Prospect Theory, Partial Liquidation and the Disposition Effect [†]

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We solve a liquidation problem for an agent with prospect theory preferences who seeks to sell a portfolio of (divisible) claims on an underlying asset. Our methodology enables us to consider different formulations of prospect preferences in the literature, and various asset price processes. We find that these differences in specification are important - for instance, with piecewise power functions (but not piecewise exponentials) the agent may voluntarily liquidate at a loss relative to break-even. Further, we find that the likelihood of liquidating at a (small) gain is much higher than liquidating at a (large) loss, consistent with the disposition effect documented in empirical and experimental studies. The ability to partially liquidate also has significant consequences. The prospect agent liquidates the entire position at once, in contrast to behavior under standard concave preferences. If the position is divisible, under piecewise exponential functions, the agent no longer liquidates at the break-even level, and even if the asset is very poor, prefers to gamble on the possibility of liquidating at a gain. Finally, the piecewise power specification remains consistent with the disposition effect, albeit where the whole portfolio is sold at once.

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In a rational world, agents evaluate risky gambles using expected utility (dating back to von Neumann and Morgenstern (1944)). However, experimental work has showed substantial violations of expected utility theory, resulting in a number of alternatives, one of which is the prospect theory of Kahneman and Tversky (1979) (and Tversky and Kahneman (1992)). Important features of prospect theory are as follows. First, utility is defined over gains and losses relative to a reference point, rather than over final wealth, an idea first proposed by Markowitz (1952). Second, the utility function (called value function in the prospect theory) exhibits concavity in the domain of gains and convexity in the domain of losses. The function is steeper for losses than for gains, a feature known as loss aversion. The final feature is the nonlinear probability transformation whereby small probabilities are overweighted. The theoretical literature on prospect theory applied to finance[1] has treated portfolio choice, see for example, Berkelaar et al (2004), Gomes (2005), Jin and Zhou (2008); asset pricing, see Benartzi and Thaler (1995), Barberis and Huang (2001), and Barberis et al (2001); and equilibrium, see De Giorgi et al (2008).

Our focus in this paper is on modeling the behavior of an agent with prospect theory preferences who seeks to liquidate a portfolio of assets, or more generally, contingent claims on an underlying asset. Our primary interest will be in the situation where the agent is simply choosing when to sell an asset, for example, her stock holdings, house, or, how to terminate a managerial project. In general, we will allow the agent to divide up her position and sell over time (but we will see that for agents with prosect theory preferences, this is in fact, not optimal behavior). At the chosen liquidation time, the agent compares the payoff to a reference level, which is taken to be the break-even level. For example, they might compare the sale price received for a stock or house to the price paid; or the proceeds of a project with the initial outlay. If there is a portfolio of identical claims, the agent compares each payoff (received a potentially different times) to a reference level, which is the same for each claim.

We will develop a general continuous-time optimal stopping model which will enable us to consider alternative specifications of prospect theory preferences, and any time-homogeneous diffusion asset price process. Specific examples will include the Tversky and Kahneman (1992) piecewise power functions, the piecewise exponential functions, and prices following both exponential Brownian motion and Brownian motion. This enables us to revisit the model of Kyle, Ou-Yang and Xiong (2006) as an important example where the asset is indivisible, the price follows Brownian motion, and the agent has exponential S-shaped preferences. A key contribution we make is to demonstrate that the qualitative features of the agent's liquidation behavior are not robust to changes in either the specification of prospect preferences, or to the assumption of asset indivisibility versus divisibility. In particular, the break-even level plays a major role in liquidation strategies under the model of Kyle, Ou-Yang and Xiong (2006), but plays no role once the asset is divisible, or when preferences are altered to the power S-shaped functions of Tversky and Kahneman (1992). Thus the behavior of "selling at the break-even level" which is a major conclusion of Kyle, Ou-Yang and Xiong (2006), is a rather special feature of their precise choice of model, and is not a typical finding of the general prospect theory setting.

[1] Barberis and Thaler (2003) present a broad overview of behavioral finance.

The main application of our prospect theory model will be the disposition effect. Many empirical studies find that investors are reluctant to sell assets trading at a loss relative to the price at which they were purchased, a phenomenon labeled the "disposition effect" by Shefrin and Statman (1985). Phrased differently, the disposition effect is the observation that agent's sell winners and hold onto losers. This effect has been found for individual investors in Ferris et al (1988) and Odean (1998), amongst others, as well as in experimental studies by Weber and Camerer (1998). Disposition effects have also been found in other settings - the real estate market (see Genesove and Mayer (2001)), traded options markets (Poteshman and Serbin (2003)) and executive stock options (Heath, Huddart and Lang (1999)). Whilst it is more difficult to observe whether corporate managers display behavioral biases such as the disposition effect, there is anecdotal and empirical evidence to suggest that managers are resistant to terminating losing projects.[2] Recently, Crane and Hartzell (2008) take advantage of the transparency of real estate investment trusts (REITs) and find results consistent with disposition in the corporate setting.

Prospect theory (together with narrow framing) has long been recognized as one potential way of understanding the disposition effect. The idea is that a gain (or loss) puts the investor in the risk-averse (risk-seeking) region and so she prefers to reduce (increase) her position in the risky asset. We borrow an example from Shefrin and Statman (1985) to illustrate this point. Consider an investor who bought a stock a month ago for \$50 and it is currently trading at \$40. The investor is deciding whether to sell the stock now (for a \$10 loss) or to wait another period. Suppose either the stock will increase to \$50 next period or decrease to \$30, with equal probability. Under prospect theory, the investor is choosing between:

A. sell the stock now and make a \$10 loss,

B. wait, and have a 50% chance of losing a further \$10 but a 50% chance of breaking even.

Shefrin and Statman (1985) conclude that since the choice between the lotteries is associated with the convex portion of the S-shaped function, the prospect theory investor would choose option B, thus waiting to gamble on the possibility of breaking even. They also recognize that this will depend on the odds of breaking even - and that if these were sufficiently unfavorable, the investor may choose lottery A, and sell for a loss today.

Our continuous-time optimal stopping model allows us to analyze how the behavior of the prospect theory agent depends upon the expected return and volatility of the asset itself (key in the intuition above), and importantly, the specification of prospect preferences and price dynamics. This will enable us to explore the ability of a particular model to give predictions consistent with the disposition effect. Common to all specifications is the structure of the solution - the agent stops when the asset price reaches some threshold(s) (which may be infinite) and hence the likelihood of sale is related to the distance between the current price and the threshold(s).

[2]Statman and Caldwell (1987) and Shefrin (2001) offer several anecdotes concerning major corporate investments that have the flavor of "throwing good money after bad", and Statman and Sepe (1989) show announcements of termination of losing projects are generally associated with positive abnormal returns on the stock of the terminating companies. Related evidence comes from Guedj and Scharfstein (2004) who find managers are reluctant to abandon their only drug candidates even when the results of clinical trials are not promising.

However, importantly, we show that whether the agent will in fact sell at a loss relative to break-even, depends on the specification of prospect theory, and in particular, the nature of the loss aversion. Clearly this has implications for whether a model can produce results consistent with the disposition effect.

We first study the situation where the agent's position is indivisible. Under exponential S-shaped preferences, we recover the model of Kyle, Ou-Yang and Xiong (2006) in which the agent never (voluntarily) sells at a loss. Rather, if the asset has extremely poor expected return, the agent would simply not have held it ex-ante (because the strategy is to liquidate at the break-even point, and we assume this is the price paid for the asset). However, more interestingly, the asset can have a relatively poor expected return and still be held ex-ante, with the agent gambling on the possibility of liquidating at a (small) gain.

The situation changes dramatically for power S-shape preferences of Tversky and Kahneman (1992), due to the locally infinite loss aversion.[3] The most important difference we find is that under the piecewise power specification of prospect theory preferences, the agent will (voluntarily) liquidate at a loss in some circumstances. The agent will never liquidate at the break-even point. As a consequence, if the asset has a poor Sharpe ratio, the agent will wait to liquidate either at a loss, or at a gain. Despite the poor Sharpe ratio, the prospect theory agent will choose ex-ante to hold the asset. This situation is consistent with the intuition of Shefrin and Statman (1985) who argue that if the odds are sufficiently bad, the agent will "give-up" and accept a loss. However, their story does not take into account whether the agent would hold the asset ex-ante, and indeed, more recent literature has remarked that this consideration would rule out the behavior we describe (see Hens and Vlcek (2005), Kaustia (2008) and Barberis and Xiong (2008a)). We demonstrate that the agent does hold the asset ex-ante and that, consistent with the disposition effect, it is more likely a (small) gain will be realized rather than a (large) loss.

The extension to allow the agent to partially liquidate the position has significant consequences. We find that common to both specifications of prospect preferences, the agent takes an "all-or-nothing" liquidation strategy, in that if liquidation occurs, the entire position is sold at once. This is in contrast to the typical strategy of liquidating the risky position over time which we would expect of an agent with a concave utility function. The "all-or-nothing" strategy occurs because due to the disposition effect, the majority of the region of interest is where the prospect function is convex, and hence, the agent behaves as if she has convex utility.

Analysis of the partial liquidation problem for the agent with exponential S-shaped functions shows that the agent no longer stops at the break-even level, and thus, this feature of the indivisible model is not robust to the generalization to divisibility. This is pertinent given the importance of the break-even level in the paper of Kyle, Ou-Yang and Xiong (2006), and as they note "the issue of partial liquidation is not addressed in the extant literature" (p284). Instead, there is a wider range of (poor and extremely poor) expected returns over which the agent will

[3]However, as the survey of Di Giorgi and Hens (2006) (and references therein) highlights, this infinite slope can also have other less desirable consequences with regard to other applications of prospect theory. In this paper, we are primarily concerned with applications to the disposition effect.

choose to hold two units of asset, and will gamble on the possibility of liquidating (both) assets at a gain. In fact, when the expected return is extremely poor, an agent with an indivisible position would not have taken it ex-ante, whilst the agent with a larger stake would choose to take the gamble. Again, this behavior is consistent with that of an agent with convex utility, and, as we already noted, this behavior dominates because the sales threshold is typically very close to the break-even level. The agent with the Tversky and Kahneman (1992) piecewise power functions will choose to take the divisible position ex-ante, and, for extremely poor Sharpe ratios, sell at either a loss or a gain. Again, this fits with the intuition of Shefrin and Statman (1985). The results are again consistent with the disposition effect, since it is more likely the agent liquidates (all) units at a (small) gain than a (large) loss. Indeed, Odean (1998) shows that the disposition effect remains strong even when the sample is limited to sales of investor's *entire* holdings of stock.

In addition to the optimal stopping model of Kyle, Ou-Yang and Xiong (2006), the papers of Kaustia (2008) and Barberis and Xiong (2008b) also treat stopping or liquidation problems for investors with prospect preferences. However, a stopping model in which agent's voluntarily sell at a loss is missing from the extant literature. As in the Shefrin and Statman (1985) example, the investor in Kaustia (2008) compares the alternatives of holding the position for a fixed time versus liquidating today at various gain/loss levels relative to break-even. As such, Kaustia (2008) gives the investor only one decision and the investor is forced to sell at the fixed horizon. In fact, his model coincides with a one-period version of ours with power S-shaped preferences, and asset prices following exponential Brownian motion. In contrast, our model is continuous-time so the agent faces a continuum of decisions and can choose to liquidate at *any* point in time.[4]

Barberis and Xiong (2008b) explore an optimal stopping model where utility is over realized gains or losses relative to break-even[5], together with a linear utility (not S-shaped) and a positive discount rate.[6] They find behavior in this model to be consistent with a number of observations in financial markets. However, unless forced by a liquidity shock, their investor only sells stock trading at a gain, never at a loss, and thus, as they acknowledge, their model is consistent only with a strong version of the disposition effect.

A number of papers consider the prospect theory investor's portfolio optimization problem and relate their findings to the disposition effect. Hens and Vlcek (2005) study the behavior of an prospect theory investor's optimal position in a risky asset in two-period binomial model. They find ex-post that the model sometimes predicts disposition type behavior. However ex-ante, taking initial purchase into account, this disappears. Barberis and Xiong (2008a) extend to a discrete-time dynamic optimization model and analyze how a prospect theory investor's

[4]Relative to our model, Kaustia's investor will underestimate the value of waiting, as his investor is not given the opportunity to wait beyond the fixed date.

[5]The idea of allowing unrealized gains and losses to affect utility was previously used by Barberis, Huang and Santos (2001).

[6]We comment further in Remark 3 on the differences between our framework and that of Barberis and Xiong (2008b).

optimal holdings vary as the asset price moves over time.[7] They also find that the model often fails to predict a disposition effect. Each of these portfolio optimization models studies the rebalancing of the prospect theory investor's portfolio up to a fixed terminal horizon, and relates this rebalancing to the disposition effect.[8] In contrast, our optimal stopping model (and that of Kyle, Ou-Yang and Xiong (2006)) asks at what time in the future would the prospect theory investor wish to sell the asset? Our extension to partial liquidation allows the agent to choose to sell-off her position over time, if she wishes. Since the disposition effect concerns the observation that investors, executives and managers choose to sell at a gain more readily than at a loss, an optimal stopping model is capturing both the sale and the timing elements.

Optimal stopping problems are usually formulated as free boundary problems via variational arguments. In most cases, the free boundary problem can be solved with the smooth-fit or smooth-pasting principle.[9] However, sometimes, smooth-fit does not apply, and this is the case here due to the non-differentiability of the utility function. Kyle, Ou-Yang and Xiong (2006) take a variational approach and lack of smooth-fit means that the problem becomes quite involved. Instead, we use a direct approach originating in the work of Dynkin (1965) and Dynkin and Yushkevich (1969).[10] Our approach is not dependent on smooth-fit, and thus is far more tractable. This is important as it allows us to investigate the robustness of our results, first, to the precise specification of prospect theory, and second, to the model for the price process. In particular, Kyle, Ou-Yang and Xiong (2006) comment that "it would be ideal to employ the power functions as suggested by Kahneman and Tversky (1979) but the agent's problem becomes intractable to solve in this case. Nevertheless we believe that our results based on the exponential functions are still robust" (p278). A key finding of our analysis is that the behavior is not robust to such a change in the S-shaped function, and in fact, the properties of the function at the reference level are crucial. An important contribution of our paper is to generalize to the partial liquidation problem. Again, properties of the function at the reference level are important - in particular, the important role of the break-even level in the piecewise exponential specification of the indivisible liquidation model does not extend to the divisible model.

1 General Framework

Let Y_t denote the asset price. We work on a filtration $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ supporting a Brownian motion $W = \{W_t, t \geq 0\}$ and assume Y_t follows a time-homogeneous diffusion process with state

[7] See also Berkelaar, Kouwenberg and Post (2004) for a continuous-time analog.

