A Galerkin Approach for Modelling the Pantograph-Catenary Interaction

John Martin and Stephen Duncan

Department of Engineering Science
University of Oxford
john.martin@eng.ox.ac.uk
stephen.duncan@eng.ox.ac.uk

Abstract. The pantograph-catenary interaction was modelled for high-speed electric and hybrid trains. A lumped-mass pantograph model was used and the overhead wires were modelled as Euler-Bernoulli beams. Each vertical and horizontal wire deflection was decomposed into an infinite series of spatial basis functions, which were chosen to be the eigenmodes of the Euler-Bernoulli PDE, and corresponding time functions. The boundary conditions were used to evaluate the spatial basis functions and reduce the PDEs to ODEs in terms of the time functions. Elimination of variables was used to remove the algebraic contact constraints and reduce the overall index-three DAE to an ODE. This linear, time-varying ODE was solved by integration and the elimination process was reversed in order to recover the original states. The Simulink model was validated against the 2002 and 2018 European Standards, BS:EN 50318:2002 and BS:EN 50318:2018 respectively. In both cases, the model produced accurate results with exceptional simulation speeds.

1 Introduction

Electric and hybrid trains are popular across Europe and many further electrification projects are under way. As train speeds increase, it is important to design pantographs and catenary systems that robustly provide a suitable contact force at higher operating speeds. If the contact force is too large, this will cause excessive wear on both the contact wire and the pantograph head; if the contact force is too small, there is a risk of loss of contact, which interrupts power delivery and causes arcing. Since many configurations must be tested and real line tests are expensive, accurate modelling of the pantograph-catenary interaction is a necessary part of overhead-line-equipment (OLE) design.

In [3], the results from ten simulation programs were compared and used to create the standard BS:EN 50318:2018. The majority of previous pantograph-catenary simulation software has used finite-element methods (FEM) [1, 7, 8]. Two prominent programs are PCaDA and OSCAR, which both use FEM to model the overhead system, lumped-mass models for the pantograph, the Euler-Bernoulli beam equation for the overhead wires and a penalty system to keep the height of the pantograph head and the contact-wire deflection in agreement. This paper takes an alternative approach by using a Galerkin method instead of FEM and an algebraic contact constraint instead of a penalty method. Similar algebraic constraints are also used to maintain contact between the registration arms and the contact wire.

Simulations based on FEM provide excellent accuracy close to the point of contact, but the speed of simulation is poor. It is desirable to reduce simulation time so that multiple OLE set-ups and ‘what if?’ scenarios can be tested. This paper presents pantograph-catenary modelling code, running in MATLAB and Simulink, that aims to greatly improve the simulation speed while maintaining high accuracy.

2 The Galerkin Approach

A Galerkin approach, also known as modal analysis or spectral analysis, has been used to solve the catenary-wire PDEs. This approach is explained in [4]: for each wire, the vertical and horizontal deflections are each separated into an infinite series of spatial basis functions and corresponding time functions, where the spatial basis functions are chosen to be the eigenfunctions of the PDE. The eigenfunctions are calculated using the boundary conditions and the PDEs are reduced to ODEs, with the amplitudes of the modes as the states. After integration, the evaluated time functions are recombined with the spatial basis functions to give vertical and horizontal wire deflections as functions of space and time.

The main incentive to use a Galerkin approach over FEM is to improve the speed of simulation. The infinite series of eigenfunctions is truncated to match the frequency response of the pantograph. Since the reciprocal of the maximum spatial frequency is typically more than an order of magnitude larger than the

distance between FEM elements for the same overhead system, the Galerkin approach typically requires fewer than one tenth of the number of states to be evaluated. This leads to improved simulation speed, allowing more overhead-system configurations to be modelled in a given timeframe.

Another advantage of this software is that it can take an arbitrary velocity or acceleration profile as an input. This means that realistic velocity profiles can be tested, rather than requiring constant velocity.

