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ABSTRACT.

The main aim of this thesis is to make some contribution to the theory of the tangent bundle of a smooth manifold. We study the possibility of reducing the group of the bundle to various specific subgroups. Such reducibility is equivalent to the existence of certain 'structures' on the manifold. In particular we deal with the trivial subgroup and (for even-dimensional manifolds) with a unitary subgroup of maximal rank; the corresponding structures are called 'parallelism' and 'almost complex structure'. The method consists of two stages: first ask the corresponding question for the stable tangent bundle; then find criteria for moving from the stable to the unstable range. The first stage uses the cohomology theories dealing with stable vector bundles over a space, while the criteria used in the second stage arise mainly from work of M.A.Kervaire. Throughout, the theory is illustrated by application to the total space of a sphere-bundle over a sphere, with its natural smooth structure.

We first ask, "For which oriented sphere-bundles over spheres is the total space parallelizable?" The answer is given in terms of the structural invariants of the bundle. In particular, if

the base dimension is not greater than the fibre dimension, then a necessary condition for parallelizability of the total space is that the bundle structure be trivial. However, there exist bundles in which the total space is parallelizable although the bundle structure is not even stably trivial. A complete answer to this question is not yet available. However, the method used applies to a wider class of manifolds. For example, it is proved that any Stiefel manifold which is not a sphere, is parallelizable.

Next we investigate the existence of almost complex structures on certain even-dimensional manifolds, namely those whose tangent bundles can have their groups lifted to the appropriate spinor group. In particular we apply this to products of spheres and to sphere-bundles over spheres. A product of spheres containing an odd-dimensional factor was previously known to be parallelizable, hence if such a product is even-dimensional, it certainly admits an almost complex structure. A necessary condition for a product of even-dimensional spheres to admit an almost complex structure is that the dimension of each sphere involved should be 2, 4 or 6. In particular, we show that the product of a 2-sphere with a 4-sphere admits an infinite number of distinct almost complex structures. For an oriented sphere-bundle over a sphere, we obtain the following: if the base and fibre are odd-dimensional, then the total space admits an almost complex structure if and only if it is parallelizable;

if the base and fibre are even-dimensional, then a necessary condition for the total space to admit an almost complex structure is that the dimensions of the base and fibre should each be 2, 4 or 6. As observed in the first paragraph, we discuss also the corresponding stable questions.

One of the main tools used in the parallelizability question is the mod 2 semi-characteristic as defined by M. A. Kervaire on p.220 of 'Courbure intégrale généralisée et homotopie' (Math. Annalen, Bd. 131, (1956) 219-252). We compute this for the total space of a differentiable fibre bundle in terms of the homology of the base and fibre and the characteristic classes of the bundle, in some simple cases. We also express the mod 2 semi-characteristic of certain (4k+1)-manifolds in terms of a secondary characteristic class of the tangent bundle of the manifold.

In Appendix F we reprove a theorem of Milnor and Massey on the non-existence of almost complex structure on the quaternionic projective plane; at the same time we prove the same result for certain analogous manifolds derived from multiplications on the 3-sphere other than the quaternionic multiplication.

THEORY AND APPLICATIONS OF ALGEBRAIC TOPOLOGY

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TABLE OF CONTENTS

Introduction.

Consider the tangent bundle of a smooth manifold. It may be possible to reduce the group of the bundle to a specified subgroup. (For the definitions of a bundle and of reducing its group, see Chapter I of [47].) For example if the group of the tangent bundle can be reduced to the trivial subgroup the manifold is said to be parallelizable ; again, if the group of the tangent bundle of an even-dimensional manifold can be reduced to a unitary subgroup of maximal rank, the manifold is said to admit an almost complex structure. The methods of algebraic topology are well suited to these questions. The study of specific structures on a manifold, and of integrability of these where relevant, will not be dealt with here.

A preliminary reduction of the group of the tangent bundle which can always be effected (manifolds being assumed paracompact) is from the full linear group to the corresponding orthogonal group. Any two such reductions are equivalent. (See §12.9 of [47].) It will always be assumed that this reduction has already been made.

An early application of algebraic topology to a reduction problem was : the group of the tangent bundle of a smooth closed n-dimensional manifold can be reduced to the $(n-1)$-dimensional orthogonal group if and only if the Euler characteristic of the manifold is zero. (Satz III on p.552 of [4]). The general reduction problem has been central for many years ; we shall not attempt to give a complete list of references. For progress and bibliography before 1956 see [47] . An outstanding example of recent progress is contained in [2] .

Our aim is to study some of these reduction questions. We shall describe a general procedure and illustrate how it works. This procedure consists of two stages : given a reducibility question for a bundle, first ask the corresponding question for the corresponding stable bundle ; then find criteria for moving back from the stable into the unstable range. The first stage leads naturally to the use of the 'extraordinary' cohomology theories dealing with the stable vector bundles over a space, see e.g. [6] . The criteria used in the second stage arise mainly from the work of M.A.Kervaire in [28] and [29] . Throughout, the method is applied to the total space of a sphere-bundle over a sphere, with its natural smooth structure. Even here the results are not yet complete. On the other hand, the method is applicable to a wider class of bundles. For example it is proved that any classical Stiefel manifold which is not a sphere, is parallelizable.

In Chapter I we establish some notation and describe the class of manifolds used as examples throughout. In Chapter II the parallelizability of these manifolds is studied. The result about

Stiefel manifolds is proved in Chapter III ; it is shown that the octonionic Stiefel manifolds $X_{m,2}$ defined in § 8 of [24] are sphere-bundles over spheres. Chapter IV contains a lemma about the extraordinary cohomology theory of certain manifolds. In Chapter V we study the existence of almost complex structures (and of 'generalised' almost complex structures as defined on p.126 of [22]) again with sphere-bundles over spheres as the main examples. In Chapter VI the mod.2 semi-characteristic of an odd-dimensional manifold is computed in some cases. Chapter VII contains miscellaneous results extending considerations of previous chapters. The appendices give details concerning Chapter I, summaries of work due to various authors which is used centrally in the thesis, numerical details for Chapter V, a proof due to M.F.Atiyah interesting in connection with Chapter III, and a proof, in the spirit of Chapter V, of a known result.

It is a pleasure to acknowledge my great indebtedness to my supervisor Dr.I.M.James ; also to Dr.M.F.Atiyah. Specific acknowledgements are included at the appropriate stages. I wish to thank also the Carnegie Trust for the Universities of Scotland, for financial support during my first three years of study in Oxford.

Chapter I. Preliminaries and Notation.

§1. Throughout we shall use the terms "smooth" and "differentiable" to mean "of class C^{∞}". "Manifold" will mean "compact manifold without boundary" unless the contrary is stated; we may repeat "without boundary" when this condition is vital.

Consider a fibre bundle ξ with projection $\pi : B \longrightarrow M$ in which the fibre F and the base space M are smooth manifolds, and the structural group is a Lie group G acting differentiably on F. Then the total space B may be given the structure of a smooth manifold as follows. First choose an equivalent bundle in which the co-ordinate transformations of the bundle structure are smooth (c.f. footnote on p.54 of [23]). Now give the total space of this new bundle a smooth structure as in § 6.4 of [47] , and transfer this structure to B by the bundle equivalence. The resulting smooth manifold B is unique up to diffeomorphism and π is now a smooth map. If F and M are orientable and G acts on F in an orientation-preserving fashion, then B is orientable. Appendix A contains details of this procedure. Henceforth we shall consider the total space of any such bundle as a smooth manifold in this particular way. Then

$$\tau(B) = \pi^{*}(\tau(M)) \oplus \hat{\xi} \qquad \ldots \qquad (1.1),$$

where τ denotes tangent vector bundle, $\pi^{*}(\ )$ the induced bundle,

$\oplus$ the Whitney summation, and $\hat{\xi}$ the tangent vector bundle along the fibres. (See for example section 7 of [11].)

Now suppose that the fibre F is a sphere and that the group G is the orthogonal group of the appropriate dimension. Let ξ^V be the vector bundle associated with ξ, and suppose that B is contained in the total space of ξ^V as the space of unit spheres in the fibres, which is well-defined since G is the orthogonal group. Then (c.f. p.86 of [53]) there is a canonical linear isomorphism of the fibre of $\pi^*(\xi^V)$ over each point of B with the direct sum of a (radial) line and the tangent space to the fibre of ξ through that point. Thus $\pi^*(\xi^V)$ is equivalent to the Whitney sum of $\hat{\xi}$ and a line-bundle. This line-bundle is trivial; for its total space consists of all pairs (x,y) where x is in B and y is on the line joining x to the zero 0_x of the fibre V_x of ξ^V through x, and the linear structure on the fibre over x of this line-bundle is induced from that on V_x by the translation which takes 0_x into x; thus a non-zero cross-section of the line-bundle is obtained by sending x to $(x,0_x)$. Thus

$$\pi^*(\xi^V) = \hat{\xi} \oplus 1 \qquad \ldots \qquad (1.2).$$

For any finite CW-complex X let $KO(X)$ denote the Grothendieck group of real vector bundles over X and $\widetilde{KO}(X)$ the corresponding reduced Grothendieck group (see for example [7]).

If α is any such vector bundle, its class in $\widetilde{KO}(X)$ will be written $\{\alpha\}$. For complex vector bundles the corresponding groups are denoted by $K(X)$ and $\widetilde{K}(X)$. With the hypotheses of the preceding paragraph, (1.1) and (1.2) show that

$$\{\tau(B)\} = \pi^*\{\tau(M)\} + \pi^*\{\xi^V\} \ . \ . \ . \qquad (1.3).$$

We shall denote the additive group of the integers by Z, the group of integers modulo n by Z_n, and the field of rational numbers by Q. The unitary group in n complex variables will be denoted by $U(n)$, the rotation group in n real variables by $SO(n)$, and their classifying spaces by $BU(n)$ and $BSO(n)$ respectively. The direct limits of the $U(n)$ and the $SO(n)$ under the standard inclusions $U(n) \subset U(n+1)$ and $SO(n) \subset SO(n+1)$ will be written U and SO, and the corresponding classifying spaces will be written BU and BSO. Let E be the total space of an N-universal, principal $SO(n+1)$-bundle (see for example §§ 2 and 18 of [9]). Thus $SO(n+1)$, and its subgroup $SO(n)$, each act on E : we denote the corresponding orbit spaces by $E/_{SO(n+1)}$, $E/_{SO(n)}$. For the discussion of bundles over complexes of dimension less than N, we may take $BSO(n) = E/_{SO(n)}$, $BSO(n+1) = E/_{SO(n+1)}$, and hence obtain a fibring

$$S^n \xrightarrow{k} BSO(n) \xrightarrow{i} BSO(n+1) \ .$$

(See for example p.124 of [9].)

We shall frequently use the following result. Let M be a smooth, compact, $(m-1)$-connected n-dimensional manifold without boundary, where $m \geq 1$. Then M may be represented up to homotopy type as a CW-complex $M' \cup_f e$, where the subcomplex M' is of dimension at most $(n-m)$ and

$$f : S^{n-1} \longrightarrow M'$$

is the attaching map of the n-cell e. This follows for example from Theorem C of [46] when $(n,m) \neq (4,2)$; since M has the homotopy type of a CW-complex (Corollary 1 of [40]), it is sufficient, for simply-connected M, to take a homology decomposition of M (see for example § 8 of [21]). When this result is applied, we shall refer to $M' \cup_f e$ as the <u>principal decomposition</u> of M.

Further notation will be introduced as the need arises. Internal references are by number of section and number within section. Section numbers disregard chapters.

Chapter II. Parallelizability of Sphere-bundles over Spheres.

In Théorème XII on p.247 of [28] it is proved that the product of two or more spheres is parallelizable if and only if at least one is odd-dimensional. We shall consider oriented sphere-bundles over spheres. The total space of such a bundle gets a smooth structure from the standard smooth structures on the base and fibre and the local product structure of the bundle (see Chapter I). We may ask, "For which oriented sphere-bundles over spheres is the total space parallelizable ?" The answer will be given in terms of the structural invariants of the bundle. In particular, if the base dimension is not greater than the fibre dimension, then a necessary condition for parallelizability of the total space is that the bundle structure be trivial. However, there exist bundles in which the total space is parallelizable although the bundle structure is not even stably trivial.

A smooth manifold is said to be <u>s-parallelizable</u> if the Whitney sum of its tangent bundle with a trivial line-bundle is trivial. A parallelizable manifold is clearly s-parallelizable. We proceed by asking two questions about the total space of any oriented sphere-bundle over a sphere.

Question 1. Is it s-parallelisable ?

Question 2. If so, is it parallelisable ?

The answer to Question 1 is simplified by a Thom isomorphism theorem ; I am indebted to M.F.Atiyah for suggesting its use here. The answer to Question 2 is contained in the work of J.F.Adams, [1], and of M.A.Kervaire , [28] and [29].

Results are outlined in §2. In §3 and §4 we give general answers to Questions 1 and 2, and in §5 we examine particular cases.

2. Outline of Results.

We use the notation introduced in Chapter I. The following preliminary theorem follows from (1.3).

Theorem 2.1. Let ξ be an oriented sphere-bundle with projection $\pi : B \longrightarrow M$, where M is an s-parallelizable manifold. Let ξ^V be the vector bundle associated with ξ. Then B is s-parallelizable if and only if the stable class $\{\xi^V\}$ is in the kernel of $\pi^* : \widetilde{KO}(M) \longrightarrow \widetilde{KO}(B)$.

Now let ξ be an oriented q-sphere bundle over the n-sphere, with projection $\pi : B \longrightarrow S^n$. Let θ in $\pi_{n-1}(S^q)$ be the image of a generator of $\pi_n(S^n)$ under the boundary homomorphism in the homotopy exact sequence for ξ, and let λ in $\pi_n(S^{q+1})$ be the Freudenthal suspension of θ.

Theorem 2.2. With notation as above, and n greater than 2, the kernel of $\pi^* : \widetilde{KO}(S^n) \longrightarrow \widetilde{KO}(B)$ is the image of $\lambda^* : \widetilde{KO}(S^{q+1}) \longrightarrow \widetilde{KO}(S^n)$.

We shall deduce from [1] [28] and [29],

Theorem 2.3. Let the total space B of an oriented q-sphere bundle over the n-sphere be s-parallelizable. Then one and only one of the following is true :

(i) B is parallelizable

(ii) both n and q are even

(iii) n is 8, q is 7 and the integral homology group $H_7(B;Z)$ is of odd order.

§3. Proof of Theorem 2.2.

We use the notation of §2. According to section 3 of [27], B has the cell structure $S^q \cup_\theta e^n \cup e^{n+q}$. Let π' be the restriction of π to $S^q \cup_\theta e^n$. Consider the injection of the $\widetilde{KO}$ exact sequence for π into that for π' :

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{KO}(T) \arrow[r, "j^*"] \arrow[d, "i^*"'] & \widetilde{KO}(S^n) \arrow[r, "\pi^*"] \arrow[d, "1"'] & \widetilde{KO}(B) \arrow[r] \arrow[d, "k^*"'] & \cdots \\
\cdots \arrow[r] & \widetilde{KO}(S^{q+1}) \arrow[r] & \widetilde{KO}(S^n) \arrow[r, "\pi'^*"] & \widetilde{KO}(S^q \cup_\theta e^n) \arrow[r] & \cdots
\end{tikzcd} \qquad (3.1)$$

The Thom complex T of ξ is the mapping cone of π. Since n is greater than 2, ξ is a spin bundle and by the Thom isomorphism theorem (see Appendix B), i^* is onto. Since π' is the projection of the cofibration $S^q \subset S^q \cup_\theta e^n$, the lower sequence is the $\widetilde{KO}$ exact sequence for θ (Satz 5 of [43]). Thus the unlabelled homomorphism in (3.1) is λ^*, and Theorem 2.2 follows by exactness of the upper sequence in (3.1).

4. The answer to Question 2.

The mod. 2 semi-characteristic of an odd-dimensional manifold M_{2r-1} is defined to be $\sum_{i=0}^{r-1} b_i$ where b_i is the rank of the mod. 2 homology group $H_i(M;Z_2)$. (See p.220 of [28]).

Theorem 4.1. (Kervaire and Adams). Let M be an s-parallelizable n-dimensional manifold without boundary.

K.1. When n is even, M is parallelizable if and only if its Euler characteristic is zero.

K.2. When n is odd, M is parallelizable if and only if either n is 1, 3 or 7, or the mod. 2 semi-characteristic of M is even.

Proof. The s-parallelizable d-manifolds are exactly those which can be embedded in Euclidean space R_{2d+1} with a continuous field of normal (d+1)-frames. (See e.g. p.10 of [34].) Thus K.1 is Théorème IX of [28], while K.2 follows from Théorème XI of [28] Lemma 1.2 of [29], Theorem 1.1.1 of [1], Theorem 9.3 of [29] and Théorème VIII of [28]. (See Appendix C for details.)

This theorem is used to deduce Theorem 2.3 as follows. Let B be the total space of an oriented q-sphere bundle over the n-sphere and suppose that B is s-parallelizable. If B is even-dimensional, our assertion in (2.3) is a corollary of K.1. If B is odd-dimensional, we distinguish two cases.

(a) The mod. 2 homology of B is the same as that of $S^n \times S^q$. Then B is parallelizable by K.2.

(b) Otherwise q is n-1 and the n-cell in B must be attached to the (n-1)-sphere by a map of odd degree. (We again use the cell structure on B given in section 3 of [27].) But the homotopy class of this map is in the image of $p_* : \pi_{n-1}(SO(n)) \longrightarrow \pi_{n-1}(S^{n-1})$ (see 3.4 of [27]). Hence p_* is onto. Thus S^{n-1} is parallelizable, so n is 1,2,4 or 8 (see e.g. [33]). By K.2, if n is 1,2 or 4, then B is parallelizable while if n is 8, then B is not parallelizable.

§5. Particular Cases.

We continue the discussion of oriented sphere-bundles over spheres with the notation introduced in §2. If the base dimension is not greater than the fibre dimension, then the bundle has a cross section and so $\pi^* : \widetilde{KO}(S^n) \longrightarrow \widetilde{KO}(B)$ is monomorphic. We use Theorem 2.2 to study the kernel of π^* in unstable cases. For $n > 1$, $\widetilde{KO}(S^n)$ is isomorphic to $\pi_n(BSO)$. We use freely the information in section 1 of [14] on the homotopy of both SO and U.

<u>Theorem 5.1. Let $q = n-1$, and $n > 2$.</u>

<u>(i) If n is not a multiple of 4, then π^* is monomorphic.</u>

<u>(ii) If n is a multiple of 4, so that $\widetilde{KO}(S^n)$ is infinite cyclic, then the kernel of π^* consists of all integral multiples of the order of $H_{n-1}(B;Z)$.</u>

<u>Proof.</u> (i). If n is congruent to 3,5,6 or 7 mod. 8, then $\pi_n(BSO)$ is zero and there is nothing to prove. If n is congruent to 1 mod. 8, then $\pi_{n-1}(SO(n))$ is finite (p.161 of [30]) so that $p_* : \pi_{n-1}(SO(n)) \longrightarrow \pi_{n-1}(S^{n-1})$ is zero and the λ of Theorem 2.2 is zero. If n is congruent to 2 mod. 8, λ is in $2\pi_n(S^n)$ while $\pi_{q+1}(BSO)$ is of order 2 and hence is annihilated by composition with λ.

(ii). If n is a multiple of 4, then $\pi_n(BSO)$ is infinite cyclic and λ is θ' times a generator of $\pi_n(S^n)$, where θ' is the order of $H_{n-1}(B;Z)$. Thus the image of λ^* is $\theta'\pi_n(BSO)$. The result now follows in all cases by Theorem 2.2.

<u>Theorem 5.2. Let $q+1 < n$. Then π^* is monomorphic except possibly when q is congruent to 7, and n is congruent to 1 or 2, mod. 8.</u>

<u>Proof.</u> We use the commutative diagram

$$\begin{array}{ccccc} \pi_{q+1}(BU) & \xrightarrow{r} & \pi_{q+1}(BSO) & \xrightarrow{c} & \pi_{q+1}(BU) \\ \downarrow \lambda_c^* & & \downarrow \lambda^* & & \downarrow \lambda_c^* \\ \pi_n(BU) & \xrightarrow{r} & \pi_n(BSO) & \xrightarrow{c} & \pi_n(BU) \end{array} \qquad (5.3)$$

Here λ_c^* is induced by composing with λ ; r and c are induced by the inclusion homomorphisms $U(p) \subset SO(2p)$ and $SO(q) \subset U(q)$. First observe that λ_c^* is zero whenever n and $q+1$ are unequal. For if either n or $q+1$ is odd there is nothing to prove ; while if n and $q+1$ are unequal and both even, then $\pi_{q+1}(S^n)$ is finite (Proposition 5 on p.498 of [44]); but $\pi_n(BU)$ is torsionfree so λ_c^* is zero.

Let $q+1$ be less than n. First let q be congruent to 1 or 3 mod. 8. By 1.7 of [14], $\pi_q(SO/U)$ is isomorphic to $\pi_{q+1}(SO)$ which in this case is zero. Hence from the homotopy exact sequence of the fibring $U \longrightarrow SO \longrightarrow SO/U$, it follows that the upper r in (5.3) is onto. Now λ^* is zero by the left-hand square of (5.3) since λ_c^* is zero.

Next let q be congruent to 7 mod. 8, and let n be a multiple of 4. In this case the lower c of (5.3) is monomorphic (see e.g. the first 12 lines on p.281 of [33]) and so λ^* is zero since λ_C^* is zero. By Theorem 2.2, π^* is monomorphic in the above cases. Finally let q be a multiple of 8. Then π^* is monomorphic by Theorem 1 of [16]. The nature of the homotopy groups of BSO shows that the proof of Theorem 5.2 is now complete.

