \rightarrow \infty} \frac{1}{N^{s^2+2h}} \int_{\mathcal{U}(N)} |Z_{\mathbf{A}}(0)|^{2s-2h} \left| \frac{dZ_{\mathbf{A}}}{d\theta} \right|_{\theta=0}^{2h} d\mu_{\text{Haar}}(\mathbf{A}) = \mathcal{P}(s, h), \quad (1)$$

for some positive $\mathcal{P}(s, h)$, a claim he was able to prove for $s, h \in \mathbb{N}$, deriving an expression for $\mathcal{P}(s, h)$ for these values of s, h . This, in turn, leads to the conjectural asymptotics for the analogous joint moments of $\mathcal{Z}(t)$, introduced by Ingham in 1926 [61],

$$\frac{1}{T} \int_0^T |\mathcal{Z}(t)|^{2s-2h} |\mathcal{Z}'(t)|^{2h} dt \sim (\log T)^{s+2h} \eta(s) \mathcal{P}(s, h)$$

where

$$\eta(s) = \prod_{\text{primes } p} (1 - p^{-1})^{s^2} \sum_{k=0}^{\infty} p^{-k} \left[\frac{\Gamma(k+s)}{\Gamma(k+1)\Gamma(s)} \right]^2.$$

We note that even though his initial work, and the majority of subsequent works, are centered around the joint moments of $Z_{\mathbf{A}}$, Hughes also conjectured analogous results for $V_{\mathbf{A}}$ and $\zeta\left(\frac{1}{2} + it\right)$. Our goal in this paper is to completely solve the most general version of this problem on the random matrix side, by considering moments involving an arbitrary number of derivatives and positive real exponents, for both $V_{\mathbf{A}}$ and $Z_{\mathbf{A}}$. First, we briefly review the literature.

1.2 A BRIEF HISTORY OF THE PROBLEM

Over the years, there has been significant interest and progress in both proving the convergence in (1), and also obtaining explicit formulae for the leading order coefficient $\mathcal{P}(s, h)$. After Hughes' proof for the case $s, h \in \mathbb{N}$, Conrey, Rubenstein and Snaith [42] gave an alternative proof for the case $s = h \in \mathbb{N}$, using representations of these moments via multiple contour integrals from [32, 33]. They also gave an alternative expression for the coefficient $\mathcal{P}(s, s)$ in terms of a determinant of Bessel functions, which was later shown to be related to the Painlevé equations by Forrester and Witte in [49]. Dehaye [44, 45] gave an independent proof for $s, h \in \mathbb{N}$, and gave another representation of $\mathcal{P}(s, h)$ as a certain combinatorial sum. Winn [89] gave an explicit expression for $s \in \mathbb{N}, h \in \frac{1}{2}\mathbb{N}$ in terms of certain sums over partitions. Basor et al. in [19] used Riemann-Hilbert problem

techniques to obtain a representation for $\mathcal{P}(s, h)$ in terms of Painlevé transcendents for $s, h \in \mathbb{N}$. Bailey et al [17] extended the methods developed in [42] to obtain the same Painlevé-representation of $\mathcal{P}(s, h)$. In [6], Altuğ et al extended the method of [42] to the orthogonal and symplectic groups, including to moments of higher derivatives, and used the results to formulate conjectures for the corresponding moments of families of number-theoretic L -functions, along the lines of [69]. In [14], Assiotis, Keating and Warren succeeded in proving the existence of $\mathcal{P}(s, h)$ for general positive real parameters s, h . Finally, we note that for the case where $Z_{\mathbf{A}}(\theta)$ is replaced by the characteristic polynomial $V_{\mathbf{A}}(\theta)$, which models the ζ -function itself, Hughes' conjecture remained unproven until now outside the parameter range $s, h \in \mathbb{N}$.

1.3 JOINT MOMENTS OF HIGHER ORDER DERIVATIVES

One of the main purposes of the present paper is to establish the $N \rightarrow \infty$ asymptotics, and study in detail the leading order coefficient, of the joint moments for arbitrary numbers of higher order derivatives for both $V_{\mathbf{A}}(\theta)$ and $Z_{\mathbf{A}}(\theta)$ with arbitrary positive real exponents. The other main purpose is to build a new connection between the general joint moments for finite matrix size N and σ -Painlevé V transcendents.

Definition 1.1. *Let $n_1 > \dots > n_k \in \mathbb{N} \cup \{0\}$, and $h_1, \dots, h_k \in \mathbb{R}_+$. Then, we define:*

$$\mathfrak{G}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k) \stackrel{\text{def}}{=} \int_{\mathbb{U}(N)} \prod_{j=1}^k |V_{\mathbf{A}}^{(n_j)}(0)|^{2h_j} d\mu_{\text{Haar}}(\mathbf{A}), \quad (2)$$

$$\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k) \stackrel{\text{def}}{=} \int_{\mathbb{U}(N)} \prod_{j=1}^k |Z_{\mathbf{A}}^{(n_j)}(0)|^{2h_j} d\mu_{\text{Haar}}(\mathbf{A}). \quad (3)$$

Here, the superscript n_j in $V_{\mathbf{A}}^{(n_j)}$ or $Z_{\mathbf{A}}^{(n_j)}$ means the n_j -th derivative of $V_{\mathbf{A}}$ or $Z_{\mathbf{A}}$.

Observe, that (1) is simply a statement for the asymptotics of $\mathfrak{F}_N^{(1,0)}(h, s-h)$. Barhoumi in his monograph [18] established an asymptotic formula for $\mathfrak{G}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)$ for $h_1, \dots, h_k \in \mathbb{N}$, using an approach based on symmetric function theory. More recently, the asymptotics for $k=2$, for $h_1, h_2 \in \mathbb{N}$, for both $\mathfrak{G}_N^{(n_1, n_1)}(h_1, h_2)$ and $\mathfrak{F}_N^{(n_1, n_2)}(h_1, h_2)$, were obtained in [71] using an intricate analysis of multiple contour integrals and an expression, different to the one in [18], for the leading order coefficient that was shown in [72], in the special case of $\mathfrak{F}_N^{(2,0)}(h_1, h_2)$, to be connected to Painlevé equations. We note that the expressions for the leading order coefficients given in [18, 72, 71] do not make sense for generic real exponents h_i .

Our first main result gives the convergence of both of these quantities, after appropriate rescalings, for the full parameter range (in particular without the integrality restrictions), and gives a probabilistic expression for the leading coefficient in terms of joint moments of certain natural random variables that are defined through a determinantal point process. Thus, in order to state our result, we need some definitions.

A determinantal point process on a Borel set $\mathfrak{X} \subset \mathbb{R}$ with correlation kernel $\mathcal{K} : \mathfrak{X} \times \mathfrak{X} \rightarrow \mathbb{C}$ is a probability measure \mathfrak{P} on $\text{Conf}(\mathfrak{X})$, the space of locally finite collections of points in \mathfrak{X} , endowed with a certain topology and corresponding Borel σ -algebra, see [23, 63] for the details, such that for any $k \geq 1$ and any measurable and bounded $F : \mathfrak{X}^k \rightarrow \mathbb{R}$ with

compact support, we have

$$\int_{\text{Conf}(\mathbb{X})} \sum_{a_{i_1}, \dots, a_{i_k} \in \mathbb{E}} F(a_{i_1}, \dots, a_{i_k}) d\mathfrak{P}(\mathbb{E}) = \int_{\mathbb{X}^k} F(x_1, \dots, x_k) \det_{1 \leq i, j \leq k} [\mathcal{K}(x_i, x_j)] dx_1 \cdots dx_k.$$

We will be interested in specific determinantal point processes defined via their correlation kernels $\mathfrak{G}^{(s)}$ (we note here that by results of [24, 74], the kernels $\mathfrak{G}^{(s)}$, completely determine their corresponding point processes), where $s \geq 0$ is a parameter.

Definition 1.2. *Let $s \in \mathbb{R}_{\geq 0}$. Then, we define $\mathbf{P}^{(s)}$ to be the determinantal point process on $(-\infty, 0) \times (0, \infty)$ with the correlation kernel $\mathfrak{G}^{(s)}$ given by,*

$$\mathfrak{G}^{(s)}(x, y) = \frac{1}{2\pi} \frac{(\Gamma(s+1))^2}{\Gamma(2s+1)\Gamma(2s+2)} \frac{G^{(s)}(x)H^{(s)}(y) - G^{(s)}(y)H^{(s)}(x)}{x-y},$$

where the functions $G^{(s)}(x), H^{(s)}(x)$ are given by the formulae

$$G^{(s)}(x) = 2^{2s-\frac{1}{2}} \Gamma\left(s + \frac{1}{2}\right) \cdot \frac{1}{|x|^{\frac{1}{2}}} J_{s-1/2}\left(\frac{1}{|x|}\right), \quad H^{(s)}(x) = \text{sgn}(x) 2^{2s+\frac{1}{2}} \Gamma\left(s + \frac{3}{2}\right) \cdot \frac{1}{|x|^{\frac{1}{2}}} J_{s+1/2}\left(\frac{1}{|x|}\right),$$

where J_ν denotes the Bessel function with parameter ν .

This random point process first appeared, in a completely different setting, in influential work of Borodin and Olshanski [24] (see also the subsequent work of Bourgade-Nikeghbali-Rouault [25]), and we will say more in the sequel. Moving on, certain elementary symmetric functions, the power sums and elementary symmetric functions, of the random points of $\mathbf{P}^{(s)}$ will be one of the main ingredients in our first main result.

Definition 1.3. *Let $s \geq 0$. Associated to the determinantal point process $\mathbf{P}^{(s)}$, we define random variables $\{q_n(s)\}_{n \geq 1}$ by, where the random point configuration \mathbb{E} is distributed according to $\mathbf{P}^{(s)}$,*

$$q_1(s) = \lim_{k \rightarrow \infty} \sum_{x \in \mathbb{E}} x \mathbf{1}_{|x| > k^{-2}}, \quad (4)$$

$$q_n(s) = \sum_{x \in \mathbb{E}} x^n, \quad \text{for } n \geq 2. \quad (5)$$

Moreover, define random variables $\{Y_n(s)\}_{n \geq 0}$ by letting $Y_0(s) \equiv 1, Y_1(s) = -q_1(s)$ and inductively, for $n \geq 2$,

$$Y_n(s) = -(n-1)! \sum_{j=1}^n \frac{1}{(n-j)!} Y_{n-j}(s) q_j(s). \quad (6)$$

It is not immediately obvious that these infinite sums are $\mathbf{P}^{(s)}$ -a.s. finite but this follows from the results of [24, 83]. We note that the principal value sum in the definition of $q_1(s)$ is required, otherwise the series does not converge. The random variables $\{Y_n(s)\}_{n \geq 1}$ are really just the elementary symmetric functions of $\mathbf{P}^{(s)}$ (up to explicit universal constants); the formula (6) is simply Newton's identity, in that it expresses the elementary symmetric functions in terms of power sums. We could have defined them directly by writing

$$Y_n(s) = (-1)^n n! \lim_{k \rightarrow \infty} \sum_{\substack{i_1 < i_2 < \dots < i_n \\ x_{i_1}, \dots, x_{i_n} \in \mathbb{E}}} x_{i_1} \cdots x_{i_n} \mathbf{1}_{\min(|x_{i_1}|, \dots, |x_{i_n}|) > k^{-2}}.$$

However, this would then require us to show directly from this definition that these random variables are almost surely finite, which would add technical details that are orthogonal to the analysis carried out in this paper. It is also worth noting that here, the normalization factor $(-1)^n n!$ is introduced to make the presentation of both Theorem 1.4 and its proof clearer (in particular, see Section 3).

Both families of random variables will appear in the results below, as in some cases one choice is better suited than the other. Below we normally denote by the symbol \mathbb{E} the expectation with respect to the underlying probability law, which will be clear from context.

With these definitions at hand, our first main result gives the leading order asymptotics of both quantities introduced in Definition 1.1 for the full parameter range.

Theorem 1.4. *Let $n_1 > n_2 > \dots > n_k \in \mathbb{N} \cup \{0\}$. Let $h_1, \dots, h_k \in \mathbb{R}_+$ and $s = \sum_{j=1}^k h_j > 0$. Then, we have the following finite limits,*

$$\lim_{N \rightarrow \infty} \frac{\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)}{N^{s^2 + 2 \sum_{j=1}^k n_j h_j}} = \frac{G(s+1)^2}{G(2s+1)} 2^{-2 \sum_{j=1}^k h_j n_j} \mathbb{E} \left[\prod_{j=1}^k |\Upsilon_{n_j}(s)|^{2h_j} \right], \quad (7)$$

$$\lim_{N \rightarrow \infty} \frac{\mathfrak{G}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)}{N^{s^2 + 2 \sum_{j=1}^k n_j h_j}} = \frac{G(s+1)^2}{G(2s+1)} 2^{-2 \sum_{j=1}^k h_j n_j} \mathbb{E} \left[\prod_{j=1}^k \left| \sum_{m_j=0}^{n_j} (-i)^{m_j} \binom{n_j}{m_j} \Upsilon_{n_j - m_j}(s) \right|^{2h_j} \right], \quad (8)$$

where $G(z)$ is the Barnes G -function, with γ the Euler-Mascheroni constant,

$$G(1+z) = (2\pi)^{\frac{z}{2}} \exp\left(-\frac{z+z^2(1+\gamma)}{2}\right) \prod_{j=1}^{\infty} \left(1 + \frac{z}{j}\right)^j \exp\left(\frac{z^2}{2j} - z\right).$$

Note that similar to the case of joint moments with the first order derivative, one can define the quantities in (2) and (3) for negative exponents $\{h_j\}$ as long as for all j , $h_j > -1/2$. However, the essential results from Section 2 that we use to deduce Theorem 1.4 require slight restrictions. Indeed, the use of the martingale argument requires an L^1 -integrability condition (see Proposition 2.11) which only holds when $s = \sum_{j=1}^k h_j > 0$. Furthermore, the uniform integrability we need, in order to obtain joint convergence as in Proposition 2.8, requires positivity of each h_j that corresponds to an $n_j \neq 0$. It is also worth observing that Theorem 1.4, in the more natural case of the characteristic polynomial V_A , is new whenever the exponent on any of the derivatives of V_A is non-integer. Even in the simplest possible case of $k=2$, $n_1=1$, $n_2=0$, this confirms the prediction of Hughes [60] from more than 20 years ago; the somewhat simpler case of the asymptotics of $\mathfrak{F}_N^{(1,0)}(h_1, h_2)$, also conjectured by Hughes, was in fact only proven in the last few years as well, see [14]. The probabilistic expressions in (7) and (8) in terms of the random variables $\{\Upsilon_n(s)\}_{n \geq 1}$ are aesthetically rather pleasing and, although initially may seem to come out of nowhere, it will be clear from the proof that they are rather natural. On the other hand, using them to compute quantities more explicitly is far from trivial. In proving the results that follow we will not make direct use of these expressions. Nevertheless, we stress that (7) and (8) are the only expressions for the leading order coefficients available in the literature which make sense for generic non-integer exponents h_i , even in the simplest case of the moments of a single derivative.

Towards our goal of computing the leading order coefficient more explicitly, our next

main result gives a representation of the expectation on the right-hand side of (7), for $h_1, \dots, h_k \in \mathbb{N}$, as a finite linear combination of explicit finite-dimensional integrals. These integrals can all be computed explicitly (one simply expands the Vandermonde determinant and these integrals boil down to one-dimensional integrals which are basically moments of a Student's t-distribution) but writing down a general expression becomes very tedious and we will not attempt to do it here; we will instead discuss an alternative recursive way to obtain such expressions via connections to integrable systems below. In particular, the formula in Theorem 1.5 gives a concrete expression for the joint moments whenever the exponent on the characteristic polynomial itself is a real number and the exponents on its derivatives are integers.

These finite-dimensional averages will be taken with respect to the Cauchy [50, 90] (also called Hua-Pickrell [24]) measures $\mathbf{M}_N^{(s)}$, with $s \geq 0$, which are probability measures supported on the Weyl chamber,

$$\mathbb{W}_N \stackrel{\text{def}}{=} \left\{ \mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 \geq x_2 \geq \dots \geq x_N \right\}$$

that are given by:

$$\mathbf{M}_N^{(s)}(dx_1, \dots, dx_N) \stackrel{\text{def}}{=} \frac{1}{\tilde{\mathbf{C}}_N^{(s)}} \prod_{i=1}^N \frac{1}{(1+x_i^2)^{s+N}} \Delta^2(x_1, \dots, x_N) dx_1 \cdots dx_N, \quad (9)$$

where we denote by Δ the Vandermonde determinant,

$$\Delta(x_1, \dots, x_N) \stackrel{\text{def}}{=} \prod_{1 \leq i < k \leq N} (x_i - x_k),$$

and the normalization constant $\tilde{\mathbf{C}}_N^{(s)}$ is given explicitly by

$$\tilde{\mathbf{C}}_N^{(s)} = \pi^N 2^{-N(N+2s-1)} \prod_{j=0}^{N-1} \frac{j! \Gamma(2s + N - j)}{\Gamma(s + N - j)^2}.$$

For the rest of the paper, we denote averages taken with respect to these measures as:

$$\mathbb{E}_N^{(s)} [f(\mathbf{x}_1^{(N)}, \dots, \mathbf{x}_N^{(N)})] \stackrel{\text{def}}{=} \int_{\mathbb{W}_N} f(x_1, \dots, x_N) \mathbf{M}_N^{(s)}(dx_1, \dots, dx_N).$$

Note that, whenever the function f is invariant under permutations of indices, and this is the case for all functions we will consider in this paper, one can equivalently write:

$$\mathbb{E}_N^{(s)} [f(\mathbf{x}_1^{(N)}, \dots, \mathbf{x}_N^{(N)})] = \frac{1}{N!} \int_{\mathbb{R}^N} f(x_1, \dots, x_N) \mathbf{M}_N^{(s)}(dx_1, \dots, dx_N). \quad (10)$$

We also observe that by analytic continuation the function $s \mapsto \mathbb{E}_N^{(s)}[\bullet]$ is holomorphic whenever the integrand is holomorphic in s and the integral on the right hand side of (10) exists. We note that although for $s \in \mathbb{C} \setminus \mathbb{R}$ the probabilistic interpretation breaks down, abusing notation we will still use the expectation sign $\mathbb{E}_N^{(s)}$ below. As the discussion preceding Theorem 1.4 suggests, throughout the rest of the paper, we will commonly use elementary symmetric polynomials and power sum polynomials, which we will denote

by $(e_0 = p_0 \equiv 1)$

$$e_n(x_1, x_2, \dots, x_m) \stackrel{\text{def}}{=} \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq m} x_{i_1} x_{i_2} \cdots x_{i_n},$$

$$p_n(x_1, x_2, \dots, x_m) \stackrel{\text{def}}{=} \sum_{i=1}^m x_i^n.$$

We are now in a position to state our second main result.

Theorem 1.5. *Let $n_1 > \dots > n_k \in \mathbb{N} \cup \{0\}$, and $h_j \in \frac{\mathbb{N}}{2}$, for $j = 1, \dots, k$. Moreover, define $L = \sum_j 2h_j n_j$. Then, for any real number $s > \sum_j h_j - \frac{1}{2}$, we have*

$$\mathbb{E} \left[\prod_{j=1}^k (Y_{n_j}(s))^{2h_j} \right] = \frac{(n_1!)^{2h_1} \cdots (n_k!)^{2h_k}}{L!} \sum_{m=n_1}^L (-1)^m \binom{L}{m} \mathbb{E}_m^{(s)} \left[\prod_{j=1}^k (e_{n_j}(x_1^{(m)}, x_2^{(m)}, \dots, x_m^{(m)}))^{2h_j} \right].$$

In particular, we obtain that the map

$$s \mapsto \mathbb{E} \left[\prod_{j=1}^k (Y_{n_j}(s))^{2h_j} \right]$$

is a rational function.

The above result also has the following application. A priori, it is not clear that the limiting random variables $Y_n(s)$ are almost surely non-zero. Although it may seem highly unlikely that they are almost surely zero, by some extreme coincidence it could happen. If this were the case, the asymptotics we established earlier would give a bound on the growth of the joint moments, but not the true order. In fact, in the literature on this problem, the proof that the leading coefficient in such asymptotics is non-zero is omitted (even when this coefficient is given as an explicit sum it is non-trivial to see that it is strictly positive as the sum involves both positive and negative terms), with the only exception, as far as we are aware, being [14]. It may be possible to use the determinantal point process representation of the random variables $Y_n(s)$ to conclude that they are not a.s. zero but this is most likely challenging. Instead, we simply obtain an explicit expression for the second moment of $Y_n(s)$ for any $n \in \mathbb{N}$, and show that it is non-zero and similarly for the expression appearing in the asymptotics in (8). Despite their simplicity, the expressions below are new for any choice of the parameter s . It is interesting to note that the expression for V_A and ζ is much simpler than the one for Z_A and \mathcal{Z} , even though its probabilistic expression is more complicated.

Corollary 1.6. *Let $n \in \mathbb{N}$, and $s \in (\frac{1}{2}, \infty)$. Then we have,*

$$\mathbb{E} [|Y_n(s)|^2] = 2^{2n} \frac{2s-1}{\prod_{l=1}^n (2s-2+l)^2} \sum_{i,j=0}^n \binom{n}{i} \binom{n}{j} \frac{(-2)^{-2n+i+j}}{2s-1+i+j} \frac{\Gamma(s+i)\Gamma(s+j)\Gamma(2s+n-1)^2}{\Gamma(2s+i-1)\Gamma(2s+j-1)\Gamma(s)^2},$$

$$\mathbb{E} \left[\left| \sum_{m=0}^n (-i)^m \binom{n}{m} Y_{n-m}(s) \right|^2 \right] = 2^{2n} \frac{2s-1}{2s-1+2n} \prod_{l=1}^n \left(\frac{l+s-1}{l+2s-2} \right)^2. \quad (11)$$

Moreover, we have that,

$$\mathbb{P}\left(\left|\sum_{m=0}^n (-i)^m \binom{n}{m} Y_{n-m}(s)\right| > 0\right) > 0 \quad \text{and} \quad \mathbb{P}(|Y_n(s)| > 0) > 0.$$

Although up until now we have restricted our attention to a rather special random matrix model, the Haar-distributed unitary matrices, as an immediate by-product of the methods we use to prove Theorems 1.4 and 1.5 we obtain the following general result on characteristic polynomials of infinite permutation-invariant random matrices (we refer to Section 2 for a precise explanation of what is meant by an infinite random matrix). In some sense, and this will be clear in the sequel, parts of Theorem 1.4 and Theorem 1.5 are special cases of the theorem below for a specific choice of infinite random matrix \mathbf{H} .

Theorem 1.7. *Let \mathbf{H} be an infinite Hermitian random matrix so that for any permutation τ of \mathbb{N} that fixes all but finitely many elements,*

$$\mathbf{Law}(\mathbf{H}) = \mathbf{Law}(\mathbf{P}_\tau \mathbf{H} \mathbf{P}_\tau^*),$$

where \mathbf{P}_τ is the permutation matrix that corresponds to τ , and $\mathbf{Law}(\mathfrak{X})$ denotes the law of a random variable \mathfrak{X} . Consider, for each $N \in \mathbb{N}$, the $N \times N$ top-left corner of \mathbf{H} , $\mathbf{H}_N = [\mathbf{H}_{ij}]_{i,j=1,\dots,N}$, and its rescaled (reverse) characteristic polynomial,

$$p_N(z) = \det\left(\mathbf{I} - \frac{z}{N} \mathbf{H}_N\right) = 1 + \sum_{k=1}^N \mathbf{a}_N^{(k)}(\mathbf{H}) z^k.$$

Then, whenever there exists $m \in \mathbb{N}$, $p \in [1, \infty)$ such that

$$\mathbb{E}\left[\left|\det(\mathbf{H}_k)\right|^p\right] < \infty$$

for $k = 1, \dots, m$, then there exists random variables $\{\mathbf{a}_k(\mathbf{H})\}_{k=1}^m$ such that, for $k = 1, \dots, m$,

$$\mathbf{a}_N^{(k)}(\mathbf{H}) \xrightarrow{N \rightarrow \infty} \mathbf{a}_k(\mathbf{H})$$

in L^p and almost surely with respect to $\mathbf{Law}(\mathbf{H})$. Furthermore, if $r_1, \dots, r_m \in \mathbb{R}_{\geq 0}$, are such that $\sum_j r_j = p$, then we have

$$\mathbb{E}\left[\prod_{k=1}^m \left|\mathbf{a}_N^{(k)}(\mathbf{H})\right|^{r_j}\right] \xrightarrow{N \rightarrow \infty} \mathbb{E}\left[\prod_{k=1}^m \left|\mathbf{a}_k(\mathbf{H})\right|^{r_j}\right].$$

Moreover, if $r_1, \dots, r_l \in \mathbb{N}$, $m \geq k_1 > \dots > k_l \geq 1$ and if we define $L \stackrel{\text{def}}{=} \sum_j r_j k_j$, we then have

$$\mathbb{E}\left[\prod_{j=1}^l \left(\mathbf{a}_{k_j}(\mathbf{H})\right)^{r_j}\right] = \frac{(k_1!)^{r_1} \cdots (k_l!)^{r_l}}{L!} \sum_{m=k_1}^L (-1)^{m+L} \binom{L}{m} \mathbb{E}\left[\prod_{j=1}^l \left(\mathbf{a}_m^{(k_j)}(\mathbf{H})\right)^{r_j}\right]. \quad (12)$$

We remark that it was recently proven in [11] that for a subclass of infinite permutation-invariant random matrices, the class of unitarily-invariant ones, which have been classified in [81, 82], $p_N(z)$ itself converges almost surely with respect to $\mathbf{Law}(\mathbf{H})$, uniformly on compact sets in \mathbb{C} , to a random analytic function $p_\infty(z)$. The theorem above extends this

convergence to convergence of the moments of Taylor coefficients and gives a formula for the joint moments of the limiting coefficients in terms of the pre-limit ones.

1.4 NUMBER THEORY CONJECTURES

Following the philosophy set out in [70, 69], our results stated thus far give rise to the following conjecture, which generalises all previous conjectures on this problem. Here, we write out the special case of Corollary 1.6 separately since the leading order coefficient admits a surprisingly simple new formula.

Conjecture 1.8. *Let $n_1 > \dots > n_k \in \mathbb{N} \cup \{0\}$, $h_1, \dots, h_k \in (0, \infty)$ and $s = \sum_{j=1}^k h_j$. Then, we have that*

$$\frac{1}{T} \int_0^T \prod_{j=1}^k \left| \zeta^{(n_j)} \left(\frac{1}{2} + it \right) \right|^{2h_j} dt \sim (\log T)^{s^2 + 2\sum_{j=1}^k h_j n_j} \beta(s) \mathbb{E} \left[\prod_{j=1}^k \left| \sum_{m_j=0}^{n_j} (-i)^{m_j} \binom{n_j}{m_j} \frac{Y_{n_j - m_j}(s)}{2^{n_j}} \right|^{2h_j} \right]$$

where

$$\beta(s) \stackrel{\text{def}}{=} \frac{G(s+1)^2}{G(2s+1)} \prod_{\text{primes } p} (1-p^{-1})^{s^2} \sum_{k=0}^{\infty} p^{-k} \left(\frac{\Gamma(k+s)}{\Gamma(k+1)\Gamma(s)} \right)^2.$$

In particular, for any $n \in \mathbb{N}$ and $s > \frac{1}{2}$ we have

$$\frac{1}{T} \int_0^T \left| \zeta^{(n)} \left(\frac{1}{2} + it \right) \right|^2 \left| \zeta \left(\frac{1}{2} + it \right) \right|^{2s-2} dt \sim (\log T)^{s^2 + 2n} \beta(s) \frac{2s-1}{2s-1+2n} \prod_{l=1}^n \left(\frac{l+s-1}{l+2s-2} \right)^2.$$

Similarly, we have the analogous relations:

$$\begin{aligned} \frac{1}{T} \int_0^T \prod_{j=1}^k |\mathcal{Z}^{(n_j)}(t)|^{2h_j} dt &\sim (\log T)^{s^2 + 2\sum_{j=1}^k h_j n_j} \beta(s) 2^{-2\sum_{j=1}^k h_j n_j} \mathbb{E} \left[\prod_{j=1}^k |Y_{n_j}(s)|^{2h_j} \right], \\ \frac{1}{T} \int_0^T |\mathcal{Z}^{(n)}(t)|^2 |\mathcal{Z}(t)|^{2s-2} dt &\sim (\log T)^{s^2 + 2n} \beta(s) \frac{2s-1}{\prod_{l=1}^n (2s-2+l)^2} \\ &\left\{ \sum_{i,j=0}^n \binom{n}{i} \binom{n}{j} \left(-\frac{1}{2}\right)^{2n-i-j} \frac{1}{2s-1+i+j} \prod_{l=0}^{i-1} (l+s) \prod_{l=0}^{j-1} (l+s) \prod_{l=i+1}^n (l+2s-2) \prod_{l=j+1}^n (l+2s-2) \right\}. \end{aligned}$$

For the original moment conjecture proposed by Keating and Snaith, specifically Conjecture 1.8 with $k = 1, n_1 = 0$, has been verified for exponents $h_1 = 1$ and 2 , but remains open for any exponent $h_1 \geq 3$. Conjecture 1.8 extends the original conjecture by generalizing the order of derivatives from 0 to any positive integer. This extension introduces additional parameters n_1, \dots, n_k , thereby providing a broader framework for investigating the moment conjecture. Specifically, for small exponents h_j , the validity of Conjecture 1.8 should be provable for various choices of n_j , the order of the derivatives. We shall list some verified cases below. Furthermore, Conjecture 1.8 is closely related to the distribution of the zeros of Hardy's \mathcal{Z} -function, which we will elaborate on below.

Some special cases of Conjecture 1.8 for higher orders of the derivative and small exponents have been verified in the past. For $k = 1, n_1 = n, h_1 = 1$, it was proved by

Ingham [61] for the Riemann zeta function and by Hall [55] for Hardy's \mathcal{Z} -function. For $k = 2, n_1 = 1, n_2 = 0$, for the Riemann zeta function, the following special exponents h_1, h_2 were proven by: Conrey [31] for $h_1 = 1, h_2 = 2$, and $h_1 = 0, h_2 = 2$; Conrey and Ghosh [34] for $h_1 = 1/2, h_2 = 1/2$ (in this case, assuming the Riemann hypothesis); and Hall [55] for $h_1 = 2, h_2 = 0$, and $h_1 = 1, h_2 = 1$. For $k = 2, n_1 = 2, n_2 = 0$, for Hardy's \mathcal{Z} -function, the following special exponents h_1, h_2 were proven by: Hall [55] for $h_1 = 1, h_2 = 1$, and Hall [57] for $h_1 = 2, h_2 = 0$. For $k = 2, n_1 = 2, n_2 = 1$, for Hardy's \mathcal{Z} -function, the exponents $h_1 = 1, h_2 = 1$ was shown by Hall [55].

