Foliation by area-constrained Willmore spheres near a non-degenerate critical point of the scalar curvature

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Abstract

Let (M,g) be a 3-dimensional Riemannian manifold. The goal of the paper it to show that if $P_0 \in M$ is a non-degenerate critical point of the scalar curvature, then a neighborhood of P_0 is foliated by area-constrained Willmore spheres. Such a foliation is *unique* among foliations by area-constrained Willmore spheres having Willmore energy less than 32π, moreover it is *regular* in the sense that a suitable rescaling smoothly converges to a round sphere in the Euclidean three-dimensional space. We also establish generic multiplicity of foliations and the first multiplicity result for area-constrained Willmore spheres with prescribed (small) area in a closed Riemannian manifold. The topic has strict links with the Hawking mass.

Key Words: Willmore functional, Foliation, Hawking mass, nonlinear fourth order partial differential equations, Lyapunov-Schmidt reduction.

AMS subject classification:
49Q10, 53C21, 53C42, 35J60, 83C99.

1 Introduction

Let Σ be a closed (compact, without boundary) two-dimensional surface, (M,g) a 3-dimensional Riemannian manifold and $f : \Sigma \to M$ a smooth immersion. The *Willmore functional* $W(f)$ is defined by

$$W(f) := \int_\Sigma H^2 \, d\sigma. \tag{1}$$

Here $d\sigma$ is the area form induced by f, $H := \bar{g}^{ij} A_{ij}$ the mean curvature, $\bar{g}_{ij}$ the induced metric and A_{ij} the second fundamental form. The immersion f is said to be a *Willmore surface* (or *Willmore immersion*) if it is a critical point of W with respect to normal variations or, equivalently, when it satisfies the Euler-Lagrange equation

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(n,n) = 0, \tag{2}$$

where $\Delta_{\bar{g}}$ is the Laplace-Beltrami operator, $\mathring{A}_{ij} := A_{ij} - \frac{1}{2}H\bar{g}_{ij}$ the trace free second fundamental form, n a unit normal to f and Ric the Ricci tensor of (M, g). The Willmore equation (2) is a fourth-order nonlinear elliptic PDE in the immersion map f.

The Willmore energy (1) appears not only in mathematics but also in various fields of science and technology. For example, in biology, it is a special case of the *Helfrich energy* ([9, 13, 36]). In general relativity, the *Hawking mass* contains the Willmore functional as the main term (see below for the definition of the Hawking mass) and in String Theory the Polyakov's extrinsic action involves the functional as well.

The Willmore functional was first introduced in the XIXth century in the Euclidean ambient space by Sophie Germain in her work on elasticity. Blaschke and Thomsen, in the 1920s-30s, detected the class of Willmore surfaces as a natural conformally invariant generalization of minimal surfaces. Indeed minimal surfaces are solutions of (2) (as $H \equiv 0$), and the Willmore functional W in the Euclidean space is conformally invariant (provided the center of the inversion does not lie on the surface, in which case one has to add a constant depending on the multiplicity of the immersion).

After that, Willmore rediscovered the topic in 1960s. He proved that round spheres are the only global minimizers of W among all closed immersed surfaces into the Euclidean space (see [37]) and he conjectured that the Clifford torus and its images under the Möbius transformations are the global minimizers among surfaces of higher genus. The Willmore conjecture was recently solved by Marques-Neves [25] through minimax techniques. Let us also mention other fundamental works on the Willmore functional in the Euclidean ambient space. Simon [35] proved the existence of a smooth genus-one minimizer of W in $\mathbb{R}^m$ and developed a general regularity theory for minimizers. The existence of a minimizer for every genus was settled by Bauer-Kuwert [2], Kusner [14] and Rivière [33, 34] who also developed an independent regularity theory holding more generally for stationary points of W. We also wish to mention the work by Kuwert-Schätzle [16] on the Willmore flow and by Bernard-Rivière [3] and Laurain-Rivière [22] on bubbling and energy-identities phenomena.

Let us emphasize that all the aforementioned results concern Willmore immersions into the *Euclidean space*, or equivalently into a round sphere due to the conformal invariance. The literature about Willmore immersions into curved Riemannian manifolds, which has interest in applications as it might model non-homogeneous environments, is much more recent. The first existence result in ambient space with non-constant sectional curvature was [26], where the third author showed the existence of embedded Willmore spheres (Willmore surface with genus equal to zero) in a perturbative setting. We also refer to [27] and [5] in collaboration with Carlotto for related results. Under the area constraint condition, the existence of Willmore type spheres and their properties have been investigated by Lamm-Metzger [17, 18], Lamm-Metzger-Schulze [19], and the third author in collaboration with Laurain [21]. The existence of area-constrained Willmore tori of small size have been recently addressed by the authors of this work in [10, 11].

The global problem, i.e. the existence of smooth immersed spheres minimizing quadratic curvature functionals in compact 3-dimensional Riemannian manifolds, was studied by the third author in collaboration with Kuwert and Schygulla in [15] (see also [30] for the non compact case). In collaboration with Rivière [28, 29], the third author developed the necessary tools for the calculus of variations of the Willmore functional in Riemannian manifolds and proved the existence of area-constrained Willmore spheres in homotopy classes (as well as the existence of Willmore spheres under various assumptions and constraints).

The present paper, as well as the aforementioned works [17, 18, 19, 21, 28, 29], concerns the existence of Willmore spheres under area constraint. Such immersions satisfy the equation

$$\Delta_{\bar{g}} H + H|\mathring{A}|^2 + H\mathrm{Ric}(n,n) = \lambda H,$$

for some $\lambda \in \mathbb{R}$ playing the role of Lagrange multiplier. These immersions are strictly related to the Hawking mass

$$m_H(f) := \frac{\sqrt{Area(f)}}{64\pi^{3/2}} \left(16\pi - W(f)\right),$$

in the sense that critical points of the Hawking mass under the area constraint condition are equivalent to the area-constrained Willmore immersions. We refer to [6, 19] and the references therein for more material about the latter topic.

In order to motivate our main theorems, let us discuss more in detail the literature which is closest to our new results.

- Lamm-Metzger [18] proved that, given a closed 3-dimensional Riemannian manofold (M, g), there exists $\varepsilon_0 > 0$ with the following property: for every $\varepsilon \in (0, \varepsilon_0]$ there exists an area-constrained Willmore sphere minimizing the Willmore functional among immersed spheres of area equal to $4\pi\varepsilon^2$. Moreover, as $\varepsilon \searrow 0$, such area-constrained Willmore spheres concentrate to a maximum point of the scalar curvature and, after suitable rescaling, they converge in $W^{2,2}$-sense to a round sphere.
- The above result has been generalized in two ways. On the one hand Rivière and the third author in [28, 29] proved that it is possible to minimize the Willmore energy among (bubble trees of possibly branched weak) immersed spheres of fixed area, for every positive value of the area. On the other hand Laurain and the third author in [21] showed that any sequence of area-constrained Willmore spheres with areas converging to zero and Willmore energy strictly below 32π (no matter if they minimize the Willmore energy) have to concentrate to a critical point of the scalar curvature and, after suitable rescaling, they converge *smoothly* to a round sphere.

Some natural questions then arise:

1. Is it true that around any critical point P_0 of the scalar curvature one can find a sequence of area-constrained Willmore spheres having area equal to $4\pi\varepsilon_n^2 \to 0$ and concentrating at P_0?
2. More precisely, can one find a foliation of a neighborhood of P_0 made by area-constrained Willmore spheres?
3. What about uniqueness/multiplicity?

The goal of the present paper is exactly to investigate the questions 1,2,3 above. More precisely, on the one hand we reinforce the assumption by asking that P_0 is a *non-degenerate* critical point of the scalar curvature (in the sense that the Hessian expressed in local coordinates is an invertible matrix); on the other hand we do not just prove the existence of area-constrained Willmore spheres concentrating at P_0 but we show that there exists a *regular foliation* of a neighborhood of P_0 made by area-constrained Willmore spheres. The precise statement is the following.

Theorem 1.1. *Let (M, g) be a 3-dimensional Riemannian manifold and let $P_0 \in M$ be a non-degenerate critical point of the scalar curvature* Sc*. Then there exist $\varepsilon_0 > 0$ and a neighborhood U of P_0 such that $U \setminus \{P_0\}$ is foliated by area-constrained Willmore spheres Σ_ε having area $4\pi\varepsilon^2$, $\varepsilon \in (0, \varepsilon_0)$. More precisely, there is a diffeomorphism $F : S^2 \times (0, \varepsilon_0) \to U \setminus \{P_0\}$ such that $\Sigma_\varepsilon := F(S^2, \varepsilon)$ is an area-constrained Willmore sphere having area equal to $4\pi\varepsilon^2$. Moreover*

- *If the index of P_0 as a critical point of* Sc *is equal to $3 - k$* [1] *, then each surface Σ_ε is an area-constrained critical point of W of index k.*
- *If $\mathrm{Sc}_{P_0} > 0$ then the surfaces Σ_ε have strictly positive Hawking mass.*
- *The foliation is* regular *at $\varepsilon = 0$ in the following sense. Fix a system of normal coordinates of U centered at P_0 and indentify U with an open subset of $\mathbb{R}^3$; then, called $F_\varepsilon := \frac{1}{\varepsilon} F(\cdot, \varepsilon) : S^2 \to \mathbb{R}^3$, as $\varepsilon \searrow 0$ the immersions F_ε converge smoothly to the round unit sphere of $\mathbb{R}^3$ centered at the origin.*
- *The foliation is* unique *in the following sense. Let $V \subset U$ be another neighborhood of $P_0 \in M$ such that $V \setminus \{P_0\}$ is foliated by area-constrained Willmore spheres Σ'_ε having area $4\pi\varepsilon^2$, $\varepsilon \in (0, \varepsilon_1)$, and satisfying $\sup_{\varepsilon \in (0,\varepsilon_1)} W(\Sigma'_\varepsilon) < 32\pi$. Then there exists $\varepsilon_2 \in (0, \min(\varepsilon_0, \varepsilon_1))$ such that $\Sigma_\varepsilon = \Sigma'_\varepsilon$ for every $\varepsilon \in (0, \varepsilon_2)$.*

Foliations by area-constrained Willmore spheres have been recently investigated by Lamm-Metzger-Schulze [19] who proved that a non-compact 3-dimensional manifold which is asymptotically Schwartzschild with positive mass is foliated at infinity by area-constrained Willmore spheres of large area. Even if both ours and theirs construction rely on a suitable application of the Implicit Function Theorem, the two results and proofs are actually quite different. Theorem 1.1 gives a local foliation in a small neighborhood of a point and the driving geometric quantity is the scalar curvature. On the other hand, the main result in [19] is a foliation at infinity and the driving geometric quantity is the ADM mass of the manifold.

Local foliations by spherical surfaces in manifolds have already appeared in the literature, but mostly by constant mean curvature spheres. In particular we have been inspired by the seminal paper of Ye [38], producing a local foliation of constant mean curvature spheres near a non-degenerate critical point of the scalar curvature. On the other hand let us stress the difference between the two problems: finding a foliation by constant mean curvature spheres is a *second* order problem since the mean curvature is a second order elliptic operator, while finding a foliation by area-constrained Willmore spheres is a *fourth* order problem.

