

The Complexity of Pattern Counting in Directed Graphs, Parameterised by the Outdegree*

Marco Bressan
Department of Computer Science
University of Milan
Italy

Matthias Lanzinger
Department of Computer Science
University of Oxford
United Kingdom

Marc Roth
Department of Computer Science
University of Oxford
United Kingdom

February 23, 2023

Abstract

We study the fixed-parameter tractability of the following fundamental problem: given two directed graphs \vec{H} and \vec{G} , count the number of copies of \vec{H} in \vec{G} . The standard setting, where the tractability is well understood, uses only $|\vec{H}|$ as a parameter. In this paper we take a step forward, and adopt as a parameter $|\vec{H}| + d(\vec{G})$, where $d(\vec{G})$ is the maximum outdegree of \vec{G} . Under this parameterization, we completely characterize the fixed-parameter tractability of the problem in both its non-induced and induced versions through two novel structural parameters, the *fractional cover number* ρ^* and the *source number* α_s . On the one hand we give algorithms with running time $f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{\rho^*(\vec{H}) + O(1)}$ and $f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{\alpha_s(\vec{H}) + O(1)}$ for counting respectively the copies and induced copies of \vec{H} in \vec{G} ; on the other hand we show that, unless the Exponential Time Hypothesis fails, for any class \vec{C} of directed graphs the (induced) counting problem is fixed-parameter tractable if and only if $\rho^*(\vec{C})$ ($\alpha_s(\vec{C})$) is bounded. These results explain how the orientation of the pattern can make counting easy or hard, and prove that a classic algorithm by Chiba and Nishizeki and its extensions (Chiba, Nishizeki SICOMP 85; Bressan Algorithmica 21) are optimal unless ETH fails.

Our proofs consist of several layers of parameterized reductions that preserve the out-degree of the host graph. To start with, we establish a tight connection between counting *homomorphisms* from \vec{H} to \vec{G} to $\#CSP$, the problem of counting solutions of constraint satisfactions problems, for special classes of patterns that we call *canonical DAGs*. To lift these results from canonical DAGs to arbitrary directed graphs, we exploit a combination of several ingredients: existing results for $\#CSPs$ (Marx JACM 13; Grohe, Marx TALG 14), an extension of graph motif parameters (Curticapean, Dell, Marx STOC 17) to our setting, the introduction of what we call *monotone reversible minors*, and careful analysis of quotients of directed graphs in order to relate their adaptive width and fractional hypertree width as a function to our novel parameters. Along the route we establish a novel bound of the integrality gap for the fractional independence number of hypergraphs based on adaptive width, which might be of independent interest.

*This research was funded in whole, or in part, by the Royal Society project "RAISON DATA" (Project reference: RP\R1\201074) and by the Google Focused Award "Algorithms and Learning for AI" (ALL4AI). For the purpose of Open Access, the authors have applied a CC BY public copyright licence to any Author Accepted Manuscript version arising from this submission. All data is provided in full in the results section of this paper.

1 Introduction

We study the complexity of the following fundamental counting problem: given two directed graphs, \vec{H} (the “pattern”) and \vec{G} (the “host”), count the number of occurrences or induced occurrences of \vec{H} in \vec{G} . This problem, known as *subgraph counting*, *motif counting*, or *pattern counting*, has gained great popularity because of its apparent ubiquity in a diverse selection of fields, from social network analysis [52] to network science [44, 43], and from database theory [27, 15, 26, 4, 30] and data mining [1, 51] to bioinformatics [3, 48], phylogeny [36], and genetics [49, 50]. For this reason, subgraph counting in general has received significant attention from the theoretical community in the last two decades, with a flurry of novel techniques and exciting results [6, 28, 21, 24, 35, 42, 23, 11, 37, 13, 7, 31].

Since subgraph counting in general is hard (think of counting cliques), it is common to *parameterise* the problem so as to allow for a “bad” dependence on some quantity that is believed to be small in practice [28, 23]. The standard parameterisation is by the size of \vec{H} , that is, $|\vec{H}| = |V(\vec{H})| + |E(\vec{H})|$. In that case, one says the problem is *fixed-parameter tractable*, or in the class FPT , if for some (computable) function f it admits an algorithm that runs in time $f(|\vec{H}|) \cdot |\vec{G}|^{O(1)}$ for all \vec{H} and \vec{G} . This means one considers as efficient an algorithm with running time, say, $2^{|\vec{H}|} \cdot |\vec{G}|$, but not one with running time $|\vec{G}|^{|\vec{H}|}$. The rationale is that in practice \vec{H} is often very small compared to \vec{G} , thus a running time of $2^{|\vec{H}|} \cdot |\vec{G}|$ is better than one of $|\vec{G}|^{|\vec{H}|}$. Under this parameterisation, the tractability of the problem is well understood: for the undirected version, both the induced and non-induced versions are in FPT if and only if certain invariants of H are bounded [18, 24, 23], and it is not hard to show that the same holds for the directed case as well (see Section 2).

While the parameterisation by $|\vec{H}|$ is standard, it is also quite restrictive. Consider for instance the problem of counting the *induced* copies of \vec{H} in \vec{G} : when parameterised by $|\vec{H}|$, it is well-known that the problem is in FPT if and only if the pattern size $|\vec{H}|$ is bounded (see [18] and Appendix B). Thus, under this parameterization, one can efficiently count the induced copies of just a *finite* number of patterns. Suppose instead the parameter is $|\vec{H}| + d(\vec{G})$, where $d(\vec{G})$ is the maximum outdegree of \vec{G} ; the problem is then considered tractable if for some (computable) function f it admits an algorithm that runs in time $f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{O(1)}$ for all \vec{H} and \vec{G} . It is not hard to see that, under this parameterization, the problem becomes FPT even for *infinite* families of patterns. Let indeed \vec{H} be the acyclic orientation of a k -clique: since \vec{H} has only one source s (a vertex of indegree 0), one can first guess the image of s in \vec{G} and then iterate over all $(k - 1)$ -vertex subsets in the out-neighbourhood of s , which yields an algorithm with running time $O(d(\vec{G})^{|\vec{H}|} \cdot |\vec{G}|)$. This idea was in fact extended to counting subgraphs in degenerate host graphs, which have orientations with bounded outdegree [8, 10, 9, 13, 32, 7] (see Section 3.4 for a detailed discussion). Thus, adopting $|\vec{H}| + d(\vec{G})$ as a parameter can open the door to a richer landscape of tractability.

The goal of the present work is to understand precisely what that landscape is; that is, to understand when the aforementioned problems, parameterised by $|\vec{H}| + d(\vec{G})$, are in FPT as a function of the pattern \vec{H} . In addition to the aforementioned example of pattern counting in degenerate graphs, there is another reason to consider $d(\vec{G})$ as part of the parameter when counting directed subgraphs: several “real-world” directed graphs that are natural “hosts” have small or constant outdegree. This is true for many web graphs or online social network graphs, where the maximum outdegree is much smaller than the average degree or the maximum indegree; and it is true by construction in graphs produced by generative models such as preferential attachment [2]. As is customary, to express the dependence on the structure of \vec{H} , we formulate the problems as a function of a *class* \vec{C} of patterns — for instance, one may let \vec{C} be the class of all directed complete graphs, or of all directed trees. Let then \vec{C} denote an arbitrary family of directed graphs, and for any \vec{H} and \vec{G} let $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ and $\#\text{IndSub}(\vec{H} \rightarrow \vec{G})$ denote respectively the number of copies and induced copies of \vec{H} in \vec{G} .

Our parameterised counting problems are formally defined as follows:

$\#\text{DIRSUB}_d(\vec{C})$	
Input:	a pair of digraphs (\vec{H}, \vec{G}) with $\vec{H} \in \vec{C}$
Output:	$\#\text{Sub}(\vec{H} \rightarrow \vec{G})$
Parameter:	$ \vec{H} + d(\vec{G})$
$\#\text{DIRINDSUB}_d(\vec{C})$	
Input:	a pair of digraphs (\vec{H}, \vec{G}) with $\vec{H} \in \vec{C}$
Output:	$\#\text{IndSub}(\vec{H} \rightarrow \vec{G})$
Parameter:	$ \vec{H} + d(\vec{G})$

The goal of the present work is to understand which structural properties of the elements of \vec{C} determine whether $\#\text{DIRSUB}_d(\vec{C})$ and $\#\text{DIRINDSUB}_d(\vec{C})$ are in FPT.

The rest of this manuscript is organised as follows. Section 2 gives a concise overview of our results and their significance. Section 3 gives a detailed overview of our proofs and their key technical insights. The complete proofs of all our claims can be found in the remaining sections.

2 Results

We give complete complexity classifications for $\#\text{DIRSUB}_d(\vec{C})$ and $\#\text{DIRINDSUB}_d(\vec{C})$, into FPT versus non-FPT cases, as a function of \vec{C} . These complexity classifications, which are formally stated below, have the succinct form “The problem is in FPT if and only if $p(\vec{C})$ is bounded”, where p is some parameter measuring the structural complexity of the graphs in \vec{C} . The definition of those parameters is not elementary and requires the introduction of some ancillary notation and definitions, which we are going to do next. In order to understand *why* those parameters are the right ones, instead, one should take the technical tour of Section 3.

Let us then introduce our structural parameters. First, we need to define *reachability hypergraphs* and *contours*. Let \vec{H} be a directed graph, and let \mathcal{S} be the set of its strongly connected components. Denote by \sim the equivalence relation over $V(\vec{H})$ given by \mathcal{S} , and let \vec{H}/\sim be the quotient of \vec{H} w.r.t. \sim ; with a little abuse of notation we let \mathcal{S} be the vertex set of \vec{H}/\sim . A strongly connected component $S \in V(\vec{H}/\sim)$ is a *source* if it has indegree 0 in \vec{H}/\sim . Let S_1, \dots, S_k be the set of all such sources. For any $S \in V(\vec{H}/\sim)$ let $R(S)$ be the set of vertices reachable from S in \vec{H} .

Definition 1. *The reachability hypergraph of \vec{H} , denoted by $\mathcal{R}(\vec{H})$, is the hypergraph with vertex set $V(\vec{H})$ and edge set $\{R(S_i) : i \in [k]\}$.*

Intuitively, $\mathcal{R}(\vec{H})$ measures the complexity of \vec{H} in terms of “reachability relationships”. However, to state our classifications correctly, we need to consider a slight modification of $\mathcal{R}(\vec{H})$.

Definition 2. *The contour of \vec{H} , denoted by $\Gamma(\vec{H})$, is the hypergraph $\mathcal{R}(\vec{H}) \setminus \cup_{i \in [k]} S_i$.*

For instance, if \vec{H}_n is obtained by orienting the edges of the 1-subdivision of the complete graph K_n towards the original vertices, then $\Gamma(\vec{H}_n) = K_n$.

Finally, we introduce our directed graph invariants, the *fractional cover number* and the *source number*. Let \mathcal{H} be a hypergraph. A function $\gamma : E(\mathcal{H}) \rightarrow [0, \infty)$ is a *fractional edge cover* of \mathcal{H} if for every $v \in V(\mathcal{H})$

$$\sum_{e \in E(\mathcal{H}) : v \in e} \gamma(e) \geq 1. \quad (1)$$

The weight of γ is $\sum_{e \in E(\mathcal{H})} \gamma(e)$. The fractional edge cover number of \mathcal{H} , denoted by $\rho^*(\mathcal{H})$, is the smallest weight of any fractional edge cover of \mathcal{H} . Then:

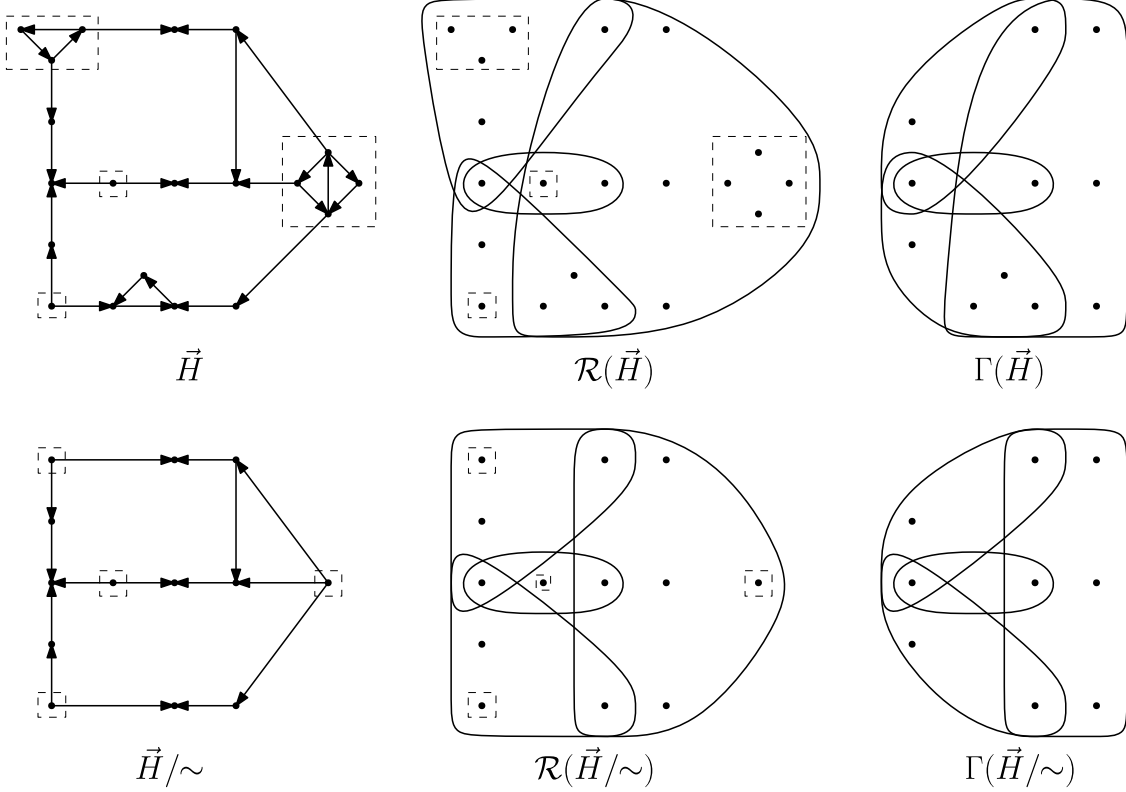


Figure 1: (*Top:*) A directed graph \vec{H} , the sources of which are highlighted in dashed boxes, its reachability hypergraph $\mathcal{R}(\vec{H})$, and its contour $\Gamma(\vec{H})$. (*Bottom:*) The same constructions for the DAG \vec{H}/\sim obtained from \vec{H} by identifying all strongly connected components. Clearly, the source number is invariant under taking the quotient w.r.t. \sim , that is, $\alpha_s(\vec{H}) = \alpha_s(\vec{H}/\sim)$. We will see that the same is true for the fractional cover number, that is, $\rho^*(\vec{H}) = \rho^*(\vec{H}/\sim)$. Consequently, it *always* suffices to consider the DAG \vec{H}/\sim for determining the complexity of counting copies and induced copies of \vec{H} .

Definition 3. The fractional cover number of \vec{H} is $\rho^*(\vec{H}) = \rho^*(\Gamma(\vec{H}))$. The source number of \vec{H} is the number of sources in \mathcal{S} , denoted by $\alpha_s(\vec{H})$; in other words, the number of strongly connected components of \vec{H} that are not reachable from any other connected component.

Intuitively, both ρ^* and α_s measure the complexity of covering \vec{H} through its sources. Our main result, the following dichotomy theorem, says that such a “covering complexity” determines precisely the fixed-parameter tractability of our problems. For any class $\vec{\mathcal{C}}$ of directed graphs let $\rho^*(\vec{\mathcal{C}}) = \sup_{\vec{H} \in \vec{\mathcal{C}}} \rho^*(\vec{H})$ and $\alpha_s(\vec{\mathcal{C}}) = \sup_{\vec{H} \in \vec{\mathcal{C}}} \alpha_s(\vec{H})$.

Theorem 4. If the Exponential Time Hypothesis holds, then:

1. $\#\text{DIRSUB}_d(\vec{\mathcal{C}}) \in \text{FPT}$ if and only if $\rho^*(\vec{\mathcal{C}}) < \infty$
2. $\#\text{DIRINDSUB}_d(\vec{\mathcal{C}}) \in \text{FPT}$ if and only if $\alpha_s(\vec{\mathcal{C}}) < \infty$

Note that ETH is used only by the “only if” direction. While the statement of Theorem 4 is simple, its proof is nontrivial — virtually all of this manuscript is devoted to it. To put the theorem into perspective, Table 1 compares it to dichotomies for the other variants of the problem. We also observe that Theorem 4 can be slightly strengthened: we can show the hardness direction even for acyclic host graphs.

As a consequence of Theorem 4, we can claim the optimality (in an FPT sense) of the well-known approach to counting the induced copies of a DAG \vec{H} in a host \vec{G} of bounded outdegree, used in several recent works on counting in hosts of bounded degeneracy [8, 10, 9, 13, 32, 7]. This approach consists in guessing the images of the sources of \vec{H} in \vec{G} , and has running time

	$\#SUB(C)$	$\#DIRSUB(\vec{C})$	$\#DIRSUB_d(\vec{C})$
FPT criterion	$vc(C) < \infty$ Curticapean, Marx [24]	$vc(\vec{C}) < \infty$ Appendix B	$\rho^*(\vec{C}) < \infty$ Theorem 4
	$\#INDSUB(C)$	$\#DIRINDSUB(\vec{C})$	$\#DIRINDSUB_d(\vec{C})$
FPT criterion	$ C < \infty$ Chen, Thurley, Weyer [18]	$ \vec{C} < \infty$ Appendix B	$\alpha_s(\vec{C}) < \infty$ Theorem 4

Table 1: Our results for $\#DIRSUB_d(\vec{C})$ and $\#DIRINDSUB_d(\vec{C})$ compared against $\#INDSUB(\vec{C})$ and $\#DIRINDSUB(\vec{C})$, which are their counterparts parameterised by $|\vec{H}|$, and against $\#SUB(C)$ and $\#INDSUB(C)$, which are their *undirected* counterparts parameterised by $|H|$. Here vc denotes the vertex cover number; for directed graphs this is just the vertex cover number of the underlying undirected graph. The results for $\#INDSUB(\vec{C})$ and $\#DIRINDSUB(\vec{C})$ are folklore in the community.

$f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{\alpha_s(\vec{H}) + O(1)}$. By Theorem 4, unless ETH fails the dependence on $\alpha_s(\vec{H})$ at the exponent cannot be avoided, hence that approach is optimal in an FPT sense.

It shall be noted that, for $\#DIRINDSUB_d(\vec{C})$, the non-FPT case in Theorem 4 also yields $\#W[1]$ -hardness (see Section 4 for a definition of $\#W[1]$). For $\#DIRSUB_d(\vec{C})$ instead we do not prove $\#W[1]$ -hardness; the reason is that our proof uses a reduction from certain families of $\#CSP$ instances which by [40] we know to be not in FPT if ETH holds, but we do not know if they are $\#W[1]$ -hard too.

