

Massey products for elliptic curves of rank 1

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Abstract

For an elliptic curve over \mathbb{Q} of analytic rank 1, we use the level-two Selmer variety and secondary cohomology products to find explicit analytic defining equations for global integral points inside the set of p -adic points.

The author must begin with an apology for writing on a topic so specific, so elementary, and so well-understood as the study of elliptic curves of rank 1. Nevertheless, it is hoped that a contribution not entirely without value or novelty is to be found within the theory of *Selmer varieties* for hyperbolic curves, applied to the complement $X = E \setminus \{e\}$ of the origin inside an elliptic curve E over \mathbb{Q} with $\text{ord}_{s=1} L(E, s) = 1$. That is, let \mathcal{E} be a regular minimal \mathbb{Z} -model for E and \mathcal{X} be the complement in \mathcal{E} of the closure of e . We choose an odd prime p of good reduction for \mathcal{E} . The main goal of the present inquiry is

to find explicit analytic equations defining $\mathcal{X}(\mathbb{Z})$ inside $\mathcal{X}(\mathbb{Z}_p)$.

The approach of this paper makes use of a *rigidified Massey product* in Galois cohomology¹. That is the étale local unipotent Albanese map

$$\mathcal{X}(\mathbb{Z}_p) \xrightarrow{j_{2,loc}^{et}} H_f^1(G_p, U_2)$$

to the level-two local Selmer variety (recalled below) associates to point z a non-abelian cocycle $a(z)$, which can be broken canonically into two components $a(z) = a_1(z) + a_2(z)$, with $a_1(z)$ taking values in $U_1 \simeq T_p(E) \otimes \mathbb{Q}_p$ and $a_2(z)$ in $U^3 \setminus U^2 \simeq \mathbb{Q}_p(1)$. Denoting by $c^p : G_p \rightarrow \mathbb{Q}_p$ the logarithm of the p -adic cyclotomic character, we use a Massey triple product

$$z \mapsto (c^p, a_1(z), a_1(z)) \in H^2(G_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p$$

to construct a function on the local points of \mathcal{X} . Recall that Massey products are secondary cohomology products arising in connection with associative differential graded algebras (A, d) : If we are given classes $[\alpha], [\beta], [\gamma] \in H^1(A)$ such that

$$[\alpha][\beta] = 0 = [\beta][\gamma],$$

we can solve the equations

$$dx = \alpha\beta, \quad dy = \beta\gamma$$

for $x, y \in A^1$. The element

$$x\gamma + \alpha y \in A^2$$

satisfies

$$d(x\gamma + \alpha y) = dx\gamma - \alpha dy = 0,$$

¹The reader is invited to consult [6] for the corresponding construction in Deligne cohomology and [22] for a \mathbb{G}_m -analogue.

so that we obtain a class

$$[x\gamma + \alpha y] \in H^2(A).$$

Of course it depends on the choice of x and y , so that the product as a function of the initial triple takes values in

$$H^2(A)/[H^1(A)[\gamma] + [\alpha]H^1(A)].$$

The elements x and y constitute a *defining system* for the Massey product. It is possible, however, to obtain a precise realization taking values in $H^2(G_p, \mathbb{Q}_p(1))$ in the present situation, starting from a non-abelian cocycle $a = a_1 + a_2$ with values in U_2 . For this, we make use, on the one hand, of the component a_2 , which is not a cocycle but satisfies the equation

$$da_2 = -(1/2)(a_1 \cup a_1).$$

That is to say, the cochain $-2a_2$ is one piece of a defining system, already included in the cocycle a . On the other, the equation for the other component b of a defining system looks like

$$db = c^p \cup a_1.$$

At this point, the assumption on the elliptic curve comes into play implying that a_1 is the localization at p of a global cocycle

$$a_1^{glob}$$

with the property that the localizations $a_1^{glob, l}$ at all primes $l \neq p$ are trivial. Since c^p is also the localization of the global p -adic log cyclotomic character c , we get an equation

$$db = c \cup a_1^{glob}$$

to which we may now seek a global solution, i.e., a cochain b on a suitable global Galois group. The main point then is the deep result of Kolyvagin [18] that can be used to deduce the existence of a global solution

$$b^{glob}.$$

Having solved the equation globally, we again localize to a cochain $b^{glob, p}$ on the local Galois group at p . It is then a simple exercise to see that the Massey product

$$b^{glob, p} \cup a_1 + c^p \cup (-2a_2)$$

obtained thereby is independent of the choice of b^{glob} , giving us a well-defined function

$$\psi^p : H_f^1(G_p, U_2) \rightarrow H^2(G_p, \mathbb{Q}_p(1)) \simeq \mathbb{Q}_p.$$

This function is in fact non-zero and algebraic with respect to the structure of $H_f^1(G_p, U_2)$ as a \mathbb{Q}_p -variety. The main theorem then says:

Theorem 0.1

$$\psi^p \circ j_{2, loc}^{et}$$

vanishes on the global points $\mathcal{X}(\mathbb{Z}) \subset \mathcal{X}(\mathbb{Z}_p)$.

This result suffices to show the finiteness of integral points. However, the matter of real interest is an explicit computation of the composition $\psi^p \circ j_{2, loc}^{et}$. At this point, the exponential map from the De Rham realization will intervene, and the third section provides a flavor of the formulas one is to expect. To get explicit expressions we choose a global regular differential form α on E and a differential β of the second kind with a pole only at $e \in E$ and with the property that $[-1]^*(\beta) = -\beta$. A tangential base-point $b \in T_e E$ for the fundamental group of X then determines analytic functions

$$\log_\alpha(z) = \int_b^z \alpha, \quad \log_\beta(z) = \int_b^z \beta, \quad D_2(z) = \int_b^z \alpha\beta,$$

on $\mathcal{X}(\mathbb{Z}_p)$ via (iterated) Coleman integration.

Corollary 0.2 *Suppose there is a point y of infinite order in $\mathcal{E}(\mathbb{Z})$. Then*

$$\mathcal{E}(\mathbb{Z}) \subset \mathcal{E}(\mathbb{Z}_p)$$

is in the zero set of

$$(\log_\alpha(y))^2(D_2(z) - (1/2)\log_\alpha(z)\log_\beta(z)) - (\log_\alpha(z))^2(D_2(y) - (1/2)\log_\alpha(y)\log_\beta(y)).$$

We obtain thereby a rather harmonious constraint on the locus of global integral points, albeit in an absurdly special situation. In fact, the theorem itself implies that the function of z

$$(D_2(z) - (1/2)\log_\alpha(z)\log_\beta(z)) - \frac{(D_2(y) - (1/2)\log_\alpha(y)\log_\beta(y))}{\log_\alpha(y)^2}(\log_\alpha(z))^2$$

is independent of the choice of y . However, in its present formulation, it requires us to have in hand one integral point before commencing the search for others.

