

Asymptotic Bias of Stochastic Gradient Search

Vladislav B. Tadić *

Arnaud Doucet †

Abstract. The asymptotic behavior of the stochastic gradient algorithm using biased gradient estimates is analyzed. Relying on arguments based on dynamic system theory (chain-recurrence) and differential geometry (Yomdin theorem and Lojasiewicz inequalities), upper bounds on the asymptotic bias of this algorithm are derived. The results hold under mild conditions and cover a broad class of algorithms used in machine learning, signal processing and statistics.

Keywords. Stochastic gradient search, biased gradient estimation, chain-recurrence, Yomdin theorem, Lojasiewicz inequalities.

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1. Introduction

Many problems in automatic control, system identification, signal processing, machine learning, operations research and statistics can be posed as a stochastic optimization problem, i.e., as a minimization (or maximization) of an unknown objective function whose values are available only through noisy observations. Such a problem can be solved efficiently by stochastic gradient search (also known as the stochastic gradient algorithm). Stochastic gradient search is a procedure of the stochastic approximation type which iteratively approximates the minima of the objective function using a statistical or Monte Carlo estimator of the gradient of the objective function. Often, the estimator is biased, since unbiased gradient estimation is usually either too computationally expensive or not available at all. As a result of using biased gradient estimates, the stochastic gradient search is also biased, i.e., the algorithm does not converge to the minima, but to their vicinity. In order to interpret the results produced by such an algorithm and to tune the algorithm's parameters (e.g., to achieve a better bias/variance tradeoff and a better convergence rate), it is important to study the asymptotic behavior and the asymptotic bias of the algorithm iterates.

Despite its practical and theoretical importance, the asymptotic behavior of stochastic gradient search using biased gradient estimates (also referred to as biased stochastic gradient search) has not attracted much attention in the literature. To the best of the authors' knowledge, this has only been analyzed in [12], [14], [15] and [16]. Although these results provide a good insight, they hold under restrictive conditions which are very hard to verify for complex stochastic gradient algorithms. Moreover, unless the objective function is of a simple form (e.g., convex or polynomial), none of these papers offers explicit bounds on the asymptotic bias of the algorithm iterates.

In this paper, we provide an original analysis of the asymptotic behavior of biased stochastic gradient search. Using arguments based on dynamic system theory (chain-recurrence) and differential geometry (Yomdin theorem and Lojasiewicz inequalities), we prove that the algorithm iterates converge to a vicinity of the set of minima. Relying on the same arguments, we also derive upper bounds on the radius of the vicinity (i.e., on the asymptotic bias of the algorithm iterates). Our results hold under mild and easily verifiable conditions and cover a broad class of complex stochastic gradient algorithms. We illustrate here how these results can be applied to the asymptotic analysis of a popular policy-gradient (reinforcement) learning algorithm proposed in [2]. In [33] (an extended version of this paper), these results have also been used to evaluate

*Department of Mathematics, University of Bristol, Bristol, United Kingdom (email: v.b.tadic@bristol.ac.uk).

†Department of Statistics, University of Oxford, Oxford, United Kingdom (email: doucet@stats.ox.ac.uk).

the asymptotic bias of an adaptive population Monte Carlo method and the asymptotic bias of recursive maximum split-likelihood estimation procedure for hidden Markov models.

The rest of this paper is organized as follows. The main results are presented in Section 2, where the biased stochastic gradient search is analyzed. In Section 3, these general results are applied to stochastic gradient algorithms with Markovian dynamics. In Section 4, we apply the results of Sections 2 and 3 to a policy-gradient algorithm. The results presented in Sections 2 – 4 are proved in Sections 5 – 8.

2. Main Results

In this section, the asymptotic behavior of the following algorithm is analyzed:

$$\theta_{n+1} = \theta_n - \alpha_n(\nabla f(\theta_n) + \xi_n), \quad n \geq 0. \quad (1)$$

Here, $f : \mathbb{R}^{d_\theta} \rightarrow \mathbb{R}$ is a differentiable function, while $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive real numbers. θ_0 is an \mathbb{R}^{d_θ} -valued random variable defined on a probability space (Ω, \mathcal{F}, P) , while $\{\xi_n\}_{n \geq 0}$ is an \mathbb{R}^{d_θ} -valued stochastic process defined on the same probability space. To allow more generality, we assume that for each $n \geq 0$, ξ_n is a random function of $\theta_0, \dots, \theta_n$. In the area of stochastic optimization, recursion (1) is known as a stochastic gradient search or stochastic gradient algorithm. The recursion minimizes the objective function $f(\cdot)$. The term $\nabla f(\theta_n) + \xi_n$ is interpreted as an estimator of the gradient $\nabla f(\theta_n)$, ξ_n representing the estimator's noise. For further details, see [27], [30] and references given therein.

Throughout the paper, the following notation is used. $\|\cdot\|$ and $d(\cdot, \cdot)$ stand for the Euclidean norm and the distance induced by the Euclidean norm (respectively). For $t \in (0, \infty)$ and $n \geq 0$, $a(n, t)$ is the integer defined as

$$a(n, t) = \max \left\{ k \geq n : \sum_{i=n}^{k-1} \alpha_i \leq t \right\}.$$

\mathcal{S} and $f(\mathcal{S})$ denote the sets of stationary points and critical values of $f(\cdot)$, i.e.,

$$\mathcal{S} = \{\theta \in \mathbb{R}^{d_\theta} : \nabla f(\theta) = 0\}, \quad f(\mathcal{S}) = \{f(\theta) : \theta \in \mathcal{S}\}. \quad (2)$$

For $\theta \in \mathbb{R}^{d_\theta}$, $\pi(\cdot; \theta)$ is the solution to the ODE $d\theta/dt = -\nabla f(\theta)$ satisfying $\pi(0; \theta) = \theta$. \mathcal{R} denotes the set of chain-recurrent points of this ODE, i.e., $\theta \in \mathcal{R}$ if and only if for any $\delta, t \in (0, \infty)$, there exist an integer $N \geq 1$, real numbers $t_1, \dots, t_N \in [t, \infty)$ and vectors $\vartheta_1, \dots, \vartheta_N \in \mathbb{R}^{d_\theta}$ (each of which can depend on θ, δ, t) such that

$$\|\vartheta_1 - \theta\| \leq \delta, \quad \|\pi(t_N; \vartheta_N) - \theta\| \leq \delta, \quad \|\vartheta_{k+1} - \pi(t_k; \vartheta_k)\| \leq \delta \quad (3)$$

for $1 \leq k < N$.

Elements of \mathcal{R} can be considered as limits to slightly perturbed solutions to the ODE $d\theta/dt = -\nabla f(\theta)$. As the piecewise linear interpolation of sequence $\{\theta_n\}_{n \geq 0}$ falls into the category of such solutions, the concept of chain-recurrence is tightly connected to the asymptotic behavior of stochastic gradient search. In [3], [4], it has been shown that for unbiased gradient estimates, all limit points of $\{\theta_n\}_{n \geq 0}$ belong to \mathcal{R} and that each element of \mathcal{R} can potentially be a limit point of $\{\theta_n\}_{n \geq 0}$ with a non-zero probability.

If $f(\cdot)$ is Lipschitz continuously differentiable, it can be established that $\mathcal{S} \subseteq \mathcal{R}$. If additionally $f(\mathcal{S})$ is of a zero Lebesgue measure (which holds when $f(\mathcal{S})$ is discrete or when $f(\cdot)$ is d_θ -times continuously differentiable), then $\mathcal{S} = \mathcal{R}$. However, if $f(\cdot)$ is only Lipschitz continuously differentiable, then it is possible to have $\mathcal{R} \setminus \mathcal{S} \neq \emptyset$ (see [18, Section 4]). Hence, in general, a limit point of $\{\theta_n\}_{n \geq 0}$ is in \mathcal{R} but not necessarily in \mathcal{S} . For more details on chain-recurrence, see [3], [4], [12] and references therein. Given these results, it will prove useful to involve both \mathcal{R} and \mathcal{S} in the asymptotic analysis of biased stochastic gradient search.

The algorithm (1) is here analyzed under the following assumptions:

Assumption 2.1. $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Assumption 2.2. $\{\xi_n\}_{n \geq 0}$ admits the decomposition $\xi_n = \zeta_n + \eta_n$ for each $n \geq 0$. $\{\zeta_n\}_{n \geq 0}$ and $\{\eta_n\}_{n \geq 0}$ are \mathbb{R}^{d_θ} -valued stochastic processes (defined on (Ω, \mathcal{F}, P)) which satisfy

$$\lim_{n \rightarrow \infty} \max_{n \leq k < a(n,t)} \left\| \sum_{i=n}^k \alpha_i \zeta_i \right\| = 0, \quad \limsup_{n \rightarrow \infty} \|\eta_n\| < \infty \quad (4)$$

almost surely on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$ for any $t \in (0, \infty)$.

Assumption 2.3.a. $\nabla f(\cdot)$ is locally Lipschitz continuous on \mathbb{R}^{d_θ} .

Assumption 2.3.b. $f(\cdot)$ is p -times differentiable on \mathbb{R}^{d_θ} , where $p > d_\theta$.

Assumption 2.3.c. $f(\cdot)$ is real-analytic on \mathbb{R}^{d_θ} .

Remark. Due to Assumption 2.1, $a(n, t)$ is well-defined and finite for all $t \in (0, \infty)$, $n \geq 0$.

Assumption 2.1 corresponds to the step-size sequence $\{\alpha_n\}_{n \geq 0}$ and is commonly used in the asymptotic analysis of stochastic gradient algorithms. It is satisfied if $\alpha_n = n^{-a}$ for $n \geq 1$, where $a \in (0, 1]$.

Assumption 2.2 is a noise condition. It can be interpreted as a decomposition of the gradient estimator's noise $\{\xi_n\}_{n \geq 0}$ into a zero-mean sequence $\{\zeta_n\}_{n \geq 0}$ (which is averaged out by step-sizes $\{\alpha_n\}_{n \geq 0}$) and the estimator's bias $\{\eta_n\}_{n \geq 0}$. Assumption 2.2 is satisfied if $\{\zeta_n\}_{n \geq 0}$ is a martingale-difference or mixingale sequence, and if $\{\eta_n\}_{n \geq 0}$ are continuous functions of $\{\theta_n\}_{n \geq 0}$. It also holds for gradient search with Markovian dynamics (see Section 3). If the gradient estimator is asymptotically unbiased (i.e., $\lim_{n \rightarrow \infty} \eta_n = 0$ almost surely), Assumption 2.2 reduces to the Kushner-Clark condition, the weakest noise assumption under which the almost sure convergence of (1) can be demonstrated.

Assumptions 2.3.a, 2.3.b and 2.3.c are related to the objective function $f(\cdot)$ and its analytical properties. Assumption 2.3.a is involved in practically any asymptotic result for stochastic gradient search (as well as in many other asymptotic and non-asymptotic results for stochastic and deterministic optimization). Although much more restrictive than Assumption 2.3.a, Assumptions 2.3.b and 2.3.c hold for a number of algorithms routinely used in engineering, statistics, machine learning and operations research. In Section 4, Assumptions 2.3.b and 2.3.c are shown to hold for a policy-gradient algorithm. In [33], the same assumptions are verified for an adaptive population Monte Carlo method and for recursive maximum split-likelihood estimation in hidden Markov models. In [31], Assumption 2.3.c (which is a special case of Assumption 2.3.b) has been shown to hold for recursive maximum (full) likelihood estimation in hidden Markov models. In [32], the same assumption has also been verified for supervised and temporal-difference learning, online principal component analysis, Monte Carlo optimization of controlled Markov chains and recursive parameter estimation in linear stochastic systems.

Compared to Assumption 2.3.a, Assumptions 2.3.b and 2.3.c allow some sophisticated results from differential geometry to be applied to the asymptotic analysis of stochastic gradient search. More specifically, Yomdin theorem (a qualitative version of Morse-Sard theorem; see [34] and Proposition 6.1 in Section 6) can be applied to functions satisfying Assumption 2.3.b, while Lojasiewicz inequalities (see [23], [24]; see also Proposition 6.2 in Section 6) hold for functions verifying Assumption 2.3.c. Using Yomdin theorem and Lojasiewicz inequalities, a more precise characterization of the asymptotic bias of the stochastic gradient search can be obtained (see Parts (ii) and (iii) of Theorem 2.1).

In order to state the main results of this section, we need some further notation. Let η denote the asymptotic magnitude of the gradient estimator's bias $\{\eta_n\}_{n \geq 0}$, i.e.,

$$\eta = \limsup_{n \rightarrow \infty} \|\eta_n\|. \quad (5)$$

Moreover, for a compact set $Q \subset \mathbb{R}^{d_\theta}$, let Λ_Q denote the event

$$\Lambda_Q = \liminf_{n \rightarrow \infty} \{\theta_n \in Q\} = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} \{\theta_k \in Q\}. \quad (6)$$

With this notation, our main result on the asymptotic bias of the recursion (1) can be stated as follows.

Theorem 2.1. *Suppose that Assumptions 2.1 and 2.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, the following is true:*

- (i) *If $f(\cdot)$ satisfies Assumption 2.3.a, there exists a (deterministic) non-decreasing function $\psi_Q : [0, \infty) \rightarrow [0, \infty)$ (independent of η and depending only on $f(\cdot)$) such that $\lim_{t \rightarrow 0} \psi_Q(t) = \psi_Q(0) = 0$ and*

$$\limsup_{n \rightarrow \infty} d(\theta_n, \mathcal{R}) \leq \psi_Q(\eta) \quad (7)$$

almost surely on Λ_Q .

- (ii) *If $f(\cdot)$ satisfies Assumption 2.3.b, there exists a real number $K_Q \in (0, \infty)$ (independent of η and depending only on $f(\cdot)$) such that*

$$\limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| \leq K_Q \eta^{q/2}, \quad \limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \leq K_Q \eta^q \quad (8)$$

almost surely on Λ_Q , where $q = (p - d_\theta)/(p - 1)$.

- (iii) *If $f(\cdot)$ satisfies Assumption 2.3.c, there exist real numbers $r_Q \in (0, 1)$, $L_Q \in (0, \infty)$ (independent of η and depending only on $f(\cdot)$) such that*

$$\limsup_{n \rightarrow \infty} d(\theta_n, \mathcal{S}) \leq L_Q \eta^{r_Q}, \quad \limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| \leq L_Q \eta^{1/2}, \quad \limsup_{n \rightarrow \infty} d(f(\theta_n), f(\mathcal{S})) \leq L_Q \eta \quad (9)$$

almost surely on Λ_Q .

Theorem 2.1 is proved in Sections 5 and 6, while its global version is provided in Appendix 1.

Remark. *If Assumption 2.3.b (or Assumption 2.3.c) is satisfied, then $\mathcal{S} = \mathcal{R}$. Hence, under Assumption 2.3.b, (7) still holds if \mathcal{R} is replaced with \mathcal{S} .*

Remark. *Function $\psi_Q(\cdot)$ depends on $f(\cdot)$ in two ways. First, it depends on $f(\cdot)$ through \mathcal{R} and its geometric properties. Second, it depends on $f(\cdot)$ through upper bounds of $\|\nabla f(\cdot)\|$ and Lipschitz constants of $\nabla f(\cdot)$. An explicit construction of $\psi_Q(\cdot)$ is provided in the proof of Part (i) of Theorem 2.1 (Section 5).*

Remark. *Like $\psi_Q(\cdot)$, constants K_Q and L_Q depend on $f(\cdot)$ through upper bounds of $\|\nabla f(\cdot)\|$ and Lipschitz constants of $\nabla f(\cdot)$. K_Q and L_Q also depend on $f(\cdot)$ through the Yomdin and Lojasiewicz constants (quantities $M_Q, M_{1,Q}, M_{2,Q}$ specified in Propositions 6.1, 6.2). Explicit formulas for K_Q and L_Q are included in the proof of Parts (ii) and (iii) of Theorem 2.1 (Section 6).*

According to the literature on stochastic optimization and stochastic approximation, stochastic gradient search with unbiased gradient estimates (the case when $\eta = 0$) exhibits the following asymptotic behavior. Under mild conditions, sequences $\{\theta_n\}_{n \geq 0}$ and $\{f(\theta_n)\}_{n \geq 0}$ converge to \mathcal{R} and $f(\mathcal{R})$ (respectively), i.e.,

$$\lim_{n \rightarrow \infty} d(\theta_n, \mathcal{R}) = 0, \quad \lim_{n \rightarrow \infty} d(f(\theta_n), f(\mathcal{R})) = 0 \quad (10)$$

almost surely on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$ (see [4, Proposition 4.1, Theorem 5.7] which hold under Assumptions 2.1, 2.2, 2.3.a). Under more restrictive conditions, sequences $\{\theta_n\}_{n \geq 0}$ and $\{f(\theta_n)\}_{n \geq 0}$ converge to \mathcal{S} and a point in $f(\mathcal{S})$ (respectively), i.e.,

$$\lim_{n \rightarrow \infty} d(\theta_n, \mathcal{S}) = 0, \quad \lim_{n \rightarrow \infty} \|\nabla f(\theta_n)\| = 0, \quad \lim_{n \rightarrow \infty} d(f(\theta_n), f(\mathcal{S})) = 0, \quad \limsup_{n \rightarrow \infty} f(\theta_n) = \liminf_{n \rightarrow \infty} f(\theta_n) \quad (11)$$

almost surely on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$ (see [4, Corollary 6.7] which holds under Assumptions 2.1, 2.2, 2.3.b). The same asymptotic behavior occurs when Assumptions 2.1, 2.3.a hold and $\{\xi_n\}_{n \geq 0}$ is a martingale-difference sequence (see [9, Proposition 1]). When the gradient estimator is biased (the case where $\eta > 0$), (10), (11) are not true any more. Now, the quantities

$$\limsup_{n \rightarrow \infty} d(\theta_n, \mathcal{R}), \quad \limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\|, \quad \limsup_{n \rightarrow \infty} d(f(\theta_n), f(\mathcal{R})), \quad \limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \quad (12)$$

are strictly positive and depend on η (it is reasonable to expect these quantities to decrease in η and to tend to zero as $\eta \rightarrow 0$). Hence, the quantities (12) and their dependence on η can be considered as a sensible

characterization of the asymptotic bias of the gradient search with biased gradient estimation. In the case of algorithm (1), such a characterization is provided by Theorem 2.1. The theorem includes relatively tight, explicit bounds on the quantities (12) in the terms of the gradient estimator's bias η and analytical properties of $f(\cdot)$.

The results of Theorem 2.1 are of a local nature. They hold only on the event where algorithm (1) is stable (i.e., where sequence $\{\theta_n\}_{n \geq 0}$ belongs to a compact set Q). Stating results on the asymptotic bias of stochastic gradient search in such a local form is quite sensible due to the following reasons. The stability of stochastic gradient search is based on well-understood arguments which are rather different from the arguments used here to analyze the asymptotic bias. Moreover (and more importantly), as demonstrated in Appendix 1, it is relatively easy to get a global version of Theorem 2.1 by combining the theorem with stability results for stochastic approximation (e.g., with the results of [11]). It is also worth mentioning that local asymptotic results are quite common in the areas of stochastic optimization and stochastic approximation (e.g., most of the results of [7, Part II], similarly as Theorem 2.1, hold only on set Λ_Q).

Stochastic gradient search with biased gradient estimation has found many applications in areas such as statistical inference, system identification and machine learning (see e.g., [8], [13], [17], [27], [28], [29] and reference cited therein). However, to the best of the authors' knowledge, the asymptotic properties of biased stochastic gradient search and biased stochastic approximation have only been studied in [12, Section 5.3], [14], [15], [16, Section 2.7]. The results obtained in these papers provide a good insight into the asymptotic behavior of the biased gradient search but are based on restrictive conditions. They only hold if $f(\cdot)$ is unimodal or if $\{\theta_n\}_{n \geq 0}$ belongs to the domain of an asymptotically stable attractor of $d\theta/dt = -\nabla f(\theta)$. Additionally, they do not provide any explicit bound on the asymptotic bias of the stochastic gradient search unless $f(\cdot)$ is of a simple form (e.g., convex or polynomial). Unfortunately, in the case of complex stochastic gradient algorithms (such as those studied in Section 4 and [33]), $f(\cdot)$ is usually multimodal with lot of unisolated local extrema and saddle points. For such algorithms, not only it is hard to verify the assumptions adopted in [12, Section 5.3], [14], [15], [16, Section 2.7], but these assumptions are likely not to hold at all.

Relying on the chain-recurrence, Yomdin theorem and Lojasiewicz inequalities, Theorem 2.1 overcomes the described difficulties. The theorem allows the objective function $f(\cdot)$ to be multimodal (with manifolds of unisolated extrema and saddle points) and does not require $d\theta/dt = -\nabla f(\theta)$ to have an asymptotically stable attractor which is infinitely often visited by $\{\theta_n\}_{n \geq 0}$. In addition to this, Theorem 2.1 provides relatively tight, explicit bounds on the asymptotic bias of algorithm (1).