When the problems in Theorem 4 are in FPT, we can show simple algorithms that solve them in time $f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{p(\vec{H}) + O(1)}$ where $p \in \{\rho^*, \alpha_s\}$. Formally, we prove:

Theorem 5. *For some computable function f there is an algorithm solving $\#DIRSUB_d(\vec{C})$ in time $f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{\rho^*(\vec{H}) + O(1)}$. The same holds for $\#DIRINDSUB_d(\vec{C})$ with α_s in place of ρ^* .*

We point out that theorems 4 and 5 remain true in the (edge or vertex) weighted setting, too.

A simple example shows Theorem 4 and Theorem 5 in action. Let Δ_1 and Δ_2 be respectively the cyclic and acyclic orientations of K_3 , and for each $k \in \mathbb{N}$ let Δ_1^k and Δ_2^k consist of k disjoint copies of respectively Δ_1 and Δ_2 . Finally, let $\vec{C}_1 = \{\Delta_1^k : k \in \mathbb{N}\}$ and $\vec{C}_2 = \{\Delta_2^k : k \in \mathbb{N}\}$. Although the patterns are rather elementary, establishing the tractability of $\#DIRSUB_d(\vec{C}_1)$ and $\#DIRSUB_d(\vec{C}_2)$ “by hand” can be laborious. Theorem 4 and Theorem 5 answer immediately: $\rho^*(\Delta_1^k) = 0$, since in Δ_1^k every vertex belongs to some source, hence $\#DIRSUB_d(\vec{C}_1)$ is fixed-parameter tractable and solvable in time $f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{O(1)}$; but $\rho^*(\Delta_2^k) = k$, since $\Gamma(\Delta_2^k)$ has k disjoint hyperedges, hence $\#DIRSUB_d(\vec{C}_2)$ is not fixed-parameter tractable unless ETH fails. One can also see that $\alpha_s(\Delta_1^k) = \alpha_s(\Delta_2^k) = k$; therefore, by Theorem 4, under ETH both $\#DIRINDSUB_d(\vec{C}_1)$ and $\#DIRINDSUB_d(\vec{C}_2)$ are not fixed-parameter tractable.

Another example helps appreciating the different between our parameterization and the standard one, as well as the necessity of ρ^* being fractional. Let H_k be the graph defined as follows. The vertices of H_k are $U_k \cup D_k$ where $U_k = \{1, \dots, 2k\}$ and $D_k = \{A \subseteq U_k \mid |A| = k\}$; and for each $i \in U_k$, there is an edge between i and $D \in D_k$ if and only if $i \in D$. Let C be the class of all H_k . It is not hard to show that H_k contains the subdivision of the k -clique as induced subgraph. Thus the vertex-cover number of C is unbounded and, assuming ETH, Table 1 yields that $\#SUB(C)$ and $\#DIRSUB(\vec{C})$ are not fixed-parameter tractable for any class \vec{C} obtained by orienting the graphs in C . However, if we parameterise also by the outdegree of the host, then the situation becomes much more subtle. Let \vec{C} be the class of digraphs obtained by orienting the edges in the H_k from U_k to D_k ; an argument similar to [33, Example 4.2] shows that $\rho^*(\vec{H}_k) \leq 2$ for each $\vec{H}_k \in \vec{C}$, thus $\#DIRSUB_d(\vec{C})$ is fixed-parameter tractable by Theorem 5. Moreover, [33, Example 4.2] show that any non-fractional cover of \vec{H}_k has super-constant weight; this proves that considering the *fractional* cover number ρ^* is crucial; its integral counterpart cannot work.

We conclude this section with a result of independent interest developed in our proofs. Let \mathcal{H} be a hypergraph. The independence number $\alpha(\mathcal{H})$ of \mathcal{H} is the size of the largest subset of $V(\mathcal{H})$ such that no two of its elements are contained in a common edge. The natural relaxation of this definition yields the *fractional* independence number $\alpha^*(\mathcal{H})$. Our result is that the ratio between $\alpha^*(\mathcal{H})$ and $\alpha(\mathcal{H})$, i.e. the integrality gap of α , is bounded by the *adaptive width*¹ of \mathcal{H} [39].

Theorem 6. *Every hypergraph \mathcal{H} satisfies $\alpha(\mathcal{H}) \geq \frac{1}{2} + \frac{\alpha^*(\mathcal{H})}{4 \text{aw}(\mathcal{H})}$.*

3 Technical Overview

This section gives an overview of the tools and techniques behind the results of Section 2. The overview focuses on $\#\text{DIRSUB}_d(\vec{C})$, but similar arguments apply to $\#\text{DIRINDSUB}_d(\vec{C})$. Before digging into the most technical part, let us give the high-level idea of our proof strategy.

At the root of all our results is a standard connection between copies and homomorphisms, explained in Section 3.1. It is well known indeed that $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ can be expressed as a linear combination of homomorphism counts, $\sum_{\vec{F}} a_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \vec{G})$, where $a_{\vec{H}}(\vec{F}) > 0$ and \vec{F} ranges over a certain set of *quotients* of \vec{H} . Here, a quotient of \vec{H} is a directed graph obtained from \vec{H} by contracting (not necessarily connected) vertex subsets into single vertices (see Section 4 for the formal definition). It is also known that the complexity of computing $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ equals, up to $f(|\vec{H}|)$ factors, that of computing the hardest $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ term. Therefore we can reduce $\#\text{DIRSUB}_d(\vec{C})$ to and from its homomorphism counting version $\#\text{DIRHOM}_d(\vec{Q})$, where \vec{Q} consists of certain quotients of \vec{C} . Armed with these results, we proceed as follows.

First, in Section 3.2 we prove that $\rho^*(\vec{C}) < \infty$ implies $\#\text{DIRSUB}_d(\vec{C}) \in \text{FPT}$. To this end we prove that if \vec{F} is a quotient of \vec{H} then the *fractional hypertreewidth* of the contour of \vec{F} satisfies $\text{fhtw}(\Gamma(\vec{F})) \leq \rho^*(\vec{H})$. Therefore, $\text{fhtw}(\Gamma(\vec{Q})) \leq \rho^*(\vec{C}) < \infty$. We then show that computing $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ can be reduced in FPT time to counting the homomorphisms from $\Gamma(\vec{F})$ to a hypergraph \mathcal{G} obtained from \vec{F} and \vec{G} . As hypergraph homomorphism counting is in FPT when the pattern has bounded fractional hypertreewidth, this proves the claim.

Next, in Section 3.3 we prove that $\rho^*(\vec{C}) = \infty$ implies $\#\text{DIRSUB}_d(\vec{C}) \notin \text{FPT}$, or ETH fails. To start with, we suppose \vec{C} contains only *canonical DAGs*, directed graphs of a particularly simple type. We can prove that the aforementioned problem of counting homomorphisms between hypergraphs can be reduced to $\#\text{DIRHOM}_d(\vec{C})$ if the considered hypergraph patterns belong to the contours of \vec{C} . By existing results this implies that, unless ETH fails, $\#\text{DIRHOM}_d(\vec{C}) \notin \text{FPT}$ whenever the contours of \vec{C} have unbounded *adaptive width* [41]. It remains to lift these results from canonical DAGs to arbitrary DAGs and, ultimately, to arbitrary directed graphs. To this end, we introduce what we call *monotone reversible minors* (MRMs). Intuitively, \vec{H}' is an MRM of \vec{H} if there exists an FPT reduction from counting copies \vec{H}' to counting copies of \vec{H} , and if \vec{H}' preserves some parameters of interest (like ρ^*). We show that *every* directed graph \vec{H} has an MRM \vec{H}' that is a canonical DAG, so counting \vec{H} is at least as hard as counting \vec{H}' . Next, we show that counting copies of \vec{H}' is hard. To this end we show that, if $\rho^*(\vec{H}')$ is large, then its reduct $\Gamma(\vec{H}')$ has large adaptive width or large independence number. By employing arguments from the homomorphism connection above and from [13], this implies that counting the copies of \vec{H}' is hard unless ETH fails, which concludes our proof.

In what follows we use standard terminology as much as possible; in any case, all concepts and terms are defined formally in Section 4.

3.1 The Directed Homomorphism Basis

The first ingredient of our work is the so-called *homomorphism basis* introduced by Curticapean, Dell, and Marx [23], which establishes a common connection between (undirected) parameterised

¹Note that adaptive width is equivalent to submodular width [40].

pattern counting problems. Although the original framework is for undirected graphs, it can be equally well be formulated for the directed case, as we are going to do. Let \vec{H} be a digraph. There is a function $\text{sub}_{\vec{H}}$ of finite support from digraphs to rationals such that for each digraph \vec{G} :

$$\#\text{Sub}(\vec{H} \rightarrow \vec{G}) = \sum_{\vec{F}} \text{sub}_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \vec{G}) \quad (2)$$

This identity follows by well-known transformations based on inclusion-exclusion and Möbius inversion (see e.g. Chapter 5.2.3. in Lovász [38]). It is also well known that $\text{sub}_{\vec{H}}(\vec{F}) \neq 0$ if and only if \vec{F} is a quotient of \vec{H} .

These facts allow us to construct a reduction from the parameterized problem of computing $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ to the parameterized problem of computing $\#\text{Hom}(\vec{H} \rightarrow \vec{G})$ and vice versa. More precisely, one can show that computing $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ is precisely as hard (in FPT-equivalence terms) as computing the hardest term $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ in the summation of (2). One direction is obvious — the time to compute $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ is the sum of the times to compute all terms $\#\text{Sub}(\vec{F} \rightarrow \vec{G})$, whose number is a function of \vec{H} . The other direction is nontrivial, and was established for multiple variants of subgraph counting over the past years [15, 23, 26, 46, 7]. Rather than extending those results to yet another variant (directed graphs), we observe that the constructive version of Dedekind's Theorem on the linear independence of characters yields a general interpolation method that subsumes all those results, including the one for directed graphs. We prove what follows (see Theorem 36 for a more complex but complete version):

Theorem 7 (Simplified Version). *Let $(G, *)$ be a semigroup. Let furthermore $(\varphi_i)_{i \in [k]}$ with $\varphi_i : G \rightarrow \mathbb{Q}$ be pairwise distinct and non-zero semigroup homomorphisms of $(G, *)$ into (\mathbb{Q}, \cdot) , that is, $\varphi_i(g_1 * g_2) = \varphi_i(g_1) \cdot \varphi_i(g_2)$ for all $i \in [k]$ and $g_1, g_2 \in G$. Let $\phi : G \rightarrow \mathbb{Q}$ be a function*

$$\phi : g \mapsto \sum_{i=1}^k a_i \cdot \varphi_i(g), \quad (3)$$

where the a_i are rational numbers. Then there is an efficient algorithm $\hat{\mathbb{A}}$ which is equipped with oracle access to ϕ and which computes the coefficients a_1, \dots, a_k .

In our setting, Theorem 7 yields what follows. First, let G be the set of all digraphs and $*$ be the directed tensor product; one can check that $(G, *)$ is indeed a semigroup. Second, for any fixed \vec{H} consider the function $\vec{G} \mapsto \#\text{Hom}(\vec{H} \rightarrow \vec{G})$; one can check this is a semigroup homomorphism into \mathbb{Q} . Using Theorem 7, we can prove:

Lemma 8. *There exists a deterministic algorithm \mathbb{A} with the following specifications:*

- *The input of \mathbb{A} is a pair (\vec{G}', ι) where \vec{G}' is a digraph and $\iota : G \rightarrow \mathbb{Q}$.*
- *\mathbb{A} is equipped with oracle access to the function*

$$\vec{G} \mapsto \sum_{\vec{F}} \iota(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \vec{G}),$$

where the sum is over all (isomorphism classes of) digraphs.

- *The output of \mathbb{A} is the list with elements $(\vec{F}, \#\text{Hom}(\vec{F} \rightarrow \vec{G}'))$ for each \vec{F} with $\iota(\vec{F}) \neq 0$.*
- *For some computable function f the running time of \mathbb{A} is bounded by $f(|\iota|) \cdot |\vec{G}'|^{O(1)}$*
- *The outdegree of every digraph \vec{G} on which \mathbb{A} invokes the oracle is at most $f(|\iota|) \cdot d(\vec{G}')$ where $d(\vec{G}')$ is the maximum outdegree of \vec{G}' .*

To understand the meaning of Lemma 8, let $\iota(\vec{F}) = \text{sub}_{\vec{H}}(\vec{F})$ for all $\vec{F} \in G$, see (2). Then Lemma 8 says that, if \mathbb{A} has oracle access to $\#\text{Sub}(\vec{H} \rightarrow \cdot)$, then \mathbb{A} can compute $\#\text{Hom}(\vec{H} \rightarrow \vec{G}')$

efficiently and by computing $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ only for \vec{G} of outdegree not larger than that of \vec{G}' . This yields a parameterised reduction from $\#\text{DIRSUB}_d(\vec{C})$ to $\#\text{DIRHOM}_d(\vec{C}')$, where \vec{C}' is the set of all digraphs \vec{F} such that $\text{sub}_{\vec{H}}(\vec{F}) \neq 0$ for some $\vec{H} \in \vec{C}$. As stated above, $\text{sub}_{\vec{H}}(\vec{F}) \neq 0$ if and only if \vec{F} is a quotient of \vec{H} . We conclude that computing $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ is at least as hard as computing $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ for each \vec{F} that is a quotient of \vec{H} . In other words we have a parameterised reduction from $\#\text{DIRSUB}_d(\vec{C})$ to $\#\text{DIRHOM}_d(\vec{C}')$ where \vec{C}' is the set of all quotients of \vec{C} . Together with the converse reduction (see above) this tells us that $\#\text{DIRSUB}_d(\vec{C})$ is *precisely as hard as* $\#\text{DIRHOM}_d(\vec{C}')$ where \vec{C}' is the set of all quotients of \vec{C} . Thus, classifying the complexity of $\#\text{DIRSUB}_d(\vec{C})$ boils down to understanding the complexity of $\#\text{DIRHOM}_d(\vec{C}')$ where \vec{C}' is again the set of all quotients of \vec{C} . Answering this question turns out to be the most challenging task in this work.

3.2 Upper bounds: a reduction to $\#\text{CSP}$

To understand the complexity of $\#\text{DIRHOM}_d(\vec{C}')$ where \vec{C}' is the set of all quotients of \vec{C} , we take two steps. First, we show that the problem can be reduced to $\#\text{CSP}$, the problem of counting the solutions to a constraint satisfaction problem. Second, we show that the fractional cover number of \vec{C} bounds the fractional hypertree width of the $\#\text{CSP}$ instances obtained from \vec{C}' , which makes the problem fixed-parameter tractable by existing results.

3.2.1 A reduction to $\#\text{CSP}$

Let \vec{H} and \vec{G} be digraphs and let d be the maximum outdegree of \vec{G} . Let furthermore $k = |\vec{H}|$ and $n = |\vec{G}|$. Recall that a source S of \vec{H} is a strongly connected component of \vec{H} such that S cannot be reached from any other strongly connected component. Let S_1, \dots, S_ℓ be the sources of \vec{H} , and let $s_i \in S_i$ for each $i \in [\ell]$. Finally, let R_i be the set of all vertices of \vec{H} that can be reached from s_i via a directed path — note that S_i is fully contained in R_i . Clearly each arc of \vec{H} is fully contained in at least one of the R_i . Writing $\vec{H}[R_i]$ for the subgraph of \vec{H} induced by R_i , one can see that every map $\varphi : V(\vec{H}) \rightarrow V(\vec{G})$ satisfies:

$$\varphi \in \text{Hom}(\vec{H} \rightarrow \vec{G}) \Leftrightarrow \forall i \in [\ell] : \varphi|_{R_i} \in \text{Hom}(\vec{H}[R_i] \rightarrow \vec{G}), \quad (4)$$

where $\varphi|_{R_i}$ is the restriction of φ on R_i . In other words, φ is a homomorphism if and only if it induces a partial homomorphism from $\vec{H}[R_i]$ for each $i \in [\ell]$.

The observation above allows us to reduce the computation of $\text{Hom}(\vec{H} \rightarrow \vec{G})$ to counting the solutions of a certain constraint satisfaction problem. Start by fixing an arbitrary order over $V(\vec{H})$, so that every R_i appears as an ordered tuple. Now, for each $i \in [\ell]$, we enumerate all partial homomorphisms $\varphi|_{R_i} \in \text{Hom}(\vec{H}[R_i] \rightarrow \vec{G})$. It is well known that this can be done in time $f(k, d) \cdot n^{O(1)}$: simply guess the image v of s_i in $V(\vec{G})$, and perform a brute force search over the $d^{O(k)}$ vertices of \vec{G} reachable from v in k steps [20, 12, 7]. Now for every $i \in [\ell]$ consider the set of all (the images of) the maps in $\text{Hom}(\vec{H}[R_i] \rightarrow \vec{G})$. This is a set of ordered tuples of vertices of \vec{G} , i.e., a relation over $V(\vec{G})$. We denote this relation by R_i . It is not hard to see that the homomorphisms from \vec{H} to \vec{G} are precisely those maps from $V(\vec{H})$ to $V(\vec{G})$ that for every $i \in [\ell]$ send R_i to an element of R_i , and that counting those maps is an instance of a counting constraint satisfaction problem ($\#\text{CSP}$).

3.2.2 Bounding the cost of solving $\#\text{CSP}$ over quotients

Recall the reachability hypergraph $\mathcal{R}(\vec{H})$ of \vec{H} : the hypergraph whose vertex set is $V(\vec{H})$ and whose edge set is $\{R_i : i \in [\ell]\}$. A well-known result due to Grohe and Marx [33] states that counting the solutions to the CSP instance above is fixed-parameter tractable whenever $\mathcal{R}(\vec{H})$ has bounded fractional hypertreewidth, where the parameter is $|\vec{H}|$; in fact, [33] shows that

there is an algorithm that solves the problem in time $f(k, d) \cdot |V(\vec{G})|^{\text{fhtw}(\mathcal{R})+O(1)}$. Now recall from Section 2 the fractional cover number $\rho^*(\vec{H})$ of \vec{H} . We prove:

Lemma 9. *Let \vec{H} be a digraph, let \vec{F} be a quotient graph of \vec{H} , and let $\mathcal{R}(\vec{F})$ be the reachability hypergraph of \vec{F} . Then $\text{fhtw}(\mathcal{R}(\vec{F})) \leq \rho^*(\vec{H})$.*

The intuition behind the proof of Lemma 9 is that (i) taking the quotient of a digraph cannot increase its fractional cover number, and (ii) the fractional hypertreewidth of a hypergraph is bounded by its fractional edge cover number (which is the fractional cover number of \vec{H}).

Together with the observations above, this implies that we can compute $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ in time $f(|\vec{H}|, d(\vec{G})) \cdot |\vec{G}|^{\rho^*(\vec{H})}$, thus proving Theorem 5 and the tractability part of Theorem 4 for $\#\text{DIRSUB}_d$. It remains to prove the intractability part of Theorem 4, which we do in the next sections.