The approach is evaluated using standardised pantograph-catenary simulations. It is shown that this Galerkin approach yields accurate solutions with good simulation speed and can handle multiple pantographs, dropper unloading and the transition from one contact wire to the next.

3 Mathematical Modelling

The mathematical model included two pantographs in contact with a single contact wire. A simplified model was used for validation against the 2002 European Standard and the full model was used for the 2018 standard.

3.1 The Pantographs

A lumped-mass pantograph model was used, as shown in Figure 1. The mass-spring-damper constants were specified by the European Standards and were chosen in order to match the frequency characteristics of the real pantograph. A constant uplift force, P_1 , is applied to the base of the pantograph by a pneumatic actuator in order to maintain contact between the pantograph and the contact wire.

Fig. 1: Diagram of lumped-mass pantograph model

$$m_1 \ddot{z}_1(t) = -(c_1 + c_2) \dot{z}_1(t) + c_2 \dot{z}_2(t) - (k_1 + k_2) z_1(t) + k_2 z_2(t) + P_1 \quad (1)$$

$$m_2 \ddot{z}_2(t) = c_2 \dot{z}_1(t) - (c_2 + c_3) \dot{z}_2(t) + c_3 \dot{z}_3(t) + k_2 z_1(t) - (k_2 + k_3) z_2(t) + k_3 z_3(t) \quad (2)$$

$$m_3 \ddot{z}_3(t) = c_3 \dot{z}_2(t) - c_3 \dot{z}_3(t) + k_3 z_2(t) - k_3 z_3(t) - \lambda(t) \quad (3)$$

where $z_i(t)$ is the height of the i^{th} pantograph block, m_i the mass of the i^{th} pantograph block, c_i the i^{th} damper constant, k_i the i^{th} spring constant, P_1 the constant uplift force and $\lambda(t)$ the contact force between the pantograph head and the contact wire.

3.2 The Overhead Wires

The contact wire and messenger wire were modelled as Euler-Bernoulli beams. The length of each wire was 550-600m and the radius was only 6.2-7.0mm, so the beams were extremely slender and bending moments were negligible.

Contact Wire - Vertical

$$\rho A \frac{\partial^2 w(x, t)}{\partial t^2} = -EI \frac{\partial^4 w(x, t)}{\partial x^4} + P \frac{\partial^2 w(x, t)}{\partial x^2} - \rho A g + \sum_{l=1}^2 (\lambda_l(t) \delta(x - x_{p,l}(t))) \quad (4)$$

where x is the horizontal position, t time, $x_{p,l}(t)$ the horizontal position of the l^{th} pantograph, $w(x, t)$ the vertical contact-wire deflection, ρ the wire density, A the cross-sectional area, E Young's modulus, I the second moment of area, P the tension, g the gravitational constant, $\lambda_l(t)$ the contact force between the l^{th} pantograph and the wire and $\delta(x)$ a Dirac Delta function.

The terms on the right-hand side, from left to right, are the bending stiffness, axial tension, self-weight and contact force. The pantograph exerts moving a point force on the contact wire, which is represented by a Dirac Delta function, because the Euler-Bernoulli beam equation is written in terms of force per unit length. The pantograph exerts no horizontal force and is never in contact with the messenger wire, so this contact-force term does not appear in the horizontal contact-wire equation or messenger-wire equation.

Contact Wire - Horizontal

$$\rho A \frac{\partial^2 w^p(x, t)}{\partial t^2} = -EI \frac{\partial^4 w^p(x, t)}{\partial x^4} + P \frac{\partial^2 w^p(x, t)}{\partial x^2} \quad (5)$$

where $w^p(x, t)$ is the horizontal contact-wire deflection.