Perhaps naively, the author has believed for some time that a satisfactory description of the set of global points is possible even for general hyperbolic curves, compact or affine, by way of a non-abelian Poitou-Tate duality of sorts, coupled to an non-abelian explicit reciprocity law (cf. [11]). As yet, a plausible formulation of such a duality is unclear, and more so the prospect of applications to Diophantine problems. What is described in the following sections is a faint projection of the phenomenon whose general nature remains elusive, a projection made possible through the most stringent assumptions that are compatible still with statements that are not entirely trivial. For the tentative nature of this exposition then, even more apologies are in order.

1 Preliminary remarks

Within the strictures of the present framework, it will be sufficient to work with the Selmer variety associated to $U_2 = U^3 \setminus U$, the first non-abelian level of the \mathbb{Q}_p -pro-unipotent fundamental group

$$U := \pi_1^{u, \mathbb{Q}_p}(\bar{X}, b)$$

of

$$\bar{X} = \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\bar{\mathbb{Q}})$$

with a rational tangential base-point b pointing out of the origin of E [4]. Here, the superscript refers to the descending central series (in the sense of pro-algebraic groups)

$$U^1 = U, \quad U^{n+1} = [U, U^n]$$

of U while the subscript denotes the corresponding quotients

$$U_n = U^{n+1} \setminus U.$$

In particular, there are canonical isomorphisms

$$U_1 \simeq H_1(\bar{X}, \mathbb{Q}_p) \simeq H_1(\bar{E}, \mathbb{Q}_p) \simeq T_p E \otimes \mathbb{Q}_p$$

and an exact sequence

$$0 \rightarrow U^3 \setminus U^2 \rightarrow U_2 \rightarrow U_1 \rightarrow 0.$$

The commutator map induces an anti-symmetric pairing

$$U_1 \otimes U_1 \rightarrow U^3 \setminus U^2,$$

which therefore leads to an isomorphism

$$U^3 \backslash U^2 \simeq \wedge^2 H_1(\bar{E}, \mathbb{Q}_p) \simeq \mathbb{Q}_p(1)$$

as representations for $G = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. The logarithm map

$$\log : U \rightarrow L := \text{Lie} U$$

is an isomorphism of schemes allowing us to identify U_2 with $L_2 = L^2 \backslash L$, which, in turn, fits into an exact sequence

$$0 \rightarrow L^3 \backslash L^2 \rightarrow L_2 \rightarrow L_1 \rightarrow 0.$$

If we choose any elements A and B of L_2 lifting a basis of L_1 , then $C := [A, B]$ is a basis for $L^3 \backslash L^2$. Using the Campbell-Hausdorff formula, we can express the multiplication in U_2 , transferred over to L via the logarithm, as

$$(aA + bB + cC) * (a'A + b'B + c'C) = (a + a')A + (b + b')B + (c + c' + (1/2)(ab' - ba'))C.$$

Given such a choice of A and B , we will also denote an element of L_2 as $l = l_1 + l_2$ where l_1 is a linear combination of A and B while l_2 is multiple of C . In this notation, the group law becomes

$$(l_1 + l_2) * (l'_1 + l'_2) = l''_1 + l''_2,$$

where $l''_1 = l_1 + l'_1$ and $l''_2 = l_2 + l'_2 + (1/2)[l_1, l'_1]$. We simplify notation a bit and put $Z := L^3 \backslash L^2$. The Lie bracket

$$[\cdot, \cdot] : L_2 \otimes L_2 \rightarrow Z$$

factors to a bilinear map

$$L_1 \otimes L_1 \rightarrow Z,$$

which we denote also by a bracket.

Lemma 1.1 *Let p be an odd prime. There is a G -equivariant vector space splitting*

$$s : L_1 \hookrightarrow L_2$$

of the exact sequence

$$0 \rightarrow Z \rightarrow L_2 \xrightarrow{f} L_1 \rightarrow 0.$$

Proof. Denote by i the involution on E that send x to $-x$ for the group law. The origin is fixed and i induces the antipode map on the tangent space $T := T_e(E)$. In particular, we get an isomorphism

$$i_* : \pi_1^{u, \mathbb{Q}_p}(\bar{X}, b) \simeq \pi_1^{u, \mathbb{Q}_p}(\bar{X}, -b).$$

Consider the $\hat{\pi}_1(\bar{X}, b)$ -torsor of paths (for the pro-finite fundamental group) $\hat{\pi}_1(\bar{X}; b, -b)$ from b to $-b$. By definition, we have

$$\hat{\pi}_1(\bar{X}; b, -b) := \text{Isom}(F_b, F_{-b}).$$

Recall briefly the definition of F_v for $v \in T^0 = T \setminus \{0\}$ ([4], section 15). F_v associates to any cover $Y \rightarrow \bar{X}$, the fiber over v of the corresponding cover ('the principal part')

$$Pr(Y) \rightarrow \bar{T}^0 = T^0 \otimes \bar{\mathbb{Q}},$$

of \bar{T}^0 . Now, choose an isomorphism $z : T \simeq \mathbb{A}^1$ that takes b to 1. Then we get an isomorphism $(T^0, b) \simeq (\mathbb{G}_m, 1)$ and the pro- p universal covering of \bar{T}^0 is the pull-back of the tower

$$(\cdot)^{p^n} : \bar{\mathbb{G}}_m \rightarrow \bar{\mathbb{G}}_m.$$

But then, the inverse image of -1 , that is $z^{-1}(-1)$, in each level of the tower gives a compatible G -invariant sequence of elements and trivializes the torsor $\hat{\pi}_1(\bar{T}^0; b, -b)$. Hence, its image in $\hat{\pi}_1(\bar{X}; b, -b)$ will also be a trivialization. We then take the unipotent image of this trivialization to get an isomorphism $t : \pi_1^{u, \mathbb{Q}_p}(\bar{X}, -b) \simeq \pi_1^{u, \mathbb{Q}_p}(\bar{X}, b)$. Note that in the abelianization, we have the canonical isomorphisms

$$\pi_1^{u, \mathbb{Q}_p}(\bar{X}, -b)^{ab} \simeq H_1(\bar{E}, \mathbb{Q}_p) \simeq \pi_1^{u, \mathbb{Q}_p}(\bar{X}, b)^{ab}.$$

But then t , being given by a path, induces the identity on $H_1(\bar{E}, \mathbb{Q}_p)$. Therefore, we have constructed an isomorphism $I := t \circ i_* : U_2 \simeq U_2$ that lifts the map $x \mapsto -x$ on U_1 . This gives a corresponding Lie algebra isomorphism

$$L_2 \simeq L_2,$$

which we will denote by the same letter I . Since $Z \subset L_2$ is generated by the bracket of two basis elements from L_1 , we see that I restricts to the identity on Z .