Let us again consider $\vec{H} = \Delta_1^k$ as a toy example, that is, \vec{H} is the disjoint union of k triangles, each of which is cyclically oriented. We can use the principle of inclusion and exclusion to reduce the computation of $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ to the computation of terms $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ where \vec{F} is a quotient of \vec{H} . Now, it can easily be observed that each quotient of \vec{H} is a disjoint union of strongly connected components S_1, \dots, S_ℓ . Unfolding our general reduction to $\#\text{CSP}$, for each of the strongly connected components S , we only have to guess the image v of one vertex $s \in S$ in \vec{G} . Then the image of each additional vertex in S must be reachable from v by a directed path of length at most k . Since the outdegree of \vec{G} is at most d , there are thus at most $d^{O(k)}$ possibilities for the images of the remaining vertices. Thus, for each strongly connected components S , we can compute $\#\text{Hom}(\vec{F}[S] \rightarrow \vec{G})$ in time $d^{O(k)} \cdot |\vec{G}|$. Finally, we have $\#\text{Hom}(\vec{F} \rightarrow \vec{G}) = \prod_{i=1}^\ell \#\text{Hom}(\vec{F}[S_i] \rightarrow \vec{G})$.

3.3 Lower bounds

The goal of this section is to prove that, roughly speaking, if $\rho^*(\vec{H})$ is large then \vec{H} has a quotient \vec{F} such that computing $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ is hard when parameterized by $|\vec{F}| + d(\vec{G})$. To this end we seek a reduction from $\#\text{CSP}$ to $\#\text{DIRHOM}_d$, i.e., in the opposite direction of Section 3.2. However, while that direction was relatively easy, since every digraph can be easily encoded as a set of relations, the direction we seek here is significantly harder. Indeed, it is not clear at all how an instance of $\#\text{CSP}$ can be “encoded” as a pair of directed graphs (\vec{H}, \vec{G}) if we can choose \vec{H} only from the class \vec{C} for which we want to prove hardness.

3.3.1 Encoding $\#\text{CSP}$ instances via canonical DAGs

To bypass the obstacle above, we start by considering classes of *canonical DAGs*. A digraph \vec{H} is a canonical DAG if it is acyclic and every vertex is either a source (i.e., it has indegree 0) or a sink (i.e., it has outdegree 0). Note that this implies that \vec{H} is bipartite, with (say) all sources on the left and all sinks on the right. If \vec{C} is a class of canonical DAGs, then it is easy to reduce $\#\text{CSP}$ to $\#\text{DIRHOM}_d(\vec{C})$ while preserving all parameters. To see why, let $(\mathcal{H}, \mathcal{G})$ be a pair of hypergraphs (the instance of $\#\text{CSP}$). Define \vec{H} by letting $V(\vec{H}) = V(\mathcal{H}) \cup \{x_e : e \in E(\mathcal{H})\}$, and adding (x_e, u) to $E(\vec{H})$ for every $e \in E(\mathcal{H})$ and every $u \in e$. Define \vec{G} similarly as a function of \mathcal{G} . One can then show, using the *color-prescribed* version of homomorphism counting (defined in Section 4), that $\#\text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$ can be computed in FPT time with $|\mathcal{H}|$ as a parameter if we can compute $\#\text{Hom}(\vec{H} \rightarrow \vec{G})$ in FPT time with $|\vec{H}| + d(\vec{G})$ as a parameter.

Recall then the *contour* $\mathcal{R}(\vec{H})$ of \vec{H} from Section 2. It is immediate to see that, if \vec{H} is a canonical DAG obtained from \mathcal{H} as described above, then $\mathcal{R}(\vec{H}) = \mathcal{H}$. This is precisely the intuitive role of the contour — to encode the structure of the reachability sets of \vec{H} (ignoring sources). Indeed, using contours we can then state our main reduction. Let \vec{C} be a class of canonical DAGs, and let $\#\text{CSP}(\Gamma(\vec{C}))$ be the restriction of $\#\text{CSP}$ to instances whose left-hand hypergraph (i.e., \mathcal{H}) is isomorphic to a contour of $\Gamma(\vec{C})$. Using as a starting point a reduction

due to Chen et al. [19], we prove that $\#\text{CSP}(\Gamma(\vec{C}))$ reduces to $\#\text{DIRHOM}_d(\vec{C})$ via parameterised Turing reductions. Now, under ETH, $\#\text{CSP}(\Gamma(\vec{C})) \notin \text{FPT}$ when the adaptive width of $\Gamma(\vec{C})$ is unbounded [40], unless ETH fails. By the reduction above, then, we obtain:

Lemma 10. *$\#\text{DIRHOM}_d(\vec{C}) \notin \text{FPT}$ for every class \vec{C} of canonical DAGs whose contours have unbounded adaptive width, unless ETH fails.*

We now seek to lift this result from canonical DAGs to arbitrary directed graphs.

3.3.2 Lifting hardness to arbitrary digraphs via Monotone Reversible Minors

Starting from Lemma 10, we prove a hardness result for $\#\text{DIRHOM}_d(\vec{C})$ for general classes of digraphs \vec{C} . To this end we need to reduce from $\#\text{DIRHOM}_d(\vec{C})$ to $\#\text{DIRHOM}_d(\vec{C}')$ where \vec{C}' is a class of canonical DAGs, so that we can apply Lemma 10; clearly, the reduction must imply that $\#\text{DIRHOM}_d(\vec{C}')$ has unbounded adaptive width.

Towards this end we introduce a kind of graph minors for digraphs, which we call *monotone reversible (MR) minors*. A digraph \vec{H}' is a MR minor of \vec{H} if it is obtained from \vec{H} by a sequence of the following operations:

- deleting a sink, i.e., a strongly connected component from which no other vertices can be reached
- deleting a loop
- contracting an arc

Note that, unlike standard minors, deletion of arbitrary vertices and arbitrary arcs are not allowed. This allows us to prove:

Lemma 11. *Let \vec{C} be a class of digraphs and let \vec{D} be a class of MR minors of \vec{C} . Then there exists a parameterised Turing reduction from $\#\text{DIRHOM}_d(\vec{D})$ to $\#\text{DIRHOM}_d(\vec{C})$.*

Lemma 11 explains the “reversible” part of MR minors—we can efficiently “revert” the operations that yielded a MR of a digraph; for the “monotone” see the next section. The heart of the proof of Lemma 11 proves the claim for the color-prescribed version of the problems; this implies the reduction for the original problems via standard interreducibility arguments. As a consequence of Lemma 11 we obtain:

Lemma 12. *Let \vec{C} be a recursively enumerable class of digraphs and let \vec{C}' be a class of canonical DAGs that are MR minors of digraphs in \vec{C} . If $\Gamma(\vec{C}')$ has unbounded adaptive width then $\#\text{DIRHOM}_d(\vec{C}) \notin \text{FPT}$, unless ETH fails.*

3.3.3 Lifting hardness from homomorphisms to subgraphs

Recall the arguments of Section 3.1: to prove that $\#\text{DIRSUB}(\vec{C})$ is hard when $\rho^*(\vec{C}) = \infty$, we essentially have to prove that every digraph \vec{H} with high fractional cover number $\rho^*(\vec{H})$ has a quotient \vec{F} that is hard. By the arguments of the previous section, to show that such a quotient \vec{F} is hard it is enough to show that \vec{F} has an MR minor \vec{F}' which is a canonical DAG whose contour $\Gamma(\vec{F}')$ has high adaptive width. We indeed prove that such a quotient exists. To this end, we consider two cases. Recall that $\alpha(\mathcal{H})$ and $\alpha^*(\mathcal{H})$ denote respectively the independence number and the fractional independence number of a hypergraph \mathcal{H} .

- (a) $\alpha(\Gamma(\vec{H}))$ is large. In this case we can show that \vec{H} contains a large matching whose edges are “isolated enough” for us to construct a quotient \vec{F} that admits, as MR minor, the 1-subdivision \vec{F}' of a large complete graph, where the arcs of \vec{F}' are directed away from the subdivision vertices. It is easy to see that \vec{F}' is a canonical DAG, and that $\Gamma(\vec{F}')$ is the complete graph itself, which has large adaptive width.

- (b) $\alpha(\Gamma(\vec{H}))$ is small. We then choose as quotient \vec{F} the graph \vec{H} itself. Recall that, by definition, $\rho^*(\vec{H}) = \rho^*(\mathcal{R}(\vec{H}))$. By LP duality the fractional cover number equals the fractional independence number, that is, $\rho^*(\mathcal{R}(\vec{H})) = \alpha^*(\mathcal{R}(\vec{H}))$. Using Theorem 6, we deduce that the adaptive width of $\mathcal{R}(\vec{H})$ is within constant factors of $\alpha^*(\mathcal{R}(\vec{H}))$, and thus of $\rho^*(\vec{H})$. By carefully exploiting this fact, we can explicitly construct an MR minor \vec{F}' of \vec{H} that is both a canonical DAG and has high adaptive width.

Thus, in both cases we can show that if $\rho^*(\vec{H})$ is large then \vec{H} admits an MR minor that is a canonical DAG of large treewidth. Formally, we obtain:

Lemma 13. *Let \vec{C} be a class of digraphs such that $\rho^*(\vec{C}) = \infty$. Then the class \vec{C}' of all canonical DAGs that are MR minors of quotients of \vec{C} has unbounded adaptive width.*

As a consequence, this implies that $\#\text{DIRSUB}_d(\vec{C}) \notin \text{FPT}$ whenever $\rho^*(\vec{C}) = \infty$, unless ETH fails. This concludes the overview of the proof of the lower bounds for $\#\text{DIRSUB}_d(\vec{C})$.

Before proceeding with the next section, we wish to point out that, in the course of our proofs, we will also see that for computing the fractional cover number and the source count of a directed graph \vec{H} , it is *always* sufficient to consider the DAG \vec{H}/\sim (see Figure 1). In other words, the complexity of $\#\text{DIRSUB}_d(\vec{C})$ and $\#\text{DIRINDSUB}_d(\vec{C})$ solely depends on the structure of the DAGs obtained by contracting the strongly connected components of the patterns in \vec{C} .

3.4 Related Work and Outlook

Our results are closely related to the recent surge of works on pattern counting in degenerate graphs [8, 10, 9, 13, 32, 7]: An undirected graph G has degeneracy d if there is an acyclic orientation \vec{G} of G with outdegree at most d . In the context of pattern counting in degenerate graphs, one is given undirected graphs H and G , and the goal is to compute the number of copies (resp. induced copies) of H in G , parameterised by the size of the pattern H and the degeneracy d of G . These problems have been completely classified with respect to linear time tractability [7] and with respect to fixed-parameter tractability [13].

The crucial difference to the results in this work is that, in our setting, the orientations of H and G are already fixed. Notably, this increases the set of tractable instances when compared to the degenerate setting: For example, counting copies of an undirected graph H in an undirected graph G , parameterised by $|H|$ and the degeneracy d of G , is FPT if and only if the induced matching number of H is small [13]. Now fix acyclic orientations \vec{H} and \vec{G} of H and G such that the outdegree of \vec{G} is at most d . One might think that the directed problem also is FPT if and only if H (i.e., the underlying undirected graph of \vec{H}) has small induced matching number. However, this is not true: We have shown in this work that we can count copies of a digraph \vec{H} in a digraph \vec{G} in FPT time (parameterised by $|\vec{H}|$ and $d(\vec{G})$) if and only if the fractional cover number of \vec{H} is small — we will see that this holds even if the host \vec{G} is a DAG. While H having small induced matching number certainly implies that the fractional cover number of \vec{H} is small, the other direction does not hold: For example, if \vec{H} contains a source that is adjacent to all non-sources, then the fractional cover number is 1, although the induced matching number can be arbitrarily large.

This work also sheds some new light on the problem of counting homomorphisms into degenerate graphs: Bressan [12] has shown that we can count homomorphisms from H to G in FPT time (parameterised by $|H|$ and the degeneracy of G) if the so-called *dag treewidth* of H is small; it is currently open whether the other direction holds as well [13, 7]. The dag treewidth of H is just the maximum (non-fractional) hypertreewidth of the reachability hypergraph of any acyclic orientation of \vec{H} . Our reduction to $\#\text{CSP}$ implies that it is sufficient for the reachability hypergraphs to have small fractional hypertreewidth, which yields a fractional version of dag treewidth. However, it is not clear whether unbounded dag treewidth and unbounded fractional dag treewidth are equivalent, the reason for which is the fact that all acyclic orientations have to be considered. We leave this as an open problem for future work.

4 Preliminaries

Given a set S , we set $S^2 = S \times S$, and we write $S^{(2)}$ for the set of all unordered pairs of distinct elements of S . Let $f : A \times B \rightarrow C$ be a function and let $a \in A$. We write $f(a, *) : B \rightarrow C$ for the function that maps $b \in B$ to $f(a, b)$.

4.1 Graphs and Directed Graphs

We denote graphs by F, G, H , and directed graphs by $\vec{F}, \vec{G}, \vec{H}$. Graphs and digraphs are encoded via adjacency lists, and we write $|G|$ (resp. $|\vec{G}|$) for the length of the encoding. In the remainder of the paper we will call directed graphs just “digraphs”. We use $\{u, v\}$ for undirected edges, and (u, v) for directed edges, which we also call *arcs*. Furthermore, we will use C to denote classes of graphs, and \vec{C} to denote classes of digraphs. Our graphs do not contain multi-edges; however, we allow forward-backward arcs (u, v) and (v, u) in digraphs. Additionally, our undirected graphs do not contain loops (edges from a vertex to itself) unless stated otherwise. For technical reasons, we will allow loops in digraphs.²

A directed acyclic graph (DAG) is a digraph without directed cycles. A *source* of a DAG is a vertex with in-degree 0, and a *sink* of a DAG is a vertex with outdegree 0 (and an isolated vertex is simultaneously a source and a sink). Given a directed (not necessarily acyclic) graph \vec{H} and a vertex $v \in V(\vec{H})$, we write $R(v)$ for the set of vertices reachable from v by a directed path; this includes v itself. Given a set of vertices $S \subseteq V(\vec{H})$, we set

$$R(S) := \bigcup_{v \in S} R(v).$$

Let H be a graph and σ a partition of $V(H)$. The *quotient graph* of H w.r.t. σ , denoted by H/σ , is defined as follows: $V(H/\sigma)$ consists of the blocks of σ , and $\{B_1, B_2\} \in E(H/\sigma)$ if and only if $\{v_1, v_2\} \in E(H)$ for some $v_1 \in B_1$ and $v_2 \in B_2$. If H/σ does not contain loops then it is called a *spasm* [23]. These definitions can be adapted in the obvious way for digraphs.

Definition 14 (The DAG \vec{H}/\sim). *Let \vec{H} be a digraph and let $x, y \in V(\vec{H})$. We denote by \sim the equivalence relation over $V(\vec{H})$ whose classes are the strongly connected components of \vec{H} . We denote by \vec{H}/\sim the DAG obtained by deleting loops from the quotient of \vec{H} with respect to the partition of $V(\vec{H})$ given by \sim .*

Next we introduce some notions that will be used in our classifications.

Definition 15 (Directed split). *The directed split \vec{H}^2 of a graph H is the digraph obtained from the 1-subdivision of H by orienting all edges towards $V(H)$.*

Homomorphisms and Colourings A *homomorphism* from H to G is a map $\varphi : V(H) \rightarrow V(G)$ such that $\{\varphi(u), \varphi(v)\} \in E(G)$ whenever $\{u, v\} \in E(H)$. The set of all homomorphisms from H to G is denoted by $\text{Hom}(H \rightarrow G)$. An *H-colouring* of G is a homomorphism $c \in \text{Hom}(G \rightarrow H)$. An *H-coloured* graph is a pair (G, c) where G is a graph and c an H -colouring of G . A homomorphism $\varphi \in \text{Hom}(H \rightarrow G)$ is *color-prescribed* (by c) if $c(\varphi(v)) = v$ for all $v \in V(H)$. We write $\text{Hom}(H \rightarrow (G, c))$ for the set of all homomorphisms from H to G color-prescribed by c . These definitions can be adapted for digraphs in the obvious way; we emphasise that a homomorphism φ between digraphs must preserve the direction of the arcs, i.e., $(\varphi(u), \varphi(v)) \in E(\vec{G})$ whenever $(u, v) \in E(\vec{H})$.

²We will see and state explicitly, that all of our hardness results will also entail corresponding hardness in the restricted case of digraphs without loops.

Subgraphs and Induced Subgraphs A *subgraph* of a graph $G = (V, E)$ is a graph with vertices $\hat{V} \subseteq V$ and edges $\hat{E} \subseteq \hat{V}^{(2)} \cap E$. We write $\text{Sub}(H \rightarrow G)$ for the set of all subgraphs of G that are isomorphic to H . Similarly, a subgraph of a digraph $\vec{G} = (V, E)$ is a digraph with vertices $\hat{V} \subseteq V$ and arcs $\hat{E} \subseteq \hat{V}^2 \cap E$, and we denote by $\text{Sub}(\vec{H} \rightarrow \vec{G})$ the set of all subgraphs of \vec{G} that are isomorphic to \vec{H} .

Given a subset of vertices S of a graph H , we write $H[S]$ for the graph *induced* by S , that is $V(H[S]) := S$ and $E(H[S]) := E(H) \cap S^{(2)}$. The subgraph $\vec{H}[S]$ of a digraph induced by a vertex set S is defined correspondingly: $V(\vec{H}[S]) := S$ and $E(\vec{H}[S]) := E(\vec{H}) \cap S^2$. We write $\text{IndSub}(H \rightarrow G) := \{S \subseteq V(G) \mid G[S] \cong H\}$ for the set of all induced subgraphs of G that are isomorphic to H . Again, the notion $\text{IndSub}(\vec{H} \rightarrow \vec{G})$ is defined similarly for digraphs.

4.2 Hypergraphs

A hypergraph is a pair $\mathcal{H} = (V, E)$ where V is a finite set and $E \subseteq 2^V \setminus \{\emptyset\}$. The elements of E are called hyperedges or simply edges. Given $X \subseteq V$, the subhypergraph of \mathcal{H} induced by X is the hypergraph $\mathcal{H}[X]$ with vertex set X and edge set $\{e \cap X : e \in E\} \setminus \{\emptyset\}$. The *arity* of a hypergraph \mathcal{G} , denoted by $a(\mathcal{G})$ is the maximum cardinality of a hyperedge. We denote hypergraphs with the symbols $\mathcal{H}, \mathcal{G}, \dots$

Definition 16 (Reachability Hypergraph). *Let \vec{H} be a digraph, let $S_1, \dots, S_k \subseteq V(\vec{H})$ be the sources of \vec{H}/\sim , and for each $i \in [k]$ let $s_i \in S_i$. The reachability hypergraph $\mathcal{R}(\vec{H})$ of \vec{H} has vertex set $V(\vec{H})$ and edge set $\{e_i = R(s_i) : i \in [k]\}$.*

Note that $\mathcal{R}(\vec{H})$ is well defined, since S_i is a strongly connected component of \vec{H} and so the choice of $s_i \in S_i$ is irrelevant. If \vec{H} is a DAG, then $\mathcal{R}(\vec{H})$ is the reachability hypergraph in the usual sense.

The following special case of DAGs, defined via reachability hypergraphs, will turn out to be crucial for our lower bounds.