Messenger Wire - Vertical

$$\rho^m A^m \frac{\partial^2 w_i^m(x, t)}{\partial t^2} = -E^m I^m \frac{\partial^4 w_i^m(x, t)}{\partial x^4} + P^m \frac{\partial^2 w_i^m(x, t)}{\partial x^2} - \rho^m A^m g \quad (6)$$

where x is the horizontal position in the messenger span's coordinate system, ρ^m the messenger-wire density, A^m the cross-sectional area, $w_i^m(x, t)$ the vertical deflection of the i^{th} messenger span, E^m Young's modulus, I^m the second moment of area and P^m the axial tension.

Messenger Wire - Horizontal

The messenger-wire's horizontal deflection was assumed to be static and piecewise linear, because no dynamic forces were exerted on the messenger wire. Each messenger-wire span stretched horizontally from one support to the next in a straight line.

3.3 The Droppers

The droppers were modelled as linear springs when in tension and were ignored when in compression. This was achieved by assuming that all droppers were in tension, then applying additional forces to remove the tensions of any unloaded droppers. The dropper tensions were given by (7) and were added to (4) and (6) as point forces, giving (8) and (9), respectively.

$$F_{d,i,j}(t) = k_{d,i,j} (w_i^m(x_{d,i,j}^m, t) - w(x_{d,i,j}, t)) \quad (7)$$

where $F_{d,i,j}(t)$ is the tension in the j^{th} dropper of the i^{th} messenger span, $k_{d,i,j}$ the spring constant of the dropper, $x_{d,i,j}$ the horizontal position of the dropper in the contact wire's coordinate system and $x_{d,i,j}^m$ the horizontal position of the dropper in the i^{th} messenger span's coordinate system.

$$\begin{aligned} \rho A \frac{\partial^2 w(x, t)}{\partial t^2} = & -EI \frac{\partial^4 w(x, t)}{\partial x^4} + P \frac{\partial^2 w(x, t)}{\partial x^2} - \rho A g + \lambda(t) \delta(x - x_p(t)) \\ & + \sum_{i=1}^{N^m} \sum_{j=1}^{N_d} (k_{d,i,j} (w_i^m(x_{d,i,j}^m, t) - w(x_{d,i,j}, t)) \delta(x - x_{d,i,j})) \end{aligned} \quad (8)$$

where N_d is the number of droppers on the messenger span.

$$\begin{aligned} \rho^m A^m \frac{\partial^2 w_i^m(x, t)}{\partial t^2} = & -E^m I^m \frac{\partial^4 w_i^m(x, t)}{\partial x^4} + P^m \frac{\partial^2 w_i^m(x, t)}{\partial x^2} - \rho^m A^m g \\ & + \sum_{j=1}^{N_d} (k_{d,i,j} (w(x_{d,i,j}, t) - w_i^m(x_{d,i,j}^m, t)) \delta(x - x_{d,i,j}^m)) \end{aligned} \quad (9)$$

3.4 The Registration Arms

Fig. 2: Model of registration arm. A is the static support, B is the centre of mass and C is the point of contact with the contact wire.

$$J_{r,k} \ddot{\theta}_k(t) = -L_{r,k} \lambda_{v,k}(t) \cos \theta_k(t) - L_{r,k} \lambda_{h,k}(t) \sin \theta_k(t) - m_{r,k} g \frac{L_{g,k}}{2} \cos(\theta_k(t) - \alpha_k) \quad (10)$$

where $J_{r,k}$ is the moment of inertia of the k^{th} registration arm, $\theta_k(t)$ the pitch of the arm, $L_{r,k}$ the length of the arm, $\lambda_{v,k}(t)$ the upward force exerted on the contact wire by the arm, $\lambda_{h,k}(t)$ the horizontal force exerted on the contact wire by the arm, $m_{r,k}$ the mass of the arm, $L_{g,k}$ the distance from the arm's support to its centre of mass and α_k the angle of the centre of mass.