[1] The index of a non-degenerate critical point P_0 of a function $h : M \to \mathbb{R}$ is the number of negative eigenvalues of the Hessian of h at P_0

Let us also discuss the relevance of Theorem 1.1 in connection with the Hawking mass. From the note of Christodoulou and Yau [6], if (M, g) has non-negative scalar curvature then isoperimetric spheres (and more generally stable CMC spheres) have positive Hawking mass; it is also known (see for instance [7] or [31]) that, if M is compact, then small isoperimetric regions converge to geodesic spheres centered at a maximum point of the scalar curvature as the enclosed volume converges to 0. Moreover, from the aforementioned paper of Ye [38] it follows that near a non-degenerate maximum point of the scalar curvature one can find a foliation by stable CMC spheres, which in particular by [6] will have positive Hawking mass. Therefore a link between Hawking mass and critical points of the scalar curvature was already present in literature; Theorem 1.1 expresses this relation precisely.

We also establish the multiplicity of area-constrained Willmore spheres and generic multiplicity of foliations. Let us mention that, despite the rich literature about existence of area-constrained Willmore spheres, this is the first multiplicity result in general Riemannian manifolds (for a prescribed value of the area constraint).

Theorem 1.2. *Let (M, g) be a closed 3-dimensional Riemannian manifold. Let*

- *k=2, if M is simply connected (i.e. if and only if M is diffeomorphic to S^3 by the recent proof of the Poincaré conjecture);*
- *k=3, if $\pi_1(M)$ is a non-trivial free group;*
- *k=4, otherwise.*

Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ there exist at least k distinct area-constrained Willmore spheres of area $4\pi\varepsilon^2$.

Remark 1.3. Examples of manifolds having a non-trivial free group as the fundamental group are $M = (S^1 \times S^2)\# \cdots \#(S^1 \times S^2)$ (the connected sum of m copies of $S^1 \times S^2$, $m \geq 1$). On the other hand, the 3-dimensional real projective space $\mathbb{RP}^3$ and the 3-torus $S^1 \times S^1 \times S^1$ are instead an example of manifold where $k = 4$. An expert reader will observe that $k = \mathrm{Cat}(M) + 1$, where $\mathrm{Cat}(M)$ is the Lusternik-Schnirelmann category of M. This is not by chance, indeed Theorem 1.2 is proved by combining a Lyapunov-Schmidt reduction with the celebrated Lusternik-Schnirelmann theory.

We conclude with a remark about generic multiplicity of foliations. To this aim note that, fixed a compact manifold M, for generic metrics the scalar curvature is a Morse function.

Remark 1.4. Let (M, g) be a closed 3-dimensional manifold such that the scalar curvature $\mathrm{Sc} : M \to \mathbb{R}$ is a Morse function and denote with $b_k(M)$ the k^{th} Betti number of M, $k = 0, \ldots, 3$. Then, by the Morse inequalities, Sc has at least $b_k(M)$ non-degenerate critical points of index k and, by Theorem 1.1, each one of these points has an associated foliation by area-constrained Willmore spheres of index $3 - k$. In particular there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, there exist $b_k(M)$ distinct area-constrained Willmore spheres of area $4\pi\varepsilon^2$ and index $3 - k$, for $k = 0, \ldots, 3$; therefore there exist at least $\sum_{k=0}^{3} b_k(M)$ distinct area-constrained Willmore spheres of area $4\pi\varepsilon^2$.

Example 1.5. Since the Morse inequalities hold by taking the Betti numbers with coefficients in any field, we are free to choose $\mathbb{R}$ or $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ depending on convenience. Let us discuss some basic examples to illustrate the multiplicity statement in Remark 1.4.

- $M = S^3$. Then $b_0(M,\mathbb{R}) = b_3(M,\mathbb{R}) = 1$, $b_1(M,\mathbb{R}) = b_2(M,\mathbb{R}) = 0$ so generically there exits 2 distinct foliations of area-constrained Willmore spheres.
- $M = S^2 \times S^1$. Then $b_k(M,\mathbb{R}) = 1$ for $k = 0,\dots,3$, so generically there exist 4 distinct foliations of area-constrained Willmore spheres.
- $M = \mathbb{RP}^3$. Then $b_k(M,\mathbb{Z}_2) = 1$ for $k = 0,\dots,3$, so generically there exist 4 distinct foliations of area-constrained Willmore spheres.
- $M = S^1 \times S^1 \times S^1$. Then $b_k(M,\mathbb{R}) = 1$ for $k = 0,3$ and $b_k(M,\mathbb{R}) = 3$ for $k = 1,2$, so generically there exist 8 distinct foliations of area-constrained Willmore spheres.

An announcement of this paper was given in [12]. In the independent work [20] the authors obtained independently some results related to ours.

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2 Preliminaries

We first recall the definition and properties of the Willmore energy. Given a 3-dimensional Riemannian manifold (M,g) and a closed surface Σ immersed in M, the Willmore energy $W_g(\Sigma)$ is defined as

$$W_g(\Sigma) := \int_\Sigma H^2 d\sigma$$

where H is the mean curvature, $H = \operatorname{tr}(A)$ where A is the second fundamental form of Σ. Here we use the following convention for A:

$$A(X,Y) = g(\nabla_X n, Y)$$

and n is a (possibly just locally defined) unit normal to Σ. We also denote by W_{g_0} the Willmore energy in the Euclidean space $(\mathbb{R}^3, g_0)$. For W_{g_0}, we have

Proposition 2.1 ([37]). *For any immersed, closed surface $\Sigma \subset \mathbb{R}^3$, one has*

$$W_{g_0}(\Sigma) \geq 16\pi = W_{g_0}(S^2)$$

where $S^2 \subset \mathbb{R}^3$ is the standard sphere of unit radius.

Next we recall the first and second variation formulas for W_g. To be more precise, let $\Sigma \subset M$ be an immersed, closed and orientable surface and $F : (-\delta,\delta) \times \Sigma \to M$ denote a perturbation of Σ satisfying $\partial_t F = \varphi n$ where $n = n(t,p)$ is a unit normal to $F(t,\Sigma)$ and $\varphi := g(n, \partial_t F)$. We write Riem for the Riemann curvature tensor of M, Ric the Ricci tensor, Sc the scalar curvature,

$\mathring{A}$ the traceless second fundamental form, $\bar{g}$ the induced metric on Σ and Δ the Laplace–Beltrami operator on $(\Sigma, \bar{g})$. For Riem, we use the following convention:

$$\mathrm{Riem}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Moreover, we define a self-adjoint elliptic operator L by

$$L\varphi := -\Delta\varphi - (|A|^2 + \mathrm{Ric}(n,n))\varphi$$

and write ϖ for the tangential component of the one-form $\mathrm{Ric}(n,\cdot)$: $\varpi = \mathrm{Ric}(n,\cdot)^t$. Finally, define the $(2,0)$ tensor T by

$$T_{ij} = \mathrm{Riem}(\partial_i, n, n, \partial_j) = \mathrm{Ric}_{ij} + G(n,n)\bar{g}_{ij}$$

where $G = \mathrm{Ric} - (\frac{1}{2}\mathrm{Sc})g$ stands for the Einstein tensor of M.

Using these notations, we have the following formulas

Proposition 2.2 (see Section 3 in [19]). *With the above notation we have*

$$\delta W_g(\Sigma)[\varphi] := \frac{d}{dt} W_g(F(t,\Sigma))\Big|_{t=0} = \int_\Sigma \left(LH + \frac{1}{2}H^3\right)\varphi \, d\sigma$$

and

$$\begin{aligned}
\delta^2 W_g(\Sigma)[\varphi,\varphi] &:= \frac{d^2}{dt^2} W_g(F(t,\Sigma))\Big|_{t=0} \\
&= 2\int_\Sigma \left[(L\varphi)^2 + \frac{1}{2}H^2|\nabla\varphi|^2 - 2\mathring{A}(\nabla\varphi,\nabla\varphi)\right] d\sigma \\
&\quad + 2\int_\Sigma \varphi^2 \Big(|\nabla H|_g^2 + 2\varpi(\nabla H) + H\Delta H + 2g(\nabla^2 H, \mathring{A}) + 2H^2|\mathring{A}|_g^2 \\
&\quad + 2Hg(\mathring{A}, T) - Hg(\nabla_n \mathrm{Ric})(n,n) - \frac{1}{2}H^2|A|_g^2 - \frac{1}{2}H^2\mathrm{Ric}(n,n)\Big) d\sigma \\
&\quad + \int_\Sigma \left(LH + \frac{1}{2}H^3\right)\left(\frac{\partial\varphi}{\partial t}\Big|_{t=0} + H\varphi^2\right) d\sigma \\
&= 2\int_\Sigma \varphi \tilde{L}\varphi \, d\sigma + \int_\Sigma \left(LH + \frac{1}{2}H^3\right)\left(\frac{\partial\varphi}{\partial t}\Big|_{t=0} + H\varphi^2\right) d\sigma
\end{aligned}$$

where the fourth-order operator $\tilde{L}$ is defined by

$$\begin{aligned}
\tilde{L}\varphi = LL\varphi &+ \frac{1}{2}H^2 L\varphi + 2Hg(\mathring{A}, \nabla^2\varphi) + 2H\varpi(\nabla\varphi) + 2\mathring{A}(\nabla\varphi, \nabla H) \\
&+ \varphi\Big(|\nabla H|_g^2 + 2\varpi(\nabla H) + H\Delta H + 2g(\nabla^2 H, \mathring{A}) \\
&\qquad + 2H^2|\mathring{A}|_g^2 + 2Hg(\mathring{A}, T) - H(\nabla_n \mathrm{Ric})(n,n)\Big).
\end{aligned}$$

For later use, we make some comments in the case $(M,g) = (\mathbb{R}^3, g_0)$ and $\Sigma = S^2$. In this case, it is easily seen that

$$\delta W_{g_0}(S^2) = 0, \qquad \delta^2 W_{g_0}(S^2)[\varphi,\varphi] = \int_{S^2} \varphi \tilde{L}_0 \varphi \, \mathrm{d}s, \quad \tilde{L}_0\varphi := \Delta(\Delta + 2)\varphi.$$

Furthermore, we see

$$\text{Ker}\,\tilde{L}_0 = \{Z_0, Z_1, Z_2, Z_3\} =: \mathcal{K}_0 \tag{3}$$

where $\Delta Z_0 = 0 = (\Delta + 2)Z_i$ $(1 \le i \le 3)$ and Z_j $(0 \le j \le 4)$ are given by

$$Z_0(q) \equiv 1 = \frac{H_0(q)}{2}, \quad Z_i(q) = g_0(\mathbf{e}_i, q) \quad \text{for } q \in S^2, \tag{4}$$

where H_0 is the mean curvature of S^2 and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ the canonical basis of $\mathbb{R}^3$. These properties will be used in Section 3.

3 Finite-dimensional reduction procedure

In this section we perform a Lyapunov–Schmidt reduction in order to reduce our problem into a finite-dimensional one, see [1] for a general introduction to this method. For most part of this paper we will work on a neighborhood of $P_0 \in M$ where P_0 is a non-degenerate critical point of Sc, so we start by analyzing this scenario. Let us fix such a neighborhood U_0 with $\overline{U}_0$ compact. Next, shrinking U_0 if necessary, we may find a local orthonormal frame $\{F_{P,1}, F_{P,2}, F_{P,3}\}_{P\in U_0}$. By using this frame, we may identify $T_P M$ with $\mathbb{R}^3$ and define the exponential map $\exp_P^g : B_{\rho_0}(0) \to M$ where $B_{\rho_0}(0)$ is a ball in the Euclidean space and $\rho_0 > 0$ independent of $P \in U_0$. We select a neighborhood V_0 of P_0 and $\varepsilon_0 > 0$ so that $\overline{V}_0 \subset U_0$, $\varepsilon S^2 \subset B_{\rho_0/2}(0)$ and $\exp_P^g(B_{2\varepsilon}(0)) \subset U_0$ for every $0 < \varepsilon \le \varepsilon_0$ and $P \in \overline{V}_0$. Moreover, since P_0 is a non-degenerate critical point of Sc, we may assume that the Hessian Hess (Sc) of Sc on $\overline{V}_0$ is invertible and P_0 is the only critical point of Sc in $\overline{V}_0$.

