Some open problems on Feynman periods

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Abstract Feynman integrals of quantum field theories that contain non-scalar particles go beyond the well-studied leading period associated to primitive Feynman graphs. It is therefore necessary to study the space of periods spanned by all convergent Feynman integrals for a given graph. Even when the leading period is known, this total space of periods is not understood and carries non-trivial structures. After reviewing the leading period, we consider all convergent integrals of a graph and related open questions.

1 The leading Feynman period

Let G be a connected graph and $\mathscr{T}(G)$ the set of its spanning trees. We associate a variable α_e to each edge of G and define its *graph polynomial* [11, 31] as

$$\psi_G := \sum_{T \in \mathscr{T}(G)} \prod_{e \notin T} \alpha_e. \tag{1}$$

The inverse of this polynomial defines the integrand of logarithmically divergent Feynman amplitudes in a scalar quantum field theory. Concretely, let N denote the number of edges in G and note that ψ_G is homogeneous of degree $h_1(G) = \dim H_1(G)$, the number of independent cycles in G. This is also known as the *loop number* of the graph. If $N = 2h_1(G)$ and

$$\mathscr{P}(G) := \int_0^\infty \cdots \int_0^\infty \frac{\mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{N-1}}{\psi_G^2\big|_{\alpha_N=1}} > 0 \tag{2}$$

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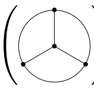
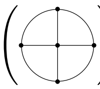
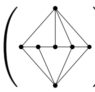
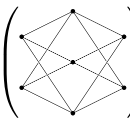
converges, then G is called *primitive log. div.* and the number $\mathscr{P}(G)$ is independent of the choice of the last edge. We will call these numbers *leading periods*. Their physical significance stems from the fact that they provide renormalization scheme independent contributions to the β-function of a four-dimensional theory.[1] Typical examples are

$\mathscr{P}\Big($  $\Big) = 6\zeta(3)$, $\mathscr{P}\Big($  $\Big) = 20\zeta(5)$ and $\mathscr{P}\Big($  $\Big) = \frac{1063}{9}\zeta(9) + 8\zeta^3(3)$.

Already thirty years ago, the periods for all primitive log. div. graphs with $h_1(G) \leq 6$ loops were evaluated in terms of Riemann zeta values $\zeta(n) = \sum_{k=1}^{\infty} k^{-n}$, their products and only one further number [13]. It took ten years [14] to identify this number,

$\mathscr{P}\Big($  $\Big) = \frac{16704}{5}\zeta(8) - \frac{6912}{5}\zeta(3,5) - 2592\zeta(3)\zeta(5), \qquad (3)$

because it involves the *multiple* zeta value $\zeta(3,5) = \sum_{1\leq k<r} 1/(k^3 r^5)$ which, conjecturally, cannot be expressed as a polynomial in Riemann zeta values with rational coefficients. More generally, multiple zeta values (MZV) are defined by

$$\zeta(n_1,\ldots,n_d) = \sum_{1\leq k_1<\cdots<k_d} \frac{1}{k_1^{n_1}\cdots k_d^{n_d}} \qquad (n_d \geq 2) \tag{4}$$

and they are of great interest in their own right. Many more occurrences of MZV in Feynman integrals have been found [39], which triggered intensive research by mathematicians and physicists. By now, Feynman integrals in general are well known to link methods from calculus, combinatorics, algebraic geometry and number theory (as illustrated in this proceedings volume; see also [5, 6, 16, 17]).

In these notes we will continue to consider only logarithmically divergent integrals, which means that they are determined by the polynomial ψ_G alone and do not depend on physical data like masses or momenta of elementary particles. For primitive log. div. graphs G, the leading periods (2) are now known for all G with $h_1(G) \leq 7$ loops [38]. In particular we now know the first explicit examples where $\mathscr{P}(G)$ is (conjecturally) not an MZV, but expressible as a linear combination of multiple polylogarithms (MPL)

$$\mathrm{Li}_{n_1,\ldots,n_d}(z) := \sum_{1\leq k_1<\cdots<k_d} \frac{z^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}} \tag{5}$$

[1] The β-function determines the running of the coupling constant under the evolution by the renormalization group and is a fundamental property of a quantum field theory [30].

at second or sixth roots of unity z [36, 38]. We also know many graphs for which there are strong indications to believe that they cannot be evaluated in terms of MPL at any algebraic arguments [21–23, 40].