Now we define the splitting by putting

$$s(x) := (1/2)(x' - I(x'))$$

for any lift x' of x to L_2 . Since another lift will differ from x' by an element of Z on which I acts as the identity, the formula is independent of the lift. We can then use this independence to check that the map is linear. If x' and y' are lifts of x and y , then $\lambda x' + \mu y'$ is a lift of $\lambda x + \mu y$. So

$$s(\lambda x + \mu y) = (1/2)(\lambda x' + \mu y' - I(\lambda x' + \mu y')) = \lambda(1/2)(x' - I(x')) + \mu(1/2)(y' - I(y')).$$

Similarly, if $g \in G$, then $g(x')$ will be a lift of $g(x)$, so that

$$s(g(x)) = (1/2)(g(x') - I(g(x'))) = (1/2)(g(x') - g(I(x'))) = g(s(x)).$$

□

We will use this splitting to write

$$L_2 = L_1 \oplus Z$$

as a G -representation, so that an arbitrary $l \in L_2$ can be decomposed into $l = l_1 + l_2$ as described above, except independently of a specific basis of L_1 . Using the identification via the log map, we will abuse notation a bit and write $u = u_1 + u_2$ also for an element of U_2 .

For any $\lambda \in \mathbb{Q}_p$, we then get a Lie algebra homomorphism

$$m(\lambda) : L_2 \simeq L_2$$

by defining

$$m(\lambda)l = \lambda l_1 + \lambda^2 l_2,$$

which is furthermore compatible with the G -action. Thus, $m(\lambda)$ can also be viewed as a G -homomorphism of U_2 . We note that the extra notation is used for the moment to distinguish this action of the multiplicative monoid \mathbb{Q}_p from the original scalar multiplication.

We recall the continuous group cohomology [12]

$$H^1(G, U_2) := U_2 \backslash Z^1(G, U_2).$$

The set $Z^1(G, U_2)$ of continuous 1-cocycles of G with values in U_2 consist of continuous maps

$$a : G \rightarrow U_2$$

such that

$$a(gh) = a(g)ga(h).$$

Using the identification of U_2 with $L_2 = L_1 \oplus Z$ just discussed, we will write such a map also as

$$a = a_1 + a_2$$

with a_1 taking values in L_1 and a_2 values in Z . The cocycle condition is then given by

$$\begin{aligned} a_1(gh) + a_2(gh) &= (a_1(g) + a_2(g)) * (ga_1(h) + ga_2(h)) \\ &= a_1(g) + ga_1(h) + a_2(g) + ga_2(h) + (1/2)[a_1(g), ga_1(h)]. \end{aligned}$$

In fact, given two cochains c, c' with values in L_1 , we define

$$(c \cup c')(g, h) = [c(g), gc'(h)],$$

a cochain with values in Z . The cocycle condition spelled out above then says that

- (1) a_1 is a cocycle with values in L_1 ;
- (2) a_2 is not a cocycle in general, but satisfies

$$ga_2(h) - a_2(gh) + a_2(h) = -(1/2)[a_1(g), ga_1(h)],$$

or, recognizing on the left hand side the differential of the cochain a_2 ,

$$da_2 = -(1/2)(a_1 \cup a_1).$$

This can be viewed as a Galois-theoretic *Maurer-Cartan equation* [7]. So even when we have split the Galois action at this first-nonabelian level, the non-abelian group structure imposes a twist on cohomology. The cohomology set is then defined by taking a quotient under the action of U_2 :

$$(u \cdot a)(g) = ua(g)g(u^{-1}).$$

This discussion carries over verbatim to various local Galois group $G_l = \text{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$ as l runs over the primes of \mathbb{Q} , which all act on U via the inclusion $G_l \rightarrow G$ induced by an inclusion $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_l$, and leading to the cohomology sets $H^1(G_l, U_2)$.

Now we choose p to be an odd place of good reduction for E and denote by T the set $S \cup \{p\}$, where S is a finite set of places that contains the infinite place and the places of bad reduction for E . We let $G_T = \text{Gal}(\mathbb{Q}_T/\mathbb{Q})$ be the Galois group of the maximal extension of \mathbb{Q} unramified outside T . In previous work [3, 12, 13, 14, 15, 16], we have made use of the Selmer variety

$$H_f^1(G, U_2) \subset H^1(G_T, U_2) \subset H^1(G, U_2).$$

By definition, $H_f^1(G, U_2)$ consists of cohomology classes that are unramified outside T and crystalline at p . Associating to a point $x \in X(\mathbb{Q})$ the torsor of paths

$$\pi_1^{u, \mathbb{Q}_p}(\bar{X}; b, x)$$

defines a map

$$X(\mathbb{Q}) \rightarrow H^1(G, U_2)$$

that takes $\mathcal{X}(\mathbb{Z}_S) \subset X(\mathbb{Q})$ to $H_f^1(G, U)$. In fact, let $l \neq p$, and $G_l := \text{Gal}(\bar{\mathbb{Q}}_l/\mathbb{Q}_l)$. Then on the globally integral points, the map

$$\mathcal{X}(\mathbb{Z}) \rightarrow H_f^1(G, U_2) \rightarrow H^1(G_l, U_2)$$

is trivial. This was essentially shown in [16]. What is shown there is that the image factors through the map

$$H^1(G_l/I_l, U_2^{I_l}) \rightarrow H^1(G_l, U_2),$$

where $I_l \subset G_l$ is the inertia subgroup. But for both U_1 and Z , the Frobenius element in G_l/I_l acts with strictly negative weights, and hence,

$$H^1(G_l/I_l, Z^I) = H^1(G_l/I_l, U_1^I) = 0,$$

from which we deduce that

$$H^1(G_l/I_l, U_2^{I_l}) = 0.$$

Denote by

$$H_{f,\mathbb{Z}}^1(G, U_2) \subset H_f^1(G, U_2)$$

the *fine Selmer variety*, defined to be the intersection of the kernels of the localization maps