[8] However, consideration of the same portfolio optimization problem with standard CRRA preferences (cf. Samuelson (1969), Merton (1969)) alone gives qualitatively similar rebalancing. (As a CRRA investor keeps a fixed proportion of wealth in the risky asset, she may reduce the quantity held after a price increase, and increase quantity after a price fall, although for some parameters, the reverse occurs). Any prospect theory based portfolio optimization model needs to distinguish their findings from such *wealth* effects obtainable in non-prospect theory models.

[9] See Oksendal (2005) or Shiryaev (1978) for expositions on optimal stopping.

[10] Direct methods for optimal stopping problems are used and developed by Dayanik and Karatzas (2003), and also Carmona and Dayanik (2007) in a multiple stopping model of swing options in energy markets.

space $\mathcal{I} \subseteq \mathbb{R}$ and

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \quad Y_0 = y$$

with Borel functions $\mu : \mathcal{I} \to \mathbb{R}$ and $\sigma : \mathcal{I} \to (0, \infty)$. We assume $\mathcal{I}$ is an interval with endpoints $-\infty \leq a_{\mathcal{I}} < b_{\mathcal{I}} \leq \infty$ and that Y is regular in $(a_{\mathcal{I}}, b_{\mathcal{I}})$.[11] Set $\tau^Y_{(a,b)} = \inf\{u : Y_u \notin (a, b)\}$, ie. the first time that Y leaves the interval (a, b). We will make use of the *scale function* $s(.)$ of the diffusion Y_t.[12] We recall the following definition (see Revuz and Yor (1999)):

Definition 1 *A locally bounded Borel function s is a scale function if and only if the process $s(Y_{t\wedge\tau^Y_{(a_{\mathcal{I}},b_{\mathcal{I}})}}); t \geq 0$ is a local martingale. Furthermore, for arbitrary but fixed $c \in \mathcal{I}$, we have $s(y) = \int_c^y \exp\left(-\int_c^x \frac{2\mu(z)}{\sigma^2(z)}dz\right)dx;\ y \in \mathcal{I}$. The function $s(y)$ is real-valued, strictly increasing, and continuous on $\mathcal{I}$. Finally, we have $\mathcal{A}s(.) = 0$ where the second order differential operator*

$$\mathcal{A}u(y) := \frac{1}{2}\sigma^2(y)\frac{d^2u}{dy^2}(y) + \mu(y)\frac{du}{dy}(y), \ \ on\ \mathcal{I}$$

is the infinitesimal generator of Y.

We consider two main examples for the dynamics of Y - exponential Brownian motion, and Brownian motion. Exponential Brownian motion is by far the most popular choice in finance (preventing financial asset prices from becoming negative); and our main motivation for also studying the Brownian motion case is to facilitate comparison with the results of Kyle, Ou-Yang and Xiong (2006). Assuming Y follows a Brownian motion may be more applicable when considering project payoffs which can be negative.

If the dynamics of Y are exponential Brownian motion, write $dY = Y(\mu dt + \sigma dW)$ for constants μ and $\sigma > 0$. The state space is $\mathcal{I} = (0, \infty)$. Define the constant parameter $\beta = 1 - 2\mu/\sigma^2$. This parameter involves the return-for-risk-per-unit-variance μ/σ^2 (by slight abuse of terminology we refer to this as the Sharpe ratio) and thus reflects the expected performance of the asset. (If $\beta < 0$ then Y_t grows to ∞ whereas, if $\beta > 0$, then Y_t tends to zero, almost surely.) The scale function is $s(y) = y^\beta$ if $\beta > 0$, $s(y) = -(y)^\beta$ if $\beta < 0$ and $s(y) = \ln(y)$ if $\beta = 0$.[13][14] If Y follows Brownian motion, we write $dY = \mu dt + \sigma dW$, again for constants μ and $\sigma > 0$. Here $\mathcal{I} = (-\infty, \infty)$. The scale function is given by $s(y) = -e^{-\frac{2\mu}{\sigma^2}y}$ if $\mu > 0$, $s(y) = e^{-\frac{2\mu}{\sigma^2}y}$ if $\mu < 0$ and $s(y) = y$ if $\mu = 0$.

For simplicity, we assume a zero interest or discount rate throughout. This also aids our direct comparison to results of Kyle, Ou-Yang and Xiong (2006) (and Barberis and Xiong (2008a)) who

[11] We assume that if Y can reach the endpoints of $\mathcal{I}$ then the boundaries are absorbing.

[12] We assume $\mu(.)$ and $\sigma(.)$ are sufficiently regular so there exists a weak solution to the SDE and for the scale function $s(.)$ to exist, see Revuz and Yor (1999).

[13] The scale function is determined up to linear multiplication (ie. if s is a scale function then so is $A + Bs$ for constants A and B). We choose a convenient version.

[14] In fact, we can easily cover the case where Y is a Bessel process of dimension $2 - \delta$, $\delta > 0$, Y hits zero with probability one, and zero is an absorbing boundary. Then Y follows $dY = dW - (\delta - 1)/2Y dt; X_0 = x > 0$ and the scale function is given by $s(y) = y^\delta$. Hence results for this process are identical to those for exponential Brownian motion with β replaced by δ. Since the process in natural scale is a CEV process, this means the results apply also to a driftless CEV process.

again have zero interest rates. In contrast, in Barberis and Xiong (2008b), a positive discount rate is important in giving the investor an incentive to realize gains today (and thus key in their paper in leading to a strong disposition effect). We want to abstract from such an incentive and concentrate on the implications of the presence of the S-shaped function and loss aversion.

The agent has $n \geq 1$ units of claim on the asset Y, each individual unit has payoff $h(Y)$, where h is a non-decreasing function. The simplest situation, and the one we focus on, is when $h(y) = y$ so the agent holds n units of the asset itself. This would be appropriate for the payout from a project, or a sale of stock, and in the situation $n = 1$, a house sale. The agent can liquidate or sell her position in the asset at any time of her choosing in the future. She is able to liquidate the units of asset at different times, if she wishes, so the asset position is (finitely or partially) divisible. Initially, we will concentrate on a special case whereby the agent is only able to sell or liquidate her entire position - the position is indivisible - to facilitate comparison with the existing literature and in particular, Kyle, Ou-Yang and Xiong (2006), who only consider this case. We return to the extension to partial sales in Section 3.

The agent has prospect theory preferences denoted by the function $U(z)$.[15] We consider two alternate specifications. The first specification is that of Tversky and Kahneman (1992) whereby power functions are used to build the S-shape. Let

$$U(z) = \begin{cases} z^{\alpha_1} & z \geq 0 \\ -\lambda(-z)^{\alpha_2} & z < 0 \end{cases} \tag{1}$$

where $\alpha_1, \alpha_2 \in (0, 1)$. Here, the parameter $\lambda > 1$ governs loss aversion and introduces an asymmetry about the origin. Note also that this specification of prospect theory has locally infinite risk aversion, $U'(0-) = U'(0+) = \infty$. Tversky and Kahneman (1992) find experimentally values of $\alpha_1 = \alpha_2 = 0.88$, and $\lambda = 2.25$.

Our second specification, as used by Kyle, Ou-Yang and Xiong (2006) and Hens and Vlcek (2005), amongst others, builds the S-shape from exponential functions. Let

$$U(z) = \begin{cases} \phi_1(1 - e^{-\gamma_1 z}) & z \geq 0 \\ \phi_2(e^{\gamma_2 z} - 1) & z < 0 \end{cases} \tag{2}$$

where $\phi_1, \phi_2, \gamma_1, \gamma_2 > 0$. The assumption $\phi_1\gamma_1 < \phi_2\gamma_2$ corresponds to ensuring loss aversion, $U'(0-) > U'(0+)$ (where now both limits are finite) so the agent is more sensitive to losses than gains around the reference point. Figure 1 displays both functions (1) and (2) for particular parameter values.

In the situation where the agent chooses when to liquidate her entire position, it is equivalent to consider a single unit of asset, so take $n = 1$, until Section 3. At the liquidation time τ of her choosing, the agent receives the payoff $h(Y_\tau)$, and compares this to her reference level, denoted

[15] In Kahneman and Tversky (1979) and much of the prospect theory literature, the function $U(z)$ is called the value function. This conflicts with the standard useage in the stochastic control literature of the terminology value function to refer to the expected payoff under optimal behavior, which we instead call the "value of the game". For this reason, we will refer to U as the S-shaped function or utility function, and avoid using the terminology of value function altogether.

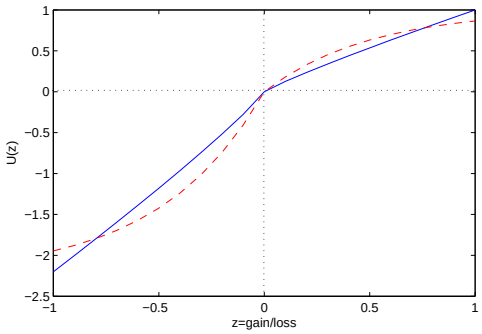


Figure 1: The Tversky and Kahneman (1992) S-shaped function $U(z)$ given in (1) with loss aversion $\lambda = 2.25$ and $\alpha_1 = \alpha_2 = 0.88$ is given by the solid line. The piecewise exponential S-shaped function $U(z)$ given in (2) with $\phi_1 = 1, \phi_2 = 2.25, \gamma_1 = \gamma_2 = 2$ is given by the dashed line.

by h_R, where $h_R > 0$. As is often the case in the literature, an appropriate interpretation of h_R is the break-even level, or equivalently, the amount (or price) the agent paid for the claim or asset itself, and we will assume this is the case throughout. For example, in the case of a stock or house, this is the price paid, and at the sale time, the agent compares the price received with the price paid. Similarly, the agent would compare the initial outlay with the eventual proceeds of a project.[16] Thus, the agent's objective is written as[17]

$$V_1(y) = \sup_{\tau} \mathbb{E}[U(h(Y_\tau) - h_R)|Y_0 = y], \quad y \in \mathcal{I} \tag{3}$$

where $U(.)$ is an increasing function.[18] (Whilst we have the two specifications of prospect theory preferences in (1) and (2) in mind, our approach works for general functions.) We will primarily be interested in the situation where the agent has an asset to sell, so that $h(Y_\tau) = Y_\tau$ and in this case we denote the corresponding reference level by $h_R = y_R$, assumed to be the price paid for the asset, or break-even level.

Remark 2 *Kyle, Ou-Yang and Xiong (2006) (also Barberis and Xiong (2008b)) include an additional exogenous stopping time at which the project is liquidated automatically, which can be interpreted in terms of exogenous liquidity needs. In fact, the inclusion of exogenous liquidity shocks in their model gives an additional incentive for the agent to realize gains because of the*

[16] Kyle, Ou-Yang and Xiong (2006) include accumulated costs K_t where $dK_t = cdt$ and compare the price received upon liquidation to the accumulated costs. As all of their results hold for the case $c = 0$, we compare to this case for simplicity. Alternatively, we might instead re-interpret our price Y to be the net profit of the project.

[17] The supremum is taken over the class of all $\{\mathcal{F}_t\}$-stopping times.

[18] We concentrate in this paper on the impact of the S-shaped preferences and loss aversion on liquidation decisions. In common with most other theoretical models (one exception is the portfolio choice model of Jin and Zhou (2008)) we do not incorporate the probability weighting aspect of prospect theory. This feature over-weights low probabilities of large gains, which has most effect when assets have highly skewed returns (see Barberis and Huang (2008)).

risk she may be forced to realize losses if an exogenous default occurs. In a similar vein, Kaustia (2008) remarks that "an exogenous factor such as a liquidity shock is needed to induce a sale. A changing perception over risk as a function of return is generally not sufficient". We do not include an exogenous liquidity shock because we want to demonstrate that in our continuous-time optimal stopping model an exogenous factor is not necessary to induce sales, and we do not want to skew the conclusions of our model by building in an extra incentive to liquidate. Finally, although the shocks complicate the calculations in Kyle, Ou-Yang and Xiong (2006), they are not needed as their (and our) problem has a well-defined solution for meaningful ranges of parameter values. Our comparison to their results takes their arrival rate $\lambda = 0$.

Remark 3 *It is worth clarifying here some similarities and differences with regard to the 'realization utility' set-up of Barberis and Xiong (2008b). Barberis and Xiong (2008b) consider a model where their investor receives "a jolt of utility right then, at the moment of sale, and where the utility depends on the size of the gain or loss realized". Note that in the case of an indivisible liquidation problem (Kyle, Ou-Yang and Xiong (2006) and our problem in (3)), these formulations collapse to become the same (modulo other significant differences in choice of functions, discounting, etc). In fact, realizing the utility (of wealth) at the time of sale is quite natural in optimal sale or exercise problems (see Henderson (2007), Carpenter, Stanton and Wallace (2008), amongst others). The novel feature of Barberis and Xiong's model is the combination of the realization utility and that gains and losses are measured relative to a reference level - and when this becomes substantially different to Kyle, Ou-Yang and Xiong (2006) and (3), is in the extension to many sales. We will comment further in Section 3. Barberis and Xiong (2008b) are emphasizing realization utility, together with linear (or piecewise linear) utility and a positive discount rate can give results consistent with various observed behavior in financial markets, including a strong version of the disposition effect.*

We now develop the general form of the solution to the agent's problem in (3), which will enable us to then consider the particular examples of price processes and functions $U(.)$ we have introduced above. Given the time-homogeneity of the problem, the structure of the solution must be to stop when the price process Y exits an interval. (A classic application in finance where this occurs is the exercise of a perpetual call or put option (Samuelson (1965)) and applied in the real options literature, eg. McDonald and Siegel (1986) and Dixit and Pindyck (1994).) Thus the approach is to consider stopping times of this form (the proof will show that this indeed gives the optimal solution) and thus choose the "best" interval. Recall the notation $\tau^Y_{(a,b)} = \inf\{u : Y_u \notin (a,b)\}$.

The key to simplifying the problem is to transform into natural scale via $\Theta_t = s(Y_t)$. Let $\Theta_0 = \theta_0 = s(y_0)$. Recall from Definition 1 that the scale function $s(.)$ is such that the transformed price Θ_t is a (local) martingale. This will allow us to transform the original stopping problem into one with a different reward function but for a martingale process. This problem is easier and more transparent to solve. Once we solve this problem, we can then transform the solution back to the original co-ordinates.

Note first that we can map exit times of price Y from an interval to exit times of Θ from a transformed interval, ie. $\tau^Y_{(a,b)} := \inf\{u : Y_u \notin (a,b)\} \equiv \inf\{u : \Theta_u \notin (s(a), s(b))\} = \inf\{u : \Theta_u \notin (\phi,\psi)\} := \tau^\Theta_{(\phi,\psi)}$, where we define the transformed interval by $\phi = s(a)$, $\psi = s(b)$. Define the function $f(.)$ via $f(y) = U(h(y) - h_R)$. where $U(.)$ is an increasing function. (The function $f(.)$ is introduced to simplify future notation and to emphasize the dependence of the reward on the price Y). Note since U and h are increasing, f is also increasing in y. Now define the function $g_1(.)$ by

$$g_1(\theta) = f(s^{-1}(\theta)) := U(h(s^{-1}(\theta)) - h_R) \tag{4}$$

and note g_1 is necessarily increasing in θ. This function represents the value of the game if the unit of claim is sold immediately, expressed in terms of the transformed price Θ.