$$\begin{aligned} \rho A \frac{\partial^2 w(x, t)}{\partial t^2} = & -EI \frac{\partial^4 w(x, t)}{\partial x^4} + P \frac{\partial^2 w(x, t)}{\partial x^2} - \rho A g + \sum_{k=1}^{N_r} (\lambda_{v,k}(t) \delta(x - x_{r,k})) \\ & + \sum_{i=1}^{N^m} \sum_{j=1}^{N_d} (k_{d,i,j} (w_i^m(x_{d,i,j}^m, t) - w(x_{d,i,j}, t)) \delta(x - x_{d,i,j})) + \lambda(t) \delta(x - x_p(t)) \end{aligned} \quad (11)$$

where $x_{r,k}$ is the horizontal position of the k^{th} registration arm in the contact wire's coordinate system.

$$\begin{aligned} \rho A \frac{\partial^2 w^p(x, t)}{\partial t^2} = & -EI \frac{\partial^4 w^p(x, t)}{\partial x^4} + P \frac{\partial^2 w^p(x, t)}{\partial x^2} \\ & + \sum_{k=1}^{N_r} (\text{sign}(s_k) \lambda_{h,k}(t) \delta(x - x_{r,k})) \end{aligned} \quad (12)$$

where s_k is the horizontal stagger of the point of contact between the registration arm and the contact wire when the arm is horizontal.

3.5 The Algebraic Constraints

The Horizontal Registration-Arm Constraints

$$w^p(x_{r,k}, t) = s_k + L_{r,k} \text{sign}(s_k) (1 - \cos \theta_k(t)) \quad (13)$$

The Vertical Registration-Arm Constraints

$$w(x_{r,k}, t) = L_{r,k} \sin \theta_k(t) \quad (14)$$

The Pantograph Position Constraints

One algebraic constraint must be satisfied for each of the two pantographs. These constraints reflect the fact that the pantographs remain in contact with the contact wire, which means that the height of each pantograph head must match the contact-wire height at that horizontal position.

$$z_{3,l}(t) + \bar{z}_{3,l} = w(x_{p,l}(t), t) \quad l = 1, 2 \quad (15)$$

where $\bar{z}_{3,l}$ is the steady-state height of the l^{th} pantograph head, with no uplift force applied.

3.6 Spatial Discretisation

Each wire deflection was decomposed into an infinite series of spatial basis functions, $\psi_n(x)$, and corresponding time functions, $q_n(t)$.

$$\begin{aligned} w(x, t) &= \sum_{n=1}^{\infty} (q_n(t) \phi_n(x)) \\ w^p(x, t) &= \sum_{n=1}^{\infty} (p_n(t) \phi_n(x)) \\ w_i^m(x, t) &= \sum_{n=1}^{\infty} (q_{i,n}^m(t) \phi_{i,n}(x)) \quad i = 1, 2, 3, \dots, N_m \end{aligned} \quad (16)$$

The ψ_n -terms were evaluated using the boundary conditions for the contact wire and for each messenger-wire span. Then, (11), (12) and (9) were rearranged in terms of $q_n(t)$, $q_n^m(t)$ and $p_n(t)$, respectively.

3.7 The Index-Three DAE

The equations of motion for the dynamics of the pantographs, overhead wires and registration arms were compiled into one ODE. This was combined with the algebraic constraints to yield (17). Since the pantograph position constraint could be differentiated three times in order to reduce this DAE to an ODE, the differential index is three.

$$\begin{aligned} \ddot{\underline{q}}(t) &= A_1 \underline{q}(t) + A_2 \dot{\underline{q}}(t) + A_3 \ddot{\underline{q}}(t) + F(t) \begin{pmatrix} \underline{\Delta}_h(t) \\ \underline{\Delta}_v(t) \\ \underline{\Delta}(t) \end{pmatrix} \\ \ddot{\underline{z}}(t) &= M^{-1} \left(C \dot{\underline{z}}(t) + K \underline{z}(t) - \begin{pmatrix} 0 \\ \underline{\Delta}(t) \end{pmatrix} \right) \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} z_{3,1}(t) \\ z_{3,2}(t) \end{pmatrix} + \begin{pmatrix} \bar{z}_{3,1}(t) \\ \bar{z}_{3,2}(t) \end{pmatrix} - \begin{pmatrix} w(x_{p,1}(t), t) \\ w(x_{p,2}(t), t) \end{pmatrix} \end{aligned} \quad (17)$$