3. Stochastic Gradient Search with Markovian Dynamics

In order to illustrate the results of Section 2 and to set up a framework for the analysis carried out in Section 4 and [33], we apply Theorem 2.1 to stochastic gradient algorithms with Markovian dynamics. These algorithms are defined by the following difference equation:

$$\theta_{n+1} = \theta_n - \alpha_n (F(\theta_n, Z_{n+1}) + \eta_n), \quad n \geq 0. \quad (13)$$

In this recursion, $F : \mathbb{R}^{d_\theta} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}^{d_\theta}$ is a Borel-measurable function, while $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive real numbers. θ_0 is an \mathbb{R}^{d_θ} -valued random variable defined on a probability space (Ω, \mathcal{F}, P) . $\{Z_n\}_{n \geq 0}$ is an \mathbb{R}^{d_z} -valued stochastic process defined on (Ω, \mathcal{F}, P) , while $\{\eta_n\}_{n \geq 0}$ is an \mathbb{R}^{d_θ} -valued stochastic process defined on the same probability space. $\{Z_n\}_{n \geq 0}$ is a Markov process controlled by $\{\theta_n\}_{n \geq 0}$, i.e., there exists a family of transition probability kernels $\{\Pi_\theta(\cdot, \cdot) : \theta \in \mathbb{R}^{d_\theta}\}$ defined on \mathbb{R}^{d_z} such that

$$P(Z_{n+1} \in B | \theta_0, Z_0, \dots, \theta_n, Z_n) = \Pi_{\theta_n}(Z_n, B) \quad (14)$$

almost surely for any Borel-measurable set $B \subseteq \mathbb{R}^{d_z}$ and $n \geq 0$. $\{\eta_n\}_{n \geq 0}$ are random functions of $\{\theta_n\}_{n \geq 0}$, i.e., η_n is a random function of $\theta_0, \dots, \theta_n$ for each $n \geq 0$. In the context of stochastic gradient search, $F(\theta_n, Z_{n+1}) + \eta_n$ represents an estimator of the gradient $\nabla f(\theta_n)$.

The algorithm (13) is analyzed under the following assumptions.

Assumption 3.1. $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Assumption 3.2. *There exist a differentiable function $f : \mathbb{R}^{d_\theta} \rightarrow \mathbb{R}$ and a Borel-measurable function $\tilde{F} : \mathbb{R}^{d_\theta} \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}^{d_\theta}$ such that $\nabla f(\cdot)$ is locally Lipschitz continuous and*

$$F(\theta, z) - \nabla f(\theta) = \tilde{F}(\theta, z) - (\Pi\tilde{F})(\theta, z) \quad (15)$$

for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, where $(\Pi\tilde{F})(\theta, z) = \int \tilde{F}(\theta, z')\Pi_\theta(z, dz')$.

Assumption 3.3. *For any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a Borel-measurable function $\varphi_Q : \mathbb{R}^{d_z} \rightarrow [1, \infty)$ such that*

$$\begin{aligned} \max\{\|F(\theta, z)\|, \|\tilde{F}(\theta, z)\|, \|(\Pi\tilde{F})(\theta, z)\|\} &\leq \varphi_Q(z), \\ \|(\Pi\tilde{F})(\theta', z) - (\Pi\tilde{F})(\theta'', z)\| &\leq \varphi_Q(z)\|\theta' - \theta''\| \end{aligned}$$

for all $\theta, \theta', \theta'' \in Q$, $z \in \mathbb{R}^{d_z}$. Moreover,

$$\sup_{n \geq 0} E(\varphi_Q^2(Z_{n+1})I_{\{\tau_Q > n\}} | \theta_0 = \theta, Z_0 = z) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, where τ_Q is the stopping time defined by $\tau_Q = \inf(\{n \geq 0 : \theta_n \notin Q\} \cup \{\infty\})$.

Assumption 3.4. $\limsup_{n \rightarrow \infty} \|\eta_n\| < \infty$ almost surely on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$.

Let \mathcal{R} , \mathcal{S} and $f(\mathcal{S})$ have the same meaning as in Section 2 for the objective function $f(\cdot)$ now specified in Assumption 3.2. Moreover, let η and Λ_Q have the same meaning as in (5), (6). Then, our results on the asymptotic behavior of the recursion (13) read as follows.

Theorem 3.1. *Suppose that Assumptions 3.1 – 3.4 hold. Let $f(\cdot)$ be the function specified in Assumption 3.2, and let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, the following is true:*

- (i) *If $f(\cdot)$ satisfies Assumption 2.3.a, Part (i) of Theorem 2.1 holds.*
- (ii) *If $f(\cdot)$ satisfies Assumption 2.3.b, Part (ii) of Theorem 2.1 holds.*
- (iii) *If $f(\cdot)$ satisfies Assumption 2.3.c, Part (iii) of Theorem 2.1 holds.*

Theorem 3.1 is proved in Section 7, while its global version is provided in Appendix 2.

Assumption 3.1 is related to the sequence $\{\alpha_n\}_{n \geq 0}$. It is satisfied if $\alpha_n = 1/n^a$ for $n \geq 1$, where $a \in (1/2, 1]$ is a constant. Assumptions 3.2 and 3.3 correspond to the stochastic process $\{Z_n\}_{n \geq 0}$ and are standard for the asymptotic analysis of stochastic approximation algorithms with Markovian dynamics. Basically, Assumptions 3.2 and 3.3 require the Poisson equation associated with algorithm (13) to have a solution which is Lipschitz continuous in θ . They hold if the following is satisfied: (i) $\Pi_\theta(\cdot, \cdot)$ is geometrically ergodic for each $\theta \in \mathbb{R}^{d_\theta}$, (ii) the convergence rate of $\Pi_\theta^n(\cdot, \cdot)$ is locally uniform in θ , and (iii) $\Pi_\theta(\cdot, \cdot)$ is locally Lipschitz continuous in θ on \mathbb{R}^{d_θ} (for further details see, [7, Chapter II.2], [26, Chapter 17] and references cited therein). Assumptions 3.2 and 3.3 have been introduced by Métivier and Priouret in [25] (see also [7, Part II]), and later generalized by Kushner and his co-workers (see [22] and references cited therein). However, none of these results cover the scenario where biased gradient estimates are used. Theorem 3.1 fills this gap in the literature on stochastic optimization and stochastic approximation.

4. Application to Reinforcement Learning

In this section, Theorems 2.1 and 3.1 are applied to the asymptotic analysis of a popular policy-gradient search algorithm for average-cost Markov decision problems introduced in [2]. Policy-gradient search is one of the most important classes of reinforcement learning (for further details, see e.g., [8], [29]).

In order to define controlled Markov chains with parametrized randomized control and to formulate the corresponding average-cost decision problems, we use the following notation. $d_\theta \geq 1$, $N_x > 1$, $N_y > 1$ are integers, while $\mathcal{X} = \{1, \dots, N_x\}$ and $\mathcal{Y} = \{1, \dots, N_y\}$. $\phi(x, y)$ is a non-negative (real-valued) function of $(x, y) \in \mathcal{X} \times \mathcal{Y}$. $p(x'|x, y)$ and $q_\theta(y|x)$ are non-negative (real-valued) functions of $(\theta, x, x', y) \in \mathbb{R}^{d_\theta} \times \mathcal{X} \times \mathcal{X} \times \mathcal{Y}$ with the following properties: $q_\theta(y|x)$ is differentiable in θ for each $\theta \in \mathbb{R}^{d_\theta}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and

$$\sum_{x' \in \mathcal{X}} p(x'|x, y) = 1, \quad \sum_{y' \in \mathcal{Y}} q_\theta(y'|x) = 1$$

for the same θ, x, y . For $\theta \in \mathbb{R}^{d_\theta}$, $\{(X_n^\theta, Y_n^\theta)\}_{n \geq 0}$ is an $\mathcal{X} \times \mathcal{Y}$ -valued Markov chain which is defined on a (canonical) probability space $(\Omega, \mathcal{F}, P_\theta)$ and satisfies

$$P_\theta(X_{n+1}^\theta = x', Y_{n+1}^\theta = y' | X_n^\theta = x, Y_n^\theta = y) = q_\theta(y'|x')p(x'|x, y)$$

for each $x, x' \in \mathcal{X}, y, y' \in \mathcal{Y}$. $f(\cdot)$ is a function defined by

$$f(\theta) = \lim_{n \rightarrow \infty} E_\theta \left(\frac{1}{n} \sum_{i=1}^n \phi(X_i^\theta, Y_i^\theta) \right) \quad (16)$$

for $\theta \in \mathbb{R}^{d_\theta}$. With this notation, an average-cost Markov decision problem with parameterized randomized control can be defined as the minimization of $f(\cdot)$. In the literature on reinforcement learning and operations research, $\{X_n^\theta\}_{n \geq 0}$ are referred to as a controlled Markov chain, while $\{Y_n^\theta\}_{n \geq 0}$ are called control actions. $p(x'|x, y)$ is referred to as the (chain) transition probability, while $q_\theta(y|x)$ is called the (control) action probability. For further details on Markov decision processes, see [8], [29], and references cited therein.

Since $f(\cdot)$ and its gradient rarely admit a closed-form expression, $f(\cdot)$ is minimized using methods based on stochastic gradient search and Monte Carlo gradient estimation. Such a method can be derived as follows. Let $s_\theta(x, y)$ be the score function defined by

$$s_\theta(x, y) = \frac{\nabla_\theta q_\theta(y|x)}{q_\theta(y|x)}$$

for $\theta \in \mathbb{R}^{d_\theta}, x \in \mathcal{X}, y \in \mathcal{Y}$. If $\{(X_n^\theta, Y_n^\theta)\}_{n \geq 0}$ is geometrically ergodic, we have $f(\theta) = \lim_{n \rightarrow \infty} E_\theta(\phi(X_n^\theta, Y_n^\theta))$ and

$$\nabla f(\theta) = \lim_{n \rightarrow \infty} E_\theta \left(\phi(X_n^\theta, Y_n^\theta) \sum_{i=0}^{n-1} s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta) \right)$$

(see the proof of Lemma 8.2 and in particular (42), (44)). Hence, quantity

$$\phi(X_n^\theta, Y_n^\theta) \sum_{i=0}^{n-1} s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta)$$

is an asymptotically unbiased estimator of $\nabla f(\theta)$. However, it can have a very large variance for large n so that the term $s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta)$ is ‘discounted’ by λ^i , where $\lambda \in [0, 1)$ is a constant referred to as the discounting factor. This leads to the following gradient estimator:

$$\phi(X_n^\theta, Y_n^\theta) \sum_{i=0}^{n-1} \lambda^i s_\theta(X_{n-i}^\theta, Y_{n-i}^\theta). \quad (17)$$

This gradient estimator (17) is biased and its bias is of the order $O(1 - \lambda)$ when $\lambda \rightarrow 1$ (see Lemma 8.2). Combining gradient search with the estimator (17), we get the policy-gradient algorithm proposed in [2]. This algorithm is defined by the following difference equations:

$$\begin{aligned} W_{n+1} &= \lambda W_n + s_{\theta_n}(X_{n+1}, Y_{n+1}), \\ \theta_{n+1} &= \theta_n - \alpha_n \phi(X_{n+1}, Y_{n+1}) W_{n+1}, \quad n \geq 0. \end{aligned} \quad (18)$$

In the recursion (18), $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive reals, while $\theta_0, W_0 \in \mathbb{R}^{d_\theta}$ are any (deterministic) vectors. $\{X_n\}_{n \geq 1}$ and $\{Y_n\}_{n \geq 1}$ are \mathcal{X} and \mathcal{Y} valued stochastic processes (respectively) generated through the following Monte Carlo simulations:

$$\begin{aligned} X_{n+1} | \theta_n, X_n, Y_n, \dots, \theta_0, X_0, Y_0 &\sim p(\cdot | X_n, Y_n), \\ Y_{n+1} | X_{n+1}, \theta_n, X_n, Y_n, \dots, \theta_0, X_0, Y_0 &\sim q_{\theta_n}(\cdot | X_{n+1}), \quad n \geq 0, \end{aligned} \quad (19)$$

where $X_0 \in \mathcal{X}, Y_0 \in \mathcal{Y}$ are deterministic quantities. Hence, $\{(X_n, Y_n)\}_{n \geq 1}$ satisfies

$$P(X_{n+1} = x, Y_{n+1} = y | \theta_n, X_n, Y_n, \dots, \theta_0, X_0, Y_0) = q_{\theta_n}(y|x)p(x|X_n, Y_n)$$

for all $x \in \mathcal{X}, y \in \mathcal{Y}, n \geq 1$.

Algorithm (18) is analyzed under the following assumptions.

Assumption 4.1. For all $\theta \in \mathbb{R}^{d_\theta}$, $\{X_n^\theta\}_{n \geq 0}$ is irreducible and aperiodic.

Assumption 4.2. For all $\theta \in \mathbb{R}^{d_\theta}$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $s_\theta(x, y)$ is well-defined (and finite). Moreover, for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $s_\theta(x, y)$ is locally Lipschitz continuous in θ on \mathbb{R}^{d_θ} .

Assumption 4.3.a. For each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $q_\theta(y|x)$ is p -times differentiable in θ on \mathbb{R}^{d_θ} , where $p > d_\theta$.

Assumption 4.3.b. For each $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $q_\theta(y|x)$ is real-analytic in θ on \mathbb{R}^{d_θ} .

Assumption 4.1 is related to the stability of the controlled Markov chain $\{X_n^\theta\}_{n \geq 0}$. In this or similar form, it is often involved in the asymptotic analysis of reinforcement learning algorithms (see e.g., [8], [29]). Assumptions 4.2, 4.3.a and 4.3.b correspond to the parameterization of the action probabilities $q_\theta(y|x)$. They are satisfied for many commonly used parameterizations (such as natural, exponential and trigonometric).

Let \mathcal{R} , \mathcal{S} and $f(\mathcal{S})$ have the same meaning as in Section 2 for the objective function $f(\cdot)$ now defined in (16). Moreover, let Λ_Q have the same meaning as in (6). Then, our results on the asymptotic behavior of the recursion (18) read as follows.

Theorem 4.1. Suppose that Assumptions 3.1, 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, the following is true:

- (i) There exists a (deterministic) non-decreasing function $\psi_Q : [0, \infty) \rightarrow [0, \infty)$ (independent of λ and depending only on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$) such that $\lim_{t \rightarrow 0} \psi_Q(t) = \psi_Q(0) = 0$ and

$$\limsup_{n \rightarrow \infty} d(\theta_n, \mathcal{R}) \leq \psi_Q(1 - \lambda)$$

almost surely on Λ_Q .

- (ii) If Assumption 4.3.a is additionally satisfied, there exists a real number $K_Q \in (0, \infty)$ (independent of λ and depending only on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$) such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| &\leq K_Q(1 - \lambda)^{q/2}, \\ \limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) &\leq K_Q(1 - \lambda)^q \end{aligned}$$

almost surely on Λ_Q , where $q = (p - d_\theta)/(p - 1)$.

- (iii) If Assumption 4.3.b is additionally satisfied, there exist real numbers $r_Q \in (0, 1)$, $L_Q \in (0, \infty)$ (independent of λ and depending only on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$) such that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(\theta_n, \mathcal{S}) &\leq L_Q(1 - \lambda)^{r_Q}, \\ \limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| &\leq L_Q(1 - \lambda)^{1/2}, \\ \limsup_{n \rightarrow \infty} d(f(\theta_n), f(\mathcal{S})) &\leq L_Q(1 - \lambda) \end{aligned}$$

almost surely on Λ_Q .

Theorem 4.1 is proved in Section 8.

Remark. Function $\psi_Q(\cdot)$ depends on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through function $f(\cdot)$ defined in (16) and its properties. It also depends on $p(x'|x, y)$, $q_\theta(y|x)$ through the properties of $\{(X_n^\theta, Y_n^\theta)\}_{n \geq 0}$ (see Lemma 8.1). Additionally, it depends on $\phi(x, y)$, $q_\theta(y|x)$ through upper bounds on $|\phi(x, y)|$, $\|s_\theta(x, y)\|$. Further details can be found in the proofs of Lemmas 8.1, 8.2 and Theorem 4.1 (Section 8).

Remark. Like $\psi_Q(\cdot)$, constants K_Q and L_Q depend on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through function $f(\cdot)$ (defined in (16)) and its properties. K_Q and L_Q also depend on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through the ergodicity properties of $\{(X_n^\theta, Y_n^\theta)\}_{n \geq 0}$. Moreover, K_Q and L_Q depend on $\phi(x, y)$, $p(x'|x, y)$, $q_\theta(y|x)$ through upper bounds on $|\phi(x, y)|$, $\|s_\theta(x, y)\|$. For further details, see the proofs of Lemmas 8.1, 8.2 and Theorem 4.1 (Section 8).

Although gradient search with ‘discounted’ gradient estimation (17) is widely used in reinforcement learning (besides policy-gradient search, temporal-difference and actor-critic learning also rely on the same approach), the available literature does not give a quite satisfactory answer to the problem of its asymptotic behavior. To the best of the present authors’ knowledge, the existing results do not offer even the guarantee that the asymptotic bias of recursion (18) goes to zero as $\lambda \rightarrow 1$ (i.e., that $\{\theta_n\}_{n \geq 0}$ converges to a vicinity of \mathcal{S} whose radius tends to zero as $\lambda \rightarrow 1$). The paper [20] can be considered as the strongest result on the asymptotic behavior of reinforcement learning with ‘discounted’ gradient estimation. However, [20] only claims that a subsequence of $\{\theta_n\}_{n \geq 0}$ converges to a vicinity of \mathcal{S} whose radius goes to zero as $\lambda \rightarrow 1$. The main difficulty stems from the fact that reinforcement learning algorithms are so complex that the existing asymptotic results for biased stochastic gradient search and biased stochastic approximation [12, Section 5.3], [14], [15], [16, Section 2.7] cannot be applied. Relying on the results presented in Sections 2 and 3, Theorem 4.1 overcomes these difficulties. Under mild and easily verifiable conditions, Theorem 4.1 guarantees that the asymptotic bias of algorithm (18) converges to zero as $\lambda \rightarrow 1$ (Part (i)). Theorem 4.1 also provides relatively tight, polynomial bounds on the rate at which the bias goes to zero (Parts (ii), (iii)).

5. Proof of Part (i) of Theorem 2.1

In this section, we rely on the following notation. For a set $A \subseteq \mathbb{R}^{d_\theta}$ and $\varepsilon \in (0, \infty)$, let $V_\varepsilon(A)$ be the ε -vicinity of A , i.e., $V_\varepsilon(A) = \{\theta \in \mathbb{R}^{d_\theta} : d(\theta, A) \leq \varepsilon\}$. For $\theta \in \mathbb{R}^{d_\theta}$ and $\gamma \in [0, \infty)$, let $F_\gamma(\theta)$ be the set defined by

$$F_\gamma(\theta) = \{-\nabla f(\theta) + \vartheta : \vartheta \in \mathbb{R}^{d_\theta}, \|\vartheta\| \leq \gamma\}$$

(notice that $F_\gamma(\theta)$ is the set-valued function of θ). For $\gamma \in [0, \infty)$, let Φ_γ be the family of solutions to the differential inclusion $d\theta/dt \in F_\gamma(\theta)$, i.e., Φ_γ is the collection of absolutely continuous functions $\varphi : [0, \infty) \rightarrow \mathbb{R}^{d_\theta}$ satisfying $d\varphi(t)/dt \in F_\gamma(\varphi(t))$ almost everywhere (in t) on $[0, \infty)$. For a compact set $Q \subset \mathbb{R}^{d_\theta}$ and $\gamma \in [0, \infty)$, let $\mathcal{H}_{Q,\gamma}$ be the largest invariant set of the differential inclusion $d\theta/dt \in F_\gamma(\theta)$ contained in Q , i.e., $\mathcal{H}_{Q,\gamma}$ is the largest set \mathcal{H} with the following property: For any $\theta \in \mathcal{H}$, there exists a solution $\varphi \in \Phi_\gamma$ such that $\varphi(0) = \theta$ and $\varphi(t) \in \mathcal{H}$ for all $t \in [0, \infty)$. For a compact set $Q \subset \mathbb{R}^{d_\theta}$ and $\gamma \in [0, \infty)$, let $\mathcal{R}_{Q,\gamma}$ be the set of chain-recurrent points of the differential inclusion $d\theta/dt \in F_\gamma(\theta)$ contained in Q , i.e., $\theta \in \mathcal{R}_{Q,\gamma}$ if and only if for any $\delta, t \in (0, \infty)$, there exist an integer $N \geq 1$, real numbers $t_1, \dots, t_N \in [t, \infty)$ and solutions $\varphi_1, \dots, \varphi_N \in \Phi_\gamma$ (each of which can depend on θ, δ, t) such that $\varphi_k(0) \in \mathcal{H}_{Q,\gamma}$ for $1 \leq k \leq N$ and

$$\|\varphi_1(0) - \theta\| \leq \delta, \quad \|\varphi_N(t_N) - \theta\| \leq \delta, \quad \|\varphi_k(t_k) - \varphi_{k+1}(0)\| \leq \delta$$

for $1 \leq k < N$. For more details on differential inclusions and their solutions, invariant sets and chain-recurrent points, see [1] and references cited therein.