Next, we introduce the following metric g_ε which is useful when we observe objects satisfying the small area constraint:

$$g_\varepsilon(P) = \frac{1}{\varepsilon^2} g(P) \quad \text{for } P \in U_0.$$

As above, $\{\varepsilon F_{P,1}, \varepsilon F_{P,2}, \varepsilon F_{P,3}\}$ is an orthonormal frame for g_ε and we may regard $\exp_P^{g_\varepsilon}$ as the map from some open neighborhood in $\mathbb{R}^3$ into M. Writing $g_P := (\exp_P^g)^* g$ and $g_{\varepsilon,P} := (\exp_P^{g_\varepsilon})^* g_\varepsilon$ for the pull-backs of g and g_ε through the exponential maps, we can check the following: (see [10, 24])

(i) Let W_{g_ε} be the Willmore functional for (U_0, g_ε) and $\Sigma \subset U_0$ be an embedded surface. Denote by H_g and H_{g_ε} the mean curvature of Σ in g and g_ε, respectively. Then one has

$$H_{g_\varepsilon} = \varepsilon H_g, \quad W_{g_\varepsilon}(\Sigma) = W_g(\Sigma), \quad W'_{g_\varepsilon}(\Sigma) = \varepsilon^3 W'_g(\Sigma).$$

Therefore, we may see that Σ is a Willmore type surface in (U_0, g) if and only if it is so is in (U_0, g_ε).

(ii) The exponential map $\exp_P^{g_\varepsilon}$ is defined in $B_{\varepsilon^{-1}\rho_0}(0)$ and satisfies

$$\exp_P^{g_\varepsilon}(z) = \exp_P^g(\varepsilon z) \quad \text{for all } |z|_{g_0} \le \varepsilon^{-1}\rho_0 \quad \text{and} \quad P \in U_0.$$

Moreover, $g_{\varepsilon,P,\alpha\beta}$ can be expanded as

$$g_{\varepsilon,P,\alpha\beta}(y) = \delta_{\alpha\beta} + \varepsilon^2 h^\varepsilon_{P,\alpha\beta}(y) \quad \text{for } |y|_{g_0} \le \varepsilon^{-1}\rho_0 \tag{5}$$

and $g_{\varepsilon,P,\alpha\beta}$ satisfies

$$|y|^{-2}|D_P^k(g_{\varepsilon,P,\alpha\beta}(y)-\delta_{\alpha\beta})|+|y|^{-1}|D_P^kD_yg_{\varepsilon,P,\alpha\beta}(y)|+\sum_{j=2}^{\ell}|D_P^kD_y^jg_{\varepsilon,P,\alpha\beta}(y)|\le C_{k,\ell}\varepsilon^2 \tag{6}$$

for any $k,\ell\ge 0$ where D_P, D_y stand for derivatives in P and y, respectively.

Using the metric g_ε, we set

$$\mathcal{T}_{\varepsilon,V_0}:=\{\exp_P^{g_\varepsilon}(S^2)\mid P\in\overline{V_0}\}.$$

Notice that $\Sigma\subset U_0$ for each $\Sigma\in\mathcal{T}_{\varepsilon,V_0}$, by the properties of U_0 and V_0. Due to the above expansion of g_ε, elements in $\mathcal{T}_{\varepsilon,V_0}$ are approximate solutions to our problem. Namely,

Lemma 3.1. *For any $k,\ell\in\mathbb{N}$ and $j=0,1$, one has*

$$\left\|D_\varepsilon^jD_P^kW'_{g_\varepsilon}(\exp_P^{g_\varepsilon}(S^2))\right\|_{C^\ell(\Sigma)}\le C_{k,\ell}\varepsilon^{2-j}\quad\text{for all } P\in\overline{V_0}\text{ and }\varepsilon\in(0,\varepsilon_0)$$

where we regard $W'_{g_\varepsilon}(\Sigma)$ as a function satisfying $\delta W_{g_\varepsilon}(\Sigma)[\varphi]=\int_\Sigma W'_{g_\varepsilon}(\Sigma)\varphi d\sigma$.

Proof. Denote by $H_{\varepsilon,P}$, $A_{\varepsilon,P}$ and $\mathrm{Ric}_{\varepsilon,P}$ the mean curvature of $\exp_P^{g_\varepsilon}(S^2)$, the second fundamental form and the Ricci curvature in $(\mathbb{R}^3,g_{\varepsilon,P})$. Remark that the unit outer normal to $\exp_P^{g_\varepsilon}(S^2)$ is given by $n_0(q)=q$ ($q\in S^2$) by Gauss' lemma. We use the notation H_0, A_0 and so on for those in the Euclidean space. Recall from Proposition 2.2 that

$$W'_{g_\varepsilon}(\exp_P^{g_\varepsilon}(S^2))=-\Delta_{g_{\varepsilon,P}}H_{\varepsilon,P}-|A_{\varepsilon,P}|^2_{g_{\varepsilon,P}}H_{\varepsilon,P}-H_{\varepsilon,P}\mathrm{Ric}_{\varepsilon,P}(n_{\varepsilon,P},n_{\varepsilon,P})+\frac{1}{2}H^3_{\varepsilon,P}.$$

By (5) and (6), we observe that

$$\begin{aligned}&\left|D_\varepsilon^jD_P^kD_y^\ell\left(g^{\alpha\beta}_{\varepsilon,P}-\delta^{\alpha\beta}\right)\right|\le C_{k,\ell}\varepsilon^{2-j},\qquad \left|D_\varepsilon^jD_P^kD_y^\ell\mathrm{Ric}_{\varepsilon,P}\right|\le C_{k,\ell}\varepsilon^{2-j},\\&\left\|D_\varepsilon^jD_P^k(A_{\varepsilon,P}-A_0)\right\|_{C^\ell(S^2)}\le C_{k,\ell}\varepsilon^{2-j},\quad \left\|D_\varepsilon^jD_P^k(\Delta_{g_{\varepsilon,P}}-\Delta_{g_0})\varphi\right\|_{C^\ell(S^2)}\le C_{k,\ell}\varepsilon^{2-j}\|\varphi\|_{C^{\ell+2}(S^2)}\end{aligned}\tag{7}$$

for all $P\in\overline{V}_0$, $k,\ell\in\mathbb{N}$, $j=0,1$, $|y|_{g_0}\le\varepsilon^{-1}\rho_0$ and $\varphi\in C^{\ell+2}(S^2)$ where $C_{k,\ell}$ depends only on k,ℓ. Since $W'_{g_0}(S^2)=0$ and $H_{\varepsilon,P},H_0\in C^\infty(S^2)$, we obtain

$$\left\|D_\varepsilon^jD_P^kW'_{g_\varepsilon}(\exp_P^{g_\varepsilon}(S^2))\right\|_{C^\ell(S^2)}=\left\|D_\varepsilon^jD_P^k\left(W'_{g_\varepsilon}(\exp_P^{g_\varepsilon}(S^2))-W'_{g_0}(S^2)\right)\right\|_{C^\ell(S^2)}\le C_{k,\ell}\varepsilon^{2-j},$$

which completes the proof. □

Next, we find a correction for each $\Sigma\in\mathcal{T}_{\varepsilon,V_0}$ such that it satisfies the area constraint condition and it solves the equation up to some finite dimensional subspace in $L^2(S^2)$.

To this aim, recall that $n_0(q)=q$ is an outer unit normal to S^2 with respect to $g_{\varepsilon,P}$. For $\varphi\in C^{4,\alpha}(S^2)$ and $P\in\overline{V}_0$, we set

$$S^2_{\varepsilon,P}[\varphi]:=\left\{(1+\varphi(q))\,q\mid q\in S^2\right\}\subset\mathbb{R}^3,\quad \Sigma_{\varepsilon,P}[\varphi]:=\exp_P^{g_\varepsilon}(S^2_{\varepsilon,P}[\varphi]).$$

Since we are interested in small $\varphi\in C^{4,\alpha}(S^2)$, we pull back all geometric quantities of $S^2_{\varepsilon,P}[\varphi]$ (or $\Sigma_{\varepsilon,P}[\varphi]$) on S^2. We denote by $\bar g_{\varepsilon,P,\varphi}$ the pull back of the tangential metric of $S^2_{\varepsilon,P}[\varphi]$ on S^2 and by

$n_{\varepsilon,P,\varphi}$ the outer unit normal. We also write $L^2_{\varepsilon,P,\varphi}(S^2)$ and $\langle\cdot,\cdot\rangle_{\varepsilon,P,\varphi}$ the L^2-space on S^2 with volume induced by $\bar{g}_{\varepsilon,P,\varphi}$ and its inner product. We use the notations $\bar{g}_{0,\varphi}, n_{0,\varphi}, \dots$ for the corresponding quantities in the case $\varepsilon = 0$, i.e. the Euclidean space case.

Next, we define the space $\mathcal{K}_{\varepsilon,P,\varphi} \subset L^2_{\varepsilon,P,\varphi}(S^2)$ corresponding to $\mathcal{K}_0$ in (3). Recalling $Z_0 = H_0/2$, we set

$$\mathcal{K}_{\varepsilon,P,\varphi} := \text{span}\,\{H_{\varepsilon,P,\varphi}, Z_1, Z_2, Z_3\} \subset L^2_{\varepsilon,P,\varphi}(S^2), \tag{8}$$

where $H_{\varepsilon,P,\varphi}$ stands for the mean curvature of $\Sigma_{\varepsilon,P}[\varphi]$.
Next, we orthonormalize $H_{\varepsilon,P,\varphi}$, Z_1, Z_2 and Z_3 in $L^2_{\varepsilon,P,\varphi}(S^2)$ as follows. We first apply a Graham–Schmidt orthogonalization to Z_1, Z_2, Z_3 in $L^2_{\varepsilon,P,\varphi}$ to obtain $Y_{1,\varepsilon,P,\varphi}, Y_{2,\varepsilon,P,\varphi}, Y_{3,\varepsilon,P,\varphi}$. Finally, we get $Y_{0,\varepsilon,P,\varphi}$ from $H_{\varepsilon,P,\varphi}$ and $Y_{i,\varepsilon,P,\varphi}$ $(1 \le i \le 3)$. Then we define the $L^2_{\varepsilon,P,\varphi}(S^2)$-projection to $(\mathcal{K}_{\varepsilon,P,\varphi})^\perp$ by

$$\Pi^\varphi_{\varepsilon,P}\psi := \psi - \sum_{i=0}^{3} \langle \psi, Y_{i,\varepsilon,P,\varphi}\rangle_{\varepsilon,P,\varphi} Y_{i,\varepsilon,P,\varphi} : L^2_{\varepsilon,P,\varphi}(S^2) \to (\mathcal{K}_{\varepsilon,P,\varphi})^\perp.$$

We denote with $Y_{i,0,\varphi}$ and Π^φ_0 the corresponding quantities in the Euclidean space.

