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Verification, Falsification, and Cancellation in KT

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Abstract The main result of this paper is that KT (=T) is closed under a cancellation principle (if LA is provably equivalent to LB and MA is provably equivalent to MB then A is provably equivalent to B). This result extends to KTG1, but it does not extend to modal systems associated with the provability interpretation of L, such as KW (=G) and KT4Grz (=S4Grz). Following Williamson, these results are applied to philosophical concerns about the proper form for theories of meaning, via the interpretation of L as some kind of verifiability. The cancellation principle can then be read as saying that verifiability conditions and falsifiability conditions jointly determine truth conditions.

A modal logic S has the *single cancellation property* just in case $A \leftrightarrow B$ is a thesis of S whenever $LA \leftrightarrow LB$ is, where L is the necessity operator. S has the *double cancellation property* just in case $A \leftrightarrow B$ is a thesis of S whenever both $LA \leftrightarrow LB$ and $MA \leftrightarrow MB$ are, where M is the possibility operator. The main result of this note is that KT (=T), the smallest normal system to contain the T axiom $Lp \rightarrow p$, has the double (but not the single) cancellation property. Although these properties are mathematically quite natural, there is also a philosophical reason for investigating them, which it may be worthwhile to mention.

One can give a nonstandard interpretation of a modal logic S by reading L as an operator expressive of some kind of verifiability rather than of necessity. One could then say that formulas A and B have the same verifiability conditions according to S if and only if $LA \leftrightarrow LB$ is a thesis of S. By the redundancy property of truth, one can also say that A and B have the same truth conditions according to S if and only if $A \leftrightarrow B$ is a thesis of S. Thus S has the single cancellation property just in case formulas with the same verifiability conditions (ac-

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cording to S) have the same truth conditions (according to S). More briefly: the single cancellation property expresses the claim that verifiability conditions determine truth conditions. In the same vein, A and B would have the same falsifiability conditions according to S if and only if $\mathrm{L}{\sim}A \leftrightarrow \mathrm{L}{\sim}B$ were a thesis of S, given that falsification is verification of the contradictory; if M is equivalent to $\sim\mathrm{L}\sim$ and the underlying propositional logic is classical, $\mathrm{L}{\sim}A \leftrightarrow \mathrm{L}{\sim}B$ is equivalent in S to $\mathrm{M}A \leftrightarrow \mathrm{M}B$. Thus S has the double cancellation property just in case formulas with both the same verifiability conditions and the same falsifiability conditions (according to S) have the same truth conditions (according to S). More briefly: the double cancellation property expresses the claim that verifiability and falsifiability conditions jointly determine truth conditions. It would follow that the potentialities of linguistically formulated propositions to be true (or false) could not transcend the capacities of relevant subjects to recognize them as true (or as false). Questions of truth and falsity would supervene on questions of verification and falsification. Some would call this a form of antirealism; however, it should be noted that these claims do not by themselves equate truth with verifiability, for a modal system can have either cancellation property without having $p \leftrightarrow \mathrm{L}p$ as a thesis ("modal collapse"). Supervenience does not entail reduction. In any case, cancellation rules in modal logic can be used to express various conceptions according to which epistemic features of propositions determine their content (this philosophical background is elaborated, with references, in Williamson [4]). One can then investigate the logics of verifiability which permit such conceptions. That is, which modal systems are closed under the corresponding cancellation rules?

Williamson [4] began such an investigation. There, as here, only normal systems are considered (see [2] for relevant background). This restriction corresponds to nontrivial philosophical assumptions: that the underlying propositional logic is classical, that $\sim\mathrm{L}\sim$ is equivalent to M, and that verifiability obeys some significant closure conditions. [4] defended these assumptions; its arguments will not be repeated here. It is also shown in [4] that, in any normal system interpreted as above, verifiability conditions determine truth conditions if and only if falsifiability conditions do.

Some known results about the cancellation properties may be mentioned. From [3] one can gather that K (the smallest normal system) and KD (the smallest normal system containing the D axiom $\mathrm{L}p \rightarrow \mathrm{M}p$) have the single and therefore the double cancellation property, but that no normal system which includes the T axiom but does not suffer modal collapse by also including $p \rightarrow \mathrm{L}p$ has the single cancellation property. In [4] it is proved that the smallest normal system with the double cancellation property which includes both $\mathrm{L}p \leftrightarrow \mathrm{LL}p$ and $\mathrm{L}p \rightarrow \mathrm{M}p$ is KT4 (=S4) + $p \rightarrow (\mathrm{ML}p \rightarrow \mathrm{L}p)$ + $\mathrm{LM}p \leftrightarrow \mathrm{ML}p$. Both KT4 and KT5 (=S5) lack the double cancellation property. The question thus arises of whether any familiar modal system has the double cancellation property without having the single one. Since KT lacks the single cancellation property, the following result answers that question in the affirmative.