Definition 17 (Canonical DAGs). *Let \mathcal{R} be a reachability hypergraph. For every $e \in E(\mathcal{R})$ fix some $s_e \in V(\mathcal{R})$ such that s_e is contained only in e . The canonical DAG \vec{H} of \mathcal{R} is defined by $V(\vec{H}) = V(\mathcal{R})$ and $E(\vec{H}) = \{(s_e, v) : e \in E(\mathcal{R}), v \in e \setminus \{s_e\}\}$.*

Note that if \vec{H} is a canonical DAG of \mathcal{R} then \mathcal{R} is the reachability hypergraph of \vec{H} . If a DAG \vec{H} is the canonical DAG of its own reachability hypergraph, then we just say \vec{H} is a canonical DAG. Equivalently, a DAG \vec{H} is a canonical DAG if every vertex is either a source or a sink.

Definition 18 (Contour). *Let \vec{H} be a digraph, let $S_1, \dots, S_k \subseteq V(\vec{H})$ be the sources of \vec{H}/\sim , and for each $i \in [k]$ let $s_i \in S_i$. The contour $\Gamma(\vec{H})$ of \vec{H} is the hypergraph obtained from $\mathcal{R}(\vec{H})$ by deleting S_i from e_i for each $i \in [k]$.*

Invariants and Width Measures In what follows, we are using the notation of [33] and [40], and we recall the most important definitions.

Definition 19 (Tree decompositions). *Let \mathcal{H} be a hypergraph. A tree decomposition of \mathcal{H} is a pair of a tree \mathcal{T} and a set of subsets of $V(\mathcal{H})$, called bags, $\mathcal{B} = \{B_t\}_{t \in V(\mathcal{T})}$ such that the following conditions are satisfied:*

1. $\bigcup_{t \in V(\mathcal{T})} B_t = V(\mathcal{H})$.
2. For every hyperedge $e \in E(\mathcal{H})$ there is a bag B_t such that $e \subseteq B_t$.
3. For every vertex $v \in V(\mathcal{H})$, the subgraph $\mathcal{T}[\{t \mid v \in B_t\}]$ of \mathcal{T} is connected.

Definition 20 (*f*-width). Let \mathcal{H} be a hypergraph, let $f : 2^{V(\mathcal{H})} \rightarrow \mathbb{R}_+$, and let $(\mathcal{T}, \mathcal{B})$ be a tree decomposition of \mathcal{H} . The *f*-width of $(\mathcal{T}, \mathcal{B})$ is defined as follows:

$$f\text{-width}(\mathcal{T}, \mathcal{B}) := \max_{t \in V(\mathcal{T})} f(B_t).$$

The *f*-width of \mathcal{H} is the minimum *f*-width of any tree decomposition of \mathcal{H} .

For example, the *treewidth* of a (hyper-)graph is just its *f*-width for the function $f(B) := |B| - 1$.

Given a hypergraph \mathcal{H} and a vertex-subset X of \mathcal{H} , the *edge cover number* $\rho_{\mathcal{H}}(X)$ of X is the minimum number of hyperedges of \mathcal{H} required to cover each vertex in X . The *edge cover number* of \mathcal{H} , denoted by $\rho(\mathcal{H})$, is defined as $\rho_{\mathcal{H}}(V(\mathcal{H}))$.

A *fractional* version of the edge cover number is defined similarly: Given \mathcal{H} and X as above, a function $\gamma : E(\mathcal{H}) \rightarrow [0, \infty]$ is a fractional edge cover of X if for each $v \in X$ we have $\sum_{e: v \in e} \gamma(e) \geq 1$. The fractional edge cover number $\rho_{\mathcal{H}}^*(X)$ of X is defined to be the minimum of $\sum_{e \in E(\mathcal{H})} \gamma(e)$ among all fractional edge covers γ of X . The *fractional edge cover number* of \mathcal{H} , denoted by $\rho^*(\mathcal{H})$, is defined as $\rho_{\mathcal{H}}^*(V(\mathcal{H}))$.

Definition 21 (Generalised and Fractional Hyper-Treewidth). The *generalised hyper-treewidth* of \mathcal{H} , denoted by $\text{htw}(\mathcal{H})$, is the $\rho_{\mathcal{H}}$ -width of \mathcal{H} . The *fractional hyper-treewidth* of \mathcal{H} , denoted by $\text{fhtw}(\mathcal{H})$, is the $\rho_{\mathcal{H}}^*$ -width of \mathcal{H} .

An *independent set* of a hypergraph \mathcal{H} is a set I of vertices such that, for each $u, v \in I$ with $u \neq v$, there is no hyperedge containing u and v . The *independence number* of \mathcal{H} , denoted by $\alpha(\mathcal{H})$, is the size of a maximum independent set of \mathcal{H} . A *fractional independent set* of a hypergraph \mathcal{H} is a mapping $\alpha^* : V(\mathcal{H}) \rightarrow [0, 1]$ such that for each $e \in E(\mathcal{H})$ we have

$$\sum_{v \in e} \alpha^*(v) \leq 1.$$

The *fractional independence number* of \mathcal{H} , denoted by $\alpha^*(\mathcal{H})$ is the maximum of $\sum_{v \in e} \alpha^*(v)$ among all fractional independent sets α^* . For a subset X of vertices in $V(\mathcal{H})$, we set $\alpha^*(X) = \sum_{v \in X} \alpha^*(v)$. We remark that, by LP duality, the fractional independence number and the fractional edge cover number are equal (see, [47]):

Fact 22. Let \mathcal{H} be a hypergraph. We have $\alpha^*(\mathcal{H}) = \rho^*(\mathcal{H})$.

We continue with the notion of adaptive width, which is equivalent³ to submodular width as shown by Marx [40].

Definition 23 (Adaptive width). Let \mathcal{H} be a hypergraph. The *adaptive width* of \mathcal{H} is

$$\text{aw}(\mathcal{H}) := \sup \{ \alpha^*\text{-width}(\mathcal{H}) \mid \alpha^* \text{ is a fractional independent set of } \mathcal{H} \}.$$

Lemma 24 ([40]). Let \mathcal{C} be a class of hypergraphs. Then

- \mathcal{C} has unbounded adaptive width
- $\Rightarrow \mathcal{C}$ has unbounded fractional hyper-treewidth
- $\Rightarrow \mathcal{C}$ has unbounded generalised hyper-treewidth.

Furthermore, all of the above implications are strict, that is, there are classes \mathcal{C}_1 and \mathcal{C}_2 such that \mathcal{C}_1 has bounded adaptive width but unbounded fractional hyper-treewidth, and \mathcal{C}_2 has bounded fractional hyper-treewidth but unbounded generalised hyper-treewidth.

³Here, “equivalent” means that a class of hypergraphs has bounded adaptive width if and only if it has bounded submodular width.

Throughout this paper, we will be interested in the independence number α , the fractional independence number α^* , and the adaptive width aw of the contours of digraphs \vec{H} . The following lemma shows that it is equivalent to consider the contour of \vec{H}/\sim for those invariants.

Lemma 25. *Let \vec{H} be a digraph. We have*

1. $\alpha(\Gamma(\vec{H})) = \alpha(\Gamma(\vec{H}/\sim))$.
2. $\alpha^*(\Gamma(\vec{H})) = \alpha^*(\Gamma(\vec{H}/\sim))$ and $\rho^*(\Gamma(\vec{H})) = \rho^*(\Gamma(\vec{H}/\sim))$.
3. $\text{aw}(\Gamma(\vec{H})) = \text{aw}(\Gamma(\vec{H}/\sim))$.

Proof. Recall that \vec{H}/\sim is obtained from \vec{H} by contracting each strongly connected component into a single vertex. In the reachability hypergraph, and thus in the contour, this corresponds to identifying blocks of vertices $B = \{v_1, \dots, v_\ell\}$ satisfying that all vertices in B are contained in the same (non-empty) set of hyperedges, that is, there is a non-empty set of hyperedges E_B such that for each $i \in [\ell]$, the set of hyperedges containing v_i is E_B .

Hence it is sufficient to show that identifying the vertices in B — we call the resulting vertex v_B — does not change any of the invariants α , α^* , and aw . For what follows, let us write \mathcal{H} for the contour $\Gamma(\vec{H})$ of \vec{H} , and let us write \mathcal{H}' for the hypergraph obtained from \mathcal{H} obtained by contracting B to v_B . Furthermore, let E'_B be the set of hyperedges that contain v_B and observe that E'_B can be obtained from E_B by contracting B to v_B for each edge $e \in E_B$.

1. Goal: $\alpha(\mathcal{H}) = \alpha(\mathcal{H}')$. Let $S \subseteq V(\mathcal{H})$ be a maximum independent set of \mathcal{H} . Note that at most one vertex in B can be contained in S . If no vertex of S is contained in B , then S is an independent set of \mathcal{H}' . Otherwise, assume $v_i \in S$ for some $i \in [\ell]$. Then $S' := (S \setminus \{v_i\}) \cup \{v_B\}$ is an independent set of \mathcal{H}' . This shows that $\alpha(\mathcal{H}) \leq \alpha(\mathcal{H}')$.

For the other direction, let S' be a maximum independent set of \mathcal{H}' . If $v_B \in S'$, then we set $S := (S' \setminus \{v_B\}) \cup \{v_1\}$. Otherwise, we set $S := S'$. Clearly, S is an independent set of \mathcal{H} and thus $\alpha(\mathcal{H}) \geq \alpha(\mathcal{H}')$.

2. Goal: $\alpha^*(\mathcal{H}) = \alpha^*(\mathcal{H}')$ (Note that this is equivalent to $\rho^*(\mathcal{H}) = \rho^*(\mathcal{H}')$ by Fact 22). Let μ be a fractional independent set of \mathcal{H} of maximum weight. Define

$$\mu'(v) := \begin{cases} \sum_{i=1}^{\ell} \mu(v_i) & v = v_B \\ \mu(v) & v \neq v_B \end{cases}.$$

We claim that μ' is a fractional independent set of \mathcal{H}' . Let $e' \in E(\mathcal{H}')$ and let e be the corresponding edge in \mathcal{H} , that is, $e = e'$ if $e' \notin E'_B$, and e' is obtained from e by contracting B into v_B otherwise. Depending on whether $e \in E_B$ we have that either $B \subseteq e$ or $B \cap e = \emptyset$. In both cases, by definition of μ' , we have that $\sum_{v' \in e'} \mu'(v') = \sum_{v \in e} \mu(v) \leq 1$. Thus μ' is a fractional independent set of \mathcal{H}' . Since, clearly, μ' has the same total weight as μ , we have that $\alpha^*(\mathcal{H}) \leq \alpha^*(\mathcal{H}')$.

For the other direction, let μ' be a fractional independent set of \mathcal{H}' of maximum weight. Define

$$\mu(v) := \begin{cases} \mu'(v_B) & v = v_1 \\ 0 & v \in B \setminus \{v_1\} \\ \mu'(v) & v \notin B \end{cases}.$$

We claim that μ is a fractional independent set of \mathcal{H} . Similarly as in the first direction, let $e \in E(\mathcal{H})$ and let e' be the corresponding edge in \mathcal{H}' . Again, depending on whether $e \in E_B$, we have that either $B \subseteq e$ or $B \cap e = \emptyset$, and in both cases, by definition of μ , we have $\sum_{v \in e} \mu(v) = \sum_{v' \in e'} \mu'(v') \leq 1$. Thus μ is a fractional independent set of \mathcal{H} . Since, clearly, μ has the same total weight as μ' , we have that $\alpha^*(\mathcal{H}) \geq \alpha^*(\mathcal{H}')$.

3. Goal: $\text{aw}(\mathcal{H}) = \text{aw}(\mathcal{H}')$. Any tree decomposition $(\mathcal{T}, \mathcal{B})$ of \mathcal{H} can be transformed to a tree decomposition $(\mathcal{T}', \mathcal{B}')$ of \mathcal{H}' as follows: In each bag that contains a vertex in B , we delete all vertices in B and add v_B . Clearly, the union over all bags in \mathcal{B}' is the set of all vertices of \mathcal{H}' , and each hyperedge of \mathcal{H}' is fully contained in some bag. For the last condition necessary for $(\mathcal{T}', \mathcal{B}')$ being a tree decomposition of \mathcal{H}' , we have to show that for each $v' \in V(\mathcal{H}')$, the subgraph $\mathcal{T}'_{v'}$ of \mathcal{T}' consisting of the bags containing v' is connected. If $v' \neq v_B$ then this property immediately follows from $(\mathcal{T}, \mathcal{B})$ being a tree decomposition. For $v' = v_B$ we use that \mathcal{T}_{v_i} is connected for each $i \in [\ell]$ since $(\mathcal{T}, \mathcal{B})$ is a tree decomposition of \mathcal{H} . Now \mathcal{T}'_{v_B} corresponds, by definition of $(\mathcal{T}', \mathcal{B}')$, to the union of all the \mathcal{T}_{v_i} . However, since $E_B \neq \emptyset$, there is a hyperedge e of \mathcal{H} fully containing B . Since e must be fully contained in a bag of $(\mathcal{T}, \mathcal{B})$, all of the \mathcal{T}_{v_i} overlap in at least one vertex, and thus \mathcal{T}'_{v_B} is connected, proving that $(\mathcal{T}', \mathcal{B}')$ is indeed a tree decomposition of \mathcal{H}' . For what follows, let τ denote the function that maps a tree decomposition $(\mathcal{T}, \mathcal{B})$ of \mathcal{H} to a tree decomposition $(\mathcal{T}', \mathcal{B}')$ of \mathcal{H}' as defined above.

In the other direction, each tree decomposition $(\mathcal{T}', \mathcal{B}')$ of \mathcal{H}' can be made a tree decomposition of \mathcal{H} by substituting v_B with the vertices in B . In this direction, it is clear that this yields a tree decomposition $(\mathcal{T}, \mathcal{B})$ of \mathcal{H} . For what follows, let τ' denote the function that maps a tree decomposition $(\mathcal{T}', \mathcal{B}')$ of \mathcal{H}' to a tree decomposition $(\mathcal{T}, \mathcal{B})$ of \mathcal{H} as defined above.

Now let $\text{aw}(\mathcal{H}) = a$. We prove that $\text{aw}(\mathcal{H}') \leq a$. Let μ' be a fractional independent set of \mathcal{H}' . We have to show that there is a tree decomposition $(\mathcal{T}', \mathcal{B}')$ of \mathcal{H}' with μ' -width $(\mathcal{T}', \mathcal{B}') \leq a$. To this end, let μ be the fractional independent of \mathcal{H} obtained from μ' as in 2. Since $\text{aw}(\mathcal{H}) = a$, there exists a tree decomposition $(\mathcal{T}, \mathcal{B})$ of \mathcal{H} with μ -width $(\mathcal{T}, \mathcal{B}) \leq a$. Let $(\mathcal{T}', \mathcal{B}') := \tau(\mathcal{T}, \mathcal{B})$. By definition of τ and μ , the μ' -width of $(\mathcal{T}', \mathcal{B}')$ is at most the μ -width of $(\mathcal{T}, \mathcal{B})$, which is at most a , concluding the first direction.

For the second direction, let $\text{aw}(\mathcal{H}') = a'$. We prove that $\text{aw}(\mathcal{H}) \leq a'$ similarly as in the first direction: Starting with a fractional independent set μ' of \mathcal{H} , we consider μ' as constructed in 2, and we obtain a tree decomposition $(\mathcal{T}', \mathcal{B}')$ of \mathcal{H}' with μ' -width $(\mathcal{T}', \mathcal{B}') \leq a'$. We set $(\mathcal{T}, \mathcal{B}) := \tau'(\mathcal{T}', \mathcal{B}')$ and observe that by definition of τ' and μ' , the μ -width of $(\mathcal{T}, \mathcal{B})$ is at most the μ' -width of $(\mathcal{T}', \mathcal{B}')$, which is at most a' , concluding the second direction and thus the proof. □

4.3 Relational Structures

A *signature* τ is a (finite) tuple of relation symbols $(R_i)_{i \in [\ell]}$ with arities $(a_i)_{i \in [\ell]}$. The arity of τ , denoted by $\mathbf{a}(\tau)$ is the maximum of the a_i . A *relational structure* \mathcal{A} of signature τ is a tuple $(V, R_1^{\mathcal{A}}, \dots, R_\ell^{\mathcal{A}})$ where V is a finite set of elements, called the *universe* of \mathcal{A} , and $R_i^{\mathcal{A}}$ is a relation on V of arity a_i for each $i \in [\ell]$. We emphasize that $R_i^{\mathcal{A}}$ is not necessarily symmetric, and that tuples might contain repeated elements. We will mainly use the symbols \mathcal{A} and \mathcal{B} to denote relational structures. Further, we assume that a structure \mathcal{A} is encoded in the standard way, i.e., the universe and the relations are encoded as lists. We denote by $|\mathcal{A}|$ the length of the encoding of \mathcal{A} .

Given two relational structures \mathcal{A} and \mathcal{B} over the same signature τ with universes U and V , a *homomorphism* from \mathcal{A} to \mathcal{B} is a mapping $\varphi : U \rightarrow V$ such that, for each $i \in [\mathbf{a}(\tau)]$ and for each tuple $t \in U^{a_i}$ we have

$$t \in R_i^{\mathcal{A}} \Rightarrow \varphi(t) \in R_i^{\mathcal{B}}.$$

We write $\text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$ for the set of homomorphisms from \mathcal{A} to \mathcal{B} .

The hypergraph $\mathcal{H}(\mathcal{A})$ of \mathcal{A} has as vertices the universe V of \mathcal{A} , and for each tuple $t = (v_1, \dots, v_a)$ of elements of V , we add an hyperedge $e_t = \{v_1, \dots, v_a\}$ if and only if t is an element

of a relation of \mathcal{A} . To avoid notational clutter, we will define the treewidth, the hypertreewidth, the fractional hypertreewidth and the submodular width of a structure as the respective width measure of its hypergraph. Similarly, a tree decomposition of a structure refers to a tree decomposition of its hypergraph.

4.4 Parameterised and Fine-Grained Complexity Theory

A *parameterised counting problem* is a pair (P, κ) of a counting problem $P : \{0, 1\}^* \rightarrow \mathbb{N}$ and a computable function $\kappa : \{0, 1\}^* \rightarrow \mathbb{N}$, called the *parameterisation*. Consider for example the parameterised clique counting problem:

#CLIQUE
Input: a pair of a graph G and a positive integer k
Output: the number of k -cliques in G
Parameter: k , that is, $\kappa(G, k) := k$

An algorithm for a parameterised (counting) problem is called a *fixed-parameter tractable* (FPT) algorithm if there is a computable function f such that, on input x , its running time can be bounded by $f(\kappa(x)) \cdot |x|^{O(1)}$. A parameterised (counting) problem is called *fixed-parameter tractable* if it can be solved by an FPT algorithm.

A *parameterised Turing-reduction* from (P, κ) to (P', κ') is an FPT algorithm for (P, κ) with oracle access to P' , additionally satisfying that there is a computable function g such that, on input x , the parameter $\kappa'(y)$ of any oracle query is bounded by $g(\kappa(x))$. We write $(P, \kappa) \leq_T^{\text{fpt}} (P', \kappa')$ if a parameterised Turing-reduction exists.

