

# Infinite-time concentration in aggregation–diffusion equations with a given potential



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## ARTICLE INFO

### Article history:

Received 23 March 2021

Available online 15 November 2021

### MSC:

35K55

35K65

35B40

35D40

35Q84

### Keywords:

Nonlinear parabolic equations

Nonlinear diffusion

Aggregation

Dirac delta formation

Blow-up in infinite time

Viscosity solutions

## ABSTRACT

Typically, aggregation–diffusion is modeled by parabolic equations that combine linear or nonlinear diffusion with a Fokker–Planck convection term. Under very general suitable assumptions, we prove that radial solutions of the evolution process converge asymptotically in time towards a stationary state representing the balance between the two effects. Our parabolic system is the gradient flow of an energy functional, and in fact we show that the stationary states are minimizers of a relaxed energy. Here, we study radial solutions of an aggregation–diffusion model that combines nonlinear fast diffusion with a convection term driven by the gradient of a potential, both in balls and the whole space. We show that, depending on the exponent of fast diffusion and the potential, the steady state is given by the sum of an explicit integrable function, plus a Dirac delta at the origin containing the rest of the mass of the initial datum. This splitting phenomenon is an uncommon example of blow-up in infinite time.

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## RÉSUMÉ

Les phénomènes d'agrégation–diffusion sont souvent modélisés par des équations paraboliques qui combinent une diffusion linéaire ou non-linéaire avec un terme de convection de type Fokker–Planck. Sous des hypothèses appropriées et très générales, nous prouvons que les solutions du problème d'évolution convergent asymptotiquement en temps vers un état stationnaire représentant l'équilibre entre les deux effets. Ce système parabolique peut s'écrire comme le flot de gradient associé à une fonctionnelle d'énergie et nous montrons que ses états stationnaires sont les minimiseurs de l'énergie relaxée. Dans cet article, nous étudions les solutions radiales d'un modèle d'agrégation–diffusion qui combine une diffusion non linéaire de type rapide avec un terme de convection entraîné par le gradient d'un potentiel, dans les boules d'abord, puis dans l'espace entier. Nous montrons qu'en fonction de l'exposant de diffusion rapide et du potentiel, l'état stationnaire est donné par la somme d'une fonction intégrable explicite plus une mesure de Dirac à l'origine

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contenant le reste de la masse de la donnée initiale. Ce phénomène de décomposition est un exemple rare d’explosion en temps infini.

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### 1. Introduction

Enormous work has been devoted over the last years to the study of mathematical models for Aggregation-Diffusion that are formulated in terms of semilinear parabolic equations combining linear or non-linear diffusion with a Fokker-Planck convection term coming either from a given potential or from an interaction potential, see [3,24,36,20,21,27,15] and the references therein, and the books [2,46]. In this paper we consider the aggregation-diffusion equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \nabla \cdot (\rho \nabla V), \quad \text{in } (0, \infty) \times \mathbb{R}^n \tag{P}$$

where the potential  $V(x)$  is given and  $0 < m < 1$ , the fast-diffusion range [44]. We take as initial data a probability measure, i. e.,

$$\rho_0 \geq 0, \quad \int_{\mathbb{R}^n} \rho_0 dx = 1. \tag{1.1}$$

We will find conditions on the radial initial data  $\rho_0$  and the radial potential  $V$  so that

- i) we provide a suitable notion of solution of the Cauchy problem defined globally in time passing through the mass (or distribution) function,
- ii) as  $t \rightarrow \infty$ , the solution undergoes one-point blow-up of the split form

$$\rho(t) \rightarrow \mu_\infty = \rho_\infty + (1 - \|\rho_\infty\|_{L^1(\mathbb{R}^n)})\delta_0,$$

where  $\rho_\infty(x) > 0$  is an explicit stationary solution of (P). The presence of the concentrated point measure is a striking fact that needs detailed understanding and is the main motivation of this work. Here and after we identify an  $L^1$  function with the absolutely continuous measure it generates.

It is known that Dirac measures are invariant by the semigroup generated by the fast-diffusion equation  $u_t = \Delta u^m$  for  $0 < m < \frac{n-2}{n}$  (see [10]), but they are never produced from  $L^1$  initial data. Here we show that the aggregation caused by the potential term might be strong enough to overcome the fast-diffusion term and produce a Dirac-delta concentration at 0 as  $t \rightarrow \infty$ , in other words, infinite-time concentration.

The case of (P) with slow diffusion  $m > 1$  was studied in [20,32], where the authors show that the steady state does not contain a Dirac delta (i.e.  $\|\rho_\infty\|_{L^1} = 1$ ). The linear diffusion case was extensively studied in [3,36,39]. The fast diffusion range  $1 > m > \frac{n-2}{n}$  with quadratic confinement potential is also well-known and its long-time asymptotics, even for Dirac initial data, is given by integrable stationary solutions, see for instance [9] and its references. See also [45] for the evolution of point singularities in bounded domains.

We will take advantage of the formal interpretation of (P) as the 2-Wasserstein flow [20,21,2] associated to the free-energy

$$\mathcal{F}[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^n} \rho(x)^m dx + \int_{\mathbb{R}^n} V(x)\rho(x)dx, \tag{1.2}$$

in order to obtain properties of this functional in terms of the Calculus of Variations. We also take advantage of this structure to obtain a priori estimates on the solution  $\rho$  of (P) due to the dissipation of the energy.

*Main assumptions and discussion of the main results.* We introduce the specific context in which point-mass concentration arises. We first examine the special stationary solutions that play a role in the asymptotics:

$$\rho_{V+h}(x) = \left(\frac{1-m}{m}(V(x) + h)\right)^{-\frac{1}{1-m}} \quad \text{for } x \in \mathbb{R}^n, \tag{1.3}$$

for  $h \geq 0$ . It is easy to check that they are solutions of (P), and they are bounded if  $h > 0$ . We now consider the class of suitable potentials. We first assume that  $V$  has a minimum at  $x = 0$  and is smooth:  $V \in W_{loc}^{2,\infty}(\mathbb{R}^n)$ ,  $V \geq 0$ ,  $V(0) = 0$ . We are interested in radial aggregating potentials, in fact we  $V$  is radially symmetric and non-decreasing. An essential assumption in the proof of formation of a point-mass concentration is the following small-mass condition for the admissible steady states:

$$a_V = \int_{\mathbb{R}^n} \rho_V(x) dx < 1. \tag{1.4}$$

As a simplifying assumption we will assume that

$$\int_{B_1} \rho_V^{1+\varepsilon}(x) dx < +\infty, \quad \text{for some } \varepsilon > 0. \tag{1.5}$$

The bounded case in which  $\rho_{V+h_1} \leq \rho_0 \leq \rho_{V+h_2}$  with  $h_1, h_2 > 0$  was studied in [14] and leads to no concentration. On the contrary, we will show that there exists a class of radial initial data  $\rho_0(x) \geq \rho_V(x)$  such that the corresponding solution converges as  $t \rightarrow \infty$  to the split measure

$$\mu_\infty = (1 - a_V)\delta_0 + \rho_V(x), \tag{1.6}$$

in the sense of mass (which will be made precise below). Moreover, under further assumptions on  $V$ , we show that  $\mu_\infty$  is the global minimizer in the space of measures of the relaxation of  $\mathcal{F}$ .

An important motivation for our paper is the current interest in the following model of aggregation diffusion with interaction potential

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m + \nabla \cdot (\rho \nabla W * \rho) \tag{1.7}$$

that has led to the discovery of some highly interesting features that have consequences for the parabolic theory and the Calculus of Variations. Recent results [17] show that, under some conditions on  $W$  the energy minimizer of the corresponding energy functional is likewise split as

$$\mu_\infty = (1 - \|\rho_\infty\|_{L^1(\mathbb{R}^n)})\delta_0 + \rho_\infty.$$

The presence of the concentrated point measure is known for specific choices of  $W$ , see [18]. To the best of our knowledge, there exist no results in the literature showing that solutions of the parabolic problem actually converge to these minimisers with a Dirac delta. In this paper, we treat as a first step the where  $V$  is known. We expect similar results to hold when  $V$  is replaced by  $W * \rho$ , but the techniques will be more difficult. For example, the non-local nature of (1.7) suggests that there might not be a comparison principle.

It was shown in [7] that for very fast diffusion,  $m < \frac{n-2}{n}$ , then the solutions of the Fast Diffusion Equation  $u_t = \Delta u^m$  with  $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  vanish in finite time, i.e.  $u(t, x) = 0$  for  $t \geq T^*$ . When  $a_V < 1$ , we construct explicit initial data that preserve the total mass, and this holds for any  $m \in (0, 1)$ .

The case of a ball of radius  $R$ . We have a more complete overall picture when we focus on the problem posed in a ball  $B_R$ , adding a no-flux condition on the boundary:

$$\begin{cases} \frac{\partial \rho}{\partial t} = \Delta \rho^m + \nabla \cdot (\rho \nabla V_R) & \text{in } (0, \infty) \times B_R, \\ (\nabla \rho^m + \rho \nabla V_R) \cdot x = 0, & \text{on } (0, \infty) \times \partial B_R, \\ \rho(0, x) = \rho_0(x). \end{cases} \tag{P_R}$$

As a convenient assumption, we require that  $V_R$  does not produce flux across the boundary

$$\nabla V_R(x) \cdot x = 0, \quad \text{on } \partial B_R. \tag{1.8}$$

We discuss this assumption on Remark 2.12. This problem is the 2-Wasserstein flow of the free energy

$$\mathcal{F}_R[\rho] = \frac{1}{m-1} \int_{B_R} \rho(x)^m dx + \int_{B_R} V_R(x) \rho(x) dx. \tag{1.9}$$

For (P<sub>R</sub>), we show that  $\mathcal{F}_R$  is bounded below and sequences of non-negative functions of fixed  $\|\rho\|_{L^1(B_R)} = \mathbf{m}$  converge weakly in the sense of measures to

$$\mu_{\infty, \mathbf{m}, R} = \begin{cases} \rho_{V_R+h} & \text{if } \exists h \geq 0 \text{ s.t. } \|\rho_{V_R+h}\|_{L^1(B_R)} = \mathbf{m}, \\ \rho_{V_R} + (\mathbf{m} - \|\rho_{V_R+h}\|_{L^1(B_R)})\delta_0 & \text{if } \|\rho_{V_R}\|_{L^1(B_R)} < \mathbf{m}. \end{cases}$$

This means that, if the mass  $a_{0,R}$  cannot be reached in the class  $\rho_{V_R+h}$ , the remaining mass is complete with a Dirac delta at 0. Notice that the mass of  $\rho_{V_R+h}$  is decreasing with  $h$ , so the largest mass is that of  $\rho_{V_R}$ .

We construct an  $L^1$ -contraction semigroup of solutions  $S_R$  of (P<sub>R</sub>) such that, if  $\rho_{V_R} \leq \rho_0 \in L^1(B_R)$  and radially symmetric, then

$$\mathcal{F}_R[S_R(t)\rho_0] \searrow \tilde{\mathcal{F}}_R[\mu_{\infty, \mathbf{m}, R}] = \mathcal{F}_R[\rho_{V_R}],$$

where  $\mathbf{m} = \|\rho_0\|_{L^1(B_R)}$  and  $\tilde{\mathcal{F}}_R$  is the relaxation of  $\mathcal{F}_R$  to the space of measures presented below (see [25]). The semigroup  $S_R$  is constructed as the limit of the semigroup of the regularised problems written below as (P<sub>Φ,R</sub>). Then, we recover our results by passing to the limit in  $\Phi$  and  $R$ .

*The mass function.* One of the main tools in this paper will be the study of the so-called mass variable, which can be applied under the assumption of radial solutions. It works as follows. First, we introduce the spatial volume variable  $v = |x|^n |B_1|$  and consider the mass function

$$M_u(t, v) = \int_{\tilde{B}_v} \rho(t, x) dx, \quad \tilde{B}_v = \left(\frac{v}{|B_1|}\right)^{\frac{1}{n}} B_1 \tag{1.10}$$

Notice that  $|\tilde{B}_v| = v$ . For convenience we define  $R_v = R^n |B_1|$ . We will prove that  $M$  satisfies the following nonlinear diffusion-convection equation in the viscosity sense

$$\frac{\partial M}{\partial t} = (n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left\{ \frac{\partial}{\partial v} \left[ \left( \frac{\partial M}{\partial v} \right)^m \right] + \frac{\partial M}{\partial v} \frac{\partial V}{\partial v} \right\}, \tag{M}$$

where  $\omega_n = |B_1|$ . The diffusion term of this equation is of  $p$ -Laplacian type, where  $p = m + 1$ . The weight will not be problematic when  $v > 0$ , as we show in Appendix A using the parabolic theory in DiBenedetto’s book [26].

Notice that the formation of a Dirac delta at 0 is equivalent to the loss of the Dirichlet boundary condition  $M(t, 0) = 0$ . Few results of loss of the Dirichlet boundary condition are known in the literature of parabolic equation. For equations of the type  $u_t = u_{xx} + |u_x|^p$ , it is known (see, e.g., [4]) that  $u_x$  may blow up on the boundary in finite or infinite time, depending on the choice of boundary conditions. The case of infinite time blow-up was revisited in [43]. The question of boundary discontinuity in finite time, loss of boundary condition, for the so-called viscous Hamilton-Jacobi equations is studied in [6,40,41,38] and does not bear a direct relation with our results. A general reference for boundary blow-up can be found in the book [42].

*Precise statement of results.* In order to approximate the problem in  $\mathbb{R}^n$ , our choice of  $V_R$  will be of the form

$$V_R(x) \begin{cases} = V(x) & |x| \leq R - \varepsilon, \\ \leq V(x) & R - \varepsilon < |x| \leq R \end{cases}$$

and with the condition  $V_R \cdot x = 0$  on  $\partial B_R$ . We also define

$$a_{V,R} = \int_{B_R} \rho_{V_R} dx \quad \text{and} \quad a_{0,R} = \int_{B_R} \rho_0 dx.$$

We will denote  $V = V_R$  until Section 7.

**Theorem 1.1** (*Infinite-time concentration of solutions of (P<sub>R</sub>)*). *Assume  $V \in W^{2,\infty}(B_R)$  is radially symmetric, strictly increasing,  $V \geq 0$ ,  $V(0) = 0$ ,  $V \cdot x = 0$  on  $\partial B_R$  and the technical assumption (1.5). Assume also that  $a_{0,R} > a_{V,R}$ ,  $\rho_0$  radially symmetric,  $\rho_0 \geq \rho_V$  and  $\rho_0 \in L^\infty(B_R \setminus B_{r_1})$  for some  $r_1 < R$ . Then, the solution  $\rho$  of (P<sub>R</sub>) constructed in Theorem 3.6 satisfies*

$$\liminf_{t \rightarrow \infty} \int_{B_r} \rho(t, x) dx \geq (a_{0,R} - a_{V,R}) + \int_{B_r} \rho_V(x) dx \quad \forall r \in [0, R]$$

(i.e., there is concentration in infinite time). Moreover, if

$$\int_{B_r} \rho_0(x) dx \leq (a_{0,R} - a_{V,R}) + \int_{B_r} \rho_V(x) dx \quad \forall r \in [0, R], \tag{1.11}$$

then for  $\mu_{\infty,R} = (a_{0,R} - a_{V,R})\delta_0 + \rho_V$  we have that

$$\lim_{t \rightarrow \infty} d_1(\rho(t), \mu_{\infty,R}) = 0,$$

where  $d_1$  denotes the 1-Wasserstein distance.

**Remark 1.2.** If we take a non-radial datum  $\rho_0 \geq \rho_{0,r}$  with  $\rho_{0,r}$  radially symmetric satisfying the hypothesis of Theorem 1.1, then the corresponding solution  $\rho(t, x)$  of (P<sub>R</sub>) constructed in Theorem 3.6 concentrates in infinite time as well, due to the comparison principle.

Through approximation as  $R \rightarrow \infty$ , we will also show that

**Corollary 1.3** (*At least infinite-time concentration of solutions of (P)*). *Under the hypothesis of Theorem 1.1 and suitable hypothesis on the initial data (specified in Section 7.1), we can show the existence of viscosity solutions of (M) in  $(0, \infty) \times (0, \infty)$  (obtained as a limit of the problems in  $B_R$ ), such that*

$$\lim_{t \rightarrow \infty} M(t, v) = (1 - a_V) + M_{\rho_V}(v)$$

for all  $v > 0$  and, furthermore, locally uniformly  $(0, \infty)$ . We also have that

$$\lim_{t \rightarrow \infty} d_1(\rho(t), (1 - a_V)\delta_0 + \rho_V) = 0.$$

Through our construction of  $M$ , we cannot guarantee in general that  $M(t, 0) = 0$  for  $t$  finite. Producing a priori estimates, we can ensure this in some cases.

**Theorem 1.4** (Infinite-time concentration for  $V$  quadratic at 0). *Let  $\rho_0 \in L^1_+(\mathbb{R}^n)$  non-increasing and assume*

$$\frac{\partial V}{\partial r}(s) \leq C_V r, \quad \text{in } B_{R_V} \text{ for some } C_V > 0. \tag{1.12}$$

Then, the viscosity mass solution constructed in Proposition 7.1 does not concentrate in finite time, i.e.  $M(t, 0) = 0$ .

The picture for power-like  $V$ . Let us discuss the case where  $V$  is of the form

$$V(x) \sim \begin{cases} |x|^{\lambda_0} & |x| \ll 1, \\ |x|^{\lambda_\infty} & |x| \gg 1. \end{cases}$$

The condition  $V \in W^{2,\infty}_{loc}(\mathbb{R}^n)$  means  $\lambda_0 \geq 2$ . In this setting, we satisfy (1.12) so concentration does not happen in finite time. The condition  $\rho_V \in L^1(\mathbb{R}^n)$  (i.e.  $a_V < \infty$ ) holds if and only if

$$\frac{n - \lambda_\infty}{n} < m < \frac{n - \lambda_0}{n}. \tag{1.13}$$

In fact, under this condition,  $\rho_V \in L^{1+\varepsilon}(\mathbb{R}^n)$ . In addition to the behaviour at 0 and  $\infty$ , the restriction  $a_V < 1$  is a condition on the intermediate profile of  $V$ . This is sufficient to construct initial data  $\rho_0$  (of the shape  $\rho_D$  present below) so that solutions converge to  $\mu_\infty$  as  $t \rightarrow \infty$ , in the sense of mass. Due to (1.12), the concentration is precisely at infinite time. But we do not know that  $\mu_\infty$  is a global minimiser of  $\mathcal{F}$ . In Remark 7.8 we prove that the energy functional is bounded below whenever  $m > \frac{n}{n+\lambda_\infty}$ . Notice that  $\frac{n-\lambda_\infty}{n} < \frac{n}{n+\lambda_\infty}$ . Therefore, if

$$\frac{n}{n + \lambda_\infty} < m < \frac{n - \lambda_0}{n}, \quad \text{and} \quad a_V < 1,$$

then  $\mu_\infty$  is the global minimiser in  $\mathcal{P}(\mathbb{R}^n)$  of the relaxation of  $\mathcal{F}$  and it is an attractor for some initial data.

*Structure of the paper.* In Section 2 we write the theory in  $B_R$  for a regularised problem where the fast-diffusion is replaced by a smooth elliptic non-linearity  $\Phi$ . In Section 3 we construct solutions of  $(P_R)$ , by passing to the limit as  $\Phi(s) \rightarrow s^m$  the solutions of Section 2. In Section 4 we show that mass functions  $M$  of the solutions of Sections 2 and 3 are solutions in a suitable sense of Problem (M), and we prove regularity and a priori estimates. In Section 5 we construct initial data  $\rho_0$  so that the mass  $M$  is non-decreasing in time as well a space. We show that these solutions  $M$  concentrate in the limit, a main goal of the paper. We recall that this means the formation of a jump at  $v = 0$  for  $t = \infty$ . Section 6 is dedicated to the minimisation of  $\mathcal{F}_R$  for functions defined in  $B_R$ . We prove that the minimisers are precisely of the form  $\mu_{\infty,m,R}$  described above. In Section 7, we pass to the limit as  $R \rightarrow \infty$  in terms of the mass. We show that the mass functions for suitable initial data still concentrate. We discuss minimisation of the function  $\mathcal{F}$ . We show the class of

potentials  $V$  that make  $\mathcal{F}$  bounded below is more restrictive than for  $\mathcal{F}_R$ , and provide suitable assumptions so that  $\mu_\infty$  is a minimiser. We list some comments and open problems in Section 8. We conclude the paper with two appendixes. The first, Appendix A recalls results from [26] and compacts them into a form we use for  $M$ . Appendix B is devoted to mixing partial space and time regularities into Hölder regularity in space and time.

*A comment on the notation.* Throughout the document, we deal with the spatial variable  $x$ , the radial variable  $r = |x|$ , and the volume variable  $v = |B_1||x|^d$ . Since it will not lead to confusion and simplifies the notation, radial functions of  $x$  will sometimes be evaluated or differentiated in  $r$  or  $v$ . This must be understood as the correct substitution.

## 2. The regularised equation in $B_R$

Following the theory of non-linear diffusion, we consider in general

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta\Phi(u) + \nabla \cdot (uE) & \text{in } (0, \infty) \times B_R \\ (\nabla\Phi(u) + uE) \cdot x = 0, & \text{on } (0, \infty) \times \partial B_R, \\ u(0, x) = u_0(x). \end{cases} \tag{P_{\Phi,R}}$$

We assume that  $\Phi \in C^1$  and elliptic we think of the problem as

$$\frac{\partial u}{\partial t} = \Delta\Phi(u) + \nabla u \cdot E + u \nabla \cdot E.$$

Furthermore, we assume

$$E(x) \cdot x = 0, \quad \text{on } \partial B_R \tag{2.1}$$

**Remark 2.1.** Our results work in a general bounded domain  $\Omega$ , where the assumption on  $E$  is that  $E \cdot n(x) = 0$  on  $\partial\Omega$ . However, we write them in a ball of radius  $R$  since our main objective is to study the long-time asymptotics of radially symmetric solutions.

The diffusion corresponds to the flux  $a(u, \nabla u) = \Phi'(u)\nabla u$ . When  $\Phi, E$  are smooth and we assume  $\Phi$  is uniformly elliptic, in the sense that there exist constants such that

$$0 < c_1 \leq \Phi'(u) \leq c_2 < \infty, \tag{2.2}$$

existence, uniqueness, and maximum principle hold from the classical theory. The literature is extensive: in  $\mathbb{R}^n$  this issue was solved at the beginning of the twentieth century (see [33]), in a bounded domain with Dirichlet boundary condition the result can be found in [29], and the case of Neumann boundary conditions was studied by the end of the twentieth century (for example [1]), where the assumptions on the lower order term were later generalised (see, e.g. [47]). Following [1], we have that, if  $u_0 \in C^2(\overline{B_R})$  then the solution  $u$  of  $(P_{\Phi,R})$  is such that

$$u \in C^1\left((0, T); C(\overline{B_R})\right) \cap C\left((0, \infty); C^2(\overline{B_R})\right) \cap C\left([0, \infty) \times \overline{B_R}\right). \tag{2.3}$$

Let us obtain further properties of the solution of  $(P_{\Phi,R})$ .