Lemma 3.2. *Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. Then there exist $r_1, \varepsilon_1, C_k > 0$ such that the maps*

$$(0,\varepsilon_1) \times \overline{V}_0 \times \overline{B_{r_1,C^{4+k,\alpha}}(0)} \ni (\varepsilon, P, \varphi) \mapsto Y_{i,\varepsilon,P,\varphi} \in C^{2+k,\alpha}(S^2) \quad (0 \le i \le 3)$$

are smooth, where $B_{r_1,C^{4+k,\alpha}}(0)$ stands for a metric ball in $C^{4+k,\alpha}(S^2)$. Moreover, the following estimates hold: for $j = 0, 1$,

$$\begin{aligned} &\left\| D^j_\varepsilon \left(Y_{i,\varepsilon,P,\varphi} - Y_{i,0,\varphi}\right)\right\|_{C^{2+k,\alpha}(S^2)} + \left\| D^j_\varepsilon D_P (Y_{i,\varepsilon,P,\varphi} - Y_{i,0,\varphi})\right\|_{\mathcal{L}(\mathbb{R}^3, C^{2+k,\alpha}(S^2))} \\ &\quad + \left\| D^j_\varepsilon D_\varphi (Y_{i,\varepsilon,P,\varphi} - Y_{i,0,\varphi})\right\|_{\mathcal{L}(C^{4+k,\alpha}(S^2), C^{2+k,\alpha}(S^2))} + \left\| D^j_\varepsilon D^2_{P,\varphi} (Y_{i,\varepsilon,P,\varphi} - Y_{i,0,\varphi})\right\|_{\mathcal{L}^2(C^{4+k,\alpha}(S^2), C^{2+k,\alpha}(S^2))} \\ &\qquad \le C_k \varepsilon^{2-j} \left(1 + \|\varphi\|_{C^{4+k,\alpha}(S^2)}\right). \end{aligned} \tag{9}$$

Proof. Let $X_\varphi(q)$ be the position vector for $S^2[\varphi]$:

$$X_\varphi(q) := (1 + \varphi(q))\, q \quad \text{for } q \in S^2. \tag{10}$$

Next, we fix small $r_1 > 0$ and $\varepsilon_1 > 0$ so that $S^2[\varphi]$ is diffeomorphic to S^2 and $\exp^{g_\varepsilon}_P(S^2_{\varepsilon,P}[\varphi])$ can be defined for all $\varepsilon \in [0, \varepsilon_1)$, $P \in \overline{V}_0$ and $\varphi \in C^{4+k,\alpha}(S^2, \mathbb{R})$ with $\|\varphi\|_{C^{4+k,\alpha}(S^2)} \le r_1$. Hereafter, we only deal with $\varphi \in \overline{B_{r_1,C^{4+k,\alpha}}(0)}$ and $\varepsilon \in [0, \varepsilon_1)$. Then it is easily seen that the map

$$\varphi \mapsto X_\varphi : \overline{B_{r_1,C^{4+k,\alpha}}(0)} \to C^{4+k,\alpha}(S^2, \mathbb{R}^3)$$

is smooth. Hence, we observe that the maps

$$\begin{aligned} (\varepsilon, P, \varphi) &\mapsto g_{\varepsilon,P}(X_\varphi) &&: (0,\varepsilon_1) \times \overline{V}_0 \times \overline{B_{r_1,C^{4+k,\alpha}}(0)} \to C^{4+k,\alpha}(S^2, (T\mathbb{R}^3)^* \otimes (T\mathbb{R}^3)^*), \\ (\varepsilon, P, \varphi) &\mapsto \bar{g}_{\varepsilon,P,\varphi},\ \bar{g}_{0,\varphi} &&: (0,\varepsilon_1) \times \overline{V}_0 \times \overline{B_{r_1,C^{4+k,\alpha}}(0)} \to C^{3+k,\alpha}(S^2, (TS^2)^* \otimes (TS^2)^*), \\ (\varepsilon, P, \varphi) &\mapsto n_{\varepsilon,P,\varphi},\ n_{0,\varphi} &&: (0,\varepsilon_1) \times \overline{V}_0 \times \overline{B_{r_1,C^{4+k,\alpha}}(0)} \to C^{3+k,\alpha}(S^2, \mathbb{R}^3), \\ (\varepsilon, P, \varphi) &\mapsto H_{\varepsilon,P,\varphi},\ H_{0,\varphi} &&: (0,\varepsilon_1) \times \overline{V}_0 \times \overline{B_{r_1,C^{4+k,\alpha}}(0)} \to C^{2+k,\alpha}(S^2, \mathbb{R}) \end{aligned}$$

are smooth. Moreover, by (5), we have

$$\left\|D_\varepsilon^j D_P^\ell D_\varphi^m \left(g_{\varepsilon,P}\left(X_\varphi\right) - g_0\left(X_\varphi\right)\right)\right\|_{\mathcal{L}^m\left(C^{4+k,\alpha}(S^2),C^{4+k,\alpha}(S^2)\right)} \le C_{k,\ell,m}\varepsilon^{2-j}\left(1+\|\varphi\|_{C^{4+k,\alpha}}\right)$$

for $j=0,1$ and $\ell,m=0,1,2$. Thus we obtain

$$\begin{aligned} &\left\|D_\varepsilon^j D_P^\ell D_\varphi^m \left(\bar{g}_{\varepsilon,P,\varphi} - \bar{g}_{0,\varphi}\right)\right\|_{C^{3+k,\alpha}(S^2)} + \left\|D_\varepsilon^j D_P^\ell D_\varphi^m \left(n_{\varepsilon,P,\varphi} - n_{0,\varphi}\right)\right\|_{C^{3+k,\alpha}(S^2)} \\ &\quad + \left\|D_\varepsilon^j D_P^\ell D_\varphi^m \left(H_{\varepsilon,P,\varphi} - H_{0,\varphi}\right)\right\|_{C^{2+k,\alpha}(S^2)} \le C_k\varepsilon^{2-j}\left(1+\|\varphi\|_{C^{4+k,\alpha}(S^2)}\right) \end{aligned} \tag{11}$$

for $j=0,1$ and $\ell,m=0,1,2$, where we used a shorthand notation to denote the norms in the space of (multi)-linear operators. Now it is easily seen that (9) holds and we complete the proof. $\square$

We next find a correction term for each element in $\mathcal{T}_{\varepsilon,V_0}$ so that the resulting surfaces solve the (area-constrained) Willmore equation up to an error in the finite dimensional subspace $\mathcal{K}_{\varepsilon,P,\varphi}$ in (8).

Proposition 3.3. *Let $\alpha\in(0,1)$. There exist $C>0$ and $\varepsilon_2>0$ so that for every $\varepsilon\in(0,\varepsilon_2)$ and $P\in\overline{V}_0$, there exists a unique $\varphi_{\varepsilon,P}\in C^{5,\alpha}(S^2)$ satisfying*

$$\text{(i)}\quad W'_{g_\varepsilon}(\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]) = \beta_0 H_{\varepsilon,P,\varphi_{\varepsilon,P}} + \sum_{i=1}^{3}\beta_i Z_i;\quad \text{(ii)}\quad |\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]|_{g_\varepsilon} = 4\pi;$$
$$\text{(iii)}\quad \langle\varphi_{\varepsilon,P}, Y_{i,\varepsilon,P}\rangle_{\varepsilon,P,\varphi_{\varepsilon,P}} = 0\ (1\le i\le 3);\qquad \text{(iv)}\quad \|\varphi_{\varepsilon,P}\|_{C^{5,\alpha}(S^2)}\le C\varepsilon^2$$

for some real numbers $\beta_0,\dots,\beta_3$ where $|\Sigma|_{g_\varepsilon}$ denotes the area of Σ in g_ε. Moreover, the map $(\varepsilon,P)\mapsto\varphi_{\varepsilon,P}:(0,\varepsilon_2)\times\overline{V}_0\to C^{5,\alpha}(S^2)$ is smooth and satisfies

$$\|D_P\varphi_{\varepsilon,P}\|_{\mathcal{L}(\mathbb{R}^3,C^{5,\alpha}(S^2))} + \|D_P^2\varphi_{\varepsilon,P}\|_{\mathcal{L}^2(\mathbb{R}^3,C^{5,\alpha}(S^2))}\le C\varepsilon^2,\quad \|D_\varepsilon\varphi_{\varepsilon,P}\|_{C^{5,\alpha}(S^2))}\le C\varepsilon\quad\text{for all } \varepsilon\in(0,\varepsilon_2).$$

Proof. Define a map $G(\varepsilon,P,\varphi):(0,\varepsilon_1)\times\overline{V}_0\times\overline{B_{r_1,C^{5,\alpha}}(0)}\to C^{1,\alpha}(S^2,\mathbb{R})$ by

$$G(\varepsilon,P,\varphi) := \Pi^\varphi_{\varepsilon,P}\left(W'_{g_\varepsilon}\left(\Sigma_{\varepsilon,P}[\varphi]\right)\right) + \left(|\Sigma_{\varepsilon,P}[\varphi]|_{g_\varepsilon} - 4\pi\right)H_{\varepsilon,P,\varphi} + \sum_{i=1}^{3}\langle Y_{i,\varepsilon,P,\varphi},\varphi\rangle_{\varepsilon,P,\varphi} Y_{i,\varepsilon,P,\varphi}.$$

By definition of $\mathcal{K}_{\varepsilon,P,\varphi}$, $Y_{j,\varepsilon,P,\varphi}$ and $\Pi^\varphi_{\varepsilon,P}$, to obtain the properties (i)–(iii) it is enough to find $\varphi_{\varepsilon,P}$ satisfying $G(\varepsilon,P,\varphi_{\varepsilon,P})=0$.

For this purpose, we show the existence of $\varepsilon_2>0$ and $r_2>0$ so that $D_\varphi G(\varepsilon,P,\varphi):C^{5,\alpha}(S^2)\to C^{1,\alpha}(S^2)$ is invertible for all $(\varepsilon,P,\varphi)\in(0,\varepsilon_2)\times\overline{V}_0\times\overline{B_{r_2,C^{5,\alpha}}(0)}$. We first remark that G is smooth in ε,P and φ. Moreover, by Lemma 3.2 together with (5)–(7), we have the following estimates

$$\sum_{k,\ell=0}^{2}\left\|D_\varepsilon^j D_P^k D_\varphi^\ell\left(G(\varepsilon,P,\varphi) - G_0(\varphi)\right)\right\|_{C^{1,\alpha}(S^2)}\le C\varepsilon^{2-j}\left(1+\|\varphi\|_{C^{5,\alpha}(S^2)}\right) \tag{12}$$

for each $j=0,1$ and $(\varepsilon,P,\varphi)\in(0,\varepsilon_1)\times\overline{V}_0\times\overline{B_{r_1,C^{5,\alpha}}(0)}$. Here $G_0(\varphi)$ is the corresponding map in the Euclidean space. Thanks to (12), it suffices to show that $D_\varphi G_0(0)$ is invertible.

For this, we recall from Proposition 2.2 and the comments below it that

$$D_\varphi W'_{g_0}(S^2[\varphi])\big|_{\varphi=0}[\psi] = \frac{d}{dt} W'_{g_0}(S^2[t\psi])\big|_{t=0} = \tilde{L}_0\psi = \Delta(\Delta+2)\psi$$

for $\psi \in C^{5,\alpha}(S^2)$. Hence, since $|S^2|_{g_0} = 4\pi$ and $W'_{g_0}(S^2) = 0$, we obtain

$$D_\varphi G_0(0)[\psi] = \Pi_0^0 \tilde{L}_0 \psi + \langle H_0, \psi\rangle_{L^2(S^2)} H_0 + \sum_{i=1}^{3} \langle Y_{i,0}, \psi\rangle_{L^2(S^2)} Y_{i,0}.$$

Noting that

$$\langle \varphi, \tilde{L}_0\psi\rangle_{L^2(S^2)} = \langle \tilde{L}_0\varphi, \psi\rangle_{L^2(S^2)}$$

holds for any $\varphi, \psi \in C^4(S^2)$ and that

$$\mathrm{Ker}\, \tilde{L}_0 = \mathcal{K}_0 = \mathrm{span}\,\{H_0, Y_{1,0}, Y_{2,0}, Y_{3,0}\},$$

we have $\Pi_0^0 \tilde{L}_0\psi = \tilde{L}_0\psi$ and

$$D_\varphi G_0(0)[\psi] = \tilde{L}_0 \psi + \langle H_0, \psi\rangle_{L^2(S^2)} H_0 + \sum_{i=1}^{3} \langle Y_{i,0}, \psi\rangle_{L^2(S^2)} Y_{i,0}. \tag{13}$$

Moreover, by Fredholm's alternative and elliptic regularity theory, we notice that

$$\tilde{L}_0\left(C^{5,\alpha}(S^2)\right) = C^{1,\alpha}(S^2) \cap \mathcal{K}_0^\perp.$$

Thus it follows from (13) and Schauder's estimates that $D_\varphi G_0(0)$ is invertible and by (12), we may find $\varepsilon_2 > 0$ and $r_2 > 0$ satisfying the desired property.