While there are still many open questions and unresolved conjectures for the leading periods, we refer to [38] for a thorough discussion. Instead, we want to look at generalizations of the periods $\mathscr{P}(G)$ from (2) which are particularly important for gauge theories.

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2 Generalized Feynman periods

Feynman integrals are essential to the perturbative formulation of quantum field theory. Namely, observables like cross-sections of scattering processes are expressed as series over a (typically infinite) set of Feynman graphs. To each of these graphs, there is an associated Feynman integral which gives a contribution to the cross-section. We refer to [30] for a detailed introduction to these concepts and to [41] for a thorough discussion of Feynman integrals (we will work exclusively in the manifoldly named Schwinger-, Feynman- or α-representation).

What is important for these notes is that different types of particles give rise to different Feynman integrals. The leading period from Eq. (2) is only a very special case of a Feynman integral, namely under the assumptions that

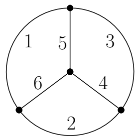
1. only scalar particles partake in the interaction,
2. the integral is logarithmically divergent,
3. there are no subdivergences and
4. the dimension of space-time is four.

The first constraint excludes all particles of the standard model (gauge bosons and fermions) with the sole exception of a scalar Higgs boson. However, there are well-known techniques which allow to reduce such Feynman integrals to linear combinations of scalar integrals [42]. The integrals arising after this procedure take the form (up to simple prefactors which are irrelevant to the following discussion)

$$I_\nu(G) := \int_0^\infty \cdots \int_0^\infty \frac{\alpha_1^{\nu_1-1}\mathrm{d}\alpha_1 \cdots \alpha_{N-1}^{\nu_{N-1}-1}\mathrm{d}\alpha_{N-1}}{\psi_G^{d/2}\Big|_{\alpha_N=1}} \tag{6}$$

and are indexed by a vector $\nu = (\nu_e)_{e=1\dots N} \in \mathbb{N}^N$ which encodes a monomial $\prod_e \alpha_e^{\nu_e-1}$ multiplying the integrand. The exponent $d/2$ in Eq. (6) is not fixed to

Table 1 Some periods of the wheel with 3 spokes graph W_3, including the leading period $\mathscr{P}(\mathrm{W}_3) = 6\zeta(3)$ from Eq. (2).

$\mathrm{W}_3 =$ 

$\prod_e \alpha_e^{\nu_e - 1}$	1	$\alpha_1\alpha_2\alpha_3$	$\alpha_1\alpha_2\alpha_4$	$\alpha_1^2\alpha_2^2\alpha_3\alpha_4$	$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$
d	4	6	6	8	8
$I_\nu(\mathrm{W}_3)$	$6\zeta(3)$	$\frac{1}{2}$	$\frac{1}{2}\zeta(3) - \frac{1}{6}$	$\frac{1}{24}$	$\frac{1}{10}\zeta(3) - \frac{7}{60}$

2; instead it can also take higher integer values subject to the condition

$$\sum_{e=1}^{N} \nu_e = \frac{d}{2} h_1(G) \tag{7}$$

which encodes that the integrand is homogeneous of degree zero (in particular, d is determined by ν). This ensures that $I_\nu(G)$ is independent of the choice of the N'th edge, corresponding to a logarithmic divergence. As a consequence, our integrals are independent of any kinematic data of the interacting particles like masses and momenta. Nevertheless these periods play a crucial role, because they compute massless propagators [35] and determine the renormalization group functions (β-functions and anomalous dimensions) in minimal subtraction schemes via standard techniques [32].

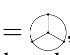
2.1 Convergence

The convergence of a generalized period can be inferred from a simple power-counting procedure [45]. In the integral representation (6), this amounts to considering the growth of the integrand as $\alpha_e \to 0$ for some subset $\gamma \ni e$ of edges.[2]

Lemma 1. *The generalized period $I_\nu(G)$ for $\nu \in \mathbb{N}^N$ converges precisely when*

$$\sum_{e \in \gamma} \nu_e > \frac{d}{2} h_1(\gamma) \tag{8}$$

holds for all non-empty proper subgraphs $\gamma \subsetneq G$. We call such indices ν convergent in d dimensions.