**Theorem 2.2** ( *$L^p$  estimates*). *Assume  $E(x) \cdot n(x) \geq 0$ . For classical solutions we have that*

$$\|u(t)_\pm\|_{L^p} \leq e^{\frac{p-1}{p}\|\nabla \cdot E\|_{L^\infty} t} \|(u_0)_\pm\|_{L^p}. \tag{2.4}$$

**Proof.** Let  $j$  be convex. We compute

$$\begin{aligned} \frac{d}{dt} \int_{B_R} j(u) &= \int_{B_R} j'(u) \nabla \cdot (\nabla \Phi(u) + uE) = - \int_{B_R} j''(u) \nabla u \cdot (\nabla \Phi(u) + uE) \\ &= - \int_{B_R} j''(u) \Phi'(u) |\nabla u|^2 - \int_{B_R} j''(u) u \nabla u \cdot E \leq - \int_{B_R} \nabla F(u) \cdot E \end{aligned}$$

where  $F'(u) = j''(u)u \geq 0$  and we can pick  $F(0) = 0$ . Hence  $F \geq 0$ . If  $j$  has a minimum at 0, then  $j' \geq 0$  and  $F \geq 0$ . Since  $E \cdot x \geq 0$ ,

$$\begin{aligned} \int_{B_R} \nabla F(u) \cdot E &= \int_{B_R} \nabla \cdot (F(u)E) - \int_{B_R} F(u) \nabla \cdot E \\ &= \int_{\partial B_R} F(u) E \cdot \frac{x}{|x|} - \int_{B_R} F(u) \nabla \cdot E \\ &\geq - \int_{B_R} F(u) \nabla \cdot E. \end{aligned}$$

Finally, we recover

$$\int_{B_R} j(u(t)) \leq \int_{B_R} j(u_0) + \int_0^t \int_{B_R} F(u) \nabla \cdot E.$$

When  $j(s) = s_{\pm}^p$ ,  $j''(s) = p(p-1)s_{\pm}^{p-2}$  we have  $F(s) = (p-1)s_{\pm}^p$ . Applying (1.8) we show that

$$\int_{B_R} u(t)_{\pm}^p \leq \int_{B_R} (u_0)_{\pm}^p + (p-1) \|\nabla \cdot E\|_{L^\infty} \int_0^t \int_{B_R} u_{\pm}^p.$$

By Gronwall’s inequality we have that

$$\int_{B_R} u(t)_{\pm}^p \leq e^{(p-1)\|\nabla \cdot E\|_{L^\infty} t} \int_{B_R} (u_0)_{\pm}^p.$$

Taking the power  $1/p$  we have (2.4) for  $p < \infty$  and letting  $p \rightarrow \infty$  we also obtain the  $L^\infty$  estimate.  $\square$

**Theorem 2.3** (Estimates on  $\nabla \Phi(u)$ ). Let  $\Psi(s) = \int_0^s \Phi(\sigma) d\sigma$ . Then, we have that

$$\frac{1}{2} \int_0^T \int_{B_R} |\nabla \Phi(u)|^2 \leq \int_{B_R} \Psi(u_0) + \frac{\|E\|_{L^\infty}^2}{2} \int_0^T \int_{B_R} u(t, x)^2 dx dt. \tag{2.5}$$

**Proof.** Multiplying by  $\Phi(u)$  and integrating

$$\int_{B_R} u_t \Phi(u) = - \int_{B_R} \nabla \Phi(u) \cdot (\nabla \Phi(u) + uE) = - \int_{B_R} |\nabla \Phi(u)|^2 - \int_{B_R} u \nabla \Phi(u) E.$$

We have that

$$\frac{d}{dt} \int_{B_R} \Psi(u) + \int_{B_R} |\nabla \Phi(u)|^2 \leq \|u(t)\|_{L^2} \|\nabla \Phi(u(t))\|_{L^2} \|E\|_{L^\infty}.$$

Applying Young’s inequality we obtain

$$\frac{d}{dt} \int_{B_R} \Psi(u) + \frac{1}{2} \int_{B_R} |\nabla \Phi(u)|^2 \leq \frac{1}{2} \|u(t)\|_{L^2}^2 \|E\|_{L^\infty}^2.$$

Notice since  $\Phi' \geq 0$  we have that  $\Psi \geq 0$ . Hence, we deduce the result.  $\square$

If  $\Psi(u_0) \in L^1$  and  $u_0 \in L^2$  then the right-hand side is finite due to (2.4).

**Remark 2.4.** When  $\Phi(s) = s^m$  then  $\Psi(s) = \frac{1}{m+1} s^{m+1}$ .

In order to get point-wise convergence, we follow the approach for the Fast Diffusion equation proposed in [44, Lemma 5.9]. Define

$$Z(s) = \int_0^s \min\{1, \Phi'(s)\} ds, \quad z(t, x) = Z(u(t, x)).$$

**Corollary 2.5.** *We have that*

$$\int_0^T \int_{B_R} |\nabla z|^2 \leq \int_{B_R} \Psi(u_0) + \frac{\|E\|_{L^\infty}^2}{2} \int_0^T \int_{B_R} u(t, x)^2 dx dt. \tag{2.6}$$

**Proof.** Notice  $|\nabla z| \leq |\Phi'(u)| |\nabla u| = |\nabla \Phi(u)|$ .  $\square$

**Lemma 2.6** (*Estimates on  $u_t$  and  $\nabla \Phi(u)$* ). *Assume  $E \cdot x = 0$  on  $\partial B_R$ ,  $u \in L^\infty(0, T; L^2(B_R))$ ,  $\Phi(u) \in L^2(0, T; H^1(B_R))$ ,  $\Phi(u_0) \in H^1(B_R)$  then*

$$\Phi(u) \in L^\infty(0, T; H^1(B_R)) \quad \text{and} \quad u_t \in L^2((0, T) \times B_R).$$

We also have, for  $z(t, x) = Z(u(t, x))$  that

$$\begin{aligned} \int_0^T \int_{B_R} |z_t|^2 \leq C & \left( \int_{B_R} |\nabla \Phi(u_0)|^2 + \frac{1}{2} \int_0^T \int_{B_R} \Phi'(u) |\nabla u|^2 |E|^2 \right. \\ & \left. + \int_{B_R} u(0) \nabla \Phi(u_0) \cdot E + \int_{B_R} |u(T)|^2 |E|^2 \right). \end{aligned} \tag{2.7}$$

**Proof.** Again we will use the notation  $w = \Phi(u)$ . When  $u$  is smooth, we can take  $w_t$  as a test function and integrate in  $B_R$ . Notice that  $w_t = \Phi'(u)u_t$ , so

$$\int_{B_R} \Phi'(u) |u_t|^2 = \int_{B_R} w_t \Delta w + \int_{B_R} w_t \nabla \cdot (uE)$$

Since  $\nabla w = 0$  on  $\partial B_R$ , then also  $\nabla w_t = 0$ . We can integrate by parts to recover

$$\int_{B_R} \Phi'(u)|u_t|^2 = -\frac{d}{dt} \int_{B_R} |\nabla w|^2 + \int_{\partial B_R} w_t u E \cdot \frac{x}{|x|} - \int_{B_R} u \nabla w_t \cdot E.$$

Using assumption (2.1) the second term on the right-hand side vanishes. Integrating in  $[0, T]$  we have

$$\int_0^T \int_{B_R} \Phi'(u)|u_t|^2 + \int_{B_R} |\nabla w(T)|^2 = \int_{B_R} |\nabla w(0)|^2 - \int_0^T \int_{B_R} u \nabla w_t \cdot E$$

Integrating by parts in time the last integral

$$\begin{aligned} & \int_0^T \int_{B_R} \Phi'(u)|u_t|^2 + \int_{B_R} |\nabla w(T)|^2 \\ &= \int_{B_R} |\nabla w(0)|^2 + \int_0^T \int_{B_R} u_t \nabla w \cdot E \\ & \quad + \int_{B_R} u(0) \nabla w(0) \cdot E - \int_{B_R} u(T) \nabla w(T) \cdot E. \end{aligned}$$

Notice that  $u_t \nabla w = \Phi'(u)^{\frac{1}{2}} u_t \Phi'(u)^{\frac{1}{2}} \nabla u$ . Applying Young’s inequality, we deduce

$$\begin{aligned} & \frac{1}{2} \int_0^T \int_{B_R} \Phi'(u)|u_t|^2 + \frac{1}{2} \int_{B_R} |\nabla w(T)|^2 \\ & \leq \int_{B_R} |\nabla w(0)|^2 + \frac{1}{2} \int_0^T \int_{B_R} \Phi'(u)|\nabla u|^2 |E|^2 \\ & \quad + \int_{B_R} u(0) \nabla w(0) \cdot E + \int_{B_R} |u(T)|^2 |E|^2. \end{aligned} \tag{2.8}$$

From the estimates above, we know that  $c_1 |\nabla u| \leq \Phi'(u) |\nabla u| = |\nabla \Phi(u)| \in L^2$ . Similarly, the result follows. Finally, we use that

$$|z_t|^2 \leq |Z'(u)|^2 |u_t|^2 \leq |\Phi'(u)| |u_t|^2.$$

using that  $Z' = \min\{1, \Phi'\}$ .  $\square$

*2.1. Free energy and its dissipation when  $E = \nabla V$*

When  $E = \nabla V$  we have, again, a variational interpretation of the equation that leads to additional a priori estimates. We can rewrite equation  $(P_{\Phi,R})$  as

$$\frac{\partial u}{\partial t} = \nabla \cdot (\Phi'(u) \nabla u + u \nabla V) = \nabla \cdot \left( u \left\{ \frac{\Phi'(u)}{u} \nabla u + \nabla V \right\} \right) = \nabla \cdot (u \nabla \{ \Theta(u) + V \}) \tag{2.9}$$

where

$$\Theta(s) = \int_1^s \frac{\Phi'(\sigma)}{\sigma} d\sigma. \tag{2.10}$$

**Remark 2.7.** Since  $c_1 \leq \Phi'(\rho) \leq c_2$  then  $\Theta(\rho) \sim \alpha \ln \rho$  so  $\Theta^{-1}(\rho) \sim e^{\alpha^{-1}\rho}$ . In particular  $\Theta(0) = +\infty$ . This is why we have to integrate from 1 in this setting. However, when  $\Phi(s) = s^m$  then  $\Theta(s) = \frac{m}{m-1}(s^{m-1} - 1)$ . So  $\Theta^{-1}(s) = (1 - \frac{1-m}{m}s)^{\frac{1}{m-1}}$ . For  $\Phi$  elliptic then  $\Theta^{-1} : \mathbb{R} \rightarrow [0, +\infty)$ . However, for the FDE passing to the limit we are restricted to  $s \leq \frac{m}{1-m}$ .

Formulation (2.9) shows that this equation is the 2-Wasserstein gradient flow of the free energy

$$\mathcal{F}_\Phi[u] = \int_{B_R} \left( \int_1^{u(x)} \Theta(s) ds + V(x)u(x) \right) dx.$$

Along the solutions of  $(P_{\Phi,R})$  it is easy to check that

$$\frac{d}{dt} \mathcal{F}_\Phi[u(t)] = - \int_{B_R} u |\nabla(\Theta(u) + V)|^2 dx \leq 0. \tag{2.11}$$

Also, by integrating in time we have that

$$0 \leq \int_0^T \int_{B_R} u |\nabla(\Theta(u) + V)|^2 dx dt = \mathcal{F}_\Phi[u_0] - \mathcal{F}_\Phi[u(t)] \tag{2.12}$$

Finally, let us take a look at the stationary states. For any  $H \in \mathbb{R}$ , the solution of  $\Theta(u) + V = -H$  is a stationary state. Since  $\Theta : [0, +\infty) \rightarrow \mathbb{R}$  is non-decreasing, we have that  $H = -\Theta(u(0))$ . We finally define

$$u_{V+H} := \Theta^{-1} \left( - (H + V) \right).$$

**Remark 2.8.** When  $\Phi$  is elliptic  $u_{V+H} \leq \Theta^{-1}(-H)$ . In the case of the FDE we have

$$u_{V+H} = \left( 1 + \frac{1-m}{m}(H + V) \right)^{\frac{1}{m-1}} = \rho_{V+h}.$$

where  $h = H + \frac{m}{1-m}$ . When  $h > 0$  we have  $\rho_{V+h}$  is bounded, but  $\rho_V$  is not bounded.

*2.2. Comparison principle and  $L^1$  contraction*

Let us present a class of solutions which have a comparison principle, and are therefore unique.

**Definition 2.9.** We define strong  $L^1$  solutions of  $(P_{\Phi,R})$  as distributional solutions such that

1.  $u \in C([0, T]; L^1(B_R))$ .
2.  $\Phi(u) \in L^1(0, T; W^{1,1}(B_R))$ ,  $\Delta\Phi(u) \in L^1((0, T) \times B_R)$ .
3.  $u_t \in L^2(0, T; L^1(B_R))$ .

**Theorem 2.10.** Assume  $E \cdot n(x) = 0$ . Let  $u, \bar{u}$  be two strong  $L^1$  solutions of  $(P_{\Phi, R})$ . Then, we have that

$$\int_{B_R} [u(t) - \bar{u}(t)]_+ \leq \int_{B_R} [u(0) - \bar{u}(0)]_+$$

In particular  $\|u(t) - \bar{u}(t)\|_{L^1(B_R)} \leq \|u(0) - \bar{u}(0)\|_{L^1(B_R)}$  and, for each  $u_0 \in L^1(B_R)$ , there exists at most one strong  $L^1$  solution.

**Proof.** We now have that  $w = \Phi(u) - \Phi(\bar{u})$ . Let  $j$  be convex and denote  $p = j'$ . We have, using the no flux condition

$$\begin{aligned} \int_0^T \int_{B_R} (u - \bar{u})_t p(w) &= \int_0^T \int_{B_R} p(w) \nabla \cdot \{ \nabla w + (u - \bar{u})E \} \\ &= - \int_0^T \int_{B_R} p'(w) |\nabla w|^2 + \int_0^T \int_{B_R} p(w) \nabla \cdot \{ (u - \bar{u})E \}. \end{aligned}$$

Notice that  $\nabla u = \frac{1}{\Phi'(u)} \nabla \Phi(u) \in L^1((0, T) \times B_R)$  due to (2.2). Expanding the divergence, we have

$$\int_0^T \int_{B_R} (u - \bar{u})_t p(w) \leq \int_0^T \int_{B_R} p(w) (\nabla(u - \bar{u})E + (u - \bar{u})\nabla \cdot E).$$

Then, as  $p \rightarrow \text{sign}_+$ , we have  $p(w) \rightarrow \text{sign}_0^+(w) = \text{sign}_0^+(u - \bar{u})$  and

$$\begin{aligned} \int_0^T \int_{B_R} ([u - \bar{u}]_+)_t &\leq \int_0^T \int_{B_R} (\nabla[u - \bar{u}]_+ E + [u - \bar{u}]_+ \nabla \cdot E) \\ &= \int_0^T \int_{B_R} \nabla \cdot ([u - \bar{u}]_+ E). \end{aligned}$$

Using again that  $E \cdot n(x) = 0$  on  $\partial B_R$ , we recover a 0 on the right hand side. This completes the proof.  $\square$

**Remark 2.11** (Uniform continuity in time). Because of the  $L^1$  contraction, and the properties of the semi-group  $\|u(t+h) - u(t)\|_{L^1} \leq \|u(h) - u(0)\|_{L^1}$ . If  $u(h) \rightarrow u(0)$  in  $L^1$ , we have uniform continuity in time  $\omega(h) = \|u(h) - u(0)\|_{L^1}$ .

**Remark 2.12** (On the assumption  $E \cdot x = 0$  on  $\partial B_R$ ). Notice that to recover the  $L^p$  estimates in Theorem 2.2 (which depend on  $\|\nabla \cdot E\|_{L^\infty}$ ) we assume only that  $E \cdot x \geq 0$  on  $\partial B_R$ . However, later (as in Lemma 2.6 and Theorem 2.10) we require  $E \cdot x = 0$  on  $\partial B_R$ . The estimates in these results do not include  $\nabla \cdot E$ , and so it seems possible to extend the results to this setting by approximation.

### 3. The aggregation-fast diffusion equation

We start this section by providing a weaker notion of solution.

**Definition 3.1.** We say that  $\rho \in L^1((0, T) \times B_R)$  is a weak  $L^1$  solution of  $(P_R)$  if  $\rho^m \in L^1(0, T; W^{1,1}(B_R))$  and, for every  $\varphi \in L^\infty(0, T; W^{2,\infty}(B_R) \cap W_0^{1,\infty}(B_R)) \cap C^1([0, T]; L^1(B_R))$  we have that

$$\begin{aligned} \int_{B_R} \rho(t)\varphi(t) - \int_0^t \int_{B_R} \rho(s)\varphi_t(s) ds \\ = - \int_0^t \int_{B_R} (\nabla \rho^m \cdot \varphi + \rho \nabla V \cdot \nabla \varphi) + \int_{B_R} \rho_0 \varphi(0) \end{aligned}$$

for a.e.  $t \in (0, T)$ .

If  $\nabla V \cdot n(x) = 0$  we then have  $\nabla \rho^m \cdot n = 0$  and we can write the notion of very weak  $L^1$  solution by integrating once more in space the diffusion term

$$\begin{aligned} \int_{B_R} \rho(t)\varphi(t) - \int_0^t \int_{B_R} \rho(s)\varphi_t(s) ds \\ = \int_0^t \int_{B_R} (\rho^m \Delta \varphi - \rho \nabla V \cdot \nabla \varphi) + \int_{B_R} \rho_0 \varphi(0). \end{aligned}$$

**Theorem 3.2** ( $L^1$  contraction for  $H^1$  solutions bounded below). Assume that  $\rho, \bar{\rho}$  are weak  $L^1$  solutions of  $(P_R)$  with initial data  $\rho_0$  and  $\bar{\rho}_0$ ,  $\rho, \bar{\rho} \in H^1((0, T) \times B_R)$ , and  $\rho, \bar{\rho} \geq c_0 > 0$ . Then

$$\int_{B_R} (\rho(t) - \bar{\rho}(t))_+ \leq \int_{B_R} (\rho_0 - \bar{\rho}_0)_+.$$

**Proof.** Since the solutions are in  $H^1$  and are bounded below, then  $\rho^m, \bar{\rho}^m \in H^1((0, T) \times B_R)$ . Let  $p$  be non-decreasing and smooth. By approximation by regularised choices, let us define  $w = \rho^m - \bar{\rho}^m$  and  $\varphi = p(w)$ . Thus we deduce

$$\int_0^t \int_{B_R} (\rho(s) - \bar{\rho}(s))_t p(w) = - \int_0^t \int_{B_R} (p'(w)|\nabla w|^2 + (\rho - \bar{\rho}) \nabla p(w) \cdot \nabla V).$$

Proceeding as in Theorem 2.10 for  $\Phi$  smooth and using (1.8) we have that

$$\int_0^t \int_{B_R} ([\rho(s) - \bar{\rho}(s)]_+)_t ds \leq 0,$$

and this proves the result.  $\square$

We can now construct a semigroup of solutions. We begin by constructing solutions for regular data, by passing to the limit in regularised problems with a sequence of smooth non-linearities  $\Phi_k(s) \rightarrow \Phi(s) = s^m$ . We consider the sequence  $\Phi_k$  of functions given by  $\Phi_k(0) = 0$  and

$$\Phi'_k(s) \sim \begin{cases} mk^{m-1} & s > k, \\ ms^{m-1} & s \in [k^{-1}, k], \\ mk^{1-m} & s < k^{-1} \end{cases} \tag{3.1}$$

up to a smoothing of the interphases. We define

$$\begin{aligned}
 Z(s) &= \int_0^s \min\{1, \Phi'(\sigma)\}d\sigma = \int_0^s \min\{1, m\sigma^{m-1}\}d\sigma \\
 &= \begin{cases} s & s < m^{-\frac{1}{1-m}} \\ C_m + s^m & s \geq m^{-\frac{1}{1-m}} \end{cases} .
 \end{aligned}$$

**Theorem 3.3** (Existence of solutions for regular initial data). Assume  $V \in W^{2,\infty}(B_R)$ ,  $V \geq 0$ ,  $V(0) = 0$ ,  $V \cdot x = 0$  on  $\partial B_R$  and the technical assumption (1.5). Let  $\rho_0$  be such that

$$0 < \varepsilon \leq \rho_0 \leq \varepsilon^{-1}, \quad \rho_0 \in H^1(B_R).$$

Then, the sequence  $u_k$  of solutions for  $(P_{\Phi,R})$  where  $\Phi = \Phi_k$  given by (3.1) is such that

$$\begin{aligned}
 u_k &\rightharpoonup \rho && \text{weakly in } H^1((0, T) \times B_R) \\
 u_k &\rightarrow \rho && \text{a.e. in } (0, T) \times B_R \\
 \Phi_k(u_k) &\rightharpoonup \rho^m && \text{weakly in } L^2(0, T, H^1(B_R))
 \end{aligned}$$

and  $\rho$  is a weak  $L^1$  solution of the problem. Moreover, we have that  $\rho \geq \omega(\varepsilon) > 0$ ,

$$\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}, \quad \|\rho(t)\|_{L^q} \leq e^{t\|\Delta V\|_{L^\infty(B_R)}} \|\rho_0\|_{L^q}.$$

In fact,  $\rho$  is the unique weak  $L^1$  solution which is  $H^1$  and bounded below.

**Proof.** First, we point out that  $\Phi_k(\rho_0) \in W^{1,\infty}(B_R)$ . Hence, for the approximation, by (2.4),  $u_k \in L^\infty((0, T) \times B_R)$  and, due to (2.5),  $\Phi_k(u_k) \in L^2(0, T; H^1(B_R))$  with uniform norm bounds. This ensures (up a subsequence)

$$\begin{aligned}
 u_k &\rightharpoonup \rho && \text{weak-}\star \text{ in } L^\infty((0, T) \times B_R), \\
 \Phi_k(u_k) &\rightharpoonup \phi && \text{weakly in } L^2((0, T) \times B_R), \\
 \nabla \Phi_k(u_k) &\rightharpoonup \nabla \phi && \text{weakly in } L^2((0, T) \times B_R), \\
 Z_k(u_k(t, x)) &\rightharpoonup Z^* && \text{weakly in } H^1((0, T) \times B_R), \\
 Z_k(u_k(t, x)) &\rightarrow Z^* && \text{a.e. in } (0, T) \times B_R.
 \end{aligned}$$

Let us characterise  $\phi$  as  $\Phi(\rho)$ . For  $k > m^{\pm\frac{1}{1-m}}$  we can compute clearly  $\min\{1, \Phi'_k\}$  from (3.1) and hence we have

$$Z'_k(s) - Z'(s) = \begin{cases} 0 & s \in [0, k], \\ m(k^{m-1} - s^{m-1}) & s > k. \end{cases}$$

Since  $u_k$  are uniformly bounded in  $L^\infty$ , taking  $k$  large enough we have that

$$Z_k(u_k) = Z(u_k).$$

Thus  $Z(u_k)$  converges pointwise to  $Z^*$ . But  $Z$  is continuous and strictly increasing, so it is invertible. Thus  $u_k \rightarrow Z^{-1}(Z^*)$  a.e. in  $(0, T) \times B_R$ . Since, if both exist, the weak  $L^2$  and a.e. limits must coincide (apply

Banach-Saks theorem and Césaro mean arguments), then  $u_k \rightarrow \rho$  a.e. in  $(0, T) \times B_R$ . Finally, due to the locally uniform convergence of  $\Phi_k \rightarrow \Phi$ ,  $\Phi_k(u_k) \rightarrow \Phi(\rho)$  a.e. and hence  $\phi = \Phi(\rho)$ .

We can now upgrade to strong convergence, using the uniform  $L^\infty$  bound  $|u_k| \leq C$ . Hence, together with the point-wise convergence, we can apply the Dominated Convergence Theorem to show that our chosen subsequence also satisfies  $u_k \rightarrow \rho$  in  $L^q((0, T) \times B_R)$ ,  $\forall q \in [1, \infty)$ .