Theorems 5.1 and 5.2 together with Theorems 2.1 and 2.2 give information on which oriented sphere-bundles over spheres have s-parallelizable total space. Theorem 2.3 tells which of the surviving candidates are parallelizable. Thus usually the total space is parallelizable if and only if the bundle is stably trivial and either the base or the fibre is odd-dimensional. However there exist bundles in which the total space is parallelizable although the bundle structure is not stably trivial ; examples are provided by Theorem 5.1 (ii) and Theorem 5.2 (see below).

The following is a genuine exception of the type allowed by Theorem 5.2. Let β be the Hopf fibring of S^{15} over S^8 with fibre S^7, and let ξ be the bundle induced from β by the essential map from S^9 to S^8. Then the total space of ξ is parallelizable, and ξ is not stably trivial. The latter assertion follows from Lemma 2 of [30], since the classifying element for β in $\pi_8(BSO(8))$ injects to a generator of $\pi_8(BSO)$ (see e.g. [50]). Now label the λ

of Theorem 2.2 by a subscript denoting the bundle for which it is defined. Then λ_β generates $\pi_8(S^8)$ so λ_ξ is the non-zero element of $\pi_9(S^8)$. Thus the stable classifying element of ξ in $\pi_9(BSO)$ is the image under λ_ξ^* of a generator of $\pi_8(BSO)$. Hence by Theorems 2.1 and 2.2 the total space of ξ is s-parallelizable, and now by Theorem 2.3 it is parallelizable.

Similar examples exist whenever q is congruent to 7 mod. 8 and n is $q+2$ or $q+3$. First let q be $8s+7$ and let n be $8s+9$ for some integer $s \geq 1$. Then $\pi_{n-1}(SO(q+1))$ is isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$, and since $\pi_{n-1}(S^q)$ is finite and $\pi_{n-2}(SO(q))$ is torsionfree, $p_* : \pi_{n-1}(SO(q+1)) \longrightarrow \pi_{n-1}(S^q)$ is non-zero. (See p.161 of [30].) The injections $\pi_{n-1}(SO(q+1)) \longrightarrow \pi_{n-1}(SO(q+2)) \longrightarrow \pi_{n-1}(SO(q+3))$ are both onto, since the last two groups are finite while $\pi_{n-1}(S^{q+1})$ and $\pi_{n-1}(S^{q+2})$ are torsionfree. Hence the injection $i_* : \pi_{n-1}(SO(q+1)) \longrightarrow \pi_{n-1}(SO)$ is non-zero. Consider $\pi_{n-1}(SO(q+1))$ as a three-dimensional vector space over Z_2. Then i_* and p_* are non-zero vectors in the dual space, so each has a null-space of dimension at most two. Hence there is χ in $\pi_{n-1}(SO(q+1))$ with $i_*(\chi)$ and $p_*(\chi)$ both non-zero. By Lemma 2 of [30], and Theorems 2.1, 2.2, and 2.3, the bundle with characteristic element χ has parallelizable total space, although its stable class is non-zero. A similar argument deals with the case q congruent to 7 mod. 8 and n equal to $q+3$.

Again, such examples exist when q is 7 and n is 17 or 18. This follows from information on the 2-components of homotopy groups of spheres in e.g. Chapter XIV of [49]. More general results are clearly desirable. The problem may be broken into two stages :

1) For which r is there an element in the stable r-stem of homotopy groups of spheres, inducing by composition a non-zero map of KO^* ?

2) When such an element exists, can it arise in the particular way required to give exceptions of the type allowed by Theorem 5.2 ?

Dr. J.F. Adams has kindly communicated by letter that the answer to 1) may be, "All r congruent to 1 or 2 mod. 8". And that this may appear in a forthcoming paper on the J-homomorphism.

Let α_0 be η, the generator of the stable 1-stem of homotopy groups of spheres, and let $\alpha_s \in \langle \alpha_{s-1}, 2\iota, 8\sigma \rangle$ where σ is the class of the Hopf fibring of S^{15} over S^8, ι is the class of the identity, and $\langle \quad \rangle$ means the Toda secondary composition as in [49]. One might conjecture that α_s (and also $\alpha_s \eta$) induces by composition a non-zero map of KO^*. If this were true, then examples of the type allowed by Theorem 5.2 exist at least whenever q is 7 and n is congruent to 1 or 2 mod. 8.

Remarks. For n less than 3, parallelisability may be decided by special arguments. When n is 1, all oriented bundles are products, and have parallelisable total space by Theorems XII of [28].
When n is 2 and q is at least 2, the opening remark of §5 applies.
When n is 2 and q is 1, either appeal to the theorem that all compact orientable 3-manifolds are parallelisable, or else note that the bundle is principal, so the bundle along the fibres is trivial. Thus $\tau(B)$, which is the Whitney sum of $\tau(S^2)$ lifted up and a trivial line-bundle, is trivial.

Suppose that the differentiable structures on S^n and S^q are changed. Then the tangent bundles remain unchanged by "(4) $\rightarrow$ (3)" on p.135 of [31], and e.g. p.8 of [34]. Thus the results of this chapter remain unchanged. I do not know what happens if the differentiable structure on the total space is changed without respecting the bundle structure.

Chapter III Parallelisability of Stiefel Manifolds.

In §6 we show that the classical Stiefel manifolds are s-parallelizable, and deduce that any Stiefel manifold which is not a sphere, is parallelisable. We show in §7 that the same method applies to the octonionic Stiefel manifolds defined in §8 of [24].

§6. The classical Stiefel manifolds.

We adopt the notation of section 1 of [25] in order to treat the real, complex and quaternionic cases simultaneously. Thus let $O_{n,r}$ denote the Stiefel manifold of orthonormal r-frames in n-dimensional co-ordinate space over any one of these fields. Consider the ladder

$$O_n = O_{n,n} \longrightarrow O_{n,n-1} \longrightarrow \cdots \longrightarrow O_{n,2} \longrightarrow O_{n,1} = S^{dn-1}$$

where d is the dimension of the base field over the reals. Any compact Lie group, in particular O_n, is parallelisable ; and S^{dn-1} is parallelizable only when dn is 1,2,4 or 8.

<u>Theorem 6.1. The Stiefel manifold $O_{n,r}$ is s-parallelizable.</u>

<u>Corollary 6.2. If r is greater than 1, then $O_{n,r}$ is parallelizable.</u>

The corollary follows using Theorem 4.1, since whenever r is greater than 1, the mod. 2 homology of $O_{n,r}$ is the same as that of a product of two or more spheres, at least one of which is odd-dimensional.

<u>Proof of 6.1.</u> Let N be greater than $\frac{1}{2}n(n+1)$ and let F_N denote the N-dimensional co-ordinate space over the base field F. Let $F_r \subset F_N$ be the subspace of vectors whose last $N-r$ components are zero, and let F_{N-n}' be the subspace of vectors whose first n components are zero. Let O_r and O_{N-n}' be the corresponding subgroups of O_N. Denote by B_s the homogeneous space $O_N/O_s \times O_{N-n}'$, where s is any positive integer up to n. Consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r, "p_{r+1}"] & O_{n,r} \arrow[r, "p_r"] \arrow[d, "\tau_r"] & O_{n,r-1} \arrow[r, "p_{r-1}"] \arrow[d, "\tau_{r-1}"] & \cdots \arrow[r, "p_3"] & O_{n,2} \arrow[r, "p_2"] \arrow[d, "\tau_2"] & O_{n,1} \arrow[d, "\tau_1"] & \\
\cdots \arrow[r, "q_{r+1}"] & B_{n-r} \arrow[r, "q_r"] & B_{n-r+1} \arrow[r, "q_{r-1}"] & \cdots \arrow[r, "q_3"] & B_{n-2} \arrow[r, "q_2"] & B_{n-1} \arrow[r, "q_1"] & B_n
\end{tikzcd}$$

Here τ_r is induced by the inclusion of O_n in O_N, p_r is obtained by suppressing the first vector of each r-frame, and q_r is induced by the similar projection of $O_{N,N-n+r}$ to $O_{N,N-n+r-1}$. Then τ_r is a classifying map for the vector bundle $\xi_r^\vee$ associated with the $(n-r-1)$-sphere bundle whose projection is p_{r+1}; and $q_1 \circ q_2 \circ \cdots \circ q_r \circ \tau_r$ is a classifying map for $\xi_r^\vee \oplus dr$ (Whitney sum with dr trivial real line-bundles). But $q_1 \circ q_2 \circ \cdots \circ q_r \circ \tau_r$ is equal to $q_1 \circ \tau_1 \circ p_2 \circ \cdots \circ p_r$, and $q_1 \circ \tau_1$ is the constant map, hence the stable class $\{\xi_r^\vee\}$ is zero.

We now prove the theorem by induction on r. First $O_{n,1}$ is the $(dn-1)$-sphere, which is s-parallelizable. Assume that $O_{n,r}$ is s-parallelizable. Then from (1.3), $\{\tau(O_{n,r+1})\}$ is equal to $p_{r+1}^{*}\left(\{\tau(O_{n,r})\} + \{\xi_r^{V}\}\right)$ which is zero by inductive hypothesis and the above. Thus $O_{n,r+1}$ is s-parallelizable and the induction is complete.

An alternative proof due to M. F. Atiyah, of a stronger result, is sketched in Appendix E.

<u>Remark</u>. For any r, $O_{n,r}$ also <u>bounds</u> a parallelizable manifold. For $r = 1$ this is clear. When r is greater than 1, consider the ball bundle over $O_{n,r-1}$ associated with ξ_{r-1}^{V}. Its total space is s-parallelizable as for $O_{n,r}$; and hence parallelizable since it is a manifold with boundary. This boundary is $O_{n,r}$.

Let $Y_{m,2}$ denote the octonionic Stiefel manifold defined in §8 of [24]. In the next section it will be shown that $Y_{m,2}$ is the total space of an oriented $(8m-9)$-sphere bundle over the $(8m-1)$-sphere. Since $\pi_{8m-1}(BSO)$ is zero, and by (1.3), (2.3), it will then follow that $Y_{m,2}$ is parallelizable.

§7. The octonionic Stiefel manifold $Y_{m,2}$ as a sphere-bundle over a sphere.

Recall the definition of octonionic Stiefel spaces from § 8 of [24]. Let F be the Cayley algebra of octonions (see e.g. § 20.5 of [47]) and let F_m be the m-fold Cartesian product of F with itself, where $m \geqslant 1$. A point of F_m, say $x = (x_1, x_2, \ldots, x_m)$, with $x_r \in F$, is referred to as an m-vector. The inner product of m-vectors x and y is defined to be the octonion $x_1\bar{y}_1 + \ldots + x_m\bar{y}_m$ where $\bar{y}_r$ denotes the conjugate of y_r. Length and orthogonality are now defined as in a proper vector space. By a k-frame in F_m we mean an ordered orthonormal set of k m-vectors. Let $Y_{m,k}$ denote the space of k-frames in F_m (with the obvious topology). The projection of $Y_{m,k}$ into $Y_{m,l}$, where $1 \leq l \leq k$, is defined by suppressing the first $k-l$ vectors of each k-frame. In particular for $k = 2$ and $l = 1$, and $m \geqslant 2$, we denote this projection by $p : Y_{m,2} \longrightarrow Y_{m,1}$. In § 8 of [24] it is indicated how to prove that $Y_{m,2}$ has a local product structure relative to p.

<u>Proposition 7.1. The Stiefel manifold $Y_{m,2}$ is the total space of an oriented sphere-bundle with projection p.</u>

As an auxiliary for the proof, let $Y_{m,k}'$ be the space of ordered orthogonal sets of k m-vectors, in which the first vector is non-zero and the others are of unit length. In particular $Y_{m,1}'$ consists of all

non-zero m-vectors. Note that $Y_{m,k}$ is contained in $Y_{m,k}'$.
The projection $p' : Y_{m,2}' \longrightarrow Y_{m,1}$ is defined as for p.
Let $GL(8m-8,R)$ denote the full linear group of dimension $8m-8$ over the real numbers.

<u>Lemma 7.2. With the above notation, $Y_{m,2}'$ is the total space of a fibre bundle ξ' with fibre $Y_{m-1,1}'$, base $Y_{m,1}$ and group $GL(8m-8,R)$.</u>

<u>Proof.</u> Following § 8 of [24], let $V_i \subset Y_{m,1}$ be the subspace of all $x = (x_1, \ldots, x_m)$ such that $x_i \neq 0$. For each i define

$$\psi_i : p'^{-1}(V_i) \longrightarrow Y_{m-1,1}' \times V_i$$

by $$\psi_i(y,x) = ((y_1, \ldots, \hat{y}_i, \ldots, y_m), x)$$

where $(y,x) \in p'^{-1}(V_i)$ and $(y_1, \ldots, \hat{y}_i, \ldots, y_m)$ is the m-vector obtained from y by omitting y_i. Then $(y_1, \ldots, \hat{y}_i, \ldots, y_m)$ <u>is</u> in $Y_{m-1,1}'$. For suppose it were the zero vector. Then since $\sum x_j \bar{y}_j$ is zero, $x_i \bar{y}_i$ would be zero. But $x_i \neq 0$, and the Cayley numbers form a division algebra, so $\bar{y}_i$, and hence y_i, would be zero. Then y would be the zero vector, which is false since $(y,x) \in Y_{m,2}'$.
By its definition ψ_i is continuous. Let

$$\phi_i : Y_{m-1,1}' \times V_i \longrightarrow p'^{-1}(V_i)$$

be defined by

$$\phi_i(z,x) = ((z_1, \ldots, z_{i-1}, \mu(z,x), z_i, \ldots, z_{m-1}), x)$$

where

$$\mu(z,x) = -(z_1\bar{x}_1 + \ldots + z_{i-1}\bar{x}_{i-1} + z_i\bar{x}_{i+1} + \ldots + z_{m-1}\bar{x}_m)\bar{x}_i^{-1}$$

Since $x \in V_i$, $\mu(z,x)$ is well-defined ; and $\phi_i(z,x)$ is an orthogonal pair, for the inner product of its m-vectors is

$$z_1\bar{x}_1 + \ldots + z_{i-1}\bar{x}_{i-1} + \mu(z,x)\bar{x}_i + z_i\bar{x}_{i+1} + \ldots + z_{m-1}\bar{x}_m, \qquad \text{or}$$

$$z_1\bar{x}_1 + \ldots + z_{i-1}\bar{x}_{i-1} + z_i\bar{x}_{i+1} + \ldots + z_{m-1}\bar{x}_m - \\ - ((z_1\bar{x}_1 + \ldots + z_{i-1}\bar{x}_{i-1} + z_i\bar{x}_{i+1} + \ldots + z_{m-1}\bar{x}_m)\bar{x}_i^{-1})\bar{x}_i .$$

This is zero since the subalgebra of F generated by any two elements is associative. The first vector in the pair $\phi_i(z,x)$ is non-zero since z is non-zero. Continuity of ϕ_i follows from its definition. Clearly $\psi_i\,\phi_i$ is the identity map of $Y_{m-1,1}' \times V_i$. If $(y,x) \in p'^{-1}(V_i)$, then

$$\phi_i\,\psi_i(y,x) = ((y_1, \ldots, y_{i-1}, \mu, y_{i+1}, \ldots, y_m), x)$$

where
$$\begin{aligned} \mu &= -(y_1\bar{x}_1 + \ldots + y_{i-1}\bar{x}_{i-1} + y_{i+1}\bar{x}_{i+1} + \ldots + y_m\bar{x}_m)\bar{x}_i^{-1} \\ &= -(-y_i\bar{x}_i)\bar{x}_i^{-1} \quad \text{since } y \text{ is orthogonal to } x \\ &= y_i \quad \text{since the subalgebra of } F \text{ generated by} \\ &\qquad \text{two elements is associative.} \end{aligned}$$

Thus ϕ_i is a homeomorphism with inverse ψ_i. Now suppose $i \neq j$. For any x in $V_i \cap V_j$, let $g_{ji}(x)$ denote the composition

$$Y_{m-1,1}' \xrightarrow{\phi_{i,x}} p'^{-1}(x) \xrightarrow{\phi_{j,x}^{-1}} Y_{m-1,1}',$$

where $\phi_{i,x}(z) = \phi_i(z,x)$. Assume for definiteness that j is less than i. Then $g_{ji}(x)$ is the homeomorphism sending $(z_1, \ldots, z_{m-1})$ to

$$(z_1, \ldots, \hat{z}_j, \ldots z_{i-1}, -(z_1\bar{x}_1 + \ldots + z_{i-1}\bar{x}_{i-1} + z_i\bar{x}_{i+1} + \ldots + z_{m-1}\bar{x}_m)\bar{x}_i^{-1}, z_i, \ldots, z_{m-1}).$$

Using Lemma 7.3, this explicit formula will show that g_{ji} is a continuous function of $V_i \cap V_j$ into $GL(8m-8, R)$. The proof of Lemma 7.2 will then be complete.

<u>Lemma 7.3. Let A be a linear algebra of dimension d over a (commutative) field K. Denote by A^m the m-fold Cartesian product of A with itself. Let $\theta: A^m \longrightarrow A^m$ be given by $\theta(a_1, \ldots, a_m) = (a_1', \ldots, a_m')$ where $a_i' = \sum_j a_j b_{ij}$ for fixed $b_{ij} \in A$. Suppose that θ is one-one onto. Identify A^m with K^{dm} by choosing a basis for A over K. Then the transformation θ' of K^{dm} induced by θ is in $GL(dm, K)$.</u>

<u>Proof</u>. By the explicit form of θ and the fact that A is a linear algebra, θ' is K-linear. It is non-singular since θ is one-one onto.

To obtain the desired reult about the g_{ji} above, apply Lemma 7.3 with F for A, R for K, and with θ_1 and θ_2 where

$$\theta_1(z_1, \ldots, z_{m-1}) = (z_1, \ldots, \hat{z}_j, \ldots, z_{i-1}, (z_1\bar{x}_1 + \ldots + z_{m-1}\bar{x}_m), z_i, \ldots, z_{m-1})$$

$$\theta_2(z_1, \ldots, z_{m-1}) = (z_1, \ldots, z_{i-2}, -z_{i-1}\bar{x}_i^{-1}, z_i, \ldots, z_{m-1}).$$

<u>Proof of 7.1.</u> Apply the first part of the theorem in § 12.9 of [47] to the bundle ξ' of Lemma 7.2. This gives a co-ordinate bundle with projection $\pi' : B' \longrightarrow Y_{m,1}$, co-ordinate neighbourhoods $\{U_j\}$ say, co-ordinate functions $\rho'_j : Y_{m-1,1}' \times U_j \longrightarrow \pi'^{-1}(U_j)$ and group $O(8m-8,R)$; and a linear bundle equivalence

$$\begin{tikzcd} B' \arrow[rr, "h'"] \arrow[dr, "\pi'"'] & & Y_{m,2}' \arrow[dl, "p'"] \\ & Y_{m,1} & \end{tikzcd}$$

Since the group of the new bundle is $O(8m-8,R)$ there is an associated unit-sphere bundle, with total space

$$B = \bigcup_j \rho'_j(Y_{m-1,1} \times U_j),$$

and projection π and co-ordinate functions ρ_j the restrictions of π' and ρ'_j. I claim that there is a homeomorphism h making the following diagram commutative : -

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "\pi"'] & & Y_{m,2} \arrow[dl, "p"] \\ & Y_{m,1} & \end{tikzcd}$$

Such an h will induce a sphere-bundle structure on $Y_{m,2}$ relative to p. To define h, identify F_m with R_{8m}. For any x in $Y_{m,1}$, let $x^{\perp}$ be the $(8m-8)$-dimensional subspace of R_{8m} orthogonal to x in the Cayley sense, and let

$$a_x : p'^{-1}(x) \longrightarrow x^{\perp}$$

be obtained by suppressing the second vector of each 2-frame. Then for each x in $V_i \cap U_j$,

$$a_x \, \phi_{i,x} : Y_{m-1,1}' \longrightarrow x^{\perp} \qquad \text{and}$$

$$\phi^{-1}_{i,x} \, h'_x \, \rho'_{j,x} : Y_{m-1,1}' \longrightarrow Y_{m-1,1}'$$

are restrictions of linear transformations of R_{8m-8} (the first by definition of $\phi_{i,x}$, the second since h' is a linear bundle equivalence), hence so is

$$a_x \, h'_x \, \rho'_{j,x} : Y_{m-1,1}' \longrightarrow x^{\perp}.$$

Now $\rho'_{j,x}(Y_{m-1,1})$ is $\pi^{-1}(x)$. Thus h'_x takes $\pi^{-1}(x)$ homeomorphically onto a subspace $\{(y,x)\}_{y \in C_x}$ of $p'^{-1}(x)$, where C_x is a compact subspace of $x^{\perp}$ such that any ray in $x^{\perp}$ from the origin meets C_x in exactly one point. Thus there is

a fibrewise homeomorphism k of $\bigcup_x \{(y,x)\}_{y \in C_x}$ onto $Y_{m,2}$.
Then $k \circ (h'|B)$ is the required homeomorphism h.

If $h_1' : B' \longrightarrow Y_{m,2}'$ is another bundle equivalence giving a fibrewise homeomorphism $h_1 : B \longrightarrow Y_{m,2}$, then $h_1 h^{-1}$ is a bundle equivalence of the two sphere-bundle structures on $Y_{m,2}$. Any two choices of reduction of the group of ξ' are equivalent in $O(8m-8,R)$, by the second part of the theorem in §12.9 of [47]. Thus the sphere-bundle structure on $Y_{m,2}$ defined above is independent of choices. Since the base $Y_{m,1}$ is homeomorphic to the $(8m-1)$-sphere, the bundle is orientable, and we get an oriented sphere-bundle structure on $Y_{m,2}$ relative to p; this structure is unique up to orientation.

Chapter IV A Lemma on Cohomology Theories of Certain Manifolds.