Conjecture 1.8 is closely related to problems in number theory concerning large gaps between zeros of the higher-order derivatives of Hardy's \mathcal{Z} -function. Let

$$\Lambda^{(m)} = \limsup_{n \rightarrow \infty} \frac{t_{n+1}^{(m)} - t_n^{(m)}}{2\pi / \log t_n^{(m)}},$$

where $\{t_n^{(m)}\}$ is the sequence of non-negative zeros of $\mathcal{Z}^{(m)}(t)$, counted according to multiplicity. For $m = 0$, the quantity $\Lambda^{(0)}$ was introduced by Hall [55]. Under the Riemann hypothesis,

$$\Lambda^{(0)} = \Lambda := \limsup_{n \rightarrow \infty} \frac{\gamma_{n+1} - \gamma_n}{2\pi / \log \gamma_n},$$

where $\{\gamma_n\}$ is the non-decreasing sequence of imaginary parts of the zeros of the Riemann zeta function in the upper half-plane, counted with multiplicity. The quantity Λ was introduced by Selberg [86], who proved that $\Lambda > 1$. Note that Λ and $\Lambda^{(m)}$ have an average value of 1 under the Riemann hypothesis. This has motivated number theorists to investigate how large these quantities can be. Conrey, Ghosh, and Gonek showed that $\Lambda > 2.337$ under the Riemann hypothesis [29] and $\Lambda > 2.68$ under the Generalized Riemann hypothesis [30]. Bui and Milinovich showed that $\Lambda > 3.18$ under the Riemann hypothesis [26]. In [56], Hall did not assume the Riemann hypothesis, but instead assumed the truth of Conjecture 1.8 for $k = 2, n_1 = 1, n_2 = 0, h_1 + h_2 \leq 6$, and proved that $\Lambda^{(0)} > 4.29$ times the average gap length. It is widely believed that $\Lambda^{(0)} = \infty$, but this problem remains open even under the assumption of the Generalized Riemann hypothesis. As far as we know, there is no specific connection between the Riemann hypothesis and the moments conjecture. Thus it is interesting to explore whether one can achieve $\Lambda^{(0)} = \infty$ under Conjecture 1.8 or under the combined assumptions of the Riemann hypothesis and Conjecture 1.8. Recently, Bui and Hall [27] obtained lower bounds of $\Lambda^{(m)}$ for some integers m with $m \geq 1$. It is hopeful to generalize Hall's method in [56] and assume Conjecture 1.8 for some special cases to obtain a sharper lower bound of $\Lambda^{(m)}$. We leave this to future research.

Finally, it is worth noting that some hard number-theoretic conjectures originating from random matrix theory considerations have been proved in the past. For example, see [9, 10, 8, 77] for recent proofs of a conjecture [53, 54] concerning the maximum of the Riemann zeta function in typical short intervals on the critical line. We also point out that the aforementioned conjectures become theorems in the function field setting over \mathbb{F}_q in the limit $q \rightarrow \infty$, using equidistribution results due to Deligne and Katz [68].

1.5 CONNECTION TO INTEGRABLE SYSTEMS

As remarked in Section 1.2, over the past two decades, the problem of joint moments was realized to be closely related to the theory of the Painlevé equations (see [19, 17, 14, 12]).

In the case of joint moments with only the first derivative, it was discovered only recently that the random variable $Y_1(s) = -q_1(s)$ is in fact responsible for this connection between joint moments and integrable systems. To be more precise, it was shown in [14] and [12] that, for any $s \geq 0$, the function $\tau^{(s)}$ defined by,

$$t \mapsto \tau^{(s)}(t) \stackrel{\text{def}}{=} t \frac{d}{dt} \log \mathbb{E} \left[e^{it \frac{Y_1(s)}{2}} \right]$$

satisfies the following form of the σ -Painlevé III' equation:

$$\left(t \frac{d^2 \tau^{(s)}}{dt^2} \right)^2 = -4t \left(\frac{d\tau^{(s)}}{dt} \right)^3 + (4s^2 + 4\tau^{(s)}) \left(\frac{d\tau^{(s)}}{dt} \right)^2 + t \frac{d\tau^{(s)}}{dt} - \tau^{(s)}, \quad (13)$$

for any $t \in \mathbb{R} \setminus \{0\}$, along with the initial conditions:

$$\begin{cases} \tau^{(s)}(0) = 0, & \text{for } s > 0, \\ \left. \frac{d}{dt} \tau^{(s)}(t) \right|_{t=0} = 0, & \text{for } s > \frac{1}{2}. \end{cases} \quad (14)$$

Here, we note that for the derivatives to exist at $t = 0$, one requires further conditions on the parameter s . That is, in order to guarantee that

$$\left. \frac{d^m}{dt^m} \mathbb{E} \left[e^{it \frac{Y_1(s)}{2}} \right] \right|_{t=0} < \infty$$

one needs to have $s > \frac{m-1}{2}$. Moving on, we note that the result above can be useful in many ways. First of all, we can see immediately that all the even integer moments (when they exist) of $Y_1(s) = -q_1(s)$ have a representation in terms of the σ -Painlevé III' equation, since we have,

$$\mathbb{E} \left[(Y_1(s))^{2h} \right] = 2^{2h} (-1)^h \frac{d^{2h}}{dt^{2h}} \left[\exp \left(\int_0^t \frac{\tau^{(s)}(u)}{u} du \right) \right] \Big|_{t=0}.$$

But more ambitiously, if one can show that the characteristic function of $Y_1(s)$ is in $L^1(\mathbb{R})$ and if it admits an explicit expression (and this is a highly non-trivial, being connected to deep results on classical solutions of Painlevé equations; see [87, 88] for what is meant by a classical solution in the context of Painlevé equations), then the expression for $\tau^{(s)}(t)$ can be used, by means of Fourier inversion, to recover an explicit density for $Y_1(s)$. This is precisely what is done in [12], for $s \in \mathbb{N}$, where the density is then used to give an explicit expression for any finite (including complex) moments of $Y_1(s)$.

Given the previous results presented in our paper, it is then natural to wonder whether certain joint moments of the random variables $\{Y_n(s)\}_{n \geq 1}$ admit representations in terms of Painlevé transcendents. We answer this question in the affirmative in Theorem 1.11 below by representing such joint moments as linear combinations, with polynomial coefficients, of derivatives of solutions to the σ -Painlevé III' equation. This is presented in terms of the allied family of random variables $\{q_n(s)\}_{n \geq 1}$ which is more convenient. In fact, we show a more general result, related to a richer object, a certain one-dimensional function, from which the joint moments can be recovered.

Proving Theorem 1.11 directly from the definition of the $\{q_n(s)\}_{n \geq 1}$ appears to be difficult. Instead, we prove it by an approximation from finite N . In order to do this, we

first need to find the correct finite-dimensional analogue of the expression on the left hand side of (16) below. We then develop a method involving a class of Hankel determinants shifted by partitions (this will be given precise meaning below) that allows us, after rather long and intricate computations, to obtain the following structure theorem for finite N . Recall that, for $s \in \mathbb{C}$, $\mathbb{E}_N^{(s)}[\bullet]$ is interpreted as explained below display (10).

Theorem 1.9. *Let $k \geq 2$, $n_2, \dots, n_k \in \mathbb{N} \cup \{0\}$ and $\sum n_j > 0$. Let $s \in \mathbb{C}$ with $\Re(s) > 2^{-1}(\sum_{q=2}^k qn_q - 1)$. Then, for all $t_1 \geq 0$,*

$$\begin{aligned} & \frac{(-2i)^{-\sum_{q=2}^k qn_q}}{N^{n_2 + \dots + n_k}} \mathbb{E}_N^{(s)} \left[e^{-it_1 \sum_{j=1}^N \frac{x_j^{(N)}}{N}} \prod_{q=2}^k \left(\sum_{j=1}^N (x_j^{(N)} - i)^q \right)^{n_q} \right] \\ &= \frac{1}{t_1^{\sum_{q=2}^k qn_q - 1}} \sum_{m=0}^{\sum_{q=2}^k (q-1)n_q} t_1^{m-1} P_m^{(s,N)}(t_1) \frac{d^m}{dt_1^m} \mathbb{E}_N^{(s)} \left[e^{-it_1 \frac{1}{N} \sum_{j=1}^N x_j^{(N)}} \right], \end{aligned} \tag{15}$$

where $t_1^{m-1} P_m^{(s,N)}(t_1)$ are polynomials of t_1 of degree at most $\sum_{q=2}^k qn_q - 1$, respectively. Moreover, the coefficients of these polynomials are polynomials in N, s , and with degrees in terms of N or s no more than $\sum_{q=2}^k (q-1)n_q$.

Analogously to $Y_1(s) = -q_1(s)$, whose characteristic function can be represented in terms of the σ -Painlevé III' equation, its pre-limit version also admits a Painlevé representation. More precisely, it was shown in [12] that the function defined by, for $s > -\frac{1}{2}$,

$$t \mapsto t \frac{d}{dt} \log \mathbb{E}_N^{(s)} \left[e^{-i \frac{t}{2N} \sum_{j=1}^N x_j^{(N)}} \right],$$

which we denote here by $\tau_N^{(s)}(t)$, satisfies, for all $t \in \mathbb{R} \setminus \{0\}$, the following form of the σ -Painlevé V equation,

$$\begin{aligned} \left(t \frac{d^2 \tau_N^{(s)}}{dt^2} \right)^2 &= -4t \left(\frac{d\tau_N^{(s)}}{dt} \right)^3 + \left(4s^2 + 4\tau_N^{(s)} + \frac{t^2}{N^2} \right) \left(\frac{d\tau_N^{(s)}}{dt} \right)^2 + t \left(1 + \frac{2s}{N} - \frac{2\tau_N^{(s)}}{N^2} \right) \frac{d\tau_N^{(s)}}{dt} \\ &\quad - \left(1 + \frac{2s}{N} - \frac{\tau_N^{(s)}}{N^2} \right) \tau_N^{(s)}. \end{aligned}$$

Now, we demonstrate an application of Theorem 1.9. By the integral representation of joint moments for any order of derivatives that we prove in Proposition 3.1, we see that general joint moments are a linear combination of derivatives of certain orders of the function appearing in the left hand-side of (15) at $t_1 = 0$. Thus, Theorem 1.9 gives an exact representation at the finite- N level for the joint moments of the characteristic polynomial and its derivatives, expressed in terms of derivatives of a σ -Painlevé V transcendent. The result holds for arbitrary non-negative real exponents of the characteristic polynomial itself and for its derivatives of any order with positive integer exponents. This result is new, and as far as we can tell, it is inaccessible by other methods. Before our work, it was only known that the joint moments of the first order derivative for finite matrix size can be represented in terms of a σ -Painlevé V transcendent [19, 12]. Using the above representation and some properties of the σ -Painlevé V transcendents, we obtain, for

the first time, an explicit recursive method to compute all the finite dimensional joint moments exactly.

It is worth pointing out that having an explicit finite- N expression is valuable in its own right: firstly, having an exact formula at the finite- N level allows one to conjecture lower order terms in the asymptotics of the joint moments of ζ , as in [33] (although we do not do this here). More importantly, the connection between random matrix theory and the ζ -function extends to other L -functions. For example, Katz and Sarnak [68] proved several results and provide heuristic evidence, that various statistics in families of L -functions (including moments over the families) can be modelled by considering analogous statistics over Haar-distributed unitary, orthogonal and symplectic matrices, depending on the symmetry type of the family (see [62, 84, 69] for a number of related results). It follows from the work of Katz and Sarnak that the explicit finite- N expressions for joint moments of characteristic polynomials of random matrices correspond *precisely* to the joint moments of ζ -functions associated with curves over finite fields of q elements in the limit $q \rightarrow \infty$. Hence, the relation between the finite- N expression and Painlevé- V gives a direct connection between analytic number theory and integrable systems in this context. To give the reader a flavour of the explicit finite- N expressions we can obtain, the following is a consequence of the machinery developed in Section 4.

Proposition 1.10. *For all $s \in (\frac{1}{2}, \infty)$,*

$$\begin{aligned} \mathfrak{F}_N^{(2,0)}(1, s-1) &= \frac{\mathfrak{F}_N^{(0)}(s)}{16} \left(\frac{N^4}{(2s+3)(2s-1)} + \frac{4sN^3}{(2s+3)(2s-1)} \right. \\ &\quad \left. + \frac{4(2s^3 + s^2 - 1)N^2}{(2s+3)(2s-1)(2s+1)} - \frac{8sN}{(2s+3)(2s-1)(2s+1)} \right). \end{aligned}$$

Returning to our main goal, the following is our main result on connections to integrable systems; again recall the characteristic function of $q_1(s) = -Y_1(s)$ is given in terms of σ -Painlevé III' transcendents. In the simplest possible case: with $k = 2$, $s \in \mathbb{N}$, and after taking the derivative with respect to t_1 and evaluating at $t_1 = 0$, Theorem 1.11 recovers (in equivalent form; the connection to the random variables $\{q_n(s)\}_{n \geq 1}$ was not clear yet in [72]) the main result of [72]. The following result will be proven by a limit transition as $N \rightarrow \infty$ of the various quantities involved in Theorem 1.9. Making these $N \rightarrow \infty$ limits rigorous from their explicit $N \times N$ (generalised) determinant representations is very difficult, but fortunately we can use our earlier probabilistic limit results as input.

Theorem 1.11. *Let $k \geq 2$, $n_2, \dots, n_k \in \mathbb{N} \cup \{0\}$ and $\sum n_j > 0$ and suppose $s \in \mathbb{R}$ with $s > 2^{-1}(\sum_{\ell=2}^k \ell n_\ell - 1)$. Then, for $t_1 \geq 0$,*

$$(-2i)^{-\sum_{\ell=2}^k \ell n_\ell} \mathbb{E} \left[e^{-it_1 q_1(s)} \prod_{j=2}^k q_j(s)^{n_j} \right] = \frac{1}{t_1^{\sum_{\ell=2}^k \ell n_\ell - 1}} \sum_{m=0}^{\sum_{\ell=2}^k (\ell-1)n_\ell} t_1^{m-1} \mathcal{A}_m^{(s)}(t_1) \frac{d^m}{dt_1^m} \mathbb{E} \left[e^{-it_1 q_1(s)} \right], \quad (16)$$

where $t_1^{m-1} \mathcal{A}_m^{(s)}(t_1)$ are polynomials in t_1 of degree at most $\sum_{\ell=2}^k (\ell-1)n_\ell - 1$ and whose coefficients are polynomials themselves of s , with degree in s at most $\sum_{\ell=2}^k (\ell-1)n_\ell$.

Even though we state Theorems 1.9 and 1.11 for $t_1 \geq 0$, by noting that

$$\text{Law}(-x_1^{(N)}, -x_2^{(N)}, \dots, -x_N^{(N)}) = \text{Law}(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)})$$

one immediately gets analogous results, both for finite N and in the $N \rightarrow \infty$ limit, for $t_1 < 0$. We omit the statements.

Moving on, the final piece of the puzzle to get Theorem 1.11, or more precisely Theorem 1.9, is an intricate induction on the indices n_2, \dots, n_k . As alluded to above, the arguments we use for the inductive step will give us a recursive way to obtain explicit formulae for the polynomials $\mathcal{A}_m^{(s)}(t_1)$, and hence for joint moments of the random variables $\mathbf{q}_j(s)$ (or equivalently, $Y_j(s)$). Namely, one can consider the asymptotic expansion of the characteristic function of $\mathbf{q}_1(s)$, as $t_1 \rightarrow 0$, and make use of its relation to the σ -Painlevé III' equation to obtain integer moments of $\mathbf{q}_1(s)$ (see for instance [19]), allowing one, by virtue of (16), to compute all the integer joint moments recursively and express them as rational functions of s efficiently. We give a few examples of these formulae in the corollary below. Despite their simplicity, these formulae are new and seem hard to obtain in any other way (and also do not follow from Theorem 1.5 since this only involves joint moments and not the characteristic function itself which is a richer object). In particular, they seem very challenging to prove directly from the determinantal point process definition.

Corollary 1.12. *For all $s \in (\frac{1}{2}, \infty)$, we have*

$$\mathbb{E}\left[e^{-it_1 \mathbf{q}_1(s)} \mathbf{q}_2(s)\right] = -\frac{2s}{t_1} \frac{d}{dt_1} \mathbb{E}\left[e^{-it_1 \mathbf{q}_1(s)}\right],$$

and for all $s \in (\frac{3}{2}, \infty)$,

$$\mathbb{E}\left[e^{-it_1 \mathbf{q}_1(s)} \mathbf{q}_2(s)^2\right] = \frac{4s^2 + 2}{t_1^2} \frac{d^2}{dt_1^2} \mathbb{E}\left[e^{-it_1 \mathbf{q}_1(s)}\right] - \frac{12s^2}{t_1^3} \frac{d}{dt_1} \mathbb{E}\left[e^{-it_1 \mathbf{q}_1(s)}\right] - \frac{2}{t_1^2} \mathbb{E}\left[e^{-it_1 \mathbf{q}_1(s)}\right].$$

The above formulae looks at first sight to be singular when $t_1 = 0$, but this is not the case: the moments of $\mathbf{q}_1(s)$ and $\mathbf{q}_2(s)$ satisfy certain relations and this guarantees that the singular t_1 powers in the denominator cancel out at $t_1 = 0$. In particular, evaluating at $t_1 = 0$ allows one to get, for instance, the following simple formulae:

$$\mathbb{E}[\mathbf{q}_2(s)] = 2s \mathbb{E}[\mathbf{q}_1^2(s)] = \frac{2s}{4s^2 - 1}.$$

Finally, we would like to emphasize that Theorem 1.11 and Corollary 1.12 not only give Painlevé representations for integer joint moments of $\mathbf{q}_n(s)$, but also for joint moments where the power of $\mathbf{q}_1(s)$ is any (suitable) real number via the elementary identity

$$|y|^p = C_p \int_0^\infty \frac{1 - \cos(ty)}{t^{p+1}} dt, \tag{17}$$

which holds for all $y \in \mathbb{R}$, $p \in (0, 2)$ with

$$\frac{1}{C_p} = \int_0^\infty \frac{1 - \cos(t)}{t^{p+1}} dt = \frac{1}{p} \cos\left(\frac{\pi p}{2}\right) \Gamma(1 - p), \tag{18}$$

where the integral is computed via a generalized Fresnel integral. Indeed, using this one

immediately gets that if $h = n_1 + p$, where $n_1 \in 2\mathbb{N}$, $p \in (0, 2)$, then

$$\mathbb{E} \left[|\mathbf{q}_1(s)|^h \prod_{j=2}^k \mathbf{q}_j(s)^{n_j} \right] = C_p \int_0^\infty \frac{\Re \left\{ \mathbb{E} \left[\prod_{j=1}^k \mathbf{q}_j(s)^{n_j} (1 - e^{-it\mathbf{q}_1(s)}) \right] \right\}}{t^{p+1}} dt,$$

where the interchange of the expectation and the integral is justified by use of the trivial inequality $1 - \cos(x) \leq \min(2, x^2)$. In particular, the average inside the integral can be written, via Theorem 1.11, in terms of Painlevé transcendents. Furthermore, using the explicit formula obtained in [12, Proposition 2.6] for $s \in \mathbb{N}$,

$$\varphi_s(t) \stackrel{\text{def}}{=} \frac{(-1)^{s(s-1)/2} G(2s+1)}{2^{s^2/2} G(s+1)^2} \times \frac{\det_{0 \leq j, k \leq s-1} [I_{j+k+1}(2\sqrt{2|t|})]}{e^{|t||t|^{s^2/2}} = \mathbb{E} [e^{-it\mathbf{q}_1(s)}], \quad (19)$$

where I_α denotes the modified Bessel function of the first kind, one can recover even further explicit formulae for joint moments of $\mathbf{q}_n(s)$, and hence of $\mathbf{Y}_n(s)$, which do not follow from the probabilistic techniques we develop to prove the results of Section 1.3. Here, we write out this formula for the cases where the polynomials $\mathcal{A}_m^{(s)}(t)$ admit rather simple expressions; namely, the cases given in Corollary 1.12. To keep our final expression relatively simple, we will assume that $n_1 = 0$ in the setting described above. Then for all $p \in (0, 2)$, and $s \geq 2$, $s \in \mathbb{N}$,

$$\mathbb{E} [|\mathbf{q}_1(s)|^p \mathbf{q}_2(s)^2] = C_p \int_0^\infty \frac{D_s \varphi_s(0) - D_s \varphi_s(t)}{t^{p+1}} dt,$$

where φ_s is as in (19), D_s is the differential operator defined, for each $f \in C^2(\mathbb{R})$ as,

$$D_s f(t) \stackrel{\text{def}}{=} \frac{4s^2 + 2}{t^2} \frac{d^2 f}{dt^2} - \frac{12s^2}{t^3} \frac{df}{dt} - \frac{2}{t^2} f,$$

and the constant C_p is as in (18).

1.6 STRATEGY OF PROOF

We give here an outline of the proofs of our main theorems and the difficulties that arise. We first note that the multiple-contour integral expressions used in [72, 71] and the symmetric function theory framework of [18] cannot be used to attack the joint moments problem in complete generality as the formulae therein do not make sense for non-integer exponents h_i . Thus, we need a different starting point. The first step is to transform the problem into one of moments over Hermitian matrices, with averages taken with respect to the Cauchy random matrix ensemble whose law of eigenvalues is given by (9). In the simplest case of joint moments with only the first derivative, namely with $k = 2$, $n_2 = 1$, $n_1 = 0$, this connection was implicit in Winn's work [89] and brought to center stage in [14]. It is not hard to prove that it extends to arbitrary joint moments. More precisely, a computation, given in Proposition 3.1, establishes that both $\mathfrak{G}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)$ and $\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)$ can be written as expectations of certain explicit symmetric polynomials in the eigenvalues of a $N \times N$ Cauchy random matrix \mathbf{H}_N , see (32).

At this stage it can be shown (this allowed us to guess the form of our first main result), by putting together general results from [81, 24, 15], that symmetric polynomials in the

eigenvalues of \mathbf{H}_N converge jointly in distribution. We stress however that convergence of the moments does not follow from the general arguments of [81, 24, 15] and one needs non-trivial estimates for this. The crux of the problem is that, in order to obtain joint moment convergence for elementary symmetric polynomials,

$$e_k(x_1^{(N)}, \dots, x_N^{(N)}) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq N} x_{i_1}^{(N)} x_{i_2}^{(N)} \cdots x_{i_k}^{(N)},$$

in points $(x_1^{(N)}, \dots, x_N^{(N)})$ distributed with respect to the eigenvalues of a Cauchy random matrix (32), one needs to take into account a crucial cancellation due to symmetry around 0. This is due to the fact that the maximal in absolute value points of $(x_1^{(N)}, \dots, x_N^{(N)})$ grow linearly with N ([24, 15]) and it can be shown that naive moment bounds that bring the absolute value inside the sum will be an order of magnitude larger than what is needed. One may attempt to exploit this cancellation due to symmetry while using the determinantal structure behind the Cauchy ensemble (32) (see [24]). However, as one considers joint moments with higher-order derivatives, or higher powers on the derivatives, the estimates one obtains this way become both more technical and importantly, suboptimal.

To take care of this issue, we take a different approach: if one looks at the problem through the lens of a hidden exchangeable structure, all technical difficulties that one would encounter using the determinantal property for the eigenvalues disappear entirely. To be more precise, we firstly observe that Cauchy random matrices can be realized as the top-left $N \times N$ corner of an infinite unitarily invariant random matrix \mathbf{H} , see Sections 2 and 3, which, when combined with unitary invariance of \mathbf{H} , allows us to reach the following new observation. Suppose $J \in \tilde{\mathbb{N}}^{(k)}$, where $\tilde{\mathbb{N}}^{(k)}$ is the set of subsets of \mathbb{N} of size at most k , with J having elements $i_1 < i_2 < \dots < i_l$. Then, if we define the random variables X_J by

$$X_J \stackrel{\text{def}}{=} \det(\mathbf{H}_{j,m})_{j,m=i_1, \dots, i_l},$$

then $(X_J)_{J \in \tilde{\mathbb{N}}^{(k)}}$ forms an exchangeable array in the sense of Definition 2.1. Furthermore, elementary symmetric polynomials in the eigenvalues $(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)})$ of \mathbf{H}_N can be rewritten in terms of these random variables via the linear-algebraic identity:

$$e_k(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)}) = \sum_{J \in \{1, \dots, N\}^{(k)}} X_J, \tag{20}$$

where $\{1, \dots, N\}^{(k)}$ denotes subsets of $\{1, \dots, N\}$ of size exactly k . We observe here that this readily brings us closer to the goal of circumventing the issue of taking into account cancellations around 0. Indeed, while the elementary symmetric polynomials, when written in terms of the eigenvalues, are sums where both the number of summands, and the summands themselves (see [24, 15]) grow with N , the distribution of the summands on the right-hand side of (20) crucially has no N -dependence: the X_J 's are, importantly, identically distributed for different J despite not being independent! Exploiting the exchangeable structure further, this allows us to prove convergence of all joint moments (and almost sure convergence) via a backward martingale convergence argument with respect to the exchangeable filtration (see Definition 2.2) $(\mathcal{E}_N)_{N \geq 1}$ and σ -algebra $\mathcal{E} = \bigcap_N \mathcal{E}_N$, see Section 2 for more details. Finally, in order to identify the limiting moments with the joint moments of the random variables $Y_n(s)$ we make use of the results of [24, 83]. The main tool employed in [24, 83] is the theory of determinantal point processes and using the determinantal structure in this part of the argument does seem unavoidable.

The more interesting use of exchangeability, however, is in proving Theorem 1.5. Our starting point is a new (as far as we can tell) formula for the joint moments of limiting U-statistics in terms of joint moments of elements in the array indexed by disjoint sets. This is given in Proposition 2.9. By taking into account certain non-trivial combinatorial cancellations it is possible to show further that this formula is equal to the formula in Theorem 1.5 in the more general Proposition 2.12. Now, in order to prove the formula in Proposition 2.9 one needs the following higher-dimensional generalisation of de-Finetti's theorem, which we were surprised not to find anywhere in the literature. The statement is simple: elements of the array indexed by disjoint sets are conditionally independent given the exchangeable σ -algebra \mathcal{E} . This statement is likely known to experts and in fact can be reduced to some deep structural results on exchangeable arrays, known as the Aldous-Hoover-Kallenberg representation theorems [67], which give a representation of the array in terms of independent uniform random variables. However, in order to make the paper self-contained and accessible to non-experts we give a direct proof from first principles which appears to be new.

Proving Theorem 1.11 on connections to integrable systems is the most challenging part of the paper, with all the machinery we built earlier along with key new ideas coming together. The method is partly inspired by the computation in [72] which treats the special case $k = 2$ and $s \in \mathbb{N}$ (after taking the derivative with respect to t_1 and evaluating at $t_1 = 0$), in a more ad-hoc way. This method essentially boils down to establishing exact relations within a class of Hankel determinants shifted (the precise meaning of this will be explained in Section 4) by partitions. However, our proof is not simply a generalisation of the computation in [72] as there are some fundamental differences. First of all, the computation in [72] is done directly in the limit and it is heavily reliant on the fact that (a weighted version of) the characteristic function of $Y_1(s) = -q_1(s)$ is explicit (we note that neither $Y_1(s)$ nor its characteristic function appear explicitly in [72], nor it is easy to guess this connection from the working in [72], but this is morally the reason the computation there works), for $s \in \mathbb{N}$, in terms of $s \times s$ determinants involving Bessel functions, see (19). For non-integer s this is no longer the case; as far as we know, there is no explicit formula of any kind for this characteristic function.

Because of this fundamental obstacle we must take a different approach. We go back to the finite matrix setting and work with N -dimensional integrals that approximate the desired quantities in the large N -limit. We first need to find what the right analogue of the left hand side of (16) is for finite N , a task for which we are inspired by the probabilistic connection to the random variables $\{q_n(s)\}_{n \geq 1}$. There are of course many choices that will approximate the left hand side of (16) in the large- N limit but a judicious choice needs to be made that will have all the exact solvability properties that we require and that is the one presented in Theorem 1.9. These exact relations that we are alluding to we first obtained by formally expanding the exponential in the characteristic function of certain random variables and proceeded to exchange sums and expectations to perform explicit computations. However, these computations are completely formal, as not only does such a series expansion not converge but in fact only finitely many terms in it even make sense, as these random variables (the reader should simply think of these as generalisations of the Cauchy random variable) have finitely many moments. Nevertheless, once we know what these relations should be, we prove them rigorously by a completely different and highly non-trivial argument. By initially restricting to $s \in \mathbb{N}$, we introduce a Hankel matrix, whose elements are given by a type of multivariate generating function of the Fourier transform of certain Cauchy-like weights. That is to say, we construct functions

$\psi_\gamma \in C^\infty(\mathbb{R}^k, \mathbb{R})$ such that if we define their Hankel determinant by,

$$\Psi_N(t_1, \dots, t_k) = \det_{1 \leq i, j \leq N} [\psi_{i+j}(t_1, \dots, t_k)],$$

then the relations we desire to obtain at the finite- N level boil down instead to proving a relation of the form:

$$\prod_{m=2}^k \frac{\partial^{\ell_m}}{\partial t_m^{\ell_m}} \Psi_N(t_1, \dots, t_k) \Big|_{t_2, \dots, t_k=0} = \sum_m P_m(t_1; N) \frac{d^m}{dt_1^m} \Psi_N(t_1, 0, \dots, 0),$$

for certain polynomials $P_m(\cdot; N)$. We then prove this relation by a delicate induction argument on the order of the derivatives ℓ_2, \dots, ℓ_k . However, to perform this induction, it transpires that we are required to introduce a generalization of the determinants Ψ_N , indexed by integer partitions λ denoted by $\Psi_{N, \lambda}$. Then, we develop a systematic way to perform the daunting computations needed for the induction argument, which requires delicate information on the N -dependence of the functions $P_m(\cdot; N)$, simultaneously for Ψ_N and also $\Psi_{N, \lambda}$ as λ ranges over a specific class of partitions. To conclude the proof of the result at the finite- N level we remove the restriction on s by an analytic continuation argument to obtain Theorem 1.9. Finally, by virtue of our previous probabilistic results, we are able to take the rigorous large- N limit of all the quantities involved in Theorem 1.9, while paying attention to which terms vanish in the limit, which from the explicit N -dimensional integral expressions is not obvious at all.

1.7 CONNECTIONS TO VARIOUS AREAS

The theory of exchangeability plays a central role in our arguments. This theory begins with de-Finetti's classical theorem [3] on sequences of random variables. The higher-dimensional structure, that of an exchangeable array and its variations, originates with the seminal works of Hoover [59] and Aldous [2, 3]. It was later developed by Kallenberg [64, 65], whose monograph [67] contains the state of the art of the general theory. There are also many interesting extensions of the notion of exchangeability in different directions, see for example [47, 46]. Through the decades exchangeability has found many applications, see [5, 4, 16]. Our work presents a novel, concrete, and rather surprising, application of exchangeability and connects this branch of probability theory to some deep questions in analytic number theory and to integrable systems. Finally, there is a body of work on exchangeability and random matrices and on permutation-invariant ensembles, see [28] and [75] for representative examples, but both the questions asked and techniques used to answer them, are different from ours.