Let us show that we maintain an upper and positive lower bound. The upper bound is uniform  $e^{t\|\Delta V\|_{L^\infty(B_R)}} \|\rho_0\|_{L^\infty(B_R)}$ . Since as  $H \rightarrow \infty$  the stationary states  $\Theta_k^{-1}(V + H)$  tend to zero uniformly, then we can choose  $H_k$  so that

$$\rho_0 \geq \Theta_k^{-1}(V + H_k) \geq \omega(\varepsilon).$$

Thus,  $u_k \geq \omega(\varepsilon)$  and, therefore, so is  $\rho$ . In fact, due to this lower bound

$$|\nabla u_k| \leq \frac{1}{Z'(\omega(\varepsilon))} |\nabla Z(u_k)|, \quad |(u_k)_t| \leq \frac{1}{Z'(\omega(\varepsilon))} |(u_t)|$$

and so the convergence  $u_k \rightharpoonup \rho$  is also weak in  $H^1$  (up to a subsequence). But then  $\rho$  is the unique weak  $L^1$  solution with this property. Since the limit is unique, the whole sequence  $u_k$  converges to  $\rho$  all the senses above.  $\square$

**Corollary 3.4** (*Approximation of the free energy*). *Under the hypothesis of Theorem 3.3 we have that*

$$\mathcal{F}_{\Phi_k}[u_k(t)] \rightarrow \mathcal{F}_R[\rho(t)], \quad \text{for a.e. } t > 0,$$

and

$$\int_0^T \int_{B_R} \rho \left| \nabla \left( \frac{m}{m-1} \rho^{m-1} + V \right) \right|^2 \leq \mathcal{F}_R[\rho_0] - \mathcal{F}_R[\rho(T)].$$

In particular,  $\mathcal{F}_R[\rho(t)]$  is a non-increasing sequence.

**Proof.** Since  $u_k \rightarrow \rho$  converges a.e. in  $(0, T) \times B_R$ , then for a.e.  $t > 0$  we have that  $u_k(t) \rightarrow \rho(t)$ . Since  $u_k$  is uniformly bounded, then the Dominated Convergence Theorem ensures the convergence of  $\mathcal{F}_{\Phi_k}[u_k]$ .

Taking into account (2.12), then the sequence  $u_k^{\frac{1}{2}} \nabla(\Theta(u_k) + V)$  is uniformly in  $L^2((0, T) \times B_R)$ . Therefore, up to a subsequence, it has limit  $\xi(x)$ . We can write

$$u_k^{\frac{1}{2}} \nabla(\Theta(u_k) + V) = \frac{\Phi'_k(u_k)}{u_k^{\frac{1}{2}}} \nabla u_k + u_k^{\frac{1}{2}} \nabla V = u_k^{-\frac{1}{2}} (\nabla \Phi_k(u_k) + u_k \nabla V).$$

We know that  $\nabla \Phi_k(u_k) + u_k \nabla V \rightharpoonup \nabla \rho^m + \rho \nabla V$  weakly in  $L^2$ . On the other hand, since we know  $u_k, \rho \geq \omega(\varepsilon)$  we can apply the intermediate value theorem to show that, up to a further subsequence,

$$\begin{aligned} \int_0^T \int_{B_R} |u_k^{-\frac{1}{2}} - \rho^{-\frac{1}{2}}|^2 dx &= \int_0^T \frac{1}{4} \int_{B_R} |\eta_k(x)|^{-3} |u_k - \rho|^2 dx \\ &\leq C \int_0^T \int_{B_R} |u_k - \rho|^2 dx \rightarrow 0, \end{aligned}$$

where the strong convergence  $L^2$  follows, up to a further subsequence, from the weak  $H^1$  convergence. Using the product of strong and weak convergence

$$u_k^{\frac{1}{2}} \nabla(\Theta(u_k) + V) \rightharpoonup \rho^{\frac{1}{2}} \nabla \left( \frac{m}{m-1} \rho^m + V \right), \quad \text{weakly in } L^1((0, T) \times B_R).$$

But this limit must coincide with  $\xi$ , so the limit holds also weakly in  $L^2$ . The weak lower-continuity of the  $L^2$  yields the result.  $\square$

We are also able to deduce from these energy estimates an  $L^1$  bound of  $\nabla \rho^m$ . Unlike (2.5) this bound can use only local boundedness of  $\nabla V$ .

**Corollary 3.5.** *In the hypothesis of Theorem 3.3 we have that, for all  $K \subset \overline{B_R}$ ,*

$$\int_0^T \int_K |\nabla \rho^m| \leq \|\rho_0\|_{L^1(B_R)}^{\frac{1}{2}} \left( \mathcal{F}_R[\rho_0] - \mathcal{F}_R[\rho(T)] + \int_0^T \int_K \rho |\nabla V|^2 \right)^{\frac{1}{2}}. \tag{3.2}$$

**Proof.** We therefore have that

$$\begin{aligned} \int_0^T \int_K |\nabla \rho^m| &= \int_0^T \int_K \rho \left| \frac{m}{m-1} \nabla \rho^{m-1} \right| \\ &\leq \int_0^T \|\rho(t)\|_{L^1(B_R)}^{\frac{1}{2}} \left( \int_K \rho \left| \nabla \frac{m}{m-1} \rho^{m-1} \right|^2 \right)^{\frac{1}{2}} dt \end{aligned}$$

Hence, we conclude the result using Corollary 3.4, Jensen’s inequality and the conservation of the  $L^1$  norm.  $\square$

Now we move to  $L^1$  data. We first point out that  $L^m(B_R) \subset L^1(B_R)$  so any  $\rho \in L^1$  has finite  $\mathcal{F}_R[\rho]$ . To be precise, by applying Hölder’s inequality with  $p = \frac{1}{m} > 1$  we have the estimate

$$\int_K \rho^m \leq |K|^{1-m} \|\rho\|_{L^1(K)}^m. \tag{3.3}$$

Now we apply density in  $L^1$  of the solutions with “good” initial data, via the comparison principle.

**Theorem 3.6** *(Existence of solution for  $L^1$  initial data).* *Under the assumptions of Theorem 3.3, there exists a semigroup  $S(t) : L^1_+(B_R) \rightarrow L^1(B_R)$  with the following properties*

1. For  $0 < \varepsilon^{-1} \leq \rho_0 \leq \varepsilon$  and  $\rho_0 \in H^1(B_R)$ ,  $S(t)\rho_0$  is the unique weak  $L^1$  solution constructed in Theorem 3.3.
2. We have  $\|S(t)\rho_0\|_{L^1(B_R)} = \|\rho_0\|_{L^1(B_R)}$ .
3. We have  $L^1$  comparison principle and contraction

$$\begin{aligned} \int_{B_R} [S(t)\rho_0 - S(t)\bar{\rho}_0]_+ &\leq \int_{B_R} [\rho_0 - \bar{\rho}_0]_+, \\ \int_{B_R} |S(t)\rho_0 - S(t)\bar{\rho}_0| &\leq \int_{B_R} |\rho_0 - \bar{\rho}_0|. \end{aligned}$$

4. If  $\rho_0 \in L^{1+\varepsilon}_+(B_R)$  is the limit of the solutions  $u_k$  of  $(P_{\Phi,R})$  with (3.1) and

$$\|\rho(t)\|_{L^{1+\varepsilon}} \leq C e^{\frac{\varepsilon}{1+\varepsilon}t\|\Delta V\|_{L^\infty}} \|\rho_0\|_{L^{1+\varepsilon}}.$$

5. If  $\rho_0 \in L^1_+(B_R)$  and (1.8), then  $\rho$  is a very weak  $L^1$  solution.

6. If  $\rho_0 \in L^1_+(B_R)$ , then  $\mathcal{F}_R[\rho(t)]$  is non-increasing and we have (3.2). Hence, it is a weak  $L^1$  solution.

**Remark 3.7.** Notice that there is no concentration in finite time. This is due the combination of the  $L^1$  contraction with the uniform  $L^{1+\varepsilon}$  estimate (2.4). By the  $L^1$  contraction, the sequence  $S(t) \max\{\rho_0, k\}$  is Cauchy in  $L^1$  and hence it has a limit in  $L^1$ . No Dirac mass may appear in finite time. In  $\mathbb{R}^n$  we do not have an equivalent guarantee that  $S(t)\rho_{0,k} \in L^1(\mathbb{R}^n)$  for some approximating sequence. We will, however, have this information in the space  $\mathcal{M}(\mathbb{R}^n)$ .

**Remark 3.8.** Notice that the construction of  $S(t)$  is unique, since for dense data it produces the unique  $H^1$  solution bounded below (which also comes as the limit of the approximations), and then it is extended into  $L^1$  by uniform continuity.

**Proof of Theorem 3.6.** We start by defining  $S(t)\rho_0 = \rho$  for the solutions constructed in Theorem 3.3. Let us construct the rest of the situations.

**Step 1.**  $0 < \varepsilon \leq \rho_0 \leq \varepsilon^{-1}$  but not necessarily in  $H^1$ . We regularise  $\rho_0$  by any procedure such that  $H^1(B_R) \ni \rho_{0,\ell} \rightarrow \rho_0$  in  $L^{1+\varepsilon}$  and a.e. Hence  $0 < \varepsilon \leq \rho_{0,\ell} \leq \varepsilon^{-1}$  for  $\ell$  large enough. By using stationary solutions and (2.4) we have that  $0 < \omega(\varepsilon) \leq \rho_\ell \leq C(t)$ . By the  $L^1$  contraction, for all  $t > 0$ ,  $S(t)\rho_{0,\ell}$  is a Cauchy sequence, and hence it has a unique  $L^1$  limit. Let

$$S(t)\rho_0 = L^1 - \lim_{\ell} S(t)\rho_{0,\ell}.$$

We have

$$\int_{B_R} |S(t)\rho_{0,\ell} - S(t)\rho_0| \leq \int_{B_R} |\rho_{0,\ell} - \rho_0|, \quad \ell > \ell_0.$$

For this subsequence  $\rho_\ell^m$  converge to  $\rho^m$  a.e. and, up to a further subsequence, in  $L^\infty$ -weak- $\star$ , and hence  $S(t)\rho_0$  is a weak  $L^1$  solution.

Taking a different  $\bar{\rho}_0$  with the same properties, and  $\bar{\rho}_{0,\ell}$  its corresponding approximation, again for  $\ell$  large,  $0 < \omega(\varepsilon) \leq \bar{\rho} \leq C(t)$ . Then we have that

$$\int_{B_R} |S(t)\rho_{0,\ell} - S(t)\bar{\rho}_{0,\ell}| \leq \int_{B_R} |\rho_{0,\ell} - \bar{\rho}_{0,\ell}|, \quad \ell > \ell_0.$$

Let  $\ell \rightarrow +\infty$  we recover the  $L^1$  contraction. Similarly for the comparison principle.

**Step 2.**  $\rho_0 \in L^1$ . **Approximation by solutions of Theorem 3.3.** We define

$$\rho_{0,K} = \max\{\rho, K\}, \quad \rho_{0,K,\varepsilon} = \max\{\rho, K\} + \varepsilon.$$

For the solutions constructed in Step 1. we have that  $\rho_{K,\varepsilon} \searrow \rho_K$  as  $\varepsilon \searrow 0$  and as  $K \nearrow +\infty$  we have  $\rho_K \nearrow \rho$ . By the  $L^1$  contraction, we have as above that the sequence are Cauchy and hence we have  $L^1$  convergence at each stage. The contraction and comparison are proven as in Step 1.

**Step 3. Item 4.** Due to the  $L^{1+\varepsilon}$  bound, we know that  $u_k \rightharpoonup \rho^*$  weakly in  $L^{1+\varepsilon}((0, T) \times B_R)$ . On the other hand, we can select adequate regularisations of the initial datum  $\rho_{0,\ell} \in H^1$  such that  $\varepsilon \leq \rho_{0,\ell} \leq \varepsilon^{-1}$ , and

the corresponding solutions  $u_{k,\ell}$  of  $(P_{\Phi,R})$  with  $\Phi = \Phi_k$  given by (3.1) satisfy the  $L^1$  contraction. Integrating in  $(0, T)$  we have that

$$\int_0^T \int_{B_R} |u_k - u_{k,\ell}| \leq T \int_{B_R} |\rho_0 - \rho_{0,\ell}|.$$

As  $k \rightarrow \infty$ , by the lower semi-continuity of the norm

$$\int_0^T \int_{B_R} |\rho^* - S(t)\rho_{0,\ell}| \leq T \int_{B_R} |\rho_0 - \rho_{0,\ell}|.$$

As  $\ell \rightarrow \infty$  we recover  $\rho^* = S(t)\rho_0$ .

**Step 4.  $\rho_0 \in L^1$ . Solutions in the very weak sense.** Finally, let us show that the solutions satisfy the equation in the very weak sense. Since we can integrate by parts,  $\rho_{K,\ell}$  satisfies the very weak formulation, and we can pass to the limit to show that so does  $\rho_K$ .

We have shown that  $\rho_K \rightharpoonup \rho$  in  $L^1$ . With the same philosophy, we prove that  $\rho_K(t) \rightharpoonup \rho(t)$  for every  $t > 0$  so  $\rho(t) \in L^1(B_R)$  for a.e. and we can pass to the limit in the weak formulation. We only need to deal with the diffusion term. We also have that  $\rho_K^m \rightharpoonup \rho^m$ . Due to (3.3) and the Monotone Convergence Theorem, we deduce that  $\rho^m \in L^1((0, T) \times B_R)$ .

**Step 5. Conservation of mass.** Since all the limits above hold in  $L^1$ , then preservation of the  $L^1$  mass follows from the properties proved in Theorem 3.3.

**Step 6. Decay of the free energy.** Since all the limits above are taken monotonously and a.e., we can pass to the limit in

$$\int_{B_R} \rho^m, \quad \int_{B_R} V\rho$$

by the Monotone Convergence Theorem. Hence, the decay of the free energy proven in Corollary 3.4 extends to  $L^1$  solutions. We can also pass to the limit in (3.2).  $\square$

#### 4. An equation for the mass

The aim of this section is to develop a well-posedness theory for the mass equation (M). We will show that the natural notion of solution in this setting is the notion of viscosity solution. We will take advantage of the construction of the solution  $\rho$  of  $(P_R)$  as the limit of the regularised problems  $(P_{\Phi,R})$ .

##### 4.1. Mass equation for the regularised problem

If  $E$  is radially symmetric and  $u$  is the solution of  $(P_{\Phi,R})$ , its mass function  $M$  satisfies

$$\frac{\partial M}{\partial t} = \kappa(v)^2 \frac{\partial}{\partial v} \Phi \left( \frac{\partial M}{\partial v} \right) + \kappa(v) \frac{\partial M}{\partial v} E(v), \quad \kappa(v) = n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}},$$

by integrating the equation for  $u = \frac{\partial M}{\partial v}$ . Notice that when  $E = \nabla V$  then  $E = \kappa(v) \frac{\partial V}{\partial v}$ . This change of variables guarantees that

$$\int_{B_R} f(t, x) dx = |\partial B_1| \int_0^R f(t, r) r^{n-1} dr = \frac{|\partial B_1|}{|B_1|} n \int_0^{R_v} f(t, v) dv$$

$$= \int_0^{R_v} f(t, v) dv,$$

for radially symmetric functions.

**Theorem 4.1** (*Comparison principle for masses*). *Let  $M_1$  and  $M_2$  be two classical solutions of the mass problem such that  $M_1(0, r) \leq M_2(0, r)$ . Then  $M_1 \leq M_2$ .*

**Proof.** For any  $\lambda > 0$ , let us consider the continuous function

$$w(t, v) = e^{-\lambda t}(M_1(t, v) - M_2(t, v)).$$

Notice that  $w \rightarrow 0$  as either  $t \rightarrow +\infty$  or  $v \rightarrow 0, R_v$ . Assume, towards a contradiction that  $w$  reaches positive values. Hence, it reaches a positive global maximum at some point  $t_0 > 0$  and  $v_0 \in (0, \infty)$ . At this maximum

$$\begin{aligned} 0 &= \frac{\partial w}{\partial t}(t_0, v_0) = e^{-\lambda t} \frac{\partial}{\partial t}(M_1 - M_2) - \lambda e^{-\lambda t}(M_1 - M_2) \\ 0 &= \frac{\partial w}{\partial v}(t_0, v_0) = e^{-\lambda t} \frac{\partial}{\partial v}(M_1 - M_2) \\ 0 &\geq \frac{\partial^2 w}{\partial v^2}(t_0, v_0) = e^{-\lambda t} \frac{\partial^2}{\partial v^2}(M_1 - M_2). \end{aligned}$$

At  $(t_0, v_0)$ , we simply write the contradictory result

$$\begin{aligned} 0 &< \lambda e^{\lambda t} w(t_0, v_0) = \lambda(M_1 - M_2) = \frac{\partial}{\partial t}(M_1 - M_2) \\ &= (n\omega_n^{\frac{1}{n}} v_0^{\frac{n-1}{n}})^2 \left\{ \Phi' \left( \frac{\partial M_1}{\partial v} \right) \frac{\partial^2 M_1}{\partial v^2} + \frac{\partial M_1}{\partial v} E \right\} \\ &\quad - (n\omega_n^{\frac{1}{n}} v_0^{\frac{n-1}{n}})^2 \left\{ \Phi' \left( \frac{\partial M_2}{\partial v} \right) \frac{\partial^2 M_2}{\partial v^2} + \frac{\partial M_2}{\partial v} E \right\} \\ &= (n\omega_n^{\frac{1}{n}} v_0^{\frac{n-1}{n}})^2 \left\{ \Phi' \left( \frac{\partial M_1}{\partial v} \right) \left( \frac{\partial^2 M_1}{\partial v^2} - \frac{\partial^2 M_2}{\partial v^2} \right) \right\} \leq 0. \quad \square \end{aligned}$$

Let us define the Hölder semi-norm for  $\alpha \in (0, 1)$

$$[f]_{C^\alpha([a,b])} = \sup_{\substack{x,y \in [a,b] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

We have the following estimate

**Lemma 4.2** (*Spatial regularity of the mass*). *If  $u(t, \cdot) \in L^q(B_R)$  for some  $q \in [1, \infty)$  then*

$$[M(t, \cdot)]_{C^{\frac{q-1}{q}}([0, R_v])} \leq \|u\|_{L^q(B_R)}. \tag{4.1}$$

*If  $q = \infty$  the same holds in  $W^{1,\infty}(0, R_v)$ .*

**Proof.** For  $v_1 \geq v_2$  we have

$$|M(t, v_1) - M(t, v_2)| = \int_{\tilde{B}_{v_1} \setminus \tilde{B}_{v_2}} u(t, x) dx \leq \|u\|_{L^q} |\tilde{B}_{v_1} \setminus \tilde{B}_{v_2}|^{\frac{q-1}{q}} = \|u\|_{L^q} (v_1 - v_2)^{\frac{q-1}{q}}. \quad \square$$

**Lemma 4.3** (Temporal regularity of the mass). *There exists a constant  $C > 0$ , independent of  $u$  or  $\Phi$ , such that*

$$\int_0^T \int_0^{R_v} |M_t|^2 dv dt \leq C \left( \int_{B_R} \Psi(u_0) + \|E\|_{L^\infty}^2 \int_0^T \int_{B_R} u(t, x)^2 dx dt \right). \tag{4.2}$$

In particular, if  $u_0 \in L^2$  and  $\Psi(u_0) \in L^1$  then  $M \in C^{\frac{1}{2}}(0, T; L^1(0, R_v))$ .

**Proof.** Let us prove first an estimate for  $\|M_t(t, \cdot)\|_{L^2(0, R_v)}$ . Since  $M = \frac{\partial u}{\partial v}$  then  $\frac{\partial M}{\partial t} = \frac{\partial u}{\partial v \partial t}$ . Applying Jensen’s inequality

$$\begin{aligned} \int_0^{R_v} \left| \frac{\partial M}{\partial t}(t, v) \right|^2 dv &= \int_0^{R_v} \left( \int_{\tilde{B}_v} \frac{\partial u}{\partial t} dx \right)^2 dv \\ &= \int_0^{R_v} \left( \int_{\tilde{B}_v} \nabla \cdot (\nabla \Phi(u) + uE) dx \right)^2 dv \\ &= \int_0^{R_v} \left( \int_{\partial \tilde{B}_v} (\nabla \Phi(u) + uE) \cdot \frac{x}{|x|} dS_x \right)^2 dv \\ &\leq \int_0^{R_v} \int_{\partial \tilde{B}_v} |\nabla \Phi(u) + uE|^2 dS_x dv. \end{aligned}$$

Making the change of variables  $v = |B_1|r^n$  we have  $|\tilde{B}_v| = v = |B_1|r^n = |B_r|$  and

$$\int_0^{R_v} \left| \frac{\partial M}{\partial t}(t, v) \right|^2 dv \leq \int_0^R \int_{\partial B_r} |\nabla \Phi(u) + uE|^2 dS_x |B_1| n r^{n-1} dr = \|\nabla \Phi(u) + uE\|_{L^2(B_R)}^2.$$

Due to (2.5) we recover (4.2). Finally

$$\begin{aligned} \|M(t_1) - M(t_2)\|_{L^1(0, R_v)} &= \int_0^{R_v} |M(t_2, v) - M(t_1, v)| dv \\ &= \int_0^{R_v} \left| \int_{t_1}^{t_2} \frac{\partial M}{\partial t}(s, v) ds \right| dv \leq \int_0^{R_v} \int_{t_1}^{t_2} \left| \frac{\partial M}{\partial t}(s, v) \right| ds dv \\ &\leq |t_2 - t_1|^{\frac{1}{2}} \left\| \frac{\partial M}{\partial t} \right\|_{L^2((0, T) \times (0, R_v))}. \quad \square \end{aligned}$$

#### 4.2. Aggregation-fast diffusion

We recall the definition of viscosity solution for the  $p$ -Laplace problem, which deals with the singular ( $p \in (1, 2)$ ) and degenerate ( $p > 2$ ) cases. We recall the definition found in many texts (see, e.g., [31, 37] and the references therein).

**Definition 4.4.** For  $p > 1$  a function  $u$  is a viscosity supersolution of  $-\Delta_p u = f(x, u, \nabla u)$  if,  $u \neq \infty$ , and for every  $\varphi \in C^2(\Omega)$  such that  $u \geq \varphi$ ,  $u(x_0) = \varphi(x_0)$  and  $\nabla\varphi(x) \neq 0$  for all  $x \neq x_0$  it holds that

$$\lim_{r \rightarrow 0} \sup_{x \in B_r(x_0) \setminus \{x_0\}} (-\Delta_p \varphi(x)) \geq f(x_0, u(x_0), \nabla\varphi(x_0)).$$

Similarly, for our problem we define

**Definition 4.5.** For  $m \in (0, 1)$  a function  $u$  is a viscosity supersolution of (M) if, for every  $t_0 > 0, v_0 \in (0, R_v)$  and for every  $\varphi \in C^2((t_0 - \varepsilon, t_0 + \varepsilon) \times (v_0 - \varepsilon, v_0 + \varepsilon))$  such that  $M \geq \varphi$ ,  $M(v_0) = \varphi(v_0)$  and  $\frac{\partial\varphi}{\partial v}(v) \neq 0$  for all  $v \neq v_0$  it holds that

$$\begin{aligned} & \frac{\partial\varphi}{\partial t}(t_0, v_0) \\ & - (n\omega^{\frac{1}{n}} v_0^{\frac{n-1}{n}})^2 \left[ \lim_{r \rightarrow 0} \sup_{0 < |v-v_0| < r} \frac{\partial}{\partial v} \left[ \left( \frac{\partial\varphi}{\partial v} \right)^m \right] + \frac{\partial\varphi}{\partial v}(t_0, v_0) \frac{\partial V}{\partial v}(v_0) \right] \geq 0. \end{aligned}$$

The corresponding definition of subsolution is made by inverting the inequalities. A viscosity solution is a function that is a viscosity sub and supersolution.