§8. Let M be a smooth, connected, compact, n-dimensional manifold without boundary, and let ν be a stable normal vector bundle of M ; so that if k is the dimension of the fibre in ν, $k > n$. Let M^{ν} denote the Thom complex of ν (see p.29 of [48]). Throughout this section, ordinary homology will be taken with integer ~~or mod 2~~ coefficients. ~~according as M is or is not orientable in the classical sense.~~ By a 'multiplicative cohomology theory' we shall mean the same as in [18]. (See also Appendix B.) If h is such a theory and P is a point, then there is a unit 1 in $h^0(P)$. When h is defined on a category of pairs of spaces in which each space X has a base-point x_0, we denote by $\tilde{h}$ the 'reduced' theory, $\tilde{h}^r(X) = h^r(X, x_0)$. We say that M is <u>h-orientable</u> if there is a fundamental class U in $\tilde{h}^k(M^{\nu})$ such that $\tilde{h}(i)(U) = \sigma^k(1)$ where σ^k denotes k-fold suspension and $i : S^k \longrightarrow M^{\nu}$ is induced by the inclusion of a fibre in the total space of ν. (See for example p.8 of [18].)

Consider the principal decomposition $M' \cup_f e$ of M (see §1).

<u>Lemma 8.1. With notation as above, suppose that M is h-orientable. Then</u> $\tilde{h}^i(f) : \tilde{h}^i(M') \longrightarrow \tilde{h}^i(S^{n-1})$ <u>is zero for any i.</u>

For the proof we require the following lemma :

Lemma 8.2. The generator of $H_{n+k}(M^{\nu})$ is a spherical class.

This lemma is well-known. For completeness a sketch proof follows. Suppose that M is smoothly embedded in a sphere S^{n+k} with $k \geq n+1$. Let A be a tubular neighbourhood of M in S^{n+k}. Let $\dot{A}$ denote the boundary of A, and B the closure of the complement of A in S^{n+k}. Now A is diffeomorphic to a neighbourhood of the zero cross-section in the normal bundle ν of the given embedding of M, so we may take M^{ν} to be $A/\dot{A}$. Consider the map $\lambda : S^{n+k} \longrightarrow S^{n+k}/B \equiv A/\dot{A}$ obtained by collapsing B to a point. This induces

$$H_{n+k}(S^{n+k}) \xrightarrow{j^*} H_{n+k}(S^{n+k},B) \underset{\text{excision}}{\xleftarrow{\approx}} H_{n+k}(A,\dot{A}).$$

But j^* is an isomorphism since B is contractible in S^{n+k}, and the lemma follows.

Proof of 8.1. Since the argument will be at the homotopy level, we shall identify M with $M' \cup e$ and ν with the bundle over $M' \cup e$ induced from it by some fixed homotopy equivalence of $M' \cup e$ with M. Subdivide M to obtain a CW-complex X such that the restriction of ν to the closure of any cell of X is trivial. Let d be an open n-cell of X contained in e. Write $N = X - d$ and let $g : S^{n-1} \longrightarrow N$ be the attaching map for d in X. We have a diagram :

$$\begin{tikzcd} S^{n-1} \arrow[rr, "g"] \arrow[dr, "f"'] & & N \arrow[dl, shift right, "r"'] \\ & M' \arrow[ur, shift right, "j"'] & \end{tikzcd}$$

where j is the inclusion, r a deformation retraction, and $r \circ g$ is homotopic to f. It is thus sufficient to prove that $\tilde{h}(g)$ is zero. Now give the Thom complex M^{ν} a cell structure in which there is one $(i+k)$-cell corresponding to each i-cell of X, and one 0-cell. Let $c \subset M^{\nu}$ be the $(n+k)$-cell corresponding to d, write $L = M^{\nu} - c$, and let $l : S^{n+k-1} \longrightarrow L$ be the attaching map for c. Then L is the Thom complex of $\nu | N$, and S^{n+k-1} is the Thom complex of the trivial bundle over S^{n-1} induced from $\nu | N$ by g. Since ν is h-orientable so are $\nu | N$ and $g^*(\nu | N)$. Applying the Thom isomorphism for h theory, (see Appendix B) we get the commutative diagram

$$\begin{tikzcd} \tilde{h}^{i+k}(L) \arrow[r, "\tilde{h}(l)"] & \tilde{h}^{i+k}(S^{n+k-1}) \\ h^{i}(N) \arrow[u, "\cong"] \arrow[r, "h(g)"] & h^{i}(S^{n-1}) . \arrow[u, "\cong"] \end{tikzcd}$$

It is now sufficient to show that $\tilde{h}(l)$ is zero. By Lemma 8.2 the composition

$$S^{n+k} \xrightarrow{\lambda} M^{\nu} \xrightarrow{p} M^{\nu}/L \cong S^{n+k}$$

is the identity, where λ is the collapsing map used in 8.2 and

p collapses L. Hence $\tilde{h}(p)$ is monomorphic. Consider the exact $\tilde{h}$ sequence of l:

$$\begin{tikzcd} \cdots \arrow[r] & \tilde{h}^{i}(L) \arrow[r, "\tilde{h}(l)"] & \tilde{h}^{i}(S^{n+k-1}) \arrow[r] \arrow[d, phantom, "\cong\ \sigma"] & \tilde{h}^{i+1}(M) \arrow[r] & \cdots \\ & & \tilde{h}^{i+1}(S^{n+k}) \arrow[ur, "\tilde{h}(p)"'] & & \end{tikzcd}$$

The triangle commutes where σ means suspension, since the $\tilde{h}$ exact sequence of l is that of p moved two places along (see Satz 5 of [43]). Now since $\tilde{h}(p)$ is monomorphic, $\tilde{h}(l)$ is zero, as required.

<u>Remarks</u>. A manifold is KO-orientable if the group of its tangent bundle may be lifted to the appropriate spinor group, or equivalently if its first and second mod 2 Stiefel-Whitney classes are zero (see Appendix B and § 26.5 of [11]).

The hypothesis of h-orientability cannot be removed from 8.1. For example if h is ordinary integral cohomology and M is non-orientable, then the lemma would imply that the pinching map $M \longrightarrow S^n$ induces a monomorphism $H^n(S^n,Z) \longrightarrow H^n(M,Z)$, which is false. Again, let P denote the complex projective plane. This is orientable but not KO-orientable. Consider the KO exact sequence of the Hopf map $\eta : S^3 \longrightarrow S^2$:

$$\cdots \longrightarrow \widetilde{KO}^{i-1}(S^2) \xrightarrow{\eta^*} \widetilde{KO}^{i-1}(S^3) \longrightarrow \widetilde{KO}^{i}(P) \longrightarrow \cdots$$

Here η^* is non-zero when $i = -5$, by Lemma 2 of [30].

Let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. Let S denote suspension.

<u>Corollary 8.3. Let M be a 2m-dimensional closed spin-manifold with principal decomposition</u> $M' \cup_f e$. <u>Then</u>

$$Sf^{\#} : [SM', BSO(2m)] \longrightarrow [S^{2m}, BSO(2m)]$$

<u>is zero.</u>

<u>Proof.</u> Write $n = 2m$. The fibring $S^n \xrightarrow{k} BSO(n) \xrightarrow{i} BSO(n+1)$ (c.f. §1) gives rise to a commutative diagram

$$\begin{tikzcd}
{[SM', S^n]} \arrow[r, "k_*"] \arrow[d, "Sf^{\natural}"] & {[SM', BSO(n)]} \arrow[r, "i_*"] \arrow[d, "Sf^{\#}"] & {[SM', BSO(n+1)]} \arrow[r, phantom, "="] \arrow[d, "Sf^*"] & \widetilde{KO}(SM') \arrow[d] \\
{[S^n, S^n]} \arrow[r, "k_*"] & {[S^n, BSO(n)]} \arrow[r, "i_*"] & {[S^n, BSO(n+1)]} \arrow[r, phantom, "="] & \widetilde{KO}(S^n)
\end{tikzcd}$$

where each horizontal sequence is exact at the middle stage. Let $\alpha \in [SM', BSO(n)]$. Then $Sf^* i_*(\alpha) = 0$ by 8.1, so $\alpha \circ Sf = k \circ \beta$ for some $\beta \in [S^n, S^n]$. Now

$$Sf^* : H^n(SM', Z) \longrightarrow H^n(S^n, Z)$$

is zero (for example by 8.1) so $k \circ \beta$ induces the zero homomorphism on n-dimensional cohomology. But k^* is non-zero on the universal Euler class, hence β is trivial and the result follows.

Chapter V Almost Complex and Stably Almost Complex Structures.

Given an n-plane bundle ξ over an n-manifold M, which n-plane bundles over M are stably equivalent to ξ ? The formal answer is easy : specialising to even-dimensional spin manifolds we obtain an effective answer which extends Théorème IX of [28]. We use this together with K and KO theory to study the existence of almost complex structures on even-dimensional spin manifolds, in particular on products of spheres and on sphere-bundles over spheres.

The 'formal answer' is contained in §9. In §10 we study Euler classes in the setting of §9. We recall in §11 the classification of 2m-plane bundles on the 2m-sphere and in §12 we extend Théorème IX of [28]. The relevance to almost complex structures is shown in §13, and in §14 and §15 we deal with products of spheres and with sphere-bundles over spheres.

§9. Stably Equivalent Bundles.

Let M be a C_σ topological space (see 11.3 of [47]) having the homotopy type of a finite CW-complex $M' \cup_f e$, where the dimension of the subcomplex M' is at most $n-1$, and $f : S^{n-1} \longrightarrow M'$ attaches the n-cell e . We shall identify

M with $M' \cup_f e$ by some fixed homotopy equivalence. As in 19.3 of [47], (equivalence classes of) bundles over M may be identified with homotopy classes of maps of M into an appropriate classifying space.

Consider the following commutative diagram of Puppe sequences :

$$\begin{array}{ccccc} [S^n, BSO(n)] & \xrightarrow{Qf^{\#}} & [M, BSO(n)] & \xrightarrow{Pf^{\#}} & [M', BSO(n)] \\ \downarrow i_* & & \downarrow i_* & & \downarrow i_* \\ [S^n, BSO(n+1)] & \xrightarrow{Qf^{*}} & [M, BSO(n+1)] & \xrightarrow{Pf^{*}} & [M', BSO(n+1)] \end{array} \quad (9.1)$$

The notation here is as in §1 of [43]: thus $Pf : M' \longrightarrow M$ is the inclusion, and $Qf : M \longrightarrow S^n$ collapses M'. Suppose that $a, b \in [M, BSO(n)]$. We say that a and b are <u>almost equivalent</u> if $Pf^{\#}(a) = Pf^{\#}(b)$, and that a and b are <u>stably equivalent</u> if $i_*(a) = i_*(b)$. Since the right-hand i_* in (9.1) is monomorphic, stable equivalence implies almost equivalence. By 4.5 of [43], a and b are almost equivalent if and only if there exists an $\alpha \in [S^n, BSO(n)]$ with $\alpha . a = b$, where the dot denotes the action defined in 4.3 of [43]. Let $I \subset [S^n, BSO(n)]$ denote the kernel of $Qf^* i_*$.

<u>Lemma 9.2. Let $a, b \in [M, BSO(n)]$. Then a and b are stably equivalent if and only if $b = \alpha . a$ for some $\alpha \in I$.</u>

Proof. Suppose that a and b are stably equivalent. Then by the above, $\alpha . a = b$ for some $\alpha \in [S^n, BSO(n)]$. By naturality of the action, $i_*(\alpha).i_*(a) = i_*(b)$. But by 15.7 of [21] for example, $i_*(\alpha).i_*(a) = Qf^* i_*(\alpha) + i_*(a)$. Hence $\alpha \in I$. Conversely if $\alpha . a = b$ for some $\alpha \in I$, then $i_*(a) = i_*(b)$.

§10. A Remark on Euler Classes.

In this section homology and cohomology will be taken with coefficients in a field F. Let M be as in §9, and suppose in addition that $H_n(M)$ has rank 1. Let Y be any space, and let $y \in H^n(Y)$ be represented by the map $y : Y \longrightarrow K_n$, where K_n denotes an Eilenberg-Maclane space of type (F, n). Let $\alpha \in [S^n, Y]$ and $a \in [M, Y]$ be represented by maps of the same names. As before let $Qf : M \longrightarrow S^n$ collapse M'. Let $p : M \longrightarrow M \vee S^n$ pinch the boundary of an n-disc in the interior of the n-cell of M. Consider the following homotopy-commutative diagram:

$$\begin{tikzcd}
M \arrow[r, "p"] \arrow[d, "\Delta"] & M \vee S^n \arrow[r, "ya \vee y\alpha"] \arrow[d, "i"] & K_n \vee K_n \arrow[r, "\nabla"] \arrow[d, "j"] & K_n \\
M \times M \arrow[r, "1 \times Qf"] & M \times S^n \arrow[r, "ya \times y\alpha"] & K_n \times K_n \arrow[ur, "m"] &
\end{tikzcd}$$

where Δ is the diagonal map, i and j are inclusions, m is the multiplication on K_n and ∇ is the folding map.

Thus (c.f. second part of proof of Theorem 15.7 of [21]),

$$(\alpha.a)^*(y) = Qf^*\alpha^*(y) + a^*(y) \quad . \quad . \quad . \quad (10.1).$$

We observe that since $H^n(M')$ is zero, Qf^* is non-zero.

For any orientable n-plane bundle ξ over M, let $E(\xi) \in H^n(M)$ denote the Euler class of ξ, where the coefficients are now rational. Apply (10.1) and the succeeding observation with n even, $BSO(n)$ for Y, rational coefficients, and y the universal Euler class. This yields :

Lemma 10.2. With the above notation, $E(\alpha.a) = E(a)$ if and only if $E(\alpha) = 0$.

§ 11. 2m-plane Bundles over S^{2m}.

We recall the following theorem, for example from [30]:

Theorem 11.1. A 2m-plane bundle over the 2m-sphere is determined by its stable class and its Euler class.

Proof. Let $i_* : \pi_{2m-1}(SO(2m)) \longrightarrow \pi_{2m-1}(SO(2m+1))$ denote the standard injection and $p_* : \pi_{2m-1}(SO(2m)) \longrightarrow \pi_{2m-1}(S^{2m-1})$ the standard projection. We have to show that any $\chi \in \pi_{2m-1}(SO(2m))$

is determined by $i_*(\chi)$ and $p_*(\chi)$. When m is even, $\pi_{2m-1}(SO(2m))$ is free Abelian of rank 2, and i_* and p_* are monomorphisms of the summands. When m is congruent to 3 mod 4, $\pi_{2m-1}(SO(2m))$ is infinite cyclic and p_* is monomorphic. When m is congruent to 1 mod 4, consider the following diagram :

$$\begin{tikzcd}
\pi_{2m}(S^{2m}) \arrow[r, "\partial"] & \pi_{2m-1}(SO(2m)) \arrow[r, "i_*"] \arrow[d, "p_*"] & \pi_{2m-1}(SO(2m+1)) \\
 & \pi_{2m-1}(S^{2m-1}) &
\end{tikzcd}$$

From [30], $\pi_{2m-1}(SO(2m))$ is isomorphic to $Z \oplus Z_2$. Then p_* is a monomorphism of the first summand; while by exactness of the horizontal sequence, i_* is non-zero on the second summand. Thus the theorem follows in all cases.

§12. The General Case.

Let $M = M' \cup_f e^n$ be as in §10, with rational coefficients and $n = 2m$.

<u>Theorem 12.1. Let ξ and η be stably equivalent orientable 2m-plane bundles over a 2m-dimensional M as above. Suppose that one of the following holds</u> :

(i) $m \not\equiv 1 \bmod 4$,

(ii) ξ is stably trivial,

(iii) M has the homotopy type of a spin manifold.

Then ξ and η are equivalent if and only if $E(\xi) = E(\eta)$.

Proof. If ξ and η are equivalent their Euler classes are certainly equal. Consider the following diagram :

$$\begin{tikzcd}
{[S^{2m},S^{2m}]} \arrow[r, "k_*"] \arrow[d, "Qf^\natural"] & {[S^{2m},BSO(2m)]} \arrow[r, "i_*"] \arrow[d, "Qf^\#"] & {[S^{2m},BSO(2m+1)]} \arrow[d, "Qf^*"] \\
{[M,S^{2m}]} \arrow[r, "k_*"] & {[M,BSO(2m)]} \arrow[r, "i"] & {[M,BSO(2m+1)]}
\end{tikzcd} \tag{12.2}$$

where notations are as in (9.1). Let $a,b \in [M,BSO(2m)]$ represent stably equivalent ξ, η. As in §9, then $\alpha.a = b$ for some $\alpha \in [S^{2m},BSO(2m)]$ with $Qf^* i_*(\alpha) = 0$. Suppose that $E(\xi) = E(\eta)$. Then by (10.2), $E(\alpha) = 0$, and in view of Theorem 11.1 it is sufficient to show that $i_*(\alpha) = 0$. If $m \equiv 3 \bmod 4$, then $[S^{2m},BSO(2m+1)]$ is zero and the result follows. If (ii) holds, then a and b are in $k_*[M,S^{2m}]$, so α may be chosen in $k_*[S^{2m},S^{2m}]$ and $i_*(\alpha) = 0$. If (iii) holds, then by Lemma 8.1 Qf^* is monomorphic so $i_*(\alpha) = 0$. Finally suppose that m is even. Consider the diagram following, where c denotes complexification, ch denotes Chern character (see 9.1 of [11]), and cohomology is taken with rational coefficients :

$$\begin{array}{ccccc} \widetilde{KO}(SM') & \xrightarrow{c} & \widetilde{K}(SM') & \xrightarrow{ch} & \widetilde{H}^*(SM') \\ \downarrow Sf^* & & \downarrow Sf^* & & \downarrow Sf^* \\ \widetilde{KO}(S^{2m}) & \xrightarrow{c} & \widetilde{K}(S^{2m}) & \xrightarrow{ch} & \widetilde{H}^*(S^{2m}) . \end{array} \qquad (12.3)$$

The lower c and ch are both monomorphic (see for example the first 12 lines on p.281 of [33], and p.19 of [6]). The right-hand Sf^* is zero; for $H^{2m}(M')$ is zero so $Qf^* : H^{2m}(S^{2m}) \longrightarrow H^{2m}(M)$, being an epimorphism between vector spaces of the same finite dimension, is monomorphic. Hence the left-hand Sf^* in (12.3) is zero, so Qf^* in (12.2) is monomorphic and $i_*(\alpha) = 0$.

Corollary 12.4. Let M satisfy the hypothesis of Theorem 12.1. Let $H \subset SO(2m)$ be a closed subgroup of rank less than m. Let ξ be a 2m-plane bundle over M which is stably equivalent to an H-bundle. Then ξ is equivalent to an H-bundle if and only if $E(\xi) = 0$.

Proof. Suppose that ξ is equivalent to an H-bundle. Then since rank $H < m$, $E(\xi) = 0$. Let $j : B_H \longrightarrow BSO(2m)$ denote the injection of classifying spaces. Suppose that $E(\xi) = 0$, and that ξ is stably equivalent to $j_*(\eta)$ for some $\eta \in [M, B_H]$. Then $E(\xi) = 0 = E(j_*(\eta))$, and the result follows by (12.1).

§ 13. Almost Complex Structures.

Let M be a smooth oriented 2m-dimensional manifold. Let $\tau(M)$ denote the tangent bundle of M, and $\{\tau(M)\}$ its stable class. Let $r : \widetilde{K}(M) \longrightarrow \widetilde{KO}(M)$ be induced by the limit of the standard inclusions $U(p) \subset SO(2p)$. If the group of $\tau(M)$ can be reduced to the subgroup $U(m)$, then M is said to admit an almost complex structure (a.c.s.). If $\{\tau(M)\}$ is in the image of r, then M is said to admit a stably almost complex structure (s.a.c.s.). (Other names for s.a.c.s. : 'generalised almost complex structure', p.126 of [22]; 'weakly almost complex structure', p.493 of [12] .)

Since the Chern classes of stably equivalent unitary bundles are equal, we can define the mth. Chern class $c_m(x)$ for $x \in \widetilde{K}(M)$.

Theorem 13.1. *Let M be a smooth oriented 2m-manifold, and if $m \equiv 1 \bmod 4$, suppose also that M is a spin manifold. Then M admits an a.c.s. if and only if there exists an $x \in r^{-1}\{\tau(M)\}$ with $c_m(x) = E(\tau(M))$.*

Proof. For the necessity of the condition see § 41.3 of [47]. Suppose that such an x exists in $r^{-1}\{\tau(M)\}$. Consider the following commutative diagram :

$$\begin{array}{ccc} [M, BU(m)] & \xrightarrow{\ i\ } & [M, BU] \\ \downarrow r & & \downarrow r \\ [M, BSO(2m)] & \xrightarrow{\ j\ } & [M, BSO] \end{array} \qquad (13.2)$$

where i and j are injections. Since M is 2m-dimensional, i is onto. Hence there is a $y \in [M, BU(m)]$ with $j \circ r(y) = j(\tau(M))$; and $E(\tau(y)) =$ $e_m(y) = e_m(x) = E(\tau(M))$. The result now follows by Theorem 12.1. Since i in (13.2) is isomorphic, there corresponds at most one a.c.s. to each s.a.c.s.

§ 14. Almost Complex Structures on Products of Spheres.

Since $S^p \times S^q$ is s-parallelizable, it admits a s.a.c.s. If p and q are odd, then $S^p \times S^q$ is parallelizable by Théorème XII of [28], so $S^p \times S^q$ admits an a.c.s. Now consider $S^{2p} \times S^{2q}$. Let

$$S^{2p} \xleftarrow{\ \pi_1\ } S^{2p} \times S^{2q} \xrightarrow{\ \pi_2\ } S^{2q}$$

denote the projections, and $\eta : S^{2p} \times S^{2q} \longrightarrow S^{2(p+q)}$ the map collapsing the axes. Write $s = p+q$. Then $\widetilde{K}(S^{2p} \times S^{2q})$ is isomorphic to $\widetilde{K}(S^{2s}) \oplus \widetilde{K}(S^{2p}) \oplus \widetilde{K}(S^{2q})$ where the direct sum decomposition is induced by η, π_1 and π_2. Let $t \in \widetilde{K}(S^{2p} \times S^{2q})$ be $x \oplus y \oplus z$ with respect to this direct sum.