Another area, which although we do not use techniques from directly, provided important intuition is the study of measures invariant under the action of inductive limit groups, see [81]. The point process $\mathbf{P}^{(s)}$ first appeared in this setting in the work of Borodin and Olshanski [24]. There is already a detailed exposition of this connection in [13, 14] and so we will be brief. The probability measures on the space of infinite Hermitian matrices $\mathbb{H}(\infty)$ which are ergodic under the action of the infinite-dimensional unitary group $\mathbb{U}(\infty)$ were classified in [81, 82] and are parametrized by an infinite-dimensional space Ω . Any invariant, not necessarily ergodic, measure on $\mathbb{H}(\infty)$, by general results, decomposes into ergodic measures via an abstract probability measure \mathfrak{m} on Ω , see [24]. The problem of ergodic decomposition amounts to finding an explicit description of the

probability measure \mathfrak{m} (see [80] for a nice exposition in a related setting). The Cauchy measures on $\mathbb{H}(\infty)$, with parameter s , are invariant, and Borodin and Olshanski began the study of their ergodic decomposition measure, denoted by $\mathfrak{m}^{(s)}$, in [24]. In the same paper they gave a partial description of $\mathfrak{m}^{(s)}$ in terms of $\mathbf{P}^{(s)}$. The description of $\mathfrak{m}^{(s)}$ was then finally completed by Qiu in [83] and we remark that this also where the interesting random variable $Y_1(s) = -q_1(s)$ first appeared.

Finally, let us say a word on relations to previous works in integrable systems. As far as we are aware, no analogue of the method we present here appeared before, beyond the special case in [72]. Nevertheless, the computations are influenced, on a conceptual level at least, by the Japanese school of integrable systems that studies Hankel determinants as τ -functions of Painlevé equations, see [78, 79], a circle of ideas that was later developed in the context of gap probabilities for random matrices by Forrester and Witte in [51, 52].

Acknowledgements. All four authors are grateful to the organisers of the conference “Random Matrices from Quantum Chaos to the Riemann Zeta Function” where this work was initiated. Part of this research was performed while TA was visiting the Institute for Pure and Applied Mathematics (IPAM), which is supported by the National Science Foundation (Grant No. DMS-1925919), and when FW was supported by NSF grant DMS-1854398. We thank Nick Simm for bringing our attention to paper [1] to compute the integral in Lemma 3.3 and David Aldous and Tim Austin for useful correspondence on the general theory of exchangeable arrays. Finally, we are grateful to an anonymous referee for carefully reading the paper and for many constructive comments and suggestions, which helped us improve its presentation.

2 EXCHANGEABLE ARRAYS AND RANDOM MATRICES

2.1 EXCHANGEABLE ARRAYS

We present some general results on exchangeable arrays that will be applied to moments of random matrices in the sequel, this application being the main novel contribution of this part of the paper. The techniques we use are (unsurprisingly) rooted in the general theory of exchangeability [67]. We first fix some notation. For each $d \in \mathbb{N}$, and $J \subseteq \mathbb{N}$, let $J^{(d)}$ denote the set of subsets of J of size d . Also, let $\tilde{J}^{(k)} = \cup_{d \leq k} J^{(d)}$. Let $[N] = \{1, 2, \dots, N\}$ and denote by Sym_N the symmetric group acting on $[N]$. Finally, recall that for an arbitrary random variable X taking values in a standard Borel space we write $\text{Law}(X)$ for its law.

An exchangeable array will be our main object of study and is defined as follows.

Definition 2.1. *Let $k \geq 1$. A collection of random variables $(X_J)_{J \in \tilde{\mathbb{N}}^{(k)}}$, with X_J taking values in a standard Borel space $(\mathcal{V}, \mathfrak{B})$, is called an exchangeable array if for any permutation τ of \mathbb{N} that fixes all but finitely many elements, we have that*

$$\text{Law} \left\{ (X_J)_{J \in \tilde{\mathbb{N}}^{(k)}} \right\} = \text{Law} \left\{ (X_{\tau(J)})_{J \in \tilde{\mathbb{N}}^{(k)}} \right\}, \quad (21)$$

where if $J = \{a_1, \dots, a_d\}$, we define $\tau(J) = \{\tau(a_1), \dots, \tau(a_d)\}$.

Let us comment on some features of this definition; for more details see Kallenberg’s monograph [67]. First, for $k = 1$ the definition is exactly that of an exchangeable sequence of random variables. In higher dimensions it is most common in the literature to index the random variables X_J by sets J of size exactly k instead of size at most k . However, the richer

object indexed by sets of sizes at most k is required if one wants to study joint moments. Finally, a more common terminology in the literature is that of a “jointly-exchangeable array” indicating that we are using a single permutation τ to act on the elements of J . This is to distinguish from a “separately-exchangeable array” for which each element is acted on by a different permutation. Since separately-exchangeable arrays will not appear in this paper we have dropped this extra word from the terminology.

Initially our random variables $(X_J)_{J \in \tilde{\mathcal{N}}^{(k)}}$ are defined on some abstract probability space $(\Omega, \mathcal{B}, \mathbb{Q})$. It will, however, be more convenient for our proofs to define them instead on the canonical space,

$$\tilde{\mathcal{V}} \stackrel{\text{def}}{=} \mathcal{V}^{\tilde{\mathcal{N}}^{(k)}},$$

endowed with the σ -algebra $\tilde{\mathfrak{B}}$ generated by all sets of the form

$$\prod_{J \in \tilde{\mathcal{N}}^{(k)}} A_J,$$

where all but finitely many $A_J \in \mathfrak{B}$ equal \mathcal{V} , as follows. For $I \in \tilde{\mathcal{N}}^{(k)}$, define the measurable function X_I from $(\tilde{\mathcal{V}}, \tilde{\mathfrak{B}})$ to $(\mathcal{V}, \mathfrak{B})$ by:

$$X_I((v_J)_{J \in \tilde{\mathcal{N}}^{(k)}}) = v_I.$$

Then, we consider the probability space $(\tilde{\mathcal{V}}, \tilde{\mathfrak{B}}, \mathbb{P})$, with \mathbb{P} the unique probability measure on $(\tilde{\mathcal{V}}, \tilde{\mathfrak{B}})$ defined via the relation:

$$\mathbb{P}(A_{J_1} \times A_{J_2} \times \cdots \times A_{J_m}) \stackrel{\text{def}}{=} \mathbb{Q}(\{X_{J_1} \in A_{J_1}\} \cap \{X_{J_2} \in A_{J_2}\} \cap \cdots \cap \{X_{J_m} \in A_{J_m}\}),$$

for $J_1, \dots, J_m \in \tilde{\mathcal{N}}^{(k)}$ and $A_{J_1}, \dots, A_{J_m} \in \mathfrak{B}$. By construction, for any measurable $B \in \tilde{\mathfrak{B}}$ we have,

$$\mathbb{P}((X_J)_{J \in \tilde{\mathcal{N}}^{(k)}} \in B) = \mathbb{Q}((X_J)_{J \in \tilde{\mathcal{N}}^{(k)}} \in B).$$

Thus, from now on we assume that we are working on the canonical space $(\tilde{\mathcal{V}}, \tilde{\mathfrak{B}}, \mathbb{P})$. We denote by \mathbb{E} , the averages taken with respect to \mathbb{P} . The following definition is fundamental for our purposes.

Definition 2.2. For each $N \geq 1$, we define the σ -algebras \mathcal{E}_N on $(\tilde{\mathcal{V}}, \tilde{\mathfrak{B}})$ by,

$$\mathcal{E}_N = \left\{ A \in \tilde{\mathfrak{B}} : (v_J)_{J \in \tilde{\mathcal{N}}^{(k)}} \in A \Rightarrow (v_{\tau(J)})_{J \in \tilde{\mathcal{N}}^{(k)}} \in A \text{ for every } \tau \in \text{Sym}_N \right\}$$

and hence define the exchangeable σ -algebra $\mathcal{E} = \bigcap_{N \geq 1} \mathcal{E}_N$.

We then have the following natural higher-dimensional extension of De-Finetti’s Theorem [66]. Surprisingly, as already mentioned in the introduction, we were not able to locate the precise statement in the literature.

Theorem 2.3. Let $(X_J)_{J \in \tilde{\mathcal{N}}^{(k)}}$ be an exchangeable array, and let $(J_l)_{l=1}^\infty \subset \tilde{\mathcal{N}}^{(k)}$ be disjoint sets. Then, conditioned on \mathcal{E} , $(X_{J_l})_{l=1}^\infty$ are independent.

We will prove Theorem 2.3 in two steps. First, we will show the claimed result when \mathcal{E} is replaced by the tail σ -algebra \mathcal{G} we define below. Then, we will show that even

though $\mathcal{G} \subseteq \mathcal{E}$, they coincide up to sets of measure zero. For the first step, we need some definitions.

Definition 2.4 (Tail σ -algebra). *Consider the map*

$$\begin{aligned} \mathcal{S} : \tilde{\mathbb{N}}^{(k)} &\rightarrow \tilde{\mathbb{N}}^{(k)}, \\ \{\gamma_1, \dots, \gamma_l\} &\mapsto \{\gamma_1 + 1, \dots, \gamma_l + 1\}, \end{aligned}$$

and with a slight abuse of notation, denote

$$\mathcal{S}^m \left((X_J)_{J \in \tilde{\mathbb{N}}^{(k)}} \right) \stackrel{\text{def}}{=} (X_{\mathcal{S}^m(J)})_{J \in \tilde{\mathbb{N}}^{(k)}}.$$

Then, for each $N \geq 1$, define the auxiliary σ -algebras:

$$\mathcal{G}_N \stackrel{\text{def}}{=} \sigma \left(\mathcal{S}^N \left((X_J)_{J \in \tilde{\mathbb{N}}^{(k)}} \right) \right)$$

and the associated tail σ -algebra \mathcal{G} by,

$$\mathcal{G} = \bigcap_N \mathcal{G}_N.$$

Observe that in proving Theorem 2.3, there is no loss of generality in assuming that each J_l is given by

$$J_l = \{j_{l-1} + 1, \dots, j_l - 1, j_l\}, \quad (22)$$

for some increasing sequence $(j_l)_{l \geq 1}$. Indeed, if one proves the result for this special form of J_l 's then using exchangeability, it follows that it holds for any disjoint sequence of J_l 's.

Next, to simplify the presentation, we define the random vector:

$$\tilde{X}_J \stackrel{\text{def}}{=} (X_I : I \subset J)$$

for each $J \in \tilde{\mathbb{N}}^{(k)}$ and also write

$$\mathcal{L} \underset{\mathcal{K}}{\perp\!\!\!\perp} \mathcal{H},$$

whenever the σ -algebras \mathcal{L} and \mathcal{H} are independent conditioned on the σ -algebra \mathcal{K} . That is to say, whenever $A \in \mathcal{L}, B \in \mathcal{H}$, we have:

$$\mathbb{E}[\mathbf{1}_A \mathbf{1}_B \mid \mathcal{K}] = \mathbb{E}[\mathbf{1}_A \mid \mathcal{K}] \mathbb{E}[\mathbf{1}_B \mid \mathcal{K}].$$

We also consider the previously defined map \mathcal{S} as a map on $\tilde{\mathcal{V}}$, defined by:

$$\begin{aligned} \mathcal{S} : \tilde{\mathcal{V}} &\rightarrow \tilde{\mathcal{V}}, \\ (v_J)_{J \in \tilde{\mathbb{N}}^{(k)}} &\mapsto (v_{\mathcal{S}(J)})_{J \in \tilde{\mathbb{N}}^{(k)}}. \end{aligned}$$

Lemma 2.5. *Let $(j_l)_{l=1}^\infty$ be a strictly increasing sequence of integers and define the corresponding J_l as in (22). Then, $(X_{J_l})_{l=1}^\infty$ are independent conditioned on \mathcal{G} .*

Proof. Firstly, note that by exchangeability, we have, for each l and $N \geq j_l$, the distribu-

tional equality of the random vectors

$$\left(\hat{X}_{J_l}, \mathcal{S}^{j_l} \left((X_J)_{J \in \tilde{\mathcal{N}}^{(k)}} \right)\right) \stackrel{d}{=} \left(\hat{X}_{J_l}, \mathcal{S}^N \left((X_J)_{J \in \tilde{\mathcal{N}}^{(k)}} \right)\right),$$

(with $\stackrel{d}{=}$ denoting equality in distribution) which, by means of Corollary 8.10 in [66], gives that

$$\hat{X}_{J_l} \perp\!\!\!\perp_{\mathcal{G}_N} \mathcal{G}_{j_l}.$$

In particular, for every $A \in \sigma(\hat{X}_{J_l})$, and all $N \geq j_l$, we have that

$$\mathbb{E}[\mathbf{1}_A | \mathcal{G}_{j_l}] = \mathbb{E}[\mathbf{1}_A | \mathcal{G}_N] \xrightarrow[\text{a.s.}]{N \rightarrow \infty} \mathbb{E}[\mathbf{1}_A | \mathcal{G}],$$

where for the equality we used Theorem 8.9 from [66]. Since the first term appearing above is independent of N , we then have that it is precisely equal to the limit, which then implies that

$$\hat{X}_{J_l} \perp\!\!\!\perp_{\mathcal{G}} \mathcal{G}_{j_l}.$$

Hence, applying this iteratively, we get that

$$\sigma(\hat{X}_{J_1}, \dots, \hat{X}_{J_l}) \perp\!\!\!\perp_{\mathcal{G}} \mathcal{G}_{j_l},$$

for each l , from which the result follows. \square

Next, we show that \mathcal{G} coincides with \mathcal{E} up to sets of measure 0 with respect to \mathbb{P} , thus proving Theorem 2.3.

Proposition 2.6. *Let $A \in \mathcal{E}$. Then, there exists $B \in \mathcal{G}$ such that*

$$\mathbb{P}(A \Delta B) = 0,$$

where $A \Delta B$ denotes the symmetric difference between the sets A and B .

Proof. We first claim that for each $A \in \mathcal{E}$, we have that

$$\mathbb{P}(A \Delta \mathcal{S}^{-1}(A)) = 0.$$

To prove the claim, we start by fixing $\delta > 0$ and noting that since $A \in \tilde{\mathfrak{B}}$, by means of Lemma 4.16 in [66], there exists $m \in \mathbb{N}$ and a set $W \in \tilde{\mathfrak{B}}$ of the form

$$W = Z \times \mathcal{V}^{K_m} \tag{23}$$

where $K_m \stackrel{\text{def}}{=} \{J \in \tilde{\mathcal{N}}^{(k)} : \max(J) > m\}$, such that

$$\mathbb{P}(A \Delta W) < \delta.$$

Here, Z is a Borel subset of \mathcal{V}^{L_m} where $L_m \stackrel{\text{def}}{=} \tilde{\mathbb{N}}^{(k)} \setminus K_m$. Next, define the map

$$\begin{aligned} \mathcal{P}_m : \tilde{\mathcal{V}} &\rightarrow \tilde{\mathcal{V}} \\ (v_J)_{J \in \tilde{\mathbb{N}}^{(k)}} &\mapsto (v_{\tau_m(J)})_{J \in \tilde{\mathbb{N}}^{(k)}}, \end{aligned}$$

where $\tau_m \in \text{Sym}_{m+1}$ is the $(m+1)$ -cycle: $(m+1 \ m \ \cdots \ 2 \ 1)$.

Then, note that \mathcal{P}_m is a bijection such that for any $A \in \mathcal{E}$, $\mathcal{P}_m(A) = A$. In particular, we have

$$\mathcal{P}_m(W) \Delta A = \mathcal{P}_m(W) \Delta \mathcal{P}_m(A) = \mathcal{P}_m(W \Delta A).$$

Combining this with the fact that \mathbb{P} is invariant under \mathcal{P}_m , we get that

$$\mathbb{P}(\mathcal{P}_m(W) \Delta A) = \mathbb{P}(W \Delta A) < \delta.$$

Now, note that if $J \in L_m$, then

$$S(J) = \tau_m^{-1}(J).$$

Hence, one easily sees that

$$\mathcal{S}^{-1}(W) = \{(v_J)_{J \in \tilde{\mathbb{N}}^{(k)}} : (v_{S(J)})_{J \in L_m} \in Z\} = \{(v_J)_{J \in \tilde{\mathbb{N}}^{(k)}} : (v_{\tau_m^{-1}(J)})_{J \in L_m} \in Z\} = \mathcal{P}_m(W)$$

Thus, we get

$$\mathbb{P}(\mathcal{S}^{-1}(W) \Delta W) = \mathbb{P}(\mathcal{P}_m(W) \Delta W) \leq \mathbb{P}(\mathcal{P}_m(W) \Delta A) + \mathbb{P}(A \Delta W) < 2\delta.$$

Combining these, and using that $\mathbb{P}(\mathcal{S}^{-1}(E)) = \mathbb{P}(E)$ for any $E \in \tilde{\mathcal{B}}$, we get that

$$\mathbb{P}(\mathcal{S}^{-1}(A) \Delta A) \leq \mathbb{P}(A \Delta W) + \mathbb{P}(W \Delta \mathcal{S}^{-1}(W)) + \mathbb{P}(\mathcal{S}^{-1}(W) \Delta \mathcal{S}^{-1}(A)) < 4\delta,$$

which proves the claim since $\delta > 0$ was arbitrary. Now, define inductively

$$\mathcal{S}^{-m}(A) \stackrel{\text{def}}{=} \mathcal{S}^{-1}(\mathcal{S}^{-(m-1)}(A)),$$

and the corresponding set,

$$B \stackrel{\text{def}}{=} \limsup_{m \rightarrow \infty} \mathcal{S}^{-m}(A).$$

Note that, $B \in \mathcal{G}$. Also note that,

$$\begin{aligned} \mathbb{P}(A \Delta \mathcal{S}^{-m}(A)) &\leq \mathbb{P}(A \Delta \mathcal{S}^{-(m-1)}(A)) + \mathbb{P}(\mathcal{S}^{-(m-1)}(A) \Delta \mathcal{S}^{-m}(A)) \\ &= \mathbb{P}(A \Delta \mathcal{S}^{-(m-1)}(A)) + \mathbb{P}(A \Delta \mathcal{S}^{-1}(A)) = 0, \end{aligned}$$

where we used induction in the last step. Hence,

$$\mathbb{P}(A \Delta B) = \mathbb{P}(A \Delta \bigcap_{n \geq 1} \bigcup_{m \geq n} \mathcal{S}^{-m}(A)) \leq \sum_{m \geq 1} \mathbb{P}(A \Delta \mathcal{S}^{-m}(A)) = 0,$$

which completes the proof. \square

From now on we assume that $\mathcal{V} = \mathbb{R}$, namely the random variables X_j are real-valued. We use the results above to first prove an analogue of the law of large numbers

for statistics of exchangeable arrays (which should not be surprising), and then give a formula for joint moments of the limiting random variables in terms of finitely many random variables in the array. This general moments formula appears to be new.

Proposition 2.7. *Let $(X_j)_{j \in \mathbb{N}^{(k)}}$ be an exchangeable array, and let $\mathcal{E}_N, \mathcal{E}$ be defined as before. Then, for each $d \leq k$ and fixed $I \in \mathbb{N}^{(d)}$, we have that*

$$\frac{d!}{N^d} \sum_{J \in [N]^{(d)}} X_J \xrightarrow[N \rightarrow \infty]{\text{a.s., } L^1} \mathbb{E}[X_I | \mathcal{E}]$$

whenever $X_I \in L^1$. If we further have $X_I \in L^p$ for $p > 1$ then convergence takes place in L^p as well.

Proof. Define the partial sums, noting that we may change $\frac{N^d}{d!}$ to $\binom{N}{d}$ without affecting the result,

$$\mathsf{T}_N^{(d)} = \frac{1}{\binom{N}{d}} \sum_{J \in [N]^{(d)}} X_J. \quad (24)$$

Then, note that T_N is \mathcal{E}_N -measurable, and for every $I, L \in [N]^{(d)}$ and $A \in \mathcal{E}_N$,

$$\mathbb{E}[X_I \mathbf{1}_A] = \mathbb{E}[X_L \mathbf{1}_A],$$

so that we get

$$\mathbb{E}[X_I | \mathcal{E}_N] = \mathbb{E}[X_L | \mathcal{E}_N].$$

In particular, we have

$$\mathsf{T}_N^{(d)} = \mathbb{E}[\mathsf{T}_N^{(d)} | \mathcal{E}_N] = \mathbb{E}[X_I | \mathcal{E}_N].$$

Thus, we see that $\mathsf{T}_N^{(d)}$ forms a backwards martingale with respect to \mathcal{E}_N . Combining this with the assumed integrability of X_I and using the backwards martingale convergence theorem, the result follows. \square

Proposition 2.8. *Let $(X_j)_{j \in \mathbb{N}^{(k)}}$ be an exchangeable array, and define, for $d \leq k$, the random variables*

$$\mathsf{T}^{(d)} \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \mathsf{T}_N^{(d)}$$

where $\mathsf{T}_N^{(d)}$ is defined as in (24). Then, if for $r_1, \dots, r_l > 0$ and $k_1, \dots, k_l \in \{1, \dots, k\}$,

$$\mathbb{E}[|X_{[k_j]}|^p] < \infty$$

for all $j = 1, \dots, l$, where we define $p \stackrel{\text{def}}{=} \sum_j r_j$, then we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N^{\sum_j r_j k_j}} \mathbb{E} \left[\prod_{j=1}^l |\mathsf{T}_N^{(k_j)}|^{r_j} \right] = \mathbb{E} \left[\prod_{j=1}^l |\mathsf{T}^{(k_j)}|^{r_j} \right].$$

Proof. Note that under the given conditions, almost sure convergence holds via Proposition 2.7. Thus, to prove the convergence of moments, it suffices to show that the sequence

$$\left\{ \prod_{j=1}^l |\mathsf{T}_N^{(k_j)}|^{r_j} \right\}_{N \geq \max(k_j)} \quad (25)$$

is uniformly integrable. To see that this follows from the assumptions, note that first applying Young's inequality, and then applying Holder's inequality with exponents $\frac{p}{r_j}$ in both cases, we get that

$$\begin{aligned} \sup_N \mathbb{E} \left[\prod_{j=1}^l |\mathbb{T}_N^{(k_j)}|^{r_j} \mathbf{1} \left\{ \prod_{j=1}^l |\mathbb{T}_N^{(k_j)}|^{r_j} \geq R \right\} \right] &\leq \sup_N \sum_{m=1}^l \mathbb{E} \left[\prod_{j=1}^l |\mathbb{T}_N^{(k_j)}|^{r_j} \mathbf{1} \left\{ |\mathbb{T}_N^{(k_m)}|^p \geq \frac{pR}{lr_m} \right\} \right] \\ &\leq \sup_N \sum_{m=1}^l \mathbb{E} \left[|\mathbb{T}_N^{(k_m)}|^p \mathbf{1} \left\{ |\mathbb{T}_N^{(k_m)}|^p \geq \frac{pR}{lr_m} \right\} \right]^{r_m/p} \prod_{j \neq m} \mathbb{E} \left[|\mathbb{T}_N^{(k_j)}|^p \right]^{r_j/p}. \end{aligned} \quad (26)$$

By Proposition 2.7, we know that for each m ,

$$\mathbb{T}_N^{(k_m)} \xrightarrow[N \rightarrow \infty]{L^p} \mathbb{T}^{(k_m)} \quad (27)$$

and hence, the sequence

$$\left\{ |\mathbb{T}_N^{(k_m)}|^p \right\}_{N \geq \max(k_j)} \quad (28)$$

is uniformly integrable. Thus, taking $R \rightarrow \infty$ in (26), we see that the sequence in (25) is also uniformly integrable, concluding the proof. \square

Next, we give a seemingly new formula for the joint moments of the random variables $\mathbb{T}^{(d)}$ in terms of joint moments of finitely many X_j 's. Its usefulness will be clear in the sequel.

Proposition 2.9. *Fix $k \in \mathbb{N}$. Let $k_1 > \dots > k_l \in \{1, \dots, k\}$ and $h_1, \dots, h_l \in \mathbb{N}$. Define $L = \sum_j h_j k_j$ and $p = \sum_j h_j$. Suppose $\{J_m\}_{m=1}^p$ is a partition of the set $\{1, \dots, L\}$ such that for each j , it contains exactly h_j sets of size k_j ; and further assume that*

$$\mathbb{E} \left[|X_{[k_j]}|^p \right] < \infty.$$

Then, with the random variables $\mathbb{T}^{(d)}$ defined as in Proposition 2.8, we have that,

$$\mathbb{E} \left[\prod_{j=1}^l (\mathbb{T}^{(k_j)})^{h_j} \right] = \mathbb{E} \left[\prod_{m=1}^p X_{J_m} \right].$$

Proof. Note that combining the integrability assumption with the fact that J_m are disjoint, we have that

$$\mathbb{E} \left[\prod_{m=1}^p X_{J_m} \mid \mathcal{E} \right] = \prod_{m=1}^p \mathbb{E} \left[X_{J_m} \mid \mathcal{E} \right] = \prod_{m=1}^p \mathbb{T}^{(|J_m|)} = \prod_{j=1}^l (\mathbb{T}^{(k_j)})^{h_j}.$$

Taking the expectation of both sides and using tower law gives the desired result. \square

2.2 CONSISTENT PERMUTATION-INVARIANT ENSEMBLES

In this section, we prove a number of new results for consistent permutation-invariant ensembles using the theory of exchangeable arrays discussed in the previous subsection.

We restrict our attention to complex Hermitian matrices because this is what we need for our applications to the problem of joint moments. However, essentially all our arguments go through verbatim if we consider ensembles over the reals or quaternions, or if we drop the self-adjoint constraint, as long as we have permutation-invariance. Before we formally define the class of permutation-invariant random matrix ensembles, let us introduce some notation that will be used throughout.

Let $\mathbb{H}(N)$ denote the space of $N \times N$ Hermitian matrices, equipped with the Borel σ -algebra $\mathcal{B}(\mathbb{H}(N))$ and the Lebesgue measure,

$$d\mathbf{H} = \prod_{i=1}^N d\mathbf{H}_{ii} \prod_{i<j} d\Re \mathbf{H}_{ij} d\Im \mathbf{H}_{ij}.$$

Define the projection maps,

$$\begin{aligned} \Lambda_N^{N+1} : \mathbb{H}(N+1) &\rightarrow \mathbb{H}(N), \\ [\mathbf{H}_{ij}]_{i,j=1,\dots,N+1} &\mapsto [\mathbf{H}_{ij}]_{i,j=1,\dots,N}, \end{aligned}$$

mapping each $\mathbf{H} \in \mathbb{H}(N+1)$ to its top-left $N \times N$ corner. Next, define $\mathbb{H}(\infty)$ to be the projective limit of $\mathbb{H}(N)$ under these maps, endowed with the projective limit topology, whose Borel sigma algebra we denote by $\mathcal{B}(\mathbb{H}(\infty))$. Finally, consider the corresponding natural projections:

$$\begin{aligned} \Lambda_N^\infty : \mathbb{H}(\infty) &\rightarrow \mathbb{H}(N), \\ [\mathbf{H}_{ij}]_{i,j \in \mathbb{N}} &\mapsto [\mathbf{H}_{ij}]_{i,j=1,\dots,N}. \end{aligned}$$

We have the following definition.

Definition 2.10. *A sequence of probability measures $\{\mu_N\}_{N \geq 1}$, with μ_N on $(\mathbb{H}(N), \mathcal{B}(\mathbb{H}(N)))$, is called a consistent permutation-invariant ensemble if:*

- For each $N \geq 1$, $\tau \in \text{Sym}_N$, and $B \in \mathcal{B}(\mathbb{H}(N))$,

$$\mu_N(\mathbf{H} \in B) = \mu_N(\mathbf{P}_\tau \mathbf{H} \mathbf{P}_\tau^* \in B),$$

where recall that \mathbf{P}_τ is the permutation matrix associated to τ .

- For each $N \geq 1$, the pushforward of μ_{N+1} under Λ_N^{N+1} is μ_N .

We denote by μ_∞ the unique probability measure on $(\mathbb{H}(\infty), \mathcal{B}(\mathbb{H}(\infty)))$ such that for all $N \geq 1$, the pushforward image of μ_∞ under Λ_N^∞ is μ_N . Note that such a measure is unique, since it is determined by the consistent finite-dimensional marginals $\{\mu_N\}_{N \geq 1}$ on the generating family of sets

$$\{\mathbf{H} \in \mathbb{H}(\infty) : \Lambda_N^\infty(\mathbf{H}) \in B\},$$

for $N \geq 1$ and $B \in \mathcal{B}(\mathbb{H}(N))$, which generates the σ -algebra $\mathcal{B}(\mathbb{H}(\infty))$.

Throughout this section, we will be considering arbitrary consistent permutation-invariant ensembles $\{\mu_N\}_{N \geq 1}$, associated with the measure μ_∞ which couples them. We denote expectation with respect to μ_∞ by \mathbb{E} . We let \mathbf{H} denote an infinite random matrix sampled according to μ_∞ and also define $\mathbf{H}_N = \Lambda_N^\infty(\mathbf{H})$ which is distributed according to

μ_N . We denote the random eigenvalues of \mathbf{H}_N by $(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)})$; how we order the eigenvalues will not be important as we will only consider symmetric functions of them.

We now prove two general results on such matrices.

Proposition 2.11. *Let $k \in \mathbb{N}$. If*

$$\mathbb{E} [|\det(\mathbf{H}_k)|] < \infty$$

then there exists a random variable $\mathsf{T}^{(k)}$ such that

$$\frac{k!}{N^k} e_k(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)}) \xrightarrow{N \rightarrow \infty} \mathsf{T}^{(k)}$$

almost surely and in L^1 , where e_k denotes the k -th elementary symmetric polynomial. If we further have that

$$\mathbb{E} [|\det(\mathbf{H}_k)|^r] < \infty,$$

for some $r > 1$, then convergence takes place in L^r as well. Furthermore, if for $r_1, \dots, r_l > 0$ and $k_1, \dots, k_l \in \mathbb{N}$, we have

$$\mathbb{E} \left[\left| \det(\mathbf{H}_{k_j}) \right|^s \right] < \infty$$

for all $1 \leq j \leq l$, where $s = \sum r_j$, then

$$\prod_{j=1}^l \frac{(k_j!)^{r_j}}{N^{k_j r_j}} \mathbb{E} \left[\left| \prod_{j=1}^l e_{k_j}(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)}) \right|^{r_j} \right] \xrightarrow{N \rightarrow \infty} \mathbb{E} \left[\prod_{j=1}^l |\mathsf{T}^{(k_j)}|^{r_j} \right]. \quad (29)$$

Proof. We start by defining, for each $J \in \tilde{\mathbb{N}}^{(k)}$, the random variables

$$X_J \stackrel{\text{def}}{=} \det(\mathbf{H}^J),$$

where if $m_1 < \dots < m_d$ are the elements of J , with $d \leq k$, we denote:

$$\mathbf{H}^J \stackrel{\text{def}}{=} [\mathbf{H}_{ij}]_{i,j=m_1, \dots, m_d}.$$

Then, note that by permutation invariance of the law of \mathbf{H} , if τ is any permutation of \mathbb{N} fixing all but finitely many elements, then

$$\text{Law}(\mathbf{H}) = \text{Law}(\mathbf{P}_\tau \mathbf{H} \mathbf{P}_\tau^*).$$

Write $\mathbf{H}_\tau = \mathbf{P}_\tau \mathbf{H} \mathbf{P}_\tau^*$. Now, observe that for any J described as above, we have that

$$\mathbf{H}_\tau^J = [\mathbf{H}_{ij}]_{i,j=\tau(m_1), \dots, \tau(m_d)}$$

so that \mathbf{H}_τ^J and $\mathbf{H}^{\tau(J)}$ are two matrices that can be obtained by one another after a reordering of their indices. In particular, they are equivalent under unitary conjugation and hence have the same determinant. Combining these two observations, we see that $(X_J)_{J \in \tilde{\mathbb{N}}^{(k)}}$ is an exchangeable array.