$$H_f^1(G, U_2) \rightarrow H^1(G_l, U_2)$$

for all $l \neq p$. Then the previous paragraph says that the image of

$$\mathcal{X}(\mathbb{Z}) \rightarrow H_f^1(G, U_2)$$

lies inside

$$H_{f,\mathbb{Z}}^1(G, U_2).$$

The action of \mathbb{Q}_p discussed above induces an action on $H_{(\cdot)}^1(\cdot, U_2)$ for any of the groups under discussion, which we will denote simply as left multiplication (since here, the danger of confusion with the original scalar multiplication does not arise). At the level of cocycles,

$$\lambda a = \lambda a_1 + \lambda^2 a_2.$$

2 Construction

(i) *Global construction*

Let $\chi : G_T \rightarrow \mathbb{Z}_p^*$ be the p -adic cyclotomic character and $c := \log \chi : G_T \rightarrow \mathbb{Q}_p$, regarded as an element of $H^1(G_T, \mathbb{Q}_p)$. For any point $x \in \mathcal{X}(\mathbb{Z}_S)$, let $a(x)$ be a cocycle representing the class of $\pi_1^{u, \mathbb{Q}_p}(\bar{X}; b, x)$ in $H^1(G_T, U_2)$. Write

$$a(x) = a_1(x)_1 + a_1(x)$$

as indicated in the previous section. Then $c \cup a_1(x)$ represents a cohomology class in $H^2(G_T, L_1)$. But if we consider the localization map

$$0 \rightarrow \text{III}_T^2(L_1) \rightarrow H^2(G_T, L_1) \rightarrow \bigoplus_{v \in T} H^2(G_v, L_1),$$

we see that

$$H^2(G_v, L_1) \simeq H^0(G_v, L_1)^* = 0$$

for all v . On the other hand, the kernel $\text{III}_T^2(L_1)$ is dual to the kernel $\text{III}_T^1(L_1)$ of the H^1 -localization

$$0 \rightarrow \text{III}_T^1(L_1) \rightarrow H^1(G_T, L_1) \rightarrow \bigoplus_{v \in T} H^1(G_v, L_1),$$

which then must lie inside the \mathbb{Q}_p -Selmer group

$$H_{f,0}^1(G, L_1) := (\varprojlim_n \text{Sel}(E)[p^n]) \otimes \mathbb{Q}_p.$$

However, because of our assumption $\text{ord}_{s=1} L(E, s) = 1$, we know that the Tate-Shafarevich group of E is finite [18], and that

$$E(\mathbb{Q}) \otimes \mathbb{Q}_p \simeq H_{f,0}^1(G, L_1).$$

Hence, there is an injection

$$H_{f,0}^1(G, L_1) \hookrightarrow H^1(G_p, L_1),$$

from which we deduce that $\text{III}_T^1(L_1) = 0$. We conclude that $H^2(G_T, L_1)$ vanishes, allowing us to find a cochain $b_x : G_T \rightarrow L_1$ such that

$$db_x = c \cup a_1(x).$$

Define a two-cochain

$$\phi_x : G_T \times G_T \rightarrow Z$$

by putting

$$\phi_x = b_x \cup a_1(x) - 2c \cup a_2(x).$$

Lemma 2.1 ϕ_x is a cocycle.

Proof. Since $a(x)_1$ and c are cocycles, we have

$$d\phi_x = db_x \cup a(x)_1 + 2c \cup da(x)_2 = c \cup a(x)_1 \cup a(x)_1 - c \cup a(x)_1 \cup a(x)_1 = 0.$$

□

Our construction of ϕ_x depends on the auxiliary cochain b_x , and hence, gives a class

$$[\phi_x] \in H^2(G_T, Z) / [H^1(G_T, L_1) \cup a_1(x)].$$

Lemma 2.2 The class $[\phi_x]$ is independent of the choice of cocycle $a(x)$.

Proof. Obviously, the subspace $H^1(G_T, L_1) \cup a_1(x)$ depends only on the class of $a_1(x)$, and hence, on the class of $a(x)$. Now we examine the action of U_2 . To reduce clutter, we will temporarily suppress x from the notation for the cochains. Write $u = u_1 + u_2$. Then

$$\begin{aligned} ua(g)g(u^{-1}) &= (u_1 + u_2) * (a_1(g) + a_2(g)) * (-gu_1 - gu_2) \\ &= (u_1 + a_1(g) + u_2 + a_2(g) + (1/2)[u_1, a_1(g)]) * (-gu_1 - gu_2) \\ &= a_1(g) + u_1 - gu_1 + a_2(g) + u_2 - gu_2 + (1/2)[u_1, a_1(g)] - (1/2)[a_1(g), gu_1] - (1/2)[u_1, gu_1]. \end{aligned}$$

The L_1 -component of this expression is $a_1 - du_1$, where we view the element $u_1 \in L_1$ as a zero-cochain. Thus, the previous choice of b_x can be changed to $b_x + c \cup u_1$, since

$$d(c \cup u_1) = dc \cup u_1 - c \cup du_1 = -cdu_1.$$

The resulting two-cocycle changes to

$$\begin{aligned} &(b_x + c \cup u_1) \cup (a_1 - du_1) - 2c \cup a_2 + 2c \cup du_2 - c \cup r + c \cup s + c \cup t \\ &= \phi_x + [c \cup u_1 \cup a_1 - b_x \cup du_1 - c \cup u_1 \cup du_1 + 2c \cup du_2 - c \cup r + c \cup s + c \cup t], \end{aligned}$$

where r, s, t are the functions $G_T \rightarrow Z$ defined by

$$r(g) = [u_1, a_1(g)], \quad s(g) = [a_1(g), gu_1], \quad t(g) = [u_1, gu_1].$$

Within the discrepancy, the term $2c \cup du_2 = -2d(c \cup u)$ is clearly a co-boundary. Also, we see that $t = u_1 \cup du_1$, ridding us of two terms. We have

$$c \cup u_1 \cup a_1(g, h) = [c \cup u_1(g), ga_1(h)] = [c(g)gu_1, ga_1(h)],$$

while

$$c \cup r(g, h) = c(g)gr(h) = c(g)g[u_1, a_1(h)] = c(g)[gu_1, ga_1(h)],$$

causing two more terms to cancel each other. Finally, it remains to analyze the difference $c \cup s - b_x \cup du_1$. But

$$d(b_x \cup u_1) = db_x \cup u_1 - b_x \cup du_1,$$

so that up to a co-boundary, we can replace $b_x \cup du_1$ by $db_x \cup u_1 = c \cup a_1 \cup u_1$. We can then compute the value

$$c \cup a_1 \cup u_1(g, h) = [c \cup a_1(g, h), gh u_1] = [c(g)ga_1(h), gh u_1],$$

which is verified to be equal to

$$c \cup s(g, h) = c(g)gs(h) = c(g)g[a_1(h), hu_1] = c(g)[ga_1(h), gh u_1].$$

Therefore, the difference $c \cup s - b_x \cup du_1$ is a coboundary. \square

Given a global cohomology class or cochain s , we will denote by s^l its localization at the prime l , that is, its restriction to G_l .