Then, for *any* fixed interval $(a,b) \in \mathcal{I}$ such that $(s(a), s(b))$ is a bounded interval,

$$\begin{aligned}\mathbb{E}[U(h(Y_{\tau^Y_{(a,b)}}) - h_R)|Y_0 = y] &= \mathbb{E}[f(Y_{\tau^Y_{(a,b)}})|Y_0 = y] = \mathbb{E}[f(s^{-1}(\Theta_{\tau^\Theta_{(\phi,\psi)}}))|\Theta_0 = \theta] \\ &= \mathbb{E}[g_1(\Theta_{\tau^\Theta_{(\phi,\psi)}})|\Theta_0 = \theta] = g_1(\phi)\frac{\psi-\theta}{\psi-\phi} + g_1(\psi)\frac{\theta-\phi}{\psi-\phi}\end{aligned}$$

where we use the probabilities of the (true) martingale Θ_t hitting each end of the interval.

The final step is to choose the "best" interval (ϕ,ψ):

$$\sup_{\phi<\theta<\psi}\left\{g_1(\phi)\frac{\psi-\theta}{\psi-\phi} + g_1(\psi)\frac{\theta-\phi}{\psi-\phi}\right\} = \bar{g}_1(\theta) \tag{5}$$

to which the solution is given by taking the smallest concave majorant $\bar{g}_1(\theta)$ of the function $g_1(\theta)$.

Figure 2 gives a stylized representation of $g_1(\theta)$ as a function of θ. We can use the graph to explain intuitively that the solution to the "best" interval problem above is indeed the smallest concave majorant. We want to choose endpoints ϕ, ψ to maximize the expression in brackets in (5). If we start at the point θ_A on the graph, then the expression in the curly brackets in (5) is maximized by taking $\phi = \psi = \theta_A$ (all other endpoints give values beneath $g_1(\theta_A)$). This corresponds to immediate stopping (recall we stop when we exit the interval (ϕ,ψ)). However, if we start at the point θ_B, the quantity in brackets is maximized if we take $\phi = \phi_B$ and $\psi = \psi_B$. In fact, for any starting point in the interval (ϕ_B, ψ_B), the endpoints ϕ_B, ψ_B are best. Thus, for any $\theta \in (\phi_B, \psi_B)$, the solution is to stop when the transformed price Θ_t reaches either endpoint of the interval. Outside the interval (ϕ_B, ψ_B), the solution is to stop immediately. The solution is to take the smallest concave majorant, which is equal to the function g_1 itself for θ outside the interval (ϕ_B, ψ_B) and the dashed straight line joining the endpoints for values of θ inside the interval. This intuition lies behind the following result, the proof of which is given in an Appendix.

Proposition 4 *On the interval $(s(a_\mathcal{I}), s(b_\mathcal{I}))$, let $\bar{g}_1(\theta)$ be the smallest concave majorant of $g_1(\theta) := f(s^{-1}(\theta))$.*

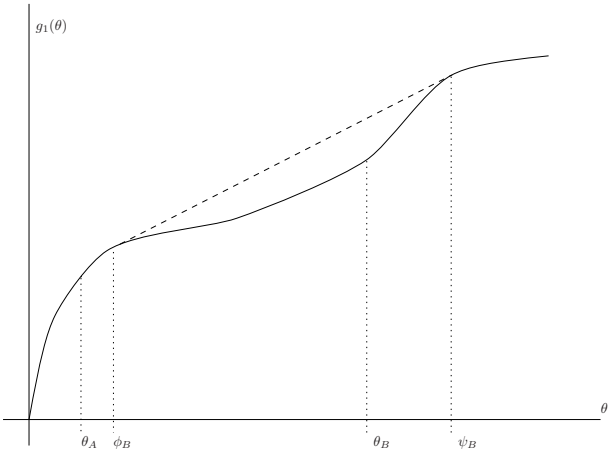


Figure 2: Stylized representation of the function $g_1(\theta)$ as a function of transformed price θ, where $\theta = s(y)$. The function $g_1(\theta)$ represents the value of the game to the agent if she sells immediately. The smallest concave majorant $\bar{g}_1(\theta)$ is formed by taking the straight dashed line for $\theta \in (\phi_B, \psi_B)$ and the function g_1 itself for $\theta \leq \phi_B$ and $\theta \geq \psi_B$.

(i) Suppose $s(a_{\mathcal{I}}) = -\infty$. *Then* $V_1(y) = f(b_I) = U(h(b_{\mathcal{I}}) - h_R)$; $\quad y \in (a_{\mathcal{I}}, b_{\mathcal{I}})$.
(ii) Suppose $s(a_{\mathcal{I}}) > -\infty$. *Then*

$$V_1(y) = \bar{g}_1(s(y)); \quad y \in (a_{\mathcal{I}}, b_{\mathcal{I}})$$

This approach allows us to study a variety of examples for the price process and the form of the S-shaped utility function. We see that the solution will depend purely on the shape of the function g_1 (which determines its smallest concave majorant) which in turn depends only on the scale function of the diffusion (ie. the choice of price dynamics) and the form of U itself.

2 Two Alternative Specifications of Prospect Theory

2.1 Piecewise Exponential functions

In this section we develop results for the model where the asset price follows Brownian motion, prospect preferences are constructed via piecewise exponentials, and the agent is choosing when to sell an indivisible asset. This is a special case of the model of Kyle, Ou-Yang and Xiong (2006), and we will compare our results to theirs. Depending on the parameter values describing the asset price, and the preferences, we find four distinct possible liquidation behaviors.

Proposition 5 *The solution to problem (3) with* $h(y) = y$, *when the asset price* Y *follows Brownian motion and* $U(z)$ *is given by piecewise exponential* S*-shape in (2) consists of four cases, depending on relative parameter values:*
(I): If $\mu \geq 0$, *the agent waits indefinitely (see Figure 3(a) and 3(b)).*

(II) If $\mu < 0$ and $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$ and $|\mu|/\sigma^2 < \frac{1}{2}\frac{\phi_1}{\phi_2}\gamma_1$, the agent stops at and above a level $\bar{y}_u^{(1)} > y_R$ which is itself greater than the break-even point. That is, the agent waits until the price is higher than the break-even level before selling, and sells at a gain. The agent waits below the level $\bar{y}_u^{(1)}$ (see Figure 3(c)). The level $\bar{y}_u^{(1)}$ is given by

$$\bar{y}_u^{(1)} = y_R - \frac{1}{\gamma_1} \ln\left(\left(\frac{2\mu}{2\mu - \gamma_1\sigma^2}\right)\left(\frac{\phi_1 + \phi_2}{\phi_1}\right)\right) \tag{6}$$

(III) If $\mu < 0$ and $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$ and $|\mu|/\sigma^2 \geq \frac{1}{2}\frac{\phi_1}{\phi_2}\gamma_1$, the agent stops everywhere at and above the break-even point y_R, but waits below the break-even point. Thus if the agent sells, she exactly breaks even (see Figure 3(d)).
(IV) If $\mu/\sigma^2 \leq -\frac{1}{2}\gamma_2$, the agent sells immediately at all price levels (see Figure 3(e)).

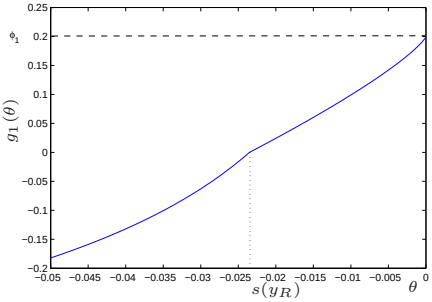
As we would anticipate, the factors influencing the optimal behavior are the Sharpe ratio of the asset (or more precisely, the return-for-risk-per-unit-variance) and the degree of loss aversion. Cases (I) and (IV) are the more extreme situations, as the expected return on the asset is either extremely high or very poor. If it is extremely high, it is optimal to continue indefinitely and never liquidate the position, regardless of where the price is relative to the break-even point, and regardless of loss aversion, see Figures 3(a) and 3(b). Conversely, if the asset's Sharpe ratio is sufficiently poor, it is optimal to liquidate immediately, again, regardless of where the price is in relation to the break-even point, and regardless of the degree of loss aversion, see Figure 3(e). Note in this case, given our interpretation of the reference level as the break-even level or price paid, liquidation will only actually occur at the break-even level. (If the current price is different to the break-even level, the agent would liquidate. However, the agent would not be in this position, because she would have liquidated immediately on receipt of the asset, when the price was y_R. Effectively, she never holds the asset.).

In cases (II) and (III) we compare the Sharpe ratio with the agent's loss aversion. In case (III), the agent will wait when the price is below the break-even level and will liquidate at the break-even level itself. This is displayed in Figure 3(d). In this situation, the loss aversion coupled with the weak Sharpe ratio causes the agent to be unwilling to wait for gains beyond just breaking even. (Note again given our interpretation of the reference level as the break-even level or price paid, the agent liquidates immediately at break-even, and thus, again, effectively does not hold the asset ex-ante.)

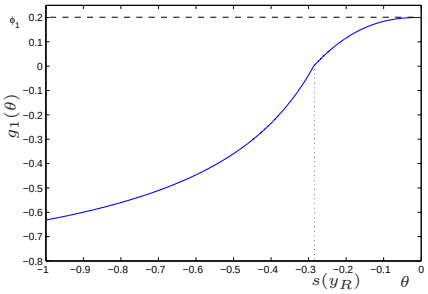
The most interesting situation is case (II), where the Sharpe ratio is better than in (III). The agent is willing to wait beyond the break-even point and will not liquidate until the position is in gains (see Figure 3(c)). We can also characterize the price at which the agent sells, given in (6).

We compare our results to those of Kyle, Ou-Yang and Xiong (2006).[19]. We see that Kyle, Ou-Yang and Xiong (2006) obtain our cases (I), (III) and (IV), but do not obtain a situation

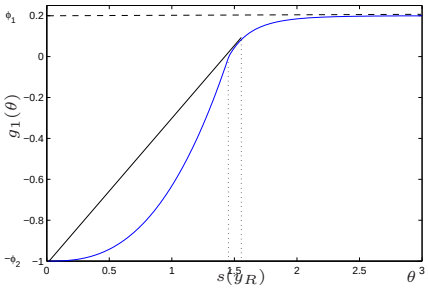
[19]Recall, our models are the same apart from the exogenous stopping and cost parameters, both of which we set to zero for comparison.



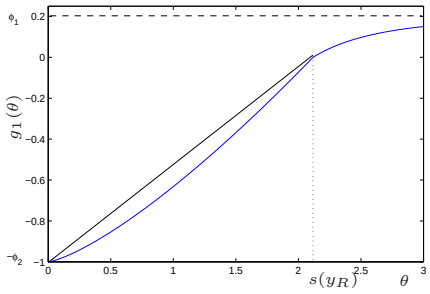
(a) (I). $\mu = 0.3$, $s(y_R) = -0.0235$. Wait at all values of θ; equivalently at all prices y.



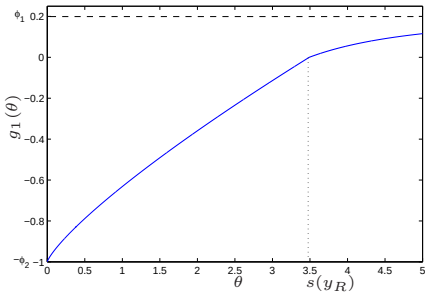
(b) (I). $\mu = 0.1$, $s(y_R) = -0.2865$. Wait at all values of θ; equivalently at all prices y.



(c) (II). $\mu = -0.03$, $s(y_R) = 1.455$. Stop for $\theta > 1.54$ (where $s(y_R) < 1.54$); or equivalently, for prices $y > s^{-1}(1.54) = 1.15 > y_R = 1$.



(d) (III). $\mu = -0.06$, $s(y_R) = 2.12$. Stop for $\theta \geq s(y_R)$, or equivalently, for prices $y \geq y_R = 1$.



(e) (IV). $\mu = -0.1$, $s(y_R) = 3.49$. Stop immediately for all θ and equivalently, all y.

Figure 3: Optimal Liquidation of an Indivisible Asset under Exponential S-shaped utility and Brownian motion price process. Each panel plots the function $g_1(\theta)$ against the transformed price θ. The transformed reference level $s(y_R)$ is given in each panel and is marked with a dotted vertical line. In panels (c), (d) and (e), the smallest concave majorant $\bar{g}_1(\theta)$ is plotted. In (d), the liquidation threshold is marked by a dotted line to the right of $s(y_R)$. (However, in (e), $\bar{g}_1(\theta) = g_1(\theta)$ and in (a) and (b), $\bar{g}_1(\theta) = \phi_1$.) Recall, the agent sells for transformed prices θ such that $\bar{g}_1(\theta) = g_1(\theta)$ and waits for θ such that $\bar{g}_1(\theta) > g_1(\theta)$. Common parameters are: $\sigma = 0.4$, $\phi_1 = 0.2$, $\phi_2 = 1$, $\gamma_1 = 3$, $\gamma_2 = 1$ and reference level, $y_R = 1$.

analogous to our case (II).[20] In their paper, they do not have a situation where the agent liquidates at a level higher than the break-even point (at a gain). Closer examination of their model reveals that they have ruled out this case by their assumptions on parameter values, perhaps because it assisted in the ease of their calculations. Since our method is more tractable, we do not need to rely on any additional assumptions and can obtain all cases.

We comment now on the applicability of this parameterization of prospect theory to the disposition effect. In cases (II), (III) and (IV), the asset has negative expected return (since we have zero interest rates, this can be interpreted as being a worse expected performance than a riskless investment). Case (II) is the "best" of these, where the expected return is negative, but smallest in magnitude. Despite the relatively poor expected return, the agent in case (II) would want to enter the game in the first place, and waits to liquidate at a gain. The fact that the agent would buy or take on this position ex-ante can be seen by looking at the value of the game at the reference level (as we assume the reference level is the break-even or the price paid) $V_1(y_R) = \bar{g}_1(s(y_R))$ which is positive.[21] Thus the agent with exponential S-shaped preferences would want to have the opportunity to gamble on what appears to be a relatively poor asset, and in doing so, will wait to liquidate at a gain.