**Remark 4.6.** Since we have a one dimensional problem, we can write the viscosity formulation equivalently by multiplying by  $(\frac{\partial\varphi}{\partial v})^{1-m}$  everywhere, to write the problem in degenerate rather than singular form.

**Remark 4.7.** Our functions  $M$  will be increasing in  $v$ . This allows to a simplification of the condition in some cases. For example, if also have a lower bound on  $\rho$ , in the sense that

$$M(t, v_2) - M(t, v_1) \geq c(v_2 - v_1), \quad \forall v_0 + \varepsilon \geq v_2 \geq v_1 \geq v_0 - \varepsilon \text{ where } c > 0$$

then we know that it suffices to take viscosity test functions  $\varphi$  such  $\frac{\partial\varphi}{\partial v} \geq \frac{c}{2}$ . In particular, we can simplify the definition of sub and super-solution by removing the limit and the supremum.

**Remark 4.8.** We can define the upper jet as

$$\begin{aligned} \mathcal{J}^{2,+} M(t_0, v_0) &= \left\{ (D\varphi(t_0, v_0), D^2\varphi(t_0, v_0)) \right. \\ &: \varphi \in C^2((t_0 - \varepsilon, t_0 + \varepsilon) \times (v_0 - \varepsilon, v_0 + \varepsilon)), \\ & \left. M(t, v) - \varphi(t, v) \leq 0 = M(t_0, v_0) - \varphi(t_0, v_0) \right\}. \end{aligned}$$

The elements of the upper jet are usually denoted by  $(p, X)$ . The lower jet  $\mathcal{J}^{2,-}$  is constructed by changing the inequality above. The definition of viscosity subsolution (resp. super-) can be written in terms of the upper jet (resp. lower).

**Theorem 4.9** (Existence from the semigroup theory for  $\rho$ ). *Let  $\rho_0 \in L^1(B_R)$ . Then*

$$M(t, v) = \int_{\tilde{B}_v} S(t)[\rho_0](x) dx$$

*is a viscosity solution of (M) with  $M(t, 0) = 0$  and  $M(t, R_v) = \|\rho_0\|_{L^1(B_R)}$ . Furthermore, for any  $v_1, v_2, T > 0$ ,  $M \in C([0, T] \times [v_1, v_2])$  with a modulus of continuity that depends only on  $n, m, v_1, v_2$ ,*

$T, \|\frac{\partial V}{\partial v}\|_{L^\infty(v_1, v_2)}$  and the modulus of continuity of  $M_{\rho_0}$  in  $[v_1, v_2]$ . Moreover, we have the following interior regularity estimate: for any  $T_1 > 0$  and  $0 < v_1 < v_2 < R_v$  there exists  $\gamma > 0$  and  $\alpha \in (0, 1)$  depending only on  $n, m, \|\frac{\partial V}{\partial v}\|_{L^\infty(v_1, v_2)}, v_1, v_2, T_1$ , such that

$$|M(t_1, v_1) - M(t_2, v_2)| \leq \gamma \left( \frac{|v_1 - v_2| + \|\rho_0\|_{L^1(B_R)}^{\frac{m-1}{m+1}} |t_1 - t_2|^{\frac{1}{m+1}}}{\min\{v_1, R_v - v_2\} + \|\rho_0\|_{L^1(B_R)}^{\frac{m-1}{m+1}} T_1^{\frac{1}{m+1}}} \right)^\alpha, \tag{4.3}$$

for all  $(t_i, v_i) \in [T_1, +\infty) \times [v_1, v_2]$ .

**Proof. Step 1.**  $\varepsilon \leq \rho_0 \leq \varepsilon^{-1}$  and  $\rho_0 \in H^1(B_R)$ . Let us show that

$$M_{u_k} \rightarrow M_\rho \quad \text{uniformly in } [0, T] \times B_R.$$

$M_\rho$  is a viscosity solution of (M) and  $M_\rho$  is a weak local solution in the sense of Appendix A.

By our construction of  $\rho$  by regularised problems in Theorem 3.3, the strong  $L^q$  convergence of  $u_k$  to  $\rho$  ensures that

$$\int_0^T \sup_{v \in [0, R_v]} |M_{u_k}(t, v) - M_\rho(t, v)| dt \leq \int_0^T \int_{\mathbb{R}^n} |u_k(t, x) - \rho(t, x)| dx dt \rightarrow 0.$$

So we know  $M_{u_k} \rightarrow M_\rho$  in  $L^1(0, T; L^\infty(0, R_v))$ , and hence (up a to a subsequence) a.e.

Through estimates (4.1), (4.2), and Theorem Appendix B.1 we have

$$|M_{u_k}(t_1, v_1) - M_{u_k}(t_2, v_2)| \leq C(\varepsilon)(|v_1 - v_2|^\alpha + |t - s|^\gamma), \quad t, s \in [0, T], v_1, v_2 \in [\varepsilon, R_v - \varepsilon]. \tag{4.4}$$

To check that  $M_\rho$  is a viscosity solution, we select  $v_0 \in (0, R_v)$ . Taking a suitable interval  $(\varepsilon, R_v - \varepsilon) \ni v_0$ , by the Ascoli-Arzelá theorem, a further subsequence is uniformly convergent. Since we have characterised the a.e. limit we have

$$\|M_{u_k} - M_\rho\|_{L^\infty([0, T] \times [\varepsilon, R_v - \varepsilon])} \rightarrow 0.$$

Due to the uniform convergence, we can pass to the limit in the sense of viscosity solutions and  $M_\rho$  is a viscosity solution at  $x_0$ .

The argument is classical and goes as follows (see [23]). Take a viscosity test function  $\varphi$  touching  $M_\rho$  from above at  $x_0$ . Then, due to the uniform convergence  $M_{u_k}$  to  $M_\rho$  in a neighbourhood of  $x_0$ , there exists points  $x_k$  where  $\varphi$  touches  $M_{u_k}$  from above. We apply the definition of viscosity solution for  $M_{u_k}$  at  $x_k$ , and pass to the limit.

Due to the pointwise convergence,  $M_\rho$  also satisfies (4.4).

**Step 2.**  $\rho_0 \in L^1$ . We pick the approximating sequence

$$\rho_{0, K, \varepsilon} = \max\{\rho_0, K\} + \varepsilon.$$

As we did in Theorem 3.6 the  $L^1$  limit of the corresponding solutions is  $S(t)\rho_0$ . Furthermore, the limits  $\varepsilon \searrow 0$  and  $K \nearrow +\infty$  are taking monotonically in  $\rho$ , so also monotonically in  $M$ . This guarantees monotone convergence in  $M$ . With the universal upper bound 1 we have  $L^1$  convergence.

Since the  $C^\alpha$  bound is uniform away from 0, we know that  $M$  maintains it and is continuous. Due to Dini’s theorem the convergence is uniform over  $[0, T] \times [\varepsilon, R_v - \varepsilon]$ , and  $M_\rho$  is a viscosity solution of the problem.

The value  $M(t, 0) = 0$  is given by  $S(t)\rho_0 \in L^1(B_R)$  and the value at  $M(t, R_v) = a_{0,R}$  by the fact that  $\|S(t)\rho_0\|_{L^1(B_R)} = \|\rho_0\|_{L^1(B_R)} = a_{0,R}$ . The uniform continuity is a direct application of Corollary [Appendix A.3](#). We point out that, since  $\rho_0 \in L^1(B_R)$ , then  $M_{\rho_0}$  is point-wise continuous, and therefore uniformly continuous over compact sets. Estimate [\(4.3\)](#) follows from Theorem [Appendix A.1](#).  $\square$

Let us now state a comparison principle, under simplifying hypothesis.

**Theorem 4.10** (*Comparison principle of viscosity solutions if  $\rho$  is bounded below*). *Let  $\underline{M}$  and  $\overline{M}$  be uniformly continuous sub and supersolution. Assume, furthermore, that there exists  $C_0 > 0$  such that*

$$\underline{M}(t, v_2) - \underline{M}(t, v_1) \geq C_0(v_2 - v_1), \quad \forall v_2 \geq v_1.$$

*Then, the solutions are ordered, i.e.  $\underline{M} \leq \overline{M}$ .*

**Proof.** Assume, towards a contradiction that

$$\sup_{t>0, v \in [0, R_v]} (\underline{M}(t, v) - \overline{M}(t, v)) = \sigma > 0.$$

Since both functions are continuous, there exists  $(t_1, v_1)$  such that  $\underline{M}(t_1, v_1) - \overline{M}(t_1, v_1) > \frac{3\sigma}{4}$ . Clearly,  $t_1, v_1 > 0$ . Let us take  $\lambda$  positive such that

$$\lambda < \frac{\sigma}{16(t_1 + 1)}.$$

With this choice, we have that

$$2\lambda t_1 < \frac{\sigma}{4}.$$

For this  $\varepsilon$  and  $\lambda$  fixed, let us construct the variable-doubling function defined as

$$\Phi(t, s, v, \xi) = \underline{M}(t, v) - \overline{M}(s, \xi) - \frac{|v - \xi|^2 + |s - t|^2}{\varepsilon^2} - \lambda(s + t).$$

This function is continuous and bounded above, so it achieves a maximum at some point. Let us name this maximum depending on  $\varepsilon$ , but not on  $\lambda$  by

$$\Phi(t_\varepsilon, s_\varepsilon, v_\varepsilon, \xi_\varepsilon) \geq \Phi(t_1, t_1, v_1, v_1) > \frac{3\sigma}{4} - 2\lambda t_1 > \frac{\sigma}{2}.$$

In particular, it holds that

$$\underline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) \geq \Phi(t_\varepsilon, s_\varepsilon, v_\varepsilon, \xi_\varepsilon) > \frac{\sigma}{2}. \tag{4.5}$$

**Step 1. Variables collapse.** As  $\Phi(t_\varepsilon, s_\varepsilon, v_\varepsilon, \xi_\varepsilon) \geq \Phi(0, 0, 0, 0)$ , we have

$$\frac{|v_\varepsilon - \xi_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2}{\varepsilon^2} + \lambda(s_\varepsilon + t_\varepsilon) \leq \underline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) - \Phi(0, 0, 0, 0) \leq C.$$

Therefore, we obtain

$$|v_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon| \leq C\varepsilon.$$

This implies that, as  $\varepsilon \rightarrow 0$ , the variable doubling collapses to a single point.

We can improve the first estimate using that  $\Phi(t_\varepsilon, s_\varepsilon, v_\varepsilon, \xi_\varepsilon) \geq \Phi(t_\varepsilon, t_\varepsilon, v_\varepsilon, v_\varepsilon)$ . This gives us

$$\begin{aligned} \frac{|v_\varepsilon - \xi_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2}{\varepsilon^2} &\leq \overline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) + \lambda(t_\varepsilon - s_\varepsilon) \\ &\leq \overline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) + C\varepsilon. \end{aligned}$$

Since  $\overline{M}$  is uniformly continuous, we have that

$$\lim_{\varepsilon \rightarrow 0} \frac{|v_\varepsilon - \xi_\varepsilon|^2 + |s_\varepsilon - t_\varepsilon|^2}{\varepsilon^2} = 0. \tag{4.6}$$

**Step 2. For  $\varepsilon > 0$  sufficiently small, the points are interior.** We show that there exists  $\mu$  such that  $t_\varepsilon, s_\varepsilon \geq \mu > 0$  for  $\varepsilon > 0$  small enough. For this, since  $\underline{M}$  and  $\overline{M}$  are uniformly continuous we can estimate as

$$\begin{aligned} \frac{\sigma}{2} &< \underline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) \\ &= \underline{M}(t_\varepsilon, v_\varepsilon) - \underline{M}(0, v_\varepsilon) + \underline{M}(0, v_\varepsilon) - \overline{M}(0, v_\varepsilon) \\ &\quad + \overline{M}(0, v_\varepsilon) - \overline{M}(t_\varepsilon, v_\varepsilon) + \overline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) \\ &\leq \omega(t_\varepsilon) + \omega(|v_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon|), \end{aligned}$$

where  $\omega \geq 0$  is a modulus of continuity (the minimum of the moduli of continuity of  $\underline{M}$  and  $\overline{M}$ ), i.e. a continuous non-decreasing function such that  $\lim_{r \rightarrow 0} \omega(r) = 0$ . For  $\varepsilon > 0$  such that

$$\omega(|v_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon|) < \frac{\sigma}{4},$$

we have  $\omega(t_\varepsilon) > \frac{\sigma}{4}$ . The reasoning is analogous for  $s_\varepsilon$ . For  $v_\varepsilon$  we can proceed much in the same manner

$$\begin{aligned} \frac{\sigma}{2} &< \underline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) \\ &= \underline{M}(t_\varepsilon, v_\varepsilon) - \underline{M}(t_\varepsilon, 0) + \underline{M}(t_\varepsilon, 0) - \overline{M}(t_\varepsilon, v_\varepsilon) + \overline{M}(t_\varepsilon, v_\varepsilon) - \overline{M}(s_\varepsilon, \xi_\varepsilon) \\ &\leq \omega(v_\varepsilon) + \omega(|v_\varepsilon - \xi_\varepsilon| + |t_\varepsilon - s_\varepsilon|). \end{aligned}$$

And analogously for  $\xi_\varepsilon$ . A similar argument holds for  $R_v - v_\varepsilon$  and  $R_v - \xi_\varepsilon$ .

**Step 3. Choosing viscosity test functions.** Unlike in the case of first order equations, there is no simple choice of  $\varphi$  that works in the viscosity formula. We have to take a detailed look at the jet sets. Due to [23, Theorem 3.2] applied to  $u_1 = \underline{M}$ ,  $u_2 = -\overline{M}$  and

$$\varphi_\varepsilon(t, s, v, \xi) = \frac{|v - \xi|^2 + |s - t|^2}{\varepsilon^2} + \lambda(s + t)$$

for any  $\delta > 0$ , there exists  $\underline{X}$  and  $\overline{X}$  in the corresponding jets such that

$$\left( \frac{\partial \varphi_\varepsilon}{\partial(t, v)}(z_\varepsilon), \underline{X} \right) \in \mathcal{J}^{2,+} \underline{M}(t_\varepsilon, v_\varepsilon), \quad \left( -\frac{\partial \varphi_\varepsilon}{\partial(s, \xi)}(z_\varepsilon), -\overline{X} \right) \in \mathcal{J}^{2,-} \overline{M}(s_\varepsilon, \xi_\varepsilon),$$

where  $z_\varepsilon = (t_\varepsilon, s_\varepsilon, v_\varepsilon, \xi_\varepsilon)$  and we have

$$-(\delta^{-1} + \|A\|)I \leq \begin{pmatrix} \underline{X} \\ -\overline{X} \end{pmatrix} \leq A + \delta A^2$$

where  $A = D^2\varphi_\varepsilon(z_\varepsilon)$ . In particular, this implies that the term of second spatial derivatives satisfies  $\underline{X}_{22} \leq \overline{X}_{22}$  (see [23]). Notice that

$$\frac{\partial\varphi}{\partial t}(z_\varepsilon) = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} + \lambda, \quad -\frac{\partial\varphi}{\partial s}(z_\varepsilon) = \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} - \lambda$$

and

$$\frac{\partial\varphi}{\partial v}(z_\varepsilon) = \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2} = -\frac{\partial\varphi}{\partial\xi}(z_\varepsilon).$$

Since  $\underline{M}(t_\varepsilon, v) - \Phi(t_\varepsilon, s_\varepsilon, v, \xi_\varepsilon)$  as a maximum at  $v = \xi_\varepsilon$  we have that, for  $v > v_\varepsilon$

$$\frac{|v - \xi_\varepsilon|^2 - |v_\varepsilon - \xi_\varepsilon|^2}{\varepsilon^2} \geq \underline{M}(t_\varepsilon, v) - \underline{M}(t_\varepsilon, v_\varepsilon) \geq C_0(v - v_\varepsilon).$$

Therefore, we conclude

$$\frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2} \geq C_0.$$

Plugging everything back into the notion of viscosity sub and super-solution

$$\begin{aligned} \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} + \lambda + H\left(v_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \underline{X}\right) &\leq 0 \\ \frac{2(t_\varepsilon - s_\varepsilon)}{\varepsilon^2} - \lambda + H\left(\xi_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \overline{X}\right) &\geq 0 \end{aligned}$$

where

$$H(v, p, X) = -(n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left\{ mp^{m-1} X_{22} + p \frac{\partial V}{\partial v}(v) \right\}.$$

**Step 4. A contradiction.** Subtracting these two equations

$$\begin{aligned} 0 &< 2\lambda \leq H\left(\xi_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \overline{X}\right) - H\left(v_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \underline{X}\right) \\ &= H\left(\xi_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \overline{X}\right) - H\left(\xi_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \underline{X}\right) \\ &\quad + H\left(\xi_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \underline{X}\right) - H\left(v_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \underline{X}\right) \\ &\leq H\left(\xi_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \underline{X}\right) - H\left(v_\varepsilon, \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2}, \underline{X}\right) \\ &= (n\omega_n^{\frac{1}{n}})^2 \frac{2(v_\varepsilon - \xi_\varepsilon)}{\varepsilon^2} \left( v_\varepsilon^{2\frac{n-1}{n}} \frac{\partial V}{\partial v}(v_\varepsilon) - \xi_\varepsilon^{2\frac{n-1}{n}} \frac{\partial V}{\partial v}(\xi_\varepsilon) \right) \rightarrow 0, \end{aligned}$$

since  $v^{2\frac{n-1}{n}} \frac{\partial V}{\partial v}(v) = r^{n-1} \frac{\partial V}{\partial r}$  is Lipschitz continuous and (4.6).  $\square$

### 5. Existence of concentrating solutions

When we now take  $F : (0, \infty) \rightarrow (0, \infty)$

$$\rho_F(x) = \left( \frac{1-m}{m} F(V(x)) \right)^{-\frac{1}{1-m}}, \quad F' \leq 1, \quad F(0) = 0 \quad F(s) > 0 \text{ for all } s > 0, \quad (5.1)$$

we have that  $\rho_F \geq \rho_V$ , so the corresponding solutions with  $\rho(0, x) = \rho_F(x)$  satisfy

$$\rho(t, x) \geq \rho_V(x), \quad \forall t \geq 0, x \in B_R.$$

We will prove that with this initial data we have  $U = \frac{\partial M}{\partial t} \geq 0$  by showing it satisfies a PDE with a comparison principle and  $U(0, \cdot) \geq 0$ . First, we prove an auxiliary result for the regularised problem.

**Theorem 5.1** (*Solutions of  $(P_{\Phi, R})$  with increasing mass*). *Let  $h \in \mathbb{R}$ ,  $F$  be such that  $F' \leq 1$ ,  $F \geq 0$ ,  $F(0) = 0$ ,*

$$u_0 = \Theta^{-1}\left(h - F(V(x))\right), \quad (5.2)$$

*$u$  be the solution of  $(P_{\Phi, R})$  and  $M$  be its mass. Then, we have that*

$$M(t + h, x) \geq M(t, x), \quad \forall h \geq 0 \quad (5.3)$$

and

$$M(t, v_2) - M(t, v_1) \geq \int_{\tilde{B}_{v_2} \setminus \tilde{B}_{v_1}} \Theta^{-1}(h - V(x)) dx, \quad \forall v_1 \leq v_2. \quad (5.4)$$

**Proof.** Notice also that  $F(s) \leq s$  so  $u_0(x) \geq \Theta^{-1}(h - V(x))$ , and this is a stationary solution. Hence this inequality holds for  $u(t)$  as well, due to Theorem 2.10. Thus (5.4) holds. Since  $u \in C^1((0, T); C(\bar{B}_R))$ , we can consider

$$U(t, v) = \int_{\tilde{B}_v} \frac{\partial u}{\partial t}(t, x) dx = \frac{\partial M}{\partial t}.$$

Due to (2.3),  $U \in C((0, T) \times [0, R_v])$ . Since we have  $M(t, 0) = 0, M(t, R_v) = 1$  the boundary conditions are  $U(t, 0) = U(t, R_v) = 0$ . Using the equation for the mass we have that

$$U(0, r) = (n\omega_n v^{\frac{n-1}{n}})^2 u_0 \frac{\partial}{\partial v} (\Theta(u_0) + V) \geq 0,$$

since  $\partial V / \partial v \geq 0$ , by hypothesis. Taking formally a time derivative in the equation of the mass, we obtain that

$$\begin{aligned} \frac{\partial U}{\partial t} &= (n\omega_n v^{\frac{n-1}{n}})^2 \left( \frac{\partial}{\partial v} \left( \Phi'(u) \frac{\partial M}{\partial v \partial t} \right) + \frac{\partial M}{\partial t} \frac{\partial V}{\partial v} \right) \\ &= (n\omega_n v^{\frac{n-1}{n}})^2 \left( \frac{\partial}{\partial v} \left( \Phi'(u) \frac{\partial U}{\partial v} \right) + \frac{\partial U}{\partial v} \frac{\partial V}{\partial v} \right) \\ &= A(v) \frac{\partial^2 U}{\partial v^2} + B(v) \frac{\partial U}{\partial v} \end{aligned}$$

where  $A(v) = (n\omega_n v^{\frac{n-1}{n}})^2 \Phi'(u) \geq 0$  and  $B(v) = (n\omega_n v^{\frac{n-1}{n}})^2 (\frac{\partial}{\partial v}[\Phi'(u)] + \frac{\partial V}{\partial v})$ . This can be justified in the weak local sense. For  $\varphi \in C_c^\infty((0, T) \times (0, R_v))$  we can write

$$\begin{aligned}
 - \int_0^T \int_0^{R_v} M \frac{\partial \varphi}{\partial t} &= - \int_0^T \int_0^{R_v} \Phi(u) \frac{\partial}{\partial v} \left( (n\omega_n v^{\frac{n-1}{n}})^2 \varphi \right) \\
 &\quad + \int_0^T \int_0^{R_v} (n\omega_n v^{\frac{n-1}{n}})^2 \frac{\partial M}{\partial v} \frac{\partial V}{\partial v} \varphi,
 \end{aligned}$$

we can simply take  $\varphi = \frac{\partial \psi}{\partial t}$  and integrating by parts in time to recover

$$\begin{aligned}
 \int_0^T \int_0^{R_v} \frac{\partial M}{\partial t} \frac{\partial \psi}{\partial t} &= \int_0^T \int_0^{R_v} \frac{\partial}{\partial t} (\Phi(u)) \frac{\partial}{\partial v} \left( (n\omega_n v^{\frac{n-1}{n}})^2 \psi \right) \\
 &\quad - \int_0^T \int_0^{R_v} (n\omega_n v^{\frac{n-1}{n}})^2 \frac{\partial M}{\partial t \partial v} \frac{\partial V}{\partial v} \psi.
 \end{aligned}$$

Since  $u$  is  $C^1$  then  $\frac{\partial}{\partial t} (\Phi(u)) = \Phi'(u) \frac{\partial U}{\partial t}$  is a continuous function. Operating with the derivatives of  $\psi$ , we recover that

$$\int_0^T \int_0^{R_v} U \left\{ \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial v} \left( A(t, v) \frac{\partial \psi}{\partial v} \right) + \frac{\partial}{\partial v} (B(v)\psi) \right\} = 0,$$

for all  $\psi \in C_c^\infty((0, T) \times (0, R_v))$ . We now show that  $U$  is a solution in the weak sense, incorporating the boundary conditions. Since  $U$  is continuous and  $U(t, 0) = U(t, R_v) = 0$ , for any  $\psi$  suitably regular we can use an approximating sequence  $\psi_k \in C_c^\infty((0, T) \times (0, R_v))$  to show that

$$\begin{aligned}
 \int_0^{R_v} U(T)\psi(T)dv + \int_0^T \int_0^{R_v} U \left\{ \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial v} \left( A(t, v) \frac{\partial \psi}{\partial v} \right) + \frac{\partial}{\partial v} (B(v)\psi) \right\} \\
 = \int_0^{R_v} U(0)\psi(0)dv.
 \end{aligned}$$

Fix  $\Psi_0$  smooth and let  $\Psi$  the solution of

$$\begin{cases} \frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial v} (A(T-t, v) \frac{\partial \Psi}{\partial v}) + \frac{\partial}{\partial v} (B(v)\Psi) & \text{in } (0, T) \times (0, R_v) \\ \Psi(t, 0) = \Psi(0, R_v) = 0, \\ \Psi(0, v) = \Psi_0. \end{cases} \tag{5.5}$$

If  $\Psi$  is a classical interior solution, then taking as a test function  $\psi(t, x) = \Psi(T-t, x)$  we have that

$$\int_0^{R_v} U(T)\Psi_0 dv = \int_0^{R_v} U(0)\Psi(T)dv.$$

Notice that  $A(0) = 0$ . Substituting  $A$  by the uniformly elliptic diffusion  $A(T - t, v) + \delta$ ,  $\delta > 0$ , and letting  $\delta \searrow 0$ , for any  $\Psi_0 \geq 0$ , we can construct a non-negative solution of (5.5). Therefore, since  $U(0) \geq 0$  we have that  $U \geq 0$  in  $(0, T) \times (0, R_v)$ , and the proof is complete.  $\square$

Before we continue, we point out that  $M_{\rho_F}$  lies between  $M_{\rho_V}$  and one of its upward translations.