Lemma 2.3 *The subspace $H^1(G_T, L_1) \cup a_1(x)$ of $H^2(G_T, Z)$ is zero.*

Proof. Because $a_1(x)$ is the class of a point and we are taking \mathbb{Q}_p -coefficients, $a_1(x)^l = 0$ for all $l \neq p$. Thus, for any $r \in H^1(G_T, L_1)$, we have $(r \cup a_1(x))^l = 0$ for all $l \neq p$. This is of course also true for the archimedean component since p is odd. Thus, the only possible component of $r \cup a_1(x)$ that survives is at p , which then must be zero since

$$\sum_v (r \cup a_1(x))_v = 0.$$

Because we also have an injection

$$0 \rightarrow H^2(G_T, \mathbb{Q}_p(1)) \hookrightarrow \bigoplus_v H^2(G_v, \mathbb{Q}_p(1)),$$

we see that $r \cup a_1(x) = 0$. \square

We actually have therefore a well-defined class

$$[\phi_x] \in H^2(G_T, Z).$$

Lemma 2.4 *Let $l \neq p$. Then*

$$[\phi_x]^l \in H^2(G_l, Z)$$

can be computed locally in the following sense: Choose any local representative $t(x)$ for the class $[a(x)]^l$, and let $s : G_l \rightarrow L_1$ be any local cochain such that $ds = c^l \cup t_1(x)$. Then

$$[\phi_x]^l = [s \cup t_1(x) - 2c^l \cup t_2(x)].$$

Proof. The class modulo

$$H^1(G_l, L_1) \cup t_1(x)$$

will clearly be independent of the choice of t and s . Since $[t_1(x)] = [a_1^l(x)]$, we have

$$H^1(G_l, L_1) \cup t_1(x) = H^1(G_l, L_1) \cup a_1(x).$$

But, as we pointed out above, $a_1^l(x)$ is the trivial class. Therefore, the class in $H^2(G_l, Z)$ is independent of the choices. In particular, local choices s and t will give the same class as the localization of the global choices a and b . \square

(ii) *Local construction*

We will now make use of a point $y \in E(\mathbb{Q})$ of infinite order. Given any class $s \in H_f^1(G_p, U_2)$, its component $s_1 \in H_f^1(G_p, U_1)$ is a \mathbb{Q}_p multiple of $a_1^p(y)$:

$$s_1 = \lambda(s)a_1^p(y),$$

for some $\lambda(s) \in \mathbb{Q}_p$. In particular, $s_1 = x_1^p$ for some cocycle $x_1 : G_T \rightarrow L_1$ such that $[x_1]^l = 0$ for all $l \neq p$. By the theorem of Kolyvagin cited earlier, the equation

$$db = c \cup (\lambda(s)x_1)$$

has a solution $b^{glob} : G_T \rightarrow L_1$. Thus, we get a class

$$\psi^p(s) := [b^{glob,p} \cup s_1 - 2c^p \cup s_2] \in H^2(G_p, Z)/[\text{loc}_p(H^1(G_T, L_1)) \cup s_1]$$

since two choices of b^{glob} will differ by an element of $H^1(G_T, L_1)$. But

$$\text{loc}_p(H^1(G_T, L_1)) \cup s_1 = \text{loc}_p(H^1(G_T, L_1)) \cup a_1^p(y) = \text{loc}_p((H^1(G_T, L_1)) \cup a_1(y)) = 0$$

by Lemma 2.3. Therefore, we have a well-defined class

$$\psi^p(s) \in H^2(G_p, Z).$$

The following lemma is straightforward from the definitions:

Lemma 2.5 *Let $x \in \mathcal{X}(\mathbb{Z}_S)$. Then*

$$\psi^p(a^p(x)) = [\phi_x]^p.$$

Now we can give the

Proof of theorem 0.1

If $x \in \mathcal{X}(\mathbb{Z})$, then we know that $[a^l(x)] = 0$ for all $l \neq p$. By lemma 2.4, this implies that $[\phi_x]^l = 0$ for all $l \neq p$, and hence,

$$\psi^p(a^p(x)) = [\phi_x]^p = 0.$$

□

Any class $s \in H_f^1(G_p, U_2)$ lies over the same point in $H_f^1(G_p, U_1)$ as $\lambda(s)a^p(y)$ ². By the exact sequence [16]

$$0 \rightarrow H_f^1(G_p, Z) \rightarrow H_f^1(G_p, U_2) \rightarrow H_f^1(G, U_1),$$

the two classes then differ by the action of an element of $H_f^1(G_p, Z)$ which we denote by

$$\lambda(s)a^p(y) - s \in H_f^1(G_p, Z).$$

Using the point y , we get the following alternative description of the function ψ :

Lemma 2.6

$$\psi^p(s) = \lambda(s)^2 \psi^p(y) + 2(c^p \cup (\lambda(s)a^p(y) - s)).$$

Proof. Let $b : G_T \rightarrow \Lambda_1$ be a solution of

$$db = c \cup a_1(y).$$

Then

$$d(\lambda(s)b) = c \cup (\lambda(s)a_1(y))$$

and

$$(\lambda(s)b)^p \cup s_1 = (\lambda(s)b)^p \cup (\lambda(s)a_1^p(y)) = \lambda(s)^2 b^p \cup a_1^p(y).$$

Therefore,

$$\begin{aligned} \psi^p(s) &= \lambda(s)^2 b^p \cup a_1^p(y) - 2c^p \cup s_2 \\ &= \lambda(s)^2 (b^p \cup a_1^p(y) - 2c^p \cup a_2^p(y)) + 2(\lambda(s)^2 c^p \cup a_2^p(y) - c^p \cup s_2) \end{aligned}$$

²Take care that this multiplication now refers to the \mathbb{Q}_p -action discussed in the previous section.