Then by the multiplicative property of the total Chern class,

$$c_s(t) = \eta^*(c_s(x)) + \pi_1^*(c_p(y)) \cdot \pi_2^*(c_q(z)).$$

Now for any $a \in \widetilde{K}(S^{2m})$, $c_m(a)$ is divisible by $(m-1)!$ (by the corollary on p. 56 of [13]). Thus $c_s(t)$ is divisible by the highest common factor (h.c.f.) of $(s-1)!$ and $(p-1)! \cdot (q-1)!$ Now the Euler characteristic of $S^{2p} \times S^{2q}$ is 4. Hence by Theorem 13.1, if $S^{2p} \times S^{2q}$ admits an a.c.s., then p and q must belong to the set $\{1, 2, 3\}$. Since S^2 and S^6 admit a.c.s. (see 41.21 of [47]), so do $S^2 \times S^2$, $S^2 \times S^6$ and $S^6 \times S^6$. The direct sum decompositions of $\widetilde{K}(S^{2p} \times S^{2q})$ and $\widetilde{KO}(S^{2p} \times S^{2q})$ given by the maps η, π_1 and π_2 are compatible with $r : \widetilde{K}(\ ) \longrightarrow \widetilde{KO}(\ )$. Thus since $S^{2p} \times S^{2q}$ is s-parallelizable, $r^{-1}\{\tau(S^{2p} \times S^{2q})\}$ is the set of all $x \oplus y \oplus z$ in $\widetilde{K}(S^{2p} \times S^{2q})$ with each of x, y, z in the kernel of the appropriate r.

Case 1 : $S^4 \times S^6$

If $t = x \oplus y \oplus z$ is a s.a.c.s., then x is divisible by 2, and $y = 0$. Hence $c_5(t)$ is divisible by $2(4!)$, and so $S^4 \times S^6$ does not admit an a.c.s.

Case 2 : $S^4 \times S^4$.

Here $r : \widetilde{K}(S^4 \times S^4) \longrightarrow \widetilde{KO}(S^4 \times S^4)$ is monomorphic, so the only s.a.c.s. is zero, which does not have the correct c_4. Thus $S^4 \times S^4$ does not admit an a.c.s.

Case 3 : $S^2 \times S^4$.

The s.a.c. structures are, all $x \oplus y \oplus 0$ with $x \in \widetilde{K}(S^6)$, and $y \in \widetilde{K}(S^2)$ divisible by 2. If x' generates $\widetilde{K}(S^6)$, then $c_3(2x' \oplus y \oplus 0) = \pm E(\tau(S^2 \times S^4))$. Thus by Theorem 13.1 there exist an infinite number of distinct a.c.s. on $S^2 \times S^4$.

We note that the a.c.s. on $S^2 \times S^2$, $S^2 \times S^6$ and $S^6 \times S^6$ are unique up to bundle equivalence, for a fixed choice of orientation of these manifolds.

Next let P be any product of spheres. Then P admits a s.a.c.s. since it is s-parallelizable; and if any one of the spheres is odd-dimensional, then P is parallelizable.

Lemma 14.2. Let P denote the product $S_1 \times S_2 \times \cdot \cdot \times S_n$, where S_i is a $2d_i$-dimensional sphere, and $d_i \geq 1$. Suppose that P admits an a.c.s. Then for each i, $d_i \in \{1,2,3\}$.

Proof. Observe that $\widetilde{K}(P)$ is isomorphic to $\sum_{\sigma} \widetilde{K}(S^{2J(\sigma)})$ where the summation is over all non-empty subsets σ of

$\{1,2,\ldots,n\}$ and $j(\sigma) = \sum_{i\in\sigma} d_i$. (This follows for example from § 3 of [52].) As before, the top Chern class of P is therefore divisible by $(d_i-1)!$ for any i. The result now follows since the Euler characteristic of P is 2^n.

<u>Theorem 14.3. Let P be the product of l copies of S^2, m copies of S^4 and n copies of S^6. Then P admits an a.c.s. if and only if there exist integers x and y such that $0 \leq x \leq l$, $0 \leq y \leq n+lm+\binom{l}{3}$, $x+y \leq l+m+n$, and $x+3y = l+2m+3n$, where the binomial coefficient $\binom{l}{3}$ is defined to be zero if $l < 3$.</u>

<u>Proof</u>. Let u_P denote the fundamental homology class of P. Consider the expansion $\widetilde{K}(P) \cong \sum_{\sigma} \widetilde{K}(S^{2j(\sigma)})$ as in the previous proof. An a.c.s. exists for P if and only if we can select

(a) a set $\{\sigma_1,\ldots,\sigma_N\}$ of subsets of $\{1,2,\ldots,n\}$ such that $j(\sigma_1)+\ldots+j(\sigma_N) = 2l+4m+6n$ (the dimension of P), and

(b) a $\xi_i \in \widetilde{K}(S^{2j(\sigma_i)})$, for $i = 1,2,\ldots,N$, in the kernel of $r : \widetilde{K}(S^{2j(\sigma_i)}) \longrightarrow \widetilde{KO}(S^{2j(\sigma_i)})$, such that

$$\left\langle c_{j(\sigma_1)}(\xi_1).\ \ldots\ .c_{j(\sigma_N)}(\xi_N)\,,\, u_P\right\rangle = 2^{l+m+n}$$

where $\langle\ \ \rangle$ denotes the Kronecker bracket. Thus each $j(\sigma_i)$ must be in $\{1,2,3\}$. Since $r : \widetilde{K}(S^4) \to \widetilde{KO}(S^4)$ is monomorphic, $j(\sigma_i)$ must be 1 or 3. The number of σ_i with $j(\sigma_i) = 1$ is l, and the number of σ_i with $j(\sigma_i) = 3$ is

$n+1m+(\frac{1}{3})$. If $\xi \in \widetilde{K}(S^2)$ and $\eta \in \widetilde{K}(S^6)$ are in the kernel of $r : \widetilde{K} \to \widetilde{KO}$, then $c_1(\xi)$ and $c_3(\eta)$ are divisible by 2. The result now follows by considering the ways of choosing x copies of S^2 and y copies of S^6 from the collection of $S^{2J(\sigma_1)}$.

§15. A.c.s. and s.a.c.s. on Sphere-bundles over Spheres.

Theorem 15.1. *Let B be the total space of an oriented q-sphere bundle ξ over the n-sphere, with q and n both odd and $n \geq 3$. Then B admits a s.a.c.s. if and only if B is parallelizable.*

Proof. Let the projection of ξ be $\varpi : B \longrightarrow S^n$. Suppose that B admits a s.a.c.s. Let ξ^V be the vector bundle associated with ξ. Then by (1.3), $\varpi^*\{\xi^V\}$ is in the image of $r : \widetilde{K}(B) \longrightarrow \widetilde{KO}(B)$. Consider the following commutative diagram :

$$\begin{tikzcd}
 & & \widetilde{KO}(S^{n+q}) \arrow[d, "p^*"] & \widetilde{K}(S^{n+q}) \arrow[l, "r"'] \arrow[d, "p^*"] \\
\widetilde{KO}(S^{q+1}) \arrow[r, "\lambda^*"] & \widetilde{KO}(S^n) \arrow[r, "\varpi^*"] \arrow[dr, "\varpi'^*"'] & \widetilde{KO}(B) \arrow[d, "k^*"] & \widetilde{K}(B) \arrow[l, "r"'] \\
 & & \widetilde{KO}(S^q \cup_\theta e^n) &
\end{tikzcd} \qquad (15.2)$$

Here we use the cell structure $B = S^q \cup_\theta e^n \cup e^{n+q}$ (section 3 of [27]) : $\pi' = \pi | S^q \cup_\theta e^n$, λ is the suspension of θ, and p collapses $S^q \cup_\theta e^n$. Since q and n are odd, the right-hand p^* is onto. Since $n \geq 3$, B is a spin manifold, and by Lemma 8.1, each p^* in (15.2) is monomorphic. Hence $\pi^*\{\xi^V\} \in$ Image p^*, so $k^*\pi^*\{\xi^V\} = 0$. It follows that $\{\xi^V\} \in \lambda^*\widetilde{KO}(S^{q+1})$, so by Theorems 2.1, 2.2 and 2.3, B is parallelisable.

The questions of existence of a.c.s. and of s.a.c.s. for such a B coincide, since if B is parallelizable, then it admits an a.c.s.

__Theorem 15.3.__ __Let B be the total space of an orientable q-sphere bundle ξ over the n-sphere. Suppose that either n or q is even, and that $n > 2$. Then B admits a s.a.c.s. if and only if one of the following is true :__

(i) __the stable class of the associated vector bundle is a multiple of two;__

(ii) __$n \not\equiv 0, 1 \bmod 8$;__

(iii) __$n \equiv 0$ or $1 \bmod 8$, $q \equiv 6 \bmod 8$, and λ_ξ (see § 2) induces a non-zero map of $\widetilde{KO}^{-1}$.__

__Proof.__ If either (i) or (ii) holds, then $\{\xi^V\}$ is in the image of $r : \widetilde{K}(S^n) \longrightarrow \widetilde{KO}(S^n)$, hence $\pi^*\{\xi^V\}$ is in $r(\widetilde{K}(B))$ and by (1.3) B admits a s.a.c.s. Let $\lambda \in \pi_n(S^{q+1})$ be as in § 2, let $p : B \longrightarrow S^{n+q}$ be the map collapsing $S^q \cup_\theta e^n$, and let $i : S^q \longrightarrow B$ be the inclusion. Then $B/_{S^q}$ has the homotopy type of $S^n \vee S^{n+q}$, and for any space Y we have an exact sequence as follows (c.f. §23):

$$[S^{q+1},Y] \xrightarrow{(\lambda^*,\mu^*)} [S^n,Y] \oplus [S^{n+q},Y] \xrightarrow{\pi^*\oplus p^*} [B,Y] \xrightarrow{i^*} [S^q,Y] \quad ,$$

where $\mu : S^{n+q} \longrightarrow S^{q+1}$ is the attaching map of the $(n+q+1)$-cell in the Thom complex of ξ. Hence when Y is BU or BSO, $\mu^* = 0$ by the Thom isomorphism theorem (c.f. 3). Consider the fibring $U \xrightarrow{r} O \xrightarrow{s} O/U$. By 1.7 of [14], O/U has the weak homotopy type of the loop space on O: and so we have the commutative diagram of exact sequences following:

$$\begin{tikzcd}
\to \widetilde{K}^{-1}(S^q) \arrow[r, "\lambda^*"] \arrow[d, "r"] & \widetilde{K}(S^n)\oplus\widetilde{K}(S^{n+q}) \arrow[r, "\pi^*\oplus p^*"] \arrow[d, "r"] & \widetilde{K}(B) \arrow[r, "i^*"] \arrow[d, "r"] & \widetilde{K}(S^q) \to \arrow[d, "r"] \\
\to \widetilde{KO}^{-1}(S^q) \arrow[r, "\lambda^*"] \arrow[d, "s"] & \widetilde{KO}(S^n)\oplus\widetilde{KO}(S^{n+q}) \arrow[r, "\pi^*\oplus p^*"] \arrow[d, "s"] & \widetilde{KO}(B) \arrow[r, "i^*"] \arrow[d, "s"] & \widetilde{KO}(S^q) \to \arrow[d, "s"] \\
\to \widetilde{KO}^{-2}(S^q) \arrow[r, "\lambda^*"] & \widetilde{KO}^{-1}(S^n)\oplus\widetilde{KO}^{-1}(S^{n+q}) \arrow[r, "\pi^*\oplus p^*"] & \widetilde{KO}^{-1}(B) \arrow[r, "i^*"] & \widetilde{KO}^{-1}(S^q) \to .
\end{tikzcd} \qquad (15.4)$$

Suppose that $n \equiv 0$ or $1 \bmod 8$. If x generates $\widetilde{KO}(S^n)$, then $s(x)$ generates $\widetilde{KO}^{-1}(S^n)$, and from (15.4), $\pi^*(x)$ is in the image of $r : \widetilde{K}(B) \to \widetilde{KO}(B)$ if and only if $s(x)$ is in the image of $\lambda^* : \widetilde{KO}^{-2}(S^q) \to \widetilde{KO}^{-1}(S^n)$. The result follows using Theorem 5.2.

<u>Remarks on (iii).</u> An example of a 6-sphere bundle over S^9 with total space admitting a s.a.c.s. although it is not s-parallelisable may be obtained as follows: since the boundary

$$\partial \; : \pi_8(S^6) \longrightarrow \pi_7(SO(6))$$

in the homotopy sequence of the

standard projection $p : SO(7) \longrightarrow S^6$ is zero (see for example p.162 of [30]), $p_* : \pi_8(SO(7)) \longrightarrow \pi_8(S^6)$ is non-zero. Also the injection $i : \pi_8(SO(7)) \longrightarrow \pi_8(SO)$ is non-zero since π_8 of the real Stiefel manifold $O_{10,3}$ is zero (see for example p.135 of [20]†). Thus there exists a χ in $\pi_8(SO(7))$ with $i_*(\chi)$ and $p_*(\chi)$ both non-zero. By (iii) above and Lemma 2 of [30], the bundle with characteristic element χ is as required.

Is there an $(8r+6)$-sphere bundle ξ over the $8s$-sphere with $\{\xi^V\}$ generating $\widetilde{KO}(S^{8s})$ and total space admitting a s.a.c.s. ? For $r = s-1$ the answer is, "No", for $p_* : \pi_{8s-1}(SO(8s-1)) \longrightarrow \pi_{8s-1}(S^{8s-2})$ is zero whenever $s \geq 2$ by Theorem 1 on p.162 of [30], while any element of $\pi_7(SO(7))$ has stable class a multiple of two (see for example [50]). If $s > 2r+1$, then $\pi_{8s-1}(SO(8r+7))$ is finite (see for example Proposition 10.2 on p.147 of [9] and Corollary 2 on p.289 of [45]) so that any $(8r+6)$-sphere bundle over the $8s$-sphere is then stably trivial. For fixed r, examples of the type asked for exist for at most one s in the range $r+1 < s \leq 2r+1$. For suppose $\chi \in \pi_{8s-1}(SO(8r+7))$ is such that $i_*(\chi)$ and $p_*(\chi)$ are non-zero. Let $\chi' \in \pi_{8s'-1}(SO(8r+7))$. Then $i_*(\chi')$ is a multiple of two if $s' > s$ while $p_*(\chi')$ is zero if $s' < s$.

† The table in [20] shows that $\pi_8(O_{9,2}) \approx Z_2$; the required result now follows e.g. from the homotopy exact sequence of the fibring $O_{10,3} \longrightarrow S^9$, and Theorem 27.11 of [47].

Theorem 15.5. Let B be the total space of an oriented q-sphere bundle over the n-sphere, with q and n even. If B admits an a.c.s., then $q, n \in \{2, 4, 6\}$.

The proof will use an integrality property of the Chern character of any element of $\widetilde{K}(B)$. For any prime p, and any integer r, let $\nu_r(p)$ denote the highest power of p dividing r. Let $m(r)$ be 2 if r is odd; if r is even let $m(r)$ be $2 \prod_p p^{\nu_r(p)+1}$ where the product is over all primes p such that $p-1$ divides r. The following theorem is a mild extension of Theorem 9.7 of [3]:

Theorem 15.6. Let X_r be a CW-complex with cell structure $e_0 \cup e_1 \cup \ldots \cup e_r$ where the dimension of e_i is $2n_i$ and $n_r > n_{r-1} > \ldots > n_0 = 0$. Let h_i denote a generator of the image of $H^{2n_i}(X_r, Z)$ in $H^{2n_i}(X_r, Q)$. Then there exist generators $\xi_1, \xi_2, \ldots, \xi_r$ for $\widetilde{K}(X_r)$ such that

$$\begin{aligned}
ch\, \xi_r &= h_r \\
ch\, \xi_{r-1} &= h_{r-1} + \lambda_{r-1,r} h_r \\
&\vdots \\
ch\, \xi_1 &= h_1 + \lambda_{1,2} h_2 + \ldots + \lambda_{1,r} h_r
\end{aligned}$$

where the $\lambda_{i,j}$ are rational numbers such that

$\prod_{r > k \geq s} m(n_r - n_k) \cdot \lambda_{s,r}$ is an integer for each pair (r, s).

Corollary 15.7. Let B be the total space of an oriented 2r-sphere bundle over the 2t-sphere, with $r < t$. Let $\xi \in \widetilde{K}(B)$. Then the following elements of $H^*(B,Q)$ lie in the image of the integral cohomology :

$$\mathrm{ch}_r(\xi)$$
$$m(t-r).\mathrm{ch}_t(\xi)$$
$$m(t).m(r).\mathrm{ch}_{r+t}(\xi).$$

For a sphere-bundle over a sphere, we can improve on Theorem 15.6.

Theorem 15.8. With the hypotheses of Corollary 15.7, $m(t).\mathrm{ch}_{r+t}(\xi)$ is in the image of the integral cohomology.

Proof of 15.6. We use a two-fold induction. When $r = 1$ the result is well-known. Let X_{r-1} denote the subcomplex $e_0 \cup e_1 \cup \cdot \cdot \cup e_{r-1}$ of X_r and suppose that generators $\xi'_1, \cdot \cdot \cdot, \xi'_{r-1}$ for $\widetilde{K}(X_{r-1})$ have been chosen to satisfy the theorem for X_{r-1}. Consider the $\widetilde{K}$ exact sequence of (X_r, X_{r-1}). Since $\widetilde{K}^1(X_{r-1})$ and $\widetilde{K}^1(X_r/X_{r-1})$ are zero, we obtain a short exact sequence, to which we apply the Chern character :

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}(X_r/X_{r-1}) \arrow[r] \arrow[d, "\mathrm{ch}"] & \widetilde{K}(X_r) \arrow[r] \arrow[d, "\mathrm{ch}"] & \widetilde{K}(X_{r-1}) \arrow[r] \arrow[d, "\mathrm{ch}"] & 0 \\
& H^*(S^{2n_r}, Q) \arrow[r] & H^*(X_r, Q) \arrow[r] & H^*(X_{r-1}, Q).
\end{tikzcd}$$

Since ch is monomorphic on S^{2n_r} and on X_{r-1}, it is clear that we may choose generators $\xi_1, \ldots, \xi_r$ for $\widetilde{K}(X_r)$ such that :

$$\begin{aligned} ch\, \xi_r &= h_r \\ ch\, \xi_{r-1} &= h_{r-1} + \lambda_{r-1,r} . h_r \\ &\vdots \\ ch\, \xi_1 &= h_1 + \lambda_{1,2} h_2 + \cdots + \lambda_{1,r} . h_r \end{aligned}$$

where the $\lambda_{i,j}$ for $j \leq r-1$ are those given by inductive assumption, and $\lambda_{1,r}, \ldots, \lambda_{r-1,r}$ are some rational numbers. We shall prove that the conclusion of the theorem holds for $\lambda_{s,r}$ by decreasing induction on s. Recall that

$$ch_q \Psi^k = k^q . ch_q$$

where Ψ^k is defined on p.615 of [2]. Suppose first that

$$\Psi^k(\xi_{r-1}) = a_1 \xi_1 + \cdots + a_r \xi_r$$

where the a_i are thus integers. Then applying ch, we obtain

$$k^{n_{r-1}} h_{r-1} + \lambda_{r-1,r} k^{n_r} h_r = a_1 h_1 + \cdots + (a_1 \lambda_{1,r} + \cdots + a_r) . h_r .$$

Hence $a_1 = \cdots = a_{r-2} = 0, \quad a_{r-1} = k^{n_{r-1}}$, and

$$a_r = \lambda_{r-1,r}(k^{n_r} - a_{r-1}) = \lambda_{r-1,r}(k^{n_r} - k^{n_{r-1}}).$$

Thus $\lambda_{r-1,r}(k^{n_r} - k^{n_{r-1}})$ is an integer for all k, and by the result from number theory recalled in Appendix D, $m(n_r - n_{r-1})\cdot\lambda_{r-1,r}$ is an integer. Suppose inductively that $\lambda_{j,r}\cdot\prod_{r>i\geq j} m(n_r - n_i)$ is an integer for all j with $r>j>s$. Let $\psi^k(\xi_s) = b_1\xi_1 + \ldots + b_r\xi_r$ where the b_i are integers. Applying ch and equating components we obtain

$$(k^{n_r} - k^{n_s})\,\lambda_{s,r} = b_{s+1}\lambda_{s+1,r} + \ldots + b_r\,.$$

Now the right-hand side multiplied by $\prod_{r>i\geq s+1} m(n_r - n_i)$ is an integer by inductive assumption. Hence

$\lambda_{s,r}(k^{n_r} - k^{n_s})\cdot\prod_{r>i\geq s+1} m(n_r - n_i)$ is an integer for all k,

and the inductive step follows by Appendix D.

Corollary 15.7 follows by the cell structure on B.

Proof of Theorem 15.8. We have the following commutative diagram (c.f. §23) :

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}(S^{2t}\vee S^{2r+2t}) \arrow[r] \arrow[d, "ch"] & \widetilde{K}(B) \arrow[r] \arrow[d, "ch"] & \widetilde{K}(S^{2r}) \arrow[r] \arrow[d, "ch"] & 0 \\
 & H^*(S^{2t}\vee S^{2r+2t}, Q) \arrow[r] & H^*(B,Q) \arrow[r] & H^*(S^{2r},Q) &
\end{tikzcd}$$

Thus there exist generators ξ, η, ζ for $\widetilde{K}(B)$ with

$$\begin{aligned}
ch\,\xi &= h_t\\
ch\,\eta &= h_{r+t}\\
ch\,\zeta &= h_r + \lambda h_t + \mu h_{r+t}
\end{aligned}$$

for some rational λ, μ. Suppose that $\psi^k\zeta = a\xi + b\eta + c\zeta$. Applying ch and equating components we obtain

$$a = \lambda(k^t - k^r)$$

$$b = \mu(k^{t+r} - k^r).$$

Thus by Appendix D, the result follows.