Next, observe that the elementary symmetric polynomials in the eigenvalues can be

rewritten as:

$$\frac{d!}{N^d} e_d(\mathbf{x}_1^{(N)}, \mathbf{x}_2^{(N)}, \dots, \mathbf{x}_N^{(N)}) = \frac{d!}{N^d} \sum_{J \in [N]^{(d)}} \det(\mathbf{H}^J) = \frac{d!}{N^d} \sum_{J \in [N]^{(d)}} \mathbf{x}_J.$$

Note that the first equality above easily follows by considering the characteristic polynomial $\det(t\mathbf{I} - \mathbf{H})$. Indeed, by means of Vieta's formulas, one easily sees that the d -th symmetric polynomial in eigenvalues equals the coefficient of t^{N-d} , which then also equals the sum of the determinants given above that one sees by expanding the characteristic polynomial as a sum over the symmetric group. Combining all of these observations, we see that the result now follows by applications of Propositions 2.7 and 2.8. \square

Proposition 2.12. *Let $k_1 \geq \dots \geq k_l$, and $r_1, \dots, r_l \in \mathbb{N}$, and $L = \sum_j k_j r_j$. Then, under the same assumptions as Proposition 2.11,*

$$\mathbb{E} \left[\prod_{j=1}^l (\mathbb{T}^{(k_j)})^{r_j} \right] = \frac{(k_1!)^{r_1} \dots (k_l!)^{r_l}}{L!} \sum_{m=k_1}^L (-1)^{m+L} \binom{L}{m} \mathbb{E} \left[\prod_{j=1}^l (e_{k_j}(\mathbf{x}_1^{(m)}, \mathbf{x}_2^{(m)}, \dots, \mathbf{x}_m^{(m)}))^{r_j} \right].$$

Proof. We start by noting that if we let

$$\{\mathbb{I}_m^{(j)}\}_{j,m} \tag{30}$$

be a partition of $[L]$ such that for each j, m , $\mathbb{I}_m^{(j)}$ contains k_j elements; then by means of Proposition 2.9, we get that our claim amounts to showing:

$$\mathbb{E} \left[\prod_{j,m} \det(\mathbf{H}^{\mathbb{I}_m^{(j)}}) \right] = \frac{(k_1!)^{r_1} \dots (k_l!)^{r_l}}{L!} \sum_{m=k_1}^L (-1)^{m+L} \binom{L}{m} \mathbb{E} \left[\prod_{j=1}^l \left(\sum_{J \in [m]^{(k_j)}} \det(\mathbf{H}^J) \right)^{r_j} \right]. \tag{31}$$

To prove the formula above, we will expand the right-hand side of the equation, and use permutation-invariance to note cancellations. With this aim, for each $k \in \{k_1, \dots, L\}$ define the sets

$$A_k = \left\{ \mathcal{J} = (J_1^{(1)}, \dots, J_{r_1}^{(1)}, J_1^{(2)}, \dots, J_{r_2}^{(2)}, \dots, J_1^{(l)}, \dots, J_{r_l}^{(l)}) : J_m^{(j)} \in [L]^{(k_j)}, \left| \bigcup_{j,m} J_m^{(j)} \right| = k \right\}.$$

Let the symmetric group Sym_L act on A_k such that for any $\tau \in \text{Sym}_L$, we have

$$\begin{aligned} \tau(J_1^{(1)}, \dots, J_{r_1}^{(1)}, J_1^{(2)}, \dots, J_{r_2}^{(2)}, \dots, J_1^{(l)}, \dots, J_{r_l}^{(l)}) \\ = (\tau(J_1^{(1)}), \dots, \tau(J_{r_1}^{(1)}), \tau(J_1^{(2)}), \dots, \tau(J_{r_2}^{(2)}), \dots, \tau(J_1^{(l)}), \dots, \tau(J_{r_l}^{(l)})). \end{aligned}$$

Next, consider the terms appearing on the expansion of the right-hand side of (31) that are given, for any $\mathcal{J} \in \bigcup_{k=k_1}^L A_k$, by:

$$M_{\mathcal{J}} \stackrel{\text{def}}{=} \mathbb{E} \left[\det(\mathbf{H}^{J_1^{(1)}}) \dots \det(\mathbf{H}^{J_{r_1}^{(1)}}) \dots \det(\mathbf{H}^{J_1^{(l)}}) \dots \det(\mathbf{H}^{J_{r_l}^{(l)}}) \right].$$

Then, note that by permutation-invariance we have that for any $\tau \in \text{Sym}_L$, the average defining $M_{\mathcal{J}}$ remains unchanged if the random matrix \mathbf{H} is replaced by $\mathbf{H}_{\tau} = \mathbf{P}_{\tau} \mathbf{H} \mathbf{P}_{\tau}^*$. In addition, noting that for any $J \subseteq [L]$, \mathbf{H}_{τ}^J and $\mathbf{H}^{\tau(J)}$ are two matrices that are equal up to re-ordering of the indices, we see that they are equivalent under unitary conjugation and hence have the same determinant. Combining these two observations, we easily see that for any $\tau \in \text{Sym}_L$, we have that $M_{\mathcal{J}} = M_{\tau(\mathcal{J})}$. Next, with a slight abuse of notation, for any $\mathcal{J} \in \bigcup_{k=k_1}^L A_k$, let $\text{Orb}_m(\mathcal{J})$ denote the number of times an average corresponding to the Sym_L -orbit of \mathcal{J} appears on the expansion of the m -th summand on the right-hand side of (31), where $m = k_1, \dots, L$. Then, realize that if $\mathcal{J} \in A_k$, it follows that the terms of the form $M_{\tau(\mathcal{J})}$ as τ runs over Sym_L only appear on the expansion of the m -th summand on the right-hand side of (31) if $k \leq m \leq L$, and we have the equality

$$\text{Orb}_m(\mathcal{J}) = \binom{m}{k} \text{Orb}_k(\mathcal{J}).$$

Thus, if $k \neq L$, we have that the contribution coming from the Sym_L -orbit of \mathcal{J} equals

$$M_{\mathcal{J}} \text{Orb}_k(\mathcal{J}) \sum_{m=k}^L \binom{L}{m} (-1)^m \binom{m}{k} = 0.$$

Furthermore, note that if $\mathcal{J} \in A_L$, then $\{J_m^{(j)}\}_{j,m}$ forms a partition of $[L]$. In particular, we get by permutation-invariance that

$$M_{\mathcal{J}} = \mathbb{E} \left[\prod_{j,m} \det \left(\mathbf{H}_m^{(j)} \right) \right],$$

for all such \mathcal{J} . Next, observe that the terms corresponding to A_L only appear in the sum in (31) when $m = L$, in which case the coefficient that multiplies the average is simply 1. Hence, combining these observations with the simple combinatorial formula

$$|A_L| = \frac{L!}{(k_1!)^{r_1} \cdots (k_l!)^{r_l}},$$

gives the desired result. \square

Remark 2.13. *Proposition 2.12 can be generalised to the setting of arbitrary real-valued exchangeable arrays and the proof is the same. In the final formula one simply replaces $e_{k_j} \left(x_1^{(m)}, x_2^{(m)}, \dots, x_m^{(m)} \right)$ by $\sum_{J \in [m]^{(k_j)}} X_J$.*

Proof of Theorem 1.7. This is a direct consequence of Propositions 2.11 and 2.12 above by virtue of Vieta's formulae. \square

3 PROOFS OF CONVERGENCE AND EXPLICIT FORMULAE

3.1 CAUCHY ENSEMBLE AND CONVERGENCE OF MOMENTS

The main results of this section will be obtained by using various results from the last section on consistent permutation-invariant ensembles. The specific ensemble that will be

used here is called the Cauchy [50, 90] (also Hua-Pickrell [24]) random matrix ensemble, which is a sequence of probability measures μ_N on $\mathbb{H}(N)$ given by

$$\mu_N^{(s)}(d\mathbf{H}) \stackrel{\text{def}}{=} \text{const} \cdot \det(\mathbf{I} + \mathbf{H}^2)^{-s-N} d\mathbf{H}, \quad (32)$$

where $s \geq 0$, and the explicit expression of the normalization constant will be omitted as it will not play a role in what follows. Note that, via Weyl's integration formula, one can compute that the eigenvalue distribution of a random matrix \mathbf{H} sampled according to $\mu_N^{(s)}$ is given by $\mathbf{M}_N^{(s)}$ from the introduction as defined in (9).

The reason these measures are closely related to the Haar measure on $\mathbb{U}(N)$ is that under the transformation $x_j = \cot(\theta_j/2)$, averages over the eigenvalues of a Haar-unitary random matrix can be written as an average against $\mathbf{M}_N^{(s)}$.

Proposition 3.1. *Let $n_1 > \dots > n_k \geq 0$ be integers. Let $h_1, \dots, h_k \in (0, \infty)$ and $s = \sum_{j=1}^k h_j$. Then, we have the following expression for $\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)$,*

$$\frac{\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)}{\mathfrak{F}_N^{(0)}(s)} = 2^{-\sum_{j=1}^k 2h_j n_j} \mathbb{E}_N^{(s)} \left[\prod_{j=1}^k \left| \Xi_{n_j}(\mathbf{x}_1^{(N)}, \mathbf{x}_2^{(N)}, \dots, \mathbf{x}_N^{(N)}) \right|^{2h_j} \right],$$

and for $\mathfrak{G}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)$,

$$\frac{\mathfrak{G}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)}{\mathfrak{G}_N^{(0)}(s)} = 2^{-\sum_{j=1}^k 2h_j n_j} \mathbb{E}_N^{(s)} \left[\prod_{j=1}^k \left| \sum_{m_j=0}^{n_j} (-i)^{m_j} \widetilde{\Xi}_{n_j-m_j} \right|^{2h_j} \right],$$

where the polynomials Ξ_m and $\widetilde{\Xi}_m$ are given by,

$$\Xi_{n_j}(x_1, x_2, \dots, x_N) = \sum_{l=0}^{n_j} a_{n_j, l} e_l(x_1, x_2, \dots, x_N), \quad (33)$$

$$\widetilde{\Xi}_{n_j-m_j}(x_1, x_2, \dots, x_N) = N^{m_j} \binom{n_j}{m_j} \Xi_{n_j-m_j}(x_1, x_2, \dots, x_N),$$

with the constants $a_{n,l}$ given as follows,

$$a_{n,l} \stackrel{\text{def}}{=} (-1)^{\frac{n+l}{2}} \sum_{\substack{\sum_{i=1}^N m_i = n \\ \text{only } m_1, m_2, \dots, m_l \text{ are odd integers}}} \binom{n}{m_1, \dots, m_N}. \quad (34)$$

Proof. By the definition of $Z_{\mathbf{A}}(\theta)$, we have

$$Z_{\mathbf{A}}(\theta) = (-2)^N \prod_{n=1}^N \sin\left(\frac{\theta - \theta_n}{2}\right). \quad (35)$$

Note that, for $l \geq 1$,

$$\frac{d^l}{d\theta^l} \sin\left(\frac{\theta - \theta_n}{2}\right) = 2^{-l} (-1)^{\frac{l-1}{2}} \cos\left(\frac{\theta - \theta_n}{2}\right) = 2^{-l} (-1)^{\frac{l-1}{2}} \sin\left(\frac{\theta - \theta_n}{2}\right) \cot\left(\frac{\theta - \theta_n}{2}\right)$$

when l is an odd integer; and

$$\frac{d^l}{d\theta^l} \sin\left(\frac{\theta - \theta_n}{2}\right) = 2^{-l} (-1)^{\frac{l}{2}} \sin\left(\frac{\theta - \theta_n}{2}\right)$$

when l is an even integer. So we can compute:

$$\begin{aligned} \frac{d^{(n_j)} Z_{\mathbf{A}}(\theta)}{d\theta^{n_j}} \Big|_{\theta=0} &= (-2)^N \sum_{m_1 + \dots + m_N = n_j} \binom{n_j}{m_1, \dots, m_N} \prod_{n=1}^N \frac{d^{m_n} \sin\left(\frac{\theta - \theta_n}{2}\right)}{d\theta^{m_n}} \Big|_{\theta=0} \\ &= \frac{Z_{\mathbf{A}}(0)}{2^{n_j}} \sum_{m_1 + \dots + m_N = n_j} \binom{n_j}{m_1, \dots, m_N} \prod_{\substack{n=1 \\ m_n \text{ is even}}}^N (-1)^{\frac{m_n}{2}} \prod_{\substack{n=1 \\ m_n \text{ is odd}}}^N (-1)^{\frac{m_n+1}{2}} \cot\left(\frac{\theta_n}{2}\right). \end{aligned} \quad (36)$$

Note that, if $m_1 + \dots + m_N = n_j$, then $\#\{i : m_i \text{ is odd}\} \leq n_j$. Hence, the above can be simplified to

$$\frac{d^{(n_j)} Z_{\mathbf{A}}(\theta)}{d\theta^{n_j}} \Big|_{\theta=0} = \frac{Z_{\mathbf{A}}(0)}{2^{n_j}} \sum_{l=0}^{n_j} \sum_{\substack{i_1, \dots, i_l=1 \\ i_1 < \dots < i_l}}^N c_{i_1, \dots, i_l} \cot\left(\frac{\theta_{i_1}}{2}\right) \cdots \cot\left(\frac{\theta_{i_l}}{2}\right), \quad (37)$$

where the constants c_{i_1, \dots, i_l} are given by,

$$c_{i_1, \dots, i_l} = (-1)^{\frac{n_j+l}{2}} \sum_{\substack{\sum_{i=1}^N m_i = n_j \\ \text{only } m_{i_1}, m_{i_2}, \dots, m_{i_l} \text{ are odd integers}}} \binom{n_j}{m_1, \dots, m_N}. \quad (38)$$

But then one immediately sees that $c_{i_1, \dots, i_l} = a_{n_j, l}$ for any $i_1 < \dots < i_l$ so that we have,

$$\begin{aligned} \frac{d^{(n_j)} Z_{\mathbf{A}}(\theta)}{d\theta^{n_j}} \Big|_{\theta=0} &= \frac{Z_{\mathbf{A}}(0)}{2^{n_j}} \sum_{l=0}^{n_j} a_{n_j, l} e_l\left(\cot\left(\frac{\theta_1}{2}\right), \dots, \cot\left(\frac{\theta_N}{2}\right)\right) \\ &= \frac{Z_{\mathbf{A}}(0)}{2^{n_j}} \Xi_{n_j}\left(\cot\left(\frac{\theta_1}{2}\right), \dots, \cot\left(\frac{\theta_N}{2}\right)\right). \end{aligned}$$

Next, by applying Weyl's integration formula for the Haar measure on $\mathbb{U}(N)$, we see that

$$\begin{aligned} \mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k) &= \int_{\mathbb{U}(N)} \prod_{j=1}^k |Z_{\mathbf{A}}^{(n_j)}(0)|^{2h_j} d\mu_{\text{Haar}}(\mathbf{A}) \\ &= \frac{1}{2^{\sum_{j=1}^k 2h_j n_j} (2\pi)^N N!} \int_{[0, 2\pi]^N} \prod_{1 \leq j < m \leq N} |e^{i\theta_j} - e^{i\theta_m}|^2 \prod_{n=1}^N |1 - e^{i\theta_n}|^{2s} \\ &\quad \prod_{j=1}^k \left| \Xi_{n_j}\left(\cot\left(\frac{\theta_1}{2}\right), \dots, \cot\left(\frac{\theta_N}{2}\right)\right) \right|^{2h_j} d\theta_1 \cdots d\theta_N. \end{aligned}$$

Then, applying the transformation $x_j = \cot(\theta_j/2)$ gives that

$$\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k) = \frac{K_N(s)}{2^{\sum_{j=1}^k 2h_j n_j}} \mathbb{E}_N^{(s)} \left[\prod_{j=1}^k \left| \Xi_{n_j}(\mathbf{x}_1^{(N)}, \mathbf{x}_2^{(N)}, \dots, \mathbf{x}_N^{(N)}) \right|^{2h_j} \right],$$

for some constant $K_N(s)$ that is independent of h_1, \dots, h_k . Plugging in $h_j = 0$ for all j to both sides of the equation now gives the desired result for $\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)$. To get the second part, simply note that

$$V_{\mathbf{A}}^{(n_j)}(0) = e^{-iN\frac{\pi}{2} + i\sum_{n=1}^N \frac{\theta_n}{2}} \sum_{m_j=0}^{n_j} \binom{n_j}{m_j} \left(-\frac{iN}{2}\right)^{m_j} \frac{d^{(n_j-m_j)} Z_{\mathbf{A}}(\theta)}{d\theta^{n_j-m_j}} \Big|_{\theta=0},$$

which gives the desired conclusion. \square

We can now prove Theorem 1.4.

Proof of Theorem 1.4. The fact that $\{\mu_N^{(s)}\}_{N \geq 1}$ forms a consistent permutation-invariant (in fact it is unitarily invariant) ensemble was shown in Proposition 3.1 in [24]. We write $\mu_\infty^{(s)}$ for the corresponding measure coupling them on infinite matrices, with respect to which all almost-sure statements below are understood. Now, using the fact that

$$\mathbb{E}^{(s)} \left[\left| \det(\mathbf{H}_{n_j}) \right|^{2s} \right] = \mathbb{E}_{n_j}^{(s)} \left[\prod_{m=1}^{n_j} \left| \mathbf{x}_m^{(n_j)} \right|^{2s} \right] < \infty,$$

where $\mathbb{E}^{(s)}$ denotes expectation with respect to $\mu_\infty^{(s)}$, we obtain, by means of Proposition 2.11, there is a sequence of random variables $\{\tilde{\Upsilon}_j(s)\}_{j \geq 0}$ with $\tilde{\Upsilon}_0(s) = 1$ such that

$$\tilde{\Upsilon}_j(s) = \lim_{N \rightarrow \infty} \frac{(-1)^j j! e_j(\mathbf{x}_1^{(N)}, \dots, \mathbf{x}_N^{(N)})}{N^j} \quad (39)$$

holds in $L^{2s}(\mu_\infty^{(s)})$ and almost surely. Now we claim that

$$\frac{\Xi_{n_j}(\mathbf{x}_1^{(N)}, \dots, \mathbf{x}_N^{(N)})}{N^{n_j}} \xrightarrow{N \rightarrow \infty} \tilde{\Upsilon}_{n_j}(s) \quad (40)$$

in $L^{2s}(\mu_\infty^{(s)})$ and almost surely. To derive this, note that it suffices to show that the highest-order contribution in the sum defining Ξ_{n_j} comes from the term with $l = n_j$. This is seen using (39) together with the combinatorial bound

$$a_{n_j, l} = \begin{cases} O(N^{\frac{n_j-l}{2}}), & \text{if } n_j - l \text{ is even,} \\ 0, & \text{otherwise,} \end{cases}$$

and $a_{n_j, n_j} = (-1)^{n_j} n_j!$.

Below, we explain how one can obtain the above estimate for $a_{n_j, l}$. By the definition of $a_{n_j, l}$ in (34), it is a sum (with coefficients depending only on n_j) over all nonnegative integer tuples (m_1, \dots, m_N) satisfying $\sum_{i=1}^N m_i = n_j$ and for which exactly m_1, \dots, m_l are odd. In this case one necessarily has that $n_j - l$ is even. Fix $k \geq l$ and a k -tuple of

positive integers $\lambda = (\lambda_1, \dots, \lambda_k)$ with the first l entries odd, the remaining entries even, and $\sum_{i=1}^k \lambda_i = n_j$. A corresponding N -tuple is understood to be any $\mathbf{m} = (m_1, \dots, m_N)$ whose multiset of nonzero entries is exactly $\{\lambda_1, \dots, \lambda_k\}$; equivalently, k positions of \mathbf{m} are assigned the values $\lambda_1, \dots, \lambda_k$, while the remaining $N - k$ coordinates are 0. To form such an N -tuple, the odd entries must occupy the first l coordinates, and we choose $k - l$ locations for the even entries among the remaining $N - l$ positions. Hence the number of such N -tuples is at most

$$l! \binom{N-l}{k-l} (k-l)! = \frac{l!(N-l)!}{(N-k)!} \leq C(n_j) N^{k-l}.$$

Since $\lambda_1, \dots, \lambda_l \geq 1$ and $\lambda_{l+1}, \dots, \lambda_k \geq 2$, we have $k - l \leq (n_j - l)/2$, independently of N . As the number of such λ depends only on n_j , summing over all of them yields

$$|a_{n_j, l}| \leq C(n_j) N^{\frac{n_j-l}{2}}.$$

In particular, this shows that the term with $l = n_j$ dominates. As claimed earlier, we thus obtain (40).

Hence, combining Proposition 3.1 with the asymptotic formula in [70, Eq. (15)],

$$\mathfrak{G}_N^{(0)}(s) = \mathfrak{F}_N^{(0)}(s) \sim N^{s^2} \frac{G(s+1)^2}{G(2s+1)},$$

and arguing as in Proposition 2.8, we obtain convergence of all joint moments

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\mathfrak{F}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)}{N^{s^2 + 2 \sum_{j=1}^k n_j h_j}} &= \frac{G(s+1)^2}{G(2s+1)} 2^{-2 \sum_{j=1}^k h_j n_j} \mathbb{E} \left[\prod_{j=1}^k |\tilde{Y}_{n_j}(s)|^{2h_j} \right], \\ \lim_{N \rightarrow \infty} \frac{\mathfrak{G}_N^{(n_1, \dots, n_k)}(h_1, \dots, h_k)}{N^{s^2 + 2 \sum_{j=1}^k n_j h_j}} &= \frac{G(s+1)^2}{G(2s+1)} 2^{-2 \sum_{j=1}^k h_j n_j} \mathbb{E} \left[\prod_{j=1}^k \left| \sum_{m_j=0}^{n_j} (-i)^{m_j} \binom{n_j}{m_j} \tilde{Y}_{n_j-m_j}(s) \right|^{2h_j} \right]. \end{aligned}$$

It only remains to identify the random variables $\tilde{Y}_j(s)$ with the random variables $Y_j(s)$ from Definition 1.3. For this we make use of some results from the integrable probability literature. Note that by the definition of $Y_j(s)$ and Newton's identities, it suffices to show that, for any $M \geq 1$, we have the convergence in distribution,

$$\left(N^{-j} \mathbf{p}_j \left(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)} \right) \right)_{j=1}^M \xrightarrow{d} \left(\mathbf{q}_j(s) \right)_{j=1}^M. \quad (41)$$

In order to prove (41), let us first define the locally compact space $\Omega \subset \mathbb{R}^{2\infty+2}$ (not to be confused with the abstract probability space used in the previous section) as follows

$$\begin{aligned} \Omega \stackrel{\text{def}}{=} \left\{ \omega = \left((x_i^+)_{i=1}^\infty, (x_i^-)_{i=1}^\infty, \gamma, \delta \right) \in \mathbb{R}_{\geq 0}^\infty \times \mathbb{R}_{\geq 0}^\infty \times \mathbb{R} \times \mathbb{R}_{\geq 0} : x_1^+ \geq x_2^+ \geq \dots \geq 0; \right. \\ \left. x_1^- \geq x_2^- \geq \dots \geq 0; \sum_{i=1}^\infty (x_i^+)^2 + (x_i^-)^2 \leq \delta \right\}, \end{aligned}$$

which is endowed with the topology of coordinate-wise convergence. As we explain

more precisely below, we consider the Ω -valued random variables

$$\omega_N^{(s)} \stackrel{\text{def}}{=} \left(\left(N^{-1} \max \{x_i^{(N)}, 0\} \right)_{i=1}^{\infty}, \left(N^{-1} \max \{-x_{N+1-i}^{(N)}, 0\} \right)_{i=1}^{\infty}, N^{-1} \sum_{i=1}^N x_i^{(N)}, N^{-2} \sum_{i=1}^N (x_i^{(N)})^2 \right),$$

where $(x_1^{(N)}, \dots, x_N^{(N)}) \in \mathbb{W}_N$ denote the eigenvalues of the top-left $N \times N$ corner of the infinite random matrix whose distribution is given by $\mu_{\infty}^{(s)}$ (that is, they follow the distribution $\mathbf{M}_N^{(s)}$). We adhere to the convention $x_i^{(N)} \equiv 0$ for $i < 1$ or $i > N$. Then, as $N \rightarrow \infty$,

$$\omega_N^{(s)} \rightarrow \omega^{(s)} \stackrel{\text{def}}{=} \left((\alpha_i^+)_{i=1}^{\infty}, (\alpha_i^-)_{i=1}^{\infty}, \lim_{k \rightarrow \infty} \sum_{i=1}^{\infty} \mathbf{1}_{|\alpha_i^{\pm}| > k^{-2}} (\alpha_i^+ - \alpha_i^-), \sum_{i=1}^{\infty} \left[(\alpha_i^+)^2 + (\alpha_i^-)^2 \right] \right), \quad (42)$$

$\mu_{\infty}^{(s)}$ -almost surely, where $\{\alpha_i^+\} \sqcup \{-\alpha_i^-\} \stackrel{d}{=} \mathbf{P}^{(s)}$. Indeed, the almost sure (a.s.) convergence of $N^{-1} \max \{x_i^{(N)}, 0\}$ and $N^{-1} \max \{-x_{N+1-i}^{(N)}, 0\}$ to the corresponding α_i^{\pm} is a consequence of [24] (see Theorem IV and the results of Section 5 therein), whereas the a.s. convergence of $N^{-1} \sum_{i=1}^N x_i^{(N)}$ and $N^{-2} \sum_{i=1}^N (x_i^{(N)})^2$ to the corresponding sums in (42) is established in Theorem 2.1 and Theorem 2.3 in [83], respectively. Now, by virtue of Proposition 2.3 and its proof in [15], we have that coordinate-wise convergence in the space Ω ,

$$\omega^{(N)} = \left((y_i^{+, (N)})_{i=1}^{\infty}, (y_i^{-, (N)})_{i=1}^{\infty}, \gamma^{(N)}, \delta^{(N)} \right) \rightarrow \omega = \left((y_i^+)_{i=1}^{\infty}, (y_i^-)_{i=1}^{\infty}, \gamma, \delta \right),$$

is equivalent to the convergence of the following, as $N \rightarrow \infty$,

$$\gamma^{(N)} \rightarrow \gamma, \delta^{(N)} \rightarrow \delta, \sum_{i=1}^{\infty} \left[(y_i^{+, (N)})^k + (-y_i^{-, (N)})^k \right] \rightarrow \sum_{i=1}^{\infty} \left[(y_i^+)^k + (-y_i^-)^k \right], \text{ for } k \geq 3. \quad (43)$$

Hence, applying this to the convergence given in (42), on the full $\mu_{\infty}^{(s)}$ -measure set where the above convergence holds, we can identify the corresponding limiting power sums with the q_j 's, thereby obtaining (41). This concludes the proof. \square

3.2 EXPLICIT FORMULAE

Next, we apply the combinatorial formulae proved in Section 2 for the even-integer moments of the limiting random variables to our current setting. Via this representation, we further deduce that our random variables $\{Y_n(s)\}_{n \geq 1}$ are non-trivial.

Proof of Theorem 1.5. The first part of the result is a direct application of Proposition 2.12. Since $m \geq n_1 > \dots > n_k$, we simply expand

$$\prod_{j=1}^k \left(e_{n_j} \left(x_1^{(m)}, x_2^{(m)}, \dots, x_m^{(m)} \right) \right)^{2n_j}$$

as a linear combination of polynomials in $x_1^{(m)}, \dots, x_m^{(m)}$, with coefficients independent of s . Therefore, by the first part of the theorem and by the definition of $\mathbb{E}_m^{(s)}[\cdot]$, to prove the

rationality of

$$s \mapsto \mathbb{E} \left[\prod_{j=1}^k (Y_{n_j}(s))^{2h_j} \right],$$

it suffices to show that

$$\frac{1}{\tilde{C}_m^{(s)}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{x_1^{r_1}}{(1+x_1^2)^{m+s}} \cdots \frac{x_m^{r_m}}{(1+x_m^2)^{m+s}} dx_1 \cdots dx_m, \quad r_i \in \mathbb{N} \cup \{0\}, \quad r_i \leq 2m - 2 + 2 \sum_{j=1}^k h_j$$

is a rational function of s . Moreover, by symmetry, the above integral vanishes whenever at least one exponent r_i is odd, so it remains to consider the case where all r_i , $i = 1, \dots, m$, are even. Note that $\sum_{j=1}^k h_j < s + \frac{1}{2}$. The desired rationality of the above integral follows directly from the argument for the same quantity as in [14, pp. 14595–14596], with N in that reference replaced by m here. \square

The rationality claim of Theorem 1.5 turns out to be very useful in proving Corollary 1.6. The strategy to prove the result above will be as follows: we will initially restrict to $s \in \mathbb{N}$ and compute the averages in (11) by calculating the corresponding integrals over $\mathbb{U}(N)$ and taking $N \rightarrow \infty$. We will then show that the expressions we obtain extend to rational functions, which, when combined with Theorem 1.5, will give us two rational functions that agree on integers, which then implies they must agree everywhere. We will then show that the explicit expressions that appear in (11) are strictly positive, and conclude the final claim of the corollary.

To complete the aforementioned first step, we state and prove a sequence of lemmas regarding certain averages of eigenvalues of Haar-unitary random matrices. Recall that via Weyl's integration formula, it is known that if $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of a Haar-unitary random matrix, ordered so that

$$\lambda_j = e^{i\theta_j} \quad \text{with} \quad \theta_j \in [0, 2\pi), \quad \theta_1 \leq \cdots \leq \theta_N,$$

then the law of the θ_j 's is given by:

$$d\nu_{\text{Haar}}(\theta_1, \dots, \theta_N) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^N} \prod_{1 \leq j < k \leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_N.$$

Thus, for the rest of this section, we denote

$$\mathbb{E}_{\mathbb{U}(N)} [f(\lambda_1, \dots, \lambda_N)] \stackrel{\text{def}}{=} \int f(e^{i\theta_1}, \dots, e^{i\theta_N}) d\nu_{\text{Haar}}(\theta_1, \dots, \theta_N).$$

Lemma 3.2. *Let $s, n \in \mathbb{N}$. Then, as $N \rightarrow \infty$, we have*

$$\mathfrak{F}_N^{(n,0)}(1, s-1) \sim \frac{\partial^n}{\partial z_1^n} \frac{\partial^n}{\partial w_1^n} \mathbb{E}_{\mathbb{U}(N)} \left[\prod_{i=1}^s (z_i w_i)^{-\frac{N}{2}} \prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i)(\bar{\lambda}_j - w_i) \right] \Bigg|_{\substack{z_1 = \cdots = z_s = 1 \\ w_1 = \cdots = w_s = 1}}, \quad (44)$$

$$\mathfrak{G}_N^{(n,0)}(1, s-1) \sim \frac{\partial^n}{\partial z_1^n} \frac{\partial^n}{\partial w_1^n} \mathbb{E}_{\mathbb{U}(N)} \left[\prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i)(\bar{\lambda}_j - w_i) \right] \Bigg|_{\substack{z_1 = \cdots = z_s = 1 \\ w_1 = \cdots = w_s = 1}}, \quad (45)$$

where $\lambda_1, \dots, \lambda_N$ are the random eigenvalues of a Haar-distributed $\mathbf{A} \in \mathbb{U}(N)$.