**Remark 5.2.** Notice that  $F(0) = 0$  and  $F' \leq 1$  then  $F(s) \leq s$ . Hence  $\rho_F \geq \rho_V$ . Integrating forward from 0 we have

$$M_{\rho_F}(v) = \int_{\tilde{B}_v} \rho_F dx \geq \int_{\tilde{B}_v} \rho_V dx = M_{\rho_V}(v).$$

On the other hand, integrating backwards from  $R_v$  we have

$$\begin{aligned} M_{\rho_F}(v) &= M_{\rho_F}(R_v) - \int_{B_R \setminus \tilde{B}_v} \rho_F dx \leq M_{\rho_F}(R_v) - \int_{B_R \setminus \tilde{B}_v} \rho_V dx \\ &= \left( M_{\rho_F}(R_v) - a_{V,R} \right) + M_{\rho_V}(v). \end{aligned} \tag{5.6}$$

Now we move to considering suitable initial data for  $(P_R)$ . We make the following construction

**Lemma 5.3.** *Let  $b_1 \in (0, V(R))$ . Assume that  $a_{V,R} < a_{V,0}$ . There exists  $\bar{b}_2(b_1) < V(R)$  such that, for all  $b_2 \in (b_1, \bar{b}_2(b_1))$  there exists  $D(b_1, b_2) < b_1$  such that the  $\rho_D$  given by*

$$F_D(s) = \begin{cases} s & \text{if } s \in [0, b_1], \\ D & \text{if } s \in [b_1, b_2], \\ D + s - b_2 & \text{if } s \geq b_2, \end{cases} \quad \rho_D(x) = \left( \frac{1-m}{m} F_D(V(x)) \right)^{-\frac{1}{1-m}} \tag{5.7}$$

satisfies  $\int_{B_R} \rho_D = a_{0,R}$ .

The function  $F_D$  can be taken as the limit of functions  $F_\varepsilon \in C^1$  in the assumptions of Theorem 5.1. Take  $F_\varepsilon(0) = 0$  and

$$F'_\varepsilon(s) = \begin{cases} 1 & \text{if } s \in [0, b_1], \\ 1 - \frac{1+\alpha}{\varepsilon/4}(s - b_1) & \text{if } s \in [b_1, b_1 + \frac{\varepsilon}{4}] \\ -\alpha & \text{if } s \in [b_1 + \frac{\varepsilon}{4}, b_1 + \frac{\varepsilon}{2}] \\ -\alpha + \frac{\alpha}{\varepsilon/2}(s - b_1 - \frac{\varepsilon}{2}) & \text{if } s \in [b_1 + \frac{\varepsilon}{2}, b_1 + \varepsilon] \\ 0 & \text{if } s \in [b_1 + \varepsilon, b_2 - \varepsilon], \\ \frac{1}{\varepsilon}(s - b_2 + \varepsilon) & \text{if } s \in [b_2 - \varepsilon, b_2], \\ 1 & \text{if } s \geq b_2, \end{cases}$$

where given  $0 < b_1 < b_2 < V(R)$  and  $\varepsilon < \frac{b_2 - b_1}{4}$  and  $0 < D \leq b_1$ , we can always select

$$\alpha(b_1, b_2, D, \varepsilon) > 0 \text{ such that } F_\varepsilon(b_1 + \varepsilon) = \int_0^{b_1 + \varepsilon} F'_\varepsilon(s) ds = D.$$

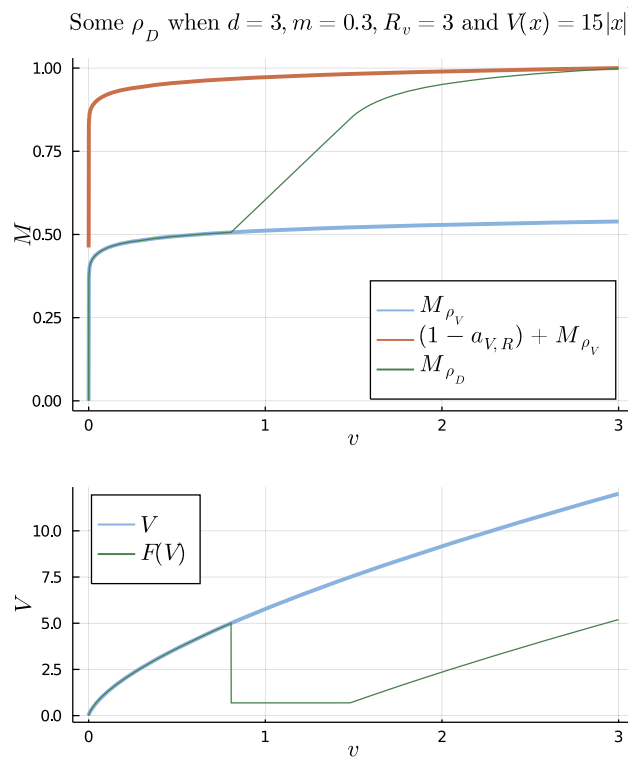


Fig. 1. Example of  $\rho_D$  for some parameters  $b_1$  and  $b_2$ . In this example  $a_{0,R} = 1$ .

Notice that  $F_\varepsilon(s) > 0$  for  $s > 0$  and  $F'_\varepsilon \leq 1$  and  $F_\varepsilon \in C^1$ . This form is rather elaborate, so we pick the limit as  $\varepsilon \searrow 0$ . Notice that  $\inf F'_\varepsilon \rightarrow -\infty$  as  $\varepsilon \searrow 0$ .

**Proof of Lemma 5.3.** We start by pointing out that  $\int_{B_R} \rho_F$  is continuous in all parameters. Taking  $D = b_1$  we have that as  $b_2 \searrow b_1$  we have that  $\int_{B_R} \rho_D \searrow \int_{B_R} \rho_V = a_{V,R} < a_{0,R}$ . Hence, when  $D = b_1$ , there exists  $\bar{b}_2(b_1)$  such that for  $b_2 < \bar{b}_2$ , and  $D = b_1$ ,  $\int_{B_R} \rho_D < a_{0,R}$ . Fixed  $b_1$  and  $b_2$ , as  $D \searrow 0$  we have  $\int_{B_R} \rho_D \nearrow \infty$ . So there exists a choice of  $D$  such that  $\int_{B_R} \rho_F = a_{0,R}$ .  $\square$

Notice that  $\rho_D \in L^{1+\varepsilon}(B_R)$  due to the assumption (1.5). We sketch the profile in Fig. 1.

**Theorem 5.4** (Solutions of  $(P_R)$  with increasing mass). *Under the hypothesis of Theorem 3.3, let  $\rho_D$  be given by (5.7). Then, the mass  $M$  of  $\rho(t) = S(t)\rho_D$  constructed in Theorem 3.6 is such that*

$$M(t, v) \nearrow (a_{0,R} - a_{V,R}) + M_{\rho_V}(v) \quad \text{uniformly in } [\varepsilon, R_v].$$

In particular,  $\rho(t, \cdot) \rightharpoonup (a_{0,R} - a_{V,R})\delta_0 + \rho_V$  weak- $\star$  in the sense of measures.

**Proof of Theorem 5.4. Step 1. Properties by approximation.** Since  $\rho_D \in L^{1+\varepsilon}$ , looking at how we constructed  $S(t)\rho_0$  in Theorems 3.3 and 3.6, it can be approximated by  $S_k(t)\rho_0$  where  $S_k$  is the semigroup of  $(P_{\Phi_k,R})$  with  $\Phi_k$  given by (3.1). Notice that, the associated  $\Theta_k$  given by (2.10) is

$$\Theta_k(s) = \frac{m}{1-m} (1 - s^{m-1}), \quad \text{for } s \in [k^{-1}, k].$$

Hence, we recover

$$\Theta_k^{-1}(s) = (1 - \frac{1-m}{m}s)^{-\frac{1}{1-m}}, \quad \text{for } s \in [\Theta_k(k^{-1}), \Theta_k(k)].$$

Taking  $h = \frac{m}{1-m}$  in (5.2) we have initial data  $u_{0,k}$  such that

$$u_{0,k} = \left(\frac{1}{1-m}F(V)\right)^{-\frac{1}{1-m}}, \quad \text{whenever } F(V(x)) \in [\Theta_k(k^{-1}), \Theta_k(k)],$$

and  $M_{u_k}$  non-decreasing in  $t$ . This corresponds to an interval of the form  $v \in [\varepsilon_k, R_v - \delta_k]$ . Let us denote  $u_k = S_k(t)u_{0,k}$ . Due to the  $L^1$  contraction we have that

$$\int_{\tilde{B}_R} |u_k(t) - S_k(t)\rho_D| dx \leq \int_{\tilde{B}_R} |u_{0,k} - \rho_D| dx.$$

Hence, by Theorem 3.6 we infer that  $S_k(t)u_{0,k} \rightarrow S(t)\rho_D$  in  $L^1(B_R)$  for a.e.  $t > 0$ . This guarantees the a.e. convergence of the masses. Hence, the mass function  $M$ , which is already a viscosity solution of (M) and  $C^\alpha$  regular, also inherits the point-wise estimate from  $M_{u_k}$  in (5.4).  $M$  is also non-decreasing in  $t$  and  $v$ . Moreover, due to (5.6) and Theorem 4.9 due to Equation (4.3), we conclude that

$$M_{\rho_D}(v) \leq M(t, v) \leq (a_{0,R} - a_{V,R}) + M_{\rho_V}(v). \tag{5.8}$$

**Step 2. Uniform convergence of  $M(t, \cdot)$  as  $t \rightarrow +\infty$ .** Since  $M$  is point-wise non-decreasing in  $t$  and bounded above by  $a_{0,R}$ , we know there exists a function  $M_\infty$  such that

$$M(t, x) \nearrow M_\infty(x), \quad t \nearrow \infty. \tag{5.9}$$

By the estimate (4.3) we know that  $M_\infty$  belongs to  $C_{loc}^\alpha((0, R_v))$  and hence continuous in interior points. On the other hand, (5.8) implies

$$M_{\rho_D}(v) \leq M_\infty(v) \leq (a_{0,R} - a_{V,R}) + M_{\rho_V}(v).$$

Hence, by the sandwich theorem,  $M_\infty(R_v) = a_{0,R}$  and it is continuous at  $R_v$  (due to the explicit formulas we can actually show rates). Since  $M_\infty$  is non-decreasing and  $M_\infty \geq 0$ , due to (5.9), there exists a limit

$$\lim_{v \rightarrow 0} M_\infty(v) \leq a_{0,R} - a_{V,R}.$$

Defining  $M_\infty(0) = \lim_{v \rightarrow 0} M_\infty(v)$ , the function is obviously continuous in  $[0, R_v]$ . Hence, applying Dini’s theorem, we know that

$$\sup_{v \in [\varepsilon, R_v]} |M(t, v) - M_\infty(v)| \rightarrow 0.$$

Due to (5.4) and our choice of  $h$ , we have that

$$M_\infty(v_2) - M_\infty(v_1) \geq (v_2 - v_1) \inf_{\tilde{B}_{v_2} \setminus \tilde{B}_{v_1}} \rho_V, \quad \forall v_1 \leq v_2. \tag{5.10}$$

**Step 3. Characterisation of  $M_\infty$  as a viscosity solution.** Let us check that  $M_\infty$  is a viscosity solution of

$$\frac{\partial^2 M_\infty}{\partial v^2} + \frac{1}{m} \left(\frac{\partial M_\infty}{\partial v}\right)^{2-m} \frac{\partial V}{\partial v} = 0. \tag{5.11}$$

Due to our lower bound (5.10),  $\frac{\partial M_\infty}{\partial v}$  is bounded below. We define the sequence of masses  $M_n : [0, 1] \times [0, R_v] \rightarrow \mathbb{R}$  given by  $M_n(t, v) = M(t - n, v)$ . These are viscosity solutions for (M) due to Theorem 4.9. We also know that

$$\sup_{(t,v) \in [0,1] \times [\varepsilon, R_v]} |M_n(t, v) - M_\infty(v)| \rightarrow 0.$$

By standard arguments of stability of viscosity solutions,  $M_\infty$  is also a solution of (M). Since it does not depend on  $t$ , we can select spatial viscosity test functions, and hence it is a solution of (5.11). Since we have removed the time dependency, we dropped also the spatial weight  $(n\omega_n v^{\frac{n-1}{n}})^2$ .

**Step 4.  $M_\infty$  is  $C^2((0, R_v))$ .**

**Step 4a. Lipschitz regularity** Since  $M_\infty$  is non-decreasing, at the point of contact of a viscosity test function touching from below, we deduce

$$-\frac{\partial^2 \varphi}{\partial v^2}(v_0) \geq \frac{1}{m} \left( \frac{\partial \varphi}{\partial v}(v_0) \right)^{2-m} \frac{\partial V}{\partial v}(v_0) \geq 0.$$

Hence,  $M_\infty$  is a viscosity super-solution of  $-\Delta M = 0$ . Due to [30], we have that  $M$  is also a distributional super-solution of  $-\Delta M = 0$ . Distributional super-solutions are concave. Since  $M_\infty$  is concave, it is  $W^{1,\infty}([\varepsilon, R_v - \varepsilon])$  of all  $\varepsilon > 0$ .

**Step 4b. Higher regularity by bootstrap.** Now we can treat the right-hand side as a datum

$$f = \frac{1}{m} \left( \frac{\partial M_\infty}{\partial v}(v_0) \right)^{2-m} \frac{\partial V}{\partial v} \in L^\infty(\varepsilon, R_v - \varepsilon).$$

Applying the regularisation results in [12] we recover that  $M_\infty \in C^{1,\alpha}(2\varepsilon, R_v - 2\varepsilon)$ . Since  $V \in W^{2,\infty} = C^{0,1}$ , then  $f \in C^{0,\beta}(2\varepsilon, R_v - 2\varepsilon)$  for some  $\beta > 0$ , so  $M_\infty \in C^{2,\beta}(4\varepsilon, R_v - 4\varepsilon)$ .

**Step 5. Explicit formula of  $M_\infty$ .** Since  $M_\infty \in C^2((0, R_v)) \cap C([0, R_v])$ , we can integrate (5.11) to show that

$$M_\infty(v) = M_\infty(0) + M_{\rho_{V+h}}$$

for some  $h \geq 0$ . Since  $M_\infty(R_v) - M_\infty(0) = a_{0,R} - M_\infty(0) \leq a_V$  then, for some  $h \geq 0$  we have that  $M_\infty(R_v) - M_\infty(0) = a_{V+h,R}$ . By the comparison principle, which holds due to (5.10), we conclude the equality  $M_\infty(v) - M_\infty(0) = M_{\rho_{V+h}}(v)$  for  $v \in [0, R_v]$ . Due to (5.10), the singularity at 0 is incompatible with  $h > 0$ . Thus  $h = 0$ .  $\square$

**Remark 5.5.** Notice that the aggregation does not occur in finite time, since we assume (1.5).

**Proof of Theorem 1.1.** To compute the  $\liminf$ , it suffices to pick a  $\rho_D$  such that  $M_{\rho_D} \geq M_{\rho_V}$ . This can be done by selecting  $b_1$  sufficiently close to  $V(R)$ . If we assume (1.11), by the comparison of masses we have

$$\int_{B_r} \rho(t, x) dx \leq (a_{0,R} - a_{V,R}) + M_{\rho_V}(v), \quad \forall r \in [0, R].$$

Then, the  $\liminf$  and  $\limsup$  coincide with this upper bound, i.e.

$$\lim_{t \rightarrow +\infty} \int_{B_r} \rho(t, x) dx = (a_{0,R} - a_{V,R}) + M_{\rho_V}(v), \quad \forall r \in [0, R].$$

To check the convergence in Wasserstein distance, we must write the convergence of the masses in  $L^1$  in radial coordinates. Let  $\mu_{\infty,R} = (a_{0,R} - a_{V,R})\delta_0 + \rho_V$ , then we have that

$$d_1(\rho(t), \mu_{\infty,R}) = n\omega_n \int_0^R \left| \int_{B_r} \rho(t, x) dx - \mu_{\infty,R}(B_r) \right| r^{n-1} dr,$$

due to the fact that the optimal transport between radial densities is radial and the characterisation of  $d_1$  in one dimension (see [46]). Since we have shown in the proof above that  $\int_{B_r} \rho(t) dx \leq \mu_\infty(B_r)$

$$d_1(\rho(t), \mu_\infty) = n\omega_n \int_0^R \left( \mu_{\infty,R}(B_r) - \int_{B_r} \rho(t, x) dx \right) r^{n-1} dr.$$

Due to the monotone convergence  $\int_{B_r} \rho(t, x) \nearrow \mu_{\infty,R}(B_r)$  for  $r \in (0, R]$ , the right-hand goes to 0 as  $t \rightarrow +\infty$ .  $\square$

### 6. Minimisation of $\mathcal{F}_R$

It is very easy to see that the free energy  $\mathcal{F}_R$  is bounded below, in particular

$$\mathcal{F}_R[\rho] \geq -\frac{1}{1-m} |B_R|^{\frac{1}{1-m}} \|\rho\|_{L^1(B_R)}^m, \tag{6.1}$$

due to (3.3) and that  $V \geq 0$ . Therefore, there exists a minimising sequence. The problem is that the functional setting does not offer sufficient compactness to guarantee its minimiser is in  $L^1(B_R)$ . However, we can define its extension to the set of measures as

$$\widetilde{\mathcal{F}}_R[\mu] = -\frac{1}{1-m} \int_{B_R} \mu_{ac}^m + \int_{B_R} V d\mu$$

This is the unique extension of  $\mathcal{F}_R$  to  $\mathcal{M}_+(B_R)$  that is lower-semicontinuous in the weak- $\star$  topology (see [25] and related results in [11]).

Since we work on a bounded domain, tightness of measures is not a limitation. For convenience, let us define for  $\rho \in L^1(B_R)$ ,

$$\mathcal{E}_{m,R}[\rho] = \frac{1}{m-1} \int_{B_R} \rho(x)^m dx.$$

Let us denote the set of non-negative measures of fixed total mass  $\mathbf{m}$  in  $B_R$  as

$$\mathcal{P}_{\mathbf{m}}(\overline{B_R}) = \{ \mu \in \mathcal{M}_+(\overline{B_R}) : \mu(B_R) = \mathbf{m} \}.$$

We have the following result

**Theorem 6.1** (*Characterisation of the unique minimiser of  $\mathcal{F}_R$* ). *Let us fix  $\mathbf{m} > 0$ ,  $V \in W^{2,\infty}(B_R)$ ,  $V(0) = 0$  and  $V$  is radially increasing. Then, any sequence  $\rho_j$  minimising  $\mathcal{F}_R$  over  $\mathcal{P}_{\mathbf{m}}(\overline{B_R}) \cap L^1(B_R)$  converges weakly- $\star$  in the sense of measures to*

$$\mu_{\infty,\mathbf{m}} = \begin{cases} \rho_{V+h} & \text{for } h \text{ such that } a_{V+h} = \mathbf{m}, \\ (\mathbf{m} - a_{V,R})\delta_0 + \rho_V & \text{if } a_{V,R} < \mathbf{m}. \end{cases}$$

Furthermore,

$$\widetilde{\mathcal{F}}_R[\mu_{\infty,\mathbf{m}}] = \inf_{\mu \in \mathcal{P}_{\mathbf{m}}(\overline{B_R})} \widetilde{\mathcal{F}}_R[\mu] = \inf_{\rho \in \mathcal{P}_{\mathbf{m}}(\overline{B_R}) \cap L^1(B_R)} \mathcal{F}_R[\rho]. \tag{6.2}$$

**Remark 6.2** (*Lieb’s trick*). Given a radially decreasing  $\rho \geq 0$ ,  $\rho^q \in L^1(B_R)$  for some  $q > 0$  (for any  $R \leq \infty$ ), using and old trick of Lieb’s (see [34,35]) we get, for  $|x| \leq R$ ,

$$\int_{B_R} \rho^q dx = n\omega_n \int_0^R \rho(r)^q r^{n-1} dr \geq n\omega_n \int_0^{|x|} \rho(r)^q r^{n-1} dr \geq n\omega_n \rho(x)^q \int_0^{|x|} r^{n-1} dr.$$

Hence, we deduce the point-wise estimate

$$\rho(x) \leq \left( \frac{\int_{B_R} \rho^q}{n\omega_n |x|^n} \right)^{\frac{1}{q}}. \tag{6.3}$$

It is easy to see that (6.3) is not sharp. However, it is useful to prove tightness for sets of probability measures. Similarly, if additionally  $V\rho \in L^1(B_R)$ , and  $V \geq 0$  we can estimate

$$\begin{aligned} \int_{B_R} V\rho dx &= n\omega_n \int_0^{|x|} V(r)\rho(r)r^{n-1} dr \geq n\omega_n \int_0^{|x|} V(r)\rho(r)r^{n-1} dr \\ &\geq n\omega_n \rho(x) \int_0^{|x|} V(r)r^{n-1} dr, \end{aligned}$$

so we recover the point-wise estimate

$$\rho(x) \leq \frac{\int_{B_R} V\rho}{\int_{B_{|x|}} V}. \tag{6.4}$$

**Proof of Theorem 6.1.** The second equality in (6.2) is due to the weak- $\star$  density of  $L^1_+(B_R)$  in the space of non-negative measures, and the construction of  $\tilde{\mathcal{F}}_R$  (see [25]). Let us consider a minimising sequence. Let us show that we can replace it by a radially-decreasing minimising sequence. Let  $\rho_j \in L^1_+(B_R)$  with  $\|\rho_j\|_{L^1} = m$ . By standard rearrangement results

$$\mathcal{E}_{m,R}[\rho_j^*] = \mathcal{E}_{m,R}[\rho_j].$$

Since  $V \geq 0$  and radially symmetric and non-decreasing then

$$\int_{B_R} V(x)\rho_j^*(x) dx \leq \int_{B_R} V(x)\rho_j(x) dx.$$

Hence, there exists minimising sequence  $\rho_j \in L^1(B_R)$  that we can assume radially non-increasing. Since  $\rho_j \in \mathcal{P}_m(\overline{B_R})$ , by Prokhorov’s theorem, this minimising sequence must have a weak- $\star$  limit in the sense of measures, denoted by  $\mu_{\infty,m}$ .