Lemma 5.1. *Suppose that Assumption 2.3.a holds. Then, given a compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a non-decreasing function $\phi_Q : [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{\gamma \rightarrow 0} \phi_Q(\gamma) = \phi_Q(0) = 0$ and $\mathcal{R}_{Q,\gamma} \subseteq V_{\phi_Q(\gamma)}(\mathcal{R})$ for all $\gamma \in [0, \infty)$.*

Proof. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Moreover, let $\phi_Q : [0, \infty) \rightarrow [0, \infty)$ be the function defined by $\phi_Q(0) = 0$ and

$$\phi_Q(\gamma) = \sup(\{d(\theta, \mathcal{R}) : \theta \in \mathcal{R}_{Q,\gamma}\} \cup \{0\})$$

for $\gamma \in (0, \infty)$. Then, it is easy to show that $\phi_Q(\cdot)$ is well-defined and satisfies $\mathcal{R}_{Q,\gamma} \subseteq V_{\phi_Q(\gamma)}(\mathcal{R})$ for all $\gamma \in [0, \infty)$. It is also easy to check that $F_\gamma(\theta) \subseteq F_\delta(\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$, $\gamma, \delta \in [0, \infty)$ satisfying $\gamma \leq \delta$. Consequently, $\Phi_\gamma \subseteq \Phi_\delta$, $\mathcal{H}_{Q,\gamma} \subseteq \mathcal{H}_{Q,\delta}$, $\mathcal{R}_{Q,\gamma} \subseteq \mathcal{R}_{Q,\delta}$ for all $\gamma, \delta \in [0, \infty)$ satisfying $\gamma \leq \delta$. Thus, $\phi_Q(\cdot)$ is non-decreasing.¹ Moreover, [6, Theorem 3.1] implies that given $\varepsilon \in (0, \infty)$, there exists a real number $\gamma_Q(\varepsilon) \in (0, \infty)$ such that $\mathcal{R}_{Q,\gamma} \subseteq V_\varepsilon(\mathcal{R})$ for all $\gamma \in [0, \gamma_Q(\varepsilon))$. Therefore, $\phi_Q(\gamma) \leq \varepsilon$ for all $\varepsilon \in (0, \infty)$, $\gamma \in [0, \gamma_Q(\varepsilon))$.² Consequently, $\lim_{\gamma \rightarrow 0} \phi_Q(\gamma) = \phi_Q(0) = 0$. \square

¹Notice that $\{d(\theta, \mathcal{R}) : \theta \in \mathcal{R}_{Q,\gamma}\} \subseteq \{d(\theta, \mathcal{R}) : \theta \in \mathcal{R}_{Q,\delta}\}$ whenever $\gamma \leq \delta$.

²Notice that $d(\theta, \mathcal{R}) \leq \varepsilon$ whenever $\theta \in \mathcal{R}_{Q,\gamma}$, $\gamma \in [0, \gamma_Q(\varepsilon))$.

Proof of Part (i) of Theorem 2.1. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set and let $\psi_Q : [0, \infty) \rightarrow [0, \infty)$ be the function defined by $\psi_Q(t) = \phi_Q(2t)$ for $t \in [0, \infty)$ ($\phi_Q(\cdot)$ is specified in the statement of Lemma 5.1). Then, due to Lemma 5.1, $\psi_Q(\cdot)$ is non-decreasing and $\lim_{t \rightarrow 0} \psi_Q(t) = \psi_Q(0) = 0$. Moreover, owing to Assumption 2.2, there exists an event $N_Q \in \mathcal{F}$ such that the following holds: $P(N_Q) = 0$ and (4) is satisfied on $\Lambda_Q \setminus N_Q$ for all $t \in (0, \infty)$. Let ω be an arbitrary sample in $\Lambda_Q \setminus N_Q$. To prove Part (i) of Theorem 2.1, it is sufficient to show (7) for ω . Notice that all formulas that follow in the proof correspond to ω .

If $\eta = 0$, then [4, Proposition 4.1, Theorem 5.7] imply that all limit points of $\{\theta_n\}_{n \geq 0}$ are included in \mathcal{R} . Hence, (7) holds when $\eta = 0$.

Now, suppose $\eta > 0$. Then, there exists $n_0 \geq 0$ (depending on ω) such that $\theta_n \in Q$, $\|\eta_n\| \leq 2\eta$ for $n \geq n_0$. Therefore,

$$\frac{\theta_{n+1} - \theta_n}{\alpha_n} + \zeta_n = -(\nabla f(\theta_n) + \eta_n) \in F_{2\eta}(\theta_n)$$

for $n \geq n_0$. Consequently, [5, Proposition 1.3, Theorem 3.6] imply that all limit points of $\{\theta_n\}_{n \geq 0}$ are contained in $\mathcal{R}_{Q, 2\eta}$. Combining this with Lemma 5.1, we conclude that the limit points of $\{\theta_n\}_{n \geq 0}$ are included in $V_{\phi_Q(2\eta)}(\mathcal{R}) = V_{\psi_Q(\eta)}(\mathcal{R})$. Thus, (7) holds when $\eta > 0$. \square

6. Proof of Parts (ii), (iii) of Theorem 2.1

In this section, the following notation is used. ϕ is the random variable defined by

$$\phi = \limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\|.$$

For $t \in (0, \infty)$ and $n \geq 0$, $\phi_{1,n}(t), \phi_{2,n}(t), \phi_n(t)$ are the random quantities defined as

$$\begin{aligned} \phi_{1,n}(t) &= -(\nabla f(\theta_n))^T \sum_{i=n}^{a(n,t)-1} \alpha_i (\nabla f(\theta_i) - \nabla f(\theta_n)), \\ \phi_{2,n}(t) &= \int_0^1 (\nabla f(\theta_n + s(\theta_{a(n,t)} - \theta_n)) - \nabla f(\theta_n))^T (\theta_{a(n,t)} - \theta_n) ds, \\ \phi_n(t) &= \phi_{1,n}(t) + \phi_{2,n}(t). \end{aligned}$$

Then, it is straightforward to show that

$$\begin{aligned} f(\theta_{a(n,t)}) - f(\theta_n) &= -\|\nabla f(\theta_n)\|^2 \sum_{i=n}^{a(n,t)-1} \alpha_i - (\nabla f(\theta_n))^T \sum_{i=n}^{a(n,t)-1} \alpha_i \xi_i + \phi_n(t) \\ &\leq -\|\nabla f(\theta_n)\| \left(\|\nabla f(\theta_n)\| \sum_{i=n}^{a(n,t)-1} \alpha_i - \left\| \sum_{i=n}^{a(n,t)-1} \alpha_i \xi_i \right\| \right) + |\phi_n(t)| \end{aligned} \quad (20)$$

for $t \in (0, \infty)$, $n \geq 0$. Moreover, Assumption 2.1 implies

$$\lim_{n \rightarrow \infty} \sum_{i=n}^{a(n,t)-1} \alpha_i = \lim_{n \rightarrow \infty} \sum_{i=n}^{a(n,t)} \alpha_i = t \quad (21)$$

for $t \in (0, \infty)$.³

We also need the following additional notation. The Lebesgue measure is denoted by $m(\cdot)$. For a compact set $Q \subset \mathbb{R}^{d_\theta}$ and $\varepsilon \in (0, \infty)$, $A_{Q,\varepsilon}$ is the set defined by

$$A_{Q,\varepsilon} = \{f(\theta) : \theta \in Q, \|\nabla f(\theta)\| \leq \varepsilon\}. \quad (22)$$

In order to treat Assumptions 2.3.b, 2.3.c in a unified way and to provide a unified proof of Parts (ii), (iii) of Theorem 2.1, we introduce the following assumption.

³Notice that $t \geq \sum_{i=n}^{a(n,t)-1} \alpha_i = \sum_{i=n}^{a(n,t)} \alpha_i - \alpha_{a(n,t)} \geq t - \alpha_{a(n,t)}$ for $t \in (0, \infty)$, $n \geq 0$.

Assumption 6.1. *There exists a real number $s \in (0, 1]$ and for any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $M_Q \in [1, \infty)$ such that $m(A_{Q,\varepsilon}) \leq M_Q \varepsilon^s$ for all $\varepsilon \in (0, \infty)$.*

Proposition 6.1. *Suppose that Assumption 2.3.b holds. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, there exists a real number $M_Q \in [1, \infty)$ (depending only on $f(\cdot)$) such that $m(A_{Q,\varepsilon}) \leq M_Q \varepsilon^q$ for all $\varepsilon \in (0, \infty)$ (q is specified in the statement of Theorem 2.1).*

Proof. The proposition is a particular case of Yomdin theorem [34, Theorem 1.2]. \square

Proposition 6.2. *Suppose that Assumption 2.3.c holds. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, the following is true:*

- (i) *There exists a real number $M_Q \in [1, \infty)$ (depending only on $f(\cdot)$) such that $m(A_{Q,\varepsilon}) \leq M_Q \varepsilon$ for all $\varepsilon \in (0, \infty)$.*
- (ii) *There exist real numbers $r_Q \in (0, 1)$, $M_{1,Q}, M_{2,Q} \in [1, \infty)$ (depending only on $f(\cdot)$) such that*

$$d(\theta, \mathcal{S}) \leq M_{1,Q} \|\nabla f(\theta)\|^{r_Q}, \quad d(f(\theta), f(\mathcal{S})) \leq M_{2,Q} \|\nabla f(\theta)\| \quad (23)$$

for all $\theta \in Q$ (\mathcal{S} and $f(\mathcal{S})$ are specified in (2)).

Proof. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Owing to Lojasiewicz (ordinary) inequality (see [10, Theorem 6.4, Remark 6.5]), there exist real numbers $r_Q \in (0, 1)$, $M_{1,Q} \in [1, \infty)$ such that the first inequality in (23) holds for all $\theta \in Q$. Moreover, due to Lojasiewicz gradient inequality (see [21, Theorem LI, Page 775]), we have the following: For any $a \in f(Q) = \{f(\theta) : \theta \in Q\}$, there exist real numbers $\delta_{Q,a} \in (0, 1)$, $\nu_{Q,a} \in (1, 2]$, $N_{Q,a} \in [1, \infty)$ such that

$$|f(\theta) - a| \leq N_{Q,a} \|\nabla f(\theta)\|^{\nu_{Q,a}} \quad (24)$$

for all $\theta \in Q$ satisfying $|f(\theta) - a| \leq \delta_{Q,a}$.

Now, we show by contradiction that $f(\mathcal{S} \cap Q) = \{f(\theta) : \theta \in \mathcal{S} \cap Q\}$ has finitely many elements. Suppose the opposite. Then, there exists a sequence $\{\vartheta_n\}_{n \geq 0}$ in $\mathcal{S} \cap Q$ such that $\{f(\vartheta_n)\}_{n \geq 0}$ contains infinitely many different elements. Since $\mathcal{S} \cap Q$ is compact, $\{\vartheta_n\}_{n \geq 0}$ has a convergent subsequence $\{\tilde{\vartheta}_n\}_{n \geq 0}$ such that $\{f(\tilde{\vartheta}_n)\}_{n \geq 0}$ also contains infinitely many different elements. Let $\vartheta = \lim_{n \rightarrow \infty} \tilde{\vartheta}_n$, $a = f(\vartheta)$. As $\delta_{Q,a} > 0$, there exists an integer $n_0 \geq 0$ such that $|f(\tilde{\vartheta}_n) - a| \leq \delta_{Q,a}$ for $n \geq n_0$. Since $\nabla f(\tilde{\vartheta}_n) = 0$ for $n \geq 0$, (24) implies $f(\tilde{\vartheta}_n) = a$ for $n \geq n_0$. However, this is impossible, since $\{f(\tilde{\vartheta}_n)\}_{n \geq 0}$ has infinitely many different elements.

Let n_Q be the number of elements in $f(\mathcal{S} \cap Q)$, while $\{a_i : 1 \leq i \leq n_Q\}$ are the elements of $f(\mathcal{S} \cap Q)$. For $1 \leq i \leq n_Q$, let

$$B_{Q,i} = \{\theta \in Q : \|\nabla f(\theta)\| < 1, f(\theta) \in (a_i - \delta_{Q,a_i}, a_i + \delta_{Q,a_i})\},$$

while $B_Q = \bigcup_{i=1}^{n_Q} B_{Q,i}$, $\varepsilon_Q = \inf\{\|\nabla f(\theta)\| : \theta \in Q \setminus B_Q\}$. As B_Q is open and $\mathcal{S} \cap Q \subset B_Q$, we have $\varepsilon_Q > 0$.

Let $\tilde{C}_{1,Q} \in [1, \infty)$ be an upper bound of $|f(\cdot)|$ on Q . Moreover, let $\tilde{C}_{2,Q} = \max_{1 \leq i \leq n_Q} N_{Q,a_i}$, $M_{2,Q} = 2 \max\{\varepsilon_Q^{-1} \tilde{C}_{1,Q}, \tilde{C}_{2,Q}\}$. Then, if $\theta \in B_Q$, we have

$$d(f(\theta), f(\mathcal{S})) = \min_{1 \leq i \leq n_Q} |f(\theta) - a_i| \leq \max_{1 \leq i \leq n_Q} N_{Q,a_i} \|\nabla f(\theta)\|^{\nu_{Q,a_i}} \leq \tilde{C}_{2,Q} \|\nabla f(\theta)\| \leq M_{2,Q} \|\nabla f(\theta)\|$$

(notice that $\|\nabla f(\theta)\| < 1$, $\nu_{Q,a_i} > 1$). If $\theta \in Q \setminus B_Q$, we get

$$d(f(\theta), f(\mathcal{S})) = \min_{1 \leq i \leq n_Q} |f(\theta) - a_i| \leq 2\tilde{C}_{1,Q} \leq 2\varepsilon_Q^{-1} \tilde{C}_{1,Q} \|\nabla f(\theta)\| \leq M_{2,Q} \|\nabla f(\theta)\|$$

(notice that $\|\nabla f(\theta)\| \geq \varepsilon_Q$). Hence, the second inequality in (23) holds for all $\theta \in Q$.

Let $M_Q = 2M_{2,Q}n_Q$. Owing to the second inequality in (23), we have

$$A_{Q,\varepsilon} \subseteq \bigcup_{i=1}^{n_Q} [f(a_i) - M_{2,Q}\varepsilon, f(a_i) + M_{2,Q}\varepsilon]$$

for each $\varepsilon \in (0, \infty)$. Consequently, $m(A_{Q,\varepsilon}) \leq 2M_{2,Q}n_Q\varepsilon = M_Q\varepsilon$ for all $\varepsilon \in (0, \infty)$. \square

Lemma 6.1. *Let Assumptions 2.1 and 2.2 hold. Then, there exists an event $N_0 \in \mathcal{F}$ such that $P(N_0) = 0$ and*

$$\limsup_{n \rightarrow \infty} \max_{n \leq k < a(n,t)} \left\| \sum_{i=n}^k \alpha_i \xi_i \right\| \leq \eta t, \quad (25)$$

$$\lim_{n \rightarrow \infty} |f(\theta_{n+1}) - f(\theta_n)| = 0 \quad (26)$$

on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \setminus N_0$ for all $t \in (0, \infty)$. Moreover, given a compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $C_{1,Q} \in [1, \infty)$ (independent of η and depending only on $f(\cdot)$) such that

$$\limsup_{n \rightarrow \infty} \max_{n \leq k \leq a(n,t)} |f(\theta_k) - f(\theta_n)| \leq C_{1,Q} t (\phi + \eta), \quad (27)$$

$$\limsup_{n \rightarrow \infty} |\phi_n(t)| \leq C_{1,Q} t^2 (\phi + \eta)^2 \quad (28)$$

on $\Lambda_Q \setminus N_0$ for all $t \in (0, \infty)$.

Proof. Owing to Assumption 2.2, there exists $N_0 \in \mathcal{F}$ such that the following holds: $P(N_0) = 0$ and (4) is satisfied on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \setminus N_0$ for all $t \in (0, \infty)$. Moreover, we have

$$\left\| \sum_{i=n}^k \alpha_i \xi_i \right\| \leq \left\| \sum_{i=n}^k \alpha_i \zeta_i \right\| + \sum_{i=n}^k \alpha_i \|\eta_i\| \leq \max_{n \leq j < a(n,t)} \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| + t \max_{j \geq n} \|\eta_j\|$$

for $0 \leq n \leq k < a(n,t)$, $t \in (0, \infty)$. Consequently,

$$\limsup_{n \rightarrow \infty} \max_{n \leq k < a(n,t)} \left\| \sum_{i=n}^k \alpha_i \xi_i \right\| \leq \limsup_{n \rightarrow \infty} \max_{n \leq k < a(n,t)} \left\| \sum_{i=n}^k \alpha_i \zeta_i \right\| + t \lim_{n \rightarrow \infty} \max_{k \geq n} \|\eta_k\| = \eta t$$

on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \setminus N_0$ for $t \in (0, \infty)$.

Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set, while $\tilde{C}_Q \in [1, \infty)$ stands for a Lipschitz constant of $f(\cdot)$, $\nabla f(\cdot)$ on Q . Moreover, let $C_{1,Q} = 2\tilde{C}_Q$, while ω is an arbitrary sample from $\Lambda_Q \setminus N_0$. In order to prove the lemma, it is sufficient to show that (26) – (28) hold for ω and any $t \in (0, \infty)$. Notice that all formulas which follow in the proof correspond to ω .

Let $\varepsilon \in (0, \infty)$ be any real number. Then, there exists $n_0 \geq 0$ (depending on ω , ε) such that $\theta_n \in Q$, $\|\nabla f(\theta_n)\| \leq \phi + \varepsilon$ for $n \geq n_0$ (notice that these relations hold for all but finitely many n). Therefore,

$$\|\theta_k - \theta_n\| \leq \sum_{i=n}^{k-1} \alpha_i \|\nabla f(\theta_i)\| + \left\| \sum_{i=n}^{k-1} \alpha_i \xi_i \right\| \leq t(\phi + \varepsilon) + \max_{n \leq j < a(n,t)} \left\| \sum_{i=n}^j \alpha_i \xi_i \right\|$$

for $n_0 \leq n \leq k \leq a(n,t)$, $t \in (0, \infty)$. Combining this with (25), we get

$$\limsup_{n \rightarrow \infty} \max_{n \leq k \leq a(n,t)} \|\theta_k - \theta_n\| \leq t(\phi + \eta + \varepsilon)$$

for $t \in (0, \infty)$. Then, the limit process $\varepsilon \rightarrow 0$ yields

$$\limsup_{n \rightarrow \infty} \max_{n \leq k \leq a(n,t)} \|\theta_k - \theta_n\| \leq t(\phi + \eta)$$

for $t \in (0, \infty)$ (notice that $\varepsilon \in (0, \infty)$ is any real number). As

$$|f(\theta_k) - f(\theta_n)| \leq \tilde{C}_Q \|\theta_k - \theta_n\|$$

for $k \geq n \geq n_0$ (notice that $\theta_n \in Q$ for $n \geq n_0$), we have

$$\limsup_{n \rightarrow \infty} \max_{n \leq k \leq a(n,t)} |f(\theta_k) - f(\theta_n)| \leq \tilde{C}_Q t (\phi + \eta) \leq C_{1,Q} t (\phi + \eta)$$

for $t \in (0, \infty)$. Since

$$|f(\theta_{n+1}) - f(\theta_n)| \leq \max_{n \leq k \leq a(n,t)} |f(\theta_k) - f(\theta_n)|$$

for $t \in (0, \infty)$ and sufficiently large n (notice that $a(n, t) \geq n + 1$ for sufficiently large n), we conclude

$$\limsup_{n \rightarrow \infty} |f(\theta_{n+1}) - f(\theta_n)| \leq \tilde{C}_Q t (\phi + \eta)$$

for $t \in (0, \infty)$. Then, the limit process $t \rightarrow 0$ implies (26). Moreover, we have

$$\begin{aligned} |\phi_{1,n}(t)| &\leq \tilde{C}_Q \|\nabla f(\theta_n)\| \sum_{i=n}^{a(n,t)-1} \alpha_i \|\theta_i - \theta_n\| \leq \tilde{C}_Q t \|\nabla f(\theta_n)\| \max_{n \leq k \leq a(n,t)} \|\theta_k - \theta_n\|, \\ |\phi_{2,n}(t)| &\leq \tilde{C}_Q \|\theta_{a(n,t)} - \theta_n\|^2 \leq \tilde{C}_Q \max_{n \leq k \leq a(n,t)} \|\theta_k - \theta_n\|^2 \end{aligned}$$

for $n \geq n_0, t \in (0, \infty)$. Therefore,

$$\limsup_{n \rightarrow \infty} |\phi_{1,n}(t)| \leq \tilde{C}_Q t^2 \phi (\phi + \eta), \quad \limsup_{n \rightarrow \infty} |\phi_{2,n}(t)| \leq \tilde{C}_Q t^2 (\phi + \eta)^2$$

for $t \in (0, \infty)$. Hence,

$$\limsup_{n \rightarrow \infty} |\phi_n(t)| \leq 2\tilde{C}_Q t^2 (\phi + \eta)^2 = C_{1,Q} t^2 (\phi + \eta)^2$$

for $t \in (0, \infty)$. □

Lemma 6.2. *Let Assumptions 2.1, 2.2 and 6.1 hold. Then, given a compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $C_{2,Q} \in [1, \infty)$ (independent of η and depending only on $f(\cdot)$) such that*

$$\limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \leq C_{2,Q} \eta^s \quad (29)$$

on $\Lambda_Q \setminus N_0$ (s is specified in Assumption 6.1).

Proof. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set, while \tilde{C}_Q stands for an upper bound of $\|\nabla f(\cdot)\|$ on Q . Moreover, let $C_{2,Q} = 4M_Q$. In order to avoid considering separately the cases $\eta = 0$ and $\eta > 0$, we show

$$\limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \leq C_{2,Q} (\varepsilon + \eta)^s \quad (30)$$

on $\Lambda_Q \setminus N_0$ for all $\varepsilon \in (0, \infty)$. Then, (29) follows directly from (30) by letting $\varepsilon \rightarrow 0$.