<u>Lemma 15.9.</u> <u>With the hypotheses of Corollary 15.7, $c_{r+t}(\xi)$ is divisible by $l(r,t)$ where</u> :

(1) <u>if $r < t$, $r \neq t/_2$, $l(r,t)$ = h.c.f. of</u> $(r+t-1)!/_{m(t)}$ <u>and</u> $(r-1)!(t-1)!/_{m(t-r)}$;

(2) <u>if $r = t/_2$, $l(r,t)$ = h.c.f. of</u> $(3r-1)!/_{m(2r)}$, $r!(2r-1)!/_{m(r)}$ <u>and</u> $[(r-1)!]^3/_6$;

(3) <u>if $r = t$, $l(r,t)$ = h.c.f. of</u> $(2t-1)!/_{m(t)}$ <u>and</u> $[(t-1)!]^2/_2$;

(4) <u>if $r = 2t$, $l(r,t) = (r+t-1)!$</u> ;

(5) <u>if $r > t$, $r \neq 2t$, $l(r,t)$ = h.c.f. of $(r-1)!(t-1)!$ and</u> $(r+t-1)!/_m$ <u>where m is the l.c.m. of $m(r)$ and $m(t)$</u>.

<u>Proof.</u> Apply Newton's relations to Theorem 15.8 and to the corresponding result for $r \geqslant t$. Recall that if the total Chern class $c(\xi)$ is $\prod(1 + x_i)$, then $ch\,\xi$ is $\sum_i e^{x_i}$. Thus Newton's relations hold between the Chern classes (elementary symmetric functions σ_j in the x_i) and the Chern character components

(symmetric power sums s_j in the x_i, up to a numerical factor). For ξ, the only possibly non-zero c_j are c_0, c_r, c_t, c_{r+t}. Suppose first that $r < t$. Then the relevant Newton's relations are:

<u>Case (a):</u> <u>$t \neq 2r$.</u>

$$s_r + (-1)^r r\sigma_r = 0$$

$$s_t + (-1)^t t\sigma_t = 0$$

$$s_{r+t} + (-1)^r \sigma_r s_t + (-1)^t \sigma_t s_r + (-1)^{r+t}(r+t)\sigma_{r+t} = 0$$

<u>Case (b):</u> <u>$t = 2r$.</u>

$$s_r + (-1)^r r\sigma_r = 0$$

$$s_t + (-1)^r \sigma_r s_r + (-1)^t t\sigma_t = 0$$

$$s_{r+t} + (-1)^r \sigma_r s_t + (-1)^t \sigma_t s_r + (-1)^{r+t}(r+t)\sigma_{r+t} = 0$$

From (a),

$$\sigma_{r+t} = (-1)^{r+t+1}/(r+t) \left\{ s_{r+t} - \left[(r+t)/rt\right] s_r s_t \right\}.$$

From (b),

$$\sigma_{3r} = (-1)^{3r+1}/3r \left\{ s_{3r} - \left(3/2r\right) s_r s_{2r} + (s_r)^3/2r^2 \right\}.$$

Now reinterpreting these in terms of Chern classes and Chern character components, and applying Theorem 15.8, we obtain (1) and (2) of Lemma 15.9. When $r \geqslant t$, the cell structure of B is $(S^{2r} \vee S^{2t}) \cup e^{2(r+t)}$; the method of proof of Theorem 15.6

and 15.8 shows that for $\xi \in \widetilde{K}(B)$, the following elements of $H^*(B, Q)$ come from integral classes:

$$\mathrm{ch}_r\, \xi\,, \quad \mathrm{ch}_t\, \xi\,, \quad m.\mathrm{ch}_{r+t}\, \xi \quad ,$$

where m is the l.c.m. of $m(r)$ and $m(t)$. Newton's relations now give (3) and (5) of Lemma 15.9. The conclusion (4) follows from a method applicable whenever the fibre dimension is divisible by 4, and which is illustrated in Appendix B. This appendix contains details of how Theorem 15.5 follows from Lemma 15.9 in most cases, and special arguments required for some low-dimensional cases.

Chapter VI. Remarks on the Mod 2 Semi-characteristic.

We compute the mod 2 semi-characteristic of the total space of a differentiable fibre bundle in terms of the homology of the base and fibre and the characteristic classes of the bundle, in simple cases. Throughout the chapter, homology and cohomology will be taken with mod 2 coefficients. We emphasise that "manifold" will mean "compact manifold without boundary".

§16. Fibre Totally Non-homologous to Zero.

Consider a differentiable fibre bundle ξ with fibre F and projection $\varpi : B \longrightarrow M$, (7.4 of [11]), where F and M are smooth connected manifolds of dimension q, n, respectively, with $q+n$ odd. Suppose that F is totally non-homologous to zero, mod 2 (see for example pp.170-171 of [10]). Whenever it is defined, denote the Euler characteristic of a space A by $\chi(A)$.

Theorem 16.1. *With hypotheses as above,*

(i) *if q is even, then* $\chi_*(B) \equiv \chi_*(M)\,\chi(F) \mod 2,$

(ii) *if q is odd, then* $\chi_*(B) \equiv \chi(M)\,\chi_*(F) \mod 2.$

Proof. Let f_i, m_i, b_i denote the ranks of $H_i(F)$, $H_i(M)$, $H_i(B)$

respectively. Note that f_i, m_i, b_i are zero when $i < 0$.
Since the fibre is totally non-homologous to zero,

$$b_i = \sum_j m_{i-j} f_j \qquad \text{(see for example p.170 of [10])}.$$

(i) Suppose $q = 2s$.

By mod 2 Poincaré duality in F, $f_{s+j} = f_{s-j}$, so

$$b_i = \sum_{j=0}^{s} m_{i-j} f_j + \sum_{j=1}^{s} m_{i-s-j} f_{s-j} .$$

Hence

$$\chi_*(B) = \sum_{i=0}^{n+q-1/2} b_i$$

$$= \sum_{i=0}^{n+q-1/2} m_{i-s} f_s + \sum_{j=0}^{s-1} f_j \left(\sum_{i=0}^{n+q-1/2} (m_{i-j} + m_{i-(q-j)}) \right).$$

Now $f_s \equiv \chi(F) \mod 2$, so $\displaystyle\sum_{i=0}^{n+q-1/2} m_{i-s} f_s \equiv \chi(F)\chi_*(M) \mod 2.$

Also, for any j,

$$\sum_{i=0}^{n+q-1/2} (m_{i-j} + m_{i-(q-j)}) = \sum_{k=-j}^{(n+q-1/2)-j} m_k + \sum_{k=j-q}^{(n-q-1/2)+j} m_k$$

$$\equiv \sum_{k=(n-q+1/2)+j}^{(n+q-1/2)-j} m_k \mod 2,$$

$$\equiv 0 \mod 2, \text{ by mod 2 Poincaré}$$

duality in M, since

$$n - (n+q-1/2)-j = (n-q+1/2)+j .$$

(ii) Suppose $q = 2s+1$.

As in the proof of (i),

$$\chi_*(B) \equiv \sum_{j=0}^{s} f_j \left(\sum_{i=0}^{n+q-1/2} (m_{i-j} + m_{i+j-q}) \right) \mod 2,$$

$$\equiv \chi_*(B)\,\chi(M) \quad \mod 2, \text{ since for any } j,$$

$$\sum_{k=(n-q+1/2)+j}^{(n+q-1/2)-j} m_k \equiv \chi(M) \mod 2.$$

§17. Case of a Sphere-bundle.

Let ξ be a q-sphere bundle with projection $\pi : B \longrightarrow M$, where M is a smooth connected n-dimensional manifold. Suppose that $n+q = 2s+1$. Let $\phi_i : H^{i-q-1}(M) \longrightarrow H^i(M)$ denote the homomorphism obtained by taking the cup-product with the Stiefel-Whitney class $w^{q+1}(\xi)$. By the rank of a homomorphism of vector spaces we mean the dimension of the image subspace.

Theorem 17.1. With the above notation,

$$\chi_*(B) \equiv \chi(M) + \operatorname{rank} \phi_{s+1} \mod 2.$$

Proof. Since the homology groups of M and B are finite-dimensional vector spaces over the coefficient field, their ranks are equal to the ranks of the corresponding cohomology groups. Consider the mod 2 cohomology Gysin sequence for ξ :

$$\cdots \to H^{i-q-1}(M) \xrightarrow{\phi_i} H^i(M) \xrightarrow{\pi_i} H^i(B) \xrightarrow{\psi_i} H^{i-q}(M) \xrightarrow{\phi_{i+1}} \cdots$$

The ranks of $H^i(M)$, $H^i(B)$ will be denoted by m_i, b_i, and the rank of ϕ_i by r_i. (So $r_i = 0$ whenever $i < q-1$.) From the Gysin exact sequence,

$$b_i = m_i - r_i + m_{i-q} - r_{i+1} .$$

Hence

$$\chi_*(B) = \sum_{i=0}^{s} (m_i + m_{i-q}) - \sum_{i=0}^{s} (r_i + r_{i+1}).$$

The first summation is congruent to $\chi(M)$ mod 2, and the second is congruent to r_{s+1} mod 2.

Examples :

(i) For the tangent sphere bundle to a connected n-manifold M, the homomorphism concerned is $\smile w^n(M) : H^0(M) \longrightarrow H^n(M)$, which has rank 0 or 1 according as $w^n(M)$ is zero or not, i.e. according as $\chi(M)$ is even or odd. Hence $\chi_*(B)$ is even.

(ii) Let ξ be the Hopf bundle $S^{n-1} \to S^{2n-1} \to S^n$ where n is 2,4 or 8. Then $w^n(\xi) \neq 0$, while $\chi(S^n)$ is even, so 17.1 says that $\chi_*(S^{2n-1})$ is odd, which is correct.

Remark. It would be sufficient for Theorem 17.1 to have a locally trivial fibre space with a sphere as fibre.

When ξ is a principal $SO(q+1)$-bundle with total space B, we write $B_i(\xi) = B/_{SO(q-i+1)}$ (c.f. §1).

Corollary 17.2. Let ξ be a principal $SO(q+1)$-bundle over a smooth manifold M, such that $w^{q-i+2}(\xi) = 0$ for some $i > 1$. If $B_i(\xi)$ is odd-dimensional, then $\chi_*(B_i(\xi))$ is even.

Proof. Write $B_i = B_i(\xi)$. Let $\pi_i : B_i \longrightarrow M$ be the natural projection. Let η_i be the bundle with projection $p_i : B_i \longrightarrow B_{i-1}$ and fibre S^{q-i+1} obtained from the inclusion $SO(q-i+1) \subset SO(q-i+2)$. Then $w^{q-i+2}(\eta_i) = \pi^*_{i-1}(w^{q-i+2}(\xi)) = 0$, and Theorem 17.1 applied to η_i yields

$$\chi_*(B_i) \equiv \chi(B_{i-1}) \mod 2 ;$$

but $$\chi(B_{i-1}) = \chi(M)\chi(O_{q+1,i-1})$$ which is even.

Corollary 17.3. Let ξ be a stably trivial principal $SO(q+1)$-bundle over an s-parallelisable n-manifold M. Then $B_i(\xi)$ is parallelizable whenever $i \geq 2$; and $B_1(\xi)$ is parallelizable unless either

(i) n and q are even and $\chi(M)$ is non-zero, or

(ii) $\chi(M)$ and q are odd and $n+q$ is not 1, 3 or 7.

Proof. By Corollary 17.2 and Theorem 4.1, it is sufficient to check that $B_i(\xi)$ is always s-parallelizable. This will follow from (19.3).

§18. One More Case.

Let ξ be a differentiable principal fibre bundle with fibre $SO(q+1)$ and projection $\pi : B \longrightarrow M$, where M is a smooth connected n-dimensional manifold. Let B_2 be the total space of the associated bundle with fibre the real Stiefel manifold $O_{q+1,2}$. Let $n = 2s$, so that the dimension of B_2 is $2s+2q-1$. Let

$$H^{i+q}(M) \xleftarrow{\ w\ } H^i(M) \xrightarrow{\ w'\ } H^{i+q+1}(M)$$

be the homomorphisms obtained by taking the cup-product with $w^q(\xi)$, $w^{q+1}(\xi)$ respectively. Let $R^{i+q+1} \subset H^{i+q+1}(M)$ denote the image of w'. Let $S^{i+q} \subset H^{i+q}(M)$ denote the image under w of the kernel of w'. Let $T^i \subset H^i(M)$ denote the intersection of the kernels of w and w'. Whenever $x \in H^i(M)$, $y \in H^j(M)$ and $z \in H^k(M)$ are such that $x \cup y$ and $y \cup z$ are zero, let $\langle x,y,z \rangle$ denote the Massey triple product (see p.366 of [51]) whose value is a coset in $H^{i+j+k-1}(M)$ of the subgroup $H^{i+j-1}(M) \cup z + x \cup H^{j+k-1}(M)$. From lines 1-12 on p.368 of [51], the union of all the cosets $\langle w^q(\xi),\ v,\ w^{q+1}(\xi) \rangle$ as v ranges over T^i is a subgroup of $H^{i+2q}(M)$; we denote it by $\langle w,\ T^i,\ w' \rangle$.

<u>**Theorem 18.1.**</u> <u>With notation as above,</u>

$$\chi_*(B_2) \equiv \operatorname{rank}\langle w, T^{s-q}, w'\rangle + \operatorname{rank} R^{s+q} + \operatorname{rank} S^s \mod 2.$$

Proof. Let η_2 be the bundle with projection $p_2 : B_2 \longrightarrow B_1$ and fibre S^{q-1}, as in (17.2). Thus $w^q(\eta_2) = \pi_1^* w^q(\xi)$. Since $\chi(B_1)$ is $\chi(S^q)\chi(M)$ which is even, $\chi_*(B_2)$ is congruent mod 2 to the rank of

$$\pi_1^* w^q(\xi) \smile \; : H^s(B_1) \longrightarrow H^{s+q}(B_1)$$

by Theorem 17.1. Consider the Gysin sequence for ξ_1 :

$$\cdots \to H^{i-q-1}(M) \xrightarrow{w'} H^i(M) \xrightarrow{\pi_1^*} H^i(B_1) \xrightarrow{\psi} H^{i-q}(M) \xrightarrow{w'} \cdots$$

According to §3 of [36]: if $x \in H(M)$, $y \in H(B_1)$, then

$$\psi(\pi_1^*(x) \smile y) = x \smile \psi(y) \quad \ldots \quad (1),$$

while if $u \in H(M)$ and $v \in H(M)$ with $u \smile v = 0$, and $w^{q+1}(\xi) \smile v = 0$, then

$$\pi_1^{*-1}\left[\pi_1^*(u) \smile \psi^{-1}(v)\right] = \langle u, v, w^{q+1}(\xi)\rangle \quad \ldots (2).$$

Denote the subgroup $\pi_1^*(w^q(\xi)) \smile H^s(B_1)$ by X. Then from (1) and the Gysin exact sequence, $\psi(X) = S^s$. It remains to add the rank of $X \cap \operatorname{kernel} \psi$. Now by exactness of the Gysin sequence,

$$\pi_1^{*-1}(X \cap \operatorname{kernel} \psi) = \pi_1^{*-1}(\pi_1^*(w^q(\xi)) \smile \psi^{-1}(T^{s-q})),$$

$$= \langle w, T^{s-q}, w'\rangle \qquad \text{by (2).}$$

Hence $X \cap \text{kernel } \psi$ is isomorphic to the quotient of $\langle w, T^{s-q}, w' \rangle$ by $\langle w, T^{s-q}, w' \rangle \cap R^{s+q}$. But R^{s+q} is part of the indeterminacy subgroup for all the cosets in $\langle w, T^{s-q}, w' \rangle$, so $R^{s+q} \subset \langle w, T^{s-q}, w' \rangle$, and $\text{rank}(X \cap \text{kernel } \psi)$ is equal to $\text{rank} \langle w, T^{s-q}, w' \rangle - \text{rank } R^{s+q}$. The result now follows.

Corollary 18.2. *With the hypotheses of Theorem 18.1, assume in addition that $w^{q+1}(\xi)$ is zero. Then*

$$\chi_*(B_2) \equiv \text{Rank}(w^q(\xi) \cup H^{s-q}(M)) + \text{rank}(w^q(\xi) \cup H^s(M)) \mod 2.$$

Proof. In this case, the rank of $\langle w, T^{s-q}, w' \rangle$ is equal to the rank of the indeterminacy subgroup $w^q(\xi) \cup H^s(M)$. Corollary 18.2 could also be deduced immediately from Part II of [36].

§19. Application.

Let ξ be a principal differentiable $SO(q+1)$-bundle over a smooth manifold M of dimension n. With the notation of §17, write $B_i = B_i(\xi)$. Recall that η_i is the $(q+1-i)$-sphere bundle with projection $p_i : B_i \longrightarrow B_{i-1}$ induced by the inclusion $SO(q+1-i) \subset SO(q+2-i)$. We obtain a commutative diagram as follows:

$$\begin{tikzcd}
B_1 \arrow[d, "\pi_1"] & B_2 \arrow[l, "p_2"'] \arrow[dl, "\pi_2"] & \cdots \arrow[l, "p_3"'] & B_{q+1} \arrow[l, "p_{q+1}"'] \arrow[dlll, "\pi_{q+1}"] \\
M & & &
\end{tikzcd}$$

As before, let ξ^V be the vector bundle associated with ξ, $\{\xi^V\}$ its stable class, and $\hat{\xi}$ the bundle along the fibres.

Lemma 19.1. *With the above notation, $\hat{\eta}_i$ is equivalent to $\varpi_i^*(\xi^V)$.*

Proof. We first observe that $\hat{\eta}_i$ is equivalent to η^V_{i+1}. For, writing $B = B_{q+1}$, consider the diagram

$$\begin{array}{ccccc}
SO(q-i+2)/SO(q-i+1) & \longleftarrow & SO(q-i+2)/SO(q-i) & \longleftarrow & S^{q-i} \\
\downarrow & & \downarrow & & \| \\
B_i & \xleftarrow{\hat{\varpi}} & \hat{B} & \longleftarrow & S^{q-i} \\
{\scriptstyle p_i}\downarrow & & & & \\
B_{i-1}, & & & &
\end{array} \qquad (19.2)$$

where $\hat{B}$, $\hat{\varpi}$ refer to the bundle associated with $\hat{\eta}_i$, with sphere fibre. Since the top line in (19.2) describes the tangent sphere-bundle to S^{q+1-i}, by definition we have

$$\begin{aligned}
\hat{B} &= B \times_{SO(q-i+1)} SO(q-i+2)/SO(q-i) \\
&= B/SO(q-i) \\
&= B_{i+1},
\end{aligned}$$

and $\hat{\varpi}$ coincides with p_{i+1}. The result now follows immediately from Satz 3.4.4 of [23].

By induction on i it follows that

$$\{\tau(B_i)\} = \pi_i^*(\{\tau(M)\} + i\{\xi^\vee\}) \quad . \quad . \quad (19.3).$$

For $i = 1$ this is (1.3). Suppose inductively that it holds for $i = k \geq 1$. Then

$$\begin{aligned}\{\tau(B_{k+1})\} &= p_{k+1}^*\{\tau(B_k)\} + \{\hat{\eta}_{k+1}\} \\ &= \pi_{k+1}^*(\{\tau(M)\} + (k+1)\{\xi^\vee\})\end{aligned}$$

by inductive hypothesis and Lemma 19.1. In particular if π is the projection of ξ, then $(q+1)\,\pi^*(\widetilde{KO}(M))$ is zero, since $\hat{\xi}$ is trivial. Suppose now that M is an almost parallelizable manifold and that ξ is its tangent bundle. Let p_r denote the rth Pontryagin class of M. Then B_i is parallelizable if either

(i) i is odd and $n \not\equiv 0 \bmod 4$, or

(ii) $n = 4m$, and $(i+1)\,p_m(M) \equiv 0 \bmod a_m((2m-1)!)E(\xi)$,

where $a_m = 1$ or 2 according as m is even or odd. For in case (i), $\widetilde{KO}(S^n)$ is zero or Z_2, so $(i+1)\,\widetilde{KO}(S^n)$ is zero, and if $p : M \rightarrow S^n$ collapses the $(n-1)$-skeleton of M, then $\{\tau(B_i)\}$ is in $\pi_i^* p^*((i+1)\,\widetilde{KO}(S^n))$ by (19.3). Thus B_i is s-parallelizable, so by Theorem 4.1 it is parallelizable. In case (ii), the criterion named ensures that $\{\tau(B_i)\}$ is in the image under $\pi_i^* p^*$ of the kernel of $\pi_i^* : \widetilde{KO}(S^n) \longrightarrow \widetilde{KO}(B_i)$, where π_S is the projection of a sphere-bundle over S^n such that there is a bundle map

$$\begin{array}{ccc} B_1 & \xrightarrow{\tilde{p}} & B_S \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_S} \\ M & \xrightarrow{p} & S^n . \end{array}$$

(See Lemma 1.1 of [32], and Theorem 5.1.) It would be interesting to find necessary conditions here.

Chapter VII. Miscellaneous Results.

We obtain partial results related to previous chapters: an analogue of Theorem 12.1 for certain (4k+1)-manifolds, and a limited relation between the mod 2 semi-characteristic and a certain secondary characteristic class; preliminary remarks on vector fields on sphere-bundles; and an approach, based on [26], to the additive KO theory of a bundle over a suspension. Manifolds will be smooth, compact, and without boundary, throughout.

§20. Odd-dimensional Manifolds.

If the standard n-sphere has r linearly independent tangent vector fields, then so has any n-dimensional s-parallelizable manifold. This may be proved following pp.245-246 of [28], by substituting homotopies into smaller Grassmannians for homotopies to a point. It also follows immediately by the method of proof of the next lemma, which is a partial converse :

Lemma 20.1. Let M be an r-connected s-parallelisable (2m+1)-manifold with $\chi_*(M)$ odd. If M admits r linearly independent tangent vector fields then so does the (2m+1)-sphere.