We use the following upper and lower bounds that follow from (3.3)

$$\int_{B_R} V\rho_j = \mathcal{F}_R[\rho_j] + \frac{1}{1-m} \int_{B_R} \rho_j^m \leq \mathcal{F}_R[\rho_j] + \frac{1}{1-m} |B_R|^{1-m} \|\rho_j\|_{L^1}.$$

Due to (6.4) we have a uniform bound in  $L^\infty(B_R \setminus B_\varepsilon)$  for any  $\varepsilon > 0$ . Thus, there exists  $\rho_\infty \in L^1_+(B_R) \cap L^\infty(B_R \setminus B_\varepsilon)$  for any  $\varepsilon \geq 0$  such that

$$\mu_{\infty, \mathbf{m}} = \left( \mathbf{m} - \|\rho_{\infty}\|_{L^1(B_R)} \right) \delta_0 + \rho_{\infty}.$$

Let us now characterise this measure. For  $\varphi \in C_c^\infty(\mathbb{R}^n)$  we take

$$\psi(x) = \left( \varphi(x) \int_{B_R} \rho_{\infty}(y) dy - \int_{B_R} \varphi(y) \rho_{\infty}(y) dy \right) \rho_{\infty}(x). \tag{6.5}$$

For  $\varphi$  fixed, there is  $\varepsilon_0 > 0$  such that for  $\varepsilon < \varepsilon_0$ ,  $\mu_{\infty, \mathbf{m}} + \varepsilon\psi \in \mathcal{P}_{\mathbf{m}}(\mathbb{R}^n)$  and, hence,

$$\mathcal{F}_R[\rho_{\infty}] = \widetilde{\mathcal{F}}_R[\mu_{\infty, \mathbf{m}}] \leq \widetilde{\mathcal{F}}_R[\mu_{\infty, \mathbf{m}} + \varepsilon\psi].$$

Hence, we get the expression

$$\mathcal{E}_{m,R}[\rho_{\infty} + \varepsilon\psi] - \mathcal{E}_{m,R}[\rho_{\infty}] + \varepsilon \int_{B_R} V(x)\psi(x) dx \geq 0.$$

We write

$$\begin{aligned} & \frac{\mathcal{E}_{m,R}[\rho_{\infty} + \varepsilon\psi] - \mathcal{E}_{m,R}[\rho_{\infty}]}{\varepsilon} \\ &= \frac{m}{m-1} \int_0^1 \left( \int_{B_R} |\rho(x) + t\varepsilon\psi(x)|^{m-2} (\rho(x) + t\varepsilon\psi(x)) \psi(x) dx \right) dt. \end{aligned}$$

Since we have the estimate

$$\left| \int_{\Omega} |\rho(x) + t\varepsilon\psi(x)|^{m-2} (\rho(x) + t\varepsilon\psi(x)) \psi(x) dx \right| \leq (\|\rho_{\infty}\|_{L^m} + \varepsilon_0 \|\psi\|_{L^m})^{m-1} \|\psi\|_{L^m},$$

we recover by the Dominated Convergence Theorem

$$\lim_{\varepsilon \rightarrow 0} \frac{\mathcal{E}_{m,R}[\rho_{\infty} + \varepsilon\psi] - \mathcal{E}_{m,R}[\rho_{\infty}]}{\varepsilon} = \frac{m}{m-1} \int_{B_R} \rho_{\infty}^{m-1} \psi.$$

Thus, as  $\varepsilon \rightarrow 0$  the following inequality holds

$$\int_{B_R} I[\rho_{\infty}] \psi \geq 0, \quad \text{with } I[\rho] := \frac{m}{m-1} \rho^{m-1} + V.$$

Applying the same reasoning for  $-\psi$  (which corresponds to taking  $-\varphi$  instead of  $\varphi$ ), we deduce the reversed inequality, and hence the equality to 0. This means that

$$\begin{aligned} 0 &= \int_{B_R} I[\rho_{\infty}](x) \varphi(x) \rho_{\infty}(x) \left( \int_{B_R} \rho_{\infty}(y) dy \right) dx \\ &\quad - \int_{B_R} \left( \int_{B_R} \varphi(y) \rho_{\infty}(y) dy \right) I[\rho_{\infty}](x) \rho_{\infty}(x) dx \end{aligned}$$

$$= \int_{B_R} \varphi(x) \rho_\infty(x) \left( I[\rho_\infty](x) \left( \int_{B_R} \rho_\infty(y) dy \right) - \int_{B_R} I[\rho_\infty](y) \rho_\infty(y) dy \right) dx$$

Since  $m\delta_0$  is not a minimiser (see Remark 6.3) then  $\rho_\infty \neq 0$ . As  $\varphi$  concentrates to a point, we recover for a.e.  $x$  either

$$\rho_\infty(x) = 0 \quad \text{or} \quad I[\rho_\infty](x) = \frac{\int_{B_R} I[\rho_\infty](y) \rho_\infty(y) dy}{\int_{B_R} \rho_\infty(y) dy} =: \mathcal{C}[\rho_\infty].$$

Notice that the right hand of the second term is a constant. Since  $\rho_\infty$  is radially decreasing then there exists  $R_\infty > 0$  such that

$$\rho_\infty(x) = \begin{cases} \left(\frac{1-m}{m}(V - \mathcal{C}[\rho_\infty])\right)^{-\frac{1}{1-m}} & |x| \leq R_\infty, \\ 0 & R_\infty < |x| < R \end{cases} = \rho_{V+h}(x) \chi_{B_{R_\infty}}(x)$$

where, by evaluating close to 0 we deduce that  $h = -\mathcal{C}[\rho_\infty] \geq 0$ . Notice that  $\rho_\infty$  is the minimiser of the two variable function

$$\begin{aligned} f(\tau, h) &= \widetilde{\mathcal{F}}_R \left[ \rho_{V+h} \chi_{B_\tau} + \left( m - \int_{B_\tau} \rho_{V+h} \right) \delta_0 \right] = \mathcal{F}_R[\rho_{V+h} \chi_{B_\tau}] \\ &= |\partial B_1| \int_0^\tau \left( -\frac{1}{1-m} \rho_{V+h}^m(r) + V(r) \rho_{V+h}(r) \right) r^{n-1} dr \end{aligned}$$

under the total mass constraint that

$$|\partial B_1| \int_0^\tau \rho_{V+h}(r) r^{n-1} dr \leq m.$$

It is not a difficult exercise to check that the minimum is achieved with  $R_\infty = R$  and  $h$  as small as possible. When  $m > a_{V,R}$  (which corresponds to  $h = 0$ ) we have to add a Dirac Delta at the origin, with the difference of the masses  $m - a_{V,R}$ .

To check this, first we point out that

$$\begin{aligned} -\frac{1}{1-m} \rho_{V+h}^m + V \rho_{V+h} &= \rho_{V+h} \left( -\frac{1}{1-m} \rho_{V+h}^{m-1} + V \right) \\ &= -\rho_{V+h} \left( \frac{1-m}{m} V + \frac{1}{m} h \right) < 0 \quad \text{for all } r > 0. \end{aligned}$$

We deduce that  $f \leq 0$  and increasing the integration domain decreases  $f$ , i.e.  $\frac{\partial f}{\partial \tau} < 0$  for all  $\tau, h > 0$ . On the other hand, since  $\frac{\partial \rho_{V+h}}{\partial h} < 0$  for  $r, h > 0$  we have that

$$\begin{aligned} \frac{\partial f}{\partial h} &= |\partial B_1| \int_0^\tau \left( \frac{m}{m-1} \rho_{V+h}^{m-1} + V \right) \frac{\partial \rho_{V+h}}{\partial h} r^{n-1} dr \\ &= -h |\partial B_1| \int_0^\tau \frac{\partial \rho_{V+h}}{\partial h} r^{n-1} dr > 0. \end{aligned}$$

Hence, the derivative is not achieved at interior points. We look at the boundaries of the domain:

1. The segment  $(\tau, h) \in \{0\} \times [0, +\infty)$ , where  $f = 0$ . These are all maximisers.
2. The segment  $(\tau, h) \in \{R\} \times [h_0, +\infty)$ , where  $h_0$  is the minimum value so that  $\int_{B_R} \rho_{V+h} \leq \mathbf{m}$ . If  $\mathbf{m} > a_{V,R}$ , then  $h_0 = 0$ . Using the derivative respect to  $h$ , the minimum in this segment is achieved at  $(R, h_0)$ .
3. A segment  $(\tau, h) \in [0, R_0] \times \{0\}$ , where  $R_0$  is such that  $\int_{B_{R_0}} \rho_V = \mathbf{m}$ . If  $\mathbf{m} > a_{V,R}$ , then  $R_0 = R$ . Using the derivative respect to  $\tau$ , the minimum in this segment is achieved at  $(R_0, 0)$ .

Hence, if  $\mathbf{m} \geq a_{V,R}$ , the minimum is achieved at  $\tau = R$  and  $h = 0$ . The remaining mass is completed with a Dirac. Lastly, if  $\mathbf{m} < a_{V,R}$  there is an extra part of the boundary, where the mass condition is achieved with equality

4. The curve  $(\tau, \bar{h}(\tau))$  such that

$$|\partial B_1| \int_0^\tau \rho_{V+\bar{h}}(r)r^{n-1}dr = \mathbf{m}.$$

Notice that this segment contains the minima of the other segments. Taking a derivative respect to  $\tau$  we deduce that

$$\frac{d\bar{h}}{d\tau} = -\frac{\rho_{V+\bar{h}}(\tau)\tau^{n-1}}{\int_0^\tau \frac{\partial \rho_{V+\bar{h}}}{\partial h}(r)r^{n-1}dr} \geq 0.$$

Therefore, using Leibniz’s rule again we recover

$$\begin{aligned} \frac{d}{d\tau} \left( f(\tau, \bar{h}(\tau)) \right) &= -|\partial B_1| \frac{d}{d\tau} \int_0^\tau \rho_{V+\bar{h}} \left( \frac{1-m}{m}V + \frac{1}{m}\bar{h} \right) r^{n-1}dr \\ &= -|\partial B_1| \rho_{V+\bar{h}}(\tau) \left( \frac{1-m}{m}V(\tau) + \frac{1}{m}\bar{h} \right) \tau^{n-1} \\ &\quad - |\partial B_1| \frac{d\bar{h}}{d\tau} \int_0^\tau \left( \frac{\partial \rho_{V+\bar{h}}}{\partial h} \left( \frac{1-m}{m}V + \frac{1}{m}\bar{h} \right) + \rho_{V+\bar{h}} \right) r^{n-1}dr \leq 0. \end{aligned}$$

Finally, the minimum is achieved for the lasted  $\tau$ , so again the minimum is  $(R, h_0)$ .  $\square$

**Remark 6.3** ( $\mathbf{m}\delta_0$  is not a minimiser). Let  $\rho \in L^1_+(B_R)$  smooth be fixed and let us consider the dilations  $\rho_s(x) = s^n \rho(sx)$  for  $s \geq 1$ . Notice that  $\rho_s \rightarrow \delta_0$  as  $s \rightarrow +\infty$  in the weak- $\star$  of  $\mathcal{M}(\overline{B_R})$ . As  $s \rightarrow \infty$  we can compute

$$\begin{aligned} \mathcal{F}_R[\rho_s] &= \frac{s^{n(m-1)}}{m-1} \int_{B_R} \rho(x)^m dx + \int_{B_R} V(s^{-1}x)\rho(x)dx \\ &\rightarrow 0 + V(0) \int_{B_R} \rho(x)dx = 0. \end{aligned}$$

It is not difficult see that  $\mathcal{F}_R$  takes negative values, so this is not a minimiser.

In [14] the authors prove that in  $\mathbb{R}^n$  if  $\rho_{V+h_1} \leq \rho_0 \leq \rho_{V+h_2}$  then  $\rho(t) \rightarrow \rho_{V+h}$  of the same initial mass. This shows that  $\mu_{\infty, \mathbf{m}} = \rho_{V+h}$  is attractive in the cases without Dirac Delta concentration at the origin.

We have constructed initial data  $\rho_0 > \rho_V$  such that  $\rho(t) \rightarrow \mu_{\infty,m}$  in the sense of their mass functions. Furthermore, we show that

**Lemma 6.4** (*Minimisation of  $\mathcal{F}_R$  through solution of  $(P_R)$* ). *Assume  $\rho_V \leq \rho_0$ , (1.11),  $a_{V,R} < a_{0,R} = \|\rho_0\|_{L^1(B_R)}$  and let  $\rho$  be constructed in Theorem 1.1. Then*

$$\mathcal{F}_R[\rho(t)] \searrow \widetilde{\mathcal{F}}_R[\mu_{\infty,a_{0,R}}] = \mathcal{F}_R[\rho_V].$$

**Proof.** From the gradient flow structure we know  $\mathcal{F}[\rho(t)]$  is non-increasing. First, we prove  $\rho(t) \rightarrow \rho_V$  in  $L^1(B_R \setminus B_\varepsilon)$  for some  $\varepsilon$  small. We know that  $\rho(t) \geq \rho_V$  so

$$\begin{aligned} \int_{B_R \setminus B_\varepsilon} |\rho(t) - \rho_V| &= \int_{B_R \setminus B_\varepsilon} (\rho(t) - \rho_V) \\ &= M(t, R) - M(t, \varepsilon) - (M_V(R) - M_V(\varepsilon)) \\ &= (a_{0,R} - a_{V,R}) + M_V(\varepsilon) - M(t, \varepsilon) \rightarrow 0 \end{aligned}$$

as  $t \rightarrow \infty$  due to Theorem 1.1. Now we can explicitly compute

$$\begin{aligned} &|\mathcal{F}[\rho(t)] - \mathcal{F}[\rho_V]| \\ &\leq \left| \frac{1}{1-m} \int_{B_\varepsilon} (\rho(t)^m - \rho_V^m) \right| + \left| \frac{1}{1-m} \int_{B_R \setminus B_\varepsilon} (\rho(t)^m - \rho_V^m) \right| \\ &\quad + \left| \int_{B_\varepsilon} V(\rho(t) - \rho_V) \right| + \left| \int_{B_R \setminus B_\varepsilon} V(\rho(t) - \rho_V) \right| \\ &\leq \frac{|B_\varepsilon|^{1-m}}{1-m} (\|\rho(t)\|_{L^1}^m + \|\rho_V\|_{L^1}^m) + \left| \frac{1}{1-m} \int_{B_R \setminus B_\varepsilon} (\rho(t)^m - \rho_V^m) \right| \\ &\quad + \left( \sup_{x \in B_\varepsilon} V(x) \right) (\|\rho(t)\|_{L^1} + \|\rho_V\|_{L^1}) \\ &\quad + \left( \sup_{x \in B_R} V(x) \right) \int_{B_R \setminus B_\varepsilon} |\rho(t) - \rho_V|. \end{aligned}$$

Due to the  $L^1$  convergence, we can extract a sequence  $t_k \rightarrow \infty$  such that  $\rho(t_k) \rightarrow \rho_V$  a.e. in  $B_R \setminus B_\varepsilon$ . For this subsequence, due to Fatou’s lemma and  $\rho(t) \geq \rho_V$  we have

$$\int_{B_R \setminus B_\varepsilon} \rho(t_k)^m \rightarrow \int_{B_R \setminus B_\varepsilon} \rho_V^m.$$

Collecting the above estimates, we conclude that

$$\begin{aligned} \limsup_{k \rightarrow \infty} |\mathcal{F}[\rho(t_k)] - \mathcal{F}[\rho_V]| &\leq \frac{1}{1-m} |B_\varepsilon|^{1-m} (\|\rho(t)\|_{L^1}^m \\ &\quad + \|\rho_V\|_{L^1}^m) + \left( \sup_{x \in B_\varepsilon} V(x) \right) (\|\rho(t)\|_{L^1} + \|\rho_V\|_{L^1}) \end{aligned}$$

for any  $\varepsilon > 0$ . Letting  $\varepsilon \rightarrow 0$  we recover that  $\limsup_k$  is actually a  $\lim_k$ , and it is equal to 0. Since  $\mathcal{F}[\rho(t)]$  is non-increasing, we recover the limit as  $t \rightarrow \infty$ .  $\square$

**7. The problem in  $\mathbb{R}^n$**

We start by showing the existence of a viscosity solution of the mass equation (M), by letting  $R \rightarrow +\infty$ . As  $R \rightarrow \infty$  we can modify  $V_R$  only on  $(R-1) < |x| < R$  to have  $\nabla V_R(x) \cdot x = 0$  for  $|x| = R$ . Fix  $\rho_0 \in L^1(\mathbb{R}^n)$  radially symmetric. Let  $M_R$  be the solution of the mass equation with this data. Consider the extension

$$\widetilde{M}_R(t, v) = \begin{cases} M_R(t, v) & v \leq R_v, \\ \|\rho_0\|_{L^1(B_R)} & v > R_v, \end{cases}$$

where, as above, we denote  $R_v = R^n|B_1|$ . Since  $\|\widetilde{M}_R\|_{L^\infty((0,\infty) \times (0,\infty))}$  we have that, up to a subsequence

$$\widetilde{M}_{R_k} \rightharpoonup M \text{ weak-}\star \text{ in } L^\infty((0, \infty)^2).$$

We can carry the estimate in  $C^\alpha([T_1, T_2] \times [v_1, v_2])$  given in (4.3), which is uniform in  $R$  since  $\|M_R\|_{L^\infty} \leq 1$ , for any  $0 < T_1 < T_2 < \infty$  and  $0 < v_1 < v_2 < R_v$ .

Now we show  $M$  is a viscosity solution. Due to the uniform continuity provided by Theorem 4.9 and the Ascoli-Arzelá theorem, for any  $K = [0, T] \times [v_1, v_2]$  with  $v_1, v_2, T > 0$ , we have a further subsequence that converges in  $C(K)$  to some function  $\widetilde{M}$  the uniform continuity. It is easy to characterise  $\widetilde{M} = M$  almost everywhere. Due to the uniform convergence, we preserve the value of  $M(0, v) = M_R(0, v)$  for  $v \leq R_v$ . Applying the same stability arguments for viscosity solutions as in Theorem 4.9,  $M$  is a viscosity solution of the mass equation (M).

**Proposition 7.1.** *Assume  $V \in W_{loc}^{2,\infty}(\mathbb{R}^n)$  is radially symmetric, strictly increasing,  $V \geq 0$ ,  $V(0) = 0$  and the technical assumption (1.5). Let  $\rho_0 \in L^1(\mathbb{R}^n)$  be radially symmetric such that  $\|\rho_0\|_{L^1} = 1$ . Then, there exists  $M \in C_{loc}([0, +\infty] \times (0, +\infty))$  a viscosity solution of (M) in  $(0, \infty) \times (0, \infty)$  that satisfies the initial condition*

$$M(0, v) = \int_{\widetilde{B}_v} \rho_0(x) dx.$$

We also have the  $C_{loc}^\alpha$  interior regularity estimate (4.3) with  $R_v = \infty$ .

Notice that, at this point, we do not check that  $M(t, 0) = 0$ , and hence concentration in finite time may, in principle, happen in  $\mathbb{R}^n$ . We also do not show, at this point, that  $M(t, \infty) = 1$ . There could, in principle, be loss of mass at infinity.

**Remark 7.2** (Conservation of total mass if  $m \in (\frac{n-2}{n}, 1)$ ). For this we use the following comparison. We consider  $\underline{u}_k$  the solution of the pure-diffusion equation

$$\begin{cases} \underline{u}_t = \Delta \Phi_k(\underline{u}) & t > 0, x \in B_R, \\ \partial_n \underline{u} = 0 & t > 0, x \in \partial B_R \\ \underline{u}(0, x) = u_0(x). \end{cases}$$

Then the associated mass satisfies the equation

$$\begin{cases} \frac{\partial M}{\partial t} = (n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \frac{\partial}{\partial v} \Phi_k \left( \frac{\partial M}{\partial v} \right) & t > 0, v \in (0, R_v), \\ \underline{M}(t, 0) = 0, & t > 0 \\ \underline{M}(t, R_v) = \|u_0\|_{L^1(B_R)} & t > 0. \end{cases}$$

If  $u_0 \geq 0$  is radially decreasing, then so is  $\frac{\partial M_k}{\partial v} = \underline{u}$ . Therefore, in the viscosity sense

$$\frac{\partial M}{\partial t} \leq (n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left\{ \frac{\partial}{\partial v} \Phi_k \left( \frac{\partial M}{\partial v} \right) + \frac{\partial M}{\partial v} \frac{\partial V}{\partial v} \right\}.$$

Let  $u$  be the solution of  $(P_{\Phi, R})$ . Due to Theorem 4.9 we have that

$$\underline{M}(t, v) \leq \int_{\tilde{B}_v} u(t, x) dx.$$

Recalling the limit through  $\Phi_k$  given by (3.1) and the limit  $R \rightarrow \infty$ , the mass constructed in Proposition 7.1 we have the estimate

$$\int_{\tilde{B}_v} \underline{u}(t, x) dx \leq M(t, v) \leq 1.$$

where  $\underline{u}$  is the solution of  $\underline{u}_t = \Delta \underline{u}^m$  in  $\mathbb{R}^n$ . When  $m \in (\frac{n-2}{n}, 1)$  we know that  $\int_{\mathbb{R}^n} \underline{u}(t, x) dx = \int_{\mathbb{R}^n} u_0(x) dx$  and, hence  $M(t, \infty) = 1$ .

*7.1. At least infinite-time concentration of the mass*

Let assume  $a_V < 1$  and that  $\rho_0$  is such that there exists  $F$  with the following properties

$$\|\rho_F\|_{L^1(\mathbb{R}^n)} = 1 \quad \text{and} \quad M_{\rho_F} \leq M_{\rho_0} \leq (1 - a_V) + M_{\rho_V}. \tag{7.1}$$

**Remark 7.3.** For example, this covers the class of initial data that satisfy the following three assumptions:

- $M_{\rho_V} \leq M_{\rho_0} \leq (1 - a_V) + M_{\rho_V}$
- $\int_{\tilde{B}_v} \rho_0(x) dx = (1 - a_V) + \int_{\tilde{B}_v} \rho_V(x) dx$  for  $v \geq v_0$
- $M_{\rho_0}$  is Lipschitz in  $(v_0 - \varepsilon, v_0 + \varepsilon)$ .

In this setting, we can take a suitable initial datum  $\rho_D$  as in the case of balls, and we are reduced to a problem in  $[0, v_0]$ , since the upper and lower bound guarantee that  $M(t, v) = (1 - a_V) + \int_{\tilde{B}_v} \rho_V(x) dx$  for all  $v \geq v_0$ . This is a Dirichlet boundary condition for the mass.

When  $\rho_0 = \rho_F$  then the associated mass  $M$  obtained in Proposition 7.1 satisfies

1.  $M$  is a viscosity solution of the mass equation and locally  $C^\alpha$ .
2.  $M(0, v) = \int_{\tilde{B}_v} \rho_F(x) dx$ .
3.  $M$  is non-decreasing in  $t$  and  $x$  (due to the properties of the approximations).
4. We have the comparison

$$M_{\rho_V}(v) \leq M_{\rho_D}(v) \leq M(t, v) \leq (1 - a_V) + M_{\rho_V}(v).$$

In particular  $M(t, \infty) = 1$  for all  $t$  finite.