Example 1. For the wheel with 3 spokes graph $\mathrm{W}_3 =$ , the convergence conditions from triangle subgraphs γ are (with respect to the edge labels in Tab. 1)

$$\nu_1 + \nu_2 + \nu_3, \nu_1 + \nu_5 + \nu_6, \nu_2 + \nu_4 + \nu_6, \nu_3 + \nu_4 + \nu_5 > \frac{d}{2}. \tag{9}$$

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[2] Note that due to the projective nature of the integral (6), this region is equivalently described by $\alpha_e \to \infty$ for all $e \notin \gamma$, which we thus do not have to consider separately.

The 2-loop subgraphs $\gamma = G \setminus e$ yield the constraint $d < (\sum_{i=1}^{6} \nu_i) - \nu_e$, which is equivalent to $\nu_e < d/2$ via Eq. (7). Together with Eq. (9) and $\nu_e > 0$ (from $\gamma = \{e\}$), these conditions are also sufficient for the convergence of $I_\nu(\mathrm{W}_3)$.[3] The vector $\nu = (1,1,1,2,2,2)$ is not convergent, because it gives $d/2 = 3$ and thus violates the first triangle condition in Eq. (9). Examples of convergent periods are given in Tab. 1.

Remark 1. One can check that a graph must be 2-connected in order to have any convergent periods (otherwise, Eq. (8) has no solutions).[4] We will therefore only consider 2-connected graphs.

Remark 2. Recall that d is a function of ν by Eq. (7). Condition (8) is equivalent to

$$h_1(G) \sum_{e \in \gamma} \nu_e > h_1(\gamma) \sum_{e \in G} \nu_e. \tag{10}$$

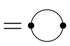
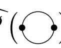
Definition 1. Given a graph G, we denote by $\widehat{\mathscr{P}}(G)$ the $\mathbb{Q}$-vector space generated by the convergent generalized Feynman periods of G in even dimensions:

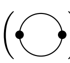
$$\widehat{\mathscr{P}}(G) := \mathrm{lin}_{\mathbb{Q}} \left\{ I_\nu(G) \colon \ \nu \text{ is convergent in an even dimension} \right\}. \tag{11}$$

Remark 3. Equivalently, $\widehat{\mathscr{P}}(G)$ is the set of all convergent integrals of the form

$$I_P(G) := \int_0^\infty \mathrm{d}\alpha_1 \cdots \int_0^\infty \mathrm{d}\alpha_{N-1} \left. \frac{P}{\psi_G^k} \right|_{\alpha_N = 1} \tag{12}$$

where $k \in \mathbb{N}$ and $P \in \mathbb{Q}[\alpha]$ is a homogeneous polynomial of degree $kh_1(G) - N$. Since we can multiply with $1 = \frac{\psi_G}{\psi_G}$, every period in dimension d is a linear combination of periods in dimension $d+2$. Also, from cohomology theory it is clear that $\widehat{\mathscr{P}}(G)$ is a finite dimensional vector space over $\mathbb{Q}$ (see Sect. 2.4.1).

Example 2. The simplest graph to consider is the bubble $G =$  with $\psi_G = \alpha_1 + \alpha_2$ and $d/2 = \nu_1 + \nu_2$. It has only rational periods, $\widehat{\mathscr{P}}\big($$\big) = \mathbb{Q}$, because

$I_\nu\Big($  $\Big)$
$$= \int_0^\infty \frac{\alpha_1^{\nu_1 - 1} \mathrm{d}\alpha_1}{(\alpha_1 + 1)^{\nu_1 + \nu_2}} = \frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1 + \nu_2)} = \frac{(\nu_1 - 1)!(\nu_2 - 1)!}{(\nu_1 + \nu_2 - 1)!} \in \mathbb{Q}. \tag{13}$$

2.2 General properties and relations

This section is a summary of very general results, while the next section will be more specific to particular examples of graphs.

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[3] More generally, for an arbitrary graph G, the only independent constraints on convergence are those arising from 2-connected subgraphs γ. Such graphs are usually called 1PI in physics. In the case of $G = \mathrm{W}_3$, the 2-connected subgraphs are precisely the edges, triangles and the graphs $G \setminus e$.

[4] G is 2-connected if it is connected and remains connected even after deletion of an arbitrary vertex.

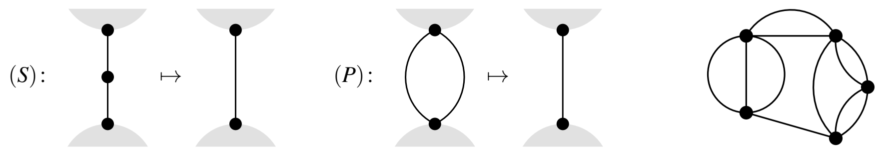


Fig. 1 The series and parallel operations (S) and (P) acting on a graph (the grey areas indicate that only a part of the actual graph is shown). On the right is an example of a series-parallel graph.