Proof. Let $\mathbf{A} \in \mathbb{U}(N)$ and let $(e^{i\theta_1}, \dots, e^{i\theta_N}) = (\lambda_1, \dots, \lambda_N)$ denote its eigenvalues. Then, define the characteristic polynomial, this time as a function on the unit circle instead of $[0, 2\pi)$ by:

$$\widetilde{V}_{\mathbf{A}}(z) \stackrel{\text{def}}{=} \prod_{j=1}^N (1 - ze^{-i\theta_j}) = \prod_{j=1}^N (1 - z\bar{\lambda}_j) = \prod_{j=1}^N \bar{\lambda}_j(\lambda_j - z)$$

and its corresponding rescaled version:

$$\widetilde{Z}_{\mathbf{A}}(z) = e^{-iN\frac{\theta_1}{2} + i\sum_{n=1}^N \frac{\theta_n}{2}} z^{-\frac{N}{2}} \widetilde{V}_{\mathbf{A}}(z).$$

Then, direct computation gives that:

$$\widetilde{Z}_{\mathbf{A}^*}^{(n)}(1) = (-1)^N \overline{\widetilde{Z}_{\mathbf{A}}^{(n)}(1)}.$$

In particular, one may compute,

$$\begin{aligned} & \frac{\partial^n}{\partial z_1^n} \frac{\partial^n}{\partial w_1^n} \prod_{i=1}^s (z_i w_i)^{-\frac{N}{2}} \prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i)(\bar{\lambda}_j - w_i) \Big|_{\substack{z_1=\dots=z_s=1 \\ w_1=\dots=w_s=1}} \\ &= \frac{\partial^n}{\partial z_1^n} z_1^{-\frac{N}{2}} \prod_{j=1}^N (\lambda_j - z_1) \frac{\partial^n}{\partial w_1^n} w_1^{-\frac{N}{2}} \prod_{j=1}^N (\bar{\lambda}_j - w_1) \prod_{i=2}^s (z_i w_i)^{-\frac{N}{2}} \prod_{j=1}^N \prod_{i=2}^s (\lambda_j - z_i)(\bar{\lambda}_j - w_i) \Big|_{\substack{z_1=\dots=z_s=1 \\ w_1=\dots=w_s=1}} \\ &= (-1)^{Ns} \widetilde{Z}_{\mathbf{A}^*}^{(n)}(1) \widetilde{Z}_{\mathbf{A}}^{(n)}(1) \left(\widetilde{Z}_{\mathbf{A}^*}(1)\right)^{s-1} \left(\widetilde{Z}_{\mathbf{A}}(1)\right)^{s-1} = \left|\widetilde{Z}_{\mathbf{A}}^{(n)}(1)\right|^2 \left|\widetilde{Z}_{\mathbf{A}}(1)\right|^{2s-2}. \end{aligned}$$

In particular, taking the average of both sides and moving the differentiation outside the integral (since all derivatives are uniformly bounded as the λ_j 's range over the unit circle):

$$\begin{aligned} & \frac{\partial^n}{\partial z_1^n} \frac{\partial^n}{\partial w_1^n} \mathbb{E}_{\mathbb{U}(N)} \left[\prod_{i=1}^s (z_i w_i)^{-\frac{N}{2}} \prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i)(\bar{\lambda}_j - w_i) \right] \Big|_{\substack{z_1=\dots=z_s=1 \\ w_1=\dots=w_s=1}} \\ &= \int_{\mathbb{U}(N)} \left|\widetilde{Z}_{\mathbf{A}}^{(n)}(1)\right|^2 \left|\widetilde{Z}_{\mathbf{A}}(1)\right|^{2s-2} d\mu_{\text{Haar}}(\mathbf{A}). \end{aligned}$$

Now, the leading coefficient in the asymptotics of

$$\int_{\mathbb{U}(N)} \left|\widetilde{Z}_{\mathbf{A}}^{(n)}(1)\right|^2 \left|\widetilde{Z}_{\mathbf{A}}(1)\right|^{2s-2} d\mu_{\text{Haar}}(\mathbf{A}) \text{ and } \mathfrak{F}_N^{(n,0)}(1, s-1),$$

are the same, since plugging in $z = e^{i\theta}$ and differentiating with respect to θ would only add terms of lower order derivatives, which, by Theorem 1.4 has strictly lower order growth. This completes the proof of the first part of the lemma. The proof of the second part follows similarly. \square

The next lemma has appeared previously in the literature; see, for instance, [1, Theorem 1] and [73, p. 2607].

Lemma 3.3. For every $s \in \mathbb{N}$, we have

$$\mathbb{E}_{\mathbb{U}(N)} \left[\prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i) (\bar{\lambda}_j - \bar{w}_i) \right] = \frac{\det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i \bar{w}_j)^l \right]}{\prod_{1 \leq i < j \leq s} (z_j - z_i) \prod_{1 \leq i < j \leq s} (\bar{w}_j - \bar{w}_i)},$$

where $\lambda_1, \dots, \lambda_N$ are the random eigenvalues of a Haar-distributed $\mathbf{A} \in \mathbb{U}(N)$.

Lemma 3.4. For $s \in \mathbb{N}, m, n \in \mathbb{N} \cup \{0\}$, with $\lambda_1, \dots, \lambda_N$ the random eigenvalues of a Haar-distributed $\mathbf{A} \in \mathbb{U}(N)$, we have,

$$\begin{aligned} & \left. \frac{\partial^n}{\partial z_1^n} \frac{\partial^m}{\partial w_1^m} \mathbb{E}_{\mathbb{U}(N)} \left[\prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i) (\bar{\lambda}_j - w_i) \right] \right|_{z_1 = \dots = z_s = w_1 = \dots = w_s = 1} \\ &= n!m! \frac{1}{(s-1+n)!} \frac{1}{(s-1+m)!} \prod_{j=2}^s \frac{1}{((s-j)!)^2} N^{s^2+n+m} \det_{1 \leq i, j \leq s} \left[\frac{1}{p_i + q_j + 1} \right] + O(N^{s^2+n+m-1}), \end{aligned}$$

where $p_1 = s-1+n, q_1 = s-1+m, p_i = q_i = s-i, i = 2, \dots, s$.

Proof. By Lemma 3.3, we have

$$\mathbb{E}_{\mathbb{U}(N)} \left[\prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i) (\bar{\lambda}_j - w_i) \right] = \frac{\det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i w_j)^l \right]}{\prod_{1 \leq i < j \leq s} (z_j - z_i) \prod_{1 \leq i < j \leq s} (w_j - w_i)}.$$

Thus, letting $z_s = w_s = 1$, we then get:

$$\begin{aligned} & \lim_{w_{s-1} \rightarrow 1} \lim_{z_{s-1} \rightarrow 1} \frac{\det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i w_j)^l \right]}{\prod_{1 \leq i < j \leq s-1} (z_j - z_i) \prod_{1 \leq i < j \leq s-1} (w_j - w_i) \prod_{1 \leq i \leq s-1} (1 - z_i) (1 - w_i)} \\ &= \lim_{w_{s-1} \rightarrow 1} \lim_{z_{s-1} \rightarrow 1} \frac{\det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i w_j)^l \right]}{(1 - z_{s-1})(1 - w_{s-1}) \prod_{1 \leq i < j \leq s-2} (z_j - z_i) (w_j - w_i) \prod_{1 \leq i \leq s-2} (1 - z_i)^2 (1 - w_i)^2} \\ &= \frac{1}{\prod_{1 \leq i < j \leq s-2} (z_j - z_i) (w_j - w_i) \prod_{1 \leq i \leq s-2} (1 - z_i)^2 (1 - w_i)^2} \\ & \quad \times \left. \frac{\partial}{\partial z_{s-1}} \frac{\partial}{\partial w_{s-1}} \det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i w_j)^l \right] \right|_{\substack{w_{s-1} = w_s = 1 \\ z_{s-1} = z_s = 1}}. \end{aligned}$$

Continuing the process above where for each $j = s, s-1, \dots, 1$, we are getting a factor of $(1 - z_j)^{s-j}$ in the denominator at the j -th step, we finally obtain:

$$\begin{aligned} & \left. \frac{\partial^n}{\partial z_1^n} \frac{\partial^m}{\partial w_1^m} \frac{\det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i w_j)^l \right]}{\prod_{1 \leq i < j \leq s} (z_j - z_i) \prod_{1 \leq i < j \leq s} (w_j - w_i)} \right|_{\substack{w_1 = \dots = w_s = 1 \\ z_1 = \dots = z_s = 1}} \\ &= (n!) (m!) \frac{1}{(s-1+n)! (s-1+m)!} \prod_{j=2}^s \frac{1}{((s-j)!)^2} \frac{\partial^{n+s-1}}{\partial z_1^{n+s-1}} \frac{\partial^{m+s-1}}{\partial w_1^{m+s-1}} \\ & \quad \prod_{i=2}^s \left(\left(\frac{\partial}{\partial z_i} \right)^{s-i} \left(\frac{\partial}{\partial w_i} \right)^{s-i} \right) \det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i w_j)^l \right] \Big|_{\substack{w_1 = \dots = w_s = 1 \\ z_1 = \dots = z_s = 1}}. \end{aligned}$$

Then, one can compute:

$$\begin{aligned} & \frac{\partial^{n+s-1}}{\partial z_1^{n+s-1}} \frac{\partial^{m+s-1}}{\partial w_1^{m+s-1}} \prod_{i=2}^s \left(\left(\frac{\partial}{\partial z_i} \right)^{s-i} \left(\frac{\partial}{\partial w_i} \right)^{s-i} \right) \det_{1 \leq i, j \leq s} \left[\sum_{l=0}^{N+s-1} (z_i w_j)^l \right] \Big|_{\substack{w_1 = \dots = w_s = 1 \\ z_1 = \dots = z_s = 1}} \\ &= \det_{1 \leq i, j \leq s} \left[\frac{\partial^{p_i+q_j}}{\partial z_i^{p_i} \partial w_j^{q_j}} \sum_{l=0}^{N+s-1} (z_i w_j)^l \right] \Big|_{\substack{w_1 = \dots = w_s = 1 \\ z_1 = \dots = z_s = 1}} = \det_{1 \leq i, j \leq s} \left(\int_0^N x^{p_i+q_j} dx \right) + O(N^{s^2+m+n-1}), \end{aligned}$$

where the last equality is obtained by a standard application of the Euler-Maclaurin formula. The result follows from the above. \square

Proof of Corollary 1.6. We start by proving the second equality in (11). Note that by Theorem 1.5, we know that

$$s \mapsto \mathbb{E} \left[\left| \sum_{m=0}^n (-i)^m \binom{n}{m} Y_{n-m}(s) \right|^2 \right]$$

is rational on $s > \frac{1}{2}$. Furthermore, by a combination of Lemmas 3.2, 3.3 and 3.4, one can see that for $s \in \mathbb{N}$, $s \geq 2$

$$\mathbb{E} \left[\left| \sum_{m=0}^n (-i)^m \binom{n}{m} Y_{n-m}(s) \right|^2 \right] = 2^{2n} \frac{G(2s+1)}{G(s+1)^2} \frac{(n!)^2}{((s-1+n)!)^2} \prod_{j=2}^s \frac{1}{((s-j)!)^2} \det_{1 \leq i, j \leq s} \left[\frac{1}{p_i + p_j + 1} \right],$$

where $p_1 = s - 1 + n$ and $p_i = s - i$, for $i = 2, \dots, s$. Thus, if one shows that the right-hand side of the above equation extends to a rational function on $s > \frac{1}{2}$, then we have an equality of two rational functions on the real line that agree on infinitely many points, and hence they must agree for all $s > \frac{1}{2}$. To see that this holds, note that by the general formula for Cauchy matrices, we have that

$$\begin{aligned} \det_{1 \leq i, j \leq s} \left[\frac{1}{p_i + p_j + 1} \right] &= \frac{\prod_{1 \leq i < j \leq s} (p_j - p_i)^2}{\prod_{1 \leq i, j \leq s} (p_i + p_j + 1)} \\ &= \frac{1}{2s-1+2n} \prod_{i=2}^s \frac{(n+i-1)^2}{(2s+n-i)^2} \frac{\prod_{i=2}^s ((s-i)!)^2}{\prod_{2 \leq i, j \leq s} (2s+1-i-j)}. \end{aligned}$$

Substituting this into the above expression and simplifying it, we obtain that

$$\mathbb{E} \left[\left| \sum_{m=0}^n (-i)^m \binom{n}{m} Y_{n-m}(s) \right|^2 \right] = 2^{2n} \frac{2s-1}{2s-1+2n} \prod_{l=1}^n \left(\frac{l+s-1}{l+2s-2} \right)^2$$

for $s \in \mathbb{N}$, $s \geq 2$ and hence for all $s > \frac{1}{2}$. This gives the second equality in (11). For the first equality, note that

$$\begin{aligned} & \frac{\partial^n}{\partial z_1^n} \frac{\partial^n}{\partial w_1^n} \mathbb{E}_{\cup(N)} \left[\prod_{i=1}^s (z_i w_i)^{-\frac{N}{2}} \prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i)(\bar{\lambda}_j - w_i) \right] \Big|_{\substack{z_1 = \dots = z_s = 1 \\ w_1 = \dots = w_s = 1}} \\ &= \sum_{i, j=0}^n \binom{n}{i} \binom{n}{j} \prod_{l=0}^{n-i-1} \left(-\frac{N}{2} - l \right) \prod_{l=0}^{n-j-1} \left(-\frac{N}{2} - l \right) \frac{\partial^i}{\partial z_1^i} \frac{\partial^j}{\partial w_1^j} \mathbb{E}_{\cup(N)} \left[\prod_{j=1}^N \prod_{i=1}^s (\lambda_j - z_i)(\bar{\lambda}_j - w_i) \right] \Big|_{\substack{z_1 = \dots = z_s = 1 \\ w_1 = \dots = w_s = 1}}. \end{aligned}$$

Hence, arguing similarly as above, we obtain the first equality in (11). Note that for the second part of the Corollary, one only needs to show positivity of the expressions that appear in (11) for $s > \frac{1}{2}$. Positivity of the second equation is obvious. To see the positivity of the first one, we use the result that a symmetric Cauchy matrix

$$\mathbf{M} = \left[\frac{1}{q_i + q_j} \right]_{i,j=0,\dots,n}$$

is positive definite if and only if all the q_i 's are positive and mutually distinct (see e.g., [48, Theorem A]). In particular, if we let $q_i = s - \frac{1}{2} + i$, $i = 0, \dots, n$, then

$$\mathbf{M} = \left[\frac{1}{2s - 1 + i + j} \right]_{i,j=0,\dots,n}$$

is positive definite. Now, consider the vector $\mathbf{x} = (x_0, x_1, \dots, x_n)^\top$ with

$$x_i = \binom{n}{i} \left(-\frac{1}{2}\right)^{n-i} \prod_{l=0}^{i-1} (l+s) \prod_{l=i+1}^n (l+2s-2).$$

Then, we see that

$$\lim_{N \rightarrow \infty} \frac{\mathfrak{F}_N^{(n,0)}(1, s-1)}{N^{s^2+2n}} = \frac{G^2(s+1)}{G(2s+1)} \frac{2s-1}{\prod_{l=1}^n (2s-2+l)^2} \mathbf{x}^\top \mathbf{M} \mathbf{x} > 0,$$

which completes the proof of the result. \square

4 CONNECTIONS TO INTEGRABLE SYSTEMS

4.1 REDUCTION TO FINITE N AND $s \in \mathbb{N}$

As explained in the introduction, Theorem 1.11 will be proven by a limiting argument: we obtain the analogue of the differential relation in (16) at the finite- N level in Theorem 1.9, and then take the $N \rightarrow \infty$ limit. It turns out that by virtue of the following analytic continuation argument, we do not lose generality by restricting to $s \in \mathbb{N}$ in Theorem 1.9. This, in turn, will allow us to represent the quantities of interest as derivatives of smooth functions, which, from their original definition, is not at all obvious. In particular, from now on we will restrict our attention to $s \in \mathbb{N}$ in proving Theorem 1.9.

Lemma 4.1. *Assume the statement of Theorem 1.9 holds for all $s \in \mathbb{N}$, with $s > 2^{-1}(\sum_{q=2}^k qn_q - 1)$. Then, the same statement holds for all $s \in \mathbb{C}$ with $\Re(s) > 2^{-1}(\sum_{q=2}^k qn_q - 1)$.*

Proof. Let $l \in \mathbb{N}$ be the smallest integer so that $l > 2^{-1}(\sum_{q=2}^k qn_q - 1)$. Then, we have for all $s \in \mathbb{C}$ with $\Re(s) \geq l$ that

$$\left| \frac{d^m}{dt_1^m} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{-i \frac{t_1}{N} x_j}}{(1+x_j^2)^{s+N}} \Delta^2(\mathbf{x}) d\mathbf{x} \right| \leq \frac{1}{N^m} \int_{\mathbb{R}^N} \left| \sum_{j=1}^N x_j \right|^m \prod_{j=1}^N \frac{1}{(1+x_j^2)^{l+N}} \Delta^2(\mathbf{x}) d\mathbf{x},$$

for $m = 0, \dots, \sum_{q=2}^k (q-1)n_q$ and also that

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \prod_{q=2}^k \left(\sum_{j=1}^N (x_j - i)^q \right)^{n_q} \prod_{j=1}^N \frac{e^{-i \frac{t_1}{N} x_j}}{(1+x_j^2)^{s+N}} \Delta^2(\mathbf{x}) d\mathbf{x} \right| \\ & \leq \int_{\mathbb{R}^N} \prod_{q=2}^k \left(\sum_{j=1}^N (1+x_j^2)^{q/2} \right)^{n_q} \prod_{j=1}^N \frac{1}{(1+x_j^2)^{t+N}} \Delta^2(\mathbf{x}) d\mathbf{x}. \end{aligned}$$

Now, observe that proving (15) amounts to showing the equality after the averages are replaced by these integrals, due to the cancellation of the normalization constants of $\mathbf{M}_N^{(s)}$. Thus, using the bounds for the integrals above and the fact that $P_m^{(s,N)}(t_1)$ are polynomials in s , we see that both sides satisfy the conditions of Carlson's lemma. Therefore, if they agree for $s \in \mathbb{N}$, $s \geq l$, then they agree for all s with $\Re(s) \geq l$. Hence, noting that both sides are analytic on $\Re(s) > 2^{-1}(\sum_{q=2}^k qn_q - 1)$ and using the identity theorem, we get the desired result. \square

The vast majority of the rest of this section will be devoted to proving Theorem 1.9. In the end, we will show that taking the $N \rightarrow \infty$ limit indeed implies Theorem 1.11, which, without the machinery we developed earlier, is highly non-trivial.

Here, we stress again that in proving Theorem 1.9, by virtue of Lemma 4.1, one may restrict to $s \in \mathbb{N}$, which we henceforth do. The strategy we will adopt can then be summarized in the following steps:

- **Step 1:** We will write the averages of the form

$$\mathbb{E}_N^{(s)} \left[e^{-it_1 \sum_{j=1}^N x_j^{(N)}} \left(\sum_{j=1}^N (x_j^{(N)} - i)^2 \right)^{\ell_2} \cdots \left(\sum_{j=1}^N (x_j^{(N)} - i)^k \right)^{\ell_k} \right], \quad (46)$$

in terms of derivatives of Hankel-determinants. To be more precise, up to a power of N , we will show that the above average can be given as

$$\prod_{m=2}^k \frac{\partial^{\ell_m}}{\partial t_m^{\ell_m}} \det_{0 \leq i, j \leq N-1} [\psi_{i+j}(t_1, \dots, t_k)] \Big|_{t_2, \dots, t_k=0} = \prod_{m=2}^k \frac{\partial^{\ell_m}}{\partial t_m^{\ell_m}} \Psi_N(t_1, \dots, t_k) \Big|_{t_2, \dots, t_k=0},$$

for some explicit functions $\psi_\gamma(t_1, \dots, t_k)$ with $\Psi_N(t_1, \dots, t_k)$ denoting their Hankel determinant. This is given in Proposition 4.2.

- **Step 2:** Next, in Section 4.3, we will introduce the associated functions $\Psi_{N,\lambda}$ where λ are integer partitions. These are basically generalizations of the Hankel determinant on the left hand-side of the equation above, such that $\Psi_{N,\emptyset} = \Psi_N$. In Lemma 4.14, we will then show that the t_m -derivatives of Ψ_N , where $m = 2, \dots, k$ are given as linear combinations of $\Psi_{N,\lambda}$ where λ ranges over a specific class of partitions. Hence, to study these derivatives, it suffices to understand the properties of $\Psi_{N,\lambda}$, which will be studied in the next step.
- **Step 3:** In Proposition 4.11, we show that the quantities $\Psi_{N,\lambda}$ satisfy certain recursive relations as λ ranges over a specific class of integer partitions.

- **Step 4:** Finally, in Section 4.4 we use the results in Steps 2 and 3 to prove Theorems 1.9 and 1.11 by a delicate induction argument.

It is also worth noting that even though considering power sums after shifting the eigenvalues by $-i$ might look unnatural, it is done to ensure that we are able to write the relations between Hankel determinants and their shifted versions in a concise manner. Indeed, the joint moments of power sums without such shifts can be written as linear combinations of the averages in (46). In addition, due to our earlier results, the additional terms one obtains from this shift do not make any contribution in the large- N limit.

4.2 JOINT MOMENTS AND HANKEL DETERMINANTS

Our starting point is the following proposition.

Proposition 4.2. *Let $N \geq 1, k \geq 2$ and $l_2, \dots, l_k \geq 0$ be integers, and let $s > (\sum_{m=2}^k ml_m - 1)/2$ be an integer. Let $\mathbf{U}(a, b; z)$ be the confluent hypergeometric function of the second kind, and let*

$$\phi_m(t_1) \stackrel{\text{def}}{=} \exp(-t_1) \mathbf{U}(1 - N - s, 2 - 2N - 2s + m; 2t_1). \quad (47)$$

Then, for $t_1 \geq 0$, we have

$$(46) = C_N^{(s)} (-2i)^{\sum_{m=2}^k ml_m} \prod_{m=2}^k \frac{\partial^{\ell_m}}{\partial t_m^{\ell_m}} \det_{0 \leq i, j \leq N-1} \left[\sum_{m_2, \dots, m_k=0}^{\infty} \frac{t_2^{m_2} \dots t_k^{m_k}}{m_2! \dots m_k!} \phi_{i+j+\sum_{l=2}^k lm_l}(t_1) \right] \Big|_{t_2=\dots=t_k=0}$$

for some constant $C_N^{(s)}$, whose explicit value will not play a role in what follows.

It will be clear, while proving the result above, that the earlier restriction to $s \in \mathbb{N}$ guarantees that the power sum considered above is well-defined, and has derivatives of all orders. This will then allow us to use the recursive relations for ϕ_γ to obtain desired relations for these series, which will be one of the main ingredients in proving various relations between Hankel determinants of concern. Moving on, to prove the proposition above, we will need two elementary lemmas.

Lemma 4.3. *Let $g(t_1, \dots, t_N)$ be a symmetric function of the form*

$$g(t_1, \dots, t_N) = \sum_{r=1}^R a_r \prod_{i=1}^N f_{r,i}(t_i)$$

for functions $f_{r,i} \in C^{2N-2}(\mathbb{R})$. Then,

$$\prod_{1 \leq i < j \leq n} \left(\frac{\partial}{\partial t_j} - \frac{\partial}{\partial t_i} \right)^2 g(t_1, \dots, t_N) \Big|_{t_1=\dots=t_N=t} = N! \sum_{r=1}^R a_r \det_{0 \leq i, j \leq N-1} [f_{r,i+1}^{(i+j)}(t)].$$

Lemma 4.4. *Let $N \geq 1, s \in \mathbb{N}$, and let $I \subset \mathbb{R}$ be a compact set. Then, there exists a constant $K = K_{N,I,s}$ so that for all $t \in I$,*

$$|\phi_m(t)| \leq K_{N,I,s} m^{N+s} \quad (48)$$

where ϕ_m is defined as in (47).

Proof. Using the well-known relation between the confluent hypergeometric function \mathbf{U} and generalised Laguerre polynomials (see, for instance, [7, p. 755]), we have

$$\mathbf{U}(1 - N - s, 2 - 2N - 2s + m; 2t_1) = (-1)^{N+s-1} (N + s - 1)! L_{N+s-1}^{(1-2N-2s+m)}(2t_1),$$

where the generalised Laguerre polynomial $L_n^{(\alpha)}$ is given by

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha} e^z}{n!} \frac{d^n}{dz^n} (e^{-z} z^{n+\alpha}).$$

Computing the derivative explicitly gives the desired bound immediately. \square

Proof of Proposition 4.2. We start by noting that (from here on the constant $C_N^{(s)}$ might change from line to line)

$$\begin{aligned} (46) &= C_N^{(s)} e^{Nt_1} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{-it_1(x_j-i)}}{(1+x_j^2)^{s+N}} \prod_{m=2}^k \left(\sum_{j=1}^N (x_j - i)^m \right)^{\ell_m} \Delta^2(x_1, \dots, x_N) dx \\ &= C_N^{(s)} e^{Nt_1} \prod_{1 \leq i < j \leq N} \left(\frac{\partial}{\partial w_j} - \frac{\partial}{\partial w_i} \right)^2 \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{-iw_j(x_j-i)}}{(1+x_j^2)^{s+N}} \prod_{m=2}^k \left(\sum_{j=1}^N (x_j - i)^m \right)^{\ell_m} dx \Big|_{w_1 = \dots = w_N = t_1}. \end{aligned}$$

We now use $g(w_1, \dots, w_N)$ to denote the integral

$$\int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{-iw_j(x_j-i)}}{(1+x_j^2)^{s+N}} \prod_{m=2}^k \left(\sum_{j=1}^N (x_j - i)^m \right)^{\ell_m} dx.$$

Then $g(w_1, \dots, w_N)$ is a symmetric function. Expanding the powers of $\sum_j (x_j - i)^m$ in the product over m , we obtain

$$g(w_1, \dots, w_N) = \sum_{\substack{\mu_{2,1} + \dots + \mu_{2,N} = \ell_2 \\ \vdots \\ \mu_{k,1} + \dots + \mu_{k,N} = \ell_k}} \prod_{m=2}^k \binom{\ell_m}{\mu_{m,1}, \dots, \mu_{m,N}} \int_{\mathbb{R}^N} \prod_{j=1}^N \frac{e^{-iw_j(x_j-i)} (x_j - i)^{\sum_{m=2}^k m \mu_{m,j}}}{(1+x_j^2)^{s+N}} dx.$$

By Lemma 4.3, we obtain that

$$(46) = C_N^{(s)} i^{\sum_{m=2}^k m \ell_m} e^{Nt_1} \sum_{\substack{\mu_{2,1} + \dots + \mu_{2,N} = \ell_2 \\ \vdots \\ \mu_{k,1} + \dots + \mu_{k,N} = \ell_k}} \prod_{m=2}^k \binom{\ell_m}{\mu_{m,1}, \dots, \mu_{m,N}} \det_{0 \leq i, j \leq N-1} [f^{(i+j+\sum_{m=2}^k m \mu_{m,j+1})}(t_1)]$$

where the function f is given, for $t \geq 0$, as

$$f(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \frac{e^{-it(x-i)}}{(1+x^2)^{s+N}} dx = \frac{2\pi}{2^{2N+2s-1} \Gamma(N+s)} e^{-2t} \mathbf{U}(1 - N - s, 2 - 2N - 2s; 2t),$$

(see [91, p.349 formula (9)]). Furthermore, by the differential relation (see [58, p. 258

formula (14))

$$\frac{d^m}{dt^m} \left(e^{-2t} \mathbf{U}(1 - N - s, 2 - 2N - 2s; 2t) \right) = (-2)^m e^{-2t} \mathbf{U}(1 - N - s, 2 - 2N - 2s + m; 2t),$$

we can rewrite (46) as follows:

$$(46) = C_N^{(s)} (-2i)^{\sum_{m=2}^k m \ell_m} \sum_{\substack{\mu_{2,1} + \dots + \mu_{2,N} = \ell_2 \\ \vdots \\ \mu_{k,1} + \dots + \mu_{k,N} = \ell_k}} \prod_{m=2}^k \binom{\ell_m}{\mu_{m,1}, \dots, \mu_{m,N}} \det_{0 \leq i, j \leq N-1} \left[\phi_{i+j+\sum_{m=2}^k m \mu_{m,j+1}}(t_1) \right] \quad (49)$$

with $\phi_n(t_1)$ as in (47). By the growth condition in Lemma 4.4, we have that, for fixed nonnegative integers n_2, \dots, n_k , the series

$$\sum_{m_2, \dots, m_k=0}^{\infty} \prod_{m=2}^k \frac{\partial^{m_m}}{\partial t_m^{m_m}} \left(\frac{t_2^{m_2} \dots t_k^{m_k}}{m_2! \dots m_k!} \right) \phi_{i+j+\sum_{l=2}^k l m_l}(t_1) = \sum_{m_2, \dots, m_k=0}^{\infty} \prod_{m=2}^k \frac{t_2^{m_2} \dots t_k^{m_k}}{m_2! \dots m_k!} \phi_{i+j+\sum_{l=2}^k l(m_l+n_l)}(t_1)$$

is uniformly convergent on compact subsets of \mathbb{R}^{k-1} as a function of t_2, \dots, t_k . Hence, the differential operator $\prod_{m=2}^k \frac{\partial^{m_m}}{\partial t_m^{m_m}}$ and the infinite sum $\sum_{m_2, \dots, m_k=0}^{\infty}$ can be interchanged. Now, by applying [71, Lemma 2.5] on derivatives of determinants of functions, we have

$$\begin{aligned} & \prod_{m=2}^k \frac{\partial^{\ell_m}}{\partial t_m^{\ell_m}} \det_{0 \leq i, j \leq N-1} \left[\sum_{m_2, \dots, m_k=0}^{\infty} \frac{t_2^{m_2} \dots t_k^{m_k}}{m_2! \dots m_k!} \phi_{i+j+\sum_{l=2}^k l m_l}(t_1) \right] \\ &= \sum_{\substack{\mu_{2,1} + \dots + \mu_{2,N} = \ell_2 \\ \vdots \\ \mu_{k,1} + \dots + \mu_{k,N} = \ell_k}} \prod_{m=2}^k \binom{\ell_m}{\mu_{m,1}, \dots, \mu_{m,N}} \det_{0 \leq i, j \leq N-1} \left[\prod_{m=2}^k \frac{\partial^{\mu_{m,j+1}}}{\partial t_m^{\mu_{m,j+1}}} \sum_{m_2, \dots, m_k=0}^{\infty} \frac{t_2^{m_2} \dots t_k^{m_k}}{m_2! \dots m_k!} \phi_{i+j+\sum_{l=2}^k l m_l}(t_1) \right]. \end{aligned}$$

By exchanging the differential operators and the infinite sum on the right-hand side of the above equation, and then evaluating both sides at $t_2 = \dots = t_k = 0$, the right-hand side becomes exactly that of (49), up to the constant factor $C_N^{(s)} (-2i)^{\sum_{m=2}^k m \ell_m}$. Therefore, the desired result follows. \square

4.3 HANKEL DETERMINANTS SHIFTED BY PARTITIONS

Following the strategy sketched earlier we define the Hankel determinants:

$$\Psi_N(t_1, \dots, t_k) = \det_{0 \leq i, j \leq N-1} \left[\sum_{m_2, \dots, m_k=0}^{\infty} \frac{t_2^{m_2} \dots t_k^{m_k}}{m_2! \dots m_k!} \phi_{i+j+\sum_{l=2}^k l m_l}(t_1) \right]. \quad (50)$$

Next, to make the presentation clearer, we fix some notation that will be used throughout the rest of this section. Firstly, let

$$\psi_\gamma(t_1, \dots, t_k) \stackrel{\text{def}}{=} \sum_{m_2, \dots, m_k=0}^{\infty} \frac{t_2^{m_2} \dots t_k^{m_k}}{m_2! \dots m_k!} \phi_{\gamma+\sum_{l=2}^k l m_l}(t_1),$$

for any $\gamma \in \mathbb{N}$. Indeed, since we are mostly concerned about the $N \rightarrow \infty$ behaviour of these determinants, it becomes more natural to consider the determinant given in (50) with the replacement $t_1 \rightarrow t_1/N, \dots, t_k \rightarrow t_k/N$. Hence, we introduce

$$\psi_\gamma(t_1, \dots, t_k) \stackrel{\text{def}}{=} \psi_\gamma\left(\frac{t_1}{N}, \dots, \frac{t_k}{N}\right),$$

and accordingly define,

$$\Psi_N(t_1, \dots, t_k) = \det_{0 \leq i, j \leq N-1} [\psi_{i+j}(t_1, \dots, t_k)].$$

Note that these functions are all well-defined due to Lemma 4.4. Next, we define shifted and weighted versions of the Hankel matrix whose determinant defines Ψ_N . For each integer partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_p > 0)$, we let

$$\mathbf{A}_{N,\lambda}(t_1, \dots, t_k) \stackrel{\text{def}}{=} [\psi_{i+j+\lambda_{N-j}}(t_1, \dots, t_k)]_{i,j=0, \dots, N-1},$$

where if $p \leq N-1$, we let $\lambda_{p+1} = \dots = \lambda_N = 0$. When $p > N$, we define $\mathbf{A}_{N,\lambda}(t_1, \dots, t_k) = 0$. Accordingly, we let

$$\Psi_{N,\lambda}(t_1, \dots, t_k) \stackrel{\text{def}}{=} \det[\mathbf{A}_{N,\lambda}].$$

Note that in the special case, $\lambda = \emptyset = (0, 0, \dots, 0)$, one has $\Psi_{N,\emptyset} = \Psi_N$. Related to these matrices, we also define their weighted version:

$$\mathbf{A}_{N,\lambda}^{(w)}(t_1, \dots, t_k) \stackrel{\text{def}}{=} [(i+j+\lambda_{N-j})\psi_{i+j+\lambda_{N-j}}(t_1, \dots, t_k)]_{i,j=0, \dots, N-1},$$

and let

$$\Psi_{N,\lambda}^{(w)}(t_1, \dots, t_k) = \det[\mathbf{A}_{N,\lambda}^{(w)}].$$

Finally, we define the main objects that will allow us to obtain recurrences for derivatives of $\Psi_{N,\lambda}$, which will become very crucial for our induction argument.