Fact *If* $\mathrm{L}A \leftrightarrow \mathrm{L}B \in \mathrm{KT}$ *and* $\mathrm{M}A \leftrightarrow \mathrm{M}B \in \mathrm{KT}$, *then* $A \leftrightarrow B \in \mathrm{KT}$.

Proof: Suppose that $\mathrm{L}A \leftrightarrow \mathrm{L}B$ and $\mathrm{M}A \leftrightarrow \mathrm{M}B \in \mathrm{KT}$ but $A \leftrightarrow B \notin \mathrm{KT}$. It can be assumed without loss of generality that $A \rightarrow B \notin \mathrm{KT}$. Since KT is complete for reflexive models, there is a model $\langle W, R, V\rangle$ (where W is a set of "worlds", R a binary accessibility relation on them, and V a valuation on the frame $\langle W, R\rangle$) such that R is reflexive and for some $w \in W$, $V(A, w) = 1$ and $V(B, w) = 0$. Choose elements $w(0), w(1), w(2), \ldots$ such that $w(0) = w$, $w(i) \notin W$ for $i > 0$, and $w(i) = w(j)$ only if $i = j$. Let $W^{\sim} = W \cup \{w(1), w(2), \ldots\}$. For $x, y \in W^{\sim}$, put $xR^{\sim}y$ iff either xRy or $x = y$ or for some i, $x = w(i+1)$ and $y = w(i)$. Let $V^{\sim}$ be the unique valuation on $\langle W^{\sim}, R^{\sim}\rangle$ such that, if $x \in W$, $V^{\sim}(C, x) = V(C, x)$ for all wff's C, and if $x \notin W$ but $x \in W^{\sim}$, $V^{\sim}(C, x) = V(C, w)$ for all atomic wff's C. Thus $V^{\sim}(A, w) = 1$ and $V^{\sim}(B, w) = 0$.

The idea of the proof is to show that $w(0), w(1), w(2), \ldots$ form a tail down which A takes alternating values, so that it always "knows" whether it is an odd or an even number of worlds below $w(0)$ – which is impossible, for the complexity of A turns out to impose an upper bound on how far up the tail it can "see", while the tail itself is homogeneous.

It will first be proved by induction on i that if i is even then $V^{\sim}(A, w(i)) = 1$ and $V^{\sim}(B, w(i)) = 0$, while if i is odd then $V^{\sim}(A, w(i)) = 0$ and $V^{\sim}(B, w(i)) = 1$. We already have the basis $i = 0$, since $w(0) = w$. Suppose, for the inductive step, that i is even, $V^{\sim}(A, w(i)) = 1$, and $V^{\sim}(B, w(i)) = 0$. For $x \in W^{\sim}$, $w(i+1)R^{\sim}x$ iff either $x = w(i+1)$ or $x = w(i)$. Suppose that $V^{\sim}(A, w(i+1)) = 1$. Then $w(i+1)R^{\sim}x$ would imply $V^{\sim}(A, x) = 1$, giving $V^{\sim}(\mathrm{L}A, w(i+1)) = 1$. Since $R^{\sim}$ is reflexive, $\langle W^{\sim}, R^{\sim}, V^{\sim}\rangle$ is a model of KT, so by hypothesis $V^{\sim}(\mathrm{L}A \leftrightarrow \mathrm{L}B, w(i+1)) = 1$; thus $V^{\sim}(\mathrm{L}B, w(i+1)) = 1$, which is absurd, since $w(i+1)R^{\sim}w(i)$ and $V^{\sim}(B, w(i)) = 0$. Thus $V^{\sim}(A, w(i+1)) = 0$. Similarly, suppose that $V^{\sim}(B, w(i+1)) = 0$. Then $w(i+1)R^{\sim}x$ would imply $V^{\sim}(B, x) = 0$, giving $V^{\sim}(\mathrm{M}B, w(i+1)) = 0$. Since $\langle W^{\sim}, R^{\sim}, V^{\sim}\rangle$ is a model of KT, $V^{\sim}(\mathrm{M}A \leftrightarrow \mathrm{M}B, w(i+1)) = 1$; thus $V^{\sim}(\mathrm{M}A, w(i+1)) = 0$, which is absurd, since $w(i+1)R^{\sim}w(i)$ and $V^{\sim}(A, w(i)) = 1$. By a symmetric argument, if i is odd, $V^{\sim}(A, w(i)) = 0$, and $V^{\sim}(B, w(i)) = 1$ then $V^{\sim}(A, w(i+1)) = 1$ and $V^{\sim}(B, w(i+1)) = 0$. This completes the induction.