Now, for $\varepsilon \in [0, \varepsilon_2)$, the Inverse Mapping Theorem ensures the existence of neighborhoods $U_{1,\varepsilon,P} \subset C^{5,\alpha}(S^2)$ of 0 and $U_{2,\varepsilon,P} \subset C^{1,\alpha}(S^2)$ of $G(\varepsilon, P, 0)$ such that $G(\varepsilon, P, \cdot) : U_{1,\varepsilon,P} \to U_{2,\varepsilon,P}$ is a diffeomorphism. Furthermore, by (12) and the proof of the Inverse Mapping Theorem (see Lang [23, Theorem 3.1 in Chapter XVIII]), shrinking $r_2 > 0$ if necessary, we may assume that

$$\overline{B_{r_2, C^{5,\alpha}(S^2)}(0)} \subset U_{1,\varepsilon,P}, \qquad \overline{B_{2r_2, C^{1,\alpha}(S^2)}(G(\varepsilon, P, 0))} \subset U_{2,\varepsilon,P}$$

for all $(\varepsilon, P) \in (0, \varepsilon_2) \times \overline{V}_0$. Noting that $\|G(\varepsilon, P, 0)\|_{C^{1,\alpha}(0)} \leq C\varepsilon^2$ holds due to Lemma 3.1 and (5)–(6), shrinking $\varepsilon_2 > 0$ enough, we have

$$\overline{B_{r_2, C^{1,\alpha}(S^2)}(0)} \subset \overline{B_{2r_2, C^{1,\alpha}(S^2)}(G(\varepsilon, P, 0))}.$$

for each $(\varepsilon, P) \in (0, \varepsilon_2) \times \overline{V}_0$. In particular, $0 \in U_{2,\varepsilon,P}$ holds and setting $\varphi_{\varepsilon,P} := (G(\varepsilon, P, \cdot))^{-1}(0)$, we see that the properties (i)–(iii) hold.
From (12) and $G(\varepsilon, P_\varepsilon, \varphi_{\varepsilon,P}) = 0$, we get $\|G_0(\varphi_{\varepsilon,P})\|_{C^{1,\alpha}(S^2)} \leq C\varepsilon^2$. Thus, by the invertibility of G_0, also (iv) holds.

The smoothness of $\varphi_{\varepsilon,P}$ in (ε, P) follows from that of G and the Implicit Function Theorem. The estimates on $D_P^k \varphi_{\varepsilon,P}$, $k = 1, 2$, (resp. on $D_\varepsilon \varphi_{\varepsilon,P}$) follow from differentiating the equation $G(\varepsilon, P, \varphi_{\varepsilon,P}) = 0$ in P (resp. in ε) and using that $\|(D_P^k G)(\varepsilon, P, \varphi_{\varepsilon,P})\|_{C^{1,\alpha}(S^2)} \leq C\varepsilon^2$ for $k = 1, 2$ due to Lemmas 3.1 and 3.2, (5)–(7), (12) and the fact that $D_\varphi G_0(0)$ is invertible (resp. using that $\|(D_\varepsilon G)(\varepsilon, P, \varphi_{\varepsilon,P})\|_{C^{1,\alpha}(S^2)} \leq C\varepsilon$). Hence, the proof of Proposition 3.3 is complete. □

Recalling the function $\varphi_{\varepsilon,P}$ given by Proposition 3.3, we set

$$\Phi_\varepsilon(P) := W_{g_\varepsilon}(\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]) \in C^2(\overline{V}_0, \mathbb{R}).$$

Proposition 3.4. *There exists $\varepsilon_3 > 0$ such that if $\varepsilon \in (0, \varepsilon_3)$ and $P_\varepsilon \in V_0$ is a critical point of Φ_ε, then $\Sigma_{\varepsilon,P_\varepsilon}[\varphi_{\varepsilon,P_\varepsilon}]$ satisfies the area-constrained Willmore equation, namely, $\beta_1 = \beta_2 = \beta_3 = 0$ hold in Proposition* 3.3 (i)*.*

Proof. We first remark that the criticality of $\Phi_\varepsilon(P)$ is independent of the choices of charts; we will use normal coordinates with respect to the metric g_ε centered at P. We will use the same notation as in the proof of Lemma 3.2: in particular recall (10).

Assume that $P_\varepsilon \in V_0$ is a critical point of Φ_ε and let (U, Ψ) be a normal coordinate system centered at P_ε. For $P \in U$ with $z = \Psi(P)$, a position vector for $\Sigma_{\varepsilon,P}[\varphi]$ in (U, Ψ) has the form

$$\tilde{X}_{\varepsilon,P,\varphi}(q) = \mathcal{X}_\varepsilon(1; z, T_\varepsilon(z)(X_{\varepsilon,P_\varepsilon,\varphi}(q))) \quad \text{for } q \in S^2,$$

where $T_\varepsilon(z) : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation with $T_\varepsilon(0) = \mathrm{Id}$, $\mathcal{X}_\varepsilon$ a solution of

$$\frac{d^2\mathcal{X}_\varepsilon^\alpha}{dt^2} + \Gamma^\alpha_{\varepsilon,\beta\gamma}(\mathcal{X}_\varepsilon)\frac{d\mathcal{X}_\varepsilon^\beta}{dt}\frac{d\mathcal{X}_\varepsilon^\gamma}{dt} = 0, \qquad \left(\mathcal{X}_\varepsilon(0; z, v), \frac{d\mathcal{X}_\varepsilon}{dt}(0; z, v)\right) = (z, v) \in \mathbb{R}^6$$

and $\Gamma^\alpha_{\varepsilon,\beta\gamma}$ stand for the Christoffel symbols of (M, g_ε) in the coordinate system (U, Ψ). Using (5) and (6), we may observe that for any $k, \ell \geq 0$,

$$\begin{aligned} &\left\|D_z^{k+1}T_\varepsilon(z)\right\|_{L^\infty} \leq C_k\varepsilon^2, && \left\|\Gamma^\alpha_{\varepsilon,\beta\gamma}\right\|_{C^k} \leq C_k\varepsilon^2, \\ &\mathcal{X}_\varepsilon(1; z, v) = z + v + R_\varepsilon(z, v), && \left\|D_z^k D_v^\ell R_\varepsilon(z, v)\right\|_{L^\infty} \leq C_{k,\ell}\varepsilon^2. \end{aligned} \tag{14}$$

Therefore, we have

$$\tilde{X}_{\varepsilon,P,\varphi}(q) = z + X_{\varepsilon,P_\varepsilon,\varphi}(q) + \tilde{R}_\varepsilon(z, X_{\varepsilon,P_\varepsilon,\varphi}(q))$$

where $\tilde{R}$ satisfies the same estimate as R_ε in (14).

We now differentiate $\tilde{X}_{\varepsilon,P,\varphi_{\varepsilon,P}}(q)$ in z^i $(1 \leq i \leq 3)$. By Proposition 3.3 and (14), we obtain

$$\frac{\partial \tilde{X}_{\varepsilon,P,\varphi_{\varepsilon,P}}}{\partial z^i} = \mathbf{e}_i + O_{C^{5,\alpha}}(\varepsilon^2)$$

where $\|O_{C^{5,\alpha}}(\varepsilon^2)\|_{C^{5,\alpha}(S^2)} \leq C\varepsilon^2$. Moreover, one has

$$n_{\varepsilon,P,\varphi_{\varepsilon,P}} - n_0 = O_{C^{4,\alpha}}(\varepsilon^2).$$

Recalling the definition of $Z_i(q)$ in Section 2 and setting

$$\psi_{i,\varepsilon,P}(q) := g_\varepsilon(\tilde{X}_{\varepsilon,P,\varphi_{\varepsilon,P}}(q))\left(\frac{\partial \tilde{X}_{\varepsilon,P,\varphi_{\varepsilon,P}}(q)}{\partial z^i}, n_{\varepsilon,P,\varphi_{\varepsilon,P}}(q)\right), \tag{15}$$

we find that $\psi_{i,\varepsilon,P} - Z_i = O_{C^{4,\alpha}}(\varepsilon^2)$. Finally, differentiating $|\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]|_{g_\varepsilon} = 4\pi$ in z^i, we get

$$\left\langle H_{\varepsilon,P,\varphi_{\varepsilon,P}}, \psi_{i,\varepsilon,P}\right\rangle_{\varepsilon,P,\varphi_{\varepsilon,P}} = 0.$$

Now, by $\partial_{z^i}\Phi_\varepsilon(P_\varepsilon) = 0$ and $\langle Z_i, Z_j\rangle_{L^2(S^2)} = \delta_{ij}\|Z_i\|^2_{L^2(S^2)}$ due to (4), we have

$$0 = \partial_{z^i}\Phi_\varepsilon(P_\varepsilon) = \left\langle W'_{g_\varepsilon}(\Sigma_{\varepsilon,P_\varepsilon}[\varphi_{\varepsilon,P_\varepsilon}]), \psi_{i,\varepsilon,P_\varepsilon}\right\rangle_{\varepsilon,P_\varepsilon,\varphi_{\varepsilon,P_\varepsilon}}$$
$$= \sum_{j=1}^{3} \beta_j \left\langle Z_j, \psi_{i,\varepsilon,P_\varepsilon}\right\rangle_{\varepsilon,P_\varepsilon,\varphi_{\varepsilon,P_\varepsilon}} = \sum_{j=1}^{3} \beta_j \left(\delta_{ij}\|Z_i\|^2_{L^2(S^2)} + O(\varepsilon^2)\right).$$

Thus there exists $\varepsilon_3 > 0$ such that if $\varepsilon \in (0, \varepsilon_3)$, then we have $\beta_1 = \beta_2 = \beta_3 = 0$ and Proposition 3.4 holds. □

Remark 3.5. If (M, g) is a 3-dimensional compact Riemannian manifold without boundary, then we can define globally the reduced functional $\Phi_\varepsilon : M \to \mathbb{R}$ as

$$\Phi_\varepsilon(P) := W_{g_\varepsilon}(\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]) \in C^2(M,\mathbb{R}) \quad \forall \varepsilon \in (0, \bar\varepsilon], \text{ for some } \bar\varepsilon = \bar\varepsilon(M) > 0.$$

Moreover P_ε is a critical point of Φ_ε if and only if the perturbed geodesic sphere $\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]$ is an area-constrained Willmore surface of area $4\pi\varepsilon^2$.
Indeed, for every $P \in M$ we can find a neighborhood $U_P \ni P$ and $\varepsilon_P > 0$ such that $\Phi_\varepsilon : U_P \to \mathbb{R}$ is well defined as above for every $\varepsilon \in (0, \varepsilon_P]$. Note that if two neighborhoods overlap, then the corresponding definitions of Φ_ε agree thanks to the uniqueness of $\varphi_{\varepsilon,P}$ in Proposition 3.3. By the compactness of M we can then find $P_1, \ldots, P_N$ so that $M = \cup_{i=1}^N U_{P_i}$. Hence, setting $\bar\varepsilon(M) = \min\{\varepsilon_{P_1}, \ldots, \varepsilon_{P_N}\} > 0$ and patching the local definitions of Φ_ε, the claim is proved.