At first, let us recall the series-parallel operations on graphs (depicted in Fig. 1):

(S) replace two sequential edges (joined at a two-valent vertex) by a single edge,
(P) replace two parallel edges by a single edge.

A 2-connected graph is called *series-parallel* if it can be reduced to the bubble (equivalently, to a single edge) by a sequence of the operations (S) and (P). The following well-known result (see [16, 41]) follows from integrations of Euler's beta function similar to Eq. (13).

Lemma 2. *If G_1 can be obtained from G_2 by series-parallel operations, then we have $\widehat{\mathscr{P}}(G_1) = \widehat{\mathscr{P}}(G_2)$. In particular, if G is series-parallel, then $\widehat{\mathscr{P}}(G) = \mathbb{Q}$.*

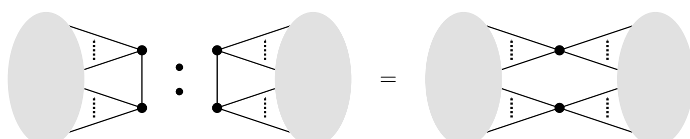


Fig. 2 The 2-vertex join of two graphs identifies a pair of vertices and deletes the edge between them.

Due to this result, it suffices to study graphs without any parallel or sequential edges. The next well-known simplification arises when G has a 2-cut, that is, there exist 2 vertices v and w such that $G \setminus \{v,w\}$ is disconnected. In such a situation we can partition the edges of G into two subgraphs G_1' and G_2' such that G_1' and G_2' intersect only at v and w (see Fig. 2). In this situation, we call G a 2-*vertex-join* of $G_1 := G_1' \dot\cup \{v,w\}$ and $G_2 := G_2' \dot\cup \{v,w\}$ and write $G = G_1 \colon G_2$. The following factorization is immediate in the momentum-space representation of Feynman integrals; a derivation directly in Schwinger parameters is given in [16, Proposition 40].

Lemma 3. *If $G = G_1 \colon G_2$ is a 2-vertex join, then $\widehat{\mathscr{P}}(G) = \widehat{\mathscr{P}}(G_1) \cdot \widehat{\mathscr{P}}(G_2)$.*

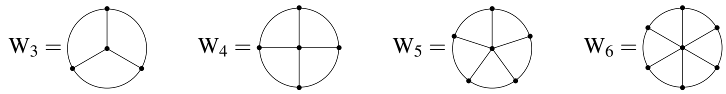


Fig. 3 The wheel graphs with three, four, five and six loops.

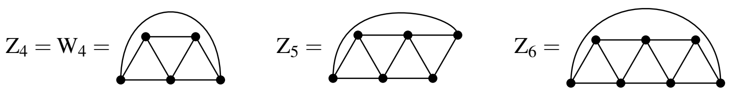
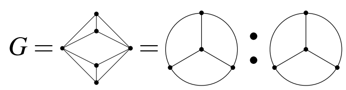


Fig. 4 The zig-zag graphs with four, five and six loops.

Example 3. The self-join of the wheel with 3 spokes gives the unique graph



(14)

with periods $\widehat{\mathscr{P}}(G) = \{p \cdot q \colon\ p,q \in \widehat{\mathscr{P}}(\mathrm{W}_3)\}$. From Tab. 1 we know $\mathbb{Q} + \mathbb{Q}\zeta(3) \subseteq \widehat{\mathscr{P}}(\mathrm{W}_3)$ and conclude that $\mathbb{Q} + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta^2(3) \subseteq \widehat{\mathscr{P}}(G)$. For instance, the leading period of G is $\mathscr{P}(G) = 36\zeta^2(3) = \mathscr{P}(\mathrm{W}_3)^2$.

Remark 4. The reduction of series-parallel graphs in Lemma 2 is actually a special case of the factorization Lemma 3 (by cutting the graph at the endpoints of the parallel or sequential edges).

We can now restrict our attention to 3-connected graphs (these are precisely the graphs *without* any 2-cuts). Recall that γ is called a *minor* of G if γ can be obtained from G by a sequence of deletions and contractions of edges. A proof of the following is given in [16, Proposition 37].