Now define the modal depth $|C|$ of a wff C in the usual way: $|C| = 0$ if C is atomic; $|{\sim}C| = |C|$; $|C \,\&\, D| = \max\{|C|, |D|\}$; $|\mathrm{L}C| = |C| + 1$. It will be proved by induction on the complexity of C that for all i and j, if $|C| \leq i$ and $|C| \leq j$ then $V^{\sim}(C, w(i)) = V^{\sim}(C, w(j))$. For the basis C is atomic, but then by definition of $V^{\sim}$, $V^{\sim}(C, w(i)) = V(C, w) = V^{\sim}(C, w(j))$. Suppose that all wff's less complex than C obey the hypothesis, and $|C| \leq i$ and $|C| \leq j$. Case (a): $C = {\sim}D$. Easy. Case (b): $C = D \,\&\, E$. Since $|D| \leq |C| \leq i$, $|D| \leq |C| \leq j$, $|E| \leq |C| \leq i$, and $|E| \leq |C| \leq j$, by the induction hypothesis $V^{\sim}(D, w(i)) = V^{\sim}(D, w(j))$ and $V^{\sim}(E, w(i)) = V^{\sim}(E, w(j))$, so $V^{\sim}(D \,\&\, E, w(i)) = V^{\sim}(D \,\&\, E, w(j))$. Case (c): $C = \mathrm{L}D$. Since $|C| = |D| + 1$, $0 < i$ and $0 < j$. By construction of $R^{\sim}$, $V^{\sim}(\mathrm{L}D, w(i)) = 1$ iff $V^{\sim}(D, w(i)) = V^{\sim}(D, w(i-1)) = 1$, and $V^{\sim}(\mathrm{L}D, w(j)) = 1$ iff $V^{\sim}(D, w(j)) = V^{\sim}(D, w(j-1)) = 1$. Now $|D| = |C| - 1 \leq i - 1$ and $|D| \leq j - 1$, so by the induction hypothesis $V^{\sim}(D, w(i-1)) = V^{\sim}(D, w(j-1))$ and $V^{\sim}(D, w(i)) = V^{\sim}(D, w(j))$, so $V^{\sim}(\mathrm{L}D, w(i)) = 1$ iff $V^{\sim}(\mathrm{L}D, w(j)) = 1$. This completes the induction

The contradiction is now obvious. Let $|A| \leq 2i$. By the former induction, $V^{\sim}(A, w(2i)) = 1$ and $V^{\sim}(A, w(2i+1)) = 0$. By the latter, $V^{\sim}(A, w(2i)) = V^{\sim}(A, w(2i+1))$.

Naturally, the proof extends to any system complete for a class of models closed under the construction just given (that of $\langle W^{\sim}, R^{\sim}, V^{\sim}\rangle$ out of $\langle W, R, V\rangle$).[1] Consider for instance the system KTG1, which results from the addition of the G1 axiom $\mathrm{ML}p \rightarrow \mathrm{LM}p$ to KT. It can be shown to be complete for the class of models $\langle W, R, V\rangle$ such that R is reflexive and *convergent*, in the sense that for any $w,x,y \in W$, if wRx and wRy then both xRz and yRz for some $z \in W$. It is not hard to show that, in the construction above, if R is reflexive and convergent then so is $R^{\sim}$. Thus if $\mathrm{L}A \leftrightarrow \mathrm{L}B$ and $\mathrm{M}A \leftrightarrow \mathrm{M}B \in$ KTG1, then $A \leftrightarrow B \in$ KTG1.

What is the philosophical importance of these results? Suppose that KT (or, for that matter, KTG1) is held to be the correct logic for some notion of verifiability—perhaps one based on a concept of knowledge that does not satisfy the claim "If you know, you know that you know". It follows that, on such a view, verifiability conditions and falsifiability conditions jointly determine truth conditions, while either set of conditions alone would not.