Definition 4.5. *Let $h \in \mathbb{N} \cup \{0\}$. For each integer partition $\lambda = (\lambda_1, \dots, \lambda_p)$ with $p \leq N$, define its translated version:*

$$\mathcal{S}_h \lambda = (\lambda_1 + h, \dots, \lambda_p + h, h, \dots, h).$$

Hence, we define the functions $\Psi_{N,\lambda,h}$ by,

$$\Psi_{N,\lambda,h}(t_1, t_2, \dots, t_k) \stackrel{\text{def}}{=} \text{Tr} [\text{adj}(\mathbf{A}_{N,\lambda}) \mathbf{A}_{N,\mathcal{S}_h \lambda}],$$

and $\Psi_{N,\lambda,h}^{(w)}$ by,

$$\Psi_{N,\lambda,h}^{(w)}(t_1, t_2, \dots, t_k) \stackrel{\text{def}}{=} \text{Tr} [\text{adj}(\mathbf{A}_{N,\lambda}) \mathbf{A}_{N,\mathcal{S}_h \lambda}^{(w)}],$$

where adj denotes the adjugate of a matrix.

It turns out that in most of our computations we will have expressions which involve a certain distinguished partition. These partitions are defined as follows:

Definition 4.6. *Let $1 \leq q \leq n$. Define the partition $\lambda_{n,q}$ of n to be the partition $(n-q+1, 1, \dots, 1)$ where the number of 1's that appear at the end is $q-1$.*

In what follows, we will first show certain recursive and differential relations satisfied by the functions ψ_γ , and then leverage these to obtain relations between derivatives of Ψ_N and its shifted versions defined above.

Lemma 4.7. *For each $\gamma \geq 0$, we have that*

$$\psi_{\gamma+1} = \frac{\psi_\gamma}{2} - \frac{N}{2} \frac{\partial \psi_\gamma}{\partial t_1}, \quad (51)$$

and also

$$\begin{aligned} \psi_{\gamma+2} &= \frac{N(N+s-1-\gamma)}{2(t_1+t_2)} \psi_\gamma + \left(\frac{N(2-2N-2s+\gamma)}{2(t_1+t_2)} + \frac{t_1}{t_1+t_2} \right) \psi_{\gamma+1} \\ &+ \frac{1}{2(t_1+t_2)} \sum_{q=3}^k ((q-1)t_{q-1} - qt_q) \psi_{\gamma+q} + \frac{kt_k}{2(t_1+t_2)} \psi_{\gamma+1+k}. \end{aligned} \quad (52)$$

Proof. This follows from the properties of the confluent hypergeometric functions

$$(b-a-1)U(a, b-1; t) - (b-1+t)U(a, b; t) + tU(a, b+1; t) = 0, \quad (53)$$

$$\frac{dU(a, b; t)}{dt} = U(a, b; t) - U(a, b+1; t), \quad (54)$$

which can be found in [58, p.257 formula (5) and p.258 formula (10)]. Indeed, the first claim is immediate from the second identity above, whereas using the first one, we obtain:

$$\begin{aligned} \psi_{\gamma+2}(t_1, \dots, t_k) &= \frac{N(N+s-1-\gamma)}{2t_1} \psi_\gamma(t_1, \dots, t_k) - \frac{1}{2t_1} \sum_{q=2}^k qt_q \psi_{\gamma+q}(t_1, \dots, t_k) \\ &+ \left(\frac{N(2-2N-2s+\gamma)}{2t_1} + 1 \right) \psi_{\gamma+1}(t_1, \dots, t_k) + \frac{1}{2t_1} \sum_{q=2}^k qt_q \psi_{\gamma+1+q}(t_1, \dots, t_k). \end{aligned}$$

Now, note that the only term $\psi_{\gamma+2}$ appearing on the right-hand side above is the $q=2$ term of the first sum. Moving that to the left-hand side we get the desired result. \square

Lemma 4.8. *Let $k \geq 2$ and $q \geq 2$. Let λ be a partition. Then we have:*

$$\begin{aligned} \Psi_{N,\lambda,q}(t_1, \dots, t_k) &= \frac{N(N+s-1)}{2(t_1+t_2)} \Psi_{N,\lambda,q-2}(t_1, \dots, t_k) - \frac{N}{2(t_1+t_2)} \Psi_{N,\lambda,q-2}^{(w)}(t_1, \dots, t_k) \\ &+ \frac{N(1-2N-2s)}{2(t_1+t_2)} \Psi_{N,\lambda,q-1}(t_1, \dots, t_k) + \frac{N}{2(t_1+t_2)} \Psi_{N,\lambda,q-1}^{(w)}(t_1, \dots, t_k) \\ &+ \frac{t_1}{t_1+t_2} \Psi_{N,\lambda,q-1}(t_1, \dots, t_k) + \frac{kt_k}{2(t_1+t_2)} \Psi_{N,\lambda,q+k-1}(t_1, \dots, t_k) \\ &+ \frac{1}{2(t_1+t_2)} \sum_{l=3}^k ((l-1)t_{l-1} - lt_l) \Psi_{N,\lambda,q+l-2}(t_1, \dots, t_k). \end{aligned} \quad (55)$$

Proof. The claim follows by applying the second recursive relation in Lemma 4.7 to $\Psi_{N,\lambda,q}$ and the linearity of the trace. \square

Lemma 4.9. For each $q = 2, \dots, k$ we have that

$$\frac{\partial \Psi_{N,\lambda}}{\partial t_q} = \frac{1}{N} \Psi_{N,\lambda,q}$$

and also that

$$\Psi_{N,\lambda,1} = -\frac{N}{2} \frac{\partial \Psi_{N,\lambda}}{\partial t_1} + \frac{N}{2} \Psi_{N,\lambda}. \quad (56)$$

Proof. The first claim follows by noting that

$$\frac{\partial \psi_\gamma}{\partial t_q} = \frac{1}{N} \psi_{\gamma+q}$$

whereas the second claim follows from (51). \square

Next, we give recursive formulae for $\Psi_{N,\lambda}$. To do this in a compact way, we first need some notation.

Definition 4.10. For integers $r \geq 0, b \geq 1$ and p with $1 \leq p \leq b$, we define

$$\Psi_N^{(r)}[b; p] \stackrel{\text{def}}{=} \left(\Psi_{N,\lambda_{b,p}}, \Psi_{N,\lambda_{b,p+1}}, \dots, \Psi_{N,\lambda_{b,b}}, 0, \dots, 0 \right)^\top,$$

where r specifies the number of 0's that appear.

Next, we give recursive formulae for the vectors defined above. These recursions will represent certain vectors as linear combinations of other allied vectors after they are multiplied by suitable matrices. These matrices do have explicit expressions, which we give in the Appendix.

Proposition 4.11. Let N, k, l be integers with $N \geq 1, k \geq 2, l \geq 3$. Then, there are matrices $\mathbf{B}^{(l)}$ and $\mathbf{Q}_m^{(l)}$ with $m = 0, 1, \dots, k+1$, all of which, except $\mathbf{Q}_2^{(l)}$, have all entries independent of N , so that:

$$\begin{aligned} \Psi_N^{(0)}[l; 1] &= \frac{kt_k(-1)^{k+1} \mathbf{B}^{(l)}}{2(t_1 + t_2)} \left(-\frac{N}{2} \frac{\partial}{\partial t_1} + \frac{N}{2} \right) \Psi_N^{(1)}[l + k - 2; k] \\ &+ \mathbf{B}^{(l)} \left(\frac{N}{2} \frac{\partial}{\partial t_1} - \frac{N}{2} + \frac{N}{2(t_1 + t_2)} \left(\sum_{m=3}^k \text{diff}_{m-1,m} \frac{\partial}{\partial t_{m-1}} + kt_k \frac{\partial}{\partial t_k} \right) \right) \Psi_N^{(1)}[l - 1; 1] \\ &+ \frac{1}{2(t_1 + t_2)} \sum_{m=1}^{k-2} \text{diff}_{m+1,m+2} (-1)^m \mathbf{Q}_{m+2}^{(l)} \Psi_N^{(0)}[l + m; 1] \\ &+ \left(\frac{t_1}{t_1 + t_2} \mathbf{Q}_1^{(l)} + \frac{N}{2(t_1 + t_2)} \mathbf{Q}_0^{(l)} \right) \Psi_N^{(0)}[l - 1; 1] + \frac{N}{2(t_1 + t_2)} \mathbf{Q}_2^{(l)} \Psi_N^{(0)}[l - 2; 1] \\ &- \frac{\mathbf{B}^{(l)}}{2(t_1 + t_2)} \sum_{m=0}^{k-3} \text{diff}_{m+2,m+3} (-1)^m \left(-\frac{N}{2} \frac{\partial}{\partial t_1} + \frac{N}{2} \right) \Psi_N^{(1)}[l + m; m + 2] \\ &+ \frac{1}{2(t_1 + t_2)} (-1)^{k-1} kt_k \mathbf{Q}_{k+1}^{(l)} \Psi_N^{(0)}[l + k - 1; 1] \end{aligned}$$

$$\begin{aligned}
 & + \frac{\mathbf{B}^{(l)} N k t_k}{2(t_1 + t_2)} \sum_{h=2}^{k-1} (-1)^{k+h} \frac{\partial}{\partial t_h} \Psi_N^{(1)}[l + k - 1 - h; k + 1 - h] \\
 & + \frac{\mathbf{B}^{(l)} N}{2(t_1 + t_2)} \sum_{m=4}^k (-1)^{m-1} \text{diff}_{m-1,m} \sum_{h=2}^{m-2} (-1)^h \frac{\partial}{\partial t_h} \Psi_N^{(1)}[l + m - 2 - h; m - h],
 \end{aligned}$$

where we denote

$$\text{diff}_{p,q} = \text{diff}_{p,q}(t_1, \dots, t_k) \stackrel{\text{def}}{=} p t_p - q t_q.$$

Proof. We start by noting that after a straightforward but lengthy computation, where we use Lemma 4.8 and [72, Theorem 17], we obtain:

$$\begin{aligned}
 \Psi_N^{(0)}[l; 1] = & \mathbf{B}^{(l)} \left(-\mathcal{R}_0 + \frac{N(N+s-1)}{2(t_1+t_2)} \mathcal{R}_2 + \frac{(1-2N-2s)N+2t_1}{2(t_1+t_2)} \mathcal{R}_1 \right. \\
 & \left. - \frac{N}{2(t_1+t_2)} \mathcal{R}_2^{(w)} + \frac{N}{2(t_1+t_2)} \mathcal{R}_1^{(w)} + \frac{1}{2(t_1+t_2)} \sum_{m=3}^k \text{diff}_{m-1,m} \mathcal{R}_m + \frac{k t_k}{2(t_1+t_2)} \mathcal{R}_{k+1} \right)
 \end{aligned}$$

where $\mathcal{R}_0 = (\Psi_{N,\lambda_{-1,1},1}, \dots, \Psi_{N,\lambda_{-l-1,l},1}, 0)^T$ with a zero appended as the last entry and for $m \geq 1$, $\mathcal{R}_m \in \mathbb{R}^l$ are given by:

$$(\mathcal{R}_m)_j \stackrel{\text{def}}{=} \begin{cases} 0, & j = 1, \\ \sum_{h=2}^j (-1)^h \Psi_{N,\lambda_{-h,j+1-h},h+\alpha(m)-2} & j = 2, \dots, l-1, \\ \Psi_{N,0,l+\alpha(m)-2} & j = l, \end{cases}$$

with $\alpha(2) = 0$, and $\alpha(m) = m$ for all other m . We also denote by $\mathcal{R}_1^{(w)}$, and $\mathcal{R}_2^{(w)}$ the vectors in \mathbb{R}^l defined in the same way as $\mathcal{R}_1, \mathcal{R}_2$ above with Ψ replaced by its weighted version $\Psi^{(w)}$. Next, for $m = 3, \dots, k+1$, we decompose $\mathcal{R}_m = \mathcal{R}'_m + \mathcal{R}^*_m \in \mathbb{R}^l$ where these latter vectors are given as:

$$(\mathcal{R}'_m)_j \stackrel{\text{def}}{=} \begin{cases} \sum_{h=1}^{j+m-2} (-1)^{h+m-2} \Psi_{N,\lambda_{+m-2-h,j+m-1-h},h} & j = 1, \dots, l-1, \\ \Psi_{N,0,l+m-2} & j = l, \end{cases}$$

$$(\mathcal{R}^*_m)_j \stackrel{\text{def}}{=} \begin{cases} \sum_{h=1}^{m-1} (-1)^{h+m-1} \Psi_{N,\lambda_{+m-2-h,j+m-1-h},h} & j = 1, \dots, l-1, \\ 0 & j = l. \end{cases}$$

Then, we apply (56) to \mathcal{R}_0 . For $m = 3, \dots, k+1$, we apply Lemma 4.9 to \mathcal{R}^*_m , where we set $t_q = t_h$ for the h -th summand with $h \geq 2$, and apply (56) to the first summand in the entries of \mathcal{R}^*_m . For the remaining terms, by virtue of [72, Theorem 17, Proposition 20], for each $j = 1, \dots, l$, the entries $(\mathcal{R}_m)_j$, $(\mathcal{R}_m^{(w)})_j$ for $m = 1, 2$, and $(\mathcal{R}'_m)_j$ for $m = 3, \dots, k+1$ are linear combinations of $\Psi_{N,\lambda_{l+\alpha(m)-2,q}}$, $q = 1, \dots, l + \alpha(m) - 2$, with explicitly computable

linear coefficients. This implies the identities:

$$\begin{aligned}\mathbf{B}^{(l)}\mathcal{R}_1 &= \mathbf{Q}_1^{(l)}\Psi_N^{(0)}[l-1;1], & \mathbf{B}^{(l)}\mathcal{R}_1^{(w)} &= \widetilde{\mathbf{Q}}_1^{(l)}\Psi_N^{(0)}[l-1;1], \\ \mathbf{B}^{(l)}\mathcal{R}_2 &= \widetilde{\mathbf{Q}}_2^{(l)}\Psi_N^{(0)}[l-2;1], & \mathbf{B}^{(l)}\mathcal{R}_2^{(w)} &= \widetilde{\mathbf{Q}}_2^{(l)}\Psi_N^{(0)}[l-2;1], \\ \mathbf{B}^{(l)}\mathcal{R}'_m &= \mathbf{Q}_m^{(l)}\Psi_N^{(0)}[l+m-2;1].\end{aligned}$$

All matrices involved above are given in the Appendix. Hence, defining

$$\mathbf{Q}_0^{(l)} \stackrel{\text{def}}{=} -2N\mathbf{Q}_1^{(l)} + \widetilde{\mathbf{Q}}_1^{(l)} + (1-2s)\mathbf{Q}_1^{(l)}, \quad \text{and} \quad \mathbf{Q}_2^{(l)} \stackrel{\text{def}}{=} (N+s-1)\widetilde{\mathbf{Q}}_2^{(l)} - \widetilde{\mathbf{Q}}_2^{(l)},$$

we obtain the desired result. We note that even though both $-2N\mathbf{Q}_1^{(l)}$ and $\widetilde{\mathbf{Q}}_1^{(l)}$ contain terms linear in N , $\mathbf{Q}_0^{(l)}$ is independent of N due to cancellations. Similarly, although both $\widetilde{\mathbf{Q}}_2^{(l)}$ and $(N+s-1)\widetilde{\mathbf{Q}}_2^{(l)}$ contain terms of order N^2 , these exactly cancel out so that entries of $\mathbf{Q}_2^{(l)}$ are $O(N)$. \square

The cancellations that take place at the end of the above proof, and the consequent bounds on the entries of the matrices therein will be very crucial while obtaining the desired order of N for the polynomials appearing in Theorem 1.9.

In order to be able to use the recursion above successively, we will also require suitable initial conditions.

Lemma 4.12. *For each $N \geq 1$, we have that*

$$\begin{aligned}\Psi_{N,\lambda_{2,1}} &= \frac{N^2}{8} \frac{\partial^2 \Psi_N}{\partial t_1^2} - \frac{N^2}{4} \frac{\partial \Psi_N}{\partial t_1} + \frac{N^2}{8} \Psi_N + \frac{N}{2} \frac{\partial \Psi_N}{\partial t_2}, \\ \Psi_{N,\lambda_{2,2}} &= \frac{N^2}{8} \frac{\partial^2 \Psi_N}{\partial t_1^2} - \frac{N^2}{4} \frac{\partial \Psi_N}{\partial t_1} + \frac{N^2}{8} \Psi_N - \frac{N}{2} \frac{\partial \Psi_N}{\partial t_2}.\end{aligned}$$

Proof. By the definition of $\Psi_{N,\lambda,h}$, we have

$$\Psi_{N,\lambda_{1,1,1}} = \Psi_{N,\lambda_{2,1}} + \Psi_{N,\lambda_{2,2}}, \quad \Psi_{N,\emptyset,2} = \Psi_{N,\lambda_{2,1}} - \Psi_{N,\lambda_{2,2}}.$$

Noting that $\Psi_{N,\emptyset,1} = \Psi_{N,\lambda_{1,1}}$, the result then follows by virtue of Lemma 4.9. \square

We then arrive at the following proposition.

Proposition 4.13. *Let $l \geq 3$ and $1 \leq q \leq l$. Let $k \geq 2$ and i_2, n_3, \dots, n_k be non-negative integers. Then define,*

$$\mathcal{L}_{N,l,q}^{(i_2, n_3, \dots, n_k)}(t_1) \stackrel{\text{def}}{=} \frac{\partial^{i_2}}{\partial t_2^{i_2}} \frac{\partial^{n_3}}{\partial t_3^{n_3}} \cdots \frac{\partial^{n_k}}{\partial t_k^{n_k}} \Psi_{N,\lambda_{l,q}} \Big|_{t_2=\dots=t_k=0}, \quad (57)$$

$$\mathcal{L}_N^{(i_2, n_3, \dots, n_k)}(t_1) \stackrel{\text{def}}{=} \frac{\partial^{i_2}}{\partial t_2^{i_2}} \frac{\partial^{n_3}}{\partial t_3^{n_3}} \cdots \frac{\partial^{n_k}}{\partial t_k^{n_k}} \Psi_N \Big|_{t_2=\dots=t_k=0}. \quad (58)$$

Let $m \geq 0$ be an integer. Denote

$$\mathbf{L}_l^{(i_2, n_3, \dots, n_k)} = \left(\mathcal{L}_{N,l,1}^{(i_2, n_3, \dots, n_k)}(t_1), \dots, \mathcal{L}_{N,l,l}^{(i_2, n_3, \dots, n_k)}(t_1) \right)^T$$

and

$$\hat{\mathbf{L}}_{l,m}^{(i_2, n_3, \dots, n_k)} = \left(\mathcal{L}_{N, l+m, m+2}^{(i_2, n_3, \dots, n_k)}(t_1), \dots, \mathcal{L}_{N, l+m, l+m}^{(i_2, n_3, \dots, n_k)}(t_1), 0 \right)^T.$$

Then, we have that

$$\mathbf{L}_l^{(i_2, n_3, \dots, n_k)} = \mathbf{B}^{(l)} \left(\frac{N}{2} \frac{d}{dt_1} - \frac{N}{2} \right) \begin{pmatrix} \mathbf{L}_{l-1}^{(i_2, n_3, \dots, n_k)} \\ 0 \end{pmatrix} + \sum_{n_2=0}^{i_2} \frac{(-1)^{i_2-n_2} \binom{i_2}{i_2-n_2} (i_2-n_2)!}{2t_1^{i_2-n_2+1}} \mathbf{V}_l^{(n_2, n_3, \dots, n_k)}. \quad (59)$$

Here the notation $\begin{pmatrix} \mathbf{L}_{l-1}^{(i_2, n_3, \dots, n_k)} \\ 0 \end{pmatrix}$ denotes a vector with 0 appended in the last entry and $\mathbf{V}_l^{(n_2, n_3, \dots, n_k)}$ is given by the complicated explicit expression

$$\begin{aligned} \mathbf{V}_l^{(n_2, n_3, \dots, n_k)} &= \mathbf{B}^{(l)} N \left(\sum_{m=3}^k (m-1) n_{m-1} \begin{pmatrix} \mathbf{L}_{l-1}^{(n_2, n_3, \dots, n_k)} \\ 0 \end{pmatrix} - \sum_{m=3}^k m n_m \begin{pmatrix} \mathbf{L}_{l-1}^{(n_2, \dots, n_{m-1}+1, n_m-1, \dots, n_k)} \\ 0 \end{pmatrix} \right. \\ &+ k n_k \begin{pmatrix} \mathbf{L}_{l-1}^{(n_2, n_3, \dots, n_k)} \\ 0 \end{pmatrix} \left. \right) + (2t_1 \mathbf{Q}_1^{(l)} + N \mathbf{Q}_0^{(l)}) \mathbf{L}_{l-1}^{(n_2, n_3, \dots, n_k)} + N \mathbf{Q}_2^{(l)} \mathbf{L}_{l-2}^{(n_2, n_3, \dots, n_k)} \\ &+ \sum_{m=1}^{k-2} (m+1) n_{m+1} (-1)^m \mathbf{Q}_{m+2}^{(l)} \mathbf{L}_{l+m}^{(n_2, \dots, n_{m+1}-1, \dots, n_k)} + (-1)^{k-1} k n_k \mathbf{Q}_{k+1}^{(l)} \mathbf{L}_{l+k-1}^{(n_2, n_3, \dots, n_k-1)} \\ &- \sum_{m=1}^{k-2} (m+2) n_{m+2} (-1)^m \mathbf{Q}_{m+2}^{(l)} \mathbf{L}_{l+m}^{(n_2, \dots, n_{m+2}-1, \dots, n_k)} - \mathbf{B}^{(l)} \left(-\frac{N}{2} \frac{d}{dt_1} + \frac{N}{2} \right) \left\{ \sum_{m=0}^{k-3} \left[(m+2) n_{m+2} (-1)^m \right. \right. \\ &\left. \hat{\mathbf{L}}_{l,m}^{(n_2, \dots, n_{m+2}-1, \dots, n_k)} - (m+3) n_{m+3} (-1)^m \hat{\mathbf{L}}_{l,m}^{(n_2, \dots, n_{m+3}-1, \dots, n_k)} \right] + (-1)^k k n_k \hat{\mathbf{L}}_{l,k-2}^{(n_2, n_3, \dots, n_k-1)} \left. \right\} \\ &+ \mathbf{B}^{(l)} N \sum_{m=4}^k \left[(-1)^{m-1} (m-1) n_{m-1} \sum_{h=2}^{m-2} (-1)^h \hat{\mathbf{L}}_{l, m-2-h}^{(n_2, \dots, n_h+1, \dots, n_{m-1}-1, \dots, n_k)} \right. \\ &\left. - (-1)^{m-1} m n_m \sum_{h=2}^{m-2} (-1)^h \hat{\mathbf{L}}_{l, m-2-h}^{(n_2, \dots, n_h+1, \dots, n_m-1, \dots, n_k)} \right] + \mathbf{B}^{(l)} N k n_k \sum_{h=2}^{k-1} (-1)^{k+h} \hat{\mathbf{L}}_{l, k-1-h}^{(n_2, \dots, n_h+1, \dots, n_k-1)}. \end{aligned}$$

Furthermore, the following initial conditions are satisfied, where we drop dependence on t_1 in the notation,

$$\begin{aligned} \mathcal{L}_{N,1,1}^{(n_2, \dots, n_k)} &= -\frac{N}{2} \frac{d \mathcal{L}_N^{(n_2, \dots, n_k)}}{dt_1} + \frac{N \mathcal{L}_N^{(n_2, \dots, n_k)}}{2}, \\ \mathcal{L}_{N,2,1}^{(n_2, \dots, n_k)} &= \frac{N^2}{8} \frac{d^2 \mathcal{L}_N^{(n_2, \dots, n_k)}}{dt_1^2} - \frac{N^2}{4} \frac{d \mathcal{L}_N^{(n_2, \dots, n_k)}}{dt_1} + \frac{N^2 \mathcal{L}_N^{(n_2, \dots, n_k)}}{8} + \frac{N \mathcal{L}_N^{(n_2+1, \dots, n_k)}}{2}, \\ \mathcal{L}_{N,2,2}^{(n_2, \dots, n_k)} &= \frac{N^2}{8} \frac{d^2 \mathcal{L}_N^{(n_2, \dots, n_k)}}{dt_1^2} - \frac{N^2}{4} \frac{d \mathcal{L}_N^{(n_2, \dots, n_k)}}{dt_1} + \frac{N^2 \mathcal{L}_N^{(n_2, \dots, n_k)}}{8} - \frac{N \mathcal{L}_N^{(n_2+1, \dots, n_k)}}{2}. \end{aligned} \quad (60)$$

Proof. The result follows via a direct but long explicit computation where we differentiate both sides of the recursion in Proposition 4.11 and the initial condition (56) and Lemma 4.12 with respect to t_2, \dots, t_k and let $t_2 = \dots = t_k = 0$.

In this computation, we use the following two facts. First, when taking the i_2 -th

partial derivative with respect to t_2 of both sides of the equation in Proposition 4.11 and evaluating at $t_2 = 0$, all the terms, except for the term $\mathbf{B}^{(l)}(\frac{N}{2}\frac{\partial}{\partial t_1} - \frac{N}{2})\Psi_N^{(1)}[l-1; 1]$, contain the multiplicative factor $\frac{1}{t_1+t_2}$. Hence, applying the product rule yields the formula (59). Second, there are terms of the form $t_m F(t_1, \dots, t_k)$, $m = 2, \dots, k$, for some smooth function F of t_1, \dots, t_k . When we take the n_m -th partial derivative with respect to t_m of these terms appearing on the right-hand side of the equation in Proposition 4.11, and then evaluate at $t_m = 0$, such terms become $n_m \frac{\partial^{n_m-1}}{\partial t_m^{n_m-1}} F(t_1, \dots, t_k) \Big|_{t_m=0}$. \square

Let us make a few comments on the specific features of the recursion above that will aid the inductive process in Section 4.4. Note that for fixed l, n_2, \dots, n_k ; all vectors $\mathbf{L}_b^{(n_2, \dots, n_k)}$ appearing therein can be classified as:

- $m_j = n_j$ for $j = 2, \dots, k$, and $b < l$,
- $m_j \leq n_j$ for $j = 2, \dots, k$ with strict inequality for some j ,
- There exist $2 \leq j < i \leq k$ such that $m_j > n_j$, but $n_i > m_i$.

Our inductive assumption will be defined via the upper indices (n_2, \dots, n_k) , in a way that for any tuple (m_2, \dots, m_k) given as in the second and third bullet points above, the corresponding vectors will have the desired properties for any b . Then, we will iterate over the sub-indices l , which will be possible due to the first bullet point above. For more details, see Section 4.4.

We also note, that in the above expression for $\mathbf{V}_l^{(n_2, n_3, \dots, n_k)}$, certain terms include negative integers in their upper index. Even though these are not defined, this only happens when $n_j = 0$ for some j , and this term is pre-multiplied by n_j , so that these terms disappear automatically. Of course, one can just define such terms to be 0, and end up with the same statement, but noting that their coefficients are 0 explains how these terms never *appear* when one runs through the computation for an index for which this situation occurs.