Although we have found that the exponential S-shaped preferences can lead to situations where the agent will liquidate at a gain relative to break-even, the agent never liquidates at a loss. Given the disposition effect says gains are realized *more often* than losses, this model only captures this in an extreme sense where losses are *never* actually taken. The piecewise exponential specification of prospect theory shares this shortcoming of a lack of (voluntary) loss realization with other theoretical models including Barberis and Xiong (2008b) and Kaustia (2008). These papers also describe models whereby the agents do not liquidate at a loss relative to break-even, and as Barberis and Xiong (2008b) acknowledge, are only consistent with a "strong" disposition effect.

Rather, in this specification, if the asset is very poor (cases (III) and (IV)), the agent sells at the break-even level, and hence, effectively, did not want to hold the asset in the first place. (Related remarks have been made albeit in very different styles of models (perhaps with different specifications of prospect theory) concerning whether the agent would hold a poor asset ex-ante (see for instance, Hens and Vlcek (2005), Barberis and Xiong (2008a), and Kaustia (2008)). Recall also the example of Shefrin and Statman (1985), which didn't consider the possibility that the asset would not be held in the first place and instead says in scenarios with poor odds,

[20] More specifically, cases (I) and (IV) correspond to Kyle, Ou-Yang and Xiong (2006)'s cases *I, II* and *IV*. Note that our case (I)(a) splits into two cases when exogenous stopping is introduced, giving Kyle, Ou-Yang and Xiong (2006) cases *I* and *II*. This occurs because waiting indefinitely is not possible once there is exogenous stopping. Our case (I)(b) is absorbed into other cases in their model with exogenous stopping, again, because waiting indefinitely is not possible. Finally, and most importantly, are our cases (II) and (III). Our case (III) corresponds to their case *III*. However, they rule out our case (II) (where $\mu/\sigma^2 < \frac{1}{2}\gamma_1$) by their assumption in equation (22). This says that for parameters $\mu/\sigma^2 = \frac{1}{2}\gamma_1$, (and thus for situations $-\frac{1}{2}\gamma_2 < \mu/\sigma^2 < \frac{1}{2}\gamma_1$ where the asset is less favorable) stopping immediately at the break-even level is preferred to never liquidating voluntarily.

[21] On Figure 3(c), $s(y_R) = 1.455$, and in original co-ordinates, $y_R = 1$. At $s(y_R)$, $g_1(s(y_R)) = 0$ but $\bar{g}_1(s(y_R)) > 0$ since for transformed prices θ in the waiting region, $\bar{g}_1(\theta) > g_1(\theta)$.

the agent will "give-up" at a loss. Here, under exponential S-shaped functions, we find two possible behaviors if an asset has a poor Sharpe ratio. Either it is very poor, and the agent doesn't hold it ex-ante, or, more surprisingly, it is poor, but the agent does hold it ex-ante and gambles on liquidating at a gain versus losing everything.

We will return to this parameterization in Section 3.1 where we extend to consider the partial liquidation problem. It will be of interest to consider how these results are altered by the ability to partially liquidate.

2.2 Piecewise Power functions

In this section, we consider the asset sale problem with exponential Brownian motion asset prices, and power S-shaped preferences, given by (1), and suggested by Tversky and Kahneman (1992).

Proposition 6 *The solution to problem (3) with $h(y) = y$, when the asset price Y follows Exponential Brownian motion and $U(z)$ is given by piecewise power S-shape in (1) consists of three cases, depending on relative parameter values. Recall the notation $\beta = 1 - \frac{2\mu}{\sigma^2}$, so β is related to the assets' Sharpe ratio (or risk-reward trade-off).*
(I): If $\beta \leq 0$; or if $0 < \beta < \alpha_1 < 1$, the agent waits indefinitely and never liquidates (see Figure 4(a) and 4(b)).
(II) If $0 < \alpha_1 < \beta \leq 1$ or $\alpha_1 = \beta < 1$, the agent stops at a level higher than the break-even point. That is, the agent waits beyond the break-even point before liquidating. Thus, if the agent liquidates, she does so at a gain (see Figure 4(c)).
(III) If $\beta > 1$, the agent stops when the price reaches either of two levels. These two levels are on either side of the break-even point. Hence the agent can liquidate either at a gain or at a loss (see Figure 4(d)).

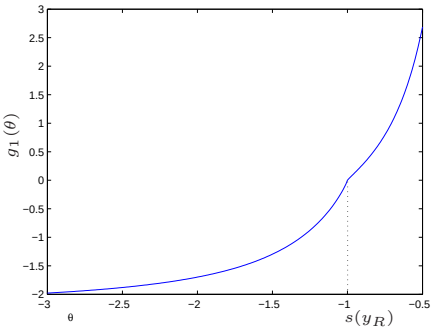
When $\alpha_2 = \alpha_1$, we can give an explicit representation of the selling thresholds in cases (II) and (III) of Proposition 6 and thus determine their behavior with respect to underlying variables.

Proposition 7 *For $\beta > 1$ (case (III) of Proposition 6), there are two selling thresholds $\bar{y}_u^{(1)} > \bar{y}_l^{(1)}$ either side of the break-even point, hence with $\bar{y}_u^{(1)} > y_R$ and $\bar{y}_l^{(1)} < y_R$. Under the additional assumption that $\alpha_2 = \alpha_1$, the thresholds can be rewritten as:*
$\bar{y}_u^{(1)} = \bar{c}_u y_R$ and $\bar{y}_l^{(1)} = \bar{c}_l y_R$ for constants $\bar{c}_l < \bar{c}_u$ with $\bar{c}_l < 1, \bar{c}_u > 1$, where the constants solve the pair of equations:

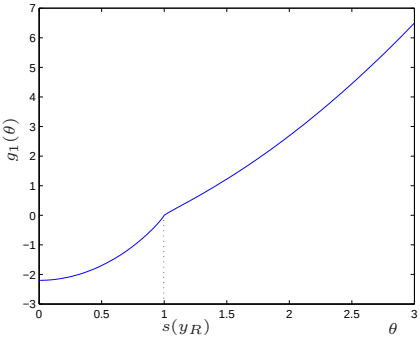
$$\frac{\alpha_1}{\beta}(\bar{c}_u - 1)^{\alpha_1 - 1}\bar{c}_u^{1-\beta} = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda(1 - \bar{c}_l)^{\alpha_1}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}} \tag{7}$$

$$\frac{\lambda\alpha_1}{\beta}(1 - \bar{c}_l)^{\alpha_1 - 1}\bar{c}_l^{1-\beta} = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda(1 - \bar{c}_l)^{\alpha_1}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}}. \tag{8}$$

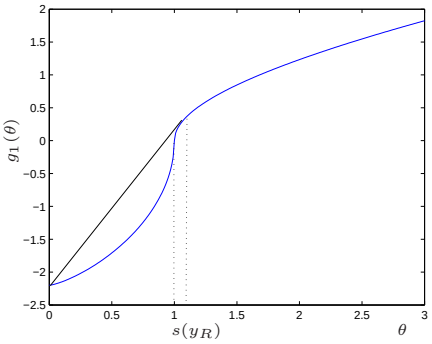
For $0 < \alpha_1 < \beta \leq 1$ (or $\alpha_1 = \beta < 1$) (case (II) of Proposition 6), there is a single selling threshold above the break-even point, hence $\bar{y}_u^{(1)} > y_R$. If $\alpha_2 = \alpha_1$, then $\bar{y}_u^{(1)} = \bar{c}_u y_R$ where $\bar{c}_u$ solves (7) with $\bar{c}_l = 0$.



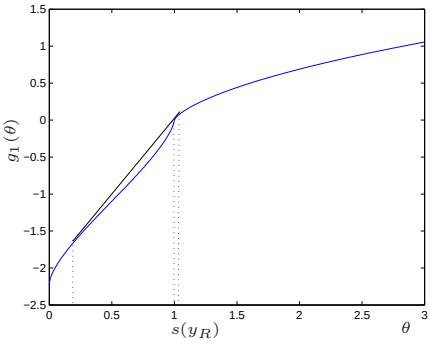
(a) (I). $\beta = -0.5$, $\alpha_1 = 0.9$, $s(y_R) = -1$. Wait for all values of θ and equivalently, for all prices y.



(b) (I). $\beta = 0.5$, $\alpha_1 = 0.9$, $s(y_R) = 1$. Waits for all values of θ and equivalently, for all prices y.



(c) (II). $\beta = 0.75$, $\alpha_1 = 0.5$, $s(y_R) = 1$. Stops for $\theta \geq \bar{\theta}_u^{(1)} = 1.06$ or equivalently for $y \geq= \bar{y}_u^{(1)} = 1.08$.



(d) (III). $\beta = 1.5$, $\alpha_1 = 0.7$, $s(y_R) = 1$. Waits for $\theta \in (\bar{\theta}_l^{(1)} = 0.1723, \bar{\theta}_u^{(1)} = 1.0105)$ and stops otherwise. Equivalently, waits for $y \in (\bar{y}_l^{(1)} = 0.31, \bar{y}_u^{(1)} = 1.007)$.

Figure 4: Optimal Liquidation of an Indivisible Asset under Power S-shaped utility and Exponential Brownian motion price process. Each panel plots the function $g_1(\theta)$ against the scaled price θ. The transformed reference level $s(y_R)$ is given in each panel and is marked with a dotted vertical line. In panels (c) and (d), the smallest concave majorant $\bar{g}_1(\theta)$ is plotted and the liquidation threshold(s) are marked by dotted lines. Recall, the agent sells for θ such that $\bar{g}_1(\theta) = g_1(\theta)$. Common parameters are: $\lambda = 2.2$, $\alpha_2 = \alpha_1$ and reference level $y_R = 1$.

The various optimal behaviors depend on the performance of the asset price itself through the Sharpe ratio (capturing the return-for-risk trade-off) and the coefficient of risk aversion $1-\alpha_1$ in the concave region above zero. In case (I), the asset has a very high Sharpe ratio (in fact the logarithm of asset price is positive in expectation) and regardless of the agent's risk preference level, it is always optimal to wait for a higher price level. This case is displayed in Figure 4(a). Waiting indefinitely is also optimal when the Sharpe ratio is good (the asset price has positive expected return) and the coefficient of risk aversion (over gains) is relatively low so that the agent is willing to wait to take advantage of the expected increased price. See Figure 4(b). In case (II), the risk aversion coefficient is high which induces the agent to sell at a gain, at some threshold level above the break-even point. Although the Sharpe ratio is good (positive expected return), the agent's risk aversion (over gains) acts as a pull against waiting for a higher price, giving a trade-off. The example shown in Figure 4(c) takes $\beta = 0.75$ (equivalently, $\mu = \sigma^2/8$) and $\alpha_1 = 0.5$ since this gives a visible threshold on the figure. If we were to increase α_1, this threshold would move closer to the break-even point, or equivalently, if a gain were realized, it would be smaller. We could also increase β (making the asset a worse proposition) so that we could use $\alpha_1 = 0.88$ as found by Tversky and Kahneman (1992).

In case (III), the Sharpe ratio is very poor (in fact the expected return is negative, relative to a zero riskless rate) and the agent has a relatively high risk aversion (over gains). As in case (II), the agent will liquidate at a threshold price above the break-even point, but additionally here, the agent liquidates at a loss, at a threshold below the break-even point.[22] Liquidation occurs at a loss in this situation because the price is transient to zero almost surely, so, if the price goes down sufficiently from the break-even level, the agent basically "gives up" and liquidates at a loss at a low threshold, rather than waiting for even larger losses.

Some of the qualitative behavior we describe here is quite different to those found by Kyle, Ou-Yang and Xiong (2006) and our re-working of their model with piecewise exponentials in the previous section. This demonstrates the behavior is not robust to changes in specification of the S-shape function and asset price process, and in fact, the properties of the function at the reference level play an important role.[23]

The most important difference we find is that under the Kahneman and Tversky (1992) specification of prospect theory preferences, the agent will liquidate at a loss in some circumstances. As we commented earlier, this has implications for relating prospect theory to the disposition effect, and is in distinct contrast to the findings of other theoretical models of Barberis and Xiong (2008b) and Kaustia (2008). This is also able to capture the intuition of Shefrin and

[22]It is worth observing that the distinction between cases (II) and (III) is solely the performance of the asset itself, and risk aversion parameters do not play a role in whether or not the agent liquidates at a level below the break-even point. This is because once the asset price is very low, the associated changes in wealth (losses relative to break-even) are small, and essentially, over such small changes, utility is approximately linear, and thus the parameters of the S-shaped function do not arise in distinguishing between the cases.

[23]However, we do not present the other combinations of exponential Brownian motion and exponential S-shaped utility, and Brownian motion and power S-shaped utility because our analysis of these show there there are no additional cases which are qualitatively different to those we have presented in our two examples.

Statman (1985) - the agent may "give-up" and accept a loss if the odds are bad. In Figure 4(d), this loss level is at $\bar{y}_l^{(1)} = 0.31$ relative to a break-even level of $y_R = 1$.

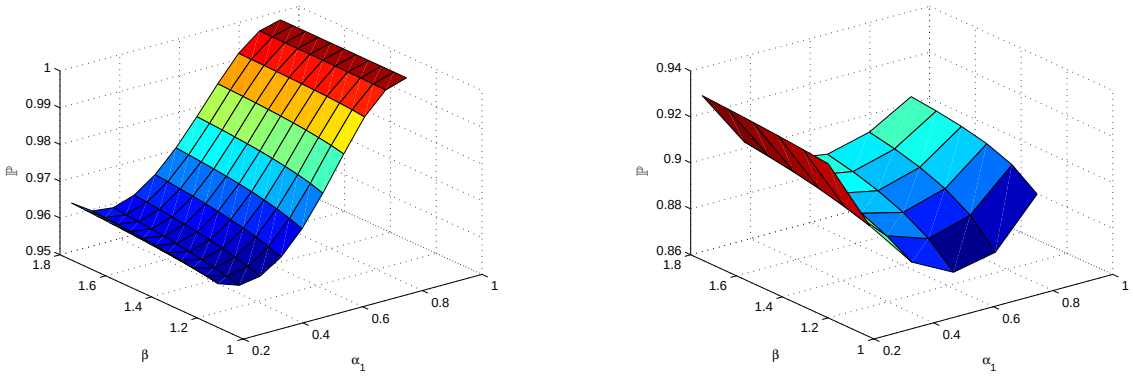
Importantly, under all parameter choices under power S-shaped preferences, the agent would have chosen to hold the asset ex-ante. This is true even in case (III), where the expected return is negative (worse than the riskless asset).[24] Even in this case, the agent is willing to enter the game and gamble on the chance of selling at a gain. In Figure 4(d), the agent would wait until the price either reached the upper threshold of $\bar{y}_u^{(1)}$=1.007 (a very small gain), or the lower threshold of $\bar{y}_l^{(1)} = 0.31$ (a much larger loss). As we commented earlier, the intuition of Shefrin and Statman (1985) does not take into account whether the agent would hold the asset ex-ante, and indeed, many papers have remarked that this consideration would rule out scenarios described in the introduction, where the agent "gives-up" and accepts a loss.