We say that (P, κ) is #W[1]-hard if #CLIQUE $\leq_T^{\text{fpt}} (P, \kappa)$. The class #W[1] can be considered a parameterised counting equivalent of NP, and we refer the interested reader to Chapter 14 in the standard textbook of Flum and Grohe [29] for a comprehensive introduction. It is known that #W[1]-hard problems are not fixed-parameter tractable unless standard assumptions, such as ETH, fail:

Definition 26 (The Exponential Time Hypothesis (ETH) [34]). *The Exponential Time Hypothesis (ETH) asserts that 3-SAT cannot be solved in time $\exp(o(n))$, where n is the number of variables.*

Theorem 27 (Chen et al. [16, 17]). *Assume that ETH holds. Then there is no function f such that #CLIQUE can be solved in time $f(k) \cdot |G|^{o(k)}$.*

Note that the previous theorem rules out an FPT algorithm for #CLIQUE (and thus all #W[1]-hard problems), unless ETH fails.

4.4.1 Parameterised Counting Problems

The following parameterized problems are central to the present work. In what follows $\vec{\mathcal{C}}$ denotes a class of directed graphs, and \mathcal{C} denotes a class of hypergraphs.

#DIRHOM_d($\vec{\mathcal{C}}$)
Input: a pair of digraphs (\vec{H}, \vec{G}) with $\vec{H} \in \vec{\mathcal{C}}$
Output: #Hom($\vec{H} \rightarrow \vec{G}$)
Parameter: $|\vec{H}| + d$ where d is the maximum outdegree of \vec{G}

#DIRSUB_d($\vec{\mathcal{C}}$)
Input: a pair of digraphs (\vec{H}, \vec{G}) with $\vec{H} \in \vec{\mathcal{C}}$
Output: #Sub($\vec{H} \rightarrow \vec{G}$)
Parameter: $|\vec{H}| + d$ where d is the maximum outdegree of \vec{G}

$\#\text{DIRINDSUB}_d(\vec{C})$

Input: a pair of digraphs (\vec{H}, \vec{G}) with $\vec{H} \in \vec{C}$

Output: $\#\text{IndSub}(\vec{H} \rightarrow \vec{G})$

Parameter: $|\vec{H}| + d$ where d is the maximum outdegree of \vec{G}

$\#\text{CP-DIRHOM}_d(\vec{C})$

Input: a digraph $\vec{H} \in \vec{C}$ and an \vec{H} -coloured digraph (\vec{G}, c)

Output: $\#\text{Hom}(\vec{H} \rightarrow (\vec{G}, c))$

Parameter: $|\vec{H}| + d$ where d is the maximum outdegree of \vec{G}

$\#\text{CSP}(\mathcal{C})$

Input: a pair of relational structures $(\mathcal{A}, \mathcal{B})$ over the same signature with $\mathcal{H}(\mathcal{A}) \in \mathcal{C}$

Output: $\#\text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$

Parameter: $|\mathcal{A}|$

It was shown by Grohe and Marx [33] that the decision version of $\#\text{CSP}(\mathcal{C})$ can be solved in polynomial time if the fractional hypertreewidth of \mathcal{C} is bounded. More precisely, they discovered an algorithm that solves the decision problem in time

$$(|\mathcal{A}| + |\mathcal{B}|)^{r+O(1)},$$

assuming that a tree decomposition of \mathcal{A} of ρ^* -width at most r is given (see Theorem 3.5 and Lemma 4.9 in [33]); recall that the ρ^* -width of a tree decomposition is the maximum fractional edge cover number of a bag. In particular, they show that the partial solutions of each bag can be enumerated in time $(|\mathcal{A}| + |\mathcal{B}|)^{r+O(1)}$. Thus the dynamic programming algorithm that solves the decision version immediately extends to counting. Finally, since computing such an optimal tree decomposition can be done in time only depending on \mathcal{A} , we obtain the following overall running time for the counting problem:

Theorem 28. *Let \mathcal{A} and \mathcal{B} be relational structures over the same signature and let r be the fractional hypertreewidth of \mathcal{A} . There is a computable function f such that we can compute $\#\text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$ in time*

$$f(|\mathcal{A}|) \cdot |\mathcal{B}|^{r+O(1)}.$$

In particular, $\#\text{CSP}(\mathcal{C})$ is fixed-parameter tractable if \mathcal{C} has bounded fractional hypertreewidth.

5 The Directed Homomorphism Basis and Dedekind's Theorem

In this section, we will revisit the interpolation technique for evaluating linear combinations of homomorphism counts due to Curticapean, Dell and Marx [23], and we will extend their framework from undirected graphs to digraphs. We wish to point out that most of the results presented in this section are easy consequences and generalisations of methods known in the literature [38, 15, 23, 26]; and we only provide the details for reasons of self-containment. For the purpose of this section, we assume that finitely supported functions ι from digraphs to rationals are encoded as a list of elements $(\vec{F}, \iota(\vec{F}))$ for all \vec{F} with $\iota(\vec{F}) \neq 0$. We write $|\iota|$ for the encoding length of ι .

To begin with, given a digraph \vec{H} , recall that $\#\text{Sub}(\vec{H} \rightarrow \star)$ and $\#\text{IndSub}(\vec{H} \rightarrow \star)$ denote the functions that map a digraph \vec{G} to $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ and $\#\text{IndSub}(\vec{H} \rightarrow \vec{G})$, respectively.

For our reductions, we express both functions $\#\text{Sub}(\vec{H} \rightarrow \star)$ and $\#\text{IndSub}(\vec{H} \rightarrow \star)$ as linear combinations of homomorphism counts from digraphs. We start with $\#\text{Sub}(\vec{H} \rightarrow \star)$ and point out that, similarly to the argument in [23], the existence of the following transformation follows from Möbius Inversion over the partition lattice as shown by Lovász (see Chapter 5.2.3 in [38]).⁴

⁴In fact, Lemma 29 and Lemma 32 are special cases of more general transformations for counting answers to conjunctive queries with disequalities and negations [26].

Lemma 29. *Let \vec{H} be a digraph. There exists a (unique and computable) function $\text{sub}_{\vec{H}}$ from digraphs to rationals such that*

$$\#\text{Sub}(\vec{H} \rightarrow \star) = \sum_{\vec{F}} \text{sub}_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \star),$$

where the sum is over all (isomorphism classes of) digraphs \vec{F} . Moreover, the function $\text{sub}_{\vec{H}}$ has finite support, and satisfies $\text{sub}_{\vec{H}}(\vec{F}) \neq 0$ if and only if \vec{F} is a quotient graph of \vec{H} .

A similar transformation is known for $\#\text{IndSub}(\vec{H} \rightarrow \star)$ which relies on *arc supergraphs*.

Definition 30 (Arc supergraphs). *Let $\vec{H}_1 = (V_1, E_1)$ and $\vec{H}_2 = (V_2, E_2)$ be digraphs without loops. We say that \vec{H}_2 is an arc supergraph of \vec{H}_1 if $V_1 = V_2$ and $E_1 \subseteq E_2$.*

In the first step, using the inclusion-exclusion principle, $\#\text{IndSub}(\vec{H} \rightarrow \star)$ can be cast as a linear combination of subgraph counts (the proof is analogous to the undirected setting; see [23] and [38, Chapter 5.2.3]):

Lemma 31. *Let \vec{H} be a digraph. There exists a (unique and computable) function $\text{indsub}_{\vec{H}}^*$ from digraphs to rationals such that*

$$\#\text{IndSub}(\vec{H} \rightarrow \star) = \sum_{\vec{F}'} \text{indsub}_{\vec{H}}^*(\vec{F}') \cdot \#\text{Sub}(\vec{F}' \rightarrow \star),$$

where the sum is over all (isomorphism classes of) digraphs \vec{F}' . Moreover, the function $\text{indsub}_{\vec{H}}^*$ has finite support and satisfies $\text{indsub}_{\vec{H}}^*(\vec{F}') \neq 0$ if and only if \vec{F}' is an arc supergraph of \vec{H} .

In combination, the previous two lemmas allow us to cast $\#\text{IndSub}(\vec{H} \rightarrow \star)$ as a linear combination of homomorphism counts.

Lemma 32. *Let \vec{H} be a digraph. There exists a (unique and computable) function $\text{indsub}_{\vec{H}}$ from digraphs to rationals such that*

$$\#\text{IndSub}(\vec{H} \rightarrow \star) = \sum_{\vec{F}} \text{indsub}_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \star),$$

where the sum is over all (isomorphism classes of) digraphs \vec{F} . Moreover, $\text{indsub}_{\vec{H}}$ satisfies the following conditions:

1. $\text{indsub}_{\vec{H}}$ has finite support.
2. If $\text{indsub}_{\vec{H}}(\vec{F}) \neq 0$ then \vec{F} is a quotient of an arc supergraph of \vec{H} .
3. If \vec{F} is an arc supergraph of \vec{H} , then $\text{indsub}_{\vec{H}}(\vec{F}) \neq 0$.

Proof. We first apply the transformation from induced subgraphs to subgraphs as in Lemma 31, second, we apply the transformation from subgraphs to homomorphisms as in Lemma 29. We obtain

$$\#\text{IndSub}(\vec{H} \rightarrow \star) = \sum_{\vec{F}'} \text{indsub}_{\vec{H}}^*(\vec{F}') \cdot \sum_{\vec{F}} \text{sub}_{\vec{F}'}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \star). \quad (5)$$

The coefficients $\text{indsub}_{\vec{H}}$ are then obtained by collecting for isomorphic terms, that is

$$\text{indsub}_{\vec{H}}(\vec{F}) = \sum_{\vec{F}'} \text{indsub}_{\vec{H}}^*(\vec{F}') \cdot \text{sub}_{\vec{F}'}(\vec{F}). \quad (6)$$

Note that 1 and 2 follow immediately from the properties of sub and indsub^* . It remains to show 3: To this end, let \vec{F} be an *arc supergraph* of \vec{H} . Then, for each arc supergraph \vec{F}' of \vec{H} , the only quotient graph of \vec{F}' that can be isomorphic to \vec{F} is \vec{F}' itself, since all other quotients have fewer vertices. By Lemma 29 we hence have that arc supergraphs \vec{F} and \vec{F}' of \vec{H} satisfy that $\text{sub}_{\vec{F}'}(\vec{F}) \neq 0$ implies $\vec{F} \cong \vec{F}'$. Finally, Lemma 31 asserts that $\text{indsub}_{\vec{H}}^*(\vec{F}') \neq 0$ if and only if \vec{F}' is an arc supergraph of \vec{H} . Thus, using that \vec{F} is an arc supergraph of \vec{H} , we have that (6) simplifies to $\text{indsub}_{\vec{H}}(\vec{F}) = \text{indsub}_{\vec{H}}^*(\vec{F})$, which is non-zero by Lemma 31. This concludes the proof. \square

A Remark on loops: Readers familiar with [23] might notice that the quotient graphs in Lemma 29 and the arc supergraphs in Lemma 32 may have loops although loops are forbidden in [23]. The reason for this is the fact that we allow digraphs to contain loops, whereas undirected graphs in [23] are not allowed to contain loops. However, we emphasize that our hardness results will also apply to the restricted case of digraphs without loops. This will be made explicit in the respective sections.

Next we show that the interpolation method in [23] for evaluating linear combinations of homomorphism counts transfers to the directed setting as well. The algorithm will be the same as in [23]; however, our correctness proof will be both more concise and more general at the same time by relying on a classical result of Dedekind.

Lemma 33. *There exists a deterministic algorithm \mathbb{A} with the following specification:*

- *The input of \mathbb{A} is a pair of a digraph \vec{G}' with outdegree d and a function ι from digraphs to rationals of finite support.*
- *\mathbb{A} is equipped with oracle access to the function*

$$\vec{G} \mapsto \sum_{\vec{F}} \iota(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \vec{G}),$$

where the sum is over all (isomorphism classes of) digraphs.

- *The output of \mathbb{A} is the list with elements $(\vec{F}, \#\text{Hom}(\vec{F} \rightarrow \vec{G}'))$ for each \vec{F} with $\iota(\vec{F}) \neq 0$.*

Additionally, for some computable function $f : \mathbb{N} \rightarrow \mathbb{N}$ the running time of \mathbb{A} is bounded by $f(|\iota|) \cdot |\vec{G}'|^{O(1)}$ and the outdegree of every oracle query \vec{G} is at most $f(|\iota|) \cdot d$.

Similar algorithms exist for the restricted cases of digraphs without loops and DAGs.

The proof of the previous lemma requires some additional set-up. For what follows, we let $\vec{\mathcal{U}}^\circ$ be the class of all digraphs, and we let $\vec{\mathcal{U}}$ be the class of all digraphs without loops, and we let $\vec{\mathcal{D}}$ be the class of all DAGs. Next, consider the following operation on digraphs:

Definition 34 (Tensor product). *The tensor product $\vec{G} \otimes \vec{F}$ of two digraphs \vec{G} and \vec{F} is the digraph with $V(\vec{G} \otimes \vec{F}) = V(\vec{G}) \times V(\vec{F})$ and with $((a, b), (c, d)) \in E(\vec{G} \otimes \vec{F})$ iff $(a, c) \in E(\vec{G})$ and $(b, d) \in E(\vec{F})$.*

A *semigroup* is a pair of a set G and an associative operation $*$ on G . Now observe that the tensor product of digraphs is clearly associative, that is, $\vec{G} \otimes (\vec{F} \otimes \vec{H}) \cong (\vec{G} \otimes \vec{F}) \otimes \vec{H}$ given by the isomorphism $(u, (v, w)) \mapsto ((u, v), w)$. Observe further that the tensor product of two digraphs without loops does not contain loops, and that the tensor product of two DAGs is again a DAG. Consequently, we obtain three semigroups:

Observation 35. *$(\vec{\mathcal{U}}^\circ, \otimes)$, $(\vec{\mathcal{U}}, \otimes)$, and $(\vec{\mathcal{D}}, \otimes)$ are semigroups.*

The fact that the tensor product induces a semigroup will allow us to invoke constructive version of Dedekind's Theorem on the linear independence of characters from Artin [5, Theorem 12].⁵

⁵Note that Artin states Dedekind's Theorem for the case of $(G, *)$ being a group, rather than a semigroup. However, the proof only needs associativity of the operation $*$ and thus applies for the more general case of semigroups as well.

Since Artin does not state the constructive version explicitly - it only follows from their proof - we provide a self-contained argument in Appendix A for the reader's convenience.

Theorem 36. *Let $(G, *)$ be a semigroup. Let $(\varphi_i)_{i \in [k]}$ with $\varphi_i : G \rightarrow \mathbb{Q}$ be pairwise distinct semigroup homomorphisms of $(G, *)$ into (\mathbb{Q}, \cdot) , that is, $\varphi_i(g_1 * g_2) = \varphi_i(g_1) \cdot \varphi_i(g_2)$ for all $i \in [k]$ and $g_1, g_2 \in G$. Let $\phi : G \rightarrow \mathbb{Q}$ be a function*

$$\phi : g \mapsto \sum_{i=1}^k a_i \cdot \varphi_i(g), \quad (7)$$

where the a_i are rational numbers. Suppose furthermore that the following functions are computable:

1. The operation $*$.
2. The mapping $(i, g) \mapsto \varphi_i(g)$.
3. A mapping $i \mapsto g_i$ such that $\varphi_i(g_i) \neq 0$.
4. A mapping $(i, j) \mapsto g_{i,j}$ such that $\varphi_i(g_{i,j}) \neq \varphi_j(g_{i,j})$ whenever $i \neq j$.

Then there is a constant B only depending on the φ_i (and not on the a_i), and an algorithm $\hat{\mathbb{A}}$ such that the following conditions are satisfied:

- $\hat{\mathbb{A}}$ is equipped with oracle access to ϕ .
- $\hat{\mathbb{A}}$ computes a_1, \dots, a_k .
- Each oracle query \hat{g} only depends on the φ_i (and not on the a_i).
- The running time of $\hat{\mathbb{A}}$ is bounded by $O\left(B \cdot \sum_{i=1}^k \log a_i\right)$

We aim to apply Theorem 36 to prove Lemma 33. However, this requires us to establish the following properties:

Lemma 37. *The following conditions are satisfied:*

1. For every $\vec{H} \in \vec{\mathcal{U}}^\circ$ the function $\#\text{Hom}(\vec{H} \rightarrow \star)$ is a homomorphism from $(\vec{\mathcal{U}}^\circ, \otimes)$ into (\mathbb{Q}, \cdot) , that is, $\#\text{Hom}(\vec{H} \rightarrow \vec{G} \otimes \vec{F}) = \#\text{Hom}(\vec{H} \rightarrow \vec{G}) \cdot \#\text{Hom}(\vec{H} \rightarrow \vec{F})$ for all $\vec{G}, \vec{F} \in \vec{\mathcal{U}}^\circ$.
2. $\#\text{Hom}(\vec{H}_1 \rightarrow \star) \neq \#\text{Hom}(\vec{H}_2 \rightarrow \star)$ whenever $\vec{H}_1 \in \vec{\mathcal{U}}^\circ$ and $\vec{H}_2 \in \vec{\mathcal{U}}^\circ$ are non-isomorphic.

The same holds true in the restricted cases of digraphs without loops and DAGs, that is, the same holds true if $\vec{\mathcal{U}}^\circ$ is substituted by $\vec{\mathcal{U}}$ or $\vec{\mathcal{D}}$.

Proof. The first claim is immediate since, by definition of \otimes , each $\varphi \in \text{Hom}(\vec{H} \rightarrow \vec{G} \otimes \vec{F})$ decomposes (via projection) to $\varphi_1 \in \text{Hom}(\vec{H} \rightarrow \vec{G})$ and $\varphi_2 \in \text{Hom}(\vec{H} \rightarrow \vec{F})$, which induces a bijection.

For the second claim we follow a classical argument by Lovász (see Chapter 5.4 in [38]). For any two $\vec{F}, \vec{G} \in \vec{\mathcal{U}}^\circ$ define:

$$\#\text{Sur}(\vec{F} \rightarrow \vec{G}) := \{\varphi \in \#\text{Hom}(\vec{F} \rightarrow \vec{G}) \mid \varphi \text{ is vertex-surjective}\} \quad (8)$$

For any $S \subseteq V(\vec{G})$ let $\vec{G}[S]$ be the subgraph of \vec{G} induced by S . By inclusion and exclusion:

$$\#\text{Sur}(\vec{F} \rightarrow \vec{G}) = \sum_{S \subseteq V(\vec{G})} (-1)^{|V(\vec{G}) \setminus S|} \cdot \#\text{Hom}(\vec{F} \rightarrow \vec{G}[S]) \quad (9)$$

Now assume for contradiction that $\#\text{Hom}(\vec{H}_1 \rightarrow \star) = \#\text{Hom}(\vec{H}_2 \rightarrow \star)$ for two non-isomorphic $\vec{H}_1, \vec{H}_2 \in \vec{\mathcal{U}}^\circ$. Then for $\vec{G} = \vec{H}_1$ (9) yields $\#\text{Sur}(\vec{H}_2 \rightarrow \vec{H}_1) = \#\text{Sur}(\vec{H}_1 \rightarrow \vec{H}_1) > 0$, and for $\vec{G} = \vec{H}_2$ it yields $\#\text{Sur}(\vec{H}_1 \rightarrow \vec{H}_2) = \#\text{Sur}(\vec{H}_2 \rightarrow \vec{H}_2) > 0$. Hence there are surjective homomorphisms of \vec{H}_1 into \vec{H}_2 and of \vec{H}_2 into \vec{H}_1 . But then \vec{H}_1 and \vec{H}_2 are isomorphic, which yields the desired contradiction. \square

We are now able to proof Lemma 33.