Inequality (30) is proved by contradiction: Suppose that there exist a sample $\omega \in \Lambda_Q \setminus N_0$ and a real number $\varepsilon \in (0, \infty)$ such that (30) does not hold for them. Notice that all formulas which follow in the proof correspond to ω .

Let $\gamma = 2(\varepsilon + \eta)$, $\delta = M_Q \gamma^s$, while

$$\mu = \delta / (C_{1,Q} (\tilde{C}_Q + \eta)), \quad \nu = \gamma^2 / (4C_{1,Q} (\tilde{C}_Q + \eta)^2), \quad \tau = \min\{\mu, \nu/2\}.$$

Since $\{\theta_n\}_{n \geq 0}$ is bounded and (30) is not satisfied, there exist real numbers $a, b \in \mathbb{R}$ (depending on ω, ε) such that $b - a > 2\delta$ and such that inequalities $f(\theta_n) < a, f(\theta_k) > b$ hold for infinitely many $n, k \geq 0$ (notice that $C_{2,Q} (\varepsilon + \eta)^s \geq 2\delta$). As $m(A_{Q,\gamma}) \leq M_Q \gamma^s = \delta$, there exists a real number c such that $c \notin A_{Q,\gamma}$ and $a < c < b - \delta$ (otherwise, $(a, b - \delta) \subset A_{Q,\varepsilon}$, which is impossible as $(b - \delta) - a > \delta$).

Let $n_0 = 0$, while

$$l_k = \min\{n \geq n_{k-1} : f(\theta_n) \leq c\}, \quad n_k = \min\{n \geq l_k : f(\theta_n) \geq b\}, \quad m_k = \max\{n \leq n_k : f(\theta_n) \leq c\}$$

for $k \geq 1$. It can easily be deduced that sequences $\{l_k\}_{k \geq 1}, \{m_k\}_{k \geq 1}, \{n_k\}_{k \geq 1}$ are well-defined and satisfy $l_k < m_k < n_k < l_{k+1}$ and

$$f(\theta_{m_k}) \leq c < f(\theta_{m_k+1}), \quad f(\theta_{n_k}) - f(\theta_{m_k}) \geq b - c, \quad \min_{m_k < n \leq n_k} f(\theta_n) > c \quad (31)$$

for $k \geq 1$. Moreover, Lemma 6.1 implies

$$\lim_{k \rightarrow \infty} |f(\theta_{m_{k+1}}) - f(\theta_{m_k})| = 0, \quad (32)$$

$$\limsup_{k \rightarrow \infty} \max_{m_k \leq j \leq a(m_k, \tau)} |f(\theta_j) - f(\theta_{m_k})| \leq C_{1,Q} \tau (\tilde{C}_Q + \eta) \leq \delta < b - c \quad (33)$$

(to get (33), notice that $\theta_n \in Q$ for all but finitely many n and that $\phi \leq \tilde{C}_Q$). Owing to (33) and the second inequality in (31), there exists $k_0 \geq 1$ such that $a(m_k, \tau) \leq n_k$ for $k \geq k_0$.⁴ Then, the last inequality in (31) implies $f(\theta_{a(m_k, \tau)}) \geq c$ for $k \geq k_0$, while $\lim_{k \rightarrow \infty} f(\theta_{m_k}) = c$ follows from (32) and the first inequality in (31). Since $\|\nabla f(\theta)\| > \gamma$ for any $\theta \in Q$ satisfying $f(\theta) = c$ (due to the way c is selected), we have $\liminf_{k \rightarrow \infty} \|\nabla f(\theta_{m_k})\| \geq \gamma$. Consequently, Lemma 6.1 and (21) yield

$$\liminf_{k \rightarrow \infty} \left(\|\nabla f(\theta_{m_k})\| \sum_{i=m_k}^{a(m_k, \tau)-1} \alpha_i - \left\| \sum_{i=m_k}^{a(m_k, \tau)-1} \alpha_i \xi_i \right\| \right) \geq \tau(\gamma - \eta) \geq \tau\gamma/2 > 0$$

(notice that $\eta < \gamma/2$). Therefore,

$$\liminf_{k \rightarrow \infty} \|\nabla f(\theta_{m_k})\| \left(\|\nabla f(\theta_{m_k})\| \sum_{i=m_k}^{a(m_k, \tau)-1} \alpha_i - \left\| \sum_{i=m_k}^{a(m_k, \tau)-1} \alpha_i \xi_i \right\| \right) \geq \tau\gamma^2/2.$$

Combining this with Lemma 6.1 and (20), we get

$$\begin{aligned} \limsup_{k \rightarrow \infty} (f(\theta_{a(m_k, \tau)}) - f(\theta_{m_k})) &\leq - \liminf_{k \rightarrow \infty} \|\nabla f(\theta_{m_k})\| \left(\|\nabla f(\theta_{m_k})\| \sum_{i=m_k}^{a(m_k, \tau)-1} \alpha_i - \left\| \sum_{i=m_k}^{a(m_k, \tau)-1} \alpha_i \xi_i \right\| \right) \\ &\quad + \limsup_{k \rightarrow \infty} |\phi_{m_k}(\tau)| \\ &\leq -\tau\gamma^2/2 + C_{1,Q} \tau^2 (\phi + \eta)^2 < 0 \end{aligned}$$

(notice that $\phi \leq \tilde{C}_Q$, $C_{1,Q} \tau (\tilde{C}_Q + \eta) \leq \gamma^2/4$). However, this is not possible, as $f(\theta_{a(m_k, \tau)}) \geq c \geq f(\theta_{m_k})$ for each $k \geq k_0$. Hence, (30) is true. \square

Lemma 6.3. *Let Assumptions 2.1 and 2.2 hold. Then, given a compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $C_{3,Q} \in (0, 1)$ (independent of η and depending only on $f(\cdot)$) such that*

$$\limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \geq C_{3,Q} \phi^2 \quad (34)$$

on $(\Lambda_Q \setminus N_0) \cap \{\phi > 2\eta\}$.

Proof. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set, while $C_{3,Q} = 1/(64C_{1,Q})$ and $\tau = 1/(16C_{1,Q})$. Moreover, let ω be an arbitrary sample from $(\Lambda_Q \setminus N_0) \cap \{\phi > 2\eta\}$. In order to prove the lemma's assertion, it is sufficient to show that (34) holds for ω . Notice that all formulas which follow in the proof correspond to ω .

Let $n_0 = 0$ and

$$n_k = \min\{n > n_{k-1} : \|\nabla f(\theta_n)\| \geq \phi - 1/k\}$$

for $k \geq 1$. Obviously, sequence $\{n_k\}_{k \geq 0}$ is well-defined and satisfies $\lim_{k \rightarrow \infty} \|\nabla f(\theta_{n_k})\| = \phi$. Then, Lemma 6.1 and (21) yield

$$\liminf_{k \rightarrow \infty} \|\nabla f(\theta_{n_k})\| \left(\|\nabla f(\theta_{n_k})\| \sum_{i=n_k}^{a(n_k, \tau)-1} \alpha_i - \left\| \sum_{i=n_k}^{a(n_k, \tau)-1} \alpha_i \xi_i \right\| \right) \geq \tau\phi(\phi - \eta) \geq \tau\phi^2/2 > 0.$$

⁴If $a(m_k, \tau) > n_k$ for infinitely many k , then (33) yields

$$\liminf_{k \rightarrow \infty} (f(\theta_{n_k}) - f(\theta_{m_k})) \leq \delta < b - c.$$

However, this contradicts the second inequality in (31).

Combining this with Lemma 6.1 and (20), we get

$$\begin{aligned} \limsup_{k \rightarrow \infty} (f(\theta_{a(n_k, \tau)}) - f(\theta_{n_k})) &\leq - \liminf_{k \rightarrow \infty} \|\nabla f(\theta_{n_k})\| \left(\|\nabla f(\theta_{n_k})\| \sum_{i=n_k}^{a(n_k, \tau)-1} \alpha_i - \left\| \sum_{i=n_k}^{a(n_k, \tau)-1} \alpha_i \xi_i \right\| \right) \\ &\quad + \limsup_{k \rightarrow \infty} |\phi_{n_k}(\tau)| \\ &\leq -\tau\phi^2/2 + C_{1,Q}\tau^2(\phi + \eta)^2 \leq -C_{3,Q}\phi^2 \end{aligned}$$

(notice that $\eta < \phi$). Consequently,

$$\limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \geq - \limsup_{k \rightarrow \infty} (f(\theta_{a(n_k, \tau)}) - f(\theta_{n_k})) \geq C_{3,Q}\phi^2.$$

Hence, (34) is true. \square

Proposition 6.3. *Suppose that Assumptions 2.1, 2.2 and 6.1 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, there exists a real number $K_Q \in [1, \infty)$ (independent of η and depending only on $f(\cdot)$) such that*

$$\limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| \leq K_Q \eta^{s/2}, \quad \limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \leq K_Q \eta^s \quad (35)$$

on $\Lambda_Q \setminus N_0$.

Proof. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set, while $\tilde{C}_Q \in [1, \infty)$ stands for an upper bound of $\|\nabla f(\cdot)\|$ on Q . Moreover, let $K_Q = \max\{2, \tilde{C}_Q, C_{2,Q}\}$. Obviously, it is sufficient to show $\phi \leq K_Q \eta^{s/2}$ on $\Lambda_Q \setminus N_0$ (notice that the second inequality in (35) is a direct consequence of Lemma 6.2).

Owing to Lemmas 6.2, 6.3, we have $C_{3,Q}\phi^2 \leq C_{2,Q}\eta^s$ on $(\Lambda_Q \setminus N_0) \cap \{\phi > 2\eta\}$. Therefore, $\phi \leq (C_{2,Q}/C_{3,Q})^{1/2}\eta^{s/2} \leq K_Q \eta^{s/2}$ on $(\Lambda_Q \setminus N_0) \cap \{\phi > 2\eta\}$. Moreover, $\phi \leq 2\eta \leq K_Q \eta^{s/2}$ on $(\Lambda_Q \setminus N_0) \cap \{\phi \leq 2\eta, \eta \leq 1\}$ (notice that $s/2 < 1$), while $\phi \leq \tilde{C}_Q \leq K_Q \eta^{s/2}$ on $(\Lambda_Q \setminus N_0) \cap \{\phi \leq 2\eta, \eta > 1\}$. Thus, $\phi \leq K_Q \eta^{s/2}$ indeed holds on $\Lambda_Q \setminus N_0$. \square

Proof of Parts (ii), (iii) of Theorem 2.1. Part (ii) of the theorem directly follows from Propositions 6.1, 6.3, while Part (iii) is a direct consequence of Propositions 6.2, 6.3. \square

7. Proof of Theorem 3.1

The following notation is used in this section. For $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $E_{\theta,z}(\cdot)$ denotes the conditional expectation given $\theta_0 = \theta$, $Z_0 = z$. For $n \geq 1$, ζ_n , ξ_n are the random variables defined by

$$\zeta_n = F(\theta_n, Z_{n+1}) - \nabla f(\theta_n), \quad \xi_n = \zeta_n + \eta_n, \quad (36)$$

while $\zeta_{1,n}$, $\zeta_{2,n}$, $\zeta_{3,n}$ are random variables defined as

$$\zeta_{1,n} = \tilde{F}(\theta_n, Z_{n+1}) - (\Pi \tilde{F})(\theta_n, Z_n), \quad \zeta_{2,n} = (\Pi \tilde{F})(\theta_n, Z_n) - (\Pi \tilde{F})(\theta_{n-1}, Z_n), \quad \zeta_{3,n} = -(\Pi \tilde{F})(\theta_n, Z_{n+1}).$$

Then, it is straightforward to verify that algorithm (13) admits the form (1). Moreover, using Assumption 3.2, it is easy to show

$$\sum_{i=n}^k \alpha_i \zeta_i = \sum_{i=n}^k \alpha_i \zeta_{1,i} + \sum_{i=n}^k \alpha_i \zeta_{2,i} + \sum_{i=n}^k (\alpha_i - \alpha_{i+1}) \zeta_{3,i} + \alpha_{k+1} \zeta_{3,k} - \alpha_n \zeta_{3,n-1} \quad (37)$$

for $1 \leq n \leq k$.

Proof of Theorem 3.1. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set and $\tilde{\Lambda}_Q$ be the event defined by $\tilde{\Lambda}_Q = \bigcap_{n=0}^{\infty} \{\theta_n \in Q\}$. Then, owing to Assumptions 3.1 and 3.3, we have

$$E_{\theta,z} \left(\sum_{n=0}^{\infty} (\alpha_n^2 + \alpha_{n+1}^2) \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}} \right) < \infty, \quad E_{\theta,z} \left(\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}} \right) < \infty \quad (38)$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$.

Let $\mathcal{F}_n = \sigma\{\theta_0, Z_0, \dots, \theta_n, Z_n\}$ for $n \geq 0$. Since $\{\tau_Q > n\} \in \mathcal{F}_n$ for $n \geq 0$, Assumption 3.2 implies

$$E_{\theta,z}(\zeta_{1,n} I_{\{\tau_Q > n\}} | \mathcal{F}_n) = \left(E_{\theta,z}(\tilde{F}(\theta_n, Z_{n+1}) | \mathcal{F}_n) - (\Pi \tilde{F})(\theta_n, Z_n) \right) I_{\{\tau_Q > n\}} = 0$$

almost surely for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $n \geq 0$. Assumption 3.3 also yields

$$\|\zeta_{1,n}\| I_{\{\tau_Q > n\}} \leq \varphi_Q(Z_n) I_{\{\tau_Q > n-1\}} + \varphi_Q(Z_{n+1}) I_{\{\tau_Q > n\}}$$

for $n \geq 0$. Combining this with (38), we get

$$E_{\theta,z} \left(\sum_{n=0}^{\infty} \alpha_n^2 \|\zeta_{1,n}\|^2 I_{\{\tau_Q > n\}} \right) \leq 2E_{\theta,z} \left(\sum_{n=0}^{\infty} (\alpha_n^2 + \alpha_{n+1}^2) \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}} \right) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. Then, using Doob theorem, we conclude that $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n} I_{\{\tau_Q > n\}}$ converges almost surely. As $\tilde{\Lambda}_Q \subseteq \{\tau_Q > n\}$ for $n \geq 0$, $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n}$ converges almost surely on $\tilde{\Lambda}_Q$.⁵

Due to Assumption 3.3, we have

$$\begin{aligned} \|\zeta_{2,n}\| I_{\tilde{\Lambda}_Q} &\leq \varphi_Q(Z_n) \|\theta_n - \theta_{n-1}\| I_{\tilde{\Lambda}_Q} \\ &\leq \alpha_{n-1} \varphi_Q(Z_n) (\|F(\theta_{n-1}, Z_n)\| + \|\eta_{n-1}\|) I_{\tilde{\Lambda}_Q} \\ &\leq \alpha_{n-1} \varphi_Q(Z_n) (\varphi_Q(Z_n) + \|\eta_n\|) I_{\tilde{\Lambda}_Q} \\ &\leq 2\alpha_{n-1} (\varphi_Q^2(Z_n) + \|\eta_{n-1}\|^2) I_{\tilde{\Lambda}_Q} \end{aligned}$$

for $n \geq 1$ (notice that $\varphi_Q(z) \geq 1$ for any $z \in \mathbb{R}^{d_z}$). Thus,

$$\begin{aligned} \sum_{n=1}^j \alpha_n \|\zeta_{2,n}\| I_{\tilde{\Lambda}_Q} &\leq 2 \sum_{n=0}^{\infty} \alpha_n \alpha_{n+1} (\varphi_Q^2(Z_{n+1}) + \|\eta_{n+1}\|^2) I_{\tilde{\Lambda}_Q} \\ &\leq \sum_{n=0}^{\infty} (\alpha_n^2 + \alpha_{n+1}^2) \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}} + \sup_{n \geq 0} \|\eta_n\|^2 I_{\tilde{\Lambda}_Q} \sum_{n=0}^{\infty} (\alpha_n^2 + \alpha_{n+1}^2) \end{aligned}$$

(notice that $2\alpha_n \alpha_{n+1} \leq \alpha_n^2 + \alpha_{n+1}^2$). Then, Assumption 3.4 and (38) imply that $\sum_{n=1}^{\infty} \alpha_n \zeta_{2,n}$ converges almost surely on $\tilde{\Lambda}_Q$.

Owing to Assumption 3.3, we have

$$\|\zeta_{3,n}\| I_{\tilde{\Lambda}_Q} \leq \varphi_Q(Z_{n+1}) I_{\tilde{\Lambda}_Q} \leq \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}}$$

for $n \geq 0$. Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha_{n+1}^2 \|\zeta_{3,n}\|^2 I_{\tilde{\Lambda}_Q} &\leq \sum_{n=0}^{\infty} \alpha_{n+1}^2 \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}}, \\ \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| \|\zeta_{3,n}\| I_{\tilde{\Lambda}_Q} &\leq \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| \varphi_Q^2(Z_{n+1}) I_{\{\tau_Q > n\}}. \end{aligned}$$

Combining this with (38), we conclude $\lim_{n \rightarrow \infty} \alpha_{n+1} \zeta_{3,n} = 0$ almost surely on $\tilde{\Lambda}_Q$. We also deduce that $\sum_{n=0}^{\infty} (\alpha_n - \alpha_{n+1}) \zeta_{3,n}$ converges almost surely on $\tilde{\Lambda}_Q$. Since $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n}$, $\sum_{n=1}^{\infty} \alpha_n \zeta_{2,n}$ converge almost surely on $\tilde{\Lambda}_Q$, (37) implies that $\sum_{n=0}^{\infty} \alpha_n \zeta_n$ also converges almost surely on $\tilde{\Lambda}_Q$. As Q is any compact set in \mathbb{R}^{d_θ} , $\sum_{n=0}^{\infty} \alpha_n \zeta_n$ converges almost surely on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$. Consequently, Assumption 3.4 yields that $\{\xi_n\}_{n \geq 0}$ defined in (36) satisfies Assumption 2.2. Then, the theorem's assertion directly follows from Theorem 2.1. \square

⁵Notice that $\sum_{n=0}^{\infty} \alpha_n \zeta_{1,n} I_{\{\tau_Q > n\}} = \sum_{n=0}^{\infty} \alpha_n \zeta_{1,n}$ on $\tilde{\Lambda}_Q$.

8. Proof of Theorem 4.1

In this section, we use the following notation. $\phi(v)$, $s_\theta(v)$ are the functions defined by

$$\phi(v) = \phi(x, y), \quad s_\theta(v) = s_\theta(x, y)$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$. For $\theta \in \mathbb{R}^{d_\theta}$, $\{V_n^\theta\}_{n \geq 0}$, $\{W_n^\theta\}_{n \geq 0}$ and $\{Z_n^\theta\}_{n \geq 0}$ are stochastic processes defined by

$$V_n^\theta = (X_n^\theta, Y_n^\theta), \quad W_{n+1}^\theta = \lambda W_n^\theta + s_\theta(V_n^\theta), \quad Z_n^\theta = (V_n^\theta, W_n^\theta)$$

for $n \geq 0$, where $W_0^\theta \in \mathbb{R}^{d_\theta}$ is a (deterministic) vector (notice that $\{V_n^\theta\}_{n \geq 0}$, $\{Z_n^\theta\}_{n \geq 0}$ are Markov chains). Moreover, for $\theta \in \mathbb{R}^{d_\theta}$, $r_\theta(\cdot|\cdot)$ and $\nu_\theta(\cdot)$ are the transition kernel and invariant probability of $\{V_n^\theta\}_{n \geq 0}$,⁶ while $\Pi_\theta(\cdot, \cdot)$ is the transition kernel of $\{Z_n^\theta\}_{n \geq 0}$.⁷ For $\theta \in \mathbb{R}^{d_\theta}$, $n \geq 0$, $r_\theta^n(\cdot|\cdot)$ is the n -th transition probability of $\{V_n^\theta\}_{n \geq 0}$, while

$$\tilde{r}_\theta^n(v'|v) = r_\theta^n(v'|v) - \nu_\theta(v')$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{X} \times \mathcal{Y}$, $n \geq 0$. Additionally, the functions $\eta(\cdot)$, $F(\cdot, \cdot)$ are defined by

$$\eta(\theta) = \sum_{n=0}^{\infty} \sum_{v, v' \in \mathcal{X} \times \mathcal{Y}} \lambda^n \phi(v') \tilde{r}_\theta^n(v'|v) s_\theta(v) \nu_\theta(v) - \nabla f(\theta), \quad F(\theta, z) = \phi(v) w - \eta(\theta)$$

for $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$.⁸ $\{Z_n\}_{n \geq 0}$, $\{\eta_n\}_{n \geq 0}$ are the stochastic processes defined as

$$Z_n = (X_n, Y_n, W_n), \quad \eta_n = \eta(\theta_n)$$

for $n \geq 0$. Then, it is straightforward to show that the algorithm (18) is of the same form as the recursion studied in Section 3 (i.e., $\{\theta_n\}_{n \geq 0}$, $\{\eta_n\}_{n \geq 0}$, $F(\cdot, \cdot)$, $\Pi_\theta(\cdot, \cdot)$ defined in Section 4 and here admit (13), (14)).