4 Concentration of area-constrained Willmore spheres

The goal of this section is to prove the next result.

Theorem 4.1. *Let (M, g) be a 3-dimensional Riemannian manifold and* Sc *denote the scalar curvature of M. Assume that $P_0 \in M$ is a non-degenerate critical point of* Sc. *Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, there exists an area-constrained Willmore sphere $\Sigma_\varepsilon \subset M$ with $|\Sigma_\varepsilon|_g = 4\pi\varepsilon^2$ such that Σ_ε concentrates at P_0. More precisely Σ_ε is the normal graph of a function $\varphi_\varepsilon \in C^{5,\alpha}(S^2)$ over a geodesic sphere centered in P_ε satisfying:*

$$\Sigma_\varepsilon := \Sigma_{\varepsilon,P_\varepsilon}[\varphi_\varepsilon] := \exp^{g_\varepsilon}_{P_\varepsilon}(S^2_{\varepsilon,P_\varepsilon}[\varphi_\varepsilon]), \quad \|\varphi_\varepsilon\|_{C^{5,\alpha}} \le C\varepsilon^2, \tag{16}$$

for some constant $C = C(P_0) > 0$ independent of ε. Moreover, if the index of P_0 as a critical point of Sc *is equal to $3 - k$, then each surface Σ_ε is an area-constrained critical point of W of index k.*

In the proof of Theorem 4.1 we will use the next lemma.

Lemma 4.2. *There exist $\varepsilon_3 > 0$ and $C > 0$ such that if $\varepsilon \in (0, \varepsilon_3)$ and a function $\Psi(P) \in C^2(\overline{V}_0, \mathbb{R})$ satisfies*

$$\frac{1}{\varepsilon^2}\left\|\Psi - 16\pi + \frac{8\pi}{3}\varepsilon^2 \mathrm{Sc}\right\|_{C^2(\overline{V}_0)} \le C\varepsilon,$$

then Ψ has a unique critical point $P_\varepsilon \in \overline{V}_0$. Moreover, as $\varepsilon \to 0$, we have $P_\varepsilon \to P_0$.

Proof. We first recall that by the choice of V_0, P_0 is the unique critical point of Sc in $\overline{V}_0$ and Hess (Sc) invertible on $\overline{V}_0$. Then it is easily seen that for sufficiently small $\zeta_0 > 0$, if $\psi \in C^2(\overline{V}_0, \mathbb{R})$ satisfies $\|\psi - \mathrm{Sc}\|_{C^2(\overline{V}_0)} \leq \zeta_0$, then Hess ($\psi$) is invertible on $\overline{V}_0$ and ψ has a unique critical point in $\overline{V}_0$. Setting $\psi_\varepsilon(P) := \varepsilon^{-2}(\Psi(P) - 16\pi)$, note that $\|\psi_\varepsilon - \frac{8\pi}{3}\mathrm{Sc}\|_{C^2(\overline{V}_0)} \leq C\varepsilon$ and $D^k_P \psi_\varepsilon = \varepsilon^{-2} D^k_P \Psi$ for $k = 1, 2$. Thus, for sufficiently small $\varepsilon > 0$, Ψ has a unique critical point $P_\varepsilon \in \overline{V}_0$. Since $\psi_\varepsilon \to \mathrm{Sc}$ in the C^2-sense and P_0 is the unique critical point of Sc, we have $P_\varepsilon \to P_0$ as $\varepsilon \to 0$. □

Proof of Theorem 4.1. By Proposition 3.4 and Lemma 4.2, it is enough to prove that

$$\frac{1}{\varepsilon^2}\left\| \Phi_\varepsilon - 16\pi + \frac{8\pi}{3}\varepsilon^2 \mathrm{Sc} \right\|_{C^2(\overline{V}_0)} \leq C\varepsilon \tag{17}$$

where $C > 0$ is independent of ε. To this aim, we decompose $\Phi_\varepsilon - 16\pi + \frac{8\pi}{3}\varepsilon^2\mathrm{Sc}$ as follows:

$$\Phi_\varepsilon(P) - 16\pi + \frac{8\pi}{3}\varepsilon^2 \mathrm{Sc}_P = (\Phi_\varepsilon(P) - W_{g_\varepsilon}(\Sigma_{\varepsilon,P}[0])) + \left(W_{g_\varepsilon}(\Sigma_{\varepsilon,P}[0]) - 16\pi + \frac{8\pi}{3}\varepsilon^2\mathrm{Sc}_P \right).$$

For the latter part, we notice that in the C^0-sense, we have

$$\frac{1}{\varepsilon^2}\left\| W_{g_\varepsilon}(S^2_{\varepsilon,(\cdot)}[0]) - 16\pi + \frac{8\pi}{3}\varepsilon^2 \mathrm{Sc}_{(\cdot)} \right\|_{L^\infty(\overline{V}_0)} \leq C\varepsilon \tag{18}$$

where $C > 0$ is independent of ε. For instance, see [26, 17, 10]. For the C^2-estimate of (18), we provide here a self-contained argument; later, in Lemma 5.1, we will give sharper estimates building on top of [27]. Recalling the expansion of g_ε in (5) and (6), setting $t = \varepsilon^2$ and

$$g_{t,P,\alpha\beta}(y) := \delta_{\alpha\beta} + t h^{\sqrt{t}}_{P,\alpha\beta}(y),$$

we can check that $t \mapsto D^k_{(P,y)} g_{t,P,\alpha\beta}$ is of class $C^{1,1/2}$ at $t = 0$ for each $k \in \mathbb{N}$. Hence, writing $W_{t,P}$ for the Willmore functional with respect to the metric $g_{t,P}$, we observe that the map $t \mapsto D^k_P W_{t,P}(S^2)$ is also of class $C^{1,1/2}$ in t. Thus we have

$$\begin{aligned} &D^k_P \left(W_{t,P}(S^2) - W_{0,P}(S^2) - \frac{\partial}{\partial s} W_{s,P}(S^2)\Big|_{s=0} t \right) \\ =&D^k_P \int_0^t \left(\frac{\partial}{\partial s} W_{s,P}(S^2) - \frac{\partial}{\partial s} W_{s,P}(S^2)\Big|_{s=0} \right) \mathrm{d}s = O(t^{3/2}). \end{aligned}$$

¿From the C^0-estimate, we deduce that

$$\frac{\partial}{\partial s} W_{s,P}(S^2)\Big|_{s=0} = -\frac{8\pi}{3}\mathrm{Sc}_P.$$

Noting that $W_{0,P}(S^2) = 16\pi$ and $t = \varepsilon^2$, it follows that

$$\frac{1}{\varepsilon^2}\left\| W_{g_\varepsilon}(S^2_{\varepsilon,(\cdot)}[0]) - 16\pi + \frac{8\pi}{3}\varepsilon^2 \mathrm{Sc}_{(\cdot)} \right\|_{C^2(\overline{V}_0)} \leq C\varepsilon. \tag{19}$$

In order to conclude the proof of Theorem 4.1, we are left with showing:

$$\frac{1}{\varepsilon^2}\left\|\Phi_\varepsilon(\cdot)-W_{g_\varepsilon}(S^2_{\varepsilon,(\cdot)}[0])\right\|_{C^2(\overline{V}_0)}\le C\varepsilon. \tag{20}$$

For this purpose, let us denote by $X_{\varepsilon,P,s}(q)$ a position vector for $S^2_{\varepsilon,P}[s\varphi_{\varepsilon,P}]$ with $0\le s\le 1$, namely,

$$X_{\varepsilon,P,s}(q):=(1+s\varphi_{\varepsilon,P}(q))\,q\quad\text{for } q\in S^2,\ 0\le s\le 1.$$

We also write $n_{\varepsilon,P,s}(q)$ $(q\in S^2)$ for the outer unit normal to $S^2_{\varepsilon,P}[s\varphi_{\varepsilon,P}]$. Recall that

$$\left\|D^k_P n_{\varepsilon,P,s}\right\|_{C^{4,\alpha}(S^2)}\le C\varepsilon^2\quad\text{for } 0\le s\le 1,\ k=0,1,2,$$

where C is independent of s and ε. Thus setting

$$\psi_{\varepsilon,P,s}(q):=g_{\varepsilon,P}(X_{\varepsilon,P,s}(q))[\varphi_{\varepsilon,P}(q),n_{\varepsilon,P,s}(q)]\quad\text{for } q\in S^2$$

and recalling the estimates of $\varphi_{\varepsilon,P}$ in Proposition 3.3, it follows that

$$\|D^k_P\psi_{\varepsilon,P,s}\|_{C^{4,\alpha}(S^2)}\le C\varepsilon^2\quad\text{for } k=0,1,2,\ 0\le s\le 1. \tag{21}$$

Furthermore, since $\|D^k_P g_{\varepsilon,P}\|_{C^\ell}\le C_\ell\varepsilon^2$ holds for every $k=1,2$ and $\ell\in\mathbb{N}$, the estimates for $\varphi_{\varepsilon,P}$ and a similar argument to the proof of Lemma 3.1 imply

$$\left\|D^k_P W'_{g_\varepsilon}(S_{\varepsilon,P}[s\varphi_{\varepsilon,P}])\right\|_{C^0(S^2)}\le C\varepsilon^2\quad\text{for } k=0,1,2,\ 0\le s\le 1.$$

Now from

$$\begin{aligned}D^k_P\left(\Phi_\varepsilon(P)-W_{g_\varepsilon}(S^2_{\varepsilon,P}[0])\right)&=D^k_P\left(W_{g_\varepsilon}(S_{\varepsilon,P}[\varphi_{\varepsilon,P}])-W_{g_\varepsilon}(S^2_{\varepsilon,P}[0])\right)\\&=\int_0^1 D^k_P\frac{d}{ds}W_{g_\varepsilon}(S_{\varepsilon,P}[s\varphi_{\varepsilon,P}])ds\\&=\int_0^1 D^k_P\left(W'_{g_\varepsilon}(S_{\varepsilon,P}[s\varphi_{\varepsilon,P}])[\psi_{\varepsilon,P,s}]\right)ds,\end{aligned} \tag{22}$$

it follows that

$$\left\|D^k_P\left(\Phi_\varepsilon(\cdot)-W_{g_\varepsilon}(S^2_{\varepsilon,(\cdot)}[0])\right)\right\|_{L^\infty(\overline{V}_0)}\le C\varepsilon^4 \tag{23}$$

for $k=0,1,2$. Hence (20) holds and the claim (17) is a consequence of (19) and (20). The combination of Proposition 3.4 and Lemma 4.2 gives directly all the claims of Theorem 4.1; we just briefly add some details regarding the index identity.

Note that the indexes of P_ε as a critical point of Φ_ε and of $-\mathrm{Sc}$ agree thanks to (17); moreover, since $W''_{g_0}(S^2)$ is positive definite on the orthogonal complement to its kernel (made of constant and affine functions) and $\|\varphi_{\varepsilon,P_\varepsilon}\|_{C^{5,\alpha}(S^2)}\le C\varepsilon^2$, it holds that $W''_{\gamma_\varepsilon}(\Sigma_\varepsilon)$ is positive-definite on the L^2-orthogonal complement to $\{\psi_{i,\varepsilon,P_\varepsilon}\}_{i=1,2,3}$ defined in (15). Observing that the index of $W''_{g_\varepsilon}(\Sigma_\varepsilon)$ in the direction of the span of $\{\psi_{i,\varepsilon,P_\varepsilon}\}_{i=1,2,3}$ coincides with the index of P_ε as a critical point of Φ_ε and that the only missing direction is fixed by area-constraint, the claim on the index identity follows. □

We next prove multiplicity of area-constrained Willmore spheres for prescribed (small) area.