Under the Kahneman and Tversky (1979,1992) power specification, the agent never stops precisely at the break-even point and as a consequence, there is no situation where the agent liquidates immediately. This is due to the locally infinite risk aversion of the power specification at the reference level. However, as we have seen, the solution in Kyle, Ou-Yang and Xiong (2006) under exponential functions is heavily focussed on liquidation at the break-even level, and under these functions, there is a finite derivative at the kink.

The model of Kaustia (2008) is essentially a one-period story of our continuous-time model under piecewise power preferences and exponential Brownian motion price dynamics. They investigate the investor's behavior numerically for a limited range of parameters, excluding situations which would lead to liquidating at a loss, as in case (III). Whilst their analysis shows that for many parameter values, the agent would have a preference for waiting (in their model, waiting until the end of the period), we show under what conditions on parameters the agent will wait, sell at a gain, or sell at a loss.

We already explained that the very existence of a case where the agent sells at a loss relative to break-even distinguishes this model from others attempting to relate to the observed disposition effect. However, of interest is the *relative likelihood* of selling at a gain versus at a loss. The disposition effect says that agents are more likely to sell at a gain than at a loss. To evaluate this likelihood in our model is straightforward. Suppose the agent has paid an amount y_R for the asset and the current price is $y = y_R$. Since we scaled the asset price so that the transformed price Θ_t is a martingale, the probability of reaching the thresholds is obtained purely from the relative distance to the threshold. Observe from Figure 4(d) that the upper threshold (above break-even) is much closer to the break-even level than the lower threshold (below break-even). This means there is a high probability the agent sells at a gain rather than a loss, consistent with the disposition effect. We can evaluate this probability in general. In case (III) where there

[24] As in the exponential specification, this can be seen by noting the value of the game at the reference level is positive.



(a) Loss aversion parameter $\lambda = 2.2$

(b) Loss aversion parameter $\lambda = 1$

Figure 5: Probability of liquidating at a gain in Case (III), as a function of β and α_1, calculated from (9). The reference level is $y_R = 1$ and the current price is $Y_0 = y = y_R = 1$.

are two selling thresholds, the probability the agent sells at a gain is given by [25]

$$\frac{\theta - \theta_l^*}{\theta_u^* - \theta_l^*} = \frac{1 - (\bar{c}_l)^\beta}{\bar{c}_u^\beta - \bar{c}_l^\beta} \tag{9}$$

Figure 5 displays this probability as a function of α_1 and β, and for the case of loss aversion versus no loss aversion. Observe that as loss aversion λ increases, the probability that the agent liquidates at a gain increases. For loss aversion of $\lambda = 2.2$, the probabilities are very high across a range of risk aversion levels $(1 - \alpha_1)$, and the parameter β capturing the Sharpe ratio of the asset, see Figure 5(a). Indeed, for the Kahneman and Tversky (1979,1992) value of $\alpha_1 = 0.88$, we see the probability of selling at a gain is very close to one. Although the probability of liquidating at a gain is lower when there is no loss aversion $\lambda = 1$, it is still relatively high, as shown in Figure 5(b). These observations seem consistent with the disposition effect. Across a wide range of parameters, there is a high probability of liquidation at a gain, and a low, but non-zero probability of liquidating at a loss. These gains tend to be quite small, and the losses, quite large.

3 Partial Liquidation

We want to extend our study to treat the corresponding partial liquidation problem. Divisibility and partial liquidation become particularly important in applications to stock portfolios, executive stock options, and in some real options applications where managerial projects may also be divisible. First we extend the general optimal stopping model and then we use this to study the two examples of exponential and power S-shaped preferences.

[25] In case (II) where there is only a single liquidation threshold (in gains), we can still compare the relative likelihood of selling at a gain relative to at a loss, albeit now the agent does not liquidate at a loss, but waits till the price goes to zero. In this case, the above probability simplifies (as $\bar{c}_l = 0$) to give the expression $\bar{c}_u^{-\beta}$.

The agent holds $n \geq 1$ units of claim with payoff $h(Y)$, and can liquidate units at possibly different times of her choosing, $\tau^n \leq ... \leq \tau^1$. (Our notation is such that the nth remaining unit is liquidated at time τ^n so the final unit is liquidated at τ^1). In this section, we need additional notation to keep track of the agent's wealth (changing due to sales over time) which was unnecessary in the indivisible model. Denote the initial wealth by x. The agent's objective is written as

$$V_n(y,x) = \sup_{\tau^n \leq ... \leq \tau^1} \mathbb{E}\left[U\left(x + \sum_{i=1}^{n} h(Y_{\tau^i}) - nh_R\right)\middle| Y_0 = y\right] \tag{10}$$

The agent compares the total payoff to the total reference level for n units, given by nh_R. The function $V_n(y,x)$ represents the value of the game for an agent with n units of claim, initial wealth x, initial asset price y and n reference levels, one for each unit of claim. As in earlier sections, we will be primarily interested in situations where the agent has assets to sell and thus typically $h(y) = y$, and denote the reference level by $h_R = y_R$. For example, the agent has paid an amount y_R for each of two identical apartments and can choose when to sell each. If she sells one apartment for amount Y_{τ_1} and the other for Y_{τ_2}, she then compares the total (including her initial fortune x) to the total reference level $2y_R$ and applies prospect preferences.

In the earlier Remark 3 we mentioned the realization utility formulation of Barberis and Xiong (2008b). As we commented, their formulation becomes dramatically different to other models in the context of many sales. Their formulation instead sums up various individual "jolts" of utility, whilst our (more standard) formulation in (10) takes the utility of the sum of the relative gains and losses. As we will see, our specification leads to the agent selling on an "all-or-nothing" basis. Whilst we might anticipate the realization utility formulation of Barberis and Xiong (2008b) to lead to partial sales, in fact, this is not the case and their model is separable over different stocks, or equivalently, over different units of the same stock.

Using conditioning, the value of the game for the agent with $n \geq 1$ units remaining can be re-expressed as

$$V_n(y,x) = \sup_{\tau^n} \mathbb{E}[V_{n-1}(Y_{\tau^n}, x + h(Y_{\tau^n}) - h_R)|Y_0 = y]$$

where define $V_0(y,x) = U(x)$. As is usual in dynamic programming, this enables us to build the solution backwards, starting with the $n = 1$ solution. Note that taking $n = 1$ and $x = 0$ in the above will recover the problem in (3) hence $V_1(y,0) \equiv V_1(y)$.

We will use the same transformation of the price Y to the martingale Θ via the scale function, $\Theta_t = s(Y_t)$, with $\Theta_0 = \theta_0 = s(y_0)$. Define $g_n(\theta,x)$ to be the value of the game with n units remaining, initial wealth x and plan to sell one unit immediately. Then

$$g_n(\theta,x) = V_{n-1}(s^{-1}(\theta), x + h(s^{-1}(\theta)) - h_R)$$

Assuming $(s(a), s(b))$ is a bounded interval, we have

$$\begin{aligned} V_n(y,x) &= \sup_{\tau^n} \mathbb{E}[V_{n-1}(Y_{\tau^n}, x + h(Y_{\tau^n}) - h_R)] = \sup_{\tau^n} \mathbb{E}[g_n(\Theta_{\tau^n}, x)] \\ &= \sup_{\phi<\theta<\psi} \left\{ g_n(\phi,x)\frac{\psi-\theta}{\psi-\phi} + g_n(\psi,x)\frac{\theta-\phi}{\psi-\phi} \right\} = \bar{g}_n(\theta,x) \end{aligned}$$

where $\bar{g}_n(\theta, x)$ is the smallest concave majorant of $g_n(\theta, x)$. Hence

$$g_n(\theta, x) = \bar{g}_{n-1}(\theta, x + h(s^{-1}(\theta)) - h_R); \quad n \geq 1$$

which gives an algorithm for obtaining the value of the game for n units from the solution (ie. the smallest concave majorant) of the game with $n-1$ units. This gives the following proposition, the proof of which follows similarly to the $n = 1$ case.

Proposition 8 *On the interval $(s(a_{\mathcal{I}}), s(b_{\mathcal{I}}))$, let $\bar{g}_n(\theta, x)$ be the smallest concave majorant of $g_n(\theta, x) := V_{n-1}(s^{-1}(\theta), x + h(s^{-1}(\theta)) - h_R)$.*
(i) Suppose $s(a_{\mathcal{I}}) = -\infty$. Then $V_n(y, x) = U(x + nh(b_{\mathcal{I}}) - nh_R); \quad y \in (a_{\mathcal{I}}, b_{\mathcal{I}})$.
(ii) Suppose $s(a_{\mathcal{I}}) > -\infty$. Then

$$V_n(y, x) = \bar{g}_n(s(y), x); \quad y \in (a_{\mathcal{I}}, b_{\mathcal{I}})$$

We can use this result to extend the two models in earlier sections to study the partial liquidation problem. Again, we will see that the different specifications of prospect theory lead to qualitatively different behavior. In each model we consider $n = 2$ units of asset. This will enable us to demonstrate the main findings whilst keeping the results fairly tractable.

3.1 Piecewise Exponential Functions

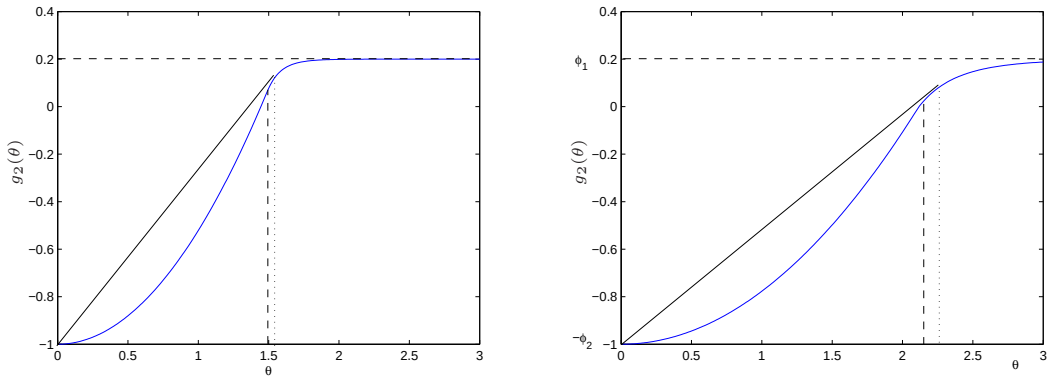
In this section, we extend the indivisible model of Section 2.1 (and Kyle, Ou-Yang and Xiong (2006)) to partial liquidation.

Proposition 9 *The solution to problem (10) with two units of asset when the asset price Y follows Brownian motion and $U(z)$ is given by piecewise exponential S-shape in (2) consists of four cases, depending on relative parameter values:*
(I): If $\mu \geq 0$, the agent waits indefinitely.
(II)/(III): If $\mu < 0$ and $\mu/\sigma^2 > -\frac{1}{2}\gamma_2$, the agent sells both units at and above a level $\bar{y}_u^{(2)}$ which is itself greater than the break-even point, y_R. That is, the agent waits until the price is higher than the break-even level before selling, and sells both units at a gain. The agent waits below the threshold $\bar{y}_u^{(2)}$ (see Figures 6(a) and 6(b)). The threshold level $\bar{y}_u^{(2)}$ is given by

$$\bar{y}_u^{(2)} = \frac{1}{2}(2y_R - x) - \frac{1}{2\gamma_1} \ln\left(\left(\frac{2\mu}{2\mu - 2\gamma_1\sigma^2}\right)\left(\frac{\phi_1 + \phi_2}{\phi_1}\right)\right) \tag{11}$$

(IV) If $\mu/\sigma^2 \leq -\frac{1}{2}\gamma_2$, the agent sells immediately at all price levels.

As we would expect, as in the indivisible model, there are no situations where the agent liquidates at a loss relative to break-even. However, we immediately notice some major differences compared with those of the indivisible setting. One change is that parameters leading to very different behavior in the indivisible model (cases (II) and (III) of Proposition 5) now both lead to the same behavior (labeled case (II)/(III) above). Recall, in case (III) of Proposition 5, the



(a) (II). $\mu = -0.03$, $\sigma = 0.4$, $\hat{\theta}_u^{(2)} = 1.4923$, $\bar{\theta}_u^{(2)} = 1.55$, $s(y_R) = 1.455$ and $\bar{y}_u^{(2)} = 1.173$.

(b) (III). $\mu = -0.06$, $\sigma = 0.4$, $\hat{\theta}_u^{(2)} = 2.117$, $\bar{\theta}_u^{(2)} = 2.23$, $s(y_R - x) = 2.117$ and $\bar{y}_u^{(2)} = 1.067$.

Figure 6: Optimal Liquidation of Two units of Asset under Exponential S-shaped utility and Brownian motion price process. Each panel plots the function $g_2(\theta, x)$ against the scaled price θ; and its smallest concave majorant $\bar{g}_2(\theta, x)$. The vertical dotted lines indicate the liquidation threshold $\bar{\theta}_u^{(2)}$. (The vertical dashed lines represent $\hat{\theta}_u^{(2)}$ at which the definition of g_2 takes a different algebraic form). The ordering of these points implies *both* units are liquidated at the threshold $\bar{\theta}_u^{(2)}$ and the agent waits below this threshold. Since $s(y_R) < \bar{\theta}_u^{(2)}$, the agent sells *both* units at a gain relative to break-even. The equivalent thresholds in terms of the price Y are given as $\bar{y}_u^{(2)}$ and should be compared with y_R. Common parameters are: $x = 0$, $\phi_1 = 0.2, \phi_2 = 1, \gamma_1 = 3, \gamma_2 = 1$, and reference level (per unit) $y_R = 1$.

expected return was very poor and the agent liquidated at the break-even level, effectively, never holding the asset ex-ante. We find in the two-unit case, under the same poor expected return, the agent is willing to hold the position ex-ante, and will gamble on being able to liquidate (both) at a gain. Thus, when the agent is faced with a larger risky stake (here, two units rather than a single unit), she becomes willing to "gamble" over a wider range of (poor) expected returns. Figure 6(b) takes parameters corresponding to case (III) (the same parameters as in Figure 3(d) earlier) and shows the liquidation threshold $\bar{y}_u^{(2)} = 1.067$ above the break-even level of $y_R = 1$.

In order to explain the finding that prospect theory is not invariant to a change of scale, we consider an agent who is offered a one-off gamble and for whom the gamble is marginal. If the agent has standard concave utility, then if she is offered two units of the gamble, she will not play. However, if the agent has a convex utility, she will accept to play a gamble twice the size. Since our agent has prospect theory preferences which are convex below the reference level and concave above it, either of these conclusions may occur. However, since we find that the price at which the agent sells is very close to the reference/break-even level, the majority of the region of interest is where the function is convex.

Allowing for divisibility or multiple units has lead to the break-even level losing the important role it played in the indivisible case. This is particularly important when we recall the emphasis in Kyle, Ou-Yang and Xiong (2006) on liquidation occurring at the reference or break-even level. We have shown that this feature of the indivisible model is not robust to the generalization to divisible assets.