We will use the following additional notation. N_v is the integer defined by $N_v = N_x N_y$, while $e \in \mathbb{R}^{N_v}$ is the vector whose all components are one. For $v \in \mathcal{X} \times \mathcal{Y}$, $e(v) \in \mathbb{R}^{N_v}$ is the vector representation of $I_v(\cdot)$, while $\phi \in \mathbb{R}^{N_v}$ is the vector representation of $\phi(\cdot)$.⁹ For $\theta \in \mathbb{R}^{d_\theta}$, $R_\theta \in \mathbb{R}^{N_v \times N_v}$ and $\nu_\theta \in \mathbb{R}^{N_v}$ are the transition matrix and the invariant probability vector of $\{V_n^\theta\}_{n \geq 0}$,¹⁰ while $\tilde{R}_\theta = R_\theta - e \nu_\theta^T$. For $\theta \in \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, $s_{\theta, j}(\cdot)$ is the j -th component of $s_\theta(\cdot)$, while $S_{\theta, j} \in \mathbb{R}^{N_v \times N_v}$ is the diagonal matrix representation of $s_{\theta, j}(\cdot)$.¹¹

Lemma 8.1. *Suppose that Assumptions 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, the following is true:*

- (i) $\{V_n^\theta\}_{n \geq 0}$ is geometrically ergodic for each $\theta \in \mathbb{R}^{d_\theta}$. Moreover, there exist real numbers $\varepsilon_Q \in (0, 1)$, $C_{1, Q} \in [1, \infty)$ (independent of λ) such that $\|\tilde{R}_\theta^n\| \leq C_{1, Q} \varepsilon_Q^n$ for all $\theta \in Q$, $n \geq 0$.
- (ii) There exists a real number $C_{2, Q} \in [1, \infty)$ (independent of λ) such that

$$\max\{\|\nu_{\theta'} - \nu_{\theta''}\|, \|R_{\theta'}^n - R_{\theta''}^n\|\} \leq C_{2, Q} \|\theta' - \theta''\|, \quad (39)$$

$$\|\tilde{R}_{\theta'}^n - \tilde{R}_{\theta''}^n\| \leq C_{2, Q} \varepsilon_Q^n \|\theta' - \theta''\| \quad (40)$$

for all $\theta', \theta'' \in Q$, $n \geq 0$.

- (iii) ν_θ is differentiable on \mathbb{R}^{d_θ} . Moreover, $\nabla_\theta \nu_\theta$ is locally Lipschitz continuous on \mathbb{R}^{d_θ} .
- (iv) If Assumption 4.3.a is satisfied, ν_θ is p times differentiable on \mathbb{R}^{d_θ} .

⁶Under Assumption 4.1, $\nu_\theta(\cdot)$ exists and is unique (the details are provided in Lemma 8.1). The transition $r_\theta(\cdot|\cdot)$ can be defined by $r_\theta(v'|v) = q_\theta(y'|x') p(x'|x, y)$ for $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x', y') \in \mathcal{X} \times \mathcal{Y}$.

⁷ $\Pi_\theta(\cdot, \cdot)$ can be defined by $\Pi_\theta(z, \{v'\} \times B) = I_B(\lambda w + s_\theta(v')) r_\theta(v'|v)$ for $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$ and a Borel-measurable set $B \subseteq \mathbb{R}^{d_\theta}$.

⁸Under Assumptions 4.1, 4.2, $f(\cdot)$ is differentiable (the details are provided in Lemma 8.2).

⁹For $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, element i of $e(v)$ is one if $i = (x-1)N_y + y$ and zero otherwise. For the same v , $\phi(v)$ is element $(x-1)N_y + y$ of ϕ .

¹⁰For $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x', y') \in \mathcal{X} \times \mathcal{Y}$, $r_\theta(v'|v)$ is entry $((x-1)N_y + y, (x'-1)N_y + y')$ of R_θ , while $\nu_\theta(v)$ is element $(x-1)N_y + y$ of ν_θ .

¹¹For $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, $s_{\theta, j}(v)$ is entry $((x-1)N_y + y, (x-1)N_y + y)$ of $S_{\theta, j}$. The off-diagonal elements of $S_{\theta, j}$ are zero.

(v) If Assumption 4.3.b is satisfied, ν_θ is real-analytic on \mathbb{R}^{d_θ} .

Proof. (i) For $\theta \in \mathbb{R}^{d_\theta}$, $n \geq 0$, let $p_\theta^n(\cdot|\cdot)$ and $\mu_\theta(\cdot)$ be the n -th transition probability and the invariant probability of $\{X_n^\theta\}_{n \geq 0}$. Moreover, for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, let $\tilde{\nu}_\theta(v) = q_\theta(y|x)\mu_\theta(x)$. Then, it is straightforward to verify

$$r_\theta^{n+1}(v'|v) - \tilde{\nu}_\theta(v') = \sum_{x'' \in \mathcal{X}} q_\theta(y'|x')(p_\theta^n(x'|x'') - \mu_\theta(x'))p(x''|x, y)$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x', y') \in \mathcal{X} \times \mathcal{Y}$, $n \geq 0$. Therefore,

$$|r_\theta^{n+1}(v'|v) - \tilde{\nu}_\theta(v')| \leq \sum_{x'' \in \mathcal{X}} q_\theta(y'|x')|p_\theta^n(x'|x'') - \mu_\theta(x')|p(x''|x, y) \leq N_x \max_{x'' \in \mathcal{X}} |p_\theta^n(x'|x'') - \mu_\theta(x')|$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, $v' = (x', y') \in \mathcal{X} \times \mathcal{Y}$, $n \geq 0$. Combining this with Assumption 4.1, we conclude that $\{V_n^\theta\}_{n \geq 0}$ is geometrically ergodic for each $\theta \in \mathbb{R}^{d_\theta}$. We also conclude that $\tilde{\nu}_\theta(\cdot)$ is the invariant probability of $\{V_n^\theta\}_{n \geq 0}$ for each $\theta \in \mathbb{R}^{d_\theta}$, i.e., $\nu_\theta(v) = \tilde{\nu}_\theta(v) = q_\theta(y|x)\mu_\theta(x)$ for $\theta \in \mathbb{R}^{d_\theta}$, $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$.

For $\theta \in \mathbb{R}^{d_\theta}$, let $\rho_\theta = \min_{v \in \mathcal{V}} \nu_\theta(v)/3$. Then, we have $0 < \rho_\theta \leq 1/(3N_v)$, $\rho_\theta \leq \nu_\theta(v)/3$ for all $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{V}$. Moreover, for any $\theta \in \mathbb{R}^{d_\theta}$, there exists an integer $n_\theta \geq 0$ such that $|r_\theta^n(v'|v) - \nu_\theta(v')| \leq \rho_\theta$ for each $v, v' \in \mathcal{V}$, $n \geq n_\theta$. Hence, $r_\theta^n(v'|v) \geq \nu_\theta(v') - \rho_\theta \geq 2\rho_\theta$ for all $\theta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$, $n \geq n_\theta$. Additionally, Assumption 4.2 implies that for each $v, v' \in \mathcal{V}$, $n \geq 0$, $r_\theta^n(v'|v)$ is locally Lipschitz continuous in θ on \mathbb{R}^{d_θ} .¹² Consequently, for any $\theta \in \mathbb{R}^{d_\theta}$, there exists a real number $\delta_\theta \in (0, 1)$ such that $|r_\theta^{n_\theta}(v'|v) - r_\theta^{n_\theta}(v'|v)| \leq \rho_\theta$ for all $\vartheta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$ satisfying $\|\vartheta - \theta\| \leq \delta_\theta$. Thus, $r_\vartheta^{n_\theta}(v'|v) \geq r_\theta^{n_\theta}(v'|v) - \rho_\theta \geq \rho_\theta$ for each $\vartheta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$ satisfying $\|\vartheta - \theta\| \leq \delta_\theta$. Since

$$r_\vartheta^n(v'|v) = \sum_{v'' \in \mathcal{V}} r_\vartheta^{n_\theta}(v'|v'')r_\vartheta^{n-n_\theta}(v''|v) \geq \rho_\theta \sum_{v'' \in \mathcal{V}} r_\vartheta^{n-n_\theta}(v''|v) = \rho_\theta$$

for any $\vartheta \in \mathbb{R}^{d_\theta}$, $v, v' \in \mathcal{V}$, $n \geq n_\theta$ satisfying $\|\vartheta - \theta\| \leq \delta_\theta$, we conclude $r_\vartheta^n(v'|v) \geq \rho_\theta$ for the same ϑ , v, v', n .

Let $B_\theta = \{\vartheta \in \mathbb{R}^{d_\theta} : \|\vartheta - \theta\| < \delta_\theta\}$ for $\theta \in \mathbb{R}^{d_\theta}$. As $\{B_\theta\}_{\theta \in Q}$ is an open covering of Q , there exists a finite set $\tilde{Q} \subseteq Q$ such that $\bigcup_{\theta \in \tilde{Q}} B_\theta \supset Q$. Let $\tilde{n}_Q = \max_{\theta \in \tilde{Q}} n_\theta$, $\tilde{\rho}_Q = \min_{\theta \in \tilde{Q}} \rho_\theta$, $\tilde{\varepsilon}_Q = (1 - \tilde{\rho}_Q)^{1/\tilde{n}_Q}$. Since each element of Q is also an element of one of $\{B_\theta\}_{\theta \in \tilde{Q}}$, we have $r_\theta^n(v'|v) \geq \tilde{\rho}_Q$ for all $\theta \in Q$, $v, v' \in \mathcal{V}$, $n \geq \tilde{n}_Q$.¹³ Then, standard results of Markov chain theory (see e.g., [26, Theorem 16.0.2]) imply

$$|r_\theta^n(v'|v) - \nu_\theta(v')| \leq (1 - \tilde{\rho}_Q N_v)^{n/\tilde{n}_Q} \leq \tilde{\varepsilon}_Q^n$$

for all $\theta \in Q$, $v, v' \in \mathcal{V}$, $n \geq 0$.

Let $\varepsilon_Q = \tilde{\varepsilon}_Q^{1/2}$, $C_{1,Q} = N_v$. Then, we have

$$\|\tilde{R}_\theta^n\| \leq N_v \max_{v, v' \in \mathcal{X} \times \mathcal{Y}} |r_\theta^n(v'|v)| \leq N_v \tilde{\varepsilon}_Q^n = C_{1,Q} \varepsilon_Q^{2n} \quad (41)$$

for all $\theta \in Q$, $n \geq 0$.

(ii) Let g be the N_v -th standard unit vector in \mathbb{R}^{N_v} (i.e., the first $N_v - 1$ elements of g are zero, while the last element of g is one) and, for $A \in \mathbb{R}^{N_v \times N_v}$, let $G(A)$ be the $N_v \times N_v$ matrix obtained when the last row of $I - A^T$ is replaced by e^T (here, I is the $N_v \times N_v$ unit matrix). Additionally, let $\mathcal{Q}_0^{N_v \times N_v} = \{A \in \mathbb{R}^{N_v \times N_v} : \det(G(A)) \neq 0\}$ and, for $A \in \mathcal{Q}_0^{N_v \times N_v}$, let $h(A) = (G(A))^{-1}g$. Then, it is easy to conclude that $\mathcal{Q}_0^{N_v \times N_v}$ is an open set (notice that $\det(G(A))$ is a polynomial function of the entries of A). It is also easy to deduce that $h(\cdot)$ is well-defined and real-analytic on $\mathcal{Q}_0^{N_v \times N_v}$ (notice that due to the Cramer's rule, all elements of $h(A)$ are rational functions of the entries of A).

Let $\mathcal{P}_0^{N_v \times N_v}$ be the set of $N_v \times N_v$ geometrically ergodic stochastic matrices. Then, each $P \in \mathcal{P}_0^{N_v \times N_v}$ has a unique invariant probability vector. Moreover, the invariant probability vector of $P \in \mathcal{P}_0^{N_v \times N_v}$ is the unique solution to the linear system of equations $G(P)x = g$, where $x \in \mathbb{R}^{N_v}$ is the unknown. Hence, $\det(G(P)) \neq 0$ for each $P \in \mathcal{P}_0^{N_v \times N_v}$ so $\mathcal{P}_0^{N_v \times N_v} \subset \mathcal{Q}_0^{N_v \times N_v}$.

¹²Notice that, due to Assumption 4.2, $q_\theta(y|x)$ is locally Lipschitz continuous in θ for each $x \in \mathcal{X}$, $y \in \mathcal{Y}$ and that $r_\theta^n(\cdot|\cdot)$ is a polynomial function of $p(\cdot|\cdot, \cdot)$, $q_\theta(\cdot|\cdot)$.

¹³If $\theta \in B_\vartheta$ and $\vartheta \in \tilde{Q}$, then $n_\theta \leq \tilde{n}_Q$ and $r_\theta^n(v'|v) \geq \rho_\theta \geq \tilde{\rho}_Q$ for $n \geq n_\theta$.

Owing to (i), $R_\theta \in \mathcal{P}_0^{N_v \times N_v}$ for each $\theta \in \mathbb{R}^{d_\theta}$. Thus, $\nu_\theta = h(R_\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$. Moreover, due to Assumption 4.2, R_θ is locally Lipschitz continuous on \mathbb{R}^{d_θ} .¹⁴ Since $h(\cdot)$ is real-analytic on $\mathcal{Q}_0^{N_v \times N_v}$ and $\mathcal{P}_0^{N_v \times N_v} \subset \mathcal{Q}_0^{N_v \times N_v}$, ν_θ is locally Lipschitz continuous on \mathbb{R}^{d_θ} .

Let $\tilde{C}_{1,Q} \in [1, \infty)$ be a Lipschitz constant of R_θ , ν_θ on Q , while $\tilde{C}_{2,Q} \in [1, \infty)$ is an upper bound of the sequence $\{n\varepsilon_Q^n\}_{n \geq 1}$. Let $C_{2,Q} = 3\varepsilon_Q^{-1}C_{1,Q}^2\tilde{C}_{1,Q}\tilde{C}_{2,Q}$. It is straightforward to verify

$$\tilde{R}_{\theta'}^{n+1} - \tilde{R}_{\theta''}^{n+1} = \sum_{i=0}^n \tilde{R}_{\theta'}^i (R_{\theta'} - R_{\theta''} - e(\nu_{\theta'} - \nu_{\theta''})^T) \tilde{R}_{\theta''}^{n-i}$$

for $\theta', \theta'' \in \mathbb{R}^{d_\theta}$, $n \geq 0$. Combining this with (41), we get

$$\begin{aligned} \|\tilde{R}_{\theta'}^{n+1} - \tilde{R}_{\theta''}^{n+1}\| &\leq \sum_{i=0}^n \|\tilde{R}_{\theta'}^i\| \|\tilde{R}_{\theta''}^{n-i}\| (\|R_{\theta'} - R_{\theta''}\| + \|\nu_{\theta'} - \nu_{\theta''}\|) \\ &\leq 2C_{1,Q}^2 \tilde{C}_{1,Q} (n+1) \varepsilon_Q^{2n} \|\theta' - \theta''\| \\ &\leq C_{2,Q} \varepsilon_Q^n \|\theta' - \theta''\| \end{aligned}$$

for each $\theta', \theta'' \in Q$, $n \geq 0$. Therefore,

$$\|R_{\theta'}^n - R_{\theta''}^n\| \leq \|\tilde{R}_{\theta'}^n - \tilde{R}_{\theta''}^n\| + \|\nu_{\theta'} - \nu_{\theta''}\| \leq \tilde{C}_{1,Q} (2C_{1,Q}^2 \tilde{C}_{2,Q} n \varepsilon_Q^{n-1} + 1) \|\theta' - \theta''\| \leq C_{2,Q} \|\theta' - \theta''\|$$

for all $\theta', \theta'' \in Q$, $n \geq 0$ (notice that $\tilde{R}_\theta^k = R_\theta^k - e\nu_\theta^T$).

(iii) Due to (i), $R_\theta \in \mathcal{P}_0^{N_v \times N_v}$ for each $\theta \in \mathbb{R}^{d_\theta}$. Hence, $\nu_\theta = h(R_\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$. Moreover, owing to Assumption 4.2, R_θ is differentiable on \mathbb{R}^{d_θ} and its first-order derivatives are locally Lipschitz continuous on the same space.¹⁵ As $h(\cdot)$ is real-analytic on $\mathcal{Q}_0^{N_v \times N_v}$ and $\mathcal{P}_0^{N_v \times N_v} \subset \mathcal{Q}_0^{N_v \times N_v}$, ν_θ is differentiable on \mathbb{R}^{d_θ} . The same arguments also imply that $\nabla_\theta \nu_\theta$ is locally Lipschitz continuous on \mathbb{R}^{d_θ} .

(iv), (v) If Assumption 4.3.a is satisfied, then R_θ is p times differentiable on \mathbb{R}^{d_θ} , and consequently, ν_θ is p times differentiable on \mathbb{R}^{d_θ} , too.¹⁶ Similarly, if Assumption 4.3.b is satisfied, then R_θ is real-analytic on \mathbb{R}^{d_θ} , and therefore, ν_θ is also real-analytic on \mathbb{R}^{d_θ} . \square

Lemma 8.2. *Suppose that Assumptions 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, the following is true:*

- (i) $f(\cdot)$ is differentiable and $\nabla f(\cdot)$ is locally Lipschitz continuous.
- (ii) There exists a real number $C_{3,Q} \in [1, \infty)$ (independent of λ) such that $\|\eta(\theta)\| \leq C_{3,Q}(1 - \lambda)$ for all $\theta \in Q$.
- (iii) If Assumption 4.3.a is satisfied, $f(\cdot)$ is p times differentiable.
- (iv) If Assumption 4.3.b is satisfied, $f(\cdot)$ is real-analytic.

Proof. (i), (iii), (iv) Owing to Lemma 8.1, we have

$$f(\theta) = \lim_{n \rightarrow \infty} E_\theta(\phi(V_n^\theta)) = \sum_{v \in \mathcal{X} \times \mathcal{Y}} \phi(v) \nu_\theta(v) = \phi^T \nu_\theta \quad (42)$$

for all $\theta \in \mathbb{R}^{d_\theta}$. Then, these parts of the lemma directly follow from Lemma 8.1.

(ii) For each $1 \leq j \leq d_\theta$, let $\tilde{C}_Q \in [1, \infty)$ be an upper bound of $\|S_{\theta,j}\|$ on Q . For $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $n \geq 0$, let also define

$$f_n(\theta, v) = \sum_{v' \in \mathcal{X} \times \mathcal{Y}} \phi(v') r_\theta^n(v'|v), \quad h(\theta) = \sum_{n=0}^{\infty} \sum_{v, v' \in \mathcal{X} \times \mathcal{Y}} \phi(v') \tilde{r}_\theta^n(v'|v) s_\theta(v) \nu_\theta(v). \quad (43)$$

Owing to Lemma 8.1, $f_n(\theta, v)$ converges to $f(\theta)$ as $n \rightarrow \infty$ uniformly in (θ, v) on $Q \times (\mathcal{X} \times \mathcal{Y})$. Due to the same lemma, $h(\cdot)$ is well-defined on Q (notice that when $\theta \in Q$, each term in the sum in (43) tends to zero at

¹⁴Notice that $r_\theta(v'|v) = q_\theta(y'|x)p(x'|x, y)$ for $v = (x, y)$, $v' = (x', y')$ and that $q_\theta(y|x)$ is locally Lipschitz continuous in θ .

¹⁵Notice that $\nabla_\theta r_\theta(v'|v) = s_\theta(x', y') q_\theta(y'|x') p(x'|x, y)$ for $v = (x, y)$, $v' = (x', y')$.