Proof of Theorem 1.2. Thanks to Remark 3.5, if M is closed, then we can define globally the reduced functional $\Phi_\varepsilon : M \to \mathbb{R}$ as

$$\Phi_\varepsilon(P) := W_{g_\varepsilon}(\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]) \in C^2(M,\mathbb{R}) \quad \forall \varepsilon \in (0,\bar{\varepsilon}], \text{ for some } \bar{\varepsilon} = \bar{\varepsilon}(M) > 0.$$

Moreover P_ε is a critical point of Φ_ε if and only if the perturbed geodesic sphere $\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}]$ is an area-constrained Willmore surface of area $4\pi\varepsilon^2$.
Note that if $P_1^\varepsilon \neq P_2^\varepsilon$ are distinct critical points of Φ_ε, then the corresponding area-constrained Willmore surfaces $\Sigma_{\varepsilon,P_1^\varepsilon}[\varphi_{\varepsilon,P_1^\varepsilon}]$ and $\Sigma_{\varepsilon,P_2^\varepsilon}[\varphi_{\varepsilon,P_2^\varepsilon}]$ are also distinct since by (iii) in Proposition 3.3 the graph function $\varphi_{\varepsilon,P}$ is L^2-orthogonal to the translations $\{Y_{j,\varepsilon,P}\}_{j=1,2,3}$.
The claim of Theorem 1.2 thus reduces to establish multiplicity of the critical points of $\Phi_\varepsilon : M \to \mathbb{R}$. ¿From the Lusternik-Schnirelman theory (see for instance [4, Theorem 1.15]), the number of critical points of a real valued C^2-function on M is bounded below by $\mathrm{Cat}(M)+1$ and, for a closed 3-dimensional manifold, the value for $\mathrm{Cat}(M)$ is computable in terms of the fundamental group [8, Corollary 4.2]:

- $\mathrm{Cat}(M) = 1$, if M is simply connected (i.e. if and only if M is diffeomorphic to S^3 by the recent proof of Poincaré's conjecture);
- $\mathrm{Cat}(M) = 2$ if $\pi_1(M)$ is a nontrivial free group;
- $\mathrm{Cat}(M) = 3$ otherwise.

Therefore, Theorem 1.2 then holds. □

5 Foliation

The goal of this section is to prove Theorem 1.1, namely the existence and uniqueness of a foliation by area-constrained Willmore spheres of a neighborhood of a non-degenerate critical point P_0 of the scalar curvature.

Before proving Theorem 1.1 in detail, let us briefly discuss what is the main geometric extra difficulty in establishing that the area-constrained Willmore spheres Σ_ε constructed in Theorem 4.1 form a foliation.
The main point is to show that the *centers* P_ε of Σ_ε converge fast enough, say at order $O(\varepsilon^2)$, to P_0 when compared to the shrinking *radius* of the spheres (which is of order $O(\varepsilon)$). This is best explained with an example: the round spheres $\Sigma_\varepsilon \subset \mathbb{R}^3$ in the Euclidean 3-dimensional space, of center $(\varepsilon, 0, 0)$ and radius ε are clearly (area-constrained) Willmore spheres concentrating at the origin $(0,0,0)$ but do not form a foliation (as they are not pairwise disjoint).
Showing that $\mathrm{d}_g(P_0, P_\varepsilon) = O(\varepsilon)$ is straightforward: just recall that P_ε is a critical point of Φ_ε and combine (17) with the assumption that P_0 is a non-degenerate critical point of Sc (so that there exists $C_{\mathrm{Sc}} > 0$ with $|\nabla \mathrm{Sc}(P)| \geq C_{\mathrm{Sc}}\, \mathrm{d}_g(P_0, P)$ near P_0). On the other hand, the estimate $\mathrm{d}_g(P_0, P_\varepsilon) = O(\varepsilon^2)$ requires more work.
The rough idea is to exploit the symmetry/anti-symmetry of the terms in the geometric expansions in order to show that the term of order $O(\varepsilon)$ vanishes. To this aim, we start by recalling the expressions of the terms involved in the Willmore equation on a small geodesic sphere, see for instance [27, Section 3.1]. For $P \in \overline{V}_0$, we set $\Sigma^0_{\varepsilon,P} := \exp^g_P(\varepsilon S^2)$ where $\varepsilon S^2 \subset T_P M \simeq \mathbb{R}^3$ is the

round sphere of radius ε parametrized by $q \in S^2$. Since in the arguments it will be enough to know whether a term is odd with respect to the antipodal map $q \mapsto -q$ of S^2, we will use the following shorthand notation:

$$\mathfrak{O} : S^2 \to \mathbb{R} \text{ will denote an arbitrary } \textit{odd} \text{ smooth function, i.e. } \mathfrak{O}(-q) = -\mathfrak{O}(q);$$

In order to keep the notation short, the functions $\mathfrak{O}$ will be allowed to vary from formula to formula and also within the same line; moreover $\mathfrak{O}$ will depend on P smoothly with $D_P^k\mathfrak{O}(-q) = -D_P^k\mathfrak{O}(q)$, but be independent of the parameter ε.

Lemma 5.1. *The following expansions hold: at $q \in S^2$ and $P \in \overline{V}_0$,*

$$H_{\Sigma^0_{\varepsilon,P}} = \frac{2}{\varepsilon} - \frac{\varepsilon}{3}\mathrm{Ric}_P(q,q) + \mathfrak{O}\varepsilon^2 + O_{C^2}(\varepsilon^3), \tag{24}$$

$$d\sigma_{\Sigma^0_{\varepsilon,P}} = \varepsilon^2\left(1 - \frac{\varepsilon^2}{6}\mathrm{Ric}_P(q,q) + \mathfrak{O}\varepsilon^3 + O_{C^2}(\varepsilon^4)\right) d\sigma_{S^2}, \tag{25}$$

$$W_g\left(\Sigma^0_{\varepsilon,P}\right) = 16\pi - \frac{8\pi}{3}\varepsilon^2\mathrm{Sc}_P + O_{C^2}(\varepsilon^4), \tag{26}$$

$$\frac{\partial}{\partial\varepsilon}W_g\left(\Sigma^0_{\varepsilon,P}\right) = -\frac{16\pi}{3}\varepsilon\mathrm{Sc}_P + O_{C^2}\left(\varepsilon^3\right) \tag{27}$$

where $d\sigma_{S^2}$ denotes the area element induced from the Euclidean metric and the terms $O_{C^2}(\varepsilon^k)$ satisfy $\sum_{i=0}^2 \|D_P^i O_{C^2}(\varepsilon^k)\|_{L^\infty(S^2)} \le C_0\varepsilon^k$.

Proof. We note that these results were essentially obtained in [27]. In fact, using a local orthonormal frame $\{F_{P,1}, F_{P,2}, F_{P,3}\}_{P\in\overline{V}_0}$ as in the beginning of section 3, $(\exp_P)^*g$ has the following expansion (see [24] and [32, Proposition 2.1]):

$$\begin{aligned}
&((\exp_P^g)^*g)(x)[F_{P,\alpha}, F_{P,\beta}] \\
=&\,\delta_{\alpha\beta} + \frac{1}{3}g_P[R_P(\Xi, F_{P,\alpha})\Xi, F_{P,\beta}] + \frac{1}{6}g_P[\nabla_\Xi R_P(\Xi, F_{P,\alpha})\Xi, F_{P,\beta}] + \frac{1}{20}g_P[\nabla_\Xi\nabla_\Xi R_P(\Xi, F_{P,\alpha})\Xi, F_{P,\beta}] \\
&+ \frac{2}{45}g_P[R_P(\Xi, F_{P,\alpha})\Xi, F_{P,\gamma}]\,g_P[R_P(\Xi, F_{P,\beta})\Xi, F_{P,\gamma}] + \mathrm{Rem}_{\alpha\beta}(x,P)
\end{aligned}$$

where $\Xi := x^\alpha F_{P,\alpha}$ and $|D_P^k\mathrm{Rem}_{\alpha\beta}(x,P)| \le C|x|^5$ for every $|x| \le \rho_0$, $P \in \overline{V}_0$ and $k \in \mathbb{N}$. Since the expansions (24) and (25) in the C^0 sense are obtained in [27, Lemmas 3.3 and 3.5], by the smooth dependence on P of the metric, we can also show (24) and (25) in the C^2 sense.

For (26), it follows from (24) that at $q = q^\alpha F_{P,\alpha}$ with $q \in S^2$,

$$H^2_{\Sigma^0_{\varepsilon,P}}(q) = \frac{1}{\varepsilon^2}\left(4 - \frac{4}{3}\varepsilon^2\mathrm{Ric}_P(q,q) + \mathfrak{O}\varepsilon^3 + O_{C^2}(\varepsilon^4)\right).$$

Noting $D_P^k\mathfrak{O}(-q) = -D_P^k\mathfrak{O}(q)$ and $\int_{S^2}\mathfrak{O}d\sigma_{S^2} = 0$, we observe from (25) that

$$W_g\left(\Sigma^0_{\varepsilon,P}\right) = \int_{S^2}\left[4 - 2\varepsilon^2\mathrm{Ric}_P(q,q) + \mathfrak{O}\varepsilon^3 + O_{C^2}(\varepsilon^4)\right]d\sigma_{S^2} = 16\pi - \frac{8\pi}{3}\varepsilon^2\mathrm{Sc}_P + O_{C^2}(\varepsilon^4).$$

Finally, for (27), we notice that

$$
\begin{aligned}
&\frac{\partial}{\partial\varepsilon}\left((\exp_P^g)^*g\right)(\varepsilon x)\left[F_{P,\alpha},F_{P,\beta}\right]\\
&=\frac{2\varepsilon}{3}g_P\left[R_P(\Xi,F_{P,\alpha})\Xi,F_{P,\beta}\right]+\frac{\varepsilon^2}{2}g_P\left[\nabla_\Xi R_P(\Xi,F_{P,\alpha})\Xi,F_{P,\beta}\right]+\frac{\varepsilon^3}{5}g_P\left[\nabla_\Xi\nabla_\Xi R_P(\Xi,F_{P,\alpha})\Xi,F_{P,\beta}\right]\\
&\quad+\frac{8\varepsilon^3}{45}g_P\left[R_P(\Xi,F_{P,\alpha})\Xi,F_{P,\gamma}\right]g_P\left[R_P(\Xi,F_{P,\beta})\Xi,F_{P,\gamma}\right]+\frac{\partial}{\partial\varepsilon}\mathrm{Rem}_{\alpha\beta}(\varepsilon x,P).
\end{aligned}
$$

We also remark that the last term satisfies

$$
\left|D_x^kD_P^\ell\frac{\partial}{\partial\varepsilon}\mathrm{Rem}_{\alpha\beta}(\varepsilon x,P)\right|\le C_{k,\ell}\varepsilon^4
$$

for any $k,\ell\in\mathbb{N}$. From these facts and the proof for (26), it is not difficult to check (27) and we complete the proof. ☐

Proof of Theorem 1.1. ¿From Theorem 4.1 we know that, for $\varepsilon_0>0$ small enough, for each $\varepsilon\in(0,\varepsilon_0)$ there exists an area-constrained Willmore sphere

$$
\Sigma_\varepsilon:=\Sigma_{\varepsilon,P_\varepsilon}[\varphi_{\varepsilon,P_\varepsilon}]:=\exp_{P_\varepsilon}^g(\varepsilon S^2[\varphi_{\varepsilon,P_\varepsilon}]),\quad|\Sigma_\varepsilon|_g=4\pi\varepsilon^2,\quad P_\varepsilon\to P_0. \tag{28}
$$

Recall also that Σ_ε is a critical point of Φ_ε. We also denote $\Sigma^0_{\varepsilon,P}:=\Sigma_{\varepsilon,P}[0]$.