Proof of Lemma 33. Let \vec{G}' and ι be the input. We apply Dedekind's Theorem (Theorem 36) to the semigroup $(\vec{\mathcal{U}}^\circ, \otimes)$ and mappings $\varphi_{\vec{F}} := \#\text{Hom}(\vec{F} \rightarrow \star)$ for the digraphs \vec{F} in the support $\text{supp}(\iota)$ of ι , that is, $k = |\text{supp}(\iota)|$. Concretely, assume that $\vec{F}_1, \dots, \vec{F}_k$ are elements of the support of ι (the \vec{F}_i are pairwise non-isomorphic) and observe that we can use our oracle to compute the following function

$$\vec{H} \mapsto \sum_{i=1}^k \iota(\vec{F}_i) \cdot \#\text{Hom}(\vec{F}_i \rightarrow \vec{G} \otimes \vec{H}).$$

Using the properties of the tensor product, this rewrites to

$$\vec{H} \mapsto \sum_{i=1}^k (\iota(\vec{F}_i) \cdot \#\text{Hom}(\vec{F}_i \rightarrow \vec{G})) \cdot \#\text{Hom}(\vec{F}_i \rightarrow \vec{H}).$$

Now we set $a_i := \iota(\vec{F}_i) \cdot \#\text{Hom}(\vec{F}_i \rightarrow \vec{G})$ and $\varphi_i := \#\text{Hom}(\vec{F}_i \rightarrow \star)$. Note that Lemma 37 makes sure that the φ_i are indeed pairwise distinct semigroup homomorphisms. Next, clearly, all functions in 1. to 4. in Theorem 36 are computable in our setting. Hence we can use Algorithm $\hat{\mathbb{A}}$ from Theorem 36 to obtain a_1, \dots, a_k . The number of steps required by $\hat{\mathbb{A}}$ is bounded by $O(B \cdot \sum_{i=1}^k \log a_i)$, where B does not depend on the a_i . Thus we can bound B by a function in $|\iota|$. Furthermore, we can generously bound $\log a_i \leq \log(\iota(\vec{F}_i) \cdot |\vec{G}'|^{|\iota|}) \leq f'(|\iota|) \cdot \log |\vec{G}'|$ for some computable function f' .

Next, when simulating an oracle query \vec{H} posed by $\hat{\mathbb{A}}$, we have to use our own oracle to query $\vec{G}' \otimes \vec{H}$. Fortunately, Theorem 36 guarantees that \vec{H} only depends on the φ_i , that is, on the \vec{F}_i , and thus only on ι . Thus, constructing $\vec{G}' \otimes \vec{H}$ can be done in time $|\vec{G}'|^{O(1)} \cdot f''(|\iota|)$ for some computable function f'' . Additionally, the outdegree of $\vec{G}' \otimes \vec{H}$ is, by definition of the tensor product, bounded by the outdegree of \vec{G}' , i.e., d , times the outdegree of \vec{H} , which only depends on ι . Finally, having obtained the a_i , we obtain the terms $\#\text{Hom}(\vec{F}_i \rightarrow \vec{G}')$ by dividing by $\iota(\vec{F}_i)$, which is well-defined, since all $\iota(\vec{F}_i)$ are non-zero.

Hence, there is a computable function f such that the following two desired properties are true:

- The total running time of our algorithm is bounded by $f(|\iota|) \cdot |\vec{G}'|^{O(1)}$, and
- the outdegree of every oracle query is bounded by $d \cdot f(|\iota|)$.

Noting that the same arguments apply in the restricted cases of digraphs without loops and DAGs, we can conclude the proof. \square

6 Counting Homomorphisms

Recall the problem $\#\text{DIRHOM}_d(\vec{C})$ from Section 4.4.1. Section 6.1 shows that, when the reachability hypergraphs of the graphs in \vec{C} have bounded fractional hypertreewidth, $\#\text{DIRHOM}_d(\vec{C})$ is fixed-parameter tractable. Section 6.2 instead gives parameterized reductions under what we call *monotone reversible minors*, a new and restricted version of digraph minors. In the next sections we will leverage these results for $\#\text{DIRSUB}_d(\vec{C})$ and $\#\text{DIRINDSUB}_d(\vec{C})$.

6.1 Upper Bounds

Let \vec{C} be a class of digraphs, and let $\mathcal{R}(\vec{C}) = \{\mathcal{R}(\vec{H}) : \vec{H} \in \vec{C}\}$. This section proves that, if $\mathcal{R}(\vec{C})$ has bounded fractional hypertreewidth, $\text{fhtw}(\mathcal{R}(\vec{C})) < \infty$, then $\#\text{DIRHOM}_d(\vec{C})$ is fixed-parameter tractable. To this end we give a parameterized reduction from $\#\text{DIRHOM}_d(\vec{C})$ to $\#\text{CSP}(\mathcal{R}(\vec{C}))$ that preserves reachability hypergraphs, and then invoke Theorem 28.

Definition 38. Let \vec{H} be a digraph, let S_1, \dots, S_ℓ be the sources of \vec{H}/\sim , and fix any $s_i \in S_i$ for each $i \in [\ell]$. Fix any ordering \prec of the vertices of \vec{H} , and, for each $i \in [\ell]$, let t_i be the tuple formed by sorting $R(s_i)$ according to \prec . Furthermore, let \vec{G} be a digraph. The relational structures $\mathcal{A}[\vec{H}]$ and $\mathcal{B}[\vec{H}, \vec{G}]$ are defined as follows:

- $\mathcal{A}[\vec{H}] = (V(\vec{H}), R_1^A, \dots, R_\ell^A)$, where $R_i^A = \{t_i\}$ for all $i \in [\ell]$.
- $\mathcal{B}[\vec{H}, \vec{G}] = (V(\vec{G}), R_1^B, \dots, R_\ell^B)$, where

$$R_i^B = \{\phi(t_i) : \phi \in \text{Hom}(\vec{H}[R(s_i)] \rightarrow \vec{G})\}$$

for all $i \in [\ell]$.

Lemma 39. $\text{Hom}(\vec{H} \rightarrow \vec{G}) = \text{Hom}(\mathcal{A}[\vec{H}] \rightarrow \mathcal{B}[\vec{H}, \vec{G}])$.

Proof. To avoid notational clutter, we set $e_i := R(s_i)$. Let $\varphi \in \text{Hom}(\vec{H} \rightarrow \vec{G})$ and fix any $i \in [\ell]$. Clearly, the restriction of φ to $\vec{H}[e_i]$ is in $\text{Hom}(\vec{H}[e_i] \rightarrow \vec{G})$. Therefore by Definition 38 we have $\varphi(t_i) \in R_i^B$. Hence φ is a homomorphism from $\mathcal{A}[\vec{H}]$ to $\mathcal{B}[\vec{H}, \vec{G}]$, that is, $\varphi \in \text{Hom}(\mathcal{A}[\vec{H}] \rightarrow \mathcal{B}[\vec{H}, \vec{G}])$.

Now let $\varphi \in \text{Hom}(\mathcal{A}[\vec{H}] \rightarrow \mathcal{B}[\vec{H}, \vec{G}])$ and fix any $i \in [\ell]$. By definition, $\varphi(t_i) \in R_i^B$. By Definition 38 this implies $\varphi(t_i) = \phi(t_i)$ for some $\phi \in \text{Hom}(\vec{H}[e_i] \rightarrow \vec{G})$, thus $(\varphi(u), \varphi(v)) \in E(\vec{G})$ for all $(u, v) \in E(\vec{H}[e_i])$. Since this holds for all $i \in [\ell]$ and since $E(\vec{H}) = \cup_{i \in [\ell]} E(\vec{H}[e_i])$, we have $(\varphi(u), \varphi(v)) \in E(\vec{G})$ for all $(u, v) \in E(\vec{H})$. Thus $\varphi \in \text{Hom}(\vec{H} \rightarrow \vec{G})$. \square

Further, we note the following immediate consequence of the definition of $\mathcal{A}[\vec{H}]$.

Observation 40. $\mathcal{H}[\mathcal{A}[\vec{H}]] = \mathcal{R}(\vec{H})$.

Next we show that $\mathcal{A}[\vec{H}]$ and $\mathcal{B}[\vec{H}, \vec{G}]$ can be constructed efficiently.

Lemma 41. *There exists a computable function f such that, given any two DAGs \vec{H} and \vec{G} , the relational structures $\mathcal{A}[\vec{H}]$ and $\mathcal{B}[\vec{H}, \vec{G}]$ can be computed in time $f(k, d) \cdot n^{O(1)}$ where $k = |V(\vec{H})|$, $n = |V(\vec{G})|$, and d is the maximum outdegree of \vec{G} . Moreover, $|\mathcal{B}[\vec{H}, \vec{G}]| \leq f(k, d) \cdot O(|\vec{G}|)$.*

Proof. It is straightforward that $\mathcal{A}[\vec{H}]$ can be constructed in time only depending on k . For what follows, we again set $e_i := R(s_i)$. Furthermore, set $a_i := |e_i|$. To construct $\mathcal{B}[\vec{H}, \vec{G}]$, and more precisely every relation \mathcal{R}_i^B , we use the following standard technique. For every $v \in V(\vec{G})$ let $N_{k-1}(v)$ be the set of vertices reachable from v by a directed path of length at most $k-1$. Note that for any $\phi \in \text{Hom}(\vec{H}[e_i] \rightarrow \vec{G})$ we have that $\vec{H}[e_i]$ is connected and contains at most k vertices. Furthermore, each vertex in $\vec{H}[e_i]$ is reachable by a directed path from s_i . Thus $\phi(t_i)$ contains only vertices of $N_{k-1}(v)$ where $v = \phi(s_i)$.

Therefore to list all $\phi(t_i)$ with $\phi \in \text{Hom}(\vec{H}[e_i] \rightarrow \vec{G})$ we take every $v \in V(\vec{G})$ in turn, we compute $N_{k-1}(v)$, and for every a_i -tuple $z_i \in (N_{k-1}(v))^{a_i-1}$ whose first element is v , we add z_i if and only if the map ϕ defined by $\phi(t_i) = z_i$ preserves all edges of $\vec{H}[e_i]$, which holds if and only if $\phi \in \text{Hom}(\vec{H}[e_i] \rightarrow \vec{G})$. Finally, recalling that d is the outdegree of \vec{G} , we have $|N_{k-1}(v)| \leq \sum_{j=0}^{k-1} d^j$, which only depends on k and d . It is thus immediate to see that we can compute all \mathcal{R}_i^B in time $f(k, d) \cdot n^{O(1)}$, and that the overall size of $\mathcal{B}[\vec{H}, \vec{G}]$ is bounded by $f(k, d) \cdot O(|\vec{G}|)$, for some computable function f . \square

Theorem 42. *For some computable function f there is an algorithm that, given any pair of digraphs (\vec{H}, \vec{G}) , computes $\#\text{Hom}(\vec{H} \rightarrow \vec{G})$ in time $f(|\vec{H}|, d) \cdot |\vec{G}|^{r+O(1)}$, where d is the maximum outdegree of \vec{G} and $r = \text{fhtw}(\mathcal{R}(\vec{H}))$. Therefore $\#\text{DIRHOM}_d(\vec{G}) \in \text{FPT}$ if $\text{fhtw}(\mathcal{R}(\vec{G})) < \infty$.*

Proof. Given an instance (\vec{H}, \vec{G}) , let $n = |V(\vec{G})|$ and $k = |V(\vec{H})|$. We compute $\mathcal{A}[\vec{H}]$ and $\mathcal{B}[\vec{H}, \vec{G}]$ as in Lemma 41. In particular, we obtain $|\mathcal{B}[\vec{H}, \vec{G}]| \leq g(k, d) \cdot O(|\vec{G}|)$ for some computable

function g . Finally, by Lemma 39, we have $\text{Hom}(\vec{H} \rightarrow \vec{G}) = \text{Hom}(\mathcal{A}[\vec{H}] \rightarrow \mathcal{B}[\vec{H}, \vec{G}])$, the latter of which can be computed, by Theorem 28, in time

$$f'(|\mathcal{A}[\vec{H}]|) \cdot |\mathcal{B}[\vec{H}, \vec{G}]|^{r+O(1)} \leq f'(|\mathcal{A}[\vec{H}]|) \cdot g(k, d)^{r+O(1)} \cdot |\vec{G}|^{r+O(1)}$$

for some computable function f' . Since $|\mathcal{A}[\vec{H}]|$ depends only on k and not on n , the proof is concluded. \square

6.2 Coloured Homomorphisms and Reductions via MR Minors

We start by introducing *monotone reversible (MR) minors* of a digraph \vec{H} .

Definition 43 (Monotone Reversible Minors). *Let \vec{H} be a digraph and consider the following operations:*

Sink deletion: delete all vertices in T for some sink T of \vec{H}/\sim . The resulting graph is denoted by $\vec{H} \setminus T$.

Contraction: identify u and v for some $uv \in E(\vec{H})$. We emphasise that a contraction does not yield a loop. The resulting graph is denoted by $\vec{H}/(u, v)$.

Loop deletion: delete a loop (u, u) . The resulting graph is denoted by $\vec{H} \setminus (u, u)$.

A monotone reversible minor (“MR minor”) of \vec{H} is a digraph that can be obtained from \vec{H} by a sequence of sink deletions, contractions and loop deletions.

Observation 44. *Let \vec{H} be a digraph.*

(M1) If \vec{H} does not have loops, then no MR minor of \vec{H} does.

(M2) \vec{H}/\sim is an MR minor of \vec{H} .

We show that the parameterized complexity of $\#\text{DIRHOM}_d$ is monotone (i.e., nonincreasing) under taking MR minors. The following three lemmas establish the fact separately for sink deletions, for contractions, and for loop deletion. For technical reasons, we will prove this property for the colour-prescribed variant; we will be able to remove the colours in our hardness reductions later.

Lemma 45. *There exists an algorithm \mathbb{A}_1 that satisfies the following constraints:*

- 1. \mathbb{A}_1 expects as input a digraph \vec{H} , a sink T of \vec{H}/\sim , and a surjectively $\vec{H} \setminus T$ -coloured digraph (\vec{G}', c') of outdegree d' .*
- 2. The running time of \mathbb{A}_1 is bounded by $\text{poly}(|\vec{H}|, |\vec{G}'|)$.*
- 3. \mathbb{A}_1 outputs a surjectively \vec{H} -coloured digraph (\vec{G}, c) of size at most $O(|H| \cdot |\vec{G}'|)$ such that the outdegree of \vec{G} is bounded by $d' + |\vec{H}|$, and*

$$\#\text{Hom}(\vec{H} \setminus T \rightarrow (\vec{G}', c')) = \#\text{Hom}(\vec{H} \rightarrow (\vec{G}, c)).$$

Proof. Let $T = \{t_1, \dots, t_k\}$. For every $i \in [k]$ let V_i be the set of vertices $v \in V(\vec{H}) \setminus T$ such that $(v, t_i) \in E(\vec{H})$. Note that, since T is a sink in \vec{H}/\sim , there are no arcs from T to $V(\vec{H}) \setminus T$ in \vec{H} .

We construct (\vec{G}, c) from (\vec{G}', c') as follows: First, we add to \vec{G}' the set T and all arcs in $T^2 \cap E(\vec{H})$. Second, for every $i \in [k]$, we add to \vec{G} all arcs from $c'^{-1}(V_i)$ to t_i — note that $c'^{-1}(V_i)$ is the set of vertices of \vec{G}' that are coloured by c' with a vertex in V_i . Finally, we let c agree with c' on all vertices of \vec{G}' , and we set $c(t_i) = t_i$ for all $i \in [k]$. Clearly, this construction can be done in time $\text{poly}(|\vec{H}|, |\vec{G}'|)$, and the resulting (\vec{G}, c) is of size at most $O(|H| \cdot |\vec{G}'|)$. Furthermore, the outdegree of \vec{G} is bounded by $d' + |\vec{H}|$. By construction, and the fact that c' is a vertex-surjective homomorphism from \vec{G}' to $\vec{H} \setminus T$, it is also immediate that c is a vertex-surjective homomorphism from \vec{G} to \vec{H} . Finally, consider the function b that maps a colour-prescribed homomorphism

$\varphi \in \text{Hom}(\vec{H} \setminus T \rightarrow (\vec{G}', c'))$ to the function ψ that agrees with φ on $V(\vec{H} \setminus T)$ and that maps t_i to t_i for each $i \in [k]$. By construction, it is easy to see that b must be a bijection from $\text{Hom}(\vec{H} \setminus T \rightarrow (\vec{G}', c'))$ to $\text{Hom}(\vec{H} \rightarrow (\vec{G}, c))$. This concludes the proof. \square

Lemma 46. *There exists an algorithm \mathbb{A}_2 that satisfies the following constraints:*

1. \mathbb{A}_2 expects as input a digraph \vec{H} , an arc (u, v) of \vec{H} , and a surjectively $\vec{H}/(u, v)$ -coloured digraph (\vec{G}', c') of outdegree d' .
2. The running time of \mathbb{A}_2 is bounded by $\text{poly}(|\vec{H}|, |\vec{G}'|)$.
3. \mathbb{A}_2 outputs a surjectively \vec{H} -coloured digraph (\vec{G}, c) of size at most $O(|H| \cdot |\vec{G}'|)$ such that the outdegree of \vec{G} is bounded by $2d' + 1$, and

$$\#\text{Hom}(\vec{H}/(u, v) \rightarrow (\vec{G}', c')) = \#\text{Hom}(\vec{H} \rightarrow (\vec{G}, c)).$$

Proof. We write uv for the vertex in $\vec{H}/(u, v)$ that corresponds to the contraction of u and v . Recall that \vec{G}' is surjectively $\vec{H}/(u, v)$ -coloured by c' , and set $V_{uv} := c'^{-1}(uv)$. Let us now provide the construction of (\vec{G}, c) :

- (a) We start with \vec{G}' and delete the set V_{uv} (including all incident arcs).
- (b) We add two copies of V_{uv} ; one is denoted by V_u , and the other one is denoted by V_v . For each vertex $w \in V_{uv}$, we denote the copy of w in V_u by w_u , and the copy of w in V_v by w_v .
- (c) For each $w \in V_{uv}$, we add an arc (w_u, w_v) .
- (d) For each $x \in V(\vec{H}) \setminus \{u, v\}$ we proceed as follows
 - If $(x, u) \in E(\vec{H})$, then, for every $y \in c'^{-1}(x)$ and $w \in V_{uv}$, we add an arc from y to $w_u \in V_u$ if and only if $(y, w) \in E(\vec{G}')$.
 - If $(u, x) \in E(\vec{H})$, then, for every $y \in c'^{-1}(x)$ and $w \in V_{uv}$, we add an arc from $w_u \in V_u$ to y if and only if $(w, y) \in E(\vec{G}')$.
 - If $(x, v) \in E(\vec{H})$, then, for every $y \in c'^{-1}(x)$ and $w \in V_{uv}$, we add an arc from y to $w_v \in V_v$ if and only if $(y, w) \in E(\vec{G}')$.
 - If $(v, x) \in E(\vec{H})$, then, for every $y \in c'^{-1}(x)$ and $w \in V_{uv}$, we add an arc from $w_v \in V_v$ to y if and only if $(w, y) \in E(\vec{G}')$.
- (e) Finally, c agrees with c' on $V(\vec{H}) \setminus \{u, v\}$, and c maps every vertex in V_u to u and it maps every vertex in V_v to v .