Theorem 1 (Minor monotonicity). *If γ is a minor of G, then $\widehat{\mathscr{P}}(\gamma) \subseteq \widehat{\mathscr{P}}(G)$.*

Example 4. Consider the family of wheel graphs depicted in Fig. 3. Their leading periods are known from [12]:

$$\mathscr{P}(\mathrm{W}_n) = \binom{2n-2}{n-1} \zeta(2n-3). \tag{15}$$

It is easy to see that each W_m with $m \leq n$ occurs as a minor of W_n. Hence their periods must appear as periods of W_n:

$$\mathbb{Q} + \mathbb{Q}\zeta(3) + \cdots + \mathbb{Q}\zeta(2n-3) \subseteq \widehat{\mathscr{P}}(\mathrm{W}_n)\,. \tag{16}$$

One can furthermore check that actually all minors of a wheel are either series-parallel or equivalent to another wheel (under series-parallel operations).

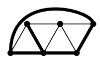
Example 5. The situation is quite different for another famous family of graphs, the zig-zags (depicted in Fig. 4). Their leading periods are also Riemann zetas [24],

$$\mathscr{P}(\mathrm{Z}_n) = 4\frac{(2n-2)!}{n!(n-1)!}\left(1-\frac{1-(-1)^n}{2^{2n-3}}\right)\zeta(2n-3), \tag{17}$$

and they contain the smaller zig-zags as minors. But they have more minors, including products. For example, we find $\mathrm{W}_3 \mathbin{:} \mathrm{W}_3$ in Z_6 (delete the middle edge of the baseline at the bottom and contract the outer arc) and hence deduce $\zeta^2(3) \in \widehat{\mathscr{P}}(\mathrm{Z}_6)$. The same reasoning shows that $\widehat{\mathscr{P}}(\mathrm{Z}_n)$ must contain many products of MZV when n gets large. We do not expect such products for the wheels (Conj. 1 below).

Remark 5. The reasoning above does not explain all products. For example, $\mathrm{W}_3 \mathbin{:} \mathrm{W}_3$ is not a minor of Z_5, but still we find $\zeta^2(3) \in \widehat{\mathscr{P}}(\mathrm{Z}_5)$. It shows up, for instance, in

$$I_\nu(\mathrm{Z}_5) = -\frac{5}{3} + \frac{161}{6}\zeta(3) + \frac{70}{3}\zeta(5) + \zeta^2(3) - \frac{441}{8}\zeta(7) \tag{18}$$

where we set $\nu_e = 2$ for the thick edges in  and $\nu_e = 1$ otherwise.

Because a minor is a quotient of a subgraph, Thm. 1 is a special case of

Theorem 2. *If γ is a subgraph of G, then $\widehat{\mathscr{P}}(\gamma) \cdot \widehat{\mathscr{P}}(G/\gamma) \subseteq \widehat{\mathscr{P}}(G)$.*

Proof. It is well-known that for a subgraph γ of G, the graph polynomial of G factorizes to leading order in the subgraph variables [5]. Concretely, if we substitute $\alpha_e = t\tilde{\alpha}_e$ for all $e \in \gamma$, then (recall that the degree of ψ is the loop number)

$$\psi_G(\alpha, \tilde{\alpha}, t) = \psi_\gamma(\tilde{\alpha})\psi_{G/\gamma}(\alpha)t^{h_1(\gamma)} + \mathscr{O}\left(t^{h_1(\gamma)+1}\right).$$

Let us label the edges of γ with $1, \ldots, N_\gamma$ and those of G/γ with $(N_\gamma + 1) \ldots N_G$. Consider a pair of convergent periods $I_{(\nu_1,\ldots,\nu_{N_\gamma})}(\gamma)$ and $I_{(\nu_{N_\gamma+1},\ldots,\nu_{N_G})}(G/\gamma)$. We may assume that they lie in the same dimension d (see Rem. 3). We claim that the period

$$\left(\prod_{e\in G}\int_0^\infty \alpha_e^{\nu_e-1}\mathrm{d}\alpha_e\right)\frac{d}{2}\frac{\left[\sum_{e\in\gamma}\alpha_e\partial_{\alpha_e} - h_1(\gamma)\right]\psi_G}{\psi_G^{d/2+1}}\delta(1-\alpha_{N_G}) \in \widehat{\mathscr{P}}(G)$$