$$\underline{q}(t) = \left(\underline{\theta}^T(t) \quad \underline{q}^T(t) \quad \underline{p}^T(t) \quad \underline{q}_1^{mT}(t) \quad \underline{q}_2^{mT}(t) \quad \dots \quad \underline{q}_{N_m}^{mT}(t) \right)^T \quad (18)$$

$$\underline{z}(t) = \left(z_{1,1}(t) \quad z_{1,2}(t) \quad z_{2,1}(t) \quad z_{2,2}(t) \quad z_{3,1}(t) \quad z_{3,2}(t) \right)^T \quad (19)$$

where the matrices $A_1, A_2, A_3, F(t), M, C$ and K were evaluated using the previously derived ODEs.

3.8 Elimination of Variables

In order to solve the DAE, the constraint equations were removed by elimination of variables. Each constraint, as well as its first and second time derivative, was rearranged to find an expression for an eliminated state in terms of the remaining states. The equations of motion for the eliminated states were rearranged to find expressions for the constraint forces in terms of the remaining states, which were substituted into the ODE equations. The equations used to eliminate variables were removed from the DAE, giving a solvable ODE.

Using elimination of variables is inadvisable for most nonlinear DAEs [6] and the GGL formulation [5] is generally preferred. However, the equation considered in this paper is a linear, time-varying DAE. Therefore, the algebraic elimination only had to be performed once and reducing the DAE to an ODE was not computationally expensive.

3.9 Reconstruction

After the ODE had been solved by integration, the solution process was reversed to recover the eliminated states. The eliminated variables were evaluated using the previously evaluated expressions, then the $\psi_n(x)$ - and $q_n(t)$ -terms were recombined to find the wire deflections.

4 Results

4.1 Validation Against BS:EN 50318:2002

Fig. 3: Diagram of single messenger span for BS:EN 50318:2002

The software was first validated against BS:EN 50318:2002. This involved modelling the interaction between a single pantograph and a simplified catenary system consisting of ten uniform messenger spans, as shown in Figure 3. Transitions and dropper unloading were not required.

The simulation was performed at 250 km/hour and at 300 km/hour. The contact-force profile for the central two messenger spans and the time profile of the vertical displacement of the contact wire under each messenger support were recorded and compared to the benchmark, as shown in the tables below.

250 km/hour	Required	Obtained
Mean contact force (N)	110-120	116.69
Standard deviation of contact force (N)	26-31	30.88
Mean contact force - 3s.d. (N)	20-40	24.06
Mean contact force + 3s.d. (N)	190-210	209.33
Minimum of contact force (N)	50-75	55.54
Maximum of contact force (N)	175-210	188.56
Maximum uplift at support (mm)	48-55	48.04
Percentage loss of contact (%)	0	0.00
Time taken (s)	-	5

Table 1: Table of 250 km/hour results for validation against BS:EN 50318:2002

300 km/hour	Required	Obtained
Mean contact force (N)	110-120	116.65
Standard deviation of contact force (N)	32-40	37.46
Mean contact force - 3s.d. (N)	-5-20	4.26
Mean contact force + 3s.d. (N)	210-230	229.04
Minimum of contact force (N)	30-55	37.17
Maximum of contact force (N)	190-225	215.47
Maximum uplift at support (mm)	55-65	55.13
Percentage loss of contact (%)	0	0.00
Time taken (s)	-	4

Table 2: Table of 300 km/hour results for validation against BS:EN 50318:2002

The results proved to be accurate, with every value falling within the required range. The simulations took 5s and 4s at 250 km/hour and 300 km/hour respectively. This is faster than real time and could be improved even further. These simulations were done in Simulink on a Windows desktop machine with a quadcore processor.