Proof. Let the principal decomposition of M be $M' \cup_f e$ (see §1) so that $\dim M' < l$, where $l = 2m+1-r$. We obtain the following commutative diagram of Puppe sequences :

$$\begin{tikzcd}
{[SM', BSO(1)]} \arrow[r, "Sf^*"] \arrow[d, "i^*"] & {[S^{2m+1}, BSO(1)]} \arrow[r, "Qf^*"] \arrow[d, "i^*"] & {[M, BSO(1)]} \arrow[r, "Pf^*"] \arrow[d, "i^*"] & {[M', BSO(1)]} \arrow[d, "i^*"] \\
{[SM', BSO(2m+1)]} \arrow[r, "Sf^*"] & {[S^{2m+1}, BSO(2m+1)]} \arrow[r, "Qf^*"] & {[M, BSO(2m+1)]} \arrow[r, "Pf^*"] & {[M', BSO(2m+1)]}
\end{tikzcd} \qquad (20.2)$$

where i is the injection. If $m = 0, 1$ or 3, the lemma is trivially true since then S^{2m+1} is parallelizable. Suppose now that $m \neq 0, 1, 3$. Let $\tau \in [M, BSO(2m+1)]$ represent the tangent bundle of M. Since M is s-parallelizable, $\tau = Qf^*(x)$ for some $x \in [S^{2m+1}, BSO(2m+1)]$. By Lemma [illegible].1, x is stably trivial. Hence x must represent the tangent bundle of S^{2m+1}, otherwise M would be parallelizable, contradicting Theorem 4.1. By hypothesis, $\tau = i^*(\tau')$ for some $\tau' \in [M, BSO(1)]$. Since $\dim M' < 1$, $\tau' = Qf^*(y)$ for some $y \in [S^{2m+1}, BSO(1)]$. By strengthened exactness of the lower Puppe sequence in (20.2), $i_*(y) - x = Sf^*(z)$ for some $z \in [SM', BSO(2m+1)]$. But the left-hand i_* in (20.2) is onto since $\dim SM' \leq 1$, hence $x \in i_*[S^{2m+1}, BSO(1)]$ as required.

Let ξ be an $(n-1)$-plane bundle over an n-manifold M, and suppose that $X(\xi) = 0$. Then the obstruction to the existence of a non-zero cross section of ξ is the secondary characteristic class $\Phi_2(\xi)$, which lies in the quotient group $H^n(M, Z_2)/(Sq^2 + w^2(\xi) \cup) H^{n-2}(M, Z_2)$. (See for example [42].)

Lemma 20.3. With the above notation, suppose that M is simply-connected and that $\xi \oplus 1$ is trivial. Then $\Phi_2(\xi) = 0$.

Proof. Consider the Puppe sequences of the principal decomposition $M' \cup_f e$ of M into the fibring $S^{n-1} \xrightarrow{k} BSO(n-1) \xrightarrow{i} BSO(n)$ (c.f. § 1) :

$$\begin{array}{ccccc}
[S^n, S^{n-1}] & \xrightarrow{k_*} & [S^n, BSO(n-1)] & & \\
\downarrow Qf^* & & \downarrow Qf^* & & \\
[M, S^{n-1}] & \xrightarrow{k_*} & [M, BSO(n-1)] & \xrightarrow{i_*} & [M, BSO(n)]
\end{array} .$$

Since $i_*(\xi) = 0$, $\xi = k_*(x)$ for some $x \in [M, S^{n-1}]$. Since $\dim M' < (n-1)$, $x = Qf^*(y)$ for some $y \in [S^n, S^{n-1}]$. Consider the homotopy sequence of the standard projection $O_{n,2} \longrightarrow S^{n-1}$: since $\pi_{n-1}(O_{n,2})$ has a Z_2 summand and $\pi_{n-1}(S^{n-1})$ is free, the injection $\pi_{n-1}(S^{n-2}) \longrightarrow \pi_{n-1}(O_{n,2})$ is non-zero, hence the boundary $\pi_n(S^{n-1}) \longrightarrow \pi_{n-1}(S^{n-2})$ is zero; i.e. the composition $\pi_n(S^{n-1}) \xrightarrow{\partial} \pi_{n-1}(SO(n)) \xrightarrow{p_*} \pi_{n-1}(S^{n-2})$ is zero, where ∂ is the boundary in the homotopy sequence of the standard projection $SO(n) \longrightarrow S^{n-1}$ and $p : SO(n-1) \longrightarrow S^{n-2}$ is the standard projection. Hence $k_*(y) \in j_*[S^n, BSO(n-2)]$ where $j : BSO(n-2) \longrightarrow BSO(n-1)$ is the injection. Thus $\xi \in j_*[M, BSO(n-2)]$, so $\Phi_2(\xi) = 0$.

<u>Theorem 20.4. Let M be a 2-connected, s-parallelizable n-manifold, where $n = 4k+1$. Denote the fundamental class of M by $u \in H_n(M, Z)$. Let the tangent bundle of M be $\tau(M) = \xi \oplus 1$. Let $x \in H^n(M, Z_2)$ be in the coset determined by $\Phi_2(\xi)$. Then $\langle x, u \rangle \equiv \chi_*(M) \bmod 2$.</u>

<u>Proof.</u> Since M is a spin manifold, the indeterminacy subgroup of $\Phi_2(\xi)$ is zero. If $\Phi_2(\xi)$ is non-zero, then $\tau(M)$ is non-trivial by Lemma 20.3, so $\chi_*(M)$ is odd by Theorem 4.1. If $\chi_*(M)$ is odd, then by Lemma 20.1 and Theorem 27.11 of [47], M does not admit 2 linearly independent tangent vector fields, so $\Phi_2(\xi)$ is non-zero.

<u>Theorem 20.5. Let M be a simply-connected $(4k+1)$-dimensional spin manifold. Let η_1, η_2 be stably equivalent spin$(4k+1)$-bundles over M. Then η_1, η_2 are equivalent if and only if there exist bundles ξ_1, ξ_2, with $\eta_1 = \xi_1 \oplus 1$, $\eta_2 = \xi_2 \oplus 1$, and $\Phi_2(\xi_1) = \Phi_2(\xi_2)$.</u>

<u>Proof.</u> If η_1 and η_2 are equivalent, such ξ_1 and ξ_2 exist; for since η_1 is an odd-dimensional bundle, $E(\eta_1)$ is of order 2. But $E(\eta_1) \in H^{4k+1}(M, Z)$ which is torsionfree, hence $E(\eta_1) = 0$.

Suppose that such ξ_1, ξ_2 exist. Write $n = 4k+1$. Let the principal decomposition of M be $M' \cup_f e$. Consider the following commutative diagram of Puppe sequences :

$$\begin{tikzcd}
{[S^n, BSO(n-1)]} \arrow[r, "j_*"] \arrow[d, "Qf^*"] & {[S^n, BSO(n)]} \arrow[d, "Qf^*"] & & \\
{[M, BSO(n-1)]} \arrow[r, "j_*"] \arrow[d, "Pf^*"] & {[M, BSO(n)]} \arrow[r, "k_*"] \arrow[d, "Pf^*"] & {[M, BSO(n+1)]} \arrow[d, "Pf^*"] & \\
{[M', BSO(n-1)]} \arrow[r, "j_*"] & {[M', BSO(n)]} \arrow[r, "k_*"] & {[M', BSO(n+1)]} &
\end{tikzcd} \qquad (20.6)$$

where j and k are inclusions. Since $k_* j_*(\xi_1) = k_* j_*(\xi_2)$, and $\dim M' < n-1$, $Pf^*(\xi_1) = Pf(\xi_2)$, so $\xi_1 = \alpha \cdot \xi_2$ for some $\alpha \in [S^n, SO(n-1)]$. By naturality and Lemma 3.1, α is stably trivial, so either $j_*(\alpha)$ is trivial, or $j_*(\alpha) = \tau(S^n)$. Since $\Phi_2(\xi_1) = \Phi_2(\xi_2)$, it follows that $\Phi_2(\alpha) = 0$ (by (10.1) and the universal definition of Φ_2 in [42]), hence α admits a cross-section. But S^n does not admit 2 linearly independent tangent vector fields when $n = 4k+1$ (Theorem 27.11 of [47]) so $j_*(\alpha) \neq \tau(S^n)$. Hence $j_*(\alpha)$ is trivial, and $\eta_1 = j_*(\alpha) \cdot \eta_2 = \eta_2$.

Remark. If M is a 2-connected $(4k+1)$-manifold, and ξ_1, ξ_2, are $4k$-plane bundles over M with $\xi_1 \oplus 1$ and $\xi_2 \oplus 1$ equivalent, then $\Phi_2(\xi_1) = \Phi_2(\xi_2)$. This follows from the next lemma:

Lemma 20.7. *Let M be a 2-connected $(4k+1)$-manifold. Let $\eta \in [M, BSO(4k+1)]$, $\beta \in [S^{4k+1}, BSO(4k+1)]$ with $\beta \cdot \eta = \eta$. Then β is trivial.*

For, under the hypotheses of the remark, and referring to (20.6), $\xi_1 = \alpha \cdot \xi_2$ and $j_*(\alpha) \cdot j_*(\xi_2) = j_*(\xi_1) = j_*(\xi_2)$. Assuming the lemma, then $j_*(\alpha)$ is trivial. Hence by (20.3), $\Phi_2(\alpha) = 0$, and $\Phi_2(\xi_1) = \Phi_2(\xi_2)$ as claimed.

Proof of Lemma 20.7. We refer to [8]. Let the principal decomposition of M be $M' \cup_f e$. Choose the K on p.57 of [8] to be M', the L to be M, $\alpha \in \pi_{4k}(M')$ the class of f, and $u : M' \longrightarrow BSO(4k+1)$ any map extending to M. Our assertion in (20.7) is a corollary of Theorem 3.3 of [8], provided that the Barcus-Barratt homomorphism

$$\alpha_u : \pi_1(BSO(4k+1)^{M'}, u) \longrightarrow \pi_{4k+1}(BSO(4k+1))$$

is zero. For any $x \in \pi_1(BSO(4k+1)^{M'}, u)$, we have a corresponding homotopy class of maps represented by, say,

$$F : S^1 \times M' \longrightarrow BSO(4k+1),$$

where $F(S^1 \times *) = *$ and $F(*, m) = u(m)$ for any $m \in M'$, $*$ denoting any base-point. Since the dimension of $S^1 \times M'$ is at most $(4k-1)$, there is a map $F' : S^1 \times M' \longrightarrow BSO(4k-1)$ making the following diagram commute up to homotopy:

$$\begin{tikzcd} S^1 \times M' \arrow[r, "F'"] \arrow[dr, "F"'] & BSO(4k-1) \arrow[d, "i"] \\ & BSO(4k+1) \end{tikzcd}$$

where i is the injection. Consider the composition

$$S^1 \times S^{4k} \xrightarrow{1 \times f} S^1 \times M' \xrightarrow{F'} BSO(4k-1).$$

This is trivial on $S^1 \times *$. On $* \times S^{4k}$ its homotopy class is stably trivial by Lemma 8.1, hence trivial in $BSO(4k)$ by Theorem 12.1 (since it factors through $* \times M'$), hence trivial in $BSO(4k-1)$ since $\partial : \pi_{4k}(S^{4k-1}) \longrightarrow \pi_{4k-1}(SO(4k-1))$ is zero (see for example p.162 of [30]). Thus if $p : S^1 \times S^{4k} \longrightarrow S^{4k+1}$ denotes the map collapsing the axes, there is a map $G' : S^{4k+1} \longrightarrow BSO(4k-1)$ making the following diagram commute up to homotopy :

$$\begin{tikzcd} S^1 \times S^{4k} \arrow[r, "1 \times f"] \arrow[d, "p"] & S^1 \times M' \arrow[r, "F'"] & BSO(4k-1) \\ S^{4k+1} \arrow[urr, "G'"] & & \end{tikzcd}$$

and such that $i \cdot G' : S^{4k+1} \longrightarrow BSO(4k+1)$ represents $\alpha_u(x)$. Now in the notation of [8], if X is an H-space then $\alpha_u(\pi_1(X^K, u)) = S^*[SK, X]$, so $\alpha_u(x)$ is stably trivial by Lemma 8.1. Also, $\alpha_u(x) \neq \tau(S^{4k+1})$ since $i \cdot G'$ factors through $BSO(4k-1)$. Hence $\alpha_u(x)$ is trivial as required.

§21. Vector Fields on Sphere-bundles.

Let ξ be an oriented q-sphere bundle over a smooth n-dimensional manifold M, with projection $\pi : B \longrightarrow M$. If q is even and $\chi(M)$ is non-zero, then $\chi(B)$ is non-zero so B admits no tangent vector fields which are everywhere non-zero.

Remark 1. Let M be s-parallelizable. Let r be the least integer such that the group of ξ is reducible to $SO(r)$, in SO. Suppose that either

(i) $\chi(M) = 0$, and the bundle of 2-frames associated with ξ admits a cross-section, or

(ii) q is odd, $\chi(M) \neq 0$, and ξ admits a cross-section.

Then B admits exactly $(n+q-r)$ linearly independent tangent vector fields. For if (i) holds, then $\tau(M)$ is $\alpha \oplus 1$ for some α, and

$$\begin{aligned} \tau(B) &= \pi^*(\alpha) \oplus \pi^*(\xi^V) \quad \text{by (1.1) and (1.2)} \\ &= \pi^*(\alpha) \oplus \pi^*(\beta \oplus 2) \quad \text{for some } \beta, \\ &= \pi^*(\tau(M) \oplus 1) \oplus \pi^*(\beta) \\ &= \pi^*(\xi^V) \oplus (n-1) \quad \text{since M is s-parallelizable,} \end{aligned}$$

whence the result. If (ii) holds, then the Euler class of $\pi^*(\tau(M))$ is non-zero. Consider $\tau(B) = \pi^* \tau(M) \oplus \hat{\xi}$. The Euler class is multiplicative with respect to $\oplus$. Thus if $E(\hat{\xi}) \neq 0$, it follows by oriented Poincare duality in B that

$\chi(B)$ is non-zero, which is false since q is odd. Thus $\hat{\xi} = \zeta \oplus 1$ for some ζ, and

$$\begin{aligned}\tau(B) &= (n+1) \oplus \zeta \quad \text{since } M \text{ is s-parallelisable,}\\ &= (n-1) \oplus \pi^*(\xi^V) \quad \text{and the result follows.}\end{aligned}$$

As a weak analogue of Corollary 12.4 and Theorem 20.5, we observe the following :

<u>Remark 2.</u> Let M be an $(r+1)$-connected spin(n)-manifold, where r is the maximum number of linearly independent tangent vector fields on the n-sphere. Let ξ be an n-plane bundle over M, whose group is reducible to $SO(n-r-1)$. Let H be a closed subgroup of $SO(n-r-1)$. Then the group of ξ is reducible to H if and only if the group of $\xi \oplus 1$ is reducible to H. For let the principal decomposition of M be $M' \cup_f e$. Let B_H denote the classifying space for H. We obtain the following commutative diagram of Puppe sequences :

$$\begin{tikzcd}
 & {[S^n, BSO(n-r-1)]} \arrow[r, "j_*"] \arrow[d, "Qf^*"] & {[S^n, BSO(n)]} \arrow[r, "k_*"] \arrow[d, "Qf^*"] & {[S^n, BSO]} \arrow[d, "Qf^*"] \\
{[M, B_H]} \arrow[r, "i_*"] & {[M, BSO(n-r-1)]} \arrow[r, "j_*"] \arrow[d, "Ef^*"] & {[M, BSO(n)]} \arrow[r, "k_*"] & {[M, BSO]} \arrow[d, "Ef^*"] \\
 & {[M', BSO(n-r-1)]} \arrow[rr, "k_* j_*"] & & {[M', BSO]}
\end{tikzcd}$$

where i, j, k are injections. If the group of ξ can be reduced to H, so can that of $\xi \oplus 1$. Now suppose the group of $\xi \oplus 1$ is reducible to H. Then $k_*(\xi) = k_* j_* i_*(x)$ for some x in $[M, B_H]$. Also $\xi = j_*(y)$ for some y in $[M, BSO(n-r-1)]$. Thus since $\dim M' < n-r-1$, $Pf^*(y) = Pf^* i_*(x)$, and so $y = \alpha . i_*(x)$ for some $\alpha \in [S^n, BSO(n-r-1)]$. This α is stably trivial by Lemma 5.1, and hence $j_*(\alpha)$ is trivial since S^n does not admit $(r+1)$ linearly independent tangent vector fields. Thus $\xi = j_*(y) = j_* i_*(x)$ as required.

§ 22. The Additive KO Theory of a Sphere-bundle over a Sphere.

We use the notation of § 3. If the bundle admits a cross-section, then the exact sequence of § 3 splits, and $\widetilde{KO}^i(B)$ is the direct sum of $\widetilde{KO}^i(S^n)$ and $\widetilde{KO}^{i-q}(S^n)$. The next simplest case is that of the tangent bundle of a $2m$-sphere. Here B has the S-type (see eg. [24]) of $S^{2m-2}(P) \vee S^{4m-1}$ where P denotes the real projective plane. Thus $\widetilde{KO}^*(B)$ is known. In the general case § 3 gives the exact sequence

$$0 \longrightarrow \widetilde{KO}^i(S^n)/\lambda^* \widetilde{KO}^i(S^{q+1}) \longrightarrow \widetilde{KO}^i(B) \longrightarrow \widetilde{KO}^i(S^{n+q}) \oplus \text{kernel}\,\lambda^* \longrightarrow 0$$

where the right-hand λ^* has domain $\widetilde{KO}^{i+1}(S^{q+1})$. The analogous exact sequence obtained by first "smashing" every space with P solves the group extension problems posed. (The Thom isomorphism still holds, c.f. Appendix D.) Thus additively $\widetilde{KO}^*(B)$ depends only on λ^*. I am indebted to R.M.Wood for the following remarks on $\lambda * 1 : S^n P \longrightarrow S^{q+1} P$: given a map $\lambda : S^m \longrightarrow S^{8k}$, then by naturality of the tensor product, and Bott periodicity in the form given in [15], there is a commutative diagram as follows :

$$\begin{tikzcd}
\widetilde{KO}^i(X) \otimes \widetilde{KO}(S^{8k}) \arrow[r, "1 \otimes \lambda^*"] \arrow[d, "\cong"] & \widetilde{KO}^i(X) \otimes \widetilde{KO}(S^m) \arrow[d, "\cong"] \\
\widetilde{KO}^i(X * S^{8k}) \arrow[r, "(1 * \lambda)^*"] & \widetilde{KO}^i(X * S^m),
\end{tikzcd}$$

where X is any finite CW-complex. The effect of $(1 * \lambda)^*$ is therefore the same as the module action of $\lambda^*(1)$ on $\widetilde{KO}^i(X)$.

Thus to compute

$$(1 * \lambda)^* \quad : \widetilde{KO}^i(S^{q+1}P) \longrightarrow \widetilde{KO}^i(S^n P)$$

we need only know λ^* on spheres and the module action of $\widetilde{KO}^*(\text{point})$ on $\widetilde{KO}^*(P)$. We do not pursue this, because of the gap in knowledge of λ^* on spheres.

§23. Bundles over a Suspension.

We follow [26]. We work in the category of special complexes with base-point : thus each object is a countable CW-complex with only one 0-cell, situated at the base-point. Maps and homotopies are to respect base-points. We denote, typically, the base-point in a complex A by a . The cone on A is denoted by CA and the suspension of A by SA. In both cases points are represented by pairs (a,t) where $a \in A$ and $0 \leq t \leq 1$. To form CA we identify

$$(a,0) = a_0 = (a_0,t), \quad (a,1) = a$$

so that $A \subset CA$; to form SA we further identify A with a_0. Consider a G-bundle ξ over SA with projection $\pi : B \to SA$ and fibre Y, where G is an effective topological transformation group of Y. Let the characteristic map for ξ be $u' : A \to G$ (c.f. § 18 of [47]) and let $u : A \times Y \to Y$ be defined by $u(a,y) = u'(a).y$ for $a \in A$, $y \in Y$. Then (§ 3 of [26]) B may be formed from the disjoint union $(Y \times CA) \cup Y$ by identifying $(y,(a,1)) = u(y,a)$. Write $Y * A = S(Y \# A)$ where $Y \# A$ denotes the space obtained from $Y \times A$ by collapsing to a point z_0 the subspace $Y \vee A = Y \times a_0 \cup y_0 \times A$. Let $\mu : Y * A \to SY$ be the 'Hopf construction' on u, and $\lambda : SA \longrightarrow SY$ be the suspension of $u|y_0 \times A$ (see §§ 2 and 7 of [26]). Let i be the composition of the identification map with the embedding of Y as the right-hand component in $(Y \times CA) \cup Y$. Write $B/_Y = B/_{i(Y)}$. Then $B/_Y$ has the homotopy type of $Y * A \vee SA$, and the pair (B,Y) yields the following $\widetilde{KO}$ exact sequence :

$$\cdots \to \widetilde{KO}^r(SY) \xrightarrow{\mu^* \oplus \lambda^*} \widetilde{KO}^r(Y * A) \oplus \widetilde{KO}^r(SA) \longrightarrow \widetilde{KO}^r(B) \longrightarrow \widetilde{KO}^{r+1}(SY) \to \cdots$$

It is not clear in general how to compute λ^*, μ^* and their mod 2 analogues.

Appendix A. The Total Space of a Bundle as a Smooth Manifold.

Consider a fibre bundle ξ with projection $\pi : B \longrightarrow X$, fibre F and group G. Suppose that X and F are smooth manifolds and that G is a compact connected Lie group acting smoothly on F. We wish to give B the structure of a smooth manifold in a canonical way (up to diffeomorphism) arising from the smooth structures on X and F and the bundle structure of ξ. It is well-known that this can be done. We sketch one approach.