is equal to the product $I_{\ldots}(\gamma)I_{\ldots}(G/\gamma)$ of our given pair of periods. To see this, we multiply with $1 = \int_0^\infty \mathrm{d}t\,\delta(\alpha_{N_\gamma} - t)$ and change variables $\alpha_e = t\tilde{\alpha}_e$ for all $e \in \gamma$ as above. The integrand then becomes

$$\left(\prod_{e\in\gamma}\tilde{\alpha}_e^{\nu_e-1}\mathrm{d}\tilde{\alpha}_e\right)\delta(1-\tilde{\alpha}_{N_\gamma})\left(\prod_{e\in G/\gamma}\alpha_e^{\nu_e-1}\mathrm{d}\alpha_e\right)\delta(1-\alpha_{N_G})(-\partial_t)\frac{t^{h_1(\gamma)d/2}}{\psi_G^{d/2}}$$

such that the integration of t becomes trivial. It produces precisely the product of the integrands of the sought-after periods of γ and G/γ, because the factorization of ψ_G mentioned above provides

□

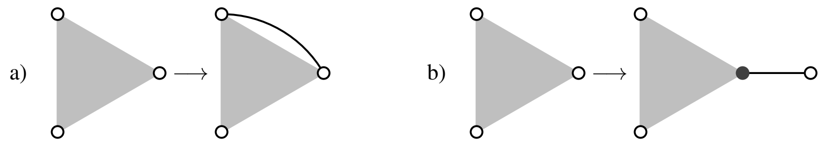
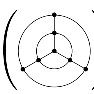


Fig. 5 The allowed steps to construct a graph of vertex-width 3 are: a) add an edge between two of the three marked vertices, b) attach a new vertex with a single (new) edge to one of the marked vertices (the new vertex becomes now marked, and the attachment point is not marked anymore).

$$\lim_{t\to 0} \frac{t^{h_1(\gamma)d/2}}{\psi_G^{d/2}} = \frac{1}{[\psi_\gamma(\tilde{\alpha})]^{d/2}} \frac{1}{[\psi_{G/\gamma}(\alpha)]^{d/2}}.$$

Example 6. Inserting W_3 into a vertex of another copy of W_3 yields a graph which, according to Thm. 2, has $\zeta^2(3)$ as a period. Note that the leading period itself [35],

$\mathscr{P}\Bigg($  $\Bigg) = 72\zeta^2(3) - \frac{189}{2}\zeta(7),$ (19)

is a combination of $\zeta^2(3)$ and $\zeta(7)$. Consequently, we know that $\zeta(7)$ on its own must also be a period of this graph.

Remark 6. This product structure in terms of sub-quotients is essential to the motivic theory of Feynman periods. In fact, a motivic version of Thm. 2 is true; see [19, Section 7.4].

2.3 Families of graphs with polylogarithmic periods

In general it is extremely hard to get a handle on all periods of a Feynman graph, because in most cases it is unknown what kind of numbers to expect. We will restrict here to very special cases where the periods can be expressed as MPL from Eq. (5). At the moment it is unknown how to decide if an arbitrary given graph belongs to this class. However, there are sufficient criteria which cover many cases of interest. In this section we summarize results from the integration of the integrals (6) with hyperlogarithms and refer to [9, 10, 15, 16, 36, 37] for a discussion of this method.

Definition 2 (from [16]). A graph G has *vertex-width* 3 if its edges can be ordered in such a way that the subgraphs formed by $\{e_1, \ldots, e_k\}$ and $\{e_{k+1}, \ldots, e_N\}$ have at most 3 vertices in common (for all $1 < k < N$).

Equivalently, G has vertex-width 3 if it can be constructed from the triangle by a sequence of the operations shown in Fig. 5 (the three white vertices mark the intersection of the subgraphs in Def. 2). Note that all wheels and zig-zags can be obtained this way, hence they are covered by the following result from [16, 36]:

Theorem 3. *If G has vertex-width 3, then all of its periods are rational linear combinations of MZV.*

Furthermore, only $\zeta(n_1,\ldots,n_r)$ of *weight* $n_1+\ldots+n_r \leq N-3$ can appear. Using the known relations among MZV, this implies for example that

$$\widehat{\mathscr{P}}(\mathrm{W}_4) \subseteq \mathbb{Q}+\mathbb{Q}\zeta(2)+\mathbb{Q}\zeta(3)+\mathbb{Q}\zeta(4)+\mathbb{Q}\zeta(5)+\mathbb{Q}\zeta(2)\zeta(3).$$