Again, there exists a point-wise limit

$$M_\infty(v) = \lim_{t \rightarrow \infty} M(t, v).$$

As in Theorem 5.4,  $M_\infty$  preserves the  $C_{loc}^\alpha$  estimates, using Dini’s theorem we can prove uniform convergence in intervals  $[\varepsilon, \varepsilon^{-1}]$ . Thus  $M_\infty$  is a viscosity solution of (5.11). Due to the sandwich theorem and monotonicity

$$M_\infty(0^+) \leq 1 - a_V, \quad M_\infty(+\infty) = 1.$$

It is easy to characterise  $M_\infty$  as we have done in the case of balls.

This proves Corollary 1.3 under hypothesis (7.1).

**Remark 7.4** (Convergence of  $\rho_R$  as  $R \rightarrow \infty$ ). Since we do not have any  $L^q$  bound for  $\rho$  for  $q > 1$ , we do not have any suitable compactness. We can extend  $\rho_R(t)$  by 0 outside  $B_R$  and we do know that  $\|\tilde{\rho}_R(t)\|_{\mathcal{M}(\mathbb{R}^n)} \leq 1$ . If we assume that (7.2) and that  $V(x) \geq c|x|^\alpha$  for  $c, \alpha > 0$ . The properties can be inherited to  $\rho_R$  so

$$\int_{B_R} |x|^\alpha \rho_R \leq C(1 + \mathcal{F}[\rho_0]).$$

For  $\rho_0$  in a suitable integrability class, we have tightness, and hence a weakly convergent subsequence such that

$$\tilde{\rho}_R \rightharpoonup \mu \quad \text{weak} - \star \text{ in } L^\infty(0, \infty; \mathcal{M}(\mathbb{R}^n))$$

We also know that  $\rho_R^m$  is uniformly integrable. However, since we cannot assure  $\rho_R^m \rightharpoonup (\mu_{ac})^m$ , we cannot characterise  $\mu$  as a solution of (P). This remark is still valid for radial initial data.

### 7.2. Minimisation of the free energy

Following the arguments in [5,13,17], we have an existence and characterisation result for the minimiser. In  $\mathbb{R}^n$  the free-energy of the FDE  $u_t = \Delta u^m$  with  $0 < m < 1$ , is not bounded below, and  $u(t) \rightarrow 0$  as  $t \rightarrow \infty$ . In fact, the mass of solutions escapes through  $\infty$  in finite time if  $m < \frac{n-2}{n}$ . We need to ask further assumptions on  $V$  so that the formal critical points  $\rho_{V+h}$  are in fact minimisers.

We show below that it suffices that  $V$  is not critical in the sense of constants, i.e.

$$\inf_{\rho \in \mathcal{P}_{ac}(\mathbb{R}^n)} \left( \frac{1}{m-1} \int_{\mathbb{R}^n} \rho^m + (1-\varepsilon) \int_{\mathbb{R}^n} V(x)\rho(x)dx \right) > -\infty, \quad \text{for some } \varepsilon > 0. \tag{7.2}$$

We provide an example of  $V$  where this property holds below. As in  $B_R$ , we define an extension of  $\mathcal{F}$  to the space of measure as

$$\tilde{\mathcal{F}}[\mu] = \mathcal{E}_m[\mu_{ac}] + \int_{\mathbb{R}^d} V(x)d\mu(x)$$

where  $\mu_{ac}$  is the absolutely continuous part of the measure  $\mu$ . Notice that, since we choose  $V(0) = 0$ , we have that  $\tilde{\mathcal{F}}[m\delta_0 + \rho] = \mathcal{F}[\rho]$ .

**Proposition 7.5.** Assume  $V \geq 0$  and  $V(0) = 0$  and (7.2). Then, we have the following:

1. There exists a constant  $C > 0$  such that

$$\int_{\mathbb{R}^n} \rho^m + \int_{\mathbb{R}^d} V\rho \leq C(1 + \mathcal{F}[\rho]).$$

If, furthermore  $V$  is radially symmetric and non-decreasing then

2. There exists  $\mu_\infty \in \mathcal{P}(\mathbb{R}^n)$  such that

$$\tilde{\mathcal{F}}[\mu_\infty] = \inf_{\mu \in \mathcal{P}(\mathbb{R}^n)} \tilde{\mathcal{F}}[\mu] = \inf_{\rho \in \mathcal{P}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)} \mathcal{F}[\rho]. \tag{7.3}$$

3. We have that

$$\mu_\infty = \begin{cases} \rho_{V+h} & \text{if } a_{V+h} = 1, \\ (1 - a_V)\delta_0 + \rho_V & \text{if } a_V < 1. \end{cases}$$

**Proof of Proposition 7.5.** Due to the lower bound, we have that

$$\frac{1}{1-m} \int_{\mathbb{R}^n} \rho^m \leq C + (1 - \varepsilon) \int_{\mathbb{R}^n} V\rho.$$

On the other hand, we get

$$\int_{\mathbb{R}^n} V\rho = \mathcal{F}[\rho] + \frac{1}{1-m} \int_{\mathbb{R}^n} \rho^m \leq F[\rho] + C + (1 - \varepsilon) \int_{\mathbb{R}^n} V\rho$$

Thus

$$\varepsilon \int_{\mathbb{R}^n} V\rho \leq \mathcal{F}[\rho] + C$$

Finally, we recover

$$\frac{1}{1-m} \int_{\mathbb{R}^n} \rho^m \leq C + \frac{(1 - \varepsilon)}{\varepsilon} (\mathcal{F}[\rho] + C).$$

This completes the proof of Item 1.

Clearly, we have that

$$\mathcal{F}[\mu] \geq \mathcal{E}_m[\rho] + (1 - \varepsilon) \int_{\mathbb{R}^n} V(x)\rho(x)dx.$$

Hence, the infimum of  $\mathcal{F}$  is finite. As in the proof of Theorem 6.1, we can consider a minimising sequence  $\rho_j$ . As in Theorem 6.1 we may assume that  $\rho_j$  are radially symmetric and non-increasing.

Let us prove Item 2. As in Theorem 6.1, the second equality of (7.3) is due to the weak- $\star$  density of  $L^1(\mathbb{R}^n)$  in the set of measures and the construction of  $\tilde{\mathcal{F}}$ . For our minimising sequence we know hence that

$$\int_{\mathbb{R}^n} \rho_j = 1, \quad \int_{\mathbb{R}^n} \rho_j^m \leq C(1 + \mathcal{F}[\rho_j]) \leq C.$$

Using Lieb’s trick in Remark 6.2, we obtain that  $\rho_j \leq C \min\{|x|^{-n}, |x|^{-n/m}\}$ . Integrating outside of any ball  $B_R$ , we can estimate

$$\int_{\mathbb{R}^n \setminus B_R} \rho_j \leq C \left( \int_R^\infty r^{-\frac{n}{m}+n-1} dr \right) \leq CR^{n(1-\frac{1}{m})}.$$

Since  $m < 1$ , this is a tight sequence of measures. By Prokhorov’s theorem, there exists a weakly- $\star$  convergent subsequence in the sense of measures. Let its limit be  $\mu_\infty$ .

For the proof of Item 3, we proceed as in Theorem 6.1. Notice that we still have the estimate

$$\rho_j(x) \leq \frac{\int_{\mathbb{R}^n} V \rho}{\int_{B_{|x|}} V}.$$

Since  $V$  is strictly increasing, this is an  $L^\infty(\mathbb{R}^n \setminus B_\kappa)$  of any  $\kappa > 0$ , and we can repeat the argument in  $B_R$ .  $\square$

Let us illustrate the previous theorem by giving sufficient conditions on  $V$  satisfying the main assumption of Proposition 7.5. We extend the argument in [16] to show a family of potentials  $V$  for which (7.2) holds.

**Theorem 7.6.** *Assume that, for some  $\alpha \in (0, m)$  we have that*

$$\chi_V = \sum_{j=1}^\infty 2^{jn} V(2^j)^{-\frac{\alpha}{1-m}} < \infty. \tag{7.4}$$

Then, (7.2) holds for any  $\varepsilon \in (0, 1)$ .

**Remark 7.7.** If the function  $r \mapsto V(r)^{-\frac{\alpha}{1-m}} r^n$  is non-increasing, then the integral criterion for series and the change of variable show that the condition becomes

$$\int_1^\infty 2^{ny} V(2^y)^{-\frac{\alpha}{1-m}} dy = \frac{1}{\ln 2} \int_2^\infty V(r)^{-\frac{\alpha}{1-m}} r^{n-1} dr \sim \int_{|x| \geq 2} \rho_V^\alpha dx < \infty.$$

We are requesting that  $\rho_V^{m-\delta} \in L^1$  for some  $\delta \in (0, m)$ . This is only slightly more restrictive than simply that  $\rho_V$  gives a finite quantity in either term of  $\mathcal{F}$ .

**Proof of Theorem 7.6.** We look first at the integral on  $B_1$ . Due to Hölder’s inequality, we have that

$$\frac{1}{m-1} \int_{B_1} \rho^m \geq \frac{|B_1|^{1-m}}{m-1} \left( \int_{B_1} \rho \right)^m.$$

On the other hand, since  $V, \rho \geq 0$  we know that  $\int_{B_1} V \rho dx \geq 0$ . Hence, we only need to care about the integration on  $\mathbb{R}^n \setminus B_1$ . We define, for  $j \geq 1$

$$\rho_j = \int_{B_{2^j} \setminus B_{2^{j-1}}} \rho(x) dx.$$

First, we point out that

$$\int_{\mathbb{R}^n \setminus B_1} V(x)\rho(x)dx \geq \sum_{j=1}^{\infty} V(2^{j-1})\rho_j.$$

Due to Jensen’s inequality

$$\begin{aligned} \int_{B_{2^j} \setminus B_{2^{j-1}}} \rho^m &\leq |B_{2^j} \setminus B_{2^{j-1}}| \left( \frac{1}{|B_{2^j} \setminus B_{2^{j-1}}|} \int_{B_{2^j} \setminus B_{2^{j-1}}} \rho(x)dx \right)^m \\ &= |B_{2^j} \setminus B_{2^{j-1}}|^{1-m} \rho_j = C_n 2^{jn(1-m)} \rho_j^m. \end{aligned}$$

Notice that  $B_{2^j} \setminus B_{2^{j-1}} = 2^j(B_1 \setminus B_{\frac{1}{2}})$ . Hence

$$\int_{\mathbb{R}^n \setminus B_1} \rho^m \leq \sum_{j=1}^{\infty} \frac{C_n 2^{jn(1-m)}}{V(2^{j-1})^\alpha} V(2^{j-1})^\alpha \rho_j^\alpha \rho_j^{m-\alpha}.$$

Applying the triple Hölder inequality with exponents  $p = (1 - m)^{-1}, q = \alpha^{-1}, r = (m - \alpha)^{-1}$  we recover

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B_1} \rho^m &\leq \left( \sum_{j=1}^{\infty} \frac{C_n 2^{jn}}{V(2^{j-1})^{\frac{\alpha}{1-m}}} \right)^{1-m} \left( \sum_{j=1}^{\infty} V(2^{j-1})\rho_j \right)^\alpha \left( \sum_{j=1}^{\infty} \rho_j \right)^{m-\alpha} \\ &\leq \chi_V^{1-m} \|\rho\|_{L^1}^{m-\alpha} \left( \int_{\mathbb{R}^n \setminus B_1} V(x)\rho(x)dx \right)^\alpha \end{aligned}$$

Lastly, using Young’s inequality we have, for any  $\varepsilon > 0$

$$\int_{\mathbb{R}^n \setminus B_1} \rho^m \leq \varepsilon(1 - m) \int_{\mathbb{R}^n \setminus B_1} V(x)\rho(x)dx + C(\varepsilon, \alpha, m)\chi_V^{\frac{1-m}{1-\alpha}} \|\rho\|_{L^1}^{\frac{m-\alpha}{1-\alpha}}$$

Therefore

$$\frac{1}{m-1} \int_{\mathbb{R}^n} \rho^m + (1 - \varepsilon) \int_{\mathbb{R}^n} V\rho \geq \frac{|B_1|^{1-m}}{m-1} \left( \int_{B_1} \rho \right)^m - \frac{C(\varepsilon, \alpha, m)}{1-m} \chi_V^{\frac{1-m}{1-\alpha}} \|\rho\|_{L^1}^{\frac{m-\alpha}{1-\alpha}}.$$

This completes the proof.  $\square$

**Remark 7.8** (The power-type case  $V(x) = C|x|^\lambda$  for  $|x| \geq R_0$ ). In this setting, (7.4) becomes  $m > \frac{n}{n+\lambda}$  (equivalently  $\frac{n(1-m)}{m} < \lambda$ ), and in this case can take any  $\alpha$  such that  $\frac{n(1-m)}{\lambda} < \alpha < m$ . This condition is sharp. Let us see that, otherwise,  $\mathcal{F}$  is not bounded below. We recall the following computation, which can be found in [16, Theorem 15] following the reasoning in [19, Theorem 4.3].

Assume  $m < \frac{n}{n+\lambda}$ . We can construct densities  $\rho$  where the energy attains  $-\infty$ . Let

$$\rho_\beta = \sum_{j=j_0}^{\infty} \frac{\rho_j}{|B_{2^{j+1}} \setminus B_{2^j}|} \chi_{B_{2^{j+1}} \setminus B_{2^j}}, \quad \text{where } \rho_j = \frac{2^{-j\beta}}{\sum_{j=j_0}^{\infty} 2^{-j\beta}},$$

where  $\beta > 0$  is a constant we will choose later, and  $j_0$  is such that  $2^{j_0} > R_0$ . We can explicitly compute

$$\int_{\mathbb{R}^n} |x|^\lambda \rho_\beta(x) dx = \frac{2^{n+\lambda} - 1}{n + \lambda} \frac{\sum_{j=j_0}^\infty 2^{-j(\beta-\lambda)}}{\sum_{j=j_0}^\infty 2^{-j\beta}}$$

This is a finite number whenever  $\beta > \lambda$ . On the other hand

$$\int_{\mathbb{R}^n} \rho_\beta(x)^m dx = C(n, \lambda) \frac{\sum_{j=j_0}^\infty 2^{-j(m\beta-n(1-m))}}{\left(\sum_{j=j_0}^\infty 2^{-j\beta}\right)^m}$$

This number is infinite if  $m\beta < n(1 - m)$ . Hence,

$$-\frac{1}{1-m} \int_{\mathbb{R}^n} \rho_\beta(x)^m dx + \int_{\mathbb{R}^n} C|x|^\lambda \rho_\beta(x) dx = -\infty, \quad \forall C \in \mathbb{R} \text{ and } \lambda < \beta < \frac{n(1-m)}{m}.$$

The case of the equality  $m = \frac{n}{n+\lambda}$  is, as usual, more delicate due to the scaling. However, we still prove that

$$\inf_{\rho \in \mathcal{P} \cap L^1} \left( - \int_{\mathbb{R}^n} \rho^{\frac{n}{n+\lambda}} + C \int_{\mathbb{R}^n} |x|^\lambda \rho \right) = -\infty, \quad \forall C \in \mathbb{R}.$$

As in the proof of [17, Proposition 4], we can take the following functions:

$$\rho_k(x) = D_k |x|^{-(n+\lambda)} \chi_{B_k \setminus B_{R_0}}, \quad \text{where } D_k = \left( \int_{B_k \setminus B_{R_0}} |x|^{-(n+\lambda)} dx \right)^{-1}.$$

It is a direct computation that

$$\frac{1}{D_k} \int_{\mathbb{R}^n} |x|^\lambda \rho_k = \int_{B_k \setminus B_{R_0}} |x|^{-n} dx = \frac{1}{D_k^{\frac{n}{n+\lambda}}} \int_{\mathbb{R}^n} \rho_k^{\frac{n}{n+\lambda}}.$$

For any  $\alpha_k > 0$ , we have that the rescaling  $\tilde{\rho}_k(x) = \alpha_k^n \rho_k(\alpha_k x)$  is such that

$$\begin{aligned} a_k &:= \frac{\int_{\mathbb{R}^n} |x|^\lambda \tilde{\rho}_k}{\left(\int_{\mathbb{R}^n} \tilde{\rho}_k^{\frac{n}{n+\lambda}}\right)^{\frac{n+\lambda}{n}}} = \frac{\int_{\mathbb{R}^n} |x|^\lambda \rho_k}{\left(\int_{\mathbb{R}^n} \rho_k^{\frac{n}{n+\lambda}}\right)^{\frac{n+\lambda}{n}}} = \left( \int_{B_k \setminus B_{R_0}} |x|^{-n} dx \right)^{1 - \frac{n+\lambda}{n}} \\ &= \left( |\partial B_1| \log \frac{k}{R_0} \right)^{1 - \frac{n+\lambda}{n}} \rightarrow 0. \end{aligned}$$

For any sequence  $b_k$  which is yet to be determined, we can pick  $\alpha_k$  so that  $\int_{\mathbb{R}^n} \tilde{\rho}_k^{\frac{n}{n+\lambda}} = b_k$  by taking

$$\alpha_k = \left( \frac{b_k}{\int_{\mathbb{R}^n} \rho_k^{\frac{n}{n+\lambda}}} \right)^{-\frac{n+\lambda}{\lambda n}}.$$

Then, passing to the notation  $m = \frac{n}{n+\lambda}$ , we recover that

$$-\int_{\mathbb{R}^n} \tilde{\rho}_k^m + C \int_{\mathbb{R}^n} |x|^\lambda \tilde{\rho}_k = -b_k + C a_k b_k^{\frac{1}{m}} = -b_k^{\frac{1}{m}} \left( b_k^{-\frac{1-m}{m}} - C a_k \right) = -b_k^{\frac{1}{m} - \varepsilon},$$

if pick the sequence  $b_k$  so that  $b_k^{-\frac{1-m}{m}} - Ca_k = b_k^{-\varepsilon}$ . Notice that the function  $g_{a,b}(s) = s^a - s^b$  is strictly increasing near 0 if  $a < b$ . Hence, for  $k$  large enough and  $\varepsilon > \frac{1-m}{m}$ , we can solve  $Ca_k = b_k^{-\frac{1-m}{m}} - b_k^{-\varepsilon}$ , and we recover  $b_k \rightarrow +\infty$  as  $k \rightarrow \infty$ . Hence, taking  $\varepsilon \in (\frac{1-m}{m}, \frac{1}{m})$ , and  $k \rightarrow \infty$ , we prove the result.

**Remark 7.9.** With the sequence  $\rho_k$  above, we can also prove that

$$\inf_{\rho \in \mathcal{P} \cap L^1} \frac{\int_{\mathbb{R}^n} |x|^\lambda \rho}{\left(\int_{\mathbb{R}^n} \rho^{\frac{n}{n+\lambda}}\right)^{\frac{n+\lambda}{n}}} = 0. \tag{7.5}$$

This corresponds to the borderline case of the Carlson type inequalities

$$\left(\int_{\mathbb{R}^n} \rho\right)^{1-\frac{n(1-m)}{\lambda m}} \left(\int_{\mathbb{R}^n} |x|^\lambda \rho\right)^{\frac{n(1-m)}{\lambda m}} \geq c_{n,\lambda,m} \left(\int_{\mathbb{R}^n} \rho^m\right)^{\frac{1}{m}},$$

for all  $\frac{n}{n+\lambda} < m < 1$  and  $\rho \geq 0$ , which are known with the explicit constant (see, e.g., [17, Lemma 5]).

*7.3. Infinite-time concentration if  $V$  is quadratic at 0*

Our aim in this section is to compare the solutions of (P) with the solutions of the pure-aggregation problem

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla \tilde{V}), \tag{7.6}$$

where  $\tilde{V}$  is a different potential. The equation for the mass can be written in radial coordinates as

$$\frac{\partial M}{\partial t} = \frac{\partial M}{\partial r} \frac{\partial \tilde{V}}{\partial r}. \tag{7.7}$$

We will show that infinite-time aggregation happens for (7.6) if and only if

$$\int_{0^+} \left(\frac{\partial \tilde{V}}{\partial r}(s)\right)^{-1} ds = +\infty. \tag{7.8}$$

Clearly, a sufficient condition that  $\frac{\partial \tilde{V}}{\partial r} \leq Cr$  near 0. This is the so-called Osgood condition used to distinguish infinite from finite time blow-up in aggregation equations [8].

**Proposition 7.10.** *Assume  $\tilde{V} \in C^2(\mathbb{R}^n)$ , is radially symmetric,  $\tilde{V}(0) = 0$ ,  $\frac{\partial \tilde{V}}{\partial r}(r) > 0$  for  $r > 0$ , (7.8) and let  $M_0$  be a continuous, non-decreasing and bounded function. Then*

1. *There exists a unique classical solution by characteristics  $M(t, r)$  of (7.7) defined for all  $t, r > 0$ .*
2. *We have  $M(t, 0) = 0$  for all  $t > 0$ , i.e. there is no concentration in finite time.*

**Proof of Proposition 7.10.** Equation (7.7) is a first order linear PDE that we can solve by characteristics. We can look at the characteristic curves of constant mass  $\bar{M}(t, r_c(t, r_0)) = \bar{M}(0, r_0)$ . Taking a derivative we recover  $\frac{dr_c}{dt}(t) = -\frac{\partial \tilde{V}}{\partial r}(r_c(t))$ . These are the same characteristics obtained when applying the method directly to (7.6). Clearly  $r_c(t, r_0) \leq r_0$ . Since  $V \in C^2(\mathbb{R}^n)$ , these characteristics exist for some time  $t(r_0) > 0$ , and are unique up to that time. Hence, let

$$t = \int_{r_c(t,r_0)}^{r_0} \left( \frac{\partial \tilde{V}}{\partial r}(s) \right)^{-1} ds. \tag{7.9}$$

Concentration will occur if  $r_c(t, r_0) = 0$  for some  $r_0 > 0$  and  $t < \infty$ , which is incompatible with (7.8). Notice that since  $0 < r_c(t, r_0) \leq r_0$ , these functions are defined for all  $t > 0$ . Let us check that  $r_c(t, r_0)$  do not cross, and hence can be used as characteristics. If two of them cross at time  $t$ , we have that

$$\int_{r_0}^{r_c(t,r_0)} \left( \frac{\partial \tilde{V}}{\partial r}(s) \right)^{-1} ds = -t = \int_{r_1}^{r_c(t,r_1)} \left( \frac{\partial \tilde{V}}{\partial r}(s) \right)^{-1} ds.$$

Since  $r_c(t, r_0) = r_c(t, r_1)$  then we get

$$\int_{r_0}^{r_1} \left( \frac{\partial \tilde{V}}{\partial r}(s) \right)^{-1} ds = 0.$$

As  $\frac{\partial \tilde{V}}{\partial r} > 0$  outside 0, then  $r_0 = r_1$  and the characteristics are the same. Due to the regularity of  $\tilde{V}$ , there is continuous dependence and, since the characteristics point inwards and do not cross, they fill the space  $[0, +\infty) \times [0, +\infty)$ .