Much of Part I of [47] holds for a sub-category of co-ordinate bundles, which we call 'pre-differentiable co-ordinate bundles' (p.c.b.). By a p.c.b. we mean a co-ordinate bundle as in 2.3 of [47], with the following additional assumptions : the base space X and the fibre F are smooth manifolds, the group G is a connected compact Lie group acting smoothly on F, and the co-ordinate transformations g_{ij} are smooth. Strict equivalence of p.c.b.s is defined as in 2.4 of [47] with smooth g_{kj}. A map $h : \mathcal{B} \longrightarrow \mathcal{B}'$ of one p.c.b. into another is defined as a continuous map of total spaces, satisfying (16), (17) and (18) of 2.5 of [47], with the $h : X \to X'$ of (16) and the g_{kj} of (18) smooth. Lemmas 2.6, 2.7, the definition of equivalence of p.c.b.s, Lemma 2. , Lemma 2.10 and the theorem in 3.2 (all references to [47]) go through with the obvious modifications.

In the bundle structure theorem corresponding to 7.4 of [47] we assume G and B to be Lie groups and that f is a smooth local section. Then the conclusion is that we can assign a p.c.b. structure to $B/_H$ relative to the projection $p : B/_H \longrightarrow B/_G$. Induced p.c.b.s are dealt with as in §10 of [47]. A homotopy of two maps of p.c.b.s $\mathcal{B}$ and $\mathcal{B}'$ is a map of $\mathcal{B}\times I$ to $\mathcal{B}'$, hence induces a smooth map of $X \times I$ to X'. The first covering homotopy theorem is stated for p.c.b.s by analogy with 11.3 of [47] ; h is a smooth homotopy of h_0. Some modification of the proof is required : the subsets of $X \times I$ over which the induction proceeds should be smooth. One requires a sequence of smooth maps $\sigma_i : X \longrightarrow R_+$ (non-negative real numbers) such that $0 = \sigma_0 \leq \sigma_1 \leq \ . \ . \ \leq \sigma_N \geq 1$, and if $(X\times R_+)_i$ denotes the set of pairs (x,t) in $X \times I$ such that $\sigma_{i-1}(x) < t \leq \sigma_i(x)$, then for each i, $\bar{h}(X \times R_+)_i$ should be contained in some co-ordinate neighbourhood of X'. One may obtain these smooth σ_i inductively, following the proof on p.51 of [47]. First 'shrink' the cover $\left\{\bar{h}^{-1}(V_j')\right\}$ of $X\times I$ once, and then use a refinement of the new cover, of the form $\left\{U_\lambda \times I_\lambda\right\}$ described in [47]. Now proceed as in [47], but smoothly instead - at the first stage use a smooth Urysohn function σ_1, and at the ith stage smooth out $\max(\sigma_{i-1}, \tau_i)$ to a smooth function i

which is nowhere less than max (σ_{i-1}, τ_i), but which is still such that $\bar{h}\,(X \times R_+)_i$ is contained in some V'_j. (This may be done since the cover was shrunk.) Also we may ensure that $(X \times R_+)_i$ is contained in $X \times (I_{\nu+1} - I_{\nu-1})$ in the Steenrod notation. We observe that $\bar{h}$ may be extended smoothly over $X \times R_+$ by defining $\bar{h}(x,t) = \bar{h}(x,1)$ whenever $t \geqslant 1$, so that there is no trouble at the end of the inductive process. Hence 11.4 and 11.5 of [47] hold for p.c.b.s. Now the universal bundles in 19.6 of [47] for a compact Lie group G are p.c.b.s by the bundle structure theorem for p.c.b.s. Thus if $f : X \longrightarrow B_G$ is a classifying map for a bundle ξ over X, we may approximate f by a smooth map, and the induced bundle is then a p.c.b. Moreover if two smooth classifying maps are continuously homotopic, then they are smoothly homotopic and hence using 11.5 of [47] for p.c.b.s, we see that the induced p.c.b.s are equivalent.

We now describe the functor from p.c.b.s to smooth manifolds, which gives the total space of a p.c.b. its natural smooth structure. Let ξ be a p.c.b. with projection $p : B \longrightarrow X$, fibre F and co-ordinate transformations $g_{ij} : V_i \cap V_j \longrightarrow G$. Suppose that the dimensions of X and F are n, q respectively. For any $b \in B$, let V be a co-ordinate neighbourhood of X containing $\pi(b)$ and which is contained in some V_i. Thus we have a chart

$h_1 : V \longrightarrow R^n$ and a homeomorphism $\theta_i : p^{-1}(V) \to V \times F$. Let $\pi_2 : V \times F \longrightarrow F$ denote projection on the second factor, and let W be a co-ordinate neighbourhood of $\pi_2\, \theta_i(b)$ in F, with chart $h_2 : W \to R^q$. Then define $h : \theta_i^{-1}(V \times W) \longrightarrow R^{n+q}$ to be the composition $(h_1 \times h_2) \circ \theta_i$. The choice of such an h for each $b \in B$ will give the atlas for a smooth structure on E. Let $h' : \theta_j^{-1}(V' \times W') \longrightarrow R^{n+q}$ be another such chart in a neighbourhood of b, constructed in a fashion similar to h. Let A denote the set $(V \times W) \cap \theta_i\, \theta_j^{-1}(V' \times W')$, and B the set $\theta_j\, \theta_i^{-1}(V \times W) \cap (V' \times W')$. We check that the composition

$$h_1 \times h_2(A) \xrightarrow{h_1^{-1} \times h_2^{-1}} A \xrightarrow{\theta_j \theta_i^{-1}} B \xrightarrow{h_1' \times h_2'} h_1' \times h_2'(B) \qquad (C)$$

is smooth. For $\theta_j \cdot \theta_i^{-1}$ is defined on $(V_i \cap V_j) \times F$ and is the composition

$$(V_i \cap V_j) \times F \xrightarrow{\Delta \times 1} (V_i \cap V_j) \times (V_i \cap V_j) \times F \xrightarrow{1 \times g_{ij} \times 1} (V_i \cap V_j) \times G \times F \xrightarrow{1 \times \mu} (V_i \cap V_j) \times F$$

where Δ is the diagonal map and $\mu : G \times F \to F$ is the smooth action. Since also g_{ij} is smooth, $\theta_j \cdot \theta_i^{-1} : A \to B$ is smooth, which means by definition that the composition (C) is smooth.

Suppose now that $\mathcal{B}$ and $\mathcal{B}'$ are two p.c.b.s with the same group and fibre, and that $h : \mathcal{B} \to \mathcal{B}'$ is a map of p.c.b.s. Then $h : B \to B'$ is smooth with respect to the smooth structures defined above. For, using the notation of 2.5 of [47], what we have to check is that $h : p^{-1}(V_j \cap \bar{h}^{-1}(V_k')) \longrightarrow p'^{-1}(V_k')$

is smooth, and this follows since the g_{kj} are smooth.

Since this assignment of smooth structure to the total space of a p.c.b. is functorial it is clear that equivalent p.c.b.s have diffeomorphic total spaces.

Appendix B. The Thom-Gysin Isomorphism.

References are : §4 of [7] , the article by A. Dold in [18] , and mimeographed notes by M. F. Atiyah and R. Bott, to appear.

Let $\mathcal{W}$ be the category of pairs of spaces which have the homotopy type of finite CW-complexes, and homotopy classes of maps. A cohomology theory h on $\mathcal{W}$ with values in an Abelian category $\mathcal{A}$ is a sequence of contravariant functors $h^q: \mathcal{W} \longrightarrow \mathcal{A}$ $(q \in Z)$ together with natural transformations $\delta^q: h^q(A) \longrightarrow h^{q+1}(X,A)$ such that

1) $h(k): h(A \cup B, A) \longrightarrow h(B, B \cap A)$ is an isomorphism, and

2) $\ldots \longrightarrow h^q(A) \xrightarrow{\delta} h^{q+1}(X,A) \xrightarrow{h(j)} h^{q+1}(X) \xrightarrow{h(i)} h^{q+1}(A) \longrightarrow \ldots$.

is exact, where i, j, k are inclusions.

The coefficient object for h in $\mathcal{A}$ is the graded object $\check{h} = h(P)$ where P is a point. If h is a cohomology theory and $\pi : E \longrightarrow B$ is a map in $\mathcal{W}$, an exact couple (A, C) is defined by

$$A^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}))$$

$$C^{p,q} = h^{p+q}(\pi^{-1}(B^{(p)}), \pi^{-1}(B^{(p-1)}))$$

where $B^{(p)}$ denotes the p-skeleton of B, and all pairs are assumed to be in $\mathcal{W}$. The spectral sequence of this exact couple converges to

$$E_\infty^{p,n-p} = {}^{p}h^{n}(E)/{}^{p+1}h^{n}(E)$$

where ${}^{p}h(E)$ is the image of $h(E, \pi^{-1}(B^{(p)}))$ in $h(E)$.
Now let π be for example a locally trivial fibre map. Then in the above spectral sequence,

$$E_1^{p,q} = C^p(B, \underline{h}^q)$$

so $\quad E_2^{p,q} = H^p(B, \underline{h}^q)$, where $\underline{h}^q$ denotes the local coefficient system given by $\{h^q(\pi^{-1}(x))\}$ for $x \in B^{(0)}$.
(These isomorphisms follow since $\pi^{-1}(B^{(p)})/\pi^{-1}(B^{(p-1)})$ is a wedge of p-fold suspensions of the fibre.)

A multiplicative cohomology theory is a cohomology theory in which $\mathfrak{A}$ is a category of Abelian groups, together with a natural map $h^i(X,A) \times h^j(Y,B) \longrightarrow h^{i+j}(X \times Y,\ X \times B \cup A \times Y)$ for all i, j, which is bilinear, associative, commutative, has a unit 1 in $h^0(S^0, \ )$, and gives a commutative diagram

$$\begin{tikzcd}
h^i(A) \times h^j(Y,B) \arrow[rrr, "\delta \times 1"] \arrow[d] & & & h^{i+1}(X,A) \times h^j(Y,B) \arrow[d] \\
h^{i+j}(A \times Y, A \times B) & h^{i+j}(X \times B \cup A \times Y, X \times B) \arrow[l, "\cong"'] \arrow[r] & h^{i+j}(X \times B \cup A \times Y) \arrow[r, "\delta"] & h^{i+j+1}(X \times Y, X \times B \cup A \times Y)
\end{tikzcd}$$

where the vertical arrows denote the multiplication and the unnamed maps are induced by inclusions. If $\pi : E \longrightarrow B$ given a spectral sequence as above, and if there exist $a_1, a_2, \ldots, a_r \in h(E)$ whose restrictions to the fibre F form a basis for $h(F)$ as a module over h, then $h(E)$ is a free $h(B)$-module (module structure defined using $h(\pi)$); for by mapping $(z_1, z_2, \ldots, z_r)$ to $\sum_i h(\pi)(z_i).a_i$ one defines an E_1-isomorphism of $C \oplus C \oplus \ldots \oplus C$ (r summands) and C', where C, C' refer to the exact couples for the identity map of B, and for π respectively. In particular let $\pi : E \longrightarrow B$ be the projection of an n-plane bundle. Let E_0 denote the complement in E of the zero cross-section, and write $F_0 = F \cap E_0$. Suppose that the bundle is <u>h-orientable</u>, i.e. that there is a $U \in h^n(E, E_0)$ whose restriction generates $h^n(F, F_0) \cong h^n(S^n, *)$. Then by the above applied to the relative fibration (E, E_0), there follows the Thom isomorphism $\psi : h^i(B) \cong h^{i+n}(E, E_0)$ where $\psi(z) = U.h(\pi)(z)$.

For K-theory or KO-theory a suitable U may be chosen universally for spin-bundles, using representation theory. Thus let $RO(Spin(n))$ denote the Grothendieck ring of real orthogonal representations of the n-dimensional spinor group. Let $BSpin(n)$ denote the classifying space of $Spin(n)$ (or a suitable finite-dimensional approximation to this classifying space). A homomorphism $\rho : RO(Spin(n)) \rightarrow \widetilde{KO}(BSpin(n))$ is induced by sending a representation r to the bundle associated by r with the universal bundle over $BSpin(n)$.

The universal sphere-bundle over $BSpin(n)$ may be taken to have total space $BSpin(n-1)$, so the Thom complex $MSpin(n)$ of this universal bundle is the quotient space of the cofibration $BSpin(n-1) \subset BSpin(n)$. The unitary representation ring $R(Spin(2m))$ is generated by the exterior power representations $\lambda^1, \lambda^2, \ldots, \lambda^{m-2}$ together with the 'half-spinor' representations Δ^+, Δ^-; and the kernel of the restriction $R(Spin(2m)) \longrightarrow R(Spin(2m-1))$ is generated by $\Delta^+ - \Delta^-$. In $Spin(8k)$, $\Delta^+ - \Delta^-$ is a real representation. Also, the natural injections $c : RO(Spin(n)) \longrightarrow R(Spin(n))$ are monomorphic. Thus $\Delta^+ - \Delta^-$ is in the kernel of $RO(Spin(8k)) \to RO(Spin(8k-1))$, and so $\rho(\Delta^+ - \Delta^-) = j^*(U_R)$ for some U_R in $\widetilde{KO}(MSpin(8k))$, where j is the inclusion. From the centre and right-hand squares of the commutative diagram below it follows that $j^* c(U_R) = \rho_c(\Delta^+ - \Delta^-)$.

$$\begin{array}{ccccccc}
\widetilde{KO}(S^{8k}) & \xleftarrow{i^*} & \widetilde{KO}(MSpin(8k)) & \xrightarrow{j^*} & \widetilde{KO}(BSpin(8k)) & \xleftarrow{\rho} & RO(Spin(8k)) \\
\downarrow c & & \downarrow c & & \downarrow c & & \downarrow c \\
\widetilde{K}(S^{8k}) & \xleftarrow{i^*} & \widetilde{K}(MSpin(8k)) & \xrightarrow{j^*} & \widetilde{K}(MSpin(8k)) & \xleftarrow{\rho_c} & R(Spin(8k)).
\end{array}$$

The extreme left-hand c in this diagram is monomorphic, so to check that $i^*(U_R)$ generates $\widetilde{KO}(S^{8k})$ it is sufficient to show that $i^* c(U_R)$ generates $\widetilde{K}(S^{8k})$. For this it is sufficient to check that $ch(c(U_R)) = \phi_*(1 + \text{higher terms})$, where ϕ_* is the Thom isomorphism in rational cohomology, since $\phi_*(1)$ restricts to a generator of the image of $H(S^{8k}, Z)$ in $H(S^{8k}, Q)$, and ch is monomorphic on spheres. Thus it is finally sufficient to check that the character of $\Delta^+ - \Delta^-$ has the correct form, and we obtain a Thom isomorphism $KO^i(B) \cong \widetilde{KO}^i(E)$ for any Spin(8k)-bundle over B with total space E. For <u>any</u> spin-bundle, one may add trivial line-bundles to the associated vector bundle to raise the fibre dimension of the latter to a multiple of 8, and since this suspends the Thom complex, we get a Thom isomorphism $KO^i(B) \cong \widetilde{KO}^{i-n}(E)$ for a Spin(n)-bundle over B with total space E. A more sophisticated approach is given in the notes of Atiyah and Bott, to appear.

Essentially the same argument gives a Thom isomorphism in the theory $KO(\ast P)$ where P denotes the real projective plane. For $KO^*(X \ast P)$ is a module over $KO^*(X)$, and again $\phi(z) = U_E \cdot \pi(z)$ will yield the necessary isomorphism.

Appendix C. The Theory of the Mod 2 Semi-characteristic.

We summarise the connection between parallelisability and the mod 2 semi-characteristic, as given in [27] and [28].

Let M be a smooth s-parallelisable, compact d-dimensional manifold without boundary. Suppose given an embedding of M in Euclidean space R_{d+n} and a continuous normal n-frame field $\mathbb{F}_n$ to this embedding. Let $O_{d+n,n}$ denote the real Stiefel manifold of orthonormal n-frames in (n+d)-space. Define

$$\phi : M \longrightarrow O_{d+n,n}$$

by translating each frame of $\mathbb{F}_n$ to the origin of R_{d+n}.
Let $c \in H_n(O_{d+n,n}, Z)$ be the image under ϕ_* of the fundamental class in $H_n(M, Z)$. Thus c is defined as an integer or as an integer mod 2 according as d is even or odd. Define $\chi_*(M)$ as follows :

(i) if d is even, $\chi_*(M)$ is half the Euler characteristic of M;

(ii) if d is odd, say $d = 2s-1$, then $\chi_*(M) = \sum_{i=0}^{s-1} b_i$ where b_i is the rank of $H_i(M, Z_2)$.

Now $\mathbb{F}_n$ specifies a homeomorphism of a tubular neighbourhood T of M with $M \times V^n$ where V^n denotes the unit n-disc.

Define $f : S^{n+d} \longrightarrow S^n$ by mapping the complement of T to the base-point in S^n, and mapping T according to the above homeomorphism followed by projection of $M \times V^n$ onto V^n, followed by a relative homeomorphism $r : (V^n, S^{n-1}) \longrightarrow (S^n, *)$. Let γ be the Hopf invariant of f. Then in [27] and [28] Kervaire proves $\gamma = c - \chi_*(M)$, where this is an equation of integers if d is even, and of integers mod 2 if d is odd.

Since if d is even $\gamma = 0$, in this case the theorem says $c = 1/_2 \chi(M)$. One first shows that $c - 1/_2 \chi$ is an invariant of the framed cobordism class of $(M, \mathbb{F}_n)$. One then considers the 'opposed' pair $(\bar{M}, \bar{\mathbb{F}}_n)$ obtained by reversing the orientations of M and $\mathbb{F}_n$. Then $c - 1/_2 \chi$ is the same for $(\bar{M}, \bar{\mathbb{F}}_n)$ as for $(M, \mathbb{F}_n)$ since c and χ are individually the same. But the opposed pair represents the inverse in the framed cobordism group, and $c - 1/_2 \chi$ gives a homomorphism of this group to a torsionfree group, hence $c - 1/_2 \chi(M)$ is zero.

When d is odd, one uses relative Stiefel-Whitney classes for the proof. Let coefficients be taken mod 2 for the remainder of this appendix. Suppose given an $O(n)$-bundle over a complex K, and a cross-section θ over a subcomplex L, in the associated bundle with fibre $V_{n,n-r}$. Then θ induces a cross-section over L in the associated bundle with fibre $O_{n,n-q}$ whenever $q \geq r$. If one attempts to extend this cross-section skeleton by skeleton over K, the first obstruction occurs in $H^{q+1}(K,L)$, and is by definition the $(q+1)$th relative Stiefel-Whitney class of the situation. This relative class has the usual naturality properties, and the following Whitney duality theorem holds : suppose given two orthogonal bundles over a complex K, with groups $O(n_i)$ $(i = 1,2)$ and cross-sections θ_i defined over subcomplexes L_i in the associated bundles with fibres $O_{n_i, n_i - r_i}$. Consider the Whitney sum of these bundles, with group $O(n)$ where $n = n_1 + n_2$. The associated bundle with fibre $O_{n,n-r}$ $(r = r_1 + r_2)$ has a cross-section θ over $L = L_1 \ L_2$, given by θ_1 and θ_2. Let $w_{R,i}^{q+1} \in H^{q+1}(K, L_i)$ denote the Stiefel-Whitney class relative to θ_i, whenever $q \geq r_i$, and also, by abuse, its image in $H^{q+1}(K,L)$. Let $w_i^p \in H^p(K)$ be the absolute Stiefel-Whitney class of the ith bundle. Let $w_R^{q+1} \in H^{q+1}(K,L)$ be the relative class with respect to the sum bundle and θ whenever $q \geq r$. Then for any $q \geq r$,

$$w_R^{q+1} = w_{R,1}^{q+1} + w_{R,1}^{q} w_2^{1} + \cdots + w_{R,1}^{r_1+1} w_2^{q-r_1} + w_1^{r_1} w_{R,2}^{q-r_1+1} + \cdots + w_{R,2}^{q+1}.$$

This is proved by re-interpreting the relative Stiefel-Whitney classes as elementary symmetric functions of suitable variables, as for example in [10] for the absolute case.

Let K be a smooth n-manifold (compact) with non-empty boundary L. Using Lefschetz-Poincare duality one defines a Wu class $U^q \in H^q(K)$ by

$$Sq^q(x) = U^q . x \quad \text{for all } x \in H^{n-q}(K,L).$$

Let M be obtained by joining two copies of K along L; and let $i : K \longrightarrow M$ be the inclusion. Then $U^q = i^*(S^q)$ where S^q denotes the absolute Wu class in $H^q(M)$. Let $\tau(M)$ denote the tangent bundle of M. By the Wu formula,

$$w(\tau(M)) = Sq(S)$$

where w denotes the total Stiefel-Whitney class, etc.