$$= \lambda(s)^2 \psi^p(y) + 2(c^p \cup (\lambda(s)a^p(y) - s)).$$

□

Fix now the isomorphism $Z \simeq \mathbb{Q}_p(1)$ induced by the Weil pairing $\langle \cdot, \cdot \rangle$, that is, that takes $[x, y]$ to $\langle x, y \rangle$, which then induces an isomorphism

$$T : H^2(G_p, Z) \simeq \mathbb{Q}_p$$

and gives us a \mathbb{Q}_p -valued function

$$T \circ \psi^p : H_f^1(G_p, U_2) \rightarrow \mathbb{Q}_p.$$

We will sometimes suppress T from the notation and simply regard ψ^p as taking values in \mathbb{Q}_p . From the definition, we see that for any $\lambda \in \mathbb{Q}_p$,

$$\psi_y^p(\lambda a^p(y)) = \lambda^2 T(\phi_y^p),$$

while for $r \in H_f^1(G_p, Z)$, we have

$$\psi_y^p(r) = -T(c^p \cup r).$$

Thus, when we take $z \in \mathbb{Z}_p^*$ with a cohomology class

$$k(z) \in H_f^1(G_p, \mathbb{Q}_p(1))$$

coming from Kummer theory that we identify with a class in $H_f^1(G_p, Z)$, then

$$\psi_y^p(z) = -\log \chi(Rec_p(z)),$$

where Rec_p is the local reciprocity map. Since $\chi(Rec_p(z)) = z$, we get

$$\psi_y^p(k(z)) = -\log z \in \mathbb{Q}_p.$$

In particular, the map is not identically zero. In fact, it is far from trivial on any fiber of

$$H_f^1(G_p, U_2) \rightarrow H_f^1(G_p, U_1).$$

With respect to the structure of $H_f^1(G_p, U_2)$ as an algebraic variety, ψ^p is in fact a non-zero algebraic function, as we see in a straightforward way by defining it for points in arbitrary \mathbb{Q}_p -algebras (as in [12]). Since there is a Coleman map

$$j_{2,loc}^{et} : \mathcal{X}(\mathbb{Z}_p) \rightarrow H_f^1(G_p, U_2)$$

with Zariski dense image for each residue disk [13], Theorem 0.1 yields the finiteness of $\mathcal{X}(\mathbb{Z})$. As mentioned in the introduction, the obvious task of importance is to compute the function

$$\psi^p \circ j_{2,loc}^{et}$$

on $\mathcal{X}(\mathbb{Z}_p)$, whose zero set is guaranteed to capture the global integral points.

3 Preliminary formulas

There is a commutative diagram [13]

$$\begin{array}{ccc} \mathcal{X}(\mathbb{Z}_p) & \xrightarrow{j_{2,loc}^{et}} & H_f^1(G, U_2) \\ & \searrow j_2^{dr/cr} & \downarrow \simeq \\ & & U_2^{DR}/F^0 \end{array}$$

bringing the De Rham fundamental group $U^{DR} = \pi_1^{DR}(X_{\mathbb{Q}_p}, b)$ and its quotient $U_2^{DR} = U^{DR}/U^3$ into our consideration. The isomorphism

$$H_f^1(G, U_2) \simeq U_2^{DR}/F^0$$

is a non-abelian analogue of the Bloch-Kato log map. There is actually a larger commutative diagram

$$\begin{array}{ccc} \mathcal{X}(\mathbb{Z}_p) & \xrightarrow{j_{loc}^{et}} & H_f^1(G, U) \\ & \searrow j^{dr/cr} & \downarrow \simeq \\ & & U^{DR}/F^0 \end{array}$$

out of which the level-two version is obtained by composing with the projection

$$U \rightarrow U_2.$$

The map j_{loc}^{et} is just a local version of the map recalled in the previous section, while $j^{dr/cr}$ associates to each point $x \in \mathcal{X}(\mathbb{Z}_p)$, the U^{DR} -torsor $\pi_1^{DR}(X_{\mathbb{Q}_p}; b, x)$, and hence, corresponds to a point $j^{dr/cr}(x) \in U^{DR}/F^0$ [13]. The point is described explicitly as follows. One chooses an element $p^H \in F^0 \pi_1^{DR}(X_{\mathbb{Q}_p}; b, x)$. On the other hand, there is a unique Frobenius invariant element $p^{cr} \in \pi_1^{DR}(X_{\mathbb{Q}_p}; b, x)$. Then $j^{dr/cr}(x)$ the coset of the element u such that $p^{cr}u = p^H$. In [13] we gave a description of a universal pro-unipotent bundle with connection on $X_{\mathbb{Q}_p}$. Let α be an invariant differential 1-form on E and let β be a differential of the second kind with a pole only at e such that $[-1]^*(\alpha) = -\alpha$ and $[-1]^*(\beta) = -\beta$. (Of course the first condition is automatic.) Let

$$R := \mathbb{Q}_p \langle\langle A, B \rangle\rangle = \varprojlim \mathbb{Q}_p \langle A, B \rangle / I^n,$$

where $\mathbb{Q}_p \langle A, B \rangle$ is the free-noncommutative \mathbb{Q}_p -algebra on the letters A, B , and $I \subset \mathbb{Q}_p \langle A, B \rangle$ is the augmentation ideal. Thus, $\mathbb{Q}_p \langle A, B \rangle$ is spanned by words w in A and B , while the elements of R are infinite formal linear combinations

$$\sum c_w w$$

in such words with coefficients $c_w \in \mathbb{Q}_p$. Using this we can construct the free $X_{\mathbb{Q}_p}$ -module $\mathcal{R} = X_{\mathbb{Q}_p} \otimes R$ together with the connection

$$\nabla f = df - (A\alpha + B\beta)f,$$

for an element $f \in \mathcal{R}$. If we choose the element $1 \in \mathcal{R}_b = R$ as the initial condition, then the element $p^{cr}(1) \in \mathcal{R}_x$ corresponding to it is given by

$$G(x) = \sum_w \int_b^x a_w w,$$

where a_w is the symbol

$$\alpha^{n_1} \beta^{m_1} \dots \alpha^{n_k} \beta^{m_k}$$

if w is the word

$$A^{n_1} B^{m_1} \dots A^{n_k} B^{m_k},$$

and the integral symbol corresponds to iterated Coleman integration ([8], [2] Prop. 4.5). This is normalized by the convention

$$d\left(\int \alpha a_w\right) = \left(\int a_w\right)\alpha, \quad d\left(\int \beta a_w\right) = \left(\int a_w\right)\beta.$$