Finally, notice also that  $\frac{\partial \tilde{V}}{\partial r}(0) = 0$  and positive otherwise, then for any  $r_0 > 0$  we have that  $\lim_{t \rightarrow +\infty} r_c(t, r_0) = 0$ . Since  $\tilde{V}$  is  $C^2$ , then we have  $\partial \tilde{V} / \partial r(0) = 0$  so  $r_c(t, 0) = 0$ , i.e.  $M(t, 0) = 0$ .  $\square$

**Proposition 7.11.** *Let  $\rho$  be a solution by characteristics of the aggregation equation (7.6), and let  $r_0(t, r)$  the foot of the characteristic through  $(t, r)$ . Then*

$$\frac{\partial \rho}{\partial r}(t, r) = \left( \frac{r_0}{r} \right)^{n-1} \rho_0(r_0) \frac{\frac{\partial \tilde{V}}{\partial r}(r_0)}{\left( \frac{\partial \tilde{V}}{\partial r}(r) \right)^2} \left( -\Delta \tilde{V}(r) + \Delta \tilde{V}(r_0) + \rho_0(r_0)^{-1} \frac{d\rho_0}{dr}(r_0) \frac{\partial \tilde{V}}{\partial r}(r_0) \right). \tag{7.10}$$

In particular, if  $\rho$  is a decreasing solution and  $\tilde{V} \in C^2(\mathbb{R}^n)$  with  $\Delta \tilde{V}(0) = 0$ , then

$$\Delta \tilde{V} + \rho_0^{-1} \frac{d\rho_0}{dr} \frac{\partial \tilde{V}}{\partial r} \leq 0, \quad \text{in } \text{supp } \rho_0. \tag{7.11}$$

**Remark 7.12.** For  $\tilde{V}(r) = r^2$  then  $\Delta \tilde{V}$  is constant, and we only have the last term, so all solutions with decreasing initial datum are decreasing. If  $\Delta \tilde{V}$  is non-increasing, then in (7.10) we have  $-\Delta \tilde{V}(r) + \Delta \tilde{V}(r_0) \leq 0$  and all solutions are decreasing. This is the case for  $\tilde{V}(r) = \gamma r^\lambda$  with  $\lambda \in (0, 2]$ . When  $\tilde{V}(r) = \gamma r^\lambda$  with  $\lambda > 2$ , let us show that decreasing solutions of (7.6) are not  $L^1(\mathbb{R}^n)$ . Hence, any decreasing integrable initial data produces a solution that losses monotonicity. Indeed, if  $\tilde{V}(r) = r^\lambda$  then  $\Delta \tilde{V} = (n + \lambda - 2)r^{\lambda-2}$  and integrating in (7.11) we recover  $\rho_0 \geq Cr^{-(n+\lambda-2)}$  which is not integrable for  $\lambda > 2$ .

**Proof of Proposition 7.11.** Taking the derivative directly on  $M(t, r) = n\omega_n r^{n-1} \frac{\partial \rho}{\partial r}$ , we recover that

$$\begin{aligned} \frac{\partial \rho}{\partial r}(t, r) &= (n\omega_n)^{-1} \frac{\partial}{\partial r} \left( r^{1-n} \frac{\partial}{\partial r}(M(t, r)) \right) = (n\omega_n)^{-1} \frac{\partial}{\partial r} \left( r^{1-n} \frac{\partial}{\partial r}(M_0(r_0(t, r))) \right) \\ &= r^{1-n} r_0^{n-1} \frac{\partial r_0}{\partial r}(t, r) \rho_0(r_0) \left( -(n-1)r^{-1} + (n-1)r_0^{-1} \frac{\partial r_0}{\partial r} \right) \end{aligned}$$

$$+ \rho_0(r_0)^{-1} \frac{d\rho_0}{dr}(r_0) \frac{\partial r_0}{\partial r} + \frac{\frac{\partial^2 r_0}{\partial r^2}}{\frac{\partial r_0}{\partial r}} \Bigg).$$

Going back to (7.9) and taking a derivative in  $r$ , we deduce

$$\frac{\partial r_0}{\partial r}(t, r) = \frac{\frac{\partial \tilde{V}}{\partial r}(r_0(t, r))}{\frac{\partial \tilde{V}}{\partial r}(r)} \geq 0.$$

Taking another derivative we have that

$$\frac{\partial^2 r_0}{\partial r^2}(t, r) = \frac{\frac{\partial \tilde{V}}{\partial r}(r_0(t, r))}{\frac{\partial \tilde{V}}{\partial r}(r)^2} \left( \frac{\partial^2 \tilde{V}}{\partial r^2}(r_0) - \frac{\partial^2 \tilde{V}}{\partial r^2}(r) \right).$$

Joining this information and collecting terms we recover (7.10). Clearly, (7.11) and the convexity of  $\rho_0$  guarantee that  $\rho(t, \cdot)$  is decreasing. Let us show that the condition holds in general. If  $\rho$  is decreasing, then this value is not positive. For  $r_0 \in \text{supp } \rho_0$  we therefore have

$$-\Delta \tilde{V}(r) + \Delta \tilde{V}(r_0) + \rho_0(r_0)^{-1} \frac{d\rho_0}{dr}(r_0) \frac{\partial \tilde{V}}{\partial r}(r_0) \leq 0.$$

The support of  $\rho_0$  is a ball. Fixing a value of  $r \in \text{supp } \rho_0$  we have that

$$-\Delta \tilde{V}(r_c(t, r)) + \Delta \tilde{V}(r) + \rho_0(r)^{-1} \frac{d\rho_0}{dr}(r) \frac{\partial \tilde{V}}{\partial r}(r) \leq 0.$$

Letting  $t \rightarrow +\infty$ , since  $r_c(t, r) \rightarrow 0$ ,  $\Delta \tilde{V}$  is continuous and  $\Delta \tilde{V}(0) = 0$ , we recover (7.11). This completes the proof.  $\square$

Now we have the tools to show that concentration does not happen in finite time if  $\frac{\partial V}{\partial r} \leq C_v r$  close to 0. We construct a super-solution using the pure-aggregation equation.

**Proof of Theorem 1.4.** Take

$$\bar{\rho}_0(x) = \rho_0(x) \frac{\int_{\mathbb{R}^n} \rho_0(x) dx}{\int_{B_{R_V}} \rho_0(x) dx} \chi_{B_{R_V}}$$

and

$$\tilde{V}(r) = \frac{C_V}{2} r^2. \tag{7.12}$$

Obtain  $\bar{M}$  as the solution by characteristics of (7.7) constructed in Proposition 7.10. Due the definition of  $\tilde{V}$ , we know that it satisfies the hypothesis of Proposition 7.11 and we have  $\Delta \tilde{V} = nC_V \geq 0$ . Thus, (7.10) shows that  $\bar{\rho}(t, \cdot)$  is decreasing, and non-negative. Therefore, it holds that, in the viscosity sense  $\frac{\partial \bar{M}}{\partial v} \geq 0$  and  $\frac{\partial^2 \bar{M}}{\partial v^2} \geq 0$ . Hence, still in the viscosity sense

$$\begin{aligned} & \frac{\partial \bar{M}}{\partial t} - (n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left\{ m \left( \frac{\partial \bar{M}}{\partial v} \right)^{m-1} \frac{\partial^2 \bar{M}}{\partial v^2} + \frac{\partial \bar{M}}{\partial v} \frac{\partial V}{\partial v} \right\} \\ & \geq \frac{\partial \bar{M}}{\partial t} - (n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left\{ \frac{\partial \bar{M}}{\partial v} \frac{\partial V}{\partial v} \right\} \end{aligned}$$

$$= \frac{\partial \overline{M}}{\partial t} - \frac{\partial \overline{M}}{\partial r} \frac{\partial V}{\partial r} = \frac{\partial \overline{M}}{\partial r} \left( C_V r - \frac{\partial V}{\partial r} \right).$$

Since characteristics retract,  $\text{supp } \frac{\partial \overline{M}}{\partial v} \subset B_{R_V}$  so the last term is non-negative by the assumption, because either  $\frac{\partial \overline{M}}{\partial v} = 0$  or  $C_V r - \frac{\partial V}{\partial r} \leq 0$ . Thus, using the comparison principle in  $B_R$  for  $R \geq R_V$  given in Theorem 4.10 we have that  $M_R \leq \overline{M}$  for all  $t \geq 0, v \in [0, R_V]$ . Since  $M$  is constructed by letting  $R \rightarrow \infty$ , we conclude  $M \leq \overline{M}$  for  $t, v \geq 0$ .  $\square$

### 8. Final comments

1. Blow-up is usually associated in the literature to superlinear nonlinearities, both in reaction diffusion or in Hamilton-Jacobi equations, cf. instance [42,28] and its many references. Here it is associated to sublinear diffusion, notice that (1.13) implies, at least,  $0 < m < 1$ . This might seem surprising but it is not, due to two facts. First, recall that  $0 < m < 1$  means that the diffusion coefficient  $mu^{m-1}$  is large when  $u$  is small, and small when  $u$  is large. This translates into fast diffusion of the support but slow diffusion of level sets with high values (see e.g. [22] for a thorough discussion). This explains why  $\delta_0$  may not be diffused for  $m$  small (see [10]). Secondly, the confinement potential  $V$  needs to be strong enough at the origin to compensate the diffusion and produce concentration. In  $B_R$ , this is translated in the assumption  $\int_{B_R} \rho_V < 1$  (recall that, for  $V(x) = |x|^{\lambda_0}$ , this implies  $0 < m < \frac{n-\lambda_0}{n} < 1$ ). In  $\mathbb{R}^n$  we need to deal with the behaviour at infinity, as mentioned in the introduction.
2. Formation of a concentrated singularity in finite time is a clear possibility in this kind of problem. In this paper, we do not consider the case  $V \notin W_{loc}^{2,\infty}(\mathbb{R}^n)$  (e.g.  $V(x) = |x|^\lambda$  with  $\lambda < 2$ ). So long as  $\frac{\partial V}{\partial v}$  is continuous (e.g.  $\lambda \geq 1$ ), it makes sense to use the theory of viscosity solutions of the mass equation (M). In principle, there could be concentration in finite time, even in  $(P_R)$ . Notice that, in our results, the estimate for  $\rho(t) \in L^q(B_R)$  depends on  $\|\Delta V\|_{L^\infty(B_R)}$ . For more general  $V$ , better estimates for  $\rho$  are needed in order to pass the limits  $\Phi_k(s) \rightarrow s^m$  and  $R \rightarrow \infty$ . Some of these issues will be studied elsewhere.
3. For  $\rho_0 \in L^1_+(B_R)$ ,  $S_R(t)\rho_0$  is constructed extending the semigroup through a density argument. We do not know whether it is the limit of the solutions  $u_k$  of  $(P_{\Phi,R})$  with (3.1). Furthermore, this question can be extended to initial data so that  $\mathcal{F}_R[\rho_0] < \infty$ .
4. Non-radial data. We provide a well-posedness theory in  $B_R$  when  $\rho_0 > 0$ , but not in  $\mathbb{R}^n$ . In  $B_R$ , as mentioned in Remark 1.2, we can show concentration in some non-radial cases, but the exact splitting of mass in the asymptotic distribution is still unknown. The asymptotic behaviour in the non-radial case is completely open.

### Acknowledgements

The authors are thankful to the anonymous referee for the detailed reading of the manuscript, and their insightful suggestions. The research of JAC and DGC was supported by the Advanced Grant Nonlocal-CPD (Nonlocal PDEs for Complex Particle Dynamics: Phase Transitions, Patterns and Synchronization) of the European Research Council Executive Agency (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 883363). JAC was partially supported by EPSRC grant number EP/T022132/1. The research of JLV was partially supported by grant PGC2018-098440-B-I00 from the Ministerio de Ciencia, Innovación y Universidades of the Spanish Government. JLV was an Honorary Professor at Univ. Complutense.

**Appendix A. Recalling some classical regularity results**

The equation for the mass of the solution of  $u_t = \nabla \cdot (\nabla \Phi(u) + u \nabla V)$  is given by

$$\frac{\partial M}{\partial t} = (n\omega^{\frac{1}{n}} v^{\frac{n-1}{n}})^2 \left[ \frac{\partial}{\partial v} \Phi \left( \frac{\partial M}{\partial v} \right) + \frac{\partial V}{\partial v} \frac{\partial M}{\partial v} \right]. \tag{M_\Phi}$$

Let us prove local regularity of bounded solutions by applying the results in [26]. To match the notation of [26], in this appendix we choose the notation  $x = v$ ,  $u = M$ , and  $a_0(x) = (n\omega^{\frac{1}{n}} v^{\frac{n-1}{n}})^2$ . We write the problem (M<sub>Φ</sub>) as

$$u_t = \nabla \cdot a(x, t, u, Du) + b(x, t, u, Du) \tag{A.1}$$

where

$$a(x, Du) = a_0(x)\Phi(Du), \quad b(x, Du) = -Da_0(x)\Phi(Du) + a_0(x)DV \cdot Du.$$

The standard hypothesis set in [26] are that for some  $p > 1$  we have

$$a(x, t, u, Du) \cdot Du \geq C_0|Du|^p - \varphi_0(t, x), \tag{A_1}$$

$$|a(x, t, u, Du)| \leq C_1|Du|^{p-1} + \varphi_1(t, x), \tag{A_2}$$

$$|b(x, Du)| \leq C_2|Du|^p + \varphi_2(t, x). \tag{A_3}$$

We set  $\Phi(s) = |s|^{m-1}s$  so  $p = m + 1 \in (1, 2)$ . We aim to recover local estimates on a set  $\Omega \Subset \mathbb{R}^n$ . We will be able to get local estimates outside 0. Hypothesis (A<sub>1</sub>) and (A<sub>2</sub>) are easy to check with  $C_0 = \inf_{\Omega} a_0$ ,  $C_1 = \sup_{\Omega} a_0$ , and  $\varphi_0 = \varphi_1 = 0$ . However, (A<sub>3</sub>) is initially not trivial. Since  $p \in (1, 2)$ ,  $|Du|^{p-1}$  is not controlled by  $|Du|^p$  but we have

$$|b(x, Du)| = (\max_{\Omega} |Da_0| + \max_{\Omega} a_0|DV|)(1 + |Du|^p),$$

so we choose  $C_2 = \max_{\Omega} |Da_0| + \max_{\Omega} a_0|DV|$  and  $\varphi_2 = \max_{\Omega} |Da_0| + \max_{\Omega} a_0|DV|$ . Our functions  $\varphi_i$  are bounded, so we also have the hypothesis

$$\varphi_0, \quad \varphi_1^{\frac{p}{p-1}}, \quad \varphi_2 \in L^{\hat{q}, \hat{r}}((0, T) \times \Omega) \tag{A_4}$$

where

$$\frac{1}{\hat{r}} + \frac{N}{p\hat{q}} = 1 - \kappa_1 \tag{A_5}$$

are trivially satisfied. A weak solution in DiBenedetto’s notation requires the regularity  $u \in C_{loc}(0, T; L^2_{loc}(\Omega))$ . The notion of sub-solution (resp. super-) of (A.1) is, for every  $K \Subset \Omega$  and  $0 < t_1 < t_2 \leq T$ , we have

$$\int_K u \varphi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K (-u \varphi_t + a(x, Du) \cdot D\varphi) dx d\tau \leq (\geq) \int_{t_1}^{t_2} \int_K b(x, Du) \varphi dx d\tau,$$

for test functions  $0 \leq \varphi \in W^{1,2}_{loc}(0, T; L^2(K)) \cap L^p_{loc}(0, T; W^{1,p}_0(K))$ . Let us denote  $\Omega_T = (0, T) \times \Omega$ . We have the following result:

**Theorem Appendix A.1** ([26] Chapter III, Theorem 1.1). Let  $p > 1$ , assume  $(A_1)$ ,  $(A_2)$ ,  $(A_3)$ ,  $(A_4)$  and  $(A_5)$  and let  $u$  be a local weak solution of (A.1). Then, there exists constants  $\gamma > 1$  and  $\alpha \in (0, 1)$  depending only on the constant of  $(A_1)$ - $(A_5)$ ,  $\|u\|_{L^\infty(\Omega_T)}$ , and  $\|\varphi_0, \varphi_1^{\frac{p-1}{p}}, \varphi_2\|_{L^{\hat{\alpha}, \hat{\tau}}(\Omega_T)}$  such that for all  $K \Subset (0, T) \times \Omega$

$$|u(t_1, x_1) - u(t_2, x_2)| \leq \gamma \|u\|_{L^\infty(\Omega_T)} \left( \frac{|x_1 - x_2| + \|u\|_{L^\infty(\Omega_T)}^{\frac{p-2}{p}} |t_1 - t_2|^{\frac{1}{p}}}{\text{dist}(K, \Gamma, p)} \right)^\alpha$$

where, for  $\Gamma = \{(y, s) : s = 0 \text{ or } y \in \partial\Omega\}$  we have

$$\text{dist}(K, \Gamma; p) = \inf_{\substack{(x,t) \in K \\ (y,s) \in \Gamma}} \left( |x - y| + \|u\|_{L^\infty(\Omega_T)}^{\frac{p-2}{p}} |t - s|^{\frac{1}{p}} \right).$$

*Regularity at  $t = 0$ .* For the regularity at  $t = 0$ , if  $\rho_0$  is only integrable, then  $M_{\rho_0}$  is continuous. In order to construct a modulus of continuity, we introduce the essential oscillation on a set  $K$  defined as  $\text{ess osc}_K u = \text{ess sup}_K u - \text{ess inf}_K u$ . Notice that the previous result in the whole space stated that, for any  $K \Subset (0, \infty) \times \Omega$  we have

$$\omega_{i,u}(K, h) = \underset{\substack{|t-s| \leq h^p \\ |x-y| \leq h \\ (t,x),(s,y) \in K}}{\text{ess osc}} u \leq \gamma \|u\|_{L^\infty(\omega_T)} \left( \frac{1 + \|u\|_{L^\infty(\Omega_T)}^{\frac{p-2}{p}}}{\text{dist}(K, \Gamma, p)} \right)^\alpha h^\alpha \rightarrow 0,$$

as  $h \rightarrow 0$ . This is an interior modulus of continuity (with scaling). A similar estimate on the essential oscillations holds near  $t = 0$ , but the modulus of continuity now depends on the one from  $u_0$ .

**Theorem Appendix A.2** ([26] Chapter III, Proposition 11.1). Fix  $x_0 \in \Omega$  and  $T_0 > 0$  and  $R_0 > 0$  so that  $B_{2R_0}(x_0) \subset \Omega$ . Then, for  $k \geq 1$ , there exist sequences  $R_k, T_k \searrow 0$  and  $\delta_k \rightarrow 0$  depending only on the constant of  $(A_1)$ - $(A_5)$ ,  $R_0$  and  $\|u\|_{L^\infty([0, T] \times B_{2R_0}(x_0))}$  such that

$$\underset{[0, T_k] \times B_{R_k/2}(x_0)}{\text{ess osc}} u \leq \max \left\{ \delta_k; C \underset{B_{R_k}(x_0)}{\text{ess osc}} u_0 \right\}. \tag{A.2}$$

As a consequence of the previous theorem we conclude that

$$\omega_{b,u}(x_0, h) = \text{ess sup}_{0 \leq t \leq h} |u(t, x_0) - u(0, x_0)| \leq \underset{[0, h] \times B_h(x_0)}{\text{ess osc}} u,$$

tends to 0 as  $h \rightarrow 0$ . This modulus of continuity depends only on the constants of  $(A_1)$ - $(A_5)$  and  $\omega_{u_0}(x_0, h) = \text{ess osc}_{|x-x_0| \leq h} u_0$ . For any  $K$  compact, there exists  $\omega_{u_0}(K, h)$  such that  $\omega_{u_0}(x_0, h) \leq \omega_{u_0}(K, h)$ , for all  $x_0 \in K$ , also going to 0 as  $h \rightarrow 0$ .

**Corollary Appendix A.3.** Let  $K \Subset \Omega$ , and  $u_0 \in C(K)$ . Then, for any  $T > 0$  and  $\varepsilon > 0$ , there exists  $\delta > 0$ , depending only on  $T, \varepsilon$ , the constants of  $(A_1)$ - $(A_5)$  and  $\omega_{u_0}(K, \cdot)$ , such that if  $(t, x), (s, y) \in [0, T] \times K$ ,  $|t - s| \leq \delta^p$  and  $|x - y| \leq \delta$  then

$$|u(t, x) - u(s, y)| \leq \varepsilon.$$

**Proof.** First, we point out that there exists  $\omega_{b,u}(K, h)$  depending only on  $(A_1)$ - $(A_5)$  and  $\omega_{u_0}(K, \cdot)$  such that  $\omega_{b,u}(x_0, h) \leq \omega_{b,u}(K, h)$ , for all  $x_0 \in K$ . Fix  $T > 0$  and  $\varepsilon > 0$ . Since we want to use the interior and boundary regularity, we first fix  $\delta_t > 0$  such that

$$0 \leq t \leq \delta_t^p \implies \operatorname{ess\,sup}_{x \in K} |u(t, x) - u(0, x)| \leq \frac{\varepsilon}{3}.$$

Due to the uniform continuity of  $u_0$ , there exists  $\delta_x > 0$  such that  $\omega_{u_0}(K, \delta_x) \leq \frac{\varepsilon}{3}$ . Lastly, we take  $h_i > 0$  such that  $\omega_{i,u}([\delta_t, T] \times K, h) \leq \frac{\varepsilon}{3}$ . We then take  $\delta = \min\{\delta_t, \delta_x, h_i\}$ . Let us now check the condition. We distinguish cases: If  $t, s > \delta_t$  then  $|u(t, x) - u(s, y)| \leq \frac{\varepsilon}{3} < \varepsilon$ . If  $t < \delta_t \leq s$  (or viceversa), then we write

$$|u(t, x) - u(s, y)| \leq |u(t, x) - u(0, x)| + |u(0, x) - u(\delta_t, x)| + |u(\delta_t, x) - u(s, y)| \leq \varepsilon.$$

Finally, if  $t, s < \delta_t$ , then we write

$$|u(t, x) - u(s, y)| \leq |u(t, x) - u(0, x)| + |u(0, x) - u(0, y)| + |u(0, y) - u(s, y)| \leq \varepsilon.$$

This completes the proof.  $\square$

### Appendix B. Relating space and time regularities

**Theorem Appendix B.1.** *Let  $I \subset \mathbb{R}$  and  $u \in L^\infty(0, T; C^\alpha(\bar{I})) \cap C^\beta(0, T; L^1(\bar{I}))$ . Then*

$$|u(t, x) - u(s, y)| \leq C(|x - y|^\alpha + |t - s|^{\frac{\alpha\beta}{\alpha+1}})$$

where  $C$  depends only on the norms of  $u$  in the spaces above.

**Proof.** We the following splitting  $|u(t, x) - u(s, y)| \leq |u(t, x) - u(s, y)| + |u(t, y) - u(s, y)|$ . The bound for the first term is evident and yields  $C|x - y|^\alpha$ . For the second term we write, for some  $h > 0$

$$\begin{aligned} |u(t, y) - u(s, y)| &\leq \left| \frac{1}{2h} \int_{y-h}^{y+h} (u(t, z) - u(s, z)) \, ds \right| + \left| \frac{1}{2h} \int_{y-h}^{y+h} (u(t, z) - u(t, y)) \, ds \right| \\ &\quad + \left| \frac{1}{2h} \int_{y-h}^{y+h} (u(s, z) - u(s, y)) \, ds \right| \\ &\leq \frac{1}{2h} \|u(t) - u(s)\|_{L^1} + C \int_{y-h}^{y+h} |z - y|^\alpha \, ds \leq C \left( \frac{|t - s|^\beta}{h} + h^\alpha \right). \end{aligned}$$

By choosing  $h = |t - s|^\gamma$ , the optimal rate is achieved when  $\beta - \gamma = \alpha$ , i.e.  $\gamma = \frac{\beta}{\alpha+1}$ . This choice yields

$$|u(t, y) - u(s, y)| \leq C|t - s|^{\frac{\alpha\beta}{\alpha+1}}. \quad \square$$

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