In fact, $\zeta(2)\zeta(3)$ can be excluded, because for graphs with $N=2h_1(G)$ the weight $N-3$ part of the periods is one-dimensional [21]. In this case it is spanned by the leading period $\mathscr{P}(\mathrm{W}_4)=20\zeta(5)$. Comparing this with the lower bound of Eq. (16), we miss the even zeta values. We make

Conjecture 1. Wheel graphs W_n ($n\geq 3$) have only odd Riemann zetas as periods:

$$\widehat{\mathscr{P}}(\mathrm{W}_n) = \mathbb{Q}+\mathbb{Q}\zeta(3)+\cdots+\mathbb{Q}\zeta(2n-3). \tag{20}$$

This conjecture is supported by explicit computation of periods of W_3, W_4 and W_5 using the methods of [36, 37]. It would also follow from several conjectures about *motivic* Feynman periods as explained in [19, Example 9.7]. Notice how far Thm. 3 still is from Eq. (20): We not only have to exclude *multiple* zeta values, but also all even Riemann zeta values $\zeta(2k)$.

Remark 7. Integrals of rational functions which evaluate to linear combinations of only odd Riemann zeta values are known from work on the irrationality of zeta values [3, 27]. This topic is nicely summarized in [18]. However, it seems unlikely that these integrals can be related to the Feynman integrals of the wheels in a straightforward way.

Note that Eq. (20) is false for the zig-zags Z_n due to the presence of products as demonstrated in Ex. 5 and Eq. (18). It is still striking that no even zeta values seem to appear in their periods, see Eq. (18). In fact, while $\zeta(12)$ is known to appear in periods [39], the even zeta values with lower weight have not been observed as periods of *any* graph.

Conjecture 2. $\zeta(2)$ is not a period of any graph: $\zeta(2)\notin\widehat{\mathscr{P}}(G)$.

Also this conjecture is supported by the motivic approach to Feynman periods [19]. Furthermore, the cosmic Galois group would imply that together with $\zeta(2)$, also all products with odd zeta values (like $\zeta(2)\zeta(3)$) would be excluded as periods of graphs. For a discussion of these ideas we refer to [19, 20], and physical interpretations of the absence of $\zeta(2)$ are given in [2, 33].

Inside the full space of 3-connected graphs, those with vertex-width 3 form a tiny subset. In particular note that graphs with vertex-width 3 are always planar. The first non-planar graph occurs at 4 loops (see Fig. 6), where the other two graphs have vertex-width 3. At 5 loops however there are also planar graphs which do not have vertex-width 3, like the cube in the bottom left of Fig. 7. The following was proved in [35].

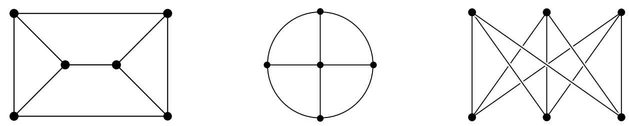


Fig. 6 All 3-connected graphs with four loops.

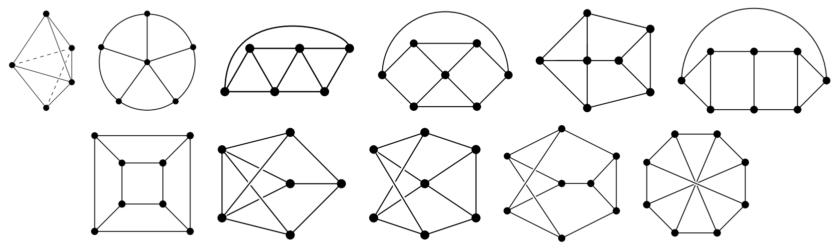


Fig. 7 All 3-connected graphs with five loops.

Theorem 4. *The periods of all planar graphs with ≤ 5 loops, except for the cube, are MZV. For the cube and the non-planar graphs with ≤ 5 loops, all periods are rational linear combinations of MPL at $z = -1$ (so-called* alternating sums*).*

The interesting observation is that all graphs with ≤ 5 loops seem to have only MZV as periods [1, 34, 35]; that is, all periods of the supposedly more complicated graphs (the non-planar ones and the cube) turn out to be MZV.

Conjecture 3. All periods of graphs with ≤ 5 loops are multiple zeta values.