Step 1. For a suitable neighborhood U of P_0 and $j=0,1$,

$$
\left\|D_\varepsilon^j\left(D_P\Phi_\varepsilon+\frac{8\pi}{3}\varepsilon^2\,D_P\mathrm{Sc}_{(\cdot)}\right)\right\|_{C^1(U)}\le C\varepsilon^{4-j}. \tag{29}
$$

To this aim, for all $P\in U\subset\overline{V}_0$, we first remark that

$$
\begin{aligned}
&\left|D_\varepsilon^j\left(D_P\Phi_\varepsilon(P)+\frac{8\pi}{3}\varepsilon^2\,D_P\mathrm{Sc}_P\right)\right|\\
&\le\left|D_\varepsilon^j\left[D_P\Phi_\varepsilon(P)-D_P(W(\Sigma^0_{\varepsilon,P}))\right]\right|+\left|D_\varepsilon^j\left[D_P(W(\Sigma^0_{\varepsilon,P}))+\frac{8\pi}{3}\varepsilon^2\,D_P\mathrm{Sc}_P\right]\right|.
\end{aligned}
$$

Recalling $W_g(\Sigma^0_{\varepsilon,P})=W_g(\exp_P^g(\varepsilon S^2))=W_{g_\varepsilon}(\exp_P^{g_\varepsilon}(S^2))$ and (22), we know that

$$
D_P\left(\Phi_\varepsilon-W_g\left(\Sigma^0_{\varepsilon,P}\right)\right)=\int_0^1 D_P\left(W'_{g_\varepsilon}\left(S_{\varepsilon,P}[s\varphi_{\varepsilon,P}]\right)[\psi_{\varepsilon,P,s}]\right)ds.
$$

By Lemmas 3.1 and 3.2, Proposition 3.3 and (21), we may observe that for $j=0,1$,

$$
\left\|D_\varepsilon^jD_P\left(\Phi_\varepsilon-W_g\left(\Sigma^0_{\varepsilon,(\cdot)}\right)\right)\right\|_{C^1(U)}\le C\varepsilon^{4-j}.
$$

Thus, in order to get (29) it is enough to prove that

$$
\left\|D_\varepsilon^jD_P\left(W_g\left(\Sigma^0_{\varepsilon,(\cdot)}\right)+\frac{8\pi}{3}\varepsilon^2\,\mathrm{Sc}_{(\cdot)}\right)\right\|_{C^1(U)}\le C\varepsilon^{4-j}
$$

for $j = 0, 1$. But this is easily seen from $D_P 16\pi = 0$ and Lemma 5.1. Thus, Step 1 holds.

Step 2. $\mathrm{d}_g(P_\varepsilon, P_0) \le C\varepsilon^2$.
Since P_ε is a critical point of Φ_ε and we may assume $P_\varepsilon \in U$, Step 1 gives

$$0 = |D_P \Phi_\varepsilon(P_\varepsilon)| \ge |D_P \mathrm{Sc}_{P_\varepsilon}|\varepsilon^2 - C\varepsilon^4 \ge C_{\mathrm{Sc}}\, \mathrm{d}_g(P_0, P_\varepsilon)\varepsilon^2 - C\varepsilon^4,$$

which yields the claim $\mathrm{d}_g(P_\varepsilon, P_0) \le (C/C_{Sc})\varepsilon^2$.

Step 3. The surfaces $\{\Sigma_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0)}$, defined in (28), form a foliation of $U \setminus P_0$.
We will work with the following parametrisation of Σ_ε:

$$S^2[\varphi_{\varepsilon,P_\varepsilon}] := \big\{\big(1+\varphi_{\varepsilon,P_\varepsilon}(q)\big)\, q \mid q \in S^2\big\} \subset \mathbb{R}^3 \simeq T_{P_\varepsilon}M, \quad \Sigma_{\varepsilon,P_\varepsilon}[\varphi_{\varepsilon,P_\varepsilon}] := \exp^g_{P_\varepsilon}(\varepsilon S^2[\varphi_{\varepsilon,P_\varepsilon}]).$$

Set

$$F : S^2 \times (0,\varepsilon_0) \to M, \quad F(q,\varepsilon) := \exp^g_{P_\varepsilon}\Big(\varepsilon\big(1+\varphi_{\varepsilon,P_\varepsilon}(q)\big)q\Big).$$

For $\varepsilon_0 > 0$ small enough, we claim that $F(S^2 \times (0,\varepsilon_0)) = U \setminus \{P_0\}$ and that F is a diffeomorphism onto its image.
First of all we show that F is smooth. Thanks to Proposition 3.3, the map $(\varepsilon, P) \mapsto \varphi_{\varepsilon,P}$ is smooth. Also the map $\varepsilon \mapsto P_\varepsilon$ is smooth. Indeed P_ε is defined as the unique solution in U of

$$0 = D_P\Phi_\varepsilon = -\frac{8\pi}{3}\varepsilon^2 D_P \mathrm{Sc} + O_{C^1}(\varepsilon^4)$$

where in the last identity we used (29); since by assumption P_0 is a non-degenerate critical point of Sc, the Implicit Function Theorem guarantees the smoothness of $\varepsilon \mapsto P_\varepsilon$. We thus conclude that F is smooth, as composition of smooth maps. Moreover, differentiating $0 = D_P\Phi_\varepsilon(P_\varepsilon)$ in ε and using (29), we obtain

$$D^2_P\Phi_\varepsilon(P_\varepsilon)\frac{dP_\varepsilon}{d\varepsilon} = -D_\varepsilon D_P\Phi_\varepsilon(P_\varepsilon) = \frac{16\pi}{3}\pi\varepsilon D_P \mathrm{Sc}_{P_\varepsilon} + O(\varepsilon^3).$$

By $D_P\mathrm{Sc}|_{P=P_0} = 0$, $\mathrm{d}_g(P_\varepsilon, P_0) = O(\varepsilon^2)$ due to Step 2 and $(D^2_P\Phi_\varepsilon)^{-1}|_{P=P_\varepsilon} = O(\varepsilon^{-2})$ thanks to (29), we observe that

$$\left|\frac{dP_\varepsilon}{d\varepsilon}\right| \le C\varepsilon \quad \text{for } \varepsilon \in (0,\varepsilon_0). \tag{30}$$

We next claim that there exists $C > 0$ (independent of ε) such that

$$\left| g\left(\frac{\partial}{\partial\varepsilon}F(q,\varepsilon), n_\varepsilon(q)\right) - 1\right| \le C\varepsilon \quad \text{for every } (q,\varepsilon) \in S^2 \times (0,\varepsilon_0), \tag{31}$$

where $n_\varepsilon(q)$ is the outer unit normal to $\Sigma_\varepsilon = F(S^2 \times \{\varepsilon\})$ at $F(q,\varepsilon)$. To this aim we compute:

$$\begin{aligned}\frac{\partial F}{\partial\varepsilon}(q,\varepsilon) &= (D_P \exp^g_P(v))\Big|_{(P,v)=(P_\varepsilon,\varepsilon(1+\varphi_{\varepsilon,P_\varepsilon})q)}\frac{dP_\varepsilon}{d\varepsilon} \\ &\quad + (D_v \exp^g_{P_\varepsilon}(v))\big|_{v=\varepsilon(1+\varphi_{\varepsilon,P_\varepsilon}(q))q}\left(\Big(1+\varphi_{\varepsilon,P} + \frac{\partial\varphi_{\varepsilon,P}}{\partial\varepsilon} + D_P\varphi_{\varepsilon,P}\frac{dP_\varepsilon}{d\varepsilon}\Big)q\right)\Big|_{P=P_\varepsilon} \\ &= O(\varepsilon) + (D_v \exp^g_{P_\varepsilon})\big|_{v=\varepsilon(1+\varphi_{\varepsilon,P_\varepsilon}(q))q}\big(q+O(\varepsilon)\big), \end{aligned} \tag{32}$$

where in the second line we used Proposition 3.3 and (30).
The claim (31) follows by combining (11) with (32).
Since from Step 2 we know that $\mathrm{d}(P_\varepsilon, P_0) \le C\varepsilon^2$, the estimate in (31) ensures that $F(S^2 \times (0,\varepsilon_0)) = U \setminus \{P_0\}$ and that F is a diffeomorphism onto its image; in other words, F induces a foliation of $U \setminus \{P_0\}$ by the area-constrained Willmore spheres $\Sigma_\varepsilon = F(S^2 \times \{\varepsilon\})$.

Step 3. The foliation is regular at $\varepsilon = 0$.
Fix a system of normal coordinates of U centered at P_0, indentify U with an open subset of $\mathbb{R}^3$ and call $F_\varepsilon := \frac{1}{\varepsilon} F(\cdot, \varepsilon) : S^2 \to \mathbb{R}^3$. Since $\mathrm{d}_g(P_\varepsilon, P_0) \le C\varepsilon^2$ by Step 2 and $\|\varphi_\varepsilon\|_{C^{5,\alpha}(S^2)} \le C\varepsilon^2$ by (16), it follows that the immersions F_ε converge in $C^{5,\alpha}$-norm to the round unit sphere of $\mathbb{R}^3$, as $\varepsilon \searrow 0$. The convergence in $C^k(S^2)$-norm, for every $k \in \mathbb{N}$, follows from a standard bootstrap argument thanks to the ellipticity of $W'(S^2)$.

Step 4. The foliation is unique among Willmore spheres of energy $< 32\pi$.
Let $V \subset U$ be another neighborhood of $P \in M$ such that $V \setminus \{P\}$ is foliated by area-constrained Willmore spheres Σ'_ε having area $4\pi\varepsilon^2$, $\varepsilon \in (0, \varepsilon_1)$, and satisfying $\sup_{\varepsilon \in (0,\varepsilon_1)} W_g(\Sigma'_\varepsilon) < 32\pi$.
By [21], for ε small enough, Σ'_ε are normal graphs over geodesic spheres, i.e. there exist $P'_\varepsilon \in V, \varphi'_\varepsilon \in C^{5,\alpha}(S^2)$ such that

$$\Sigma'_\varepsilon = \Sigma_{\varepsilon, P'_\varepsilon}[\varphi'_\varepsilon] := \exp^g_{P'_\varepsilon}(\varepsilon S^2[\varphi'_\varepsilon]).$$

Since by assumption Σ'_ε is an area-constrained Willmore sphere, by the uniqueness statement in the Implicit Function Theorem, Proposition 3.3 implies that $\varphi'_\varepsilon = \varphi_{\varepsilon, P'_\varepsilon}$. Using again that Σ'_ε is an area-constrained Willmore sphere, we infer that P'_ε is a critical point of the reduced functional $V \ni P \mapsto \Phi_\varepsilon(P) := W_g(\Sigma_{\varepsilon,P}[\varphi_{\varepsilon,P}])$. But since by assumption P_0 is a non-degenerate critical point of Sc, the arguments above (30) yield that Φ_ε has a unique critical point in V. We conclude that $P'_\varepsilon = P_\varepsilon$ and thus $\Sigma_\varepsilon = \Sigma'_\varepsilon$ for ε small enough.

□

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