It is immediate that c is a surjective \vec{H} -colouring of \vec{G} . Furthermore, it is clear that the running time is bounded by $\text{poly}(|\vec{H}|, |\vec{G}'|)$, and that the size of (\vec{G}, c) is at most $O(|H| \cdot |\vec{G}'|)$. Let us consider the outdegree: For $x \in V(\vec{H}) \setminus \{u, v\} = V(\vec{H}/(u, v)) \setminus \{uv\}$, the outdegree of every vertex $v \in c^{-1}(x) = c'^{-1}(x)$ in \vec{G} is bounded by twice the outdegree of v in \vec{G}' (see (d) above). Furthermore, the outdegree of every vertex $w_u \in c^{-1}(u)$ in \vec{G} is bounded by the outdegree of w in \vec{G}' plus 1 (see (d) above and note that the arcs added in (c) above can increase it by 1). Finally, the outdegree of every vertex $w_v \in c^{-1}(v)$ in \vec{G} is bounded by the outdegree of w in \vec{G}' (see (d) above). Consequently, we can bound the outdegree of \vec{G} by $2d' + 1$.

Finally, note that (c) above makes sure that any homomorphism ψ in $\text{Hom}(\vec{H} \rightarrow (\vec{G}, c))$ must map u and v to the same copy of a vertex $w \in V_{uv}$. Furthermore, (d) makes sure that the mapping φ that agrees with ψ on $V(\vec{H}) \setminus \{u, v\}$ and that maps uv to w (where w_u and w_v are the images of u and v under ψ), is a homomorphism in $\text{Hom}(\vec{H}/(u, v) \rightarrow (\vec{G}', c'))$. On the other hand every homomorphism $\varphi \in \text{Hom}(\vec{H}/(u, v) \rightarrow (\vec{G}', c'))$ corresponds to the homomorphism $\psi \in \text{Hom}(\vec{H} \rightarrow (\vec{G}, c))$ that agrees with φ on $V(\vec{H}) \setminus \{u, v\}$, and that maps u and v to w_u and w_v , where $w = \varphi(uv)$. Concretely, we obtain the desired bijection between $\text{Hom}(\vec{H}/(u, v) \rightarrow (\vec{G}', c'))$ and $\text{Hom}(\vec{H} \rightarrow (\vec{G}, c))$, concluding the proof. \square

Lemma 47. *There exists an algorithm \mathbb{A}_3 that satisfies the following constraints:*

1. \mathbb{A}_3 expects as input a digraph \vec{H} , a loop (u, u) of \vec{H} , and a surjectively $\vec{H} \setminus (u, u)$ -coloured digraph (\vec{G}', c') of outdegree d' .
2. The running time of \mathbb{A}_3 is bounded by $\text{poly}(|\vec{H}|, |\vec{G}'|)$.
3. \mathbb{A}_3 outputs a surjectively \vec{H} -coloured digraph (\vec{G}, c) of size at most $O(|\vec{G}'|)$ such that the outdegree of \vec{G} is bounded by $d' + 1$, and

$$\#\text{Hom}(\vec{H} \setminus (u, u) \rightarrow (\vec{G}', c')) = \#\text{Hom}(\vec{H} \rightarrow (\vec{G}, c)).$$

Proof. This is a very easy case: Obtain \vec{G} from \vec{G}' by adding a loop to each vertex of \vec{G}' coloured by c' with u . Furthermore, we set $c := c'$. Clearly, c is a surjective \vec{H} -colouring of \vec{G} , and the outdegree of \vec{G} can increase by at most 1. Furthermore, the construction immediately yields that $\text{Hom}(\vec{H} \setminus (u, u) \rightarrow (\vec{G}', c')) = \text{Hom}(\vec{H} \rightarrow (\vec{G}, c))$, that is, a mapping $\varphi : V(\vec{H}) (= V(\vec{H}')) \rightarrow V(\vec{G}) (= V(\vec{G}'))$ is a homomorphism from $\vec{H} \setminus (u, u)$ to (\vec{G}', c') if and only if it is a homomorphism from \vec{H} to (\vec{G}, c) \square

In combination, the three lemmas above yield the following:

Lemma 48. *Let \vec{C} be a recursively enumerable class of digraphs and let \vec{C}' be a class of MR minors of graphs in \vec{C} . Then*

$$\#\text{CP-DirHom}_d(\vec{C}') \leq_{\text{T}}^{\text{fpt}} \#\text{CP-DirHom}_d(\vec{C}).$$

Proof. Let \vec{H}' and (\vec{G}', c') be an input instance of $\#\text{CP-DirHom}_d(\vec{C}')$. We start by searching a graph $\vec{H} \in \vec{C}$ such that \vec{H}' is an MR minor of \vec{H} . Note that this can be done in time only depending on \vec{H}' . Since \vec{H}' can be obtained from \vec{H} by a sequence of ℓ sink-deletions, contractions, and loop deletions, we can use algorithms \mathbb{A}_1 , \mathbb{A}_2 and \mathbb{A}_3 from the previous three lemmas for a total of ℓ times. Note that the crucial property of the \mathbb{A}_i is that the oracle queries always have size bounded by $f(|\vec{H}|) \cdot O(|\vec{G}'|)$. Hence, even after ℓ applications of the constructions, the total size will still be bounded by $f(|\vec{H}|) \cdot O(|\vec{G}'|)$. Since furthermore each individual application takes only polynomial time, we obtain, as desired a parameterised Turing-reduction. \square

The final part of this subsection is the following lemma for removing the colours. Note that its proof is a simple application of the inclusion-exclusion principle and transfers verbatim from e.g. [45, Lemma 2.49] (see also [22, Lemma 1.34]). We emphasise that the reduction only requires oracle queries for subgraphs of the input host-graph which cannot increase the outdegree; thus the reduction applies to our setting.

Lemma 49. *Let \vec{C} be a class of digraphs. We have*

$$\#\text{CP-DirHom}_d(\vec{C}) \leq_{\text{T}}^{\text{fpt}} \#\text{DirHom}_d(\vec{C}).$$

6.3 Lower Bounds

We start by extending the notion of adaptive width from hypergraphs to digraphs.

Definition 50 (Adaptive width of digraphs). *The adaptive width of a digraph \vec{H} , denoted by $\text{aw}(\vec{H})$, is defined as the adaptive width of its contour. That is $\text{aw}(\vec{H}) := \text{aw}(\Gamma(\vec{H}))$.*

We proceed by proving intractability of $\#\text{CP-DirHom}_d(\vec{C})$ for classes \vec{C} of canonical DAGs of unbounded adaptive width.

Lemma 51. *Let \vec{C} be a recursively enumerable class of canonical DAGs. If the adaptive width of \vec{C} is unbounded then $\#\text{CP-DirHom}_d(\vec{C})$ is not fixed-parameter tractable, unless ETH fails.*

The proof of Lemma 51 uses a careful reduction from a version of the parameterised constraint satisfaction problem and is encapsulated in the following subsection.

6.3.1 Reduction from a Constraint Satisfaction problem

The starting point of our reduction is the following *decision* problem and the corresponding hardness result proven by Chen et al. [14] (building upon Marx [40]); below, \mathcal{S} denotes a class of relational structures.

$\text{HOM}(\mathcal{S})$
 Input: a pair of relational structures $(\mathcal{A}, \mathcal{B})$ with $\mathcal{A} \in \mathcal{S}$
 Output: TRUE iff there exists a homomorphism from \mathcal{A} to \mathcal{B}
 Parameter: $|\mathcal{A}|$

Let us emphasise the subtle difference between the input restrictions of $\text{HOM}(\mathcal{S})$ and $\#\text{CSP}(\mathcal{C})$: In the former, we enforce that \mathcal{A} is contained in \mathcal{S} . In the latter, we only restrict the hypergraphs of the relational structure \mathcal{A} by requiring that the hypergraph $\mathcal{H}(\mathcal{A})$ is contained in \mathcal{C} .

For what follows, we call a relational structure \mathcal{A} *minimal under homomorphic equivalence* if there is no homomorphism from \mathcal{A} to a proper substructure of \mathcal{A} .

Theorem 52 ([14]). *Let \mathcal{S} be a recursively enumerable class of relational structures that are minimal under homomorphic equivalence, and assume that ETH holds. If \mathcal{S} has unbounded adaptive width then $\text{HOM}(\mathcal{S})$ is not fixed-parameter tractable.*

To obtain hardness of $\#\text{DIRHOM}_d(\vec{\mathcal{C}})$ from Theorem 52, we show a chain of parameterized Turing reductions using two intermediate problems; one of them is $\#\text{CP-DIRHOM}_d(\vec{\mathcal{C}})$, and for the second one we extend the notion of colour-prescribed homomorphisms to hypergraphs: Let \mathcal{H} be a hypergraph. An \mathcal{H} -colouring of a hypergraph \mathcal{G} is a homomorphism $c \in \text{Hom}(\mathcal{G} \rightarrow \mathcal{H})$. An \mathcal{H} -coloured hypergraph is a pair (\mathcal{G}, c) where \mathcal{G} is a hypergraph and c is an \mathcal{H} -colouring of \mathcal{G} . Given a \mathcal{H} -coloured hypergraph (\mathcal{G}, c) , a map $\psi : V(\mathcal{H}) \rightarrow V(\mathcal{G})$ is c -colour-prescribed if $c(\psi(v)) = v$ for every $v \in V(\mathcal{H})$. The set of all c -colour-prescribed homomorphisms from \mathcal{H} to \mathcal{G} is denoted by $\text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))$.

We can now introduce the problem $\#\text{CP-HOM}_a(\mathcal{C})$; here \mathcal{C} is a class of hypergraphs:

$\#\text{CP-HOM}_a(\mathcal{C})$
 Input: a hypergraph $\mathcal{H} \in \mathcal{C}$ and an \mathcal{H} -coloured hypergraph (\mathcal{G}, c)
 Output: $\#\text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))$
 Parameter: $|\mathcal{H}| + a(\mathcal{G})$ (recall that $a(\mathcal{G})$ is arity of \mathcal{G})

In the rest of this subsection we prove:

$$\text{HOM}(\mathcal{S}) \leq_{\text{T}}^{\text{fpt}} \#\text{CP-HOM}_a(\mathcal{C}) \leq_{\text{T}}^{\text{fpt}} \#\text{CP-DIRHOM}_d(\vec{\mathcal{C}}) \quad (10)$$

where \mathcal{S} and \mathcal{C} are carefully constructed from $\vec{\mathcal{C}}$ so to preserve the (un)boundedness of adaptive width. The next two paragraphs prove the reductions of (10) in order.

$\text{HOM}(\mathcal{S}) \leq_{\text{T}}^{\text{fpt}} \#\text{cp-HOM}_a(\mathcal{C})$ We reduce $\text{HOM}(\mathcal{S})$ to $\#\text{CP-HOM}_a(\mathcal{C})$ for a certain class $\mathcal{S} = \mathcal{S}(\mathcal{C})$ described below. To this end we convert every hypergraph into a structure; this structure is the same of Definition 38 – only for hypergraphs. Without loss of generality, in what follows we assume $V(\mathcal{H}) = \{1, \dots, k\}$ where $k = |V(\mathcal{H})|$.

Definition 53. *Let \mathcal{H} be any hypergraph. The structure $\mathcal{A}[\mathcal{H}]$ has universe $V(\mathcal{H})$ and, for every $e \in E(\mathcal{H})$, contains a relation $R_e^{\mathcal{A}}$ whose only tuple is the set e sorted in nondecreasing order.*

Since we use individual relation symbols for each $e \in E(\mathcal{H})$, the following is straightforward to prove:

Claim 54. *Let \mathcal{H} be any hypergraph. Then $\mathcal{A}[\mathcal{H}]$ is minimal under homomorphic equivalence, and the hypergraph of $\mathcal{A}[\mathcal{H}]$ is \mathcal{H} .*

For any class \mathcal{C} of hypergraphs let $\mathcal{S}[\mathcal{C}] = \{A[\mathcal{H}] \mid \mathcal{H} \in \mathcal{C}\}$. We prove:

Lemma 55. *For every recursively enumerable class \mathcal{C} of hypergraphs without isolated vertices, $\text{HOM}(\mathcal{S}[\mathcal{C}]) \leq_{\text{T}}^{\text{fpt}} \#\text{CP-HOM}_a(\mathcal{C})$.*

Proof. Let $(\mathcal{A}, \mathcal{B})$ be an instance of $\text{HOM}(\mathcal{S}[\mathcal{C}])$, and let $U(\mathcal{A})$ and $U(\mathcal{B})$, respectively, be the universes of \mathcal{A} and \mathcal{B} . If \mathcal{A} and \mathcal{B} have different signatures then clearly the solution is NO. Otherwise, since $\mathcal{A} \in \mathcal{S}[\mathcal{C}]$ and \mathcal{C} is recursively enumerable, in time $f(|\mathcal{A}|)$ for some computable f we find $\mathcal{H} \in \mathcal{C}$ such that $\mathcal{A} = \mathcal{A}[\mathcal{H}]$. Then we construct the hypergraph \mathcal{G} defined by:

- $V(\mathcal{G}) = U(\mathcal{B}) \times V(\mathcal{H})$
- $E(\mathcal{G}) = \{(x_1, i_1), \dots, (x_\ell, i_\ell)\} : e = \{i_1, \dots, i_\ell\} \in E(\mathcal{H}), i_1 < \dots < i_\ell, (x_1, \dots, x_\ell) \in R_e^{\mathcal{B}}\}$

Let $c : V(\mathcal{G}) \rightarrow V(\mathcal{H})$ be defined by $c((x, i)) = i$ for all $(x, i) \in V(\mathcal{G})$. Note that c is an \mathcal{H} -colouring of \mathcal{G} . Consider the instance $(\mathcal{H}, (G, c))$ of $\#\text{CP-HOM}_a(\mathcal{C})$. We claim:

$$\text{Hom}(\mathcal{A} \rightarrow \mathcal{B}) \neq \emptyset \Leftrightarrow \text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c)) \neq \emptyset \quad (11)$$

To prove that $\text{Hom}(\mathcal{A} \rightarrow \mathcal{B}) \neq \emptyset \Rightarrow \text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c)) \neq \emptyset$, suppose $\varphi \in \text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$. Define $\psi : V(\mathcal{H}) \rightarrow V(\mathcal{G})$ by $\psi(i) = (\varphi(i), i)$ for all $i \in V(\mathcal{H})$. We claim that $\psi \in \text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))$. Let $e = \{i_1, \dots, i_\ell\} \in E(\mathcal{H})$ with $i_1 < \dots < i_\ell$. By definition of $\mathcal{A}[\mathcal{H}]$ we have $(i_1, \dots, i_\ell) \in R_e^{\mathcal{A}}$; and since φ is a homomorphism, $(\varphi(i_1), \dots, \varphi(i_\ell)) \in R_e^{\mathcal{B}}$. By definition of \mathcal{G} this implies that $\{(\varphi(i_1), i_1), \dots, (\varphi(i_\ell), i_\ell)\} \in E(\mathcal{G})$. Hence, $\psi \in \text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$. To see that ψ is c -colour-prescribed, note that $c(\psi(i)) = c((\varphi(i), i)) = i$ for all $i \in V(\mathcal{H})$.

To prove that $\text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c)) \neq \emptyset \Rightarrow \text{Hom}(\mathcal{A} \rightarrow \mathcal{B}) \neq \emptyset$, suppose $\psi \in \text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))$. Since ψ is c -colour-prescribed, for each $i \in V(\mathcal{H})$ we have $\psi(i) = (x_i, i)$ for some $x_i \in U(\mathcal{B})$. Define $\varphi : U(\mathcal{A}) \rightarrow U(\mathcal{B})$ by letting $\varphi(i) = x_i$ for all $i \in U(\mathcal{A})$. We claim that $\varphi \in \text{Hom}(\mathcal{A} \rightarrow \mathcal{B})$. Let indeed $(i_1, \dots, i_\ell) \in R_e^{\mathcal{A}}$. By definition of \mathcal{A} this implies $e = \{i_1, \dots, i_\ell\} \in E(\mathcal{H})$ and $i_1 < \dots < i_\ell$. Since ψ is c -color-prescribed, then $\psi(i_j) = (x_j, i_j)$ for all $j = 1, \dots, \ell$; and since $\psi \in \text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$, then $\{(x_1, i_1), \dots, (x_\ell, i_\ell)\} \in E(\mathcal{G})$. By definition of \mathcal{G} this implies $(x_1, \dots, x_\ell) \in R_e^{\mathcal{B}}$, and since $\varphi(i_1, \dots, i_\ell) = (x_1, \dots, x_\ell)$, then $\varphi(i_1, \dots, i_\ell) \in R_e^{\mathcal{B}}$.

Finally, note that $|\mathcal{H}| = f(|\mathcal{A}|)$ and $a(\mathcal{G}) \leq a(\mathcal{H}) \leq |\mathcal{H}|$. Therefore $|\mathcal{H}| + a(\mathcal{G})$ is a function of $|\mathcal{A}|$, hence the reduction preserves the parameter. \square

$\#\text{cp-HOM}_a(\mathcal{C}) \leq_{\text{T}}^{\text{fpt}} \#\text{cp-DirHom}_d(\vec{\mathcal{C}})$ Recall that the contour of a digraph \vec{H} , denoted by $\Gamma(\vec{H})$, is obtained by deleting from the reachability hypergraph $\mathcal{R}(\vec{H})$ the vertices corresponding to sources in \vec{H}/\sim (see Definition 18). For our reduction we need to prove that the adaptive width of a contour is at least that of the original hypergraph.

Lemma 56. *Let \vec{H} be a canonical DAG and let \mathcal{R} be its reachability hypergraph. Then $\text{aw}(\vec{H}) \geq \text{aw}(\mathcal{R})$.*

Proof. Recall that, by definition, $\text{aw}(\vec{H}) = \text{aw}(\Gamma(\vec{H}))$. Set $\mathcal{F} := \Gamma(\vec{H})$ and let s_1, \dots, s_ℓ be the sources of \vec{H} and e_1, \dots, e_ℓ the corresponding hyperedges in \mathcal{R} . Let $\mathcal{F}_0 = \mathcal{R}$, and for $i = 1, \dots, \ell$ let \mathcal{F}_i be the hypergraph obtained from \mathcal{F}_{i-1} by first deleting s_i and then removing a copy of e_i if more than one exist. Note that $\mathcal{F}_\ell = \mathcal{F}$. We prove that $\text{aw}(\mathcal{F}_i) \geq \text{aw}(\mathcal{F}_{i-1})$ for all $i = 1, \dots, \ell$, which implies $\text{aw}(\mathcal{F}_\ell) \geq \text{aw}(\mathcal{F}_0)$, that is, $\text{aw}(\mathcal{F}) \geq \text{aw}(\mathcal{R})$. Since removing a copy of a multi-edge leaves adaptive width unchanged, we can assume \mathcal{F}_i is obtained from \mathcal{F}_{i-1} by just deleting s_i .