To recall the detailed description, we need to choose a local coordinate z at $e \in E$ such that $d/dz = b$. Furthermore, fix Iwasawa's branch of the p -adic log. This determines a ring $A^{col}(\mathbb{I}_e)(\log z)$ of logarithmic Coleman functions in the residue disk \mathbb{I}_e of the origin of E , and there is a unique element (see op. cit.) $G^b \in A^{col}(\mathbb{I}_e)(\log z) \otimes R$ characterized by the property $\nabla G = 0$, together with the initial condition specifying that when we write

$$G^b = G_0^b + G_1^b \log z + G_2^b (\log z)^2 + \cdots,$$

we have $G_0^b(0) = 1$. Then analytic continuation along the Frobenius produces an element

$$G^x \in A^{col}(\mathbb{I}_x) \otimes R$$

in the residue disk \mathbb{I}_x compatible with G^b , and

$$G(x) := G^x(x).$$

There is a co-multiplication

$$\Delta : R \rightarrow R \otimes R$$

determined by

$$\Delta(A) = A \otimes 1 + 1 \otimes A, \quad \Delta(B) = B \otimes 1 + 1 \otimes B,$$

with respect to which $G(x)$ is group-like, i.e., satisfies

$$\Delta(G(x)) = G(x) \otimes G(x).$$

Thus, $G(x)$ corresponds to a \mathbb{Q}_p point of $\pi_1^{DR}(X_{\mathbb{Q}_p}; b, x) = \text{Spec}(R^*)$, where

$$R^* := \varinjlim \text{Hom}(R/I^n, \mathbb{Q}_p).$$

The structure of R^* can also be elucidated as the free \mathbb{Q}_p -vector space generated by the functions α_w such that $\alpha_w(w') = \delta_{ww'}$. Another description of R^* is as the H^0 of the circular reduced bar construction ([10], section 3)

$$H^0(B(TW(G(\Omega_{E_{\mathbb{Q}_p}}^*(\log e))), \mathbb{Q}_p)),$$

where $TW(G(\Omega_{E_{\mathbb{Q}_p}}^*(\log e)))$ is the Thom-Whitney construction $TW(\cdot)$ on the Godement resolution $G(\cdot)$ of $\Omega_{E_{\mathbb{Q}_p}}(\log e)$ [17, 19], the sheaf of differential forms on $E_{\mathbb{Q}_p}$ with log poles along e , and the \mathbb{Q}_p is the $TW(G(\Omega_{E_{\mathbb{Q}_p}}^*(\log e)))$ -bi-module obtained by evaluation at b and x . We have a multiplicative quasi-isomorphism of sheaves of commutative differential graded algebras [5]

$$\Omega_{E_{\mathbb{Q}_p}}^*(\log e) \rightarrow j_* \Omega_{X_{\mathbb{Q}_p}}^*,$$

where $j : X \hookrightarrow E$ is the inclusion. The Hodge filtration F on the left is compatible with the pole-filtration P on the right, where

$$P^0 j_* \Omega_{X_{\mathbb{Q}_p}}^* = [_{E_{\mathbb{Q}_p}}(e) \rightarrow \Omega_{E_{\mathbb{Q}_p}}(2e)],$$

and

$$P^1 j_* \Omega_{X_{\mathbb{Q}_p}}^* = [0 \rightarrow \Omega_{E_{\mathbb{Q}_p}}^1(e)].$$

Thus, we have filtered quasi-isomorphisms

$$TW(G(\Omega_{E_{\mathbb{Q}_p}}^*(\log e))) \simeq TW(G(j_* \Omega_{X_{\mathbb{Q}_p}}^*)).$$

But the latter admits a filtered quasi-isomorphism

$$\Omega_{X_{\mathbb{Q}_p}}^*(X_{\mathbb{Q}_p}) \rightarrow TW(G(j_* \Omega_{X_{\mathbb{Q}_p}}^*)),$$

where the pole filtration on the left hand side has simply

$$P^1\Omega_{X_{\mathbb{Q}_p}}^*(X_{\mathbb{Q}_p}) = [0 \rightarrow \Omega_{E_{\mathbb{Q}_p}}^1(E_{\mathbb{Q}_p})].$$

Therefore, F^1 of the Hodge filtration on R^* is generated by the α_w such that w contains at least one A . Hence, the Hodge filtration on R has F^0 generated by the elements B^n . Therefore, with respect to the basis $\{A, A^2, AB, BA\}$ of U_2^{DR}/F^0 , we can express $j_2^{dr/cr}$ as

$$j_2^{dr/cr}(x) = 1 + \int_b^x \alpha A + \int_b^x \alpha^2 A^2 + \int_b^x \alpha \beta AB + \int_b^x \beta \alpha BA.$$

This map as well is conveniently expressed in terms of the logarithm $\log U^{DR} \rightarrow L^{DR}$ as

$$\log j_2^{dr/cr}(x) = \int_b^x \alpha A + \left(\int_b^x \alpha \beta - (1/2) \left(\int_b^x \alpha \right) \left(\int_b^x \beta \right) \right) [A, B].$$

Introducing the notation

$$\log_\alpha(x) = \int_b^x \alpha, \quad \log_\beta(x) = \int_b^x \beta, \quad D_2(x) = \int_b^x \alpha \beta,$$

we can also write this as

$$\log j^{DR}(x) = \log_\alpha(x)A + (D_2(x) - (1/2) \log_\alpha(x) \log_\beta(x)) [A, B].$$

Regarding the \mathbb{Q}_p -action, our choice of α and β implies that the automorphism of L_2^{DR} induced by the involution of section 1 is simply

$$A \mapsto -A, \quad B \mapsto -B,$$

so that A and B are basis elements compatible with the grading on L_2 . In particular, the \mathbb{Q}_p -action can be described as

$$m(\lambda)A = \lambda A, \quad m(\lambda)B = \lambda B, \quad m(\lambda)[A, B] = \lambda^2[A, B].$$

Given the class $a(x) \in H_f^1(G_p, U_2)$ of a point $x \in \mathcal{X}(\mathbb{Z}_p)$, the number λ such that $a_1(x) = \lambda a_1^p(y)$ can be written as

$$\lambda = \log_\alpha(x) / \log_\alpha(y),$$

since the logarithm is a group homomorphism. On the other hand, we have

$$\begin{aligned} \log j_2^{dr/cr}(x) &= \log_\alpha(x)A + (D_2(x) - (1/2) \log_\alpha(x) \log_\beta(x)) [A, B] \\ \log \lambda j_2^{dr/cr}(y) &= \lambda \log_\alpha(y)A + \lambda^2 (D_2(y) - (1/2) \log_\alpha(y) \log_\beta(y)) [A, B] \\ &= \log_\alpha(x)A + \lambda^2 (D_2(y) - (1/2) \log_\alpha(y) \log_\beta(y)) [A, B]. \end{aligned}$$