Hence $w(\tau(M)|K) = SqU$. Suppose given a cross-section over L in the bundle associated with $\tau(M)|K$ with fibre $O(n)$, so that corresponding relative classes $w_R^q \in H^q(K,L)$ are defined for all q. Add formally a unit 1 of degree zero to the graded ring $H^*(K,L)$, with $Sq(1) = 1$, to form a new ring H_1. Then a total relative class $w_R = 1 + w_R^1 + \ldots$ is defined in H_1, and Sq is a ring automorphism of H_1, so one may define a Wu class U_R by

$$w_R = Sq(U_R) \quad \ldots \quad (C.1)$$

If $h : (K, \emptyset) \longrightarrow (K,L)$ is the inclusion, $h^*(U_R) = U$ and hence for any $x \in H^{n-q}(K,L)$

$$Sq^q(x) = U_R^q . x \quad \ldots \quad (C.2)$$

Now suppose that $n = 2s$. Using (C.1) and (C.2) one proves :

$$w_R^n = U_R^s . U_R^s + U_R^{2s} \quad . \; . \; . \qquad (C.3)$$

Let r be the rank of the cup-product form $H^s(K,L) \times H^s(K,L) \rightarrow H^n(K,L)$. Apply (C.2) to U_R^s expressed in terms of a basis for $H^s(K,L)$ with respect to which this form becomes diagonal; this leads to

$$U_R^s . U_R^s = r . A^n \quad . \; . \; . \; . \; . \qquad (C.4)$$

where A^n generates $H^n(K,L)$. Define dual relative classes $\bar{w}_R$ and $\bar{U}_R$ as the inverses of w_R and U_R in H_1. By uniqueness of inverses and (C.1),

$$\bar{w}_R = Sq(\bar{U}_R) \quad . \; . \; . \; . \; . \qquad (C.5) .$$

Using (C.2), (C.5) and the definition of U_R, $\bar{U}_R$, one obtains

$$\bar{w}_R^n = U_R^n \quad . \; . \; . \; . \; . \qquad (C.6),$$

and so

$$\begin{aligned} r.A^n &= U_R^s . U_R^s \\ &= w_R^n + U_R^n \qquad \text{by (C.3)} \\ &= w_R^n + \bar{w}_R^n \qquad \text{by (C.6)}. \end{aligned}$$

Thus by definition of $\bar{w}_R$,

$$r.A^n = w^{n-1} . \bar{w}_R^1 + . \; . \; . + w_R^1 . \bar{w}_R^{n-1} \quad . \; . \qquad (C.7).$$

To prove $\gamma = c - \chi_*(M)$ it is sufficient to consider the case of a d-dimensional M embedded in R_{2d+1}, since both sides of the equation are stable under suspension. Thus let

M (with d odd) be embedded in R_{2d+1} with a continuous field $\mathbb{F}_{d+1}$ of normal $(d+1)$-frames. Let $f : S^{2d+1} \longrightarrow S^{d+1}$ be the "Thom" construction on $\mathbb{F}_{d+1}$ as described above, and let q be the base-point on S^{d+1}. Then $M = f^{-1}(q)$. If for any $x \in M$, $v_1(x)$ denotes the first vector of the frame at x, and if $\varepsilon > 0$ is small enough, then the manifold M' traced out by $\varepsilon\, v_1(x)$ is $f^{-1}(q')$ for suitable q' on S^{d+1}. Then γ is the looping number of M and M'.

Since all Stiefel-Whitney numbers of M vanish, M is a mod 2 boundary. By standard use of Whitney theorems, one immerses in R_{2d+2} a smooth $(d+1)$-manifold X with regular boundary M, in such a way that X meets the hyperplane R_{2d+1} orthogonally in precisely M embedded in the given fashion, and the rest of X lies on one side of R_{2d+1}. Now add at the beginning of $\mathbb{F}_{d+1}$ the unit vector $v_0(x)$ in the positive direction of the last co-ordinate axis in R_{2d+2}, to obtain a continuous normal $(d+2)$-frame field $\mathbb{F}_{d+2}$ over M. Thus we get a map $\psi : M \longrightarrow O_{2d+2,\,d+2}$ by analogy with ϕ, and ψ_* on the fundamental class of M gives c times the generator of $H_d(O_{2d+2,\,d+2})$. Now $\mathbb{F}_{d+1}$ provides a cross-section over M of the principal normal bundle to the given immersion of X, so that classes $\bar{w}_R^{\,i} \in H^i(X,M)$ are defined for all $i > 0$. Similarly the vector field v_0 gives rise to $w_R^{d+1} \in H^{d+1}(X,M)$. Finally $\mathbb{F}_{d+2}$ provides a cross-section over M of the bundle of $(d+2)$-frames associated with the Whitney sum of the tangent and normal bundles to X, and this leads to relative Stiefel-Whitney classes $s_R^i \in H^i(X,M)$ whenever $i \geq d+1$. Now by relative Whitney duality,

$$s_R^{d+1} = w_R^{d+1} + w^d . \bar{w}_R^1 + \ldots + w^1 . \bar{w}_R^d + \bar{w}_R^{d+1} \quad \ldots \quad (C.8).$$

Since s_R^{d+1} is the obstruction to extending $\mathbb{F}_{d+2}$ over X as a continuous frame-field,

$$s_R^{d+1} = c.A^{d+1} \quad \text{where } A^{d+1} \text{ generates } H^{d+1}(X,M) . \quad (C.9).$$

Also, w_R^{d+1} is the obstruction to extending v_0 over X as a tangent vector field, hence

$$w_R^{d+1} = \chi(X).A^{d+1} \quad \ldots \quad (C.10).$$

Finally $\bar{w}_R^{d+1}$ is the obstruction to extending v_1 over X as a normal vector field. This turns out to be congruent mod 2 to the looping number of M and M', thus

$$\bar{w}_R^{d+1} = \gamma .A^{d+1} \quad \ldots \quad (C.11).$$

Now from (C.7), (C.8), (C.9), (C.10) and (C.11),

$$c \equiv \chi(X) + r + \gamma \quad , \bmod 2. \quad \ldots \quad (C.12).$$

By considering the cohomology sequence of (X,M) and using

Lefschetz-Poincare duality, one shows

$$\chi_*(M) \equiv \chi(X) + r \mod 2,$$

and now the result follows from (C.12). By Theorem 1.1.1 of [1] it follows that if d is odd and unequal to 1,3 or 7, then $c \equiv \chi_*(M) \mod 2.$

Theorem 4.1 is a corollary of the above, in the case when M is a d-dimensional s-parallelizable compact manifold without boundary, with d odd. For suppose that $\chi_*(M)$ is even. First if $d = 1,3$ or 7, then M is parallelizable by Théorème IX of [28]. Now suppose that $d \neq 1,3,7$. Let $\phi : M \to O_{2d+1,d+1}$ be the map given by some normal framing of some embedding of M in R_{2d+1}. The only obstruction to a null-homotopy of ϕ is c times the generator of $H_d(O_{2d+1,d+1})$. But c is now even, while $H_d(O_{2d+1,d+1})$ is of order two, hence ϕ is null-homotopic. Let $\pi : O_{2d+1,d+1} \longrightarrow G_{d,d+1}$ be the map projecting each $(d+1)$-frame to the orthogonal d-plane. Then $\pi \circ \phi$ is also null-homotopic; but this map classifies the tangent bundle of M, hence M is parallelizable. Conversely if M is parallelizable, then the classifying map for the tangent bundle of M is null-homotopic; lift the constant end of this homotopy into $O_{2d+1,d+1}$ and apply the homotopy covering theorem to obtain a normal framing of M for which the corresponding ϕ is null-homotopic. Then the c for this framing is even, and hence if $d \neq 1,3,7$, then $\chi_*(M)$ is even.

Appendix D. Numerical Details.

In §15 we used the following result : For any prime p and integer r let $\nu_r(p)$ denote the highest power of p dividing r. Let $m(r) = 2$ if r is odd; if r is even, let $m(r) = 2 \prod_p p^{\nu_r(p)+1}$ where the product is over all primes p such that $(p-1)$ divides r. Let μ be a rational number such that for some fixed integers n and q and any integer k, $(k^n - k^q)\mu$ is an integer. Then $m(n-q)\mu$ is an integer.

We summarise the proof, following [41].
Let $\phi(m)$ be the number of positive integers less than m which are relatively prime to m. If p is a prime, $\phi(p) = p-1$ and $\phi(p^a) = p^{a-1}(p-1)$ for $a \geq 1$. If m has prime decomposition $p_1^{a_1} . p_2^{a_2} . \; . \; . \, p_r^{a_r}$, one shows by induction on r that $\phi(m) = \phi(p_1^{a_1})\,\phi(p_2^{a_2}) . \; . \; \phi(p_r^{a_r})$. Euler's theorem states that if a and m are relatively prime, then $a^{\phi(m)} \equiv 1 \mod m$. For any relatively prime a and m define the exponent n of a with respect to m to be the least n such that $a^n \equiv 1 \mod m$. Then if N is any integer such that $a^N \equiv 1 \mod m$, n divides N; in particular n divides $\phi(m)$. If the exponent of a with respect to m is $\phi(m)$, we call a a primitive root for m. There exist primitive roots for all

powers of odd primes, and for 2 and 4. However if $m = 2^{a}$ with $a \geq 3$, the highest exponent is $1/_2\phi(m)$. Define a <u>universal exponent</u> for m to be an integer M such that $a^M \equiv 1 \mod m$, for every a relatively prime to m. For example $\phi(m)$ is a universal exponent for m. Let $\lambda(m)$ be the least universal exponent for m. If m has a primitive root, then $\lambda(m) = \phi(m)$. Thus $\lambda(p^a) = p^{a-1}(p-1)$ for an odd prime p, $\lambda(2) = 1$, $\lambda(4) = 2$, and $\lambda(2^a) = 2^{a-2}$ whenever $a \geq 3$.

Suppose $\mu = a/_{\beta p}i$ where this is in lowest terms and p does not divide β. Then since $(k^n - k^q)/\mu$ is an integer for all integers k, p^i must divide $(k^n - k^q)$ for all integers k, hence p^i divides $(k^{n-q} - 1)$ for all k not divisible by p. Thus $(n-q)$ is a universal exponent for p^i. Hence if either p is odd, or $p^i = 2$ or 4, then $p^{i-1}(p-1)$ divides $(n-q)$; while if p is 2 and $i \geq 3$, then 2^{i-2} divides $(n-q)$. Thus if $(n-q)$ is odd, the only prime on the denominator of μ is 2, since $(p-1)$ is even for odd p; and the highest power of 2 which can appear on the denominator is the first, hence 2μ is an integer. If $(n-q)$ is even, the primes which can occur on the denominator of μ are those such that $(p-1)$ divides $(n-q)$. And if p is odd, the highest power of p which can appear is the ith, where $(i-1)$ is the highest power of p dividing $(n-q)/(p-1)$; while the highest power of 2 which can occur is 2^i where

$(n-q)/_{2}i-2$ is odd $(i \geq 3)$. Hence the result. Note that $m(r)$ divides $2r.\prod_p p$ where again the product is over those primes p such that $(p-1)$ divides r.

<u>Proof of Theorem 15.5</u>. We use the notation of § 15. We shall say that a rational cohomology class is <u>integral</u> if it lies in the image of the integral cohomology under the coefficient homomorphism induced by the inclusion $Z \subset Q$. Suppose first that $r < t$, and that $t \neq 2r$. Then the $l(r,t)$ of Theorem 15.9 is the h.c.f. of $(r+t-1)!/_{m(t)}$ and $(r-1)!(t-1)!/_{m(t-r)}$.

<u>Case 1</u>: r even, t odd.

Then $m(t)$ and $m(t-r)$ are each 2, and $l(r,t)$ is a multiple of 3 whenever r or t is at least 4.

<u>Case 2</u>: r odd, t even.

The only possibly prime p greater than $(t/_2 + 1)$ and such that $(p-1)$ divides t is $(t+1)$. Hence $(r+t-1)!/_{m(t)}$ is a multiple of $(r+t-1)!/_{(t/_2 + 1)!(t+1)\,2t}$. Thus whenever $r > 1$ and $t \neq 2,6$, then $l(r,t)$ is a multiple of 3. If $r > 1$ and $t = 6$, then $l(r,t)$ is a multiple of 5. To deal with the case when $r = 1$ and $(t+1)$ is prime, recall from § 15 the following relation between the Chern classes and Chern character of any bundle over the type of space under discussion :

$$c_{r+t} = (-1)^{r+t+1}\left\{(r+t-1)!\,ch_{r+t} - (r-1)!(t-1)!\,ch_r . ch_t\right\} .$$

If $(t+1)$ is prime, then t is not prime (unless $r = 1$, $t = 2$), and $(t-1)!\,ch_t$ is integral whenever $t \geqslant 4$, since then $(t-1)!$ is a multiple of $m(t-1)$. Thus $(r+t-1)!\,ch_{r+t}$ is integral since ch_r and c_{r+t} are integral. Thus $l(r,t)$ is divisible by 3 whenever $t \geqslant 8$, while if $t = 6$, then $l(r,t)$ is divisible by 5.

Case 3: r and t even.

By similar arguments, $(r-1)!(t-1)!/_{m(t-r)}$ is a multiple of 3 whenever $r \geqslant 4$, or $r = 2$ and $t \geqslant 10$, while $(r+t-1)!/_{m(t)}$ is a multiple of 3 under the same conditions.

Case 4: r and t odd.

Then $(r+t-1)!/_{m(t)}$ is divisible by 3, while $(r-1)!(t-1)!/_{m(t-r)}$ is divisible by 3 whenever $r > 3$ or when $r = 3$ and $t \geqslant 7$. When $r = 1$, we argue as in Case 2 to show that $l(r,t)$ is a multiple of 3 whenever $t \geqslant 9$, and a multiple of 5 when $t = 7$.

The above analysis of cases has left open the possibility of a.c.s. in some low-dimensional cases; we now eliminate these. First suppose $r = 2$ and $t = 3, 6$ or 8. Consider the following commutative diagram (c.f. § 23) :

$$\begin{tikzcd}
\widetilde{K}(S^{2t} \vee S^{2t+4}) \arrow[r, "j^*"] & \widetilde{K}(B) \arrow[r, "i^*"] \arrow[d, "r"] & \widetilde{K}(S^4) \arrow[d, "r"] \\
 & \widetilde{KO}(B) \arrow[r] & \widetilde{KO}(S^4).
\end{tikzcd}$$

The right-hand r is monomorphic. Hence if $r(\xi) = \{\tau(B)\}$, then $\xi \in$ image j^*, so $c_{r+t}(\xi)$ is divisible by $(r+t-1)!$, which is a multiple of 3.

Next suppose $r = 3$, $t = 5$. Consider the standard projection $p_* : \pi_9(SO(7)) \longrightarrow \pi_9(S^6)$. Since $2.\pi_9(SO(7))$ is zero, and $\pi_9(S^6)$ is cyclic of order 24, any element of $p_*(\pi_9(SO(7)))$ is a multiple of 12 times the generator of $\pi_9(S^6)$. The method of proof of Theorem 15.6 now shows that if $\xi \in \widetilde{K}(B)$, $c_8(\xi)$ is divisible by the h.c.f. of $7!/_{m(2).m(5)}$ and $12.(4!)/_{m(2)}$, which is a multiple of 3.

Now suppose $r = 1$, $t = 5$. The attaching map of the 10-cell to S^2 in B is a composition, hence its e invariant (see Lecture 6 of [3]) is zero, and by the method of proof of Theorem 15.6, for any $\xi \in \widetilde{K}(B)$, $c_6(\xi)$ is divisible by the h.c.f. of $5!/_{\text{l.c.m. of } m(1),\, m(5)}$ and $4!$. Thus $c_6(\xi)$ is divisible by 3.

Finally suppose $r = 1$, $t = 4$. Then B has the S-type of $S^2 \vee S^8 \vee S^{10}$, for the stable 5-stem of homotopy groups of spheres is zero, and $J(\pi_7(SO(3)))$ is stably zero since $\pi_7(SO(3))$ is finite while $\pi_7(SO)$ is torsionfree. Hence for $\xi \in \widetilde{K}(B)$,

$c_5(\xi)$ is divisible by 4

The 'only if' part of Theorem 13.1 now shows that for the total space of a 2r-sphere bundle over the 2t-sphere to admit an a.c.s., where $r < t$ and $t \neq 2r$, it is necessary that $r, t \in \{1, 2, 3\}$. The proofs of Theorem 15.5 in the other cases are similar, and we omit the details.

Appendix E. Alternative Proof of Theorem 6.1.

The idea for the following alternative proof is due to M.F.Atiyah.

Recall that if G is a compact Lie group and H is a closed subgroup of G, then the tangent bundle of the quotient space $G/_H$ is the bundle associated with the fibring $H \to G \to G/_H$ by the adjoint representation of H, $\rho : H \longrightarrow \mathrm{Aut}(g/_h)$, where g, h denote the Lie algebras of G, H respectively. (See e.g. p.33 of [5]). Thus the total space of $\tau(G/_H)$ is $G \times_\rho (g/_h)$, in the notation of [9]. If ρ is induced by a representation σ of G, so that we have a commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\rho} & \\ {\scriptstyle i}\downarrow & & \mathrm{Aut}(g/_h) \\ G & \xrightarrow{\sigma} & \end{array}$$

where i denotes the inclusion, then $\tau(G/_H)$ is trivial; for a bundle map ϕ from $\tau(G/_H)$ to the product bundle is defined as follows: let (g,x) represent an element $\overline{(g,x)}$ of $G \times_\rho (g/_h)$, and define $\phi\overline{(g,x)} = (gH, \sigma(g).x)$. Similarly if the class of ρ in the representation ring $RO(H)$ is the restriction of an element of $RO(G)$, then the stable tangent bundle of $G/_H$ is trivial. (In fact the first property is stronger than the second, and says that the tangent bundle of $G/_H$ is homogeneously stably trivial.) Consider also the adjoint representations $\alpha : H \longrightarrow \mathrm{Aut}\ h$ and

$\beta : G \longrightarrow \mathrm{Aut}\ g$. Then β is an extension of the representation $\alpha \oplus \rho : H \longrightarrow \mathrm{Aut}(h \oplus g/_h) \cong \mathrm{Aut}\ g$. Thus there is an element of $RO(G)$ extending the class of ρ in $RO(H)$ if and only if there is an element extending the class of α. It can readily be verified by examining characters that this latter condition is satisfied when G is an orthogonal group over the real, complex or quaternionic fields, and H is an orthogonal subgroup over the same field.

Appendix F. A Theorem of Milnor and Massey.

Consider the multiplication on S^3 given by $(q_1, q_2) \longrightarrow q_1^{\,h} \cdot q_2 \cdot q_1^{\,j}$ where q_1 and q_2 are quaternions of unit norm, the dot refers to quaternion multiplication, and h and j are integers such that $h+j = 1$. Write $k = h-j$. We denote by P^k the projective plane associated with this multiplication (i.e. the second stage in the construction of the corresponding classifying space, see [38]). Then P^k admits a differentiable structure if and only if $h(h-1)$ is a multiple of 56, i.e. if and only if $k^2 \equiv 1 \bmod 224$. (See [35]).

Proposition. The projective plane P^k does not admit a s.a.c.s. for any k.

This has been proved by J.Milnor (see p.126 of [22]) and by W.S.Massey [37] when $k = 1$.

For any $\alpha \in \pi_{2m-1}(S^{2n})$ let $e(\alpha)$ denote the invariant defined in Lecture 6 of [3].

Lemma. Let α generate the Z_{12} summand in $\pi_7(S^4)$. Then $e(\alpha) \equiv 1/_6 \bmod 1$.

Proof. First $e(3\alpha) \equiv \frac{1}{2} \bmod 1$. For 3α is the suspension of the Toda secondary composition $\langle \eta_4, 2\iota, \eta_3 \rangle$ where η_i generates $\pi_{i+1}(S^i)$; thus $3\alpha = \langle \eta_5, 2\iota, \eta_4 \rangle$ so by Lecture 6 of [3],

$e(3\sigma) = -2\, e(\eta_4)\, e(\eta_6)$; but $e(\eta_6) = e(\eta_4) = e(\eta_2) = \frac{1}{2}$.
Also, $e(4\alpha) \equiv -1/3 \mod 1$, (Lecture 6 of [3]) since 4α generates the 3-component of the stable 3-stem. Hence the Lemma.

It follows that if $\beta \in \pi_7(S^4)$ is the class of the attaching map in P^k (which has the cell structure $S^4 \cup_\beta e^8$), then

$$e(\beta) \equiv 1/_{12} - j/_6 \mod 1.$$

<u>Proof of Proposition.</u> Consider the following commutative diagram :

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K}(S^8) \arrow[r] \arrow[d, "r"] & \widetilde{K}(P^k) \arrow[r] \arrow[d, "r"] & \widetilde{K}(S^4) \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & \widetilde{KO}(S^8) \arrow[r] & \widetilde{KO}(P^k) \arrow[r] & \widetilde{KO}(S^4) \arrow[r] & 0
\end{tikzcd},$$

where the horizontal sequences are for the inclusion $S^4 \subset P^k$ and r is induced by forgetting the complex structure on a complex vector bundle. We can choose generators ξ, η for $\widetilde{K}(P^k)$, by the lemma, such that

$$\mathrm{ch}\, \xi = u + (1/_{12} - j/_6)\, u^2$$

$$\mathrm{ch}\, \eta = u^2,$$

where u generates the image of $H^4(P^k, Z)$ in $H^4(P^k, Q)$.
Let α, β be generators of $\widetilde{KO}(P^k)$ such that

$$2\alpha = r(\eta)$$

$$\beta = r(\xi).$$

Let $c(\ \cdot\ )$ denote the total Chern class. Then

$$c(\eta) = 1 - 6u^2$$

$$c(\xi) = 1 - u + ju^2.$$

(These are computed as in § 15 using Newton's relations.)

Hence, denoting by $p(\ \ )$ the total Pontryagin class,

$$p(\alpha) = 1 - 6u^2$$

$$p(\beta) = 1 + 2u + (1+2j)u^2.$$

Now from Lemma 3 on p. 402 of [39], $p_1(P^k) = 2ku$. Since the Hopf invariant of β is 1, the index of P^k is +1, and by Hirzebruch's index theorem, (see p. 85 of [23]),

$$p_2(P^k) = (45+4k^2/7)u^2.$$

Now $$p(m\alpha + n\beta) = 1 + 2nu + \left[n(2n-1) + 2jn - 6m\right] u^2.$$

Hence in order that

$$m\alpha + n\beta = \tau(P^k)$$

we must have

$$\begin{cases} n = k \\ 6m = n(2n-1) + 2jn - (45+4k^2/7), \end{cases}$$

whence $$m = 1/6 \left\{ k^2 - (45+4k^2/7). \right\}$$

Let $k^2 = 224l + 1$ for some integer l, so that P^k does admit a smooth structure by [35]. Then $m = 16l - 1$ which is odd.

Hence if $m\alpha + n\beta = \tau(P^k)$, then $m\alpha + n\beta \notin r(\widetilde{\ }(P^k))$, as required.

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