As discussed earlier, our main goal is to obtain an invertible way to go between the t_2, \dots, t_q -derivatives of Ψ_N and $\Psi_{N,\mathcal{A}}$. To write their relation in a more compact way, we now introduce the following functions:

$$D_i(t_1, \dots, t_k) = N \left\{ (N + s - 1) \Psi_{N, \emptyset, i} - \Psi_{N, \emptyset, i}^{(w)} + (1 - 2N - 2s) \Psi_{N, \emptyset, i+1} + \Psi_{N, \emptyset, i+1}^{(w)} + 2 \frac{t_1}{N} \Psi_{N, \emptyset, i+1} \right\}.$$

where the parameter s in the above is the same parameter that is used to define $\Psi_{N,\mathcal{A},h}$ and $\Psi_{N,\mathcal{A},h}^{(w)}$. As before, we will be considering the t_2, \dots, t_k derivatives of these functions evaluated at $t_2 = \dots = t_k = 0$. Thus, we define

$$D_i^{(n_2, \dots, n_k)}(t_1) \stackrel{\text{def}}{=} \frac{\partial^{n_2}}{\partial t_2^{n_2}} \cdots \frac{\partial^{n_k}}{\partial t_k^{n_k}} D_i(t_1, \dots, t_k) \Big|_{t_2 = \dots = t_k = 0}.$$

Then, using the following identities, which can be proved directly through their definition,

$$\Psi_{N, \emptyset, h} = \sum_{j=1}^h (-1)^{j-1} \Psi_{N, \mathcal{A}_{h,j}}, \quad \Psi_{N, \emptyset, h}^{(w)} = \sum_{j=1}^h (-1)^{j-1} (2N - 2j + h) \Psi_{N, \mathcal{A}_{h,j}}, \quad (h \geq 1),$$

$$\Psi_{N,\lambda,0} = N\Psi_{N,\lambda}, \quad \Psi_{N,\lambda,0}^{(w)} = (N(N-1) + n)\Psi_{N,\lambda}, \quad (\lambda \text{ is a partition of } n),$$

we have the expressions

$$\begin{aligned} D_0^{(n_2, \dots, n_k)} &= N^2 s \mathcal{L}_N^{(n_2, \dots, n_k)} - 2sN \mathcal{L}_{N,1,1}^{(n_2, \dots, n_k)} + 2t_1 \mathcal{L}_{N,1,1}^{(n_2, \dots, n_k)} \\ &= (sN^2 - Nt_1) \frac{d\mathcal{L}_N^{(n_2, \dots, n_k)}}{dt_1} + Nt_1 \mathcal{L}_N^{(n_2, \dots, n_k)}, \end{aligned} \quad (61)$$

and, for $j \geq 1$,

$$\begin{aligned} D_j^{(n_2, \dots, n_k)} &= \sum_{i=1}^{j+1} (-1)^{i-1} \left((2-2s+j-2i)N \mathcal{L}_{N,j+1,i}^{(n_2, \dots, n_k)} + 2t_1 \mathcal{L}_{N,j+1,i}^{(n_2, \dots, n_k)} \right) \\ &\quad + \sum_{i=1}^j (-1)^{i-1} (-N+s+2i-j-1)N \mathcal{L}_{N,j,i}^{(n_2, \dots, n_k)}. \end{aligned} \quad (62)$$

Lemma 4.14. *Let $N \geq 1$. If $k = 2$, then for any $m \in \mathbb{N}$, we have*

$$N \frac{\partial \Psi_N}{\partial t_2} = \frac{D_0}{2(t_1 + t_2)} + \sum_{i=2}^m \frac{D_{i-1} t_2^{i-1}}{2(t_1 + t_2)^i} + \frac{t_2^m}{(t_1 + t_2)^m} \Psi_{N,0,m+2}, \quad (63)$$

If $k \geq 3$, then whenever $2 \leq q \leq k$, $m \in \mathbb{N}$ we have:

$$\begin{aligned} N \frac{\partial \Psi_N}{\partial t_q} &= \sum_{i=2}^{q-1} \frac{t_i}{(t_1 + t_2)^m} R_i(t_1, \dots, t_k) + \frac{D_{q-2}}{2(t_1 + t_2)} + \sum_{i=2}^m \frac{1}{(2(t_1 + t_2))^i} \sum_{j=0}^{i-1} \binom{i-1}{j} (kt_k)^j \\ &\quad \sum_{\substack{h_2 + \dots + h_k = i-1-j \\ h_2 = \dots = h_{q-1} = 0 \\ h_q \geq 0, \dots, h_k \geq 0}} \prod_{n=q}^k (nt_n)^{h_n} \left(\sum_{\substack{h'_3 \leq h_3 \\ \dots \\ h'_{k-1} \leq h_{k-1}}} a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(i,j)} D_{q + \sum_{n=1}^{k-2} n h_{n+1} + (k-2)h_k - \sum_{n=3}^{k-1} h'_n + (k-1)j-2} \right) + \frac{1}{(2(t_1 + t_2))^m} \\ &\quad \sum_{j=0}^m \binom{m}{j} (kt_k)^j \sum_{\substack{h_2 + \dots + h_k = m-j \\ h_2 = \dots = h_{q-1} = 0 \\ h_q \geq 0, \dots, h_k \geq 0}} \prod_{n=q}^k (nt_n)^{h_n} \left(\sum_{\substack{h'_3 \leq h_3 \\ \dots \\ h'_{k-1} \leq h_{k-1}}} a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(m+1,j)} \Psi_{N,0,q + \sum_{n=1}^{k-2} n h_{n+1} + (k-2)h_k - \sum_{n=3}^{k-1} h'_n + (k-1)j} \right), \end{aligned}$$

where R_2, \dots, R_{q-1} are smooth functions of t_1, \dots, t_k whose explicit forms are not relevant in what follows, and

$$a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(i,j)} = \frac{(i-1-j)! (-1)^{h'_3 + \dots + h'_{k-1} + h_k}}{h_2! h_k! \prod_{n=3}^{k-1} (h_n - h'_n)! h'_n!}. \quad (64)$$

Proof. We set $\lambda = \emptyset$ and iteratively replace $\Psi_{N,\lambda,l+q-2}$ for $l = 3, \dots, k+1$ via formula (55). Then also making use of Lemma 4.9, we obtain for $q = 2, \dots, k$ and for $m \geq 1$,

$$N \frac{\partial \Psi_N}{\partial t_q} = \Psi_{N,0,q}$$

$$\begin{aligned}
 &= \frac{D_{q-2}}{2(t_1 + t_2)} + \sum_{i=2}^m \frac{1}{(2(t_1 + t_2))^i} \sum_{j=0}^{i-1} \binom{i-1}{j} (kt_k)^j \\
 &\quad \sum_{l_1=1}^{k-2} \sum_{l_2=1}^{k-2} \cdots \sum_{l_{i-1-j}=1}^{k-2} \left(\prod_{n=1}^{i-1-j} ((l_n + 1)t_{l_n+1} - (l_n + 2)t_{l_n+2}) \right) D_{q+l_1+\cdots+l_{i-1-j}+(k-1)j-2} \\
 &\quad + \frac{1}{(2(t_1 + t_2))^m} \sum_{j=0}^m \binom{m}{j} (kt_k)^j \\
 &\quad \sum_{l_1=1}^{k-2} \sum_{l_2=1}^{k-2} \cdots \sum_{l_{m-j}=1}^{k-2} \left(\prod_{n=1}^{m-j} ((l_n + 1)t_{l_n+1} - (l_n + 2)t_{l_n+2}) \right) \Psi_{N,0,q+l_1+\cdots+l_{m-j}+(k-1)j}.
 \end{aligned}$$

Here, we use the convention that when $j = m$,

$$\sum_{l_1=1}^{k-2} \sum_{l_2=1}^{k-2} \cdots \sum_{l_{m-j}=1}^{k-2} \left(\prod_{n=1}^{m-j} ((l_n + 1)t_{l_n+1} - (l_n + 2)t_{l_n+2}) \right) \Psi_{N,0,q+l_1+\cdots+l_{m-j}+(k-1)j} = \Psi_{N,0,q+(k-1)m},$$

and when $j \leq m - 1$ and $k = 2$,

$$\sum_{l_1=1}^{k-2} \sum_{l_2=1}^{k-2} \cdots \sum_{l_{m-j}=1}^{k-2} \prod_{n=1}^{m-j} ((l_n + 1)t_{l_n+1} - (l_n + 2)t_{l_n+2}) \Psi_{N,0,q+l_1+\cdots+l_{m-j}+(k-1)j} = 0.$$

We also use a similar convention for the second summand in terms of D_b 's. Then, it is not hard to check that

$$\begin{aligned}
 &\sum_{l_1=1}^{k-2} \sum_{l_2=1}^{k-2} \cdots \sum_{l_{i-1-j}=1}^{k-2} \left(\prod_{n=1}^{i-1-j} ((l_n + 1)t_{l_n+1} - (l_n + 2)t_{l_n+2}) \right) D_{q+l_1+\cdots+l_{i-1-j}+(k-1)j-2} \\
 &= \sum_{\substack{h_2+\cdots+h_k=i-1-j \\ h_2 \geq 0, \dots, h_k \geq 0}} \prod_{n=2}^k (nt_n)^{h_n} \left(\sum_{\substack{h'_3 \leq h_3 \\ h'_{k-1} \leq h_{k-1}}} a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(i,j)} D_{q+\sum_{n=1}^{k-2} nh_{n+1}+(k-2)h_k-\sum_{n=3}^{k-1} h'_n+(k-1)j-2} \right),
 \end{aligned}$$

and also that

$$\begin{aligned}
 &\sum_{l_1=1}^{k-2} \sum_{l_2=1}^{k-2} \cdots \sum_{l_{m-j}=1}^{k-2} \left(\prod_{n=1}^{m-j} ((l_n + 1)t_{l_n+1} - (l_n + 2)t_{l_n+2}) \right) \Psi_{N,0,q+l_1+\cdots+l_{m-j}+(k-1)j} \\
 &= \sum_{\substack{h_2+\cdots+h_k=m-j \\ h_2 \geq 0, \dots, h_k \geq 0}} \prod_{n=2}^k (nt_n)^{h_n} \left(\sum_{\substack{h'_3 \leq h_3 \\ h'_{k-1} \leq h_{k-1}}} a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(m+1,j)} \Psi_{N,0,q+\sum_{n=1}^{k-2} nh_{n+1}+(k-2)h_k-\sum_{n=3}^{k-1} h'_n+(k-1)j} \right),
 \end{aligned}$$

where $a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(i,j)}$ is given as in (64). The above two equalities do not exist when $k = 2$, so we state the case $k = 2$ separately. In addition, we move all terms appearing in the above two equalities, except when $h_2 = 0, \dots, h_{q-1} = 0$, into $R_i(t_1, \dots, t_k)$ with $i = 2, \dots, q - 1$. Combining these observations, the result follows. \square

As in for the previous relations, we will take appropriate t_2, \dots, t_k -derivatives of the above result. But first, let us note that writing the t_q derivative of Ψ_N in the specific way above, as linear combinations of D_j and $\Psi_{N,0,l}$ with coefficients that are powers of t_n for $n = q, \dots, k$; allows one to have control over which of these terms appear with which upper indices after suitable derivatives are evaluated at $t_2 = \dots = t_k = 0$. In particular, when one evaluates these derivatives after choosing m suitably in the above result, only the D_j 's with upper indices having desirable features will remain. Namely, during the inductive process, the upper and lower indices on the remaining terms will help ensure the specific degree conditions on the polynomials $P_m^{(s,N)}(t_1)$ given in Theorem 1.9.

Proposition 4.15. *Let $a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(i,j)}$ be given as in (64). Then we have, for $k = 2, n_2 \geq 1$, that*

$$\mathcal{L}_N^{(n_2)}(t_1) = \frac{1}{2N} \sum_{i=0}^{n_2-1} (n_2-1)! \sum_{j=0}^{n_2-1-i} \binom{n_2-1-j}{i} \frac{(-1)^{n_2-1-i-j}}{j!} \frac{D_i^{(j)}(t_1)}{t_1^{n_2-j}}, \quad (65)$$

and for $k \geq 3, n_2 \geq 1$, that

$$\begin{aligned} \mathcal{L}_N^{(n_2, \dots, n_k)}(t_1) &= \frac{1}{2N} \sum_{i_2=0}^{n_2-1} (-1)^{n_2-1-i_2} \frac{(n_2-1)!}{i_2!} \frac{D_0^{(i_2, n_3, \dots, n_k)}(t_1)}{t_1^{n_2-i_2}} \\ &+ \sum_{i=1}^{n_2+\dots+n_k-1} \frac{1}{2^{i+1}N} \sum_{j=0}^i \binom{i}{j} \sum_{h_2+\dots+h_k=i-j} \binom{n_2-1}{h_2} \binom{n_k}{h_k+j} \binom{n_3}{h_3} \dots \binom{n_{k-1}}{h_{k-1}} (h_k+j)! \\ &k^{j+h_k} \prod_{n=2}^{k-1} n^{h_n} h_n! \sum_{i_2=0}^{n_2-1-h_2} \binom{n_2-1-h_2}{i_2} \frac{(i+n_2-1-h_2-i_2)! (-1)^{n_2-1-h_2-i_2}}{t_1^{i+n_2-h_2-i_2} i!} \\ &\sum_{\substack{h'_3 \leq h_3 \\ \dots \\ h'_{k-1} \leq h_{k-1}}} a_{h_2, h'_3, \dots, h'_{k-1}, h_k}^{(i+1, j)} D_{\sum_{n=1}^{k-2} n h_{n+1} + (k-2) h_k - \sum_{n=3}^{k-1} h'_n + (k-1) j}^{(i_2, n_3-h_3, \dots, n_{k-1}-h_{k-1}, n_k-j-h_k)}(t_1). \quad (66) \end{aligned}$$

When $n_2 = \dots = n_{q-1} = 0$ and $n_q \geq 1$ for some $3 \leq q \leq k-1$, we have:

$$\begin{aligned} \mathcal{L}_N^{(0, \dots, 0, n_q, \dots, n_k)}(t_1) &= \frac{D_{q-2}^{(0, \dots, 0, n_q-1, \dots, n_k)}(t_1)}{2N t_1} + \sum_{i=1}^{n_q+\dots+n_k-1} \frac{1}{(2t_1)^{i+1}N} \sum_{j=0}^i \binom{i}{j} \\ &\sum_{h_q+\dots+h_k=i-j} \binom{n_q-1}{h_q} \binom{n_k}{h_k+j} \binom{n_{q+1}}{h_{q+1}} \dots \binom{n_{k-1}}{h_{k-1}} k^{j+h_k} (h_k+j)! \prod_{n=q}^{k-1} n^{h_n} h_n! \\ &\sum_{\substack{h'_q \leq h_q \\ \dots \\ h'_{k-1} \leq h_{k-1}}} a_{0, \dots, 0, h'_q, \dots, h'_{k-1}, h_k}^{(i+1, j)} D_{q-2+\sum_{n=q-1}^{k-2} n h_{n+1} + (k-2) h_k - \sum_{n=q}^{k-1} h'_n + (k-1) j}^{(0, \dots, 0, n_q-1-h_q, n_{q+1}-h_{q+1}, \dots, n_{k-1}-h_{k-1}, n_k-j-h_k)}(t_1). \end{aligned}$$

When $n_2 = \dots = n_{k-1} = 0$ and $n_k \geq 1$, we have

$$\mathcal{L}_N^{(0, \dots, 0, n_k)}(t_1) = \frac{D_{k-2}^{(0, \dots, 0, n_k-1)}(t_1)}{2N t_1} + \sum_{i=1}^{n_k-1} \frac{k^i}{(2t_1)^{i+1}N} \frac{(n_k-1)!}{(n_k-1-i)!} \sum_{j=0}^i \binom{i}{j} a_{0, \dots, 0, i-j}^{(i+1, j)} D_{k-2+(k-1)i-(i-j)}^{(0, \dots, 0, n_k-1-i)}(t_1).$$

Proof. The result follows, in each case, by first using Lemma 4.14 with $m = n_q + \dots + n_k$, and q being the smallest integer so that $n_q > 0$; and then evaluating the partial derivative $\frac{\partial^{n_q-1}}{\partial t_q^{n_q-1}} \prod_{i=q+1}^k \frac{\partial^{n_i}}{\partial t_i^{n_i}}$ on both sides of the expressions for $N \frac{\partial \Psi_N}{\partial t_q}$, $q = 2, \dots, k$, in Lemma 4.14 at $t_2 = \dots = t_k = 0$.

In fact, on the right-hand side of the expression for $N \frac{\partial \Psi_N}{\partial t_q}$, the terms involving $\Psi_{N,0,l}$ are each preceded by a factor $t_q^{i_q} \dots t_k^{i_k}$ with $i_q + \dots + i_k = m = n_q + \dots + n_k$. If $i_q \leq n_q - 1$, then some $j \in [q+1, k]$ satisfies $i_j \geq n_j + 1$. Thus, $\frac{\partial^{n_q-1}}{\partial t_q^{n_q-1}} \prod_{i=q+1}^k \frac{\partial^{n_i}}{\partial t_i^{n_i}}$ annihilates such a term when evaluated at $t_q = \dots = t_k = 0$. For $i_q > n_q - 1$, the same differential operator also eliminates the term after setting $t_q = 0$. Moreover, any term involving R_i for $i = 2, \dots, q-1$ vanishes upon evaluation at $t_2 = \dots = t_{q-1} = 0$, since it is preceded by a factor of t_i . Consequently, only the D_j -terms contribute in the expression for $N \frac{\partial \Psi_N}{\partial t_q}$. Applying the above differential operator to these terms and evaluating at $t_2 = \dots = t_k = 0$ yields the desired conclusion. \square

4.4 PROOFS OF THEOREMS 1.9 & 1.11

We will use induction on k , the number of variables t_2, \dots, t_k , to prove Theorem 1.9. But before doing so, we will require some preliminary lemmas to set up the induction. To this end, let us first fix some notation to make the presentation clearer. Note that our end goal amounts to showing

$$\mathcal{L}_N^{(n_2, \dots, n_k)}(t_1) = \frac{1}{t_1^{\sum_{q=2}^k q n_q - 1}} \sum_{m=0}^{\sum_{q=2}^k (q-1)n_q} t_1^{m-1} P_m^{(s,N)}(t_1) \frac{d^m}{dt_1^m} \Psi_N(t_1, 0, \dots, 0), \quad (67)$$

for polynomials $P_m^{(s,N)}$ as described in the statement of Theorem 1.9 when $t_1 \geq 0$. In fact, this is because, by Proposition 4.2 and by the definition of $\Psi_N(t_1, \dots, t_k)$, the left-hand side and right-hand side of (67) correspond, respectively, to those of (15), up to the same constant. Note that the left hand-side of above equation is continuous at $t_1 = 0$, so the right-hand side has a limit when $t_1 = 0$. Thus, it suffices to prove (67) holds for $t_1 > 0$.

The following definition is then crucial for our inductive argument.

Definition 4.16. Let $k \geq 2$ be an integer. We define $\mathfrak{S}^{(k)} \subset (\mathbb{N} \cup \{0\})^{k-1}$ to be the set consisting of $(0, \dots, 0)$ and all (n_2, \dots, n_k) such that (67) holds for appropriate polynomials $P_m^{(s,N)}$ as described in the statement of Theorem 1.9. Similarly, we define $\mathfrak{S}_{l,j}^{(k)}$ to be the set consisting of all (n_2, \dots, n_k) such that

$$\mathcal{L}_{N,l,j}^{(n_2, n_3, \dots, n_k)}(t_1) = \frac{1}{t_1^{l + \sum_{q=2}^k q n_q - 1}} \sum_{m=0}^{l + \sum_{q=2}^k (q-1)n_q} t_1^{m-1} P_{l,j,m}^{(s,N)}(t_1) \frac{d^m}{dt_1^m} \Psi_N(t_1, 0, \dots, 0)$$

holds, where $P_{l,j,m}^{(s,N)}(t_1)$, $m \geq 1$, and $t_1^{-1} P_{l,j,0}^{(s,N)}(t_1)$ are polynomials of t_1 of degree at most $l + \sum_{q=2}^k q n_q - m$ and $l + \sum_{q=2}^k q n_q - 1$, respectively, and the coefficients of these polynomials are polynomials in N and s with degrees at most $l + \sum_{q=2}^k (q-1)n_q$ and $l + \sum_{q=2}^k (q-1)n_q - 1$ respectively. Finally,

we define the intersection of all these sets:

$$\widetilde{\mathfrak{E}}^{(k)} \stackrel{\text{def}}{=} \bigcap_{1 \leq j \leq l} \mathfrak{E}_{l,j}^{(k)} \cap \mathfrak{E}^{(k)}. \quad (68)$$

We can now begin the proof.

Lemma 4.17. *Let $k \geq 3$ be an integer. Suppose $(n_3, \dots, n_k) \in (\mathbb{N} \cup \{0\})^{k-2} \setminus (0, \dots, 0)$ is such that for any $j = 3, \dots, k$, with $n_j \geq 1$,*

$$(m_2, m_3, \dots, m_{j-1}, m_j, n_{j+1}, \dots, n_k) \in \widetilde{\mathfrak{E}}^{(k)}$$

for all $m_2, m_3, \dots, m_{j-1} \in \mathbb{N} \cup \{0\}$ and $0 \leq m_j \leq n_j - 1$. Then, we also have

$$(n_2, \dots, n_k) \in \widetilde{\mathfrak{E}}^{(k)}, \quad (69)$$

for all $n_2 \in \mathbb{N} \cup \{0\}$.

Proof. We will prove the claim (69) by inducting on n_2 . By Definition 4.16, to show that a $(k-1)$ -tuple (n_2, \dots, n_k) belongs to $\widetilde{\mathfrak{E}}^{(k)}$, one needs to verify that the degrees of N , t_1 , and s satisfy the required bounds in the definitions of $\mathcal{L}_N^{(n_2, \dots, n_k)}(t_1)$ and $\mathcal{L}_{N,l,j}^{(n_2, \dots, n_k)}(t_1)$. In the argument below we check and keep track of only the degree in N , as this determines the N -leading order of $P_m^{(s,N)}(t_1)$, which is crucial for the proof of Theorem 1.11. The other degree conditions are verified in the same way, with the minor modification that, whenever $\deg_N(D_j^{(m_2, \dots, m_j, n_{j+1}, \dots, n_k)})$ is computed, we instead examine the powers of t_1 in the denominator of $D_j^{(\cdot)}$, the highest derivative order of $\Psi_N(t_1, 0, \dots, 0)$ in its numerator, and the degrees in t_1 and s of the coefficients appearing in front of these derivatives, viewed as polynomials in t_1 and s , respectively.

We start by noting that whenever (n_3, \dots, n_k) satisfies the assumptions of the lemma, $D_j^{(m_2, \dots, m_j, n_{j+1}, \dots, n_k)}$ is a polynomial in N for all $j \geq 0$, for any such $(m_2, \dots, m_j, n_{j+1}, \dots, n_k)$ as in the statement of the lemma, by virtue of (61) and (62). In particular, whenever this is the case, we have that

$$\deg_N(D_j^{(m_2, \dots, m_j, n_{j+1}, \dots, n_k)}) \leq j + 2 + \sum_{q=2}^j (q-1)m_q + \sum_{q=j+1}^k (q-1)n_q \quad (70)$$

where $\deg_N(\cdot)$ denotes the degree of a polynomial in N . Next, we start our induction by showing that (69) holds in the base case $n_2 = 0$. Indeed, in this case, by means of Proposition 4.15, and also (70), we have that under the assumptions of the lemma,

$$(0, n_3, \dots, n_k) \in \mathfrak{E}^{(k)}. \quad (71)$$

where we note that the pre-factor N^{-1} appearing in Proposition 4.15 guarantees that the degree requirement we imposed previously is satisfied. Next, by (60), we see that

$$(0, n_3, \dots, n_k) \in \mathfrak{E}_{1,1}^{(k)}. \quad (72)$$

Hence, combining (61), (62), (71), (72), and applying (66), it is clear that

$$(1, n_3, \dots, n_k) \in \mathfrak{S}^{(k)},$$

provided that for all integers h_2, h_3, \dots, h_k, j appearing on the RHS of (66), one has

$$(0, n_3 - h_3, \dots, n_{k-1} - h_{k-1}, n_k - j - h_k) \in \mathfrak{S}_{l,i}^{(k)} \cap \mathfrak{S}_{l+1,i'}^{(k)}$$

where $1 \leq i \leq l, 1 \leq i' \leq l+1$, and

$$l = \sum_{n=2}^{k-2} nh_{n+1} + (k-2)h_k - \sum_{n=3}^{k-1} h'_n + (k-1)j$$

for integers $h'_n \leq h_n$. Then, note that in case $n_2 = 1$, we must have $h_2 = 0$, so that one cannot have $h_3 = \dots = h_{k-1} = h_k = j = 0$. Thus, by the assumption of the lemma, we have

$$(0, n_3 - h_3, \dots, n_{k-1} - h_{k-1}, n_k - j - h_k) \in \widetilde{\mathfrak{S}}^{(k)} \subseteq \mathfrak{S}_{l,j}^{(k)} \cap \mathfrak{S}_{l+1,j}^{(k)}.$$

Here, once again, we use the bound

$$\deg_N \left(\mathbb{D}_{\sum_{n=1}^{k-2} nh_{n+1} + (k-2)h_k - \sum_{n=3}^{k-1} h'_n + (k-1)j}^{(0, n_3 - h_3, \dots, n_{k-1} - h_{k-1}, n_k - j - h_k)} \right) \leq 2 + \sum_{q=3}^k (q-1)n_q.$$

for the degree in N , where, as previously, we note the pre-factor N^{-1} appearing in (66). From this, it follows, again by means of (60) that

$$(1, n_3, \dots, n_k) \in \mathfrak{S}_{1,1}^{(k)}$$

and also

$$(0, n_3, \dots, n_k) \in \mathfrak{S}_{2,1}^{(k)} \cap \mathfrak{S}_{2,2}^{(k)}.$$

Furthermore, applying the recursive relation in (59) iteratively, while using the assumption of the lemma, we see that

$$(0, n_3, \dots, n_k) \in \widetilde{\mathfrak{S}}^{(k)},$$

where we crucially use the fact that out of the matrices that appear in equation (59), only $\mathbf{Q}_2^{(l)}$ has entries that depend on N , whose entries are $O(N)$, and that it only appears as a multiplicand of $\mathbf{L}_{l-2}^{(n_2, \dots, n_k)}$. This completes the proof of the base case.

For the inductive step, assume (69) holds for all $n_2 \leq a_2 - 1$ for some $a_2 \geq 1$. Then, observe that by means of Proposition 4.15, (60) and the assumption of this lemma, we immediately have that

$$(a_2, n_3, \dots, n_k) \in \mathfrak{S}^{(k)} \cap \mathfrak{S}_{1,1}^{(k)}. \quad (73)$$

Then, we claim that applying Proposition 4.15, (60) and (73) give that,

$$(a_2 + 1, n_3, \dots, n_k) \in \mathfrak{S}^{(k)} \cap \mathfrak{S}_{1,1}^{(k)}. \quad (74)$$

Indeed, the integer tuples that appear in the upper index of \mathbb{D} in the second term on the

right-hand side of (66) can be classified as:

- (n_2, n_3, \dots, n_k) with $n_2 \leq a_2$,
- (m_2, m_3, \dots, m_k) such that there exists $j \in \{3, \dots, k\}$ with $m_j \leq n_j - 1$, $m_l = n_l$ for all $l = j + 1, \dots, k$.