When parameters are slightly better, (corresponding to case (II) in the indivisible model), qualitatively similar behavior occurs. The agent waits to liquidate (both) units at a gain. Figure 6(a) displays the liquidation threshold $\bar{y}_u^{(2)} = 1.173$ for parameters corresponding to case (II) (again, the same as in Figure 3(c)).

The second significant finding is that although we allow the agent to partially liquidate, in fact, the agent chooses to follow an "all-or-nothing" strategy. When the agent liquidates, she always liquidates both units. We will return to discuss this observation further following the results of the piecewise power function specification in the next section.

3.2 Piecewise Power functions

In this section we extend the indivisible liquidation model with power S-shaped functions presented in Section 2.2 to allow for divisible positions.

Proposition 10 *The solution to problem (10) with two units of asset when the asset price Y follows Exponential Brownian motion and $U(z)$ is given by piecewise power S-shape in (1) with $\alpha_2 = \alpha_1$, consists of three cases, depending on relative parameter values. Recall the notation $\beta = 1 - \frac{2\mu}{\sigma^2}$.*
Case (I): If $\beta \geq 0$; or if $0 < \beta < \alpha_1 < 1$, the agent waits indefinitely and never liquidates.
Case (II): If $0 < \alpha_1 < \beta \leq 1$ or $\alpha_1 = \beta < 1$, the agent waits in the region $\theta < \bar{\theta}_u^{(2)}$ and sells both units of asset in the region $\theta > \bar{\theta}_u^{(2)}$ (see Figure 7(a) and 7(b)). There are no asset values

for which the agent sells a single unit of asset.
Case (III): If $\beta > 1$, the agent sells both units of asset at either of two levels $\bar{\theta}_l^{(2)}, \bar{\theta}_u^{(2)}$ on either side of the break-even point (see Figure 7(c) and 7(d)). There are no asset values for which the agent sells a single unit of asset.

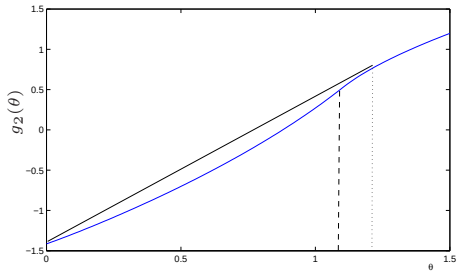
In all parameter regimes, the agent would choose to hold the position and thus would enter the game ex-ante. Figure 7 depicts the interesting cases described in the Proposition. Figure 7(a) and 7(b) take parameters corresponding to case (II) where there is no liquidation threshold below the break-even level. Recall the Sharpe ratio is good for these parameters, and the agent with a single unit of asset waited for a liquidation level above the break-even level (case (II) of Proposition 6). With two units of asset, the agent will wait for a liquidation level $\bar{\theta}_u^{(2)}$ above break-even (marked by the dotted vertical line) but at this level, will sell both units of the asset. That is, as we found in the piecewise exponential model of the previous section, the agent with piecewise power preferences will never hold a single unit of the asset. We depict this in Figure 7(a) for the situation without loss aversion, and in Figure 7(b) for loss aversion $\lambda = 2.2$. The liquidation threshold is $\bar{y}_u^{(2)} = 1.32$ in Figure 7(a) and $\bar{y}_u^{(2)} = 1.08$ in Figure 7(b), relative to break-even $y_R = 1$.

Figure 7(c) and 7(d) take parameters corresponding to case (III) where there are liquidation levels both above and below the break-even level, and so the agent liquidates (both) units either at a gain or at a loss. This is qualitatively similar to the behavior found in the indivisible model, in the sense that there are a pair of thresholds above and beneath the break-even level. Recall, in this scenario, the Sharpe ratio was poor, and the agent may "give-up" and sell at a loss. Again, we show this in Figure 7(c) for the situation without loss aversion, and in Figure 7(d) for loss aversion $\lambda = 2.2$. The liquidation thresholds are $\bar{y}_l^{(2)} = 0.65, \bar{y}_u^{(2)} = 1.03$ in Figure 7(c) and $\bar{y}_l^{(2)} = 0.31, \bar{y}_u^{(2)} = 1.007$ in Figure 7(d), relative to break-even $y_R = 1$.

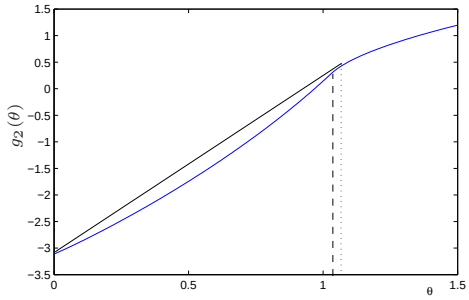
We could analyze again the relative likelihood of selling at a gain (relative to a loss) in case (III), however, we can see from Figures 7(c) and 7(d) that the upper theshold is much closer to the break-even level (note $s(y_R) = 1$) than the lower threshold. Similar arguments to earlier give that this means there is a high probability that the agent liquidates at a (small) gain. Given in situations with better Sharpe ratios (case (II)), the agent liquidates only at a gain, our divisible liquidation model with power S-shaped preferences predicts that sales at gains will be more likely than sales at losses, consistent again with the disposition effect. In fact, supportive evidence of Odean (1998) shows that the disposition effect remains strong even when the sample is limited to sales of investor's *entire* holdings of stock.

We return now to the finding that under both specifications of prospect theory, conditional on liquidating, an agent will choose to liquidate her entire position. This is in contrast to the behavior usually found under standard concave preferences. In situations where the objective is the expected utility of wealth with a standard concave utility function (and thus no reference level), agents tend to liquidate their position gradually over time[26] (in the context of executive

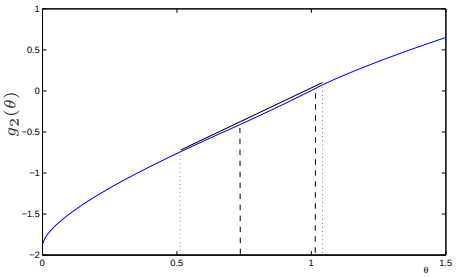
[26]In fact, we can use the same methods to study the problem with, for example, CRRA or CARA preferences. For reasons of space, we do not report the results here.



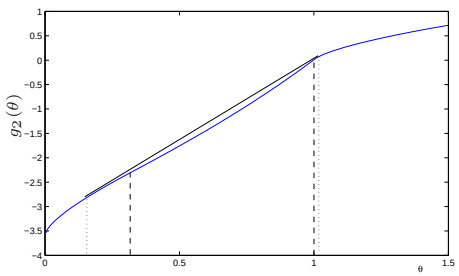
(a) (II). $\beta = 0.75$, $\alpha_1 = 0.5$, $\lambda = 1$, $\hat{\theta}_u^{(2)} = 1.1$, $\bar{\theta}_u^{(2)} = 1.23$, $s(y_R) = 1$ and $\bar{y}_u^{(2)} = 1.32$.



(b) (II). $\beta = 0.75$, $\alpha_1 = 0.5 < \beta$, $\lambda = 2.2$, $\hat{\theta}_u^{(2)} = 1.0302$, $\bar{\theta}_u^{(2)} = 1.06$, $s(y_R) = 1$ and $\bar{y}_u^{(2)} = 1.08$.



(c) (III). $\beta = 1.5$, $\alpha_1 = 0.88$, $\lambda = 1$, $\hat{\theta}_l^{(2)} = 0.6971, \hat{\theta}_u^{(2)} = 1.0252$, $\bar{\theta}_l^{(2)} = 0.52, \bar{\theta}_u^{(2)} = 1.05$, $s(y_R) = 1$. Also $\bar{y}_l^{(2)} = 0.65, \bar{y}_u^{(2)} = 1.03$.



(d) (III). $\beta = 1.5$, $\alpha_1 = 0.7$, $\lambda = 2.2$, $\hat{\theta}_l^{(2)} = 0.3252, \hat{\theta}_u^{(2)} = 1.0052$, $\bar{\theta}_l^{(2)} = 0.17, \bar{\theta}_u^{(2)} = 1.01$, and $s(y_R) = 1$. Also $\bar{y}_l^{(2)} = 0.31, \bar{y}_u^{(2)} = 1.007$

Figure 7: Optimal Liquidation of Two units of Asset under Power S-shaped utility and Exponential Brownian motion price process. Each panel plots the function $g_2(\theta, x)$ against the transformed price θ; and its smallest concave majorant $\bar{g}_2(\theta, x)$. Consider panels (c) and (d). The vertical dotted lines indicate the liquidation thresholds $\bar{\theta}_l^{(2)}, \bar{\theta}_u^{(2)}$. The vertical dashed lines represent points $\hat{\theta}_l^{(2)}, \hat{\theta}_u^{(2)}$ at which g_2 takes a different algebraic form. The ordering of these points implies both units are liquidated at the thresholds $\bar{\theta}_l^{(2)}, \bar{\theta}_u^{(2)}$, and the agent waits for price levels within this band. Since $\bar{\theta}_l^{(2)} < s(y_R) < \bar{\theta}_u^{(2)}$, the agent either sells both at a gain or at a loss. In panels (a) and (b), the interpretation is similar, so the agent liquidates both units at threshold $\bar{\theta}_u^{(2)}$ and waits below this level. Since $\bar{\theta}_u^{(2)} > s(y_R)$, the agent sells both at a gain. The equivalent thresholds in terms of the price Y are given as $\bar{y}_l^{(2)}, \bar{y}_u^{(2)}$ and should be compared with y_R. Common parameters are: $\alpha_2 = \alpha_1$, $x = 0$, and reference level (per unit) $y_R = 1$.

stock options: Jain and Subramanian (2004), Grasselli and Henderson (2009), amongst others).

We can give a simple argument for partial liquidation in the situation of a concave utility over wealth.[27] Imagine an agent is holding two units of asset and is indifferent to retaining this position, and liquidating both units of asset. Each unit of asset is worth random amount Y in one-period. The agent has preferences described by the utility function $U(x)$ and this will be applied to total wealth. Indifference implies that $\mathbb{E}U(x) = \mathbb{E}U(x+2Y)$. Now consider whether the agent would prefer to liquidate only one unit of risky asset. We see[28]

$$\begin{aligned} & \mathbb{E}U(x+Y) - \frac{1}{2}\mathbb{E}[U(x) + U(x+2Y)] \\ \approx \; & U(x) + U'(x)\mathbb{E}Y + \frac{1}{2}U''(x)\mathbb{E}(Y^2) - \frac{1}{2}U(x) - \frac{1}{2}[U(x) + 2U'(x)\mathbb{E}Y + 2U''(x)\mathbb{E}(Y^2)] \\ = \; & -\frac{1}{2}U''(x)\mathbb{E}(Y^2) \end{aligned}$$

which says the agent prefers to sell a single unit if $U''(x) < 0$, ie. if the utility function is everywhere concave. Conversely, if the utility function were everywhere convex, the agent would not prefer to sell a single unit of asset. If we consider the S-shaped prospect preferences which are convex below the reference level and concave above this level, then in principle, either behavior in terms of selling the assets could occur. However, recall that in both models, the upper sale threshold was very close to the break-even level (or even at the break-even in some cases) and therefore very little of the region of interest falls into the concave part of the function. The majority of the region of interest is where the function is convex, and hence, it is not surprising that the agent behaves like she has convex utility.

This analysis implies that *any* situation where agent's liquidate their entire position (a stock holding, a managerial project) when they could have retained a stake is consistent with our prospect theory model but inconsistent with equivalent models under standard concave utilities. In particular, it is believable that when managers do abandon a losing project, they abandon the whole project and do not merely scale it back.

4 Further Remarks and Conclusions

We have presented an optimal stopping framework which is used to analyze a liquidation problem for an agent with prospect theory preferences. Our methodology enables us to highlight the importance of differences in the piecewise exponential and piecewise power specifications of prospect theory, in particular, the loss aversion at the reference level leads them to give very different liquidation strategies. We show the agent will liquidate voluntarily at a loss under power S-shaped functions but not under the piecewise exponentials of Kyle, Ou-Yang and Xiong (2006).

[27] Other alternative arguments specific to the utility function can be given. For eg. if U is DARA, the agent would keep a constant proportion of wealth in the risky asset (Merton (1969)). As the asset price moves, the agent adjusts his holdings in order to keep this proportion constant, and this involves selling when the asset becomes more valuable.

[28] With $\approx$ denoting taking the first terms of an expansion with Y small.

Indeed, the extant literature has not provided a model under which voluntary liquidation at a loss occurs, and instead, often remarks that if the asset's odds are poor, the agent would not take the position ex-ante. The agent will be much more likely to liquidate at a small gain than at a large loss, consistent with the disposition effect. This remains true in our model when we extend to allow the agent to sell-off assets over time. Under this extension, common to both specifications is the fact that the prospect theory agent prefers to take an "all-or-nothing" sales strategy. This is in sharp contrast to the typical behavior of an agent with concave utility who will liquidate a position gradually over time.

The main application of our model is to the disposition effect. Whilst there is much evidence across markets and agents that gains are realized more readily than losses, there is less evidence on whether this effect holds for agent's sales of entire positions (for stocks, see Odean (1998)), as predicted by our model. This points to a direction for future empirical research. We also make a further remark concerning the applicability of our model(s) and others to explaining empirical findings linking historical price highs and high sales (or option exercise) volumes (Grinblatt and Keloharju (2001), Heath, Huddart and Lang (1999)). Barberis and Xiong (2008b) observe that this is consistent with their implementation of realization utility since agent's sell at price thresholds above their break-even points, and these sales occur the first time the price reaches a new high level. Clearly, our model (and that of Kyle, Ou-Yang and Xiong (2006)) also explains the same phenomena, since in either specification of the S-shaped function, under certain parameter values, the agent will sell at a threshold level above the break-even point. In fact, our specification with piecewise power functions will additionally explain the high sales volumes observed at historical price lows (observed by Grinblatt and Keloharju (2001) for households), since the agent in this case also sells below the break-even level. This hints at the fact that the ability of a model to explain sales at new historical highs (or lows) has little to do with the realization utility formulation of Barberis and Xiong (2008b). Of course, the location of the threshold above break-even is linked to the reference level of prospect theory. However, the main driver of the all of these models ability to explain high volume at new price maximia is simply their time-homogeneous structure.[29]

Our current model treats the liquidation of a divisible position of identical assets. It would be interesting to extend to a portfolio of different assets or claims on different assets. This would involve questions of how agents treat different assets or portfolios in terms of the narrow framing and more generally, mental accounting (Thaler (1980), Tversky and Kahneman (1981)). Another potentially fruitful line of future research beyond the scope of the current paper would be to consider the probability weighting aspect of prospect theory.