¹⁶Notice that $R_\theta \in \mathcal{P}_0^{N_v \times N_v} \subset \mathcal{Q}_0^{N_v \times N_v}$, $\nu_\theta = h(R_\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$. Notice also that $h(\cdot)$ is real-analytic on $\mathcal{Q}_0^{N_v \times N_v}$.

the rate ε_Q^n). Moreover, it is straightforward to show

$$\begin{aligned}
\nabla_\theta f_n(\theta, v_0) &= \nabla_\theta \left(\sum_{v_1, \dots, v_n \in \mathcal{X} \times \mathcal{Y}} \phi(v_n) \left(\prod_{i=1}^n r_\theta(v_i | v_{i-1}) \right) \right) \\
&= \sum_{v_1, \dots, v_n \in \mathcal{X} \times \mathcal{Y}} \phi(v_n) \left(\sum_{i=1}^n \frac{\nabla_\theta r_\theta(v_i | v_{i-1})}{r_\theta(v_i | v_{i-1})} \right) \left(\prod_{i=1}^n r_\theta(v_i | v_{i-1}) \right) \\
&= \sum_{i=1}^n \sum_{v', v'' \in \mathcal{X} \times \mathcal{Y}} \phi(v'') r_\theta^{n-i}(v'' | v') s_\theta(v') r_\theta^i(v' | v_0)
\end{aligned} \tag{44}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $v_0 \in \mathcal{X} \times \mathcal{Y}$, $n \geq 1$. Therefore,

$$\partial_\theta^j f_n(\theta, v) = \sum_{i=1}^n e^T(v) R_\theta^i S_{\theta, j} R_\theta^{n-i} \phi = \sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta, j} R_\theta^i \phi \tag{45}$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$, where $\partial_\theta^j f_n(\theta, v)$ is the j -th component of $\nabla_\theta f_n(\theta, v)$. We also have

$$\sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta, j} R_\theta^i e = 0 \tag{46}$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$.¹⁷ Hence,

$$\sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta, j} e \nu_\theta^T \phi = \nu_\theta^T \phi \sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta, j} R_\theta^i e = 0$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$ (notice that $R_\theta^i e = e$). Therefore,

$$\partial_\theta^j f_n(\theta, v) = \sum_{i=0}^{n-1} e^T(v) R_\theta^{n-i} S_{\theta, j} \tilde{R}_\theta^i \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Additionally, we have

$$h_j(\theta) = \sum_{n=0}^{\infty} \nu_\theta^T S_{\theta, j} \tilde{R}_\theta^n \phi = \sum_{i=0}^{n-1} e^T(v) e \nu_\theta^T S_{\theta, j} \tilde{R}_\theta^i \phi + \sum_{i=n}^{\infty} \nu_\theta^T S_{\theta, j} \tilde{R}_\theta^i \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$ (notice that $e^T(v) e = 1$), where $h_j(\theta)$ is the j -th component of $h(\theta)$. Thus,

$$\partial_\theta^j f_n(\theta, v) - h_j(\theta) = \sum_{i=0}^{n-1} e^T(v) \tilde{R}_\theta^{n-i} S_{\theta, j} \tilde{R}_\theta^i \phi - \sum_{i=n}^{\infty} \nu_\theta^T S_{\theta, j} \tilde{R}_\theta^i \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Then, Lemma 8.1 implies

$$\begin{aligned}
|\partial_\theta^j f_n(\theta, v) - h_j(\theta)| &\leq \|\phi\| \|e(v)\| \|S_{\theta, j}\| \sum_{i=0}^{n-1} \|\tilde{R}_\theta^i\| \|\tilde{R}_\theta^{n-i}\| + \|\phi\| \|\nu_\theta\| \|S_{\theta, j}\| \sum_{i=n}^{\infty} \|\tilde{R}_\theta^i\| \\
&\leq \tilde{C}_Q C_{1, Q}^2 \|\phi\| n \varepsilon_Q^n + \frac{\tilde{C}_Q C_{1, Q} \|\phi\| \varepsilon_Q^n}{1 - \varepsilon_Q}
\end{aligned}$$

for all $\theta \in Q$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Hence, $\nabla_\theta f_n(\theta, v)$ converges to $h(\theta)$ as $n \rightarrow \infty$ uniformly in (θ, v) on $Q \times (\mathcal{X} \times \mathcal{Y})$. Therefore, $\nabla f(\theta) = h(\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$ (notice that Q is any compact set). Consequently,

$$\eta_j(\theta) = \sum_{n=0}^{\infty} \lambda^n \nu_\theta^T S_{\theta, j} \tilde{R}_\theta^n \phi - h_j(\theta) = - \sum_{n=0}^{\infty} (1 - \lambda^n) \nu_\theta^T S_{\theta, j} \tilde{R}_\theta^n \phi$$

¹⁷If $\phi = e$, then $f_n(\theta, v)$ is identically one, while $\nabla_\theta f_n(\theta, v)$ is identically zero. Hence, (45) reduces to (46) when $\phi = e$.

for $\theta \in \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, where $\eta_j(\theta)$ is the j -th component of $\eta(\theta)$. Combining this with Lemma 8.1, we get

$$|\eta_j(\theta)| \leq \|\phi\| \|\nu_\theta\| \|S_{\theta,j}\| \sum_{n=0}^{\infty} (1 - \lambda^n) \|\tilde{R}_\theta^n\| \leq \tilde{C}_Q C_{1,Q} \|\phi\| \sum_{n=0}^{\infty} (1 - \lambda^n) \varepsilon_Q^n \leq \frac{\tilde{C}_Q C_{1,Q} \|\phi\| (1 - \lambda)}{(1 - \varepsilon_Q)^2}$$

for all $\theta \in Q$, $1 \leq j \leq d_\theta$. Then, we conclude that there exists a real number $C_{3,Q} \in [1, \infty)$ with the properties specified in (ii). \square

Lemma 8.3. *Suppose that Assumptions 4.1 and 4.2 hold. Let $Q \subset \mathbb{R}^{d_\theta}$ be any compact set. Then, the following is true:*

(i) *There exist real numbers $\delta_Q \in (0, 1)$, $C_{4,Q} \in [1, \infty)$ (possibly depending on λ) such that*

$$\begin{aligned} \|(\Pi^n F)(\theta, z) - \nabla f(\theta)\| &\leq C_{4,Q} n \delta_Q^n (1 + \|w\|), \\ \|((\Pi^n F)(\theta', z) - \nabla f(\theta')) - ((\Pi^n F)(\theta'', z) - \nabla f(\theta''))\| &\leq C_{4,Q} n \delta_Q^n \|\theta' - \theta''\| (1 + \|w\|), \end{aligned}$$

for all $\theta, \theta', \theta'' \in Q$, $z = (x, y, w) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^{d_\theta}$, $n \geq 0$.

(ii) *There exists a real number $C_{5,Q} \in [1, \infty)$ (possibly depending on λ) such that*

$$\|W_{n+1}\| I_{\{\tau_Q > n\}} \leq C_{5,Q} (1 + \|W_0\|)$$

for all $n \geq 0$ (τ_Q is specified in Assumption 3.3).

Proof. (i) For each $1 \leq j \leq d_\theta$, let $\tilde{C}_{1,Q} \in [1, \infty)$ be an upper bound of $\|S_{\theta,j}\|$ on Q and a Lipschitz constant of $S_{\theta,j}$ on the same set. Moreover, let $\tilde{C}_{2,Q} = 3\tilde{C}_{1,Q} C_{1,Q} C_{2,Q} N_v$, $\tilde{C}_{3,Q} = 2\tilde{C}_{2,Q} (1 - \varepsilon_Q)^{-1}$, while $\delta_Q = \max\{\lambda, \varepsilon_Q\}$.

Owing to Lemma 8.1, we have

$$\|\tilde{R}_\theta^k S_{\theta,j} R_\theta^l\| \leq \|\tilde{R}_\theta^k\| \|S_{\theta,j}\| \|R_\theta^l\| \leq \tilde{C}_{2,Q} \varepsilon_Q^k, \quad (47)$$

$$\|\nu_\theta^T S_{\theta,j} R_\theta^l\| \leq \|\nu_\theta^T\| \|S_{\theta,j}\| \|R_\theta^l\| \leq \tilde{C}_{2,Q} \quad (48)$$

for all $\theta \in Q$, $1 \leq j \leq d_\theta$, $k, l \geq 1$. Due to the same lemma, we also have

$$\begin{aligned} \|\tilde{R}_{\theta'}^k S_{\theta',j} R_{\theta'}^l - \tilde{R}_{\theta''}^k S_{\theta'',j} R_{\theta''}^l\| &\leq \|\tilde{R}_{\theta'}^k - \tilde{R}_{\theta''}^k\| \|S_{\theta',j}\| \|R_{\theta'}^l\| + \|\tilde{R}_{\theta''}^k\| \|S_{\theta',j} - S_{\theta'',j}\| \|R_{\theta'}^l\| \\ &\quad + \|\tilde{R}_{\theta''}^k\| \|S_{\theta'',j}\| \|R_{\theta'}^l - R_{\theta''}^l\| \\ &\leq \tilde{C}_{2,Q} \varepsilon_Q^k \|\theta' - \theta''\| \end{aligned} \quad (49)$$

for all $\theta', \theta'' \in Q$, $1 \leq j \leq d_\theta$, $k, l \geq 1$. In addition to this, Lemma 8.1 implies

$$\begin{aligned} \|\nu_{\theta'}^T S_{\theta',j} R_{\theta'}^l - \nu_{\theta''}^T S_{\theta'',j} R_{\theta''}^l\| &\leq \|\nu_{\theta'}^T - \nu_{\theta''}^T\| \|S_{\theta',j}\| \|R_{\theta'}^l\| + \|\nu_{\theta''}^T\| \|S_{\theta',j} - S_{\theta'',j}\| \|R_{\theta'}^l\| \\ &\quad + \|\nu_{\theta''}^T\| \|S_{\theta'',j}\| \|R_{\theta'}^l - R_{\theta''}^l\| \\ &\leq \tilde{C}_{2,Q} \|\theta' - \theta''\| \end{aligned} \quad (50)$$

for each $\theta', \theta'' \in Q$, $1 \leq j \leq d_\theta$, $l \geq 1$. Moreover, it is straightforward to show

$$\begin{aligned} (\Pi^n F)(\theta, z) &= -\eta(\theta) + E_\theta(\phi(V_n^\theta) W_n^\theta | V_0^\theta = v, W_0^\theta = w) \\ &= -\eta(\theta) + E_\theta\left(\phi(V_n^\theta) \left(\lambda^n w + \sum_{i=0}^{n-1} \lambda^i s_\theta(V_{n-i}^\theta)\right) \middle| V_0^\theta = v\right) \\ &= -\eta(\theta) + \sum_{i=0}^{n-1} \sum_{v', v'' \in \mathcal{X} \times \mathcal{Y}} \lambda^i \phi(v'') r_\theta^i(v'' | v') s_\theta(v') r_\theta^{n-i}(v' | v) + \lambda^n w \sum_{v' \in \mathcal{X} \times \mathcal{Y}} \phi(v') r_\theta^n(v' | v) \end{aligned}$$

for $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $n \geq 1$. Therefore,

$$(\Pi F_j)(\theta, z) = -\eta_j(\theta) + \sum_{i=0}^{n-1} \lambda^i e^T(v) R_\theta^{n-i} S_{\theta,j} R_\theta^i \phi + \lambda^n e_j^T w e^T(v) R_\theta^n \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Here, $F_j(\theta, z)$, $\eta_j(\theta)$ are the j -th components of $F(\theta, z)$, $\eta(\theta)$, while e_j is the j -th standard unit vector in \mathbb{R}^{d_θ} . Moreover, we have

$$\partial^j f(\theta) = -\eta_j(\theta) + \sum_{n=0}^{\infty} \lambda^n \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^n \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, where $\partial^j f(\theta)$ is the j -th component of $\nabla f(\theta)$. Since $e^T(v)e = 1$, $\tilde{R}_\theta^n = R_\theta^n - e\nu_\theta^T$ and

$$\nu_\theta^T S_{\theta,j} e = \sum_{v \in \mathcal{X} \times \mathcal{Y}} \nu_\theta(v) s_{\theta,j}(v) = \sum_{x \in \mathcal{X}} \left(\sum_{y \in \mathcal{Y}} \partial_\theta^j q_\theta(y|x) \right) \mu_\theta(x) = 0$$

for $\theta \in \mathbb{R}^{d_\theta}$, $v \in \mathcal{X} \times \mathcal{Y}$, $1 \leq j \leq d_\theta$, $n \geq 0$,¹⁸ we get

$$\begin{aligned} \partial^j f(\theta) &= -\eta_j(\theta) + \sum_{i=0}^{n-1} \lambda^i \nu_\theta^T S_{\theta,j} R_\theta^i \phi + \sum_{i=0}^{n-1} \lambda^i \nu_\theta^T S_{\theta,j} e \nu_\theta^T \phi + \sum_{i=n}^{\infty} \lambda^i \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi \\ &= -\eta_j(\theta) + \sum_{i=0}^{n-1} \lambda^i e^T(v) e \nu_\theta^T S_{\theta,j} R_\theta^i \phi + \sum_{i=n}^{\infty} \lambda^i \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi \end{aligned}$$

for the same θ , v , j , n . Consequently,

$$(\Pi^n F_j)(\theta, z) - \partial^j f(\theta) = \sum_{i=0}^{n-1} \lambda^i e^T(v) \tilde{R}_\theta^{n-i} S_{\theta,j} R_\theta^i \phi - \sum_{i=n}^{\infty} \lambda^i \nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i \phi + \lambda^n e_j^T w e^T(v) R_\theta^n \phi$$

for $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Then, (47), (48) imply

$$\begin{aligned} |(\Pi^n F_j)(\theta, z) - \partial^j f(\theta)| &\leq \|\phi\| \|e(v)\| \sum_{i=0}^{n-1} \lambda^i \|\tilde{R}_\theta^{n-i} S_{\theta,j} R_\theta^i\| + \|\phi\| \sum_{i=n}^{\infty} \lambda^i \|\nu_\theta^T S_{\theta,j} \tilde{R}_\theta^i\| + \lambda^n \|\phi\| \|e(v)\| \|R_\theta^n\| \|w\| \\ &\leq \tilde{C}_{2,Q} \left(\sum_{i=1}^n \lambda^i \varepsilon_Q^{n-i} + \sum_{i=n}^{\infty} \lambda^i \varepsilon_Q^i + \lambda^n \|w\| \right) \\ &\leq C_{3,Q} n \delta_Q^n (1 + \|w\|) \end{aligned} \tag{51}$$

for all $\theta \in Q$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Similarly, (49), (50) yield

$$\begin{aligned} &|((\Pi^n F_j)(\theta', z) - \partial^j f(\theta')) - ((\Pi^n F_j)(\theta'', z) - \partial^j f(\theta''))| \\ &\leq \|\phi\| \|e(v)\| \sum_{i=0}^{n-1} \lambda^i \|\tilde{R}_{\theta'}^{n-i} S_{\theta',j} R_{\theta'}^i - \tilde{R}_{\theta''}^{n-i} S_{\theta'',j} R_{\theta''}^i\| + \|\phi\| \sum_{i=n}^{\infty} \lambda^i \|\nu_{\theta'}^T S_{\theta',j} \tilde{R}_{\theta'}^i - \nu_{\theta''}^T S_{\theta'',j} \tilde{R}_{\theta''}^i\| \\ &\quad + \lambda^n \|\phi\| \|e(v)\| \|w\| \|R_{\theta'}^n - R_{\theta''}^n\| \\ &\leq \tilde{C}_{2,Q} \|\theta' - \theta''\| \left(\sum_{i=1}^n \lambda^i \varepsilon_Q^{n-i} + \sum_{i=n}^{\infty} \lambda^i \varepsilon_Q^i + \lambda^n \|w\| \right) \\ &\leq C_{3,Q} n \delta_Q^n \|\theta' - \theta''\| (1 + \|w\|) \end{aligned} \tag{52}$$

for all $\theta', \theta'' \in Q$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, $1 \leq j \leq d_\theta$, $n \geq 1$. Using (51), (52), we conclude that there exist real numbers δ_Q , $C_{4,Q}$ with properties specified in (i).

(ii) Let $C_{5,Q} = \tilde{C}_{1,Q}(1 - \lambda)^{-1}$ ($\tilde{C}_{1,Q}$ is specified in the proof of (i)). Then, due to Assumption 4.2, we have

$$\begin{aligned} \|W_{n+1}\| I_{\{\tau_Q > n\}} &= \left\| \lambda^{n+1} W_0 + \sum_{i=0}^n \lambda^{n-i} s_{\theta_i}(X_{i+1}, Y_{i+1}) \right\| I_{\{\tau_Q > n\}} \\ &\leq \lambda^{n+1} \|W_0\| + \tilde{C}_{1,Q} \sum_{i=0}^n \lambda^{n-i} \\ &\leq C_{5,Q} (1 + \|W_0\|) \end{aligned}$$

¹⁸Notice that $\sum_{y \in \mathcal{Y}} \partial_\theta^j q_\theta(y|x) = \partial_\theta^j \left(\sum_{y \in \mathcal{Y}} q_\theta(y|x) \right) = 0$. Notice also that $\nu_\theta(v) = q_\theta(y|x) \mu_\theta(x)$ for $v = (x, y) \in \mathcal{X} \times \mathcal{Y}$, where $\mu_\theta(x)$ is the invariant probability of $\{X_n^\theta\}_{n \geq 0}$ (see the proof of Part (i) of Lemma 8.1).

for $n \geq 0$. □

Proof of Theorem 4.1. For $\theta \in \mathbb{R}^{d_\theta}$, $z = (v, w) \in (\mathcal{X} \times \mathcal{Y}) \times \mathbb{R}^{d_\theta}$, let

$$\tilde{F}(\theta, z) = \sum_{n=0}^{\infty} ((\Pi^n F)(\theta, z) - \nabla f(\theta)), \quad \varphi(z) = 1 + \|w\|.$$

Then, using Lemma 8.3, we conclude that for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^{d_\theta}$, $\tilde{F}(\theta, z)$ is well-defined and satisfies $(\Pi \tilde{F})(\theta, z) = \sum_{n=1}^{\infty} ((\Pi^n F)(\theta, z) - \nabla f(\theta))$. Thus, Assumption 3.2 holds. Relying on Lemma 8.3, we also deduce that for any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists a real number $\tilde{C}_Q \in [1, \infty)$ (possibly depending on λ) such that

$$\begin{aligned} \max\{\|F(\theta, z)\|, \|\tilde{F}(\theta, z)\|, \|(\Pi \tilde{F})(\theta, z)\|\} &\leq \tilde{C}_Q \varphi(z), \\ \|(\Pi \tilde{F})(\theta', z) - (\Pi \tilde{F})(\theta'', z)\| &\leq \tilde{C}_Q \varphi(z) \|\theta' - \theta''\|, \\ E(\varphi^2(Z_{n+1}) I_{\{\tau_Q > n\}} | \theta_0 = \theta, Z_0 = z) &\leq \tilde{C}_Q \varphi^2(z) \end{aligned}$$

for all $\theta, \theta', \theta'' \in Q$, $z \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^{d_\theta}$. Hence, Assumptions 3.3 is satisfied, too. Moreover, Lemma 8.2 yields

$$\eta = \limsup_{n \rightarrow \infty} \|\eta_n\| \leq C_{3,Q}(1 - \lambda)$$

on Λ_Q (notice that $C_{3,Q}$ does not depend on λ). Then, the theorem's assertion directly follows from Theorem 3.1 and Parts (i), (iii), (iv) of Lemma 8.2. □

Appendix 1

In this section, a global version of Theorem 2.1 is presented. This result is based on the following assumptions.

Assumption A1.1. $f(\cdot)$ is uniformly lower bounded (i.e., $\inf_{\theta \in \mathbb{R}^{d_\theta}} f(\theta) > -\infty$), and $\nabla f(\cdot)$ is (globally) Lipschitz continuous. Moreover, there exist real numbers $c \in (0, 1)$, $\rho \in [1, \infty)$ such that $\|\nabla f(\theta)\| \geq c$ for all $\theta \in \mathbb{R}^{d_\theta}$ satisfying $\|\theta\| \geq \rho$.

Assumption A1.2. $\{\xi_n\}_{n \geq 0}$ admits the decomposition $\xi_n = \zeta_n + \eta_n$ for each $n \geq 0$, where $\{\zeta_n\}_{n \geq 0}$ and $\{\eta_n\}_{n \geq 0}$ are \mathbb{R}^{d_θ} -valued stochastic processes satisfying

$$\lim_{n \rightarrow \infty} g(\theta_n) \max_{n \leq j < a(n,t)} \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| = 0, \quad \limsup_{n \rightarrow \infty} g(\theta_n) \|\eta_n\| < \infty \quad (53)$$

almost surely for any $t \in (0, \infty)$. In addition, there exists a real number $\delta \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} h(\theta_n) \|\eta_n\| < \delta \quad (54)$$

almost surely. Here, $g, h : \mathbb{R}^{d_\theta} \rightarrow (0, \infty)$ are the (scaling) functions defined by

$$g(\theta) = (\|\nabla f(\theta)\| + 1)^{-1}, \quad h(\theta) = \begin{cases} \|\nabla f(\theta)\|^{-1}, & \text{if } \|\theta\| \geq \rho \\ 0, & \text{otherwise} \end{cases}$$

for $\theta \in \mathbb{R}^{d_\theta}$ (ρ is specified in Assumption A1.1).

Assumption A1.1 is a stability condition. In this or a similar form, it is involved in practically any stability analysis of stochastic gradient search and stochastic approximation (see e.g., [7], [12], [16] and references cited therein). This assumption is restrictive, as it requires $\nabla^2 f(\cdot)$ to be uniformly bounded. Assumption A1.1 also requires $\nabla f(\cdot)$ to grow at most linearly as $\theta \rightarrow \infty$. Using random projections, these restrictive conditions can considerably be relaxed (see [16], [33]).

Assumption A1.2 is a noise condition and can be considered as a global version of Assumption 2.2. Assumption A1.2 requires the gradient of the objective function $f(\cdot)$ (asymptotically) to cancel the effect of the gradient estimator's error $\{\xi_n\}_{n \geq 0}$. Assumption A1.2 is true whenever (4) holds almost surely. It is also satisfied for stochastic gradient search with Markovian dynamics (see Theorem A2.1, Appendix 2). Assumption A1.2 and the results based on it (Theorem A1.1, below) are motivated by the scaled ODE approach to the stability analysis of stochastic approximation [11].¹⁹

Our results on the stability and asymptotic bias of algorithm (1) are provided in the next theorem.

Theorem A1.1. *Suppose that Assumptions 2.1, A1.1 and A1.2 hold. Then, the following is true:*

- (i) *There exists a compact (deterministic) set $Q \subset \mathbb{R}^{d_\theta}$ such that $P(\Lambda_Q) = 1$ (Λ_Q is specified in (6)).*
- (ii) *There exists a (deterministic) non-decreasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ (independent of η and depending only on $f(\cdot)$) such that $\lim_{t \rightarrow 0} \psi(t) = \psi(0) = 0$ and*

$$\limsup_{n \rightarrow \infty} d(\theta_n, \mathcal{R}) \leq \psi(\eta)$$

almost surely.