So the element in Z^{DR} representing the difference is

$$[(\log_\alpha(x) / \log_\alpha(y))^2 (D_2(y) - (1/2) \log_\alpha(y) \log_\beta(y)) - (D_2(x) - (1/2) \log_\alpha(x) \log_\beta(x))] [A, B].$$

Using the (Bloch-Kato) exponential notation for the isomorphism

$$\text{Exp} : U_2^{DR}/F^0 \simeq H_f^1(G_p, U_2),$$

we see then that a formula for the function $\psi^p \circ j_{loc}^{et}$ is given by

$$\psi^p \circ j_{loc}^{et}(x) = (\log_\alpha(x) / \log_\alpha(y))^2 T(\psi^p(y)) +$$

$$2[(\log_\alpha(x)/\log_\alpha(y))^2(D_2(y)-(1/2)\log_\alpha(y)\log_\beta(y))-(D_2(x)-(1/2)\log_\alpha(x)\log_\beta(x))]T(c\cup\text{Exp}([A, B])).$$

Therefore, obtaining a ‘concrete’ expression for this function reduces to the computation of a single

$$T(\psi^P(y))$$

and

$$T(c\cup\text{Exp}([A, B])).$$

The latter is found in a rather straightforward manner. The key point is that the map

$$H_1^{et}(\overline{\mathbb{G}}_{m,\mathbb{Q}_p}, \mathbb{Q}_p) \simeq Z \hookrightarrow U_2^{et}$$

is induced by the restriction functor

$$r : \text{Cov}(\bar{X}) \rightarrow \text{Cov}(\bar{T}^0)$$

mentioned in section 1. Similarly, the map

$$H_1^{DR}(\mathbb{G}_{m,\mathbb{Q}_p}) \simeq Z^{DR} \hookrightarrow U_2^{DR}$$

is induced by a restriction functor

$$r^{DR} : \text{Un}(X) \rightarrow \text{Un}(T^0),$$

on categories of unipotent bundles with connection. The construction of r^{DR} takes a bundle (V, ∇) and associates to it first the canonical extension

$$(\bar{V}, \bar{\nabla})$$

which is a log connection

$$\bar{\nabla} : \bar{V} \rightarrow \Omega_E(\log e) \otimes \bar{V},$$

on E , and then its residue

$$\text{Res}(\nabla) : V_e \rightarrow V_e,$$

which is an endomorphism of the fiber V_e . The value $r^{DR}(V, \nabla)$ is then just the trivial bundle V_e on T^0 equipped with the connection

$$d - Nd/dt$$

for any choice of linear coordinate on T such that $t(0) = 0$. Let us compute r^{DR} for \mathcal{R}/I^3 . In a formal neighborhood of e , we can solve the equation

$$dv = \beta$$

and make the gauge transformation induced by $1 - vB$. Then the connection form with respect to this gauge becomes

$$\begin{aligned} & (1 - vB)(\alpha A + \beta B)(1 + vB + v^2 B^2) + (-dvB)(1 + vB + vB^2) \\ &= (\alpha A + \beta B - v\alpha BA - v\beta B^2)(1 + vB + v^2 B^2) - dvB - vdvB^2 \\ &= \alpha A + \beta B - v\alpha BA - v\beta B^2 + v\alpha AB + v\beta B^2 - dvB - vdvB^2 \\ &= \alpha A + v\alpha[A, B] - vdvB^2. \end{aligned}$$

modulo I^3 . From this, we see that the residue is

$$\text{Res}(v\alpha)[A, B]$$

with $Res(v\alpha) \in \mathbb{Z}_p^*$. This residue can be easily identified with the Serre duality pairing $\langle \alpha, \beta \rangle$. In any case, the connection $r^{DR}(\mathcal{R}/I^3)$ is the bundle $T^0 \otimes R/I^3$ with connection

$$d - Res(v\alpha)[A, B]d/dt.$$

The universal unipotent connection on T^0 is

$$T^0 \otimes \mathbb{Q}_p[[C]]$$

with connection form

$$d - Cd/dt.$$

So the map

$$C \mapsto Res(v\alpha)[A, B]$$

realizes the map

$$\pi^{DR}(T^0, b) \rightarrow U_2^{DR}.$$

In particular, this map fits into the commutative diagram:

$$\begin{array}{ccc} \pi^{DR}(T_{\mathbb{Q}_p}^0, b) & \longrightarrow & U_2^{DR}/F^0 \\ \text{Exp} \downarrow & & \downarrow \text{Exp} \\ H_f^1(G_p, \pi_1^{\mathbb{Q}_p, et}(\bar{T}^0, b)) & \longrightarrow & H_f^1(G_p, U_2). \end{array}$$

We analyze the left vertical arrow. Fix the linear coordinate $t : T \rightarrow \mathbb{A}^1$ such that $t(0) = 0$ and $t(b) = 1$. This induces isomorphisms

$$\begin{aligned} \pi^{\mathbb{Q}_p, et}(\bar{T}^0, b) &\simeq \mathbb{Q}_p(1) \\ \pi^{DR}(T_{\mathbb{Q}_p}^0, b) &\simeq H_1^{DR}(\mathbb{G}_{m, \mathbb{Q}_p}). \end{aligned}$$

The isomorphism

$$H_f^1(G_p, \mathbb{Q}_p(1)) \simeq H_1^{DR}(\mathbb{G}_{m, \mathbb{Q}_p})$$

takes the class $k(x)$ of $x \in \mathbb{Z}_p^*$ to the class of

$$\int_1^x dt/tC = \log(x)C.$$

On the other hand,

$$T(c \cup k(x)) = c(Rec_p(x)) = \log \chi(x) = \log x.$$

Thus, we see that $T(c \cup \text{Exp}(C)) = 1$. Therefore, from the previous commutative diagram, we deduce:

Proposition 3.1

$$T(c \cup \text{Exp}[A, B]) = Res(v\alpha)^{-1}.$$

The computation of $T(\psi^p(y))$ in general appears to be somewhat difficult. Perhaps some progress is possible through the theory of p -adic uniformization when p is a split-semi-stable prime, for which a generalization of the local Selmer theory will be necessary. In that case, it will be natural to take y the trace of a Heegner point coming from a Shimura curve uniformization, and some relationship between the various quantities and L -functions should emerge, such as

$$(\log_\alpha y)^2 = (1/2)(d^2/dk^2)L_p(E, k, k/2)_{k=2},$$

which appears in [1].

One case that is tractable right now is when we already have an integral point y of infinite order in hand, because then $T(\psi^p(y)) = 0$. Corollary 0.2 is an immediate consequence.

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