[29] As such, features such as a fixed time horizon, or time-inhomogeneous price process would destroy a model's ability to explain activity at new highs or lows.

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6 Appendix

Proofs:

Proof of Proposition 4:
Although this follows from optimal stopping theory (see Oksendal (2005), and specifically Dynkin (1965), Dynkin and Yushkevich (1969) and more recently, Dayanik and Karatzas (2003)) it is straightforward to prove the result directly in our problem.
(i) Trivially $V_1(y) \leq f(b_{\mathcal{I}})$. Let $b_n \uparrow b_{\mathcal{I}}$ and let $\tau_n = \tau^Y_{(a_{\mathcal{I}},b_n)}$. Then the local martingale Θ always leaves the interval $(s(a_{\mathcal{I}}), s(b_{\mathcal{I}})) = (-\infty, s(b_{\mathcal{I}}))$ on the right, and $V_1(y) \geq f(b_n) \uparrow f(b_{\mathcal{I}})$.
(ii) Let $\tilde{g}_1$ be any increasing, concave majorant of g_1. By definition,

$$V_1(y) = \sup_\tau \mathbb{E}[f(Y_\tau)|Y_0 = y] = \sup_\tau \mathbb{E}[g_1(\Theta_\tau)|\Theta_0 = \theta]$$

But

$$\mathbb{E}[g_1(\Theta_\tau)|\Theta_0 = \theta] \leq \mathbb{E}[\tilde{g}_1(\Theta_\tau)|\Theta_0 = \theta] \leq \tilde{g}_1(\mathbb{E}[\Theta_\tau|\Theta_0 = \theta])$$

where we use the fact $\tilde{g}_1$ is a concave majorant of g_1 and Jensen's inequality. Finally we use that $\tilde{g}_1$ is increasing, and that a local martingale bounded below is a supermartingale to give

$$\tilde{g}_1(\mathbb{E}[\Theta_\tau|\Theta_0 = \theta]) \leq \tilde{g}_1(\theta)$$

and hence $V_1(y) \leq \bar{g}_1(\theta)$ where $\bar{g}_1$ is the smallest concave majorant of g_1. Thus we have shown we cannot do better than $\bar{g}_1$, allowing for general stopping rules. It remains to show that there is a stopping rule which attains this bound.

Suppose first that $\limsup_{\psi \uparrow s(b_{\mathcal{I}})} \frac{g_1(\psi)}{\psi - s(a_{\mathcal{I}})} = \infty$, for instance if $s(b_{\mathcal{I}}) < \infty$ and $g_1(s(b_{\mathcal{I}})) = \infty$. In this case $\bar{g}_1(\theta) = \infty$ for $\theta \in (s(a_{\mathcal{I}}), s(b_{\mathcal{I}}))$. We have there exists $b_n \uparrow b_{\mathcal{I}}$ such that $\frac{\bar{g}_1(s(b_n))}{s(b_n) - s(a_{\mathcal{I}})} \uparrow \infty$. Then, for any $s(a_{\mathcal{I}}) \leq \hat{\phi} < \theta$ we have for $\tau^*_n = \tau^\Theta_{(\hat{\phi}, s(b_n))} = \tau^Y_{(s^{-1}(\hat{\phi}), b_n)}$

$$\mathbb{E}[f(Y_{\tau^*_n})|Y_0 = s^{-1}(\theta)] \geq \bar{g}_1(\hat{\phi})\frac{s(b_n) - \theta}{s(b_n) - \hat{\phi}} + \bar{g}_1(s(b_n))\frac{\theta - \hat{\phi}}{s(b_n) - \hat{\phi}} \uparrow \infty.$$

Now suppose $\limsup_{\psi \uparrow s(b_{\mathcal{I}})} \frac{g_1(\psi)}{\psi - s(a_{\mathcal{I}})} < \infty$. In this case $\bar{g}_1$ is a finite function. Fix θ and let $\Upsilon = \{\upsilon : \bar{g}_1(\upsilon) = g_1(\upsilon)\}$. Suppose $\theta \in \Upsilon$. Then with $\tau = 0$, $\mathbb{E}[f(Y_0)|\Theta_0 = \theta] = g_1(\theta) = \bar{g}_1(\theta)$ and we are done. Otherwise define

$$\begin{aligned} \phi^* &= \sup\{\xi < \theta : \xi \in \Upsilon\}, \\ \psi^* &= \inf\{\xi > \theta : \xi \in \Upsilon\}. \end{aligned}$$

and set $\phi^* = s(a_{\mathcal{I}})$ if the set $\{\xi < \theta : \xi \in \Upsilon\}$ is empty, and $\psi^* = \infty$ if $\{\xi > \theta : \xi \in \Upsilon\}$ is empty.

Suppose $\psi^* < \infty$. Then $\bar{g}_1(\theta)$ is linear on the interval $\theta \in (\phi^*, \psi^*)$ and $\Theta_{t \wedge \tau_{(\phi^*, \psi^*)}}$ is a martingale. Then

$$\mathbb{E}[f(Y_{\tau^\Theta_{(\phi^*,\psi^*)}})|\Theta_0 = \theta] = \mathbb{E}[g_1(\Theta_{\tau^\Theta_{(\phi^*,\psi^*)}})] = \mathbb{E}[\bar{g}_1(\Theta_{\tau^\Theta_{(\phi^*,\psi^*)}})] = \bar{g}_1(\theta).$$

Conversely, if $\psi^* = \infty$, then choose $\theta_n > \theta$ so that

$$\bar{g}_1(\phi^*)\frac{\theta_n - \theta}{\theta_n - \phi^*} + \bar{g}_1(\theta_n)\frac{\theta - \phi^*}{\theta_n - \phi^*} > \bar{g}_1(\theta) - \epsilon.$$

Then, for the stopping time $\tau^* = \tau^{\Theta}_{(\phi^*,\theta_n)} = \tau^{Y}_{(s^{-1}(\phi^*),s^{-1}(\theta_n))}$ we get that $\mathbb{E}[f(Y_{\tau^*})|Y_0 = s^{-1}(\theta)] \geq \bar{g}_1(\theta) - \epsilon$.

□

Proof of Proposition 5:

From Proposition 4, we know we just have to identify the smallest concave majorant $\bar{g}_1$ of g_1. If $\mu > 0$, then $s(\mathcal{I}) = (-\infty, 0)$ and $s^{-1}(\theta) = -\frac{\sigma^2}{2\mu}\ln(-\theta); \theta \in (-\infty, 0)$. Note the transformed reference level is $s(y_R) = -e^{-2\mu/\sigma^2 y_R}$. Substitution gives

$$g_1(\theta) = \begin{cases} \phi_1 - \phi_1 e^{\gamma_1 y_R}(-\theta)^{\gamma_1\sigma^2/2\mu}; & \theta \geq -e^{-2\mu/\sigma^2 y_R} \\ \phi_2 e^{-\gamma_2 y_R}(-\theta)^{-\gamma_2\sigma^2/2\mu} - \phi_2; & \theta < -e^{-2\mu/\sigma^2 y_R} \end{cases}$$

Calculations show that $g_1'(-\infty) = 0$; and if $2\mu/\sigma^2 > \gamma_1$ (case (I), Figure 3(a)), then $g_1'(0) = \infty$. If $2\mu/\sigma^2 < \gamma_1$ (case (I), Figure 3(b)), then $g_1'(0) = 0$. In both of these situations, $s(a_{\mathcal{I}}) = -\infty$, so by (i) of the Proposition, $V_1(y) = f(b_{\mathcal{I}}) = U(b_{\mathcal{I}} - y_R) = \phi_1$, since $b_{\mathcal{I}} = s^{-1}(0) = \infty$. Since Y is drifting Brownian motion, it does not reach infinity in finite time, so the agent waits indefinitely. The case $\mu = 0$ can be treated similarly.

All remaining cases involve $\mu < 0$. Then $s(\mathcal{I}) = (0, \infty)$. We have $s^{-1}(\theta) = -\frac{\sigma^2}{2\mu}\ln\theta; \theta \in (0, \infty)$ and $s(y_R) = e^{-2\mu/\sigma^2 y_R}$. Substitution results in

$$g_1(\theta) = \begin{cases} \phi_1 - \phi_1 e^{\gamma_1 y_R}\theta^{\gamma_1\sigma^2/2\mu}; & \theta \geq e^{-2\mu/\sigma^2 y_R} \\ \phi_2 e^{-\gamma_2 y_R}\theta^{-\gamma_2\sigma^2/2\mu} - \phi_2; & \theta < e^{-2\mu/\sigma^2 y_R} \end{cases}$$

Calculations show that $g_1'(0) = \infty$ if $2\mu/\sigma^2 < -\gamma_2$, $g_1'(0) = \phi_2 e^{-\gamma_2 y_R}$ if $2\mu/\sigma^2 = -\gamma_2$, whereas $g_1'(0) = 0$ if $2\mu/\sigma^2 > -\gamma_2$. In both cases, $g_1'(\infty) = 0$. Case (IV) takes $2\mu/\sigma^2 \leq -\gamma_2$, see Figure 3(e). In this case, it is clear from the figure that the smallest concave majorant is the function itself so $\bar{g}_1(\theta) = g_1(\theta); \forall\theta$ and hence the agent stops immediately at any price level.

The situation where $0 > 2\mu/\sigma^2 > -\gamma_2$ further divides into two scenarios depending on where the agent stops relative to the break-even point. In each case, $g_1(\theta)$ is convex over the range $(0, e^{-2\mu/\sigma^2 y_R})$ (to the left of break-even point) and concave to the right of the break-even point. What remains is to establish the location of the stopping point relative to the break-even point. Consider the stylized representation of the possibilities in Figure 8. The two (dashed line) curves represent possible continuations for $g_1(\theta)$ beyond the point marked with a "B" which represents the break-even point. Compare the slope of the chord from the origin to point B to the slope of g_1 to the right of point B. There are clearly two possibilities. Either the slope of the chord is less than $g_1^{'}(B+)$ (on the higher of the two curves to the right of B) and thus the stopping point lies to the right of the point B; or the slope of the chord is greater (or equal to) $g_1^{'}(B+)$ (on the lower of the two curves), and thus the stopping point lies exactly at the point B. This

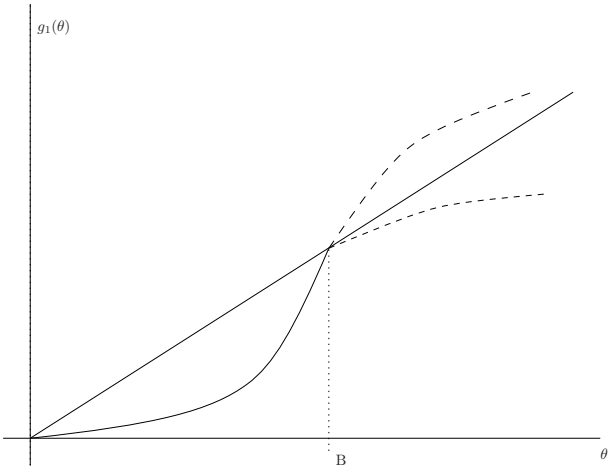


Figure 8: The solid curve and its two possible dashed continuations give a stylized representation of possible functions $g_1(\theta)$ as a function of scaled price θ, where $\theta = s(y)$. The function $g_1(\theta)$ represents the value of the game to the agent if he sells immediately. The point B marks the break-even point. To decide whether the agent stops at or to the right of point B involves comparing the slope of the chord from the origin to B with the slope of the function to the right of B.

comparison of slopes gives that if $2|\mu|/\sigma^2 \geq \frac{\phi_1}{\phi_2}\gamma_1$ (case (III)), then the stopping point is at the break-even point. This is shown in Figure 3(d). Case (II) occurs when $2|\mu|/\sigma^2 < \frac{\phi_1}{\phi_2}\gamma_1$, so the stopping point must lie to the right of the break-even point. This is shown in Figure 3(c). In case (II), the smallest concave majorant $\bar{g}_1(\theta)$ is formed by taking the chord from $(0, -\phi_2)$ to a point $\bar{\theta}_u^{(1)} > e^{-2\mu/\sigma^2 y_R}$, where the slope of the chord matches the slope of g_1 to the right of $e^{-2\mu/\sigma^2 y_R}$. Equating slopes gives $\bar{\theta}_u^{(1)}$ solves

$$\theta^{\gamma_1\sigma^2/2\mu} = \left(\frac{2\mu}{2\mu - \gamma_1\sigma^2}\right)\left(\frac{\phi_1 + \phi_2}{\phi_1}\right)e^{-\gamma_1 y_R}$$

The equivalent price threshold $\bar{y}_u^{(1)} = s^{-1}(\bar{\theta}_u^{(1)})$ solves (6). The smallest concave majorant $\bar{g}_1(\theta)$ is given by

$$\bar{g}_1(\theta) = \begin{cases} \phi_1 - \phi_1 e^{\gamma_1 y_R}\theta^{\gamma_1\sigma^2/2\mu}; \theta \geq \bar{\theta}_u^{(1)} \\ \theta\left[e^{2\mu/\sigma^2 y_R}(\phi_1)^{2\mu/(\gamma_1\sigma^2)}(-\gamma_1\sigma^2/2\mu)(\frac{2\mu}{2\mu-\gamma_1\sigma^2})^{(1-\frac{2\mu}{\gamma_1\sigma^2})}(\phi_1+\phi_2)^{(1-\frac{2\mu}{\gamma_1\sigma^2})}\right] - \phi_2; \theta < \bar{\theta}_u^{(1)} \end{cases}$$

□

Proof of Proposition 6:

Again, from Proposition 4, we simply have to identify the smallest concave majorant $\bar{g}_1(\theta)$. If

$\beta < 0$, $s^{-1}(\theta) = (-\theta)^{1/\beta}; \theta \in (-\infty, 0)$ and $s(y_R) = -(y_R)^\beta$, giving

$$g_1(\theta) = \begin{cases} (-y_R + (-\theta)^{1/\beta})^{\alpha_1}; & \theta \geq -(y_R)^\beta \\ -\lambda(y_R - (-\theta)^{1/\beta})^{\alpha_2}; & \theta < -(y_R)^\beta \end{cases}$$

As expected, $g_1'(-(y_R)^\beta-) = g_1'(-(y_R)^\beta+) = \infty$. Calculations show that $g_1'(-\infty) = 0$; and $g_1'(0) = \infty$. This situation is case (I), shown in Figure 4(a). In this case, the agent waits indefinitely. All other cases take $\beta > 0$, so $s^{-1}(\theta) = (\theta)^{1/\beta}; \theta \in (0, \infty)$ and $s(y_R) = (y_R)^\beta$, giving

$$g_1(\theta) = \begin{cases} (-y_R + \theta^{1/\beta})^{\alpha_1}; & \theta \geq (y_R)^\beta \\ -\lambda(y_R - \theta^{1/\beta})^{\alpha_2}; & \theta < (y_R)^\beta \end{cases}$$