The phenomenon that suitable linear combinations of alternating sums can combine to MZV is well-known. Such alternating sums are also called *honorary MZV*; one example is

$$\mathrm{Li}_{1,2,3}(-1) + \frac{7}{2}\mathrm{Li}_{1,5}(-1) = \frac{17}{64}\zeta^2(3) - \frac{1}{15}\zeta^3(2). \tag{21}$$

Very recently a basis of MZV in terms of such honorary alternating sums was constructed [29], using tools of motivic periods [28]. At the moment, however, we can only use these to show that a given individual computed period is an MZV (for this purpose it is also sufficient to look up the corresponding entry in the *datamine* [7]). The difficulty is to understand why this miracle has to happen for *all* periods of the graphs with ≤ 5 loops.

2.4 Further remarks

Finally we briefly mention further approaches and ideas which might help to understand the spaces of periods of Feynman graphs.

2.4.1 Integration by parts and master integrals

The periods of a graph fulfill a myriad of relations, because $\widehat{\mathscr{P}}(G)$ is finite-dimensional (over $\mathbb{Q}$). This comes about because the de Rham cohomology of the complement of the graph hypersurface $\{\alpha\colon\ \psi_G(\alpha)=0\}$ is finite dimensional. This cohomology was first studied in [5] and has been further explored since then [21, 26]. One might hope that at least for some interesting family of graphs, an algorithm to construct a finite generating set of cohomology classes might be devised. It would then suffice to compute each of the corresponding periods, which is manageable at a large scale because powerful integration algorithms have been implemented [8, 37].

Similar period relations coming from integration by parts (IBP) in momentum space have been studied systematically since [25] and have been exceedingly successful; currently they form an essential part of almost all perturbative calculations of loop corrections. However, the application to the high loop numbers we are interested in is extremely difficult. Only very recently it has been achieved at the 5-loop level [43]. Since this approach works in dimensional regularization (see Sect. 2.4.2), this alone does not yet solve our problem, because one needs to control the order of poles in ε.

For the wheels W_n the IBP identities were solved explicitly in [13] and yield

Lemma 4. *All periods of wheels are polynomials in Riemann zeta values:*

$$\widehat{\mathscr{P}}(\mathrm{W}_n) \subseteq \mathbb{Q}[\zeta(k)\colon\ k \geq 2]. \tag{22}$$

This follows because IBP identities express every period of a wheel as a linear combination of products of coefficients of the ε-expansion of the Γ function,

$$\Gamma(1-\varepsilon) = \exp\left[\varepsilon\gamma_E + \sum_{n\geq 2} \frac{\zeta(n)}{n}\varepsilon^n\right], \tag{23}$$

and one can show that the Euler-Mascheroni constant γ_E cancels in the end. Hence, MZV like $\zeta(3,5)$ are excluded from $\widehat{\mathscr{P}}(\mathrm{W}_n)$ and Lemma 4 gives a much better bound than the result of Thm. 3, but it is still a long way from Conj. 1.

2.4.2 Logarithmic periods and regularization

A further generalization of the period integrals $I_\nu(G)$ from Eq. (6) are the so called *logarithmic periods*

$$\int_0^\infty \cdots \int_0^\infty \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_{N-1}\, \frac{P(\alpha_1,\ldots,\alpha_N,\log(\alpha_1),\ldots,\log(\alpha_N),\log(\psi_G))}{\psi_G^{d/2}}\bigg|_{\alpha_N=1} \tag{24}$$

where the polynomial P is a homogeneous function of α. These integrals arise as coefficients if one expands $I_\nu(G)$ as a function of the ν_e and the dimension d. In physics, this *dimensional regularization* is a popular method to regularize divergent integrals (which appear abundantly before renormalization) and also essential to some renormalization schemes like minimal subtraction [32].

The vector space of logarithmic periods of a graph is infinite dimensional: Already the bubble  generates all Riemann zeta values and products from Eq. (13) via Eq. (23). But still there are interesting structures and several relations from Sect. 2.2 also hold for logarithmic periods [16]. In particular, it is known that all logarithmic periods of graphs with vertex-width 3 or loop order ≤ 5 are MZV or alternating sums and Conj. 3 is expected to hold also for the logarithmic periods.

However, as soon as logarithmic periods are considered, even zeta values do appear (so Conj. 2 does not hold for the spaces of logarithmic periods).

2.4.3 Motivic Feynman periods

Recently, the theory of motivic periods [20] was applied to Feynman integrals and appears very promising to understand some of the observed phenomena, as we already mentioned in several places above. Unfortunately, an adequate discussion of these ideas would be too lengthy to fit in here. Instead we advocate the comprehensive notes [19].

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