Finally, it may be of interest to compare KT with the modal system KW in respect of the double cancellation property, since KW is associated with the reading of L as a different kind of verifiability: provability in a given formal system (cf. Boolos [1], where KW is called G). The property would then express the claim that provability and refutability conditions jointly determine truth conditions. KW is the smallest normal system to contain the W axiom $\mathrm{L}(\mathrm{L}p \rightarrow p) \rightarrow \mathrm{L}p$. Let S be any extension of KW with the double cancellation property, and let t be any thesis of KW. Now $\mathrm{MM}t \leftrightarrow \mathrm{M}(\mathrm{M}t \ \&\ \mathrm{L}{\sim}\mathrm{M}t) \in$ KW because $\mathrm{M}p \leftrightarrow \mathrm{M}(p \ \&\ \mathrm{L}{\sim}p) \in$ KW and $\mathrm{LM}t \leftrightarrow \mathrm{L}(\mathrm{M}t \ \&\ \mathrm{L}{\sim}\mathrm{M}t) \in$ KW because $\mathrm{LM}p \rightarrow \mathrm{L}{\sim}t \in$ KW (cf. [1], p. 31). By double cancellation, $\mathrm{M}t \leftrightarrow (\mathrm{M}t \ \&\ \mathrm{L}{\sim}\mathrm{M}t) \in S$, so $\mathrm{M}t \rightarrow \mathrm{L}{\sim}\mathrm{M}t \in S$. But $\mathrm{MM}t \rightarrow \mathrm{M}t \in$ KW since $\mathrm{L}p \rightarrow \mathrm{LL}p \in$ KW ([1], p. 30) and $\mathrm{L}{\sim}\mathrm{M}t \rightarrow {\sim}\mathrm{MM}t \in$ KW, so $\mathrm{MM}t \rightarrow {\sim}\mathrm{MM}t \in S$, so ${\sim}\mathrm{MM}t \in S$. Since ${\sim}\mathrm{MM}{\sim}t \in$ KW, $\mathrm{MM}t \leftrightarrow \mathrm{MM}{\sim}t \in S$. Moreover, $\mathrm{LM}t \leftrightarrow \mathrm{LM}{\sim}t \in$ KW. By double cancellation again, $\mathrm{M}t \leftrightarrow \mathrm{M}{\sim}t \in S$. Consequently, $\mathrm{L}t \leftrightarrow \mathrm{L}{\sim}t \in S$. By a third double cancellation, $t \leftrightarrow {\sim}t \in S$. In other words, no consistent extension of KW has the double cancellation property; it follows that no consistent extension of KW has the single cancellation property either. This result illustrates the inability of a radical interpreter, who can recognize only formal provability and refutability in a mathematical language, to make sense of its sentences.

The last point does not depend on the failure of provability in a formal system to guarantee truth. For if L is reinterpreted as 'provable and true', the corresponding modal system is KT4Grz (=S4Grz), which results from the addition of the Grz axiom $\mathrm{L}(\mathrm{L}(p \rightarrow \mathrm{L}p) \rightarrow p) \rightarrow p$ to KT4 ([1], pp. 159–167). Now the smallest normal system with the double cancellation property which includes KT4 is KT4 + $p \rightarrow (\mathrm{ML}p \rightarrow \mathrm{L}p)$ + $\mathrm{LM}p \leftrightarrow \mathrm{ML}p$; since Grz is a thesis of this system, the closure of KT4Grz under double cancellation is not the contradictory system but KT4 + $p \rightarrow (\mathrm{ML}p \rightarrow \mathrm{L}p)$ + $\mathrm{LM}p \leftrightarrow \mathrm{ML}p$ (this follows from results in [4]). However, $p \rightarrow (\mathrm{ML}p \rightarrow \mathrm{L}p)$ is not a thesis of KT4Grz ([1], p. 166), so KT4Grz is not closed under double cancellation. One could not rehabilitate the claim that provability and refutability conditions jointly determine truth conditions in mathematics merely by restricting proof to true propositions and refutation to false ones.

In summary, if the determination of truth conditions by verifiability and falsifiability conditions has a chance, it may lie with a view on which verification is neither transparent nor formal.

NOTE

1. A more delicate analysis of the proof shows that it is not necessary to be able to hang an infinite tail from the world *w*; the proof can easily be adapted to the case in which one can hang only finite tails of arbitrary length from *w*.

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