Calculations show $g_1(0) = -\lambda(y_R)^{\alpha_2}$ and that $g_1'(0) = 0$ for $0 < \beta < 1$, $g_1'(0) = \infty$ for $\beta > 1$. We can also see that $g_1'(\infty) = 0$ for $\alpha_1 < \beta$ and $g_1'(\infty) = \infty$ for $\alpha_1 > \beta$. These result in three possible shapes for $g_1(\theta)$. The combination $0 < \beta < 1$ and $\alpha_1 > \beta$ gives a further situation where the agent waits indefinitely. This is displayed in Figure 4(b). If $0 < \beta < 1$ and $\alpha_1 < \beta$, then g_1 is convex for $\theta < (y_R)^\beta$ (left of the break-even level) and concave for $\theta > (y_R)^\beta$, resulting in case (II) displayed in Figure 4(c). The smallest concave majorant $\bar{g}_1(\theta)$ is formed by taking the chord from $(0, -\lambda(y_R)^{\alpha_2})$ to a point on the function $g_1(\theta)$ for $\theta > (y_R)^\beta$ (to the right of the break-even point). Hence the agent stops at a price level above the break-even point. Case (III) results from the choice $\beta > 1$. In this case g_1 switches from concave to convex for $\theta < (y_R)^\beta$, see Figure 4(d). The smallest concave majorant will consist of a chord which touches the function at two points, $\bar{\theta}_l^{(1)} \in (0, (y_R)^\beta)$ and $\bar{\theta}_u^{(1)} \in ((y_R)^\beta, \infty)$. Hence, the agent stops at two levels - one which is above the break-even and another which is below the break-even. The special cases $\beta = 0, \beta = 1$ and $\alpha_1 = \beta$ can be treated similarly. □

Proof of Proposition 7:

The existence of the thresholds as described follows from the proof of Proposition 6. Take $\beta > 1$ (Case (III)). To find the smallest concave majorant we solve for pair of critical points, $\bar{\theta}_l^{(1)} < \bar{\theta}_u^{(1)}$ at which the slope of g_1 matches the slope of the chord between them:

$$g_1'(\bar{\theta}_l^{(1)}) = (g_1(\bar{\theta}_u^{(1)}) - g_1(\bar{\theta}_l^{(1)}))/(\bar{\theta}_u^{(1)} - \bar{\theta}_l^{(1)}) = g_1'(\bar{\theta}_u^{(1)}).$$

The analysis when $\alpha_2 = \alpha_1$ is much simpler as scaling reduces the dimension of the problem. Then setting $\bar{\theta}_l^{(1)} = (\bar{c}_l)^\beta (y_R)^\beta$, $\bar{\theta}_u^{(1)} = (\bar{c}_u)^\beta (y_R)^\beta$ gives the constants $\bar{c}_l, \bar{c}_u$ solve (7) and (8). The transformation $y = s^{-1}(\theta)$ gives $\bar{y}_l^{(1)}$ and $\bar{y}_u^{(1)}$ as given. The smallest concave majorant $\bar{g}_1(\theta)$ is given by

$$\bar{g}_1(\theta) = \begin{cases} (\theta^{1/\beta} - y_R)^{\alpha_1}; & \theta \geq \bar{c}_u^\beta (y_R)^\beta \\ (y_R)^{\alpha_1 - \beta} \delta(\theta - \bar{c}_l^\beta (y_R)^\beta) - \lambda(1 - \bar{c}_l)^{\alpha_1} (y_R)^{\alpha_1}; & \bar{c}_l^\beta (y_R)^\beta < \theta < \bar{c}_u^\beta (y_R)^\beta \\ -\lambda(y_R - \theta^{1/\beta})^{\alpha_1}; \theta \leq \bar{c}_l^\beta (y_R)^\beta & \end{cases}$$

where the constant δ is given by $\delta = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda(1 - \bar{c}_l)^{\alpha_1}}{\bar{c}_u^\beta - \bar{c}_l^\beta}$. Case (II) is obtained from the above with $\bar{c}_l^{(1)} = 0$ and hence $\bar{\theta}_l^{(1)} = \bar{y}_l^{(1)} = 0$. □

Proof of Proposition 9:

We first need to extend the results of Section 2.1 to include initial wealth x. Replacing the reference level y_R with $y_R - x$ in the expressions for $g_1(\theta)$, $\bar{g}_1(\theta)$ (in proof of Proposition 5)) and in the definition of the thresholds gives the required results. Consider first case (II). We have

$$\begin{aligned} & g_2(\theta, x) = \bar{g}_1(\theta, x + s^{-1}(\theta) - y_R) \\ = & \begin{cases} \phi_1 - \phi_1 e^{\gamma_1(2y_R - x)}(\theta)^{2(\gamma_1\sigma^2/2\mu)}; & \theta \geq \hat{\theta}_u^{(2)} \\ (\theta)^2 \left[e^{-\frac{2\mu}{\sigma^2}(x-2y_R)}(\phi_1+\phi_2)^{(1-\frac{2\mu}{\gamma_1\sigma^2})}(-\gamma_1\sigma^2/2\mu)(\phi_1)^{(\frac{2\mu}{\gamma_1\sigma^2})}(\frac{2\mu}{2\mu-\gamma_1\sigma^2})^{(1-\frac{2\mu}{\gamma_1\sigma^2})} \right] - \phi_2; & \theta < \hat{\theta}_u^{(2)} \end{cases} \end{aligned}$$

where $\hat{\theta}_u^{(2)}$ solves

$$\theta^{2(\gamma_1\sigma^2/2\mu)} = \left(\frac{2\mu}{2\mu - \gamma_1\sigma^2}\right)\left(\frac{\phi_1+\phi_2}{\phi_1}\right)e^{\gamma_1(x-2y_R)}$$

It is straightforward to show that that g_2 is concave to the right of $\hat{\theta}_u^{(2)}$. To the left of this point, the function is either convex, or is convex for small θ and concave for larger θ. Thus it is sufficient to consider again the stylized Figure 8 but re-interpret the figure so that the curves represent $g_2(\theta, x)$ and the point marked "B" represents the point $\theta = \hat{\theta}_u^{(2)}$. Again, we see that to decide whether the smallest concave majorant $\bar{g}_2$ tangents to g_2 to the right or to the left of this point, we compare the slope of the chord between $(0, -\phi_2)$ and $(\hat{\theta}_u^{(2)}, g_2(\hat{\theta}_u^{(2)}))$, to the slope of the tangent to g_2 at $\hat{\theta}_u^{(2)}$. Consider the ratio of the chord slope to the tangent slope. If the ratio is less than or equal to one, then the smallest concave majorant of g_2 must touch at or to the right of $\hat{\theta}_u^{(2)}$. This ratio is given by

$$\frac{2\mu}{2\gamma_1\sigma^2}\left[1 - \frac{\phi_1+\phi_2}{\phi_1}e^{-\gamma_1(2y_R-x)}(\hat{\theta}_u^{(2)})^{-2(\gamma_1\sigma^2/2\mu)}\right] = 1/2$$

So, we must have $\hat{\theta}_u^{(2)} < \bar{\theta}_u^{(2)}$. Then for $\theta < \bar{\theta}_u^{(2)}$, $\bar{g}_2(\theta, x) > g_2(\theta, x)$ so the agent waits in this region. For $\theta \geq \bar{\theta}_u^{(2)}$, $\bar{g}_2 = g_2$. The agent sells one unit at threshold $\bar{\theta}_u^{(2)}$. However, given the ordering $\hat{\theta}_u^{(2)} \leq \bar{\theta}_u^{(2)}$, we know for $\theta \geq \bar{\theta}_u^{(2)}$, we must have $\theta \geq \bar{\theta}_u^{(1)}$ also, and hence the agent sells the remaining unit at the same threshold $\bar{\theta}_u^{(2)}$. Both units are sold at the threshold $\bar{\theta}_u^{(2)} > s(y_R)$ at a gain relative to break-even. See Figure 6(a). Equating slopes gives $\bar{\theta}_u^{(2)}$ solves

$$\theta^{2(\gamma_1\sigma^2/2\mu)} = \left(\frac{2\mu}{2\mu - 2\gamma_1\sigma^2}\right)\left(\frac{\phi_1+\phi_2}{\phi_1}\right)e^{-\gamma_1(2y_R-x)}$$

and the equivalent price threshold $\bar{y}_u^{(2)} = s^{-1}(\bar{\theta}_u^{(2)})$ solves (11).

Consider now case (III). In this case, the expression for g_2 simplifies to become

$$\begin{aligned} & g_2(\theta, x) = \bar{g}_1(\theta, x + s^{-1}(\theta) - y_R) \\ = & \begin{cases} \phi_1 - \phi_1 e^{\gamma_1(2y_R - x)}(\theta)^{2(\gamma_1\sigma^2/2\mu)}; & \theta \geq \hat{\theta}_u^{(2)} \\ (\theta)^2\phi_2 e^{-\frac{2\mu}{\sigma^2}(x-2y_R)} - \phi_2; & \theta < \hat{\theta}_u^{(2)} \end{cases} \end{aligned}$$

where $\hat{\theta}_u^{(2)} = e^{-\frac{\mu}{\sigma^2}(2y_R-x)}$. In this case, the ratio of the chord to the tangent at $\hat{\theta}_u^{(2)}$ can also be shown to be $1/2$, and so we again have the ordering $\hat{\theta}_u^{(2)} < \bar{\theta}_u^{(2)}$. Equating slopes gives $\bar{\theta}_u^{(2)}$ solves

(note this is identical to case (II) above)

$$\theta^{2(\gamma_1\sigma^2/2\mu)} = \left(\frac{2\mu}{2\mu - 2\gamma_1\sigma^2}\right)\left(\frac{\phi_1+\phi_2}{\phi_1}\right)e^{-\gamma_1(2y_R-x)}$$

and again, the equivalent price threshold $\bar{y}_u^{(2)} = s^{-1}(\bar{\theta}_u^{(2)})$ solves (11). Both units are sold at the threshold $\bar{\theta}_u^{(2)} > s(y_R)$ at a gain relative to break-even. See Figure 6(b).

□

Proof of Proposition 10:

We first need to extend the results of Section 2.2 to include the initial wealth x. Replacing the reference level y_R with $y_R - x$ in the expressions for $g_1(\theta)$, $\bar{g}_1(\theta)$ and in the definition of the thresholds gives the required results. Assume $\alpha_2 = \alpha_1$. Then

$$\begin{aligned} & g_2(\theta, x) = \bar{g}_1(\theta, x + s^{-1}(\theta) - y_R) \\ = & \begin{cases} (x + 2\theta^{1/\beta} - 2y_R)^{\alpha_1}; & \theta \geq \hat{\theta}_u^{(2)} \\ \delta(\theta - \bar{c}_l^{\beta}(y_R - (\theta^{1/\beta} - y_R))^{\beta})(y_R - (\theta^{1/\beta} - y_R))^{\alpha_1 - \beta} & \\ \qquad -\lambda(1-\bar{c}_l)^{\alpha_1}(y_R - (\theta^{1/\beta} - y_R))^{\alpha_1}; & \hat{\theta}_l^{(2)} < \theta < \hat{\theta}_u^{(2)} \\ -\lambda(2y_R - 2\theta^{1/\beta} - x))^{\alpha_1}; & \theta \leq \hat{\theta}_l^{(2)} \end{cases} \end{aligned}$$

where $\hat{\theta}_u^{(2)} = (\frac{\bar{c}_u}{1+\bar{c}_u})^{\beta}(2y_R - x)^{\beta}$ and $\hat{\theta}_l^{(2)} = (\frac{\bar{c}_l}{1+\bar{c}_l})^{\beta}(2y_R - x)^{\beta}$; again the constant δ is given by $\delta = \frac{(\bar{c}_u - 1)^{\alpha_1} + \lambda(1-\bar{c}_l)^{\alpha_1}}{\bar{c}_u^{\beta} - \bar{c}_l^{\beta}}$.

Consider case (II) where $0 < \alpha_1 < \beta \leq 1$. In this case, the expression for g_2 simplifies (see comments in the proof of Proposition 7.) It is straightforward to show that that g_2 is concave to the right of $\hat{\theta}_u^{(2)}$. To the left of this point, the function is either convex, or is convex for small θ and concave for larger θ. As in Proposition 9, we compare the slope of the chord between $(0, -\lambda(y_R - (x + \theta^{1/\beta} - y_R))^{\alpha_1})$ and $(\hat{\theta}_u^{(2)}, g_2(\hat{\theta}_u^{(2)}))$, to the slope of the tangent to g_2 at $\hat{\theta}_u^{(2)}$. Consider the ratio $R(\bar{c}_u)$ of the chord slope to the tangent slope. If $R(\bar{c}_u) \leq 1$ then the smallest concave majorant of g_2 must touch at or to the right of $\hat{\theta}_u^{(2)}$. After some calculations, the ratio is found to be

$$R(\bar{c}_u) = \frac{1}{2}(1+\bar{c}_u)^{\alpha_1} + \frac{(1 - 1/\bar{c}_u)[1 - (1+\bar{c}_u)^{\alpha_1}]}{2\alpha_1/\beta}$$

Since $R(1) \leq 1$, it is enough to show that $R(\bar{c}_u)$ is decreasing in $\bar{c}_u$, which can be shown. Thus $\hat{\theta}_u^{(2)} \leq \bar{\theta}_u^{(2)}$. Then for $\theta < \bar{\theta}_u^{(2)}$, $\bar{g}_2(\theta, x) > g_2(\theta, x)$ so the agent waits in this region. For $\theta \geq \bar{\theta}_u^{(2)}$, $\bar{g}_2 = g_2 = (x + 2\theta^{1/\beta} - 2y_R)^{\alpha_1}$. The agent sells one unit at threshold $\bar{\theta}_u^{(2)}$. However, given the ordering $\hat{\theta}_u^{(2)} \leq \bar{\theta}_u^{(2)}$, we know for $\theta \geq \bar{\theta}_u^{(2)}$, we must have $\theta \geq \bar{\theta}_u^{(1)}$ also, and hence the agent sells the remaining unit at the same threshold $\bar{\theta}_u^{(2)}$. Both units are sold at the threshold $\bar{\theta}_u^{(2)} > s(y_R)$ at a gain relative to break-even. Similar arguments apply in case (III). In this case, (see panels (c) and (d)), we find ordering $\hat{\theta}_u^{(2)} \leq \bar{\theta}_u^{(2)}$ and $\hat{\theta}_l^{(2)} \geq \bar{\theta}_l^{(2)}$ so by similar reasoning, the agent sells both units at thresholds $\bar{\theta}_l^{(2)}, \bar{\theta}_u^{(2)}$. Since $\bar{\theta}_l^{(2)} < s(y_R) < \bar{\theta}_u^{(2)}$, the agent sells both units either at a gain, or at a loss, relative to break-even.

□