But then observe that (a_2, n_3, \dots, n_k) does not appear in the upper index of \mathbb{D} in the second sum on the right-hand side of (66) since this is only possible for the terms corresponding to $h_2 = \dots = h_k = j = 0$, in which case we must have $i = 0$ in (66). So either $n_2 \leq a_2 - 1$ or the second bullet point above applies. Hence, first combining all these observations with (61), (62), and then using (60) we obtain (74). Thus, using (60) again, one can then see,

$$(a_2, n_3, \dots, n_k) \in \mathfrak{E}^{(k)} \cap \mathfrak{E}_{1,1}^{(k)} \cap \mathfrak{E}_{2,1}^{(k)} \cap \mathfrak{E}_{2,2}^{(k)}.$$

Now, using the above and the recursive relation (59) iteratively as before, while using the inductive assumption, one sees that

$$(a_2, n_3, \dots, n_k) \in \bigcap_{1 \leq m \leq l} \mathfrak{E}_{l,m'}^{(k)}$$

which completes the inductive step and hence the proof of the result. \square

Proof of Theorem 1.9. To prove the theorem, we will show that for any integer $k \geq 2$,

$$\widetilde{\mathfrak{E}}^{(k)} = (\mathbb{N} \cup \{0\})^{k-1}.$$

This will be done by inducting on k . First, we can use a similar argument to the proof of Lemma 4.17 to prove the case $k = 2$ by replacing (66) with (65) whenever it appears. Suppose we have proved the case $k - 1$ for some $k \geq 3$, i.e., $\widetilde{\mathfrak{E}}^{(k-1)} = (\mathbb{N} \cup \{0\})^{k-2}$. Since $\mathcal{L}_N^{(n_2, \dots, n_{k-1}, 0)} = \mathcal{L}_N^{(n_2, \dots, n_{k-1})}$ and $\mathcal{L}_{N,l,j}^{(n_2, \dots, n_{k-1}, 0)} = \mathcal{L}_{N,l,j}^{(n_2, \dots, n_{k-1})}$, we have that $(n_2, \dots, n_{k-1}, 0) \in \widetilde{\mathfrak{E}}^{(k)}$. Hence, for the sake of induction, suppose

$$(n_2, \dots, n_k) \in \widetilde{\mathfrak{E}}^{(k)} \tag{75}$$

for any $n_2, \dots, n_{k-1} \in \mathbb{N} \cup \{0\}$ and $n_k \leq a_k - 1$ for some $a_k \geq 1$. Note that combining this assumption with Lemma 4.17, we have that

$$(n_2, 0, \dots, 0, a_k) \in \widetilde{\mathfrak{E}}^{(k)}$$

for all $n_2 \in \mathbb{N} \cup \{0\}$. Next, observe that by applying iteratively Lemma 4.17 on the n_3 -variable, one gets,

$$(n_2, n_3, 0, \dots, 0, a_k) \in \widetilde{\mathfrak{E}}^{(k)} \tag{76}$$

for all $n_2, n_3 \in \mathbb{N} \cup \{0\}$. This, in turn, implies, by virtue of Lemma 4.17, (75) and (76), that

$$(n_2, 0, 1, 0, \dots, 0, a_k) \in \widetilde{\mathfrak{E}}^{(k)}.$$

Hence, again applying the induction and Lemma 4.17 on the n_3 -variable, we see that

$$(n_2, n_3, 1, 0, \dots, 0, a_k) \in \widetilde{\mathfrak{E}}^{(k)}$$

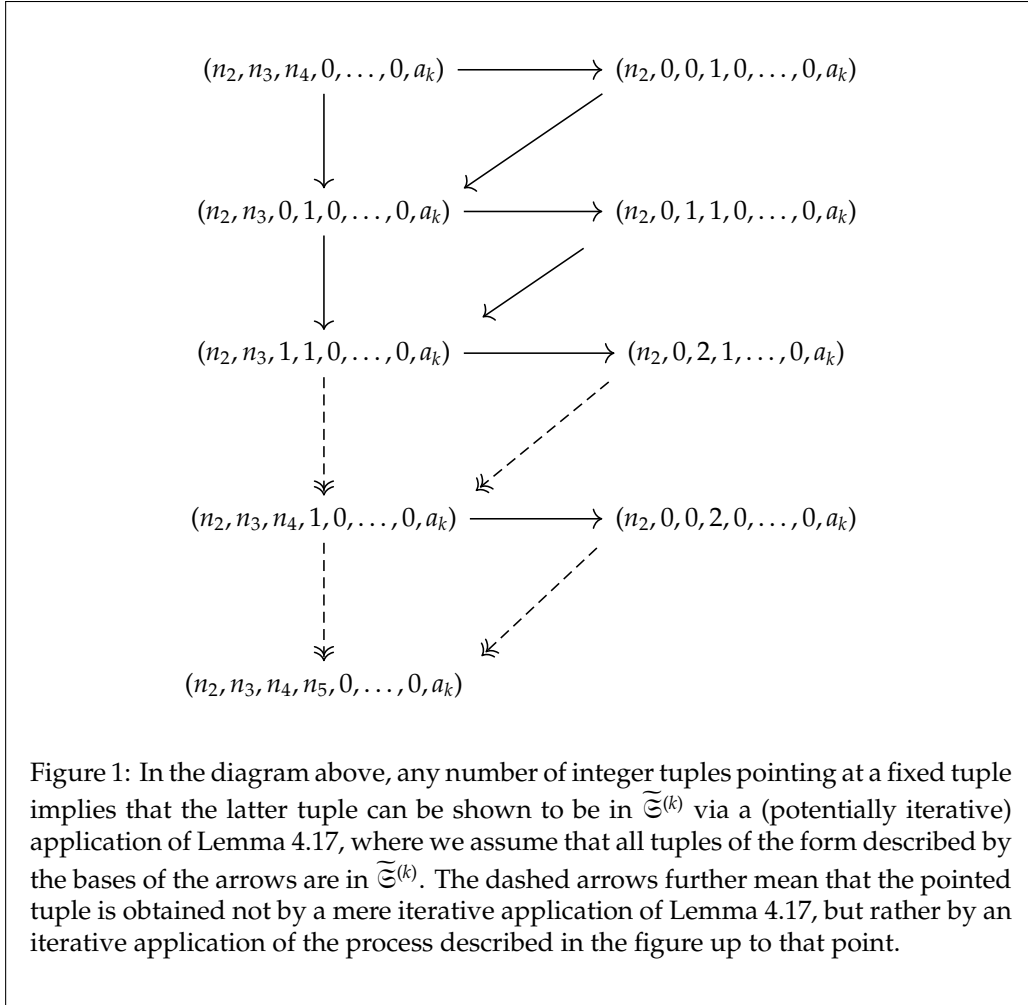
for any $n_2, n_3 \in \mathbb{N} \cup \{0\}$, which then implies

$$(n_2, 0, 2, 0, \dots, 0, a_k) \in \widetilde{\mathfrak{S}}^{(k)}.$$

Continuing this, where we increase the n_4 -variable by 1 at each step, we conclude that

$$(n_2, n_3, n_4, 0, \dots, 0, a_k) \in \widetilde{\mathfrak{S}}^{(k)}$$

for any $n_2, n_3, n_4 \in \mathbb{N} \cup \{0\}$. This process can then be continued to conclude the inductive step that was set up in (75), which is best described via the following figure:



Continuing this process, we reach at its conclusion that

$$(n_2, n_3, n_4, n_5, \dots, n_{k-1}, a_k) \in \widetilde{\mathfrak{S}}^{(k)},$$

which completes the inductive step set up in (75) and hence the proof of the theorem. \square

Proof of Theorem 1.11. First, note that by an application of Lemma 4.1, the differential

equality (15) holds for any $s \in \mathbb{R}$ with $s > 2^{-1}(\sum_{q=2}^k qn_q - 1)$, so that it now suffices to prove that for all $t_1 \geq 0$:

$$\begin{aligned} \frac{1}{N \sum_{m=2}^k m n_m} \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \prod_{m=2}^k \left(\sum_{j=1}^N (x_j^{(N)} - i)^m \right)^{n_m} \right] &\xrightarrow{N \rightarrow \infty} \mathbb{E} \left[e^{-it_1 q_1(s)} \prod_{m=2}^k (\mathbf{q}_m(s))^{n_m} \right], \\ \frac{d^m}{dt_1^m} \mathbb{E}_N^{(s)} \left[e^{-i t_1 \sum_{j=1}^N \frac{x_j^{(N)}}{N}} \right] &\xrightarrow{N \rightarrow \infty} \frac{d^m}{dt^m} \mathbb{E} \left[e^{-it_1 q_1(s)} \right]. \end{aligned} \quad (77)$$

The second limit follows immediately, for any $m < 2s + 1$, from our results in Section 3. To prove the first one, on the other hand, note that one can write the power sums in terms of a combination of smaller power sums and elementary symmetric polynomials via Newton's formula:

$$p_m(x_1, x_2, \dots, x_n) = (-1)^{m-1} m e_m(x_1, \dots, x_n) + \sum_{j=1}^{m-1} (-1)^{m-1+j} e_{m-j}(x_1, \dots, x_n) p_j(x_1, \dots, x_n).$$

Hence, using the convergence results from Section 3 for $\frac{1}{N^k} e_k(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)})$ and applying the identity above iteratively, we see that for all $m = 2, \dots, k$,

$$\frac{p_m(x_1^{(N)}, x_2^{(N)}, \dots, x_N^{(N)})}{N^m} \xrightarrow{N \rightarrow \infty} \mathbf{q}_m(s),$$

almost surely and in $L^{n_m}(\mu_\infty^{(s)})$. In particular, the relation above still holds if $p_m(x_1^{(N)}, \dots, x_N^{(N)})$ is replaced by $\sum_{j=1}^N (x_j^{(N)} - i)^m$. Thus, arguing as in the proof of Proposition 2.8 we see that (77) indeed holds, where the polynomials $\mathcal{A}_m^{(s)}(t_1)$ are given by,

$$\mathcal{A}_m^{(s)}(t_1) = \lim_{N \rightarrow \infty} \frac{P_m^{(s,N)}(t_1)}{N^{\sum_{q=2}^k (q-1)n_q}}.$$

This proves the result, except for the justification for the t_1 -degrees of the polynomials $t_1^{m-1} \mathcal{A}_m^{(s)}(t_1)$ being at most $\sum_{q=2}^k (q-1)n_q - 1$ (which is smaller than the maximal degree that is given immediately by taking the limit of Theorem 1.9). This is simply due to the fact that certain terms vanish while taking the limit. To be more precise, if one writes the expansion in N ,

$$\mathcal{L}_{N,l,q}^{(n_2, \dots, n_k)}(t_1) = N^{l + \sum_{m=2}^k (m-1)n_m} \mathcal{M}_{N,l,q}^{(n_2, \dots, n_k)}(t_1) + O(N^{l + \sum_{m=2}^k (m-1)n_m - 1}),$$

for each $1 \leq q \leq l$, (and do the same for $\mathcal{L}_N^{(n_2, \dots, n_k)}(t_1)$ and correspondingly denote the leading coefficients in powers of N by $\mathcal{M}_N^{(n_2, \dots, n_k)}(t_1)$), then by comparing the leading order coefficients in N in Propositions 4.13 and 4.15, (61) and (62), one obtains analogous relations for $\mathcal{M}_{N,l,q}^{(n_2, \dots, n_k)}$ as well as analogous expressions for $\mathcal{M}_N^{(n_2, \dots, n_k)}(t_1)$ in terms of $\mathcal{M}_{N,l,q}^{(n_2, \dots, n_k)}(t_1)$. Then, by carrying out the same inductive process as in Theorem 1.9, and noting that some terms vanish because they contribute only to sub-leading orders in N , we observe that there are no contributions from $t_1 \mathcal{L}_{N,1,1}^{(n_2, \dots, n_k)}(t_1)$ in (61), and no contribution from the terms $t_1 \mathcal{L}_{N,j+1,i}^{(n_2, \dots, n_k)}(t_1)$ in (62). This conclusion follows from the degrees in N

described by formula (68) and from the claim in the second line of the proof of Theorem 1.9. This explains why the degrees in t_1 stated in this theorem are smaller than those in Theorem 1.9. Consequently, the polynomials in t_1 appearing in $\mathcal{M}_N^{(n_2, \dots, n_k)}(t_1)$, and hence in $\mathcal{A}_m^{(s)}(t_1)$, have the desired degrees described in Theorem 1.11. This completes the proof of the theorem. \square

Remark 4.18. *We could have defined $\mathcal{M}_{N,l,q}^{(n_2, \dots, n_k)}$ earlier, and have run the inductive argument specifically for this term. However, we would also like to have a recursive algorithm to compute the joint moments at the finite- N level, for various reasons explained in the introduction, for which we need information about the lower-order terms as well.*

As mentioned in the introduction, the procedure described above not only proves the existence of a relation as in Theorem 1.11, but provides a recursive way to compute the polynomials $P_m^{(s,N)}(t_1)$ (and hence $\mathcal{A}_m^{(s)}(t_1)$).

Proof of Proposition 1.10 and Corollary 1.12. Both results are mere consequences of the relations

$$\begin{aligned} \frac{1}{N^2} \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \mathfrak{p}_2(x_1^{(N)}, \dots, x_N^{(N)}) \right] &= -\frac{2s}{t_1} \frac{d}{dt_1} \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \right] - \frac{1}{N} \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \right]. \\ \frac{1}{N^4} \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \mathfrak{p}_2^2(x_1^{(N)}, \dots, x_N^{(N)}) \right] &= \frac{4s^2 + 2}{t_1^2} \frac{d^2}{dt_1^2} \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \right] \\ &\quad + \left(\frac{4s}{Nt_1} - \frac{12s^2}{t_1^3} \right) \frac{d}{dt_1} \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \right] + \left(\frac{1}{N^2} - \frac{2}{t_1^2} - \frac{4s}{Nt_1^2} \right) \mathbb{E}_N^{(s)} \left[e^{-i \frac{t_1}{N} \sum_{j=1}^N x_j^{(N)}} \right]. \end{aligned}$$

that hold for $s \in (\frac{1}{2}, \infty)$ and $s \in (\frac{3}{2}, \infty)$ respectively. Indeed, Corollary 1.12 follows immediately after taking the $N \rightarrow \infty$ limit of these relations, whereas Proposition 1.10 follows by evaluating these relations at $t = 0$, and then using the expressions for the first few moments of the statistic $\sum_{j=1}^N x_j^{(N)}$, which were obtained in [19, Lemma 2]. \square

REFERENCES

- [1] G. AKEMANN AND G. VERNIZZI, *Characteristic polynomials of complex random matrix models*, Nuclear Physics B, 660 (2003), p. 532–556.
- [2] D. J. ALDOUS, *Representations for partially exchangeable arrays of random variables*, J. Multivariate Anal., 11 (1981), pp. 581–598.
- [3] D. J. ALDOUS, *Exchangeability and related topics*, École d’été de probabilités de Saint-Flour, XIII—1983, Lecture Notes in Math., 1117 (1985), pp. 1–198.
- [4] D. J. ALDOUS, *Exchangeability and continuum limits of discrete random structures*, in Proceedings of the International Congress of Mathematicians. Volume I, Hindustan Book Agency, New Delhi, 2010, pp. 141–153.
- [5] ———, *More uses of exchangeability: representations of complex random structures*, in Probability and mathematical genetics, vol. 378 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2010, pp. 35–63.

- [6] S. A. ALTUČ, S. BETTIN, I. PETROW, RISHIKESH, AND I. WHITEHEAD, *A recursion formula for moments of derivatives of random matrix polynomials*, *The Quarterly Journal of Mathematics*, 65 (2014), pp. 1111–1125.
- [7] G. ARFKEN, *Mathematical methods for physicists*, Academic Press, New York-London, 1966.
- [8] L.-P. ARGUIN, D. BELIUS, P. BOURGADE, M. RADZIWIŁŁ, AND K. SOUNDARARAJAN, *Maximum of the riemann zeta function on a short interval of the critical line*, *Communications on Pure and Applied Mathematics*, 72 (2019), pp. 500–535.
- [9] L.-P. ARGUIN, P. BOURGADE, AND M. RADZIWIŁŁ, *The fyodorov-hiary-keating conjecture. i*, arXiv preprint arXiv:2007.00988, (2020).
- [10] L.-P. ARGUIN, P. BOURGADE, AND M. RADZIWIŁŁ, *The fyodorov-hiary-keating conjecture. ii*, 2023.
- [11] T. ASSIOTIS, *Random entire functions from random polynomials with real zeros*, *Adv. Math.*, 410 (2022), pp. Paper No. 108701, 28.
- [12] T. ASSIOTIS, B. BEDERT, M. A. GUNES, AND A. SOOR, *On a distinguished family of random variables and painlevé equations*, *Probability and Mathematical Physics*, 2 (2021), p. 613–642.
- [13] T. ASSIOTIS, M. A. GUNES, AND A. SOOR, *Convergence and an explicit formula for the joint moments of the circular jacobi β -ensemble characteristic polynomial*, *Mathematical Physics, Analysis and Geometry*, 25 (2022), p. 15.
- [14] T. ASSIOTIS, J. P. KEATING, AND J. WARREN, *On the joint moments of the characteristic polynomials of random unitary matrices*, *International Mathematics Research Notices*, 2022 (2022), pp. 14564–14603.
- [15] T. ASSIOTIS AND J. NAJNUDEL, *The boundary of the orbital beta process*, *Moscow Mathematical Journal*, 21 (2021), p. 659–694.
- [16] T. AUSTIN, *On exchangeable random variables and the statistics of large graphs and hypergraphs*, *Probab. Surv.*, 5 (2008), pp. 80–145.
- [17] E. C. BAILEY, S. BETTIN, G. BLOWER, J. B. CONREY, A. PROKHOROV, M. O. RUBINSTEIN, AND N. C. SNAITH, *Mixed moments of characteristic polynomials of random unitary matrices*, *Journal of Mathematical Physics*, 60 (2019).
- [18] Y. BARHOUMI-ANDRÉANI, *A new approach to the characteristic polynomial of a random unitary matrix*, arXiv preprint arXiv:2011.02465, (2020).
- [19] E. BASOR, P. BLEHER, R. BUCKINGHAM, T. GRAVA, A. ITS, E. ITS, AND J. P. KEATING, *A representation of joint moments of cue characteristic polynomials in terms of painlevé functions*, *Nonlinearity*, 32 (2019), p. 4033–4078.
- [20] E. B. BOGOMOLNY AND J. P. KEATING, *Random matrix theory and the Riemann zeros. I. Three-and four-point correlations*, *Nonlinearity*, 8 (1995), pp. 1115–1131.
- [21] ———, *Gutzwiller’s trace formula and spectral statistics: beyond the diagonal approximation*, *Physical Review Letters*, 77 (1996), pp. 1472–1475.

- [22] ———, *Random matrix theory and the Riemann zeros II: n -point correlations*, *Nonlinearity*, 9 (1996), pp. 911–935.
- [23] A. BORODIN, *Determinantal point processes*, in *The Oxford handbook of random matrix theory*, Oxford Univ. Press, Oxford, 2011, pp. 231–249.
- [24] A. BORODIN AND G. OLSHANSKI, *Infinite random matrices and ergodic measures*, *Communications in Mathematical Physics*, 223 (2001), p. 87–123.
- [25] P. BOURGADE, A. NIKEGHBALI, AND A. ROUAULT, *Ewens measures on compact groups and hypergeometric kernels*, in *Séminaire de Probabilités XLIII*, vol. 2006 of *Lecture Notes in Math.*, Springer, Berlin, 2011, pp. 351–377.
- [26] H. BUI AND M. MILINOVICH, *Gaps between zeros of the Riemann zeta-function*, *The Quarterly Journal of Mathematics*, 69 (2018), pp. 403–423.
- [27] H. M. BUI AND R. R. HALL, *A note on the zeros of the derivatives of Hardy’s function $z(t)$* , *Mathematika*, 69 (2023), pp. 780–794.
- [28] S. CHATTERJEE, *A generalization of the Lindeberg principle*, *Ann. Probab.*, 34 (2006), pp. 2061–2076.
- [29] J. CONREY, A. GHOSH, AND S. GONEK, *A note on gaps between zeros of the zeta function*, *Bulletin of the London Mathematical Society*, 16 (1984), pp. 421–424.
- [30] ———, *Large gaps between zeros of the zeta-function*, *Mathematika*, 33 (1986), pp. 212–238.
- [31] J. B. CONREY, *The fourth moment of derivatives of the Riemann zeta-function*, *Quart. J. Math. Oxford Ser. (2)*, 39 (1988), pp. 21–36.
- [32] J. B. CONREY, D. W. FARMER, J. P. KEATING, M. O. RUBINSTEIN, AND N. C. SNAITH, *Autocorrelation of random matrix polynomials*, *Comm. Math. Phys.*, 237 (2003), pp. 365–395.
- [33] ———, *Integral moments of L -functions*, *Proc. London Math. Soc. (3)*, 91 (2005), pp. 33–104.
- [34] J. B. CONREY AND A. GHOSH, *On mean values of the zeta-function. II*, *Acta Arith.*, 52 (1989), pp. 367–371.
- [35] ———, *A conjecture for the sixth power moment of the Riemann zeta-function*, *Internat. Math. Res. Notices*, (1998), pp. 775–780.
- [36] J. B. CONREY AND S. M. GONEK, *High moments of the Riemann zeta-function*, *Duke Math. J.*, 107 (2001), pp. 577–604.
- [37] J. B. CONREY AND J. P. KEATING, *Moments of zeta and correlations of divisor-sums: I*, *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 373 (2015), p. 20140313.
- [38] ———, *Moments of zeta and correlations of divisor-sums: II*, in *Advances in the Theory of Numbers*, 373 (2015), pp. 75–85.
- [39] ———, *Moments of zeta and correlations of divisor-sums: III*, *Indagationes Mathematicae*, 26 (2015), pp. 736–747.

- [40] ———, *Moments of zeta and correlations of divisor-sums: IV*, *Research in Number Theory*, 2 (2016), p. 24.
- [41] ———, *Moments of zeta and correlations of divisor-sums: V*, *Proceedings of the London Mathematical Society*, 118 (2019), pp. 729–752.
- [42] J. B. CONREY, M. O. RUBINSTEIN, AND N. C. SNAITH, *Moments of the derivative of characteristic polynomials with an application to the riemann zeta function*, *Communications in Mathematical Physics*, 267 (2006), pp. 611–629.
- [43] J. B. CONREY AND N. C. SNAITH, *In support of n -correlation*, *Comm. Math. Phys.*, 330 (2014), pp. 639–653.
- [44] P.-O. DEHAYE, *Joint moments of derivatives of characteristic polynomials*, *Algebra Number Theory* 2, 31–68, (2008).
- [45] P.-O. DEHAYE, *A note on moments of derivatives of characteristic polynomials*, 22nd International Conference on Formal Power Series and Algebraic Combinatorics, 681–692, *Discrete Math. Theor. Comput. Sci. Proc. AN*, (2010).
- [46] P. DIACONIS AND D. FREEDMAN, *de Finetti’s theorem for Markov chains*, *Ann. Probab.*, 8 (1980), pp. 115–130.
- [47] ———, *Finite exchangeable sequences*, *Ann. Probab.*, 8 (1980), pp. 745–764.
- [48] M. FIEDLER, *Notes on hilbert and cauchy matrices*, *Linear algebra and its Applications*, 432 (2010), pp. 351–356.
- [49] P. FORRESTER AND N. WITTE, *Application of the τ -function theory of Painlevé equations to random matrices: PV, PIII, the LUE, JUE, and CUE*, *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 55 (2002), pp. 679–727.
- [50] P. J. FORRESTER, *Log-gases and random matrices*, vol. 34 of *London Mathematical Society Monographs Series*, Princeton University Press, Princeton, NJ, 2010.
- [51] P. J. FORRESTER AND N. S. WITTE, *Application of the τ -function theory of Painlevé equations to random matrices: PIV, PII and the GUE*, *Comm. Math. Phys.*, 219 (2001), pp. 357–398.
- [52] ———, *Application of the τ -function theory of Painlevé equations to random matrices: PV, PIII, the LUE, JUE, and CUE*, *Comm. Pure Appl. Math.*, 55 (2002), pp. 679–727.
- [53] Y. V. FYODOROV, G. A. HIARY, AND J. P. KEATING, *Freezing transition, characteristic polynomials of random matrices, and the Riemann zeta function*, *Physical review letters*, 108 (2012), p. 170601.
- [54] Y. V. FYODOROV AND J. P. KEATING, *Freezing transitions and extreme values: random matrix theory, and disordered landscapes*, *Philosophical Transactions of the Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, 372 (2014), pp. 20120503, 32.
- [55] R. HALL, *The behaviour of the Riemann zeta-function on the critical line*, *Mathematika*, 46 (1999), pp. 281–313.

- [56] ———, *Large spaces between the zeros of the Riemann zeta-function and random matrix theory*, *Journal of Number Theory*, 109 (2004), pp. 240–265.
- [57] ———, *On the stationary points of Hardy's function $z(t)$* , *Acta Arithmetica*, 111 (2004), pp. 125–140.
- [58] C. HAMMER, *Higher transcendental functions, volume I: by harry bateman (compiled by the staff of the bateman manuscript project)*. 302 pages, new york, mcgraw-hill book co., inc., 1953., 1953.
- [59] D. N. HOOVER, *Relations on probability spaces and arrays of random variables*, Preprint, (1979).
- [60] C. HUGHES, *On the characteristic polynomial of a random unitary matrix and the Riemann zeta function*, PhD Thesis, University of Bristol, (2001).
- [61] A. E. INGHAM, *Mean-Value Theorems in the Theory of the Riemann Zeta-Function*, *Proceedings of the London Mathematical Society*, 27 (1927), pp. 273–300.
- [62] H. IWANIEC, W. LUO, AND P. SARNAK, *Low lying zeros of families of L-functions*, *Institut des Hautes Études Scientifiques. Publications Mathématiques*, (2000), pp. 55–131.
- [63] K. JOHANSSON, *Random matrices and determinantal processes*, in *Mathematical statistical physics*, Elsevier B. V., Amsterdam, 2006, pp. 1–55.
- [64] O. KALLENBERG, *On the representation theorem for exchangeable arrays*, *Journal of Multivariate Analysis*, 30 (1989), pp. 137–154.
- [65] ———, *Symmetries on random arrays and set-indexed processes*, *Journal of Theoretical Probability*, 5 (1992), pp. 727–765.
- [66] ———, *Foundations of modern probability*, *Probability and its Applications (New York)*, Springer-Verlag, New York, second ed., 2002.
- [67] ———, *Probabilistic symmetries and invariance principles*, *Probability and its Applications (New York)*, Springer, New York, 2005.
- [68] N. M. KATZ AND P. SARNAK, *Random matrices, Frobenius eigenvalues, and monodromy*, vol. 45 of *American Mathematical Society Colloquium Publications*, American Mathematical Society, Providence, RI, 1999.
- [69] J. P. KEATING AND N. C. SNAITH, *Random matrix theory and L-functions at $s = 1/2$* , *Communications in Mathematical Physics*, 214 (2000), pp. 91–110.
- [70] ———, *Random matrix theory and $\zeta(1/2+it)$* , *Communications in Mathematical Physics*, 214 (2000), pp. 57–89.
- [71] J. P. KEATING AND F. WEI, *Joint moments of higher order derivatives of CUE characteristic polynomials I: asymptotic formulae*, *International Mathematics Research Notices*, 2024 (2024), pp. 9607–9632.
- [72] J. P. KEATING AND F. WEI, *Joint moments of higher order derivatives of CUE characteristic polynomials II: Structures, recursive relations, and applications*, *Nonlinearity*, 37 (2024), p. 085009.

- [73] H. KÖSTERS, *On the occurrence of the sine kernel in connection with the shifted moments of the Riemann zeta function*, *Journal of Number Theory*, 130 (2010), pp. 2596–2609.
- [74] A. LENARD, *Correlation functions and the uniqueness of the state in classical statistical mechanics*, *Communications in Mathematical Physics*, 30 (1973), pp. 35–44.
- [75] C. MALE, *Traffic distributions and independence: permutation invariant random matrices and the three notions of independence*, *Memoirs of the American Mathematical Society*, 267 (2020), pp. v+88.
- [76] H. L. MONTGOMERY, *The pair correlation of zeros of the zeta function*, vol. XXIV of *Proceedings of Symposia in Pure Mathematics*, Amer. Math. Soc., Providence, RI, 1973, pp. 181–193.
- [77] J. NAJNUDEL, *On the extreme values of the riemann zeta function on random intervals of the critical line*, *Probability Theory and Related Fields*, 172 (2018), pp. 387–452.
- [78] K. OKAMOTO, *Studies on the Painlevé equations. II. Fifth Painlevé equation P_V* , *Japanese Journal of Mathematics. New Series*, 13 (1987), pp. 47–76.
- [79] ———, *Studies on the Painlevé equations. IV. Third Painlevé equation P_{III}* , *Funkcialaj Ekvacioj. Serio Internacia*, 30 (1987), pp. 305–332.
- [80] G. OLSHANSKI, *The problem of harmonic analysis on the infinite-dimensional unitary group*, *Journal of Functional Analysis*, 205 (2003), pp. 464–524.
- [81] G. OLSHANSKI AND A. VERSHIK, *Ergodic unitary invariant measures on the space of infinite Hermitian matrices*, *American Mathematical Society Translations, Ser. 2*, 175 (1996), pp. 137–175.
- [82] D. PICKRELL, *Mackey analysis of infinite classical motion groups*, *Pacific Journal of Mathematics*, 150 (1991), pp. 139–166.
- [83] Y. QIU, *Infinite random matrices & ergodic decomposition of finite and infinite Hua-Pickrell measures*, *Advances in Mathematics*, 308 (2017), pp. 1209–1268.
- [84] M. RUBINSTEIN, *Low-lying zeros of L-functions and random matrix theory*, *Duke Mathematical Journal*, 109 (2001), pp. 147–181.
- [85] Z. RUDNICK AND P. SARNAK, *Zeros of principal L-functions and random matrix theory*, *Duke Mathematical Journal*, 81 (1996), pp. 269–322.
- [86] A. SELBERG, *The zeta-function and the Riemann hypothesis*, *Skandinaviske Matematikerkongres*, 10 (1946), pp. 187–200.
- [87] H. UMEMURA, *Second proof of the irreducibility of the first differential equation of Painlevé*, *Nagoya Mathematical Journal*, 117 (1990), pp. 125–171.
- [88] H. UMEMURA AND H. WATANABE, *Solutions of the third Painlevé equation. I*, *Nagoya Mathematical Journal*, 151 (1998), pp. 1–24.
- [89] B. WINN, *Derivative moments for characteristic polynomials from the cue*, *Communications in Mathematical Physics*, 315 (2012), pp. 531—562.
- [90] N. S. WITTE AND P. J. FORRESTER, *Gap probabilities in the finite and scaled Cauchy random matrix ensembles*, *Nonlinearity*, 13 (2000), pp. 1965–1986.

[91] D. ZWILLINGER AND A. JEFFREY, *Table of Integrals, Series, and Products*, Elsevier Science, 2007.

APPENDIX

As mentioned before, here we give explicit formulae for the matrices that were used in the proof of Theorem 1.9. Through the rest of this section, we fix $l \in \mathbb{N}$. Then, we define, for $i, j = 1, \dots, l$,

$$(\mathbf{B}^{(l)})_{ij} = \begin{cases} (-1)^{i+j-1}/j(j+1) & j \geq i; \\ -1/i & j = i-1; \\ 0 & j < i-1; \\ (-1)^{i-1}/l & j = l. \end{cases}$$

For $i = 1, \dots, l, j = 1, \dots, l-1$, we define

$$(\mathbf{Q}_0^{(l)})_{ij} = \begin{cases} (-1)^{i+j} \frac{j^{l-j-2} + 1 - 2s}{j^{(j+1)}} & \text{if } i \leq j \leq l-1; \\ \frac{j(j+2s)-l+1}{j+1} & \text{if } j = i-1; \\ 0 & \text{if } j < i-1. \end{cases}$$

$$(\mathbf{Q}_1^{(l)})_{ij} = \begin{cases} (-1)^{i+j} \frac{1}{j^{(j+1)}} & \text{if } i \leq j \leq l-1; \\ -\frac{j}{j+1} & \text{if } j = i-1; \\ 0 & \text{if } j < i-1. \end{cases}$$

$$(\widetilde{\mathbf{Q}}_1^{(l)})_{ij} = \begin{cases} \frac{(-1)^{i+j}}{j+1} \left(\frac{2N}{j} - j - 2 + l \right) & i \leq j \leq l-1; \\ -\frac{j(2N-j)+l-j-1}{j+1} & j = i-1; \\ 0 & 1 \leq j < i-2. \end{cases}$$

For $i = 1, \dots, l, j = 1, \dots, l-2$, we define

$$(\mathbf{Q}_2^{(l)})_{ij} = \begin{cases} (-1)^{i+j} \frac{(s-2)N + j(j+3) - (j+2)(l-2) + 2(s-1)}{(j+1)(j+2)} & i-1 \leq j \leq l-2; \\ \frac{(j+s)(j-N)}{j+2} & j = i-2; \\ 0 & 1 \leq j < i-2. \end{cases}$$

$$(\widehat{\mathbf{Q}}_2^{(l)})_{ij} = \begin{cases} (-1)^{i+j} \frac{N+2}{(j+1)(j+2)} & i-1 \leq j \leq l-2; \\ \frac{j-N}{j+2} & j = i-2; \\ 0 & 1 \leq j < i-2. \end{cases}$$

$$(\widetilde{\mathbf{Q}}_2^{(l)})_{ij} = \begin{cases} (-1)^{i+j} \frac{N^2 + 3N - j(j+3) + (j+2)(l-2)}{(j+1)(j+2)} & i-1 \leq j \leq l-2; \\ \frac{-N^2 + N(2j+1) - j(j+1)}{j+2} & j = i-2; \\ 0 & 1 \leq j < i-2. \end{cases}$$

For $i = 1, \dots, l, j = 1, \dots, l + m - 2, m \geq 3$, we define

$$\left(\mathbf{Q}_m^{(l)}\right)_{ij} = \begin{cases} (-1)^{m-j-1} & i = 1, j = 1, \dots, m - 1; \\ \frac{(-1)^{i+j+m}(2-m)}{(j-m+1)(j-m+2)} & j > i + m - 2; \\ \frac{i+m-2}{i} & i \neq 1, j = i + m - 2; \\ 0 & i \neq 1, j \leq i + m - 3. \end{cases}$$