4.2 Validation Against BS:EN 50318:2018

The new standard requires the modelling of two pantographs in contact with the same wire, as well as dropper unloading. It does not explicitly require contact-wire transitions to be modelled, but these were found to be necessary in order to accelerate the train before reaching the analysis section, which is ten messenger spans in the middle of the central contact wire. A third contact wire was also included to allow the front pantograph to leave the central wire while the back pantograph was still in the analysis section.

Details of modelling wire transitions are given in [2]. The software must detect contact events, when a pantograph comes into or out of contact with a wire, and update the equations accordingly.

Conservation of momentum was used to resolve velocity discontinuities when coming into contact and these events were successfully incorporated into the model, giving the results in the tables below.

275 km/hour	Required		Obtained	
	Pan 1	Pan 2	Pan 1	Pan 2
Mean contact force (N)	137-146	138-147	143.33	143.29
s.d. of contact force: 0-20Hz (N)	31.9-34.8	50.0-54.5	32.59	51.70
s.d. of contact force: 0-5Hz (N)	25.6-28.9	41.2-45.4	26.50	44.50
s.d. of contact force: 5-20Hz (N)	16.2-23.1	25.2-34.7	18.97	26.32
Maximum of contact force (N)	211-244	241-290	235.07	244.84
Minimum of contact force (N)	71-86	14-50	72.52	25.32
Contact-point displacement range (mm)	38-49	53-70	43.18	68.97
Maximum uplift at support (mm)	37-54		53.22	
Percentage loss of contact (%)	0	0	0.00	0.00
Time taken (s)	-		173	

Table 3: Table of 275 km/hour results for validation against BS:EN 50318:2018

320 km/hour	Required		Obtained	
	Pan 1	Pan 2	Pan 1	Pan 2
Mean contact force (N)	150-173	158.5-172.5	164.93	164.55
s.d. of contact force: 0-20Hz (N)	49-63	30-44	49.96	42.19
s.d. of contact force: 0-5Hz (N)	38-44	14-23	40.62	29.45
s.d. of contact force: 5-20Hz (N)	29-46	25-38	29.09	30.22
Maximum of contact force (N)	280-343	247-317	336.16	264.96
Minimum of contact force (N)	50-82	21-86	77.48	62.10
Contact-point displacement range (mm)	39-51	18-35	50.46	31.10
Maximum uplift at support (mm)	52-64		62.01	
Percentage loss of contact (%)	0	0	0.00	0.00
Time taken (s)	-		91	

Table 4: Table of 320 km/hour results for validation against BS:EN 50318:2018

As shown, only one value lay outside the required range. When these benchmark results were created, the dispersal of the indicator values for the back pantograph was relatively large [3], so the larger value shown here is not unrealistic. Furthermore, the benchmarking software dealt with the run-in differently from this software, which could explain the discrepancy in low-frequency dynamics.

Fig. 4: Plot of contact forces against time for BS:EN 50318:2018 at 275 km/hour

5 Conclusions

The pantograph-catenary interaction was modelled using a Galerkin approach instead of FEM. This yielded high simulation speed while still producing accurate results. Horizontal deflections, dropper unloading and contact-wire transitions are all included in the model. The model allows for multiple pantographs and can take any velocity profile as an input.

The model was validated against BS:EN 50318:2002 and BS:EN 50318:2018. There is further work to be done on this model, but this serves as a proof of concept of the Galerkin approach for modelling the pantograph-catenary interaction.

Current work involves updating the Simulink model to simulate a real train journey. This entails generalising the model to accept catenary systems with different lengths, layouts, material properties, as well as modelling pantograph transitions from one wire to the next. The final model will be validated against line-test data from the Old Dalby track.

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