- (iii) *If $f(\cdot)$ satisfies Assumption 2.3.b, there exists a real number $K \in (0, \infty)$ (independent of η and depending only on $f(\cdot)$) such that*

$$\limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| \leq K\eta^{q/2}, \quad \limsup_{n \rightarrow \infty} f(\theta_n) - \liminf_{n \rightarrow \infty} f(\theta_n) \leq K\eta^q$$

almost surely (q is specified in the statement of Theorem 2.1).

- (iv) *If $f(\cdot)$ satisfies Assumption 2.3.c, there exist real numbers $r \in (0, 1)$, $L \in (0, \infty)$ (independent of η and depending only on $f(\cdot)$) such that*

$$\limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| \leq L\eta^{1/2}, \quad \limsup_{n \rightarrow \infty} d(f(\theta_n), f(\mathcal{S})) \leq L\eta, \quad \limsup_{n \rightarrow \infty} d(\theta_n, \mathcal{S}) \leq L\eta^r$$

almost surely.

Proof. Owing to Assumption A1.1, there exists a real number $\tilde{C}_1 \in [1, \infty)$ such that the following is true:

- (i) $f(\theta) > -\tilde{C}_1$ for all $\theta \in \mathbb{R}^{d_\theta}$, and (ii) $f(\theta) \leq \tilde{C}_1$ for any $\theta \in \mathbb{R}^{d_\theta}$ satisfying $\|\theta\| \leq \rho + 1$. Moreover, due to Assumption A1.2, there also exists an event $N_0 \in \mathcal{F}$ with the following properties: (i) $P(N_0) = 0$, and (ii) (53), (54) hold on N_0^c for all $t \in (0, \infty)$.

Let $\varepsilon = (1 - \delta)/6$, $T = 2\tilde{C}_1\varepsilon^{-1}c^{-2}$ and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be the function defined by

$$\phi(z) = \sup\{\|\nabla f(\theta)\| : \theta \in \mathbb{R}^{d_\theta}, \|\theta\| \leq z\}$$

for $z \in [0, \infty)$. As $\nabla f(\cdot)$ is locally Lipschitz continuous, $\phi(\cdot)$ is locally Lipschitz continuous, too. $\phi(\cdot)$ is also non-negative and satisfies $\|\nabla f(\theta)\| \leq \phi(\|\theta\|)$ for all $\theta \in \mathbb{R}^{d_\theta}$.

For $z \in [0, \infty)$, let $\lambda(\cdot; z)$ be the solution to the ODE $dz/dt = 2\phi(z)$ satisfying $\lambda(0; z) = z$. As $2\phi(\cdot)$ is non-negative and locally Lipschitz continuous, $\lambda(\cdot; \cdot)$ is well-defined and locally Lipschitz continuous (in both arguments) on $[0, \infty) \times [0, \infty)$. We also have

$$\lambda(t; z) = z + 2 \int_0^t \phi(\lambda(s; z)) ds \tag{55}$$

for all $t, z \in [0, \infty)$. Then, there exists $\rho_1 \in [1, \infty)$ such that $\rho_1 \geq \rho + 1$ and such that $|\lambda(t; z)| \leq \rho_1$ for all $t \in [0, T]$, $z \in [0, \rho + 1]$.

Let $\rho_2 = \rho_1 + 1$, $Q = \{\theta \in \mathbb{R}^{d_\theta} : \|\theta\| \leq \rho_2\}$, while Λ is the event defined by

$$\Lambda = \limsup_{n \rightarrow \infty} \{\|\theta_n\| < \rho\} = \bigcap_{m=0}^{\infty} \bigcup_{n=m}^{\infty} \{\|\theta_n\| < \rho\}.$$

¹⁹The main difference between [11] and the results presented here is the choice of the scaling functions. The scaling adopted in [11] is (asymptotically) proportional to $\|\theta\|$. In this paper, the scaling is (asymptotically) proportional to $\|\nabla f(\theta)\|$.

Let also $\tilde{C}_2 \in [1, \infty)$ stand for a (global) Lipschitz constant of $\nabla f(\cdot)$ and for an upper bound of $\|\nabla f(\cdot)\|$ on Q . Finally, let $\tilde{C}_3 = 2\tilde{C}_2 \exp(2\tilde{C}_2)$, $\tilde{C}_4 = 12\tilde{C}_1\tilde{C}_2\tilde{C}_3$, while $\tau = 4^{-1}\tilde{C}_4^{-1}\varepsilon c^2$.

In order to prove the theorem's assertion, it is sufficient to show $N_0^c \subseteq \Lambda$ (i.e., to establish that on N_0^c , $\|\theta_n\| \leq \rho_2$ for all, but finitely many n).²⁰ To prove this, we use contradiction. We assume that $\|\theta_n\| > \rho_2$ for infinitely many n and some $\omega \in N_0^c$. Notice that all formulas which follow in the proof correspond to ω .

Owing to (53), (54), there exists an integer $k_1 \geq 0$ (depending on ω) such that

$$g(\theta_n) \max_{n \leq j < a(n, T)} \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| \leq \tau^2, \quad h(\theta_n) \|\eta_n\| \leq \delta \quad (56)$$

for $n \geq k_1$. Due to Assumption 2.1 and (53), we also have

$$\lim_{n \rightarrow \infty} g(\theta_n) \|\alpha_n \zeta_n\| = \lim_{n \rightarrow \infty} g(\theta_n) \|\alpha_n \eta_n\| = 0. \quad (57)$$

Since

$$g(\theta_n) \|\theta_{n+1} - \theta_n\| \leq \alpha_n + g(\theta_n) \|\alpha_n \zeta_n\| + g(\theta_n) \|\alpha_n \eta_n\|$$

for $n \geq 0$, Assumption 2.1 and (57) imply $\lim_{n \rightarrow \infty} g(\theta_n) \|\theta_{n+1} - \theta_n\| = 0$. Then, (21) implies that there exists an integer $k_2 \geq 0$ (depending on ω) such that

$$\sum_{i=n}^{a(n, \tau)-1} \alpha_i \geq (1 - \varepsilon)\tau, \quad g(\theta_n) \|\theta_{n+1} - \theta_n\| \leq \tau \quad (58)$$

for $n \geq k_2$.

Let $k_0 = \max\{k_1, k_2\}$. Moreover, let l_0, m_0, n_0 be the integers defined as follows. If $\omega \in \Lambda$ (i.e., if $\|\theta_n\| < \rho$ for infinitely many n), let

$$l_0 = \min\{n > k_0 : \|\theta_{n-1}\| < \rho\}, \quad m_0 = \min\{n > l_0 : \|\theta_n\| > \rho_2\}, \quad n_0 = \max\{n \leq m_0 : \|\theta_{n-1}\| < \rho\}. \quad (59)$$

Otherwise, if $\omega \in \Lambda^c$ (i.e., if $\|\theta_n\| < \rho$ for finitely many n), let

$$l_0 = \max\{n > 0 : \|\theta_{n-1}\| < \rho\}, \quad m_0 = \infty, \quad n_0 = \max\{k_0, l_0\}.$$

Then, we have $k_0 < n_0 \leq m_0$ and $\|\theta_n\| \geq \rho$ for $n_0 \leq n < m_0$.

Let $\phi_n(\tau), \phi_{1,n}(\tau), \phi_{2,n}(\tau)$ have the same meaning as in Section 6. Now, the asymptotic properties of $\phi_n(\tau)$ are analyzed. As $\|\theta_n\| \geq \rho$ for $n_0 \leq n < m_0$, (56) implies

$$\left\| \sum_{i=n}^j \alpha_i \xi_i \right\| \leq \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| + \sum_{i=n}^j \alpha_i \|\eta_i\| \leq \tau^2 g^{-1}(\theta_n) + \delta \sum_{i=n}^j \alpha_i \|\nabla f(\theta_i)\| \quad (60)$$

for $n_0 \leq n \leq j < \min\{m_0, a(n, T)\}$ (notice that $\|\eta_i\| \leq \delta \|\nabla f(\theta_i)\|$ when $\|\theta_i\| \geq \rho$). Therefore,

$$\begin{aligned} \|\nabla f(\theta_j)\| &\leq \|\nabla f(\theta_n)\| + \|\nabla f(\theta_j) - \nabla f(\theta_n)\| \\ &\leq \|\nabla f(\theta_n)\| + \tilde{C}_2 \|\theta_j - \theta_n\| \\ &\leq \|\nabla f(\theta_n)\| + \tilde{C}_2 \sum_{i=n}^{j-1} \alpha_i \|\nabla f(\theta_i)\| + \tilde{C}_2 \left\| \sum_{i=n}^{j-1} \alpha_i \xi_i \right\| \\ &\leq \|\nabla f(\theta_n)\| + \tilde{C}_2 \tau^2 g^{-1}(\theta_n) + 2\tilde{C}_2 \sum_{i=n}^{j-1} \alpha_i \|\nabla f(\theta_i)\| \end{aligned}$$

²⁰Assumption 2.2 is a consequence of Assumption A1.2, and therefore, Parts (ii) – (iv) directly follow from Part (i) and Theorem 2.1.

for $n_0 \leq n < j \leq \min\{m_0 - 1, a(n, \tau)\}$.²¹ Combining this with Bellman-Gronwall inequality (see e.g. [12, Appendix B]), we deduce

$$\begin{aligned}\|\nabla f(\theta_j)\| &\leq \left(\|\nabla f(\theta_n)\| + \tilde{C}_2\tau^2g^{-1}(\theta_n)\right) \exp\left(2\tilde{C}_2\sum_{i=n}^{j-1}\alpha_i\right) \\ &\leq \left(\|\nabla f(\theta_n)\| + \tilde{C}_2\tau^2g^{-1}(\theta_n)\right) (1 + \tilde{C}_3\tau) \\ &\leq \|\nabla f(\theta_n)\| + (\tilde{C}_3\tau + \tilde{C}_2\tau^2 + \tilde{C}_2\tilde{C}_3\tau^3)g^{-1}(\theta_n) \\ &\leq \|\nabla f(\theta_n)\| + \tilde{C}_4\tau g^{-1}(\theta_n)\end{aligned}$$

for $n_0 \leq n \leq j \leq \min\{m_0 - 1, a(n, \tau)\}$.²² Then, (60) implies

$$\left\|\sum_{i=n}^j\alpha_i\xi_i\right\| \leq \tau^2g^{-1}(\theta_n) + \delta\left(\|\nabla f(\theta_n)\| + \tilde{C}_4\tau g^{-1}(\theta_n)\right)\sum_{i=n}^j\alpha_i \leq \delta\tau\|\nabla f(\theta_n)\| + 2\tilde{C}_4\tau^2g^{-1}(\theta_n) \quad (61)$$

for $n_0 \leq n \leq j < \min\{m_0, a(n, \tau)\}$. Consequently,

$$\begin{aligned}\|\theta_j - \theta_n\| &\leq \sum_{i=n}^{j-1}\alpha_i\|\nabla f(\theta_i)\| + \left\|\sum_{i=n}^{j-1}\alpha_i\xi_i\right\| \\ &\leq \left(\|\nabla f(\theta_n)\| + \tilde{C}_4\tau g^{-1}(\theta_n)\right)\left(\sum_{i=n}^{j-1}\alpha_i + \delta\tau\right) + 2\tilde{C}_4\tau^2g^{-1}(\theta_n) \\ &\leq 3\tau g^{-1}(\theta_n)\end{aligned} \quad (62)$$

for $n_0 \leq n \leq j \leq \min\{m_0 - 1, a(n, \tau)\}$ (notice that $\delta < 1$, $\tilde{C}_4\tau \leq 1/4$). Therefore,

$$|\phi_{1,n}(\tau)| \leq \tilde{C}_2\|\nabla f(\theta_n)\|\sum_{i=n}^{a(n,\tau)-1}\alpha_i\|\theta_i - \theta_n\| \leq 3\tilde{C}_2\tau g^{-1}(\theta_n)\|\nabla f(\theta_n)\|\sum_{i=n}^{a(n,\tau)-1}\alpha_i \leq 3\tilde{C}_2\tau^2g^{-2}(\theta_n)$$

for $n \geq n_0$ satisfying $a(n, \tau) < m_0$. We also have

$$|\phi_{2,n}(\tau)| \leq \tilde{C}_2\|\theta_{a(n,\tau)} - \theta_n\|^2 \leq 9\tilde{C}_2\tau^2g^{-2}(\theta_n)$$

for $n \geq n_0$ satisfying $a(n, \tau) < m_0$. Thus,

$$|\phi_n(\tau)| \leq \tilde{C}_4\tau^2g^{-2}(\theta_n) \quad (63)$$

when $n \geq n_0$, $a(n, \tau) < m_0$. Additionally, as a result of (58), (61), we get

$$\begin{aligned}\|\nabla f(\theta_n)\|\sum_{i=n}^{a(n,\tau)-1}\alpha_i - \left\|\sum_{i=n}^{a(n,\tau)-1}\alpha_i\xi_i\right\| &\geq (1 - \delta - \varepsilon)\tau\|\nabla f(\theta_n)\| - 2\tilde{C}_4\tau^2g^{-1}(\theta_n) \\ &= 5\varepsilon\tau\|\nabla f(\theta_n)\| - 2\tilde{C}_4\tau^2g^{-1}(\theta_n) \\ &\geq 3\varepsilon\tau\|\nabla f(\theta_n)\|\end{aligned}$$

when $n \geq n_0$, $a(n, \tau) < m_0$.²³ Then, (20), (63) imply

$$f(\theta_{a(n,\tau)}) - f(\theta_n) \leq -3\varepsilon\tau\|\nabla f(\theta_n)\|^2 + \tilde{C}_4\tau^2g^{-2}(\theta_n) \leq -\varepsilon\tau\|\nabla f(\theta_n)\|^2 \leq -\varepsilon\tau c^2 \quad (64)$$

for $n \geq n_0$ satisfying $a(n, \tau) < m_0$.²⁴

²¹Notice that τ, T are defined as $\tau = 4^{-1}\tilde{C}_4^{-1}\varepsilon c^2$, $T = 2\tilde{C}_1\varepsilon^{-1}c^{-2}$. Notice also $\tau < 1 < T$ since $\tilde{C}_1, \tilde{C}_4 \in [1, \infty)$, $\varepsilon, c \in (0, 1)$.

²²Notice that $\sum_{i=n}^{j-1}\alpha_i \leq \tau < 1$ when $n \leq j \leq a(n, \tau)$. Notice also $g^{-1}(\theta_n) > \|\nabla f(\theta_n)\|$ and $\exp(2\tilde{C}_2\tau) \leq 2\tilde{C}_2\tau \exp(2\tilde{C}_2) = \tilde{C}_3\tau$.

²³Notice that $1 - \delta = 6\varepsilon$, $\varepsilon \geq \varepsilon c \geq 2\tilde{C}_4\tau$. Notice also that $2\varepsilon\tau\|\nabla f(\theta_n)\| \geq \varepsilon\tau\|\nabla f(\theta_n)\| + \varepsilon\tau c \geq 2\tilde{C}_4\tau^2g^{-1}(\theta_n)$ for $n_0 \leq n < m_0$.

²⁴Notice that $2\varepsilon\|\nabla f(\theta_n)\|^2 \geq \varepsilon\|\nabla f(\theta_n)\|^2 + \varepsilon c^2 \geq \tilde{C}_4\tau g^{-2}(\theta_n)$ for $n_0 \leq n < m_0$.

Let $\{n_k\}_{k \geq 0}$ be the sequence recursively defined by $n_{k+1} = a(n_k, \tau)$ for $k \geq 0$. Now, we show by contradiction $\omega \in \Lambda$ (i.e., $\|\theta_n\| < \rho$ for infinitely many n). We assume the opposite. Then, $m_0 = \infty$ and $\|\theta_n\| \geq \rho$ for $n \geq n_0$, while (64) implies $f(\theta_{n_{k+1}}) - f(\theta_{n_k}) \leq -\varepsilon\tau c^2$ for $k \geq 0$. Hence, $\lim_{k \rightarrow \infty} f(\theta_{n_k}) = -\infty$. However, this is impossible due to Assumption A1.1. Thus, $\omega \in \Lambda$ (i.e., $\|\theta_n\| < \rho$ for infinitely many n). Therefore, m_0, n_0 are defined through (59), and thus, $\|\theta_{n_0-1}\| < \rho$, $\|\theta_{m_0}\| > \rho_2$. Combining this with (58), we conclude

$$\|\theta_{n_0} - \theta_{n_0-1}\| \leq \tau g^{-1}(\theta_{n_0-1}) \leq \tau(\tilde{C}_2 + 1) \leq 1/2$$

(notice that $\|\nabla f(\theta_{n_0-1})\| \leq \tilde{C}_2$, $\tilde{C}_2\tau \leq 1/4$). Consequently,

$$\|\theta_{n_0}\| \leq \|\theta_{n_0-1}\| + \|\theta_{n_0} - \theta_{n_0-1}\| \leq \rho + 1/2 < \rho_2. \quad (65)$$

Hence, $n_0 < m_0$, $f(\theta_{n_0}) \leq \tilde{C}_1$.

Let i_0, j_0 be the integers defined by $j_0 = \max\{j \geq 0 : n_j < m_0\}$, $i_0 = n_{j_0}$. Then, we have $n_0 \leq i_0 = n_{j_0} < n_{j_0+1} = m_0 \leq a(i_0, \tau)$. As a result of this and (62), we get

$$\|\theta_{m_0} - \theta_{i_0}\| \leq 3\tau g^{-1}(\theta_{i_0}) \leq 3\tau(\tilde{C}_2 + 1) \leq 1/2$$

(notice that $\|\nabla f(\theta_{i_0})\| \leq \tilde{C}_2$, $\tilde{C}_2\tau \leq 1/12$). Consequently,

$$\|\theta_{i_0}\| \geq \|\theta_{m_0}\| - \|\theta_{m_0} - \theta_{i_0}\| \geq \rho_2 - 1/2 > \rho_1. \quad (66)$$

Let $\{\gamma_n\}_{n \geq 0}$, $\theta_0(\cdot)$ have the same meaning as in Section 5. Now, we show by contradiction that $\gamma_{i_0} - \gamma_{n_0} \geq T$. We assume the opposite. Then, (60), (65) imply

$$\begin{aligned} \|\theta_0(t)\| &= \|\theta_j\| \leq \|\theta_{n_0}\| + \sum_{i=n_0}^{j-1} \alpha_i \|\nabla f(\theta_i)\| + \left\| \sum_{i=n_0}^{j-1} \alpha_i \xi_i \right\| \\ &\leq \|\theta_{n_0}\| + \tau^2 g^{-1}(\theta_{n_0}) + 2 \sum_{i=n_0}^{j-1} \alpha_i \|\nabla f(\theta_i)\| \\ &\leq \rho + 1 + 2 \sum_{i=n_0}^{j-1} \alpha_i \phi(\|\theta_i\|) \\ &\leq \rho + 1 + 2 \int_{\gamma_{n_0}}^t \phi(\|\theta_0(s)\|) ds \end{aligned} \quad (67)$$

for $t \in [\gamma_j, \gamma_{j+1})$, $n_0 \leq j \leq i_0$.²⁵ Applying the comparison principle (see [19, Section 3.4]) to (55), (67), we conclude $\|\theta_0(t)\| \leq \lambda(t - \gamma_{n_0}; \rho + 1) \leq \rho_1$ for all $t \in [\gamma_{n_0}, \gamma_{i_0}]$. Thus, $\|\theta_{i_0}\| = \|\theta_0(\gamma_{i_0})\| \leq \rho_1$. However, this is impossible, due to (66). Hence, $\gamma_{i_0} - \gamma_{n_0} \geq T$. Consequently,

$$T \leq \gamma_{i_0} - \gamma_{n_0} = \sum_{j=0}^{j_0-1} (\gamma_{n_{j+1}} - \gamma_{n_j}) \leq j_0\tau \quad (68)$$

(notice that $n_{j_0} = i_0$, $\gamma_{n_{j+1}} - \gamma_{n_j} = \sum_{i=n_j}^{n_{j+1}-1} \alpha_i \leq \tau$).

Owing to (64), we have $f(\theta_{n_{j+1}}) - f(\theta_{n_j}) \leq -\varepsilon\tau c^2$ for $0 \leq j \leq j_0$. Combining this with (68), we get

$$f(\theta_{i_0}) = f(\theta_{n_{j_0}}) \leq f(\theta_{n_0}) - j_0\varepsilon\tau c^2 \leq \tilde{C}_1 - \varepsilon c^2 T \leq -\tilde{C}_1.$$

However, this is impossible, since $f(\theta) > -\tilde{C}_1$ for all $\theta \in \mathbb{R}^{d_\theta}$. Hence, $\|\theta_n\| > \rho_2$ for finitely many n . \square

²⁵As $j \leq i_0 < m_0$, we have $\gamma_j - \gamma_{n_0} \leq \gamma_{i_0} - \gamma_{n_0} \leq T$ and $j \leq \min\{m_0 - 1, a(n_0, T)\}$. We also have $\tau^2 g^{-1}(\theta_{n_0}) \leq \tau^2(\tilde{C}_2 + 1) \leq 1/2$.

Appendix 2

In this section, a global version of Theorem 3.1 is presented. This result is based the following assumptions.

Assumption A2.1. *There exists a Borel-measurable function $\varphi : \mathbb{R}^{d_z} \rightarrow [1, \infty)$ such that*

$$\begin{aligned} \max\{\|F(\theta, z)\|, \|\tilde{F}(\theta, z)\|, \|(\Pi\tilde{F})(\theta, z)\|\} &\leq \varphi(z)(\|\nabla f(\theta)\| + 1), \\ \|(\Pi\tilde{F})(\theta', z) - (\Pi\tilde{F})(\theta'', z)\| &\leq \varphi(z)\|\theta' - \theta''\| \end{aligned}$$

for all $\theta, \theta', \theta'' \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. In addition, one has

$$\sup_{n \geq 0} E(\varphi^2(Z_n) | \theta_0 = \theta, Z_0 = z) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$.

Assumption A2.2. $\eta_n = \eta(\theta_n)$ for $n \geq 0$, where $\eta : \mathbb{R}^{d_\theta} \rightarrow \mathbb{R}^{d_\theta}$ is a continuous function. Moreover, there exists a real number $\delta \in (0, 1)$ such that $\|\eta(\theta)\| \leq \delta\|\nabla f(\theta)\|$ for all $\theta \in \mathbb{R}^{d_\theta}$ satisfying $\|\theta\| \geq \rho$ (ρ is specified in Assumption A1.1).

Assumption A2.1 is a global version of Assumption 3.3. In a similar form, it is involved in the stability analysis of stochastic approximation carried out in [7, Section II.1.9]. Assumption A2.2 is related to the bias of the gradient estimator. It requires the bias $\{\eta_n\}_{n \geq 0}$ to be a deterministic function of the algorithm iterates $\{\theta_n\}_{n \geq 0}$. As demonstrated in Section 4 and [33], this is often satisfied in practice. Assumption A2.2 can be considered as one of the weakest conditions under which the stability of the perturbed ODE $d\theta/dt = -(\nabla f(\theta) + \eta(\theta))$ can be shown.

Our results on the stability and asymptotic bias of algorithm (13) are provided in the next theorem.

Theorem A2.1. *Suppose that Assumptions 3.1, 3.2, A1.1, A2.1 and A2.2 hold. Let $f(\cdot)$ be the function specified in Assumption 3.2. Then, the following is true:*

- (i) *If $f(\cdot)$ satisfies Assumption 2.3.a, Part (i) of Theorem A1.1 holds.*
- (ii) *If $f(\cdot)$ satisfies Assumption 2.3.b, Part (ii) of Theorem A1.1 holds.*
- (iii) *If $f(\cdot)$ satisfies Assumption 2.3.c, Part (iii) of Theorem A1.1 holds.*

Proof. Let $g(\cdot)$, $h(\cdot)$ be the functions defined in Assumption A1.2. Then, due to Assumption A2.2, $g(\theta)\eta(\theta)$ is uniformly bounded in $\theta \in \mathbb{R}^{d_\theta}$, while $h(\theta)\eta(\theta) \leq \delta$ for all $\theta \in \mathbb{R}^{d_\theta}$ satisfying $\|\theta\| \geq \rho$. Let $C \in [1, \infty)$ stand for a (global) Lipschitz constant of $\nabla f(\cdot)$ and for an (global) upper bound of $g(\cdot)\eta(\cdot)$. Define $\tau = 1/(18C^2)$ and let $\{\zeta_n\}_{n \geq 0}$, $\{\zeta_{1,n}\}_{n \geq 0}$, $\{\zeta_{2,n}\}_{n \geq 0}$, $\{\zeta_{3,n}\}_{n \geq 0}$ have the same meaning as in the proof of Theorem 3.1, while τ_n is the stopping time defined by

$$\tau_n = \min(\{j \geq n : g(\theta_n)g^{-1}(\theta_j) > 3\} \cup \{\infty\})$$

for $n \geq 0$. Finally, for $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, let $E_{\theta,z}(\cdot)$ denote the conditional mean given $\theta_0 = \theta$, $Z_0 = z$.

As a direct consequence of Assumptions 3.1, A2.1, we get

$$E_{\theta,z} \left(\sum_{n=0}^{\infty} \alpha_n^2 \varphi^2(Z_{n+1}) \right) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. We also have

$$g(\theta_n)\|\zeta_n\| \leq \varphi(Z_{n+1}) + 1 \leq 2\varphi(Z_{n+1})$$

for $n \geq 0$. Consequently,

$$\lim_{n \rightarrow \infty} \alpha_n \varphi(Z_{n+1}) = \lim_{n \rightarrow \infty} \alpha_n g(\theta_n) \|\zeta_n\| = 0 \tag{69}$$

almost surely.

Let $\{m_k\}_{k \geq 0}$ be the sequence recursively defined by $m_0 = 0$ and $m_{k+1} = a(m_k, \tau)$ for $k \geq 0$. Moreover, let $\mathcal{F}_n = \sigma\{\theta_0, Z_0, \dots, \theta_n, Z_n\}$ for $n \geq 0$. Due to Assumption 3.2, we have

$$E_{\theta, z}(g(\theta_n)\zeta_{1,j}I_{\{\tau_n > j\}}|\mathcal{F}_j) = g(\theta_n)\left(E_{\theta, z}(\tilde{F}(\theta_j, Z_{j+1})|\mathcal{F}_j) - (\Pi\tilde{F})(\theta_j, Z_j)\right)I_{\{\tau_n > j\}} = 0$$

almost surely for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $0 \leq n \leq j$ (notice that $\{\tau_n > j\}$ is measurable with respect to \mathcal{F}_j). Moreover, Assumption A2.1 implies

$$g(\theta_n)\|\zeta_{1,j}\|I_{\{\tau_n > j\}} \leq g(\theta_n)g^{-1}(\theta_j)(\varphi(Z_j) + \varphi(Z_{j+1}))I_{\{\tau_n > j\}} \leq 3(\varphi(Z_j) + \varphi(Z_{j+1}))$$

for $0 \leq n \leq j$. Then, as a result of Doob inequality, we get

$$\begin{aligned} E_{\theta, z}\left(\max_{n < j < a(n, \tau)} \left\| \sum_{i=n+1}^j \alpha_i g(\theta_n) \zeta_{1,i} \right\|^2 I_{\{\tau_n > j\}}\right) &\leq E_{\theta, z}\left(\max_{n < j < a(n, \tau)} \left\| \sum_{i=n+1}^j \alpha_i g(\theta_n) \zeta_{1,i} I_{\{\tau_n > i\}} \right\|^2\right) \\ &\leq 4E_{\theta, z}\left(\sum_{i=n+1}^{a(n, \tau)-1} \alpha_i^2 g^2(\theta_n) \|\zeta_{1,i}\|^2 I_{\{\tau_n > i\}}\right) \\ &\leq 72E_{\theta, z}\left(\sum_{i=n+1}^{a(n, \tau)} \alpha_i^2 (\varphi^2(Z_i) + \varphi^2(Z_{i+1}))\right) \end{aligned}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $n \geq 0$. Combining this with Assumptions 3.1, A2.1, we deduce

$$E_{\theta, z}\left(\sum_{k=0}^{\infty} g^2(\theta_{m_k}) \max_{m_k < j < m_{k+1}} \left\| \sum_{i=m_k}^j \alpha_i \zeta_{1,i} \right\|^2 I_{\{\tau_{m_k} > j\}}\right) \leq 72E_{\theta, z}\left(\sum_{n=0}^{\infty} (\alpha_n^2 + \alpha_{n+1}^2) \varphi^2(Z_{n+1})\right) < \infty$$

for each $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$, $n \geq 0$. Therefore,

$$\lim_{k \rightarrow \infty} g(\theta_{m_k}) \max_{m_k < j < m_{k+1}} \left\| \sum_{i=m_k}^j \alpha_i \zeta_{1,i} \right\| I_{\{\tau_{m_k} > j\}} = 0 \quad (70)$$

almost surely.

Since $\alpha_n \alpha_{n+1} = O(\alpha_n^2)$, $\alpha_n - \alpha_{n+1} = O(\alpha_n^2)$ for $n \rightarrow \infty$ (see the proof of Theorem 3.1), Assumptions 3.1, A2.1 yield

$$E_{\theta, z}\left(\sum_{n=0}^{\infty} \alpha_n \alpha_{n+1} \varphi^2(Z_{n+1})\right) < \infty, \quad E_{\theta, z}\left(\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| \varphi^2(Z_{n+1})\right) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $z \in \mathbb{R}^{d_z}$. Additionally, due to Assumptions A2.1, A2.2, we have

$$\begin{aligned} g(\theta_n)\|\zeta_{2,j}\|I_{\{\tau_n > j\}} &\leq g(\theta_n)\varphi(Z_j)\|\theta_j - \theta_{j-1}\|I_{\{\tau_n > j-1\}} \\ &\leq \alpha_{j-1}g(\theta_n)\varphi(Z_j)(\|F(\theta_{j-1}, Z_j)\| + \|\eta_{j-1}\|)I_{\{\tau_n > j\}} \\ &\leq \alpha_{j-1}g(\theta_n)g^{-1}(\theta_{j-1})\varphi(Z_j)(\varphi(Z_j) + C)I_{\{\tau_n > j\}} \\ &\leq 6C\alpha_{j-1}\varphi^2(Z_j) \end{aligned}$$

for $0 \leq n < j$ (notice that $\varphi(z) \geq 1$ for any $z \in \mathbb{R}^{d_z}$). We also have

$$g(\theta_n)\|\zeta_{3,j}\|I_{\{\tau_n > j\}} \leq g(\theta_n)g^{-1}(\theta_j)\varphi(Z_{j+1})I_{\{\tau_n > j\}} \leq 3\varphi(Z_{j+1}) \leq 3\varphi^2(Z_{j+1})$$

for $0 \leq n \leq j$. Hence,

$$\begin{aligned} g(\theta_n)\left\| \sum_{i=n+1}^j \alpha_i \zeta_{2,i} \right\| I_{\{\tau_n > j\}} &\leq \sum_{i=n+1}^j \alpha_i g(\theta_n) \|\zeta_{2,i}\| I_{\{\tau_n > i\}} \leq 6C \sum_{i=n}^j \alpha_i \alpha_{i+1} \varphi^2(Z_{i+1}), \\ g(\theta_n)\left\| \sum_{i=n+1}^j (\alpha_i - \alpha_{i+1}) \zeta_{3,i} \right\| I_{\{\tau_n > j\}} &\leq \sum_{i=n+1}^j |\alpha_i - \alpha_{i+1}| g(\theta_n) \|\zeta_{3,i}\| I_{\{\tau_n > i\}} \leq 3 \sum_{i=n+1}^j |\alpha_i - \alpha_{i+1}| \varphi^2(Z_{i+1}) \end{aligned}$$

for $0 \leq n < j$. Consequently,

$$\lim_{n \rightarrow \infty} g(\theta_n) \max_{j > n} \left\| \sum_{i=n+1}^j \alpha_i \zeta_{2,i} \right\| I_{\{\tau_n > j\}} = \lim_{n \rightarrow \infty} g(\theta_n) \max_{j > n} \left\| \sum_{i=n+1}^j (\alpha_i - \alpha_{i+1}) \zeta_{3,i} \right\| I_{\{\tau_n > j\}} = 0 \quad (71)$$

almost surely (notice that $\alpha_{j+1}/\alpha_j = O(1)$ for $j \rightarrow \infty$). Moreover, (69) yields

$$\lim_{n \rightarrow \infty} g(\theta_n) \max_{j \geq n} \alpha_{j+1} \|\zeta_{3,j}\| I_{\{\tau_n > j\}} = 0 \quad (72)$$

almost surely. Combining (69) – (72) with (37), we deduce

$$\lim_{k \rightarrow \infty} g(\theta_{n_k}) \max_{m_k \leq j < m_{k+1}} \left\| \sum_{i=m_k}^j \alpha_i \zeta_i \right\| I_{\{\tau_{m_k} > j\}} = 0 \quad (73)$$

almost surely.

Owing to Assumptions A1.1, A2.2, we have

$$\begin{aligned} g^{-1}(\theta_{j+1}) I_{\{\tau_n > j\}} &\leq g^{-1}(\theta_n) + \|\nabla f(\theta_{j+1}) - \nabla f(\theta_n)\| I_{\{\tau_n > j\}} \\ &\leq g^{-1}(\theta_n) + C \|\theta_{j+1} - \theta_n\| I_{\{\tau_n > j\}} \\ &\leq g^{-1}(\theta_n) + C \sum_{i=n}^j \alpha_i \|\nabla f(\theta_i)\| I_{\{\tau_n > j\}} + C \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| I_{\{\tau_n > j\}} + C \sum_{i=n}^j \alpha_i \|\eta_i\| I_{\{\tau_n > j\}} \\ &\leq g^{-1}(\theta_n) + C \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| I_{\{\tau_n > j\}} + 2C^2 \sum_{i=n}^j \alpha_i g^{-1}(\theta_i) I_{\{\tau_n > j\}} \end{aligned}$$

for $0 \leq n \leq j$ (notice that $\|\eta(\theta)\| \leq Cg^{-1}(\theta)$ for each $\theta \in \mathbb{R}^{d_\theta}$). Combining this with Bellman-Gronwall inequality (see e.g., [12, Appendix B]), we conclude

$$\begin{aligned} g^{-1}(\theta_{j+1}) I_{\{\tau_n > j\}} &\leq \left(g^{-1}(\theta_n) + C \max_{n \leq j < a(n, \tau)} \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| I_{\{\tau_n > j\}} \right) \exp \left(2C^2 \sum_{i=n}^{j-1} \alpha_i \right) \\ &\leq 2g^{-1}(\theta_n) \left(1 + Cg(\theta_n) \max_{n \leq j < a(n, \tau)} \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| I_{\{\tau_n > j\}} \right) \end{aligned}$$

for $0 \leq n \leq j \leq a(n, \tau)$.²⁶ Then, (73) yields

$$\limsup_{k \rightarrow \infty} g(\theta_{m_k}) \max_{m_k \leq j < m_{k+1}} g^{-1}(\theta_{j+1}) I_{\{\tau_{m_k} > j\}} \leq 2 \quad (74)$$

almost surely.

Let N_0 be the event where (73) or (74) does not hold. Then, in order to prove the theorem's assertion, it is sufficient to show that (53), (54) are satisfied on N_0^c for any $t \in (0, \infty)$. Let ω be any sample in N_0^c , while $t \in (0, \infty)$ is any real number. Notice that all formula which follow in the proof correspond to ω .

Due to Assumption A2.2, we have

$$\limsup_{n \rightarrow \infty} g(\theta_n) \|\eta_n\| \leq C < \infty, \quad \limsup_{n \rightarrow \infty} h(\theta_n) \|\eta_n\| \leq \delta < 1.$$

Moreover, Assumption 3.1 and (21), (74) imply that there exists an integer $k_0 \geq 0$ (depending on ω) such that

$$\sum_{i=m_k}^{m_{k+1}-1} \alpha_i \geq \tau/2, \quad g(\theta_{m_k}) \left\| \sum_{i=m_k}^j \alpha_i \zeta_i \right\| I_{\{\tau_{m_k} > j\}} \leq \tau, \quad g(\theta_{m_k}) g^{-1}(\theta_{j+1}) I_{\{\tau_{m_k} > j\}} \leq 3 \quad (75)$$

²⁶Notice that $\sum_{i=n}^{j-1} \alpha_i \leq \tau$ for $n \leq j \leq a(n, \tau)$. Notice also that $\exp(2C^2\tau) \leq \exp(1/2) \leq 2$.

for $k \geq k_0$, $m_k \leq j < m_{k+1}$. As $\tau_n > n$ for $n \geq 0$, we conclude $\tau_{m_k} > m_{k+1}$ for $k \geq k_0$.²⁷ Consequently, $I_{\{\tau_{m_k} > j\}} = 1$ for $k \geq k_0$, $m_k \leq j \leq m_{k+1}$. Combining this with (75), we get $g(\theta_{m_k}) \leq 3g(\theta_{j+1})$ and

$$\begin{aligned}
g^{-1}(\theta_{j+1}) &\geq g^{-1}(\theta_{m_k}) - \|\nabla f(\theta_{j+1}) - \nabla f(\theta_n)\| \\
&\geq g^{-1}(\theta_{m_k}) - C\|\theta_{j+1} - \theta_n\| \\
&\geq g^{-1}(\theta_{m_k}) - C \sum_{i=m_k}^j \alpha_i \|\nabla f(\theta_i)\| - C \left\| \sum_{i=m_k}^j \alpha_i \zeta_i \right\| - C \sum_{i=m_k}^j \alpha_i \|\eta_i\| \\
&\geq g^{-1}(\theta_{m_k}) - 2C^2 \sum_{i=m_k}^j \alpha_i g^{-1}(\theta_i) - C \left\| \sum_{i=m_k}^j \alpha_i \zeta_i \right\| \\
&\geq g^{-1}(\theta_{m_k})(1 - 6C^2\tau - C\tau) \\
&\geq 3^{-1}g^{-1}(\theta_{m_k})
\end{aligned} \tag{76}$$

for $k \geq k_0$, $m_k \leq j < m_{k+1}$.²⁸ Hence, $3^{-1}g(\theta_{m_k}) \leq g(\theta_j) \leq 3g(\theta_{m_k})$ for $k \geq k_0$, $m_k \leq j \leq m_{k+1}$.

Let $n_0 = m_{k_0}$, while $k(n) = \max\{k \geq 0 : m_k \leq n\}$, $m(n) = m_{k(n)}$ for $n \geq 0$. Then, (76) implies $g(\theta_n) \leq 3g(\theta_{m(n)})$, $g(\theta_{m_k}) \leq 3g(\theta_{m_{k+1}})$ for $n \geq n_0$, $k \geq k_0$ (notice that $k(n) \geq k_0$, $m_{k(n)} \leq n < m_{k(n)+1}$ when $n \geq n_0$). Hence, $g(\theta_n) \leq C_{n,k} g(\theta_{m_k})$ for $n \geq n_0$, $k \geq m(n)$, where $C_{n,k} = 3^{k-k(n)+1}$.²⁹ Since

$$2^{-1}(k(j) - k(n))\tau \leq \sum_{k=k(n)+1}^{k(j)} \sum_{i=m_k}^{m_{k+1}-1} \alpha_i \leq \sum_{i=n}^j \alpha_i \leq t$$

for $n_0 \leq n \leq j \leq a(n, \tau)$, we conclude $k(j) - k(n) \leq 2t/\tau$ for the same n, j . Consequently,

$$\begin{aligned}
g(\theta_n) \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| &= g(\theta_n) \left\| \sum_{k=k(n)}^{k(j)} \sum_{i=m_k}^{m_{k+1}-1} \alpha_i \zeta_i - \sum_{i=m(n)}^{n-1} \alpha_i \zeta_i + \sum_{i=m(j)}^j \alpha_i \zeta_i \right\| \\
&\leq \sum_{k=k(n)}^{k(j)-1} C_{n,k} g(\theta_{m_k}) \left\| \sum_{i=m_k}^{m_{k+1}-1} \alpha_i \zeta_i \right\| + C_{n,k(n)} g(\theta_{m(n)}) \left\| \sum_{i=m(n)}^{n-1} \alpha_i \zeta_i \right\| \\
&\quad + C_{n,k(j)} g(\theta_{m(j)}) \left\| \sum_{i=m(j)}^j \alpha_i \zeta_i \right\| \\
&\leq C(t) \max_{\substack{m_k \leq l < m_{k+1} \\ k(n) \leq k}} g(\theta_{m_k}) \left\| \sum_{i=m_k}^l \alpha_i \zeta_i \right\|
\end{aligned}$$

for $n_0 \leq n \leq j \leq a(n, t)$,³⁰ where $C(t) = (2t/\tau + 3)3^{2t/\tau+3}$. Since $\tau_{m_k} > m_{k+1}$ for $k \geq k_0$ (i.e., $I_{\{\tau_{m_k} > j\}} = 1$ for $k \geq k_0$, $m_k \leq j \leq m_{k+1}$), (73) implies

$$\lim_{n \rightarrow \infty} g(\theta_n) \max_{n \leq j < a(n, t)} \left\| \sum_{i=n}^j \alpha_i \zeta_i \right\| = 0$$

(notice that $\lim_{n \rightarrow \infty} k(n) = \infty$). Hence, (53), (54) hold. \square

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²⁷If $\tau_{m_k} \leq m_{k+1}$, then $\tau_{m_k} = j$ and $g(\theta_{m_k})g^{-1}(\theta_j)I_{\{\tau_{m_k} > j-1\}} = g(\theta_{m_k})g^{-1}(\theta_j) > 3$ for some j satisfying $m_k < j \leq m_{k+1}$.

²⁸Notice that $g^{-1}(\theta_i) \leq 3g^{-1}(\theta_{m_k})$, $\sum_{i=m_k}^{m_{k+1}-1} \alpha_i \leq \tau$ when $k \geq k_0$, $m_k \leq i < m_{k+1}$. Notice also that $6C^2\tau = 1/3$, $C\tau \leq 1/3$.

²⁹Notice that $g(\theta_n)g^{-1}(\theta_{m(n)}) \leq 3$, $g(\theta_{m(n)})g^{-1}(\theta_{m_k}) \leq 3^{k-k(n)}$ when $n \geq n_0$, $k \geq m(n)$. Notice also $g(\theta_n) = (g(\theta_n)g^{-1}(\theta_{m(n)}))(g(\theta_{m(n)})g^{-1}(\theta_{m_k}))g(\theta_{m_k})$.

³⁰Here, the following convention is used: If the lower limit of a sum is (strictly) greater than the upper limit, then the sum is zero.

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