We rephrase and prove the claim $\text{aw}(\mathcal{F}_i) \geq \text{aw}(\mathcal{F}_{i-1})$ as follows. Let \mathcal{H} be a hypergraph, let $e \in E(\mathcal{H})$, and let \mathcal{H}' be the hypergraph obtained from \mathcal{H} by adding a new vertex v' and replacing e with $e \cup \{v'\}$. We claim $\text{aw}(\mathcal{H}) \geq \text{aw}(\mathcal{H}')$. To this end, we show that for every fractional independent set μ' of \mathcal{H}' there exists a tree decomposition $(\mathcal{T}', \mathcal{B}')$ of \mathcal{H}' such that μ' -width $(\mathcal{T}', \mathcal{B}') \leq \text{aw}(\mathcal{H})$. Let then μ' be a fractional independent set of \mathcal{H}' , and let μ be the restriction of μ' to $V(\mathcal{H}) = V(\mathcal{H}') \setminus \{v'\}$. Note that μ is a fractional independent set of

\mathcal{H} ; thus, by definition of adaptive width, there is a tree decomposition $(\mathcal{T}, \mathcal{B})$ of \mathcal{H} such that $\mu\text{-width}(\mathcal{T}, \mathcal{B}) \leq \text{aw}(\mathcal{H})$. By definition of tree decomposition, there exists $B_e \in \mathcal{B}$ such that $e \subseteq B_e$. Let $(\mathcal{T}', \mathcal{B}')$ be obtained from $(\mathcal{T}, \mathcal{B})$ by appending the bag $B_{e'} = e'$ to B_e .

We claim that $(\mathcal{T}', \mathcal{B}')$ is a tree decomposition of \mathcal{H}' and that $\mu'\text{-width}(\mathcal{T}', \mathcal{B}') \leq \text{aw}(\mathcal{H})$. To see that $(\mathcal{T}', \mathcal{B}')$ is a tree decomposition of \mathcal{H}' note that:

- every edge of \mathcal{H}' is a subset of some bag of \mathcal{B}' . Indeed, $e' \subseteq B_{e'}$, while every other edge is in $E(\mathcal{H})$, and is thus a subset of some bag of \mathcal{B} since $(\mathcal{T}, \mathcal{B})$ is a tree decomposition of \mathcal{H}
- for every $u \in V(\mathcal{H}')$ the subgraph \mathcal{T}'_u of \mathcal{T}' induced by the bags containing u is connected. Indeed, if $u = v'$ then \mathcal{T}'_u has only one vertex. Else, \mathcal{T}'_u equals \mathcal{T}_u if $u \notin e$ and \mathcal{T}_u with an appended vertex otherwise.

To prove $\mu'\text{-width}(\mathcal{T}', \mathcal{B}') \leq \text{aw}(\mathcal{H})$ consider any $B' \in \mathcal{B}'$. If $B' = e'$ then $\mu'(B') \leq 1$ since μ' is a fractional independent set of \mathcal{H}' ; and since $\text{aw} \geq 1$, then $\mu'(B') \leq \text{aw}(\mathcal{H}')$. If $B' \neq e'$ then $B' \in \mathcal{B}$ and $v' \notin B'$, so $\mu'(B') = \mu(B')$. But $\mu(B') \leq \mu\text{-width}(\mathcal{T}, \mathcal{B}) \leq \text{aw}(\mathcal{H})$, thus $\mu'(B') \leq \text{aw}(\mathcal{H})$. \square

We are ready for our parameterised Turing-reduction.

Lemma 57. *Let \vec{C} be a recursively enumerable class of canonical DAGs and define $\hat{\mathcal{C}} = \{\Gamma(\vec{H}) \mid \vec{H} \in \vec{C}\}$. Then $\#\text{CP-HOM}_a(\hat{\mathcal{C}}) \leq_{\text{fpt}} \#\text{CP-DIRHOM}_d(\vec{C})$.*

Proof. Let $(\mathcal{H}, (\mathcal{G}, c))$ be the input to $\#\text{CP-HOM}_a(\hat{\mathcal{C}})$. As $\mathcal{H} \in \{\Gamma(\vec{H}) \mid \vec{H} \in \vec{C}\}$ and \vec{C} is recursively enumerable, for some computable f in time $f(|\mathcal{H}|)$ we find $\vec{H} \in \vec{C}$ such that $\mathcal{H} = \Gamma(\vec{H})$. Let s_1, \dots, s_ℓ be the sources of \vec{H} , and for every $s \in \{s_1, \dots, s_\ell\}$ let e_s be the set of non-source vertices reachable from s . Note that $e_i \in E(\mathcal{H})$.

We construct an \vec{H} -coloured DAG (\vec{G}, c') such that $d(\vec{G}) \leq a(\mathcal{G})$ and

$$|\text{Hom}(\Gamma \rightarrow (\mathcal{G}, c))| = |\text{Hom}(\vec{H} \rightarrow (\vec{G}, c'))|.$$

First, since $c \in \text{Hom}(\mathcal{G} \rightarrow \mathcal{H})$, then $c(e) \in E(\mathcal{H})$ for every $e \in E(\mathcal{G})$. Let $S_{c(e)}$ contain every $s \in \{s_1, \dots, s_\ell\}$ such that $\{s\} \cup c(e)$ is the reachable set of s in \vec{H} . This implies that $(s, v) \in E(\vec{H})$ for all $s \in S_{c(e)}$ and $v \in c(e)$. Then define:

$$V(\vec{G}) = V(\mathcal{G}) \cup \{x_{e,s} : e \in E(\mathcal{G}), s \in S_{c(e)}\} \quad (12)$$

$$E(\vec{G}) = \{(x_{e,s}, v) : e \in E(\mathcal{G}), s \in S_{c(e)}, v \in c(e)\} \quad (13)$$

and:

$$c'(v) = c(v) : v \in V(\mathcal{G}) \quad (14)$$

$$c'(x_{e,s}) = s : e \in E(\mathcal{G}), s \in S_{c(e)} \quad (15)$$

Observe that $d(\vec{G}) \leq a(\mathcal{G})$ and that \vec{G} and c' can be constructed in FPT time.

Let us show that $c' \in \text{Hom}(\vec{G} \rightarrow \vec{H})$. Let $(u, v) \in E(\vec{G})$. By construction $u = x_{e,s}$ for $e \in E(\mathcal{G})$, $s \in S_{c(e)}$, $v \in c(e)$. By definition $c'(x_{e,s}) = s$ and $c'(v) = c(v)$. Thus $(c'(x_{e,s}), c'(v)) = (s, c(v))$, and as observed above $(s, c(v)) \in E(\vec{H})$. Thus $(c(u), c(v)) \in E(\vec{H})$ as desired.

Now we give a bijection between $\text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))$ and $\text{Hom}(\vec{H} \rightarrow (\vec{G}, c'))$, proving that $|\text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))| = |\text{Hom}(\vec{H} \rightarrow (\vec{G}, c'))|$. First, let $\varphi \in \text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))$, and define the following extension $\psi : V(\vec{H}) \rightarrow V(\vec{G})$ of φ :

$$\psi(v) = \varphi(v) : v \in V(\mathcal{H}) \quad (16)$$

$$\psi(s) = x_{\varphi(e_s), s} : s \in \{s_1, \dots, s_\ell\} \quad (17)$$

We claim that $\psi \in \text{Hom}(\vec{H} \rightarrow (\vec{G}, c'))$. First, let us show that $\psi \in \text{Hom}(\vec{H} \rightarrow \vec{G})$. Let $(s, v) \in E(\vec{H})$. Then $(\psi(s), \psi(v)) = (x_{\varphi(e_s), s}, \varphi(v))$. Since $e_s \in E(\mathcal{H})$ and $\varphi \in \text{Hom}(\mathcal{H} \rightarrow \mathcal{G})$, then

$\varphi(e_s) = e$ for some $e \in E(\mathcal{G})$; let us then write $(\psi(s), \psi(v)) = (x_{e,s}, \varphi(v))$. Now, since $(s, v) \in E(\vec{H})$, then $v \in e_s$, which implies $\varphi(v) \in \varphi(e_s) = e$. By construction of \vec{G} this implies $(x_{e,s}, \varphi(v)) \in E(\vec{G})$. Therefore $\psi \in \text{Hom}(\vec{H} \rightarrow \vec{G})$. To show that $\psi \in \text{Hom}(\vec{H} \rightarrow (\vec{G}, c'))$, note that for all $v \in V(\mathcal{H})$ we have $c'(\psi(v)) = c(\varphi(v)) = v$ by definition of c' and c ; while for all $s \in \{s_1, \dots, s_\ell\}$ we have $c'(\psi(s)) = c'(x_{\varphi(e_s), s}) = s$. Hence $\psi \in \text{Hom}(\vec{H} \rightarrow (\vec{G}, c'))$.

Next, let $\psi \in \text{Hom}(\vec{H} \rightarrow (\vec{G}, c'))$, and define φ as the restriction of ψ to $V(\mathcal{H})$. Note that this is the inverse of the extension defined above, and as a consequence it is also c -prescribed. Therefore to establish our bijection we need only to prove that $\varphi \in \text{Hom}(\mathcal{H} \rightarrow (\mathcal{G}, c))$. Consider any edge $\epsilon \in E(\mathcal{H})$. By construction, $\epsilon = e_s$ for some $s \in \{s_1, \dots, s_\ell\}$. Since $\psi \in \text{Hom}(\vec{H} \rightarrow \vec{G})$, then $(\psi(s), \psi(v)) \in E(\vec{G})$ for all $v \in e_s$. By construction of \vec{G} and since ψ is c' -prescribed, this implies that $\psi(s) = x_{e,s}$ and $\psi(e_s) \subseteq e$ for some $e \in E(\mathcal{G})$. The injectivity of c and thus c' on e , however, implies that $|e| \leq |\psi(e_s)|$. Therefore, $\psi(e_s) = e$ and thus $\varphi(e_s) = \psi(e_s) \in E(\mathcal{G})$. \square

We are now able to prove Lemma 51; recall that we are required to show that $\#\text{CP-DirHom}_d(\vec{C})$ is (fixed-parameter) intractable whenever \vec{C} is a class of canonical DAGs of unbounded adaptive width.

Proof of Lemma 51. Let $\hat{\mathcal{C}} := \{\Gamma(\vec{H}) \mid \vec{H} \in \vec{C}\}$ and note that $\hat{\mathcal{C}}$ does not have isolated vertices (a vertex contained in a hyperedge of cardinality 1 is not isolated). Recall further that $\mathcal{S}[\hat{\mathcal{C}}] = \{\mathcal{A}[\mathcal{H}] \mid \mathcal{H} \in \hat{\mathcal{C}}\}$. By Lemma 55 and Lemma 57, we have

$$\text{Hom}(\mathcal{S}[\hat{\mathcal{C}}]) \leq_{\text{T}}^{\text{fpt}} \#\text{CP-HOM}_a(\hat{\mathcal{C}}) \leq_{\text{T}}^{\text{fpt}} \#\text{CP-DirHom}_d(\vec{C}) \quad (18)$$

\square

Since \vec{C} has unbounded adaptive width, we obtain by Lemma 56 that $\hat{\mathcal{C}}$ and thus $\mathcal{S}[\hat{\mathcal{C}}]$ has unbounded adaptive width as well. By Claim 54, each structure in $\mathcal{S}[\hat{\mathcal{C}}]$ is furthermore minimal under homomorphic equivalence. By Theorem 52, the problem $\text{Hom}(\mathcal{S}[\hat{\mathcal{C}}])$ is thus not fixed-parameter tractable, unless ETH fails. The proof can thus be concluded by applying the chain of reductions (18).

6.4 The Intractability Result

In combination with our MR minor operations and the removal of colours, we obtain the following result, which will be the basis of every intractability result in the forthcoming sections.

Lemma 58. *Let \vec{C} be a recursively enumerable class of digraphs and let \vec{C}' be a class of canonical DAGs that are MR minors of digraphs in \vec{C} . If \vec{C}' has unbounded adaptive width, then $\#\text{DirHom}_d(\vec{C})$ is not fixed-parameter tractable, unless ETH fails.*

Proof. By Lemma 51, the problem $\#\text{CP-DirHom}_d(\vec{C}')$ is fixed-parameter intractable, unless ETH fails. By Lemma 48 and Lemma 49, we have that

$$\#\text{CP-DirHom}_d(\vec{C}') \leq_{\text{T}}^{\text{fpt}} \#\text{CP-DirHom}_d(\vec{C}) \leq_{\text{T}}^{\text{fpt}} \#\text{DirHom}_d(\vec{C}),$$

which concludes the proof. \square

7 Counting Subgraphs

The following invariant will turn out to precisely capture the complexity for counting directed subgraphs in bounded outdegree graphs.

Definition 59 (Fractional Cover Number). A fractional cover of a DAG \vec{H} with sources S is a function $\psi : S \rightarrow [0, \infty]$ such that for every vertex $v \in V(\vec{H}) \setminus S$ we have

$$\sum_{\substack{s \in S \\ v \in R(s)}} \psi(s) \geq 1.$$

The weight of a fractional cover is $\sum_{s \in S} \psi(s)$, and we also require that the weight is at least 1.⁶ The fractional cover number of \vec{H} , denoted by $\rho^*(\vec{H})$, is the minimum weight of a fractional cover of \vec{H} . The fractional cover number of a digraph \vec{H} (not necessarily acyclic) is $\rho^*(\vec{H}) = \rho^*(\vec{H}/\sim)$.

In the previous definition we overloaded the symbol ρ^* , which also denotes the fractional edge cover number of hypergraphs. The motivation for reusing the symbol stems from the following observation:

Observation 60. Let \vec{H} be a digraph. We have $\rho^*(\vec{H}) = \rho^*(\Gamma(\vec{H}/\sim))$, that is, the fractional cover number of \vec{H} is equal to the fractional edge cover number of the contour of the DAG \vec{H}/\sim .

7.1 Upper Bounds

To avoid notational clutter, given a fractional cover ψ of \vec{H}/\sim , we will call ψ also a fractional cover of \vec{H} .

Lemma 61. Let \vec{H} be a digraph and let \vec{H}' be a quotient of \vec{H} . Then $\rho^*(\vec{H}') \leq \rho^*(\vec{H})$.

Proof. It suffices to show the following: Let u and v be (not necessarily adjacent) vertices of \vec{H} and let \vec{H}' be the graph obtained by identifying u and v , that is, $\vec{H}' = \vec{H}/\sigma$, where σ is the partition of $V(\vec{H})$ containing a block $\{u, v\}$ and singleton blocks $\{w\}$ for each $w \notin \{u, v\}$. Then $\rho^*(\vec{H}') \leq \rho^*(\vec{H})$.

To prove the previous claim, let S_1, \dots, S_k be the sources of \vec{H}/\sim and let W_1, \dots, W_ℓ be the non-sources of \vec{H}/\sim , that is $V(\vec{H}/\sim) = \{S_1, \dots, S_k, W_1, \dots, W_\ell\}$. Thus $V(\vec{H}) = \mathcal{S} \dot{\cup} \mathcal{W}$ where $\mathcal{S} := S_1 \cup \dots \cup S_k$ and $\mathcal{W} := W_1 \cup \dots \cup W_\ell$. In particular, the S_i and the W_i are the strongly connected components of \vec{H} .

Now let ψ be a fractional cover of \vec{H}/\sim of weight $\rho^*(\vec{H})$.

We will proceed with a case-distinction on whether u and v (or both) are contained in the set of sources $\mathcal{S} := S_1 \cup \dots \cup S_k$. In each case, we will define a fractional cover ψ' of \vec{H}'/\sim of total weight at most $\rho^*(\vec{H})$. Note that we will always assume that u and v are in distinct strongly connected components, since otherwise, $\vec{H}/\sim = \vec{H}'/\sim$ and thus the fractional cover number does not change.

To avoid confusion, we denote $\mathbf{S} = \{S_1, \dots, S_k\}$ as the set of sources of \vec{H}/\sim and we denote \mathbf{S}' as the set of sources of \vec{H}'/\sim . Similarly, we denote $\mathbf{W} = \{W_1, \dots, W_\ell\}$ as the set of non-sources of \vec{H}/\sim and we denote \mathbf{W}' as the set of non-sources of \vec{H}'/\sim . Furthermore, given $S \in \mathbf{S}$ we write $R(S)$ for the set of vertices reachable from S in \vec{H}/\sim , and, given $S' \in \mathbf{S}'$ we write $R'(S')$ for the set of vertices reachable from S' in \vec{H}'/\sim .

($u \in \mathcal{S}, v \in \mathcal{S}$) Assume w.l.o.g. that $u \in S_1$ and $v \in S_2$. By identifying u and v we merge the strongly connected components S_1 and S_2 ; let us denote the resulting component by \hat{S} . Thus, \vec{H}'/\sim is obtained from \vec{H}/\sim by identifying S_1 and S_2 , and calling the resulting vertex \hat{S} . Hence $\mathbf{S}' = \{\hat{S}, S_3, \dots, S_k\}$ and $\mathbf{W} = \mathbf{W}'$. Now, for $S' \in \mathbf{S}'$, we define

$$\psi'(S') := \begin{cases} \psi(S') & S' \in \{S_3, \dots, S_k\} \\ \psi(S_1) + \psi(S_2) & S' = \hat{S} \end{cases}$$

⁶The technical reason for this is the corner case of $V(\vec{H}) \setminus S$ being empty.

Clearly, we have

$$\sum_{S' \in \mathbf{S}'} \psi'(S') = \psi'(\hat{S}) + \sum_{i=3}^k \psi'(S_i) = \psi(S_1) + \psi(S_2) + \sum_{i=3}^k \psi'(S_i) = \sum_{S \in \mathbf{S}} \psi(S) = \rho^*(\vec{H}).$$

It remains to show that ψ' is a fractional cover of \vec{H}'/\sim . To this end, let W be a non-source of \vec{H}'/\sim . Note that $W \in R(S_i)$ if and only if $W \in R'(S_i)$ for all $i \in \{3, \dots, k\}$. If W is not reachable from \hat{S} , then W is not reachable from either of S_1 or S_2 in \vec{H}/\sim . Thus

$$\sum_{\substack{S' \in \mathbf{S}' \\ W \in R'(S')}} \psi'(S') = \sum_{\substack{S \in \mathbf{S} \\ W \in R(S)}} \psi'(S) = \sum_{\substack{S \in \mathbf{S} \\ W \in R(S)}} \psi(S) \geq 1.$$

Otherwise, we have

$$\sum_{\substack{S' \in \mathbf{S}' \\ W \in R'(S')}} \psi'(S') = \psi(S_1) + \psi(S_2) + \sum_{\substack{S \in \mathbf{S} \\ W \in R(S) \setminus \{S_1, S_2\}}} \psi'(S) \geq \sum_{\substack{S \in \mathbf{S} \\ W \in R(S)}} \psi(S) \geq 1,$$

($u \in \mathcal{S}, v \notin \mathcal{S}$) Assume w.l.o.g. that $u \in S_1$ and let V be the strongly connected component containing v . We have to consider the following two subcases:

Case 1: The only source in \vec{H}'/\sim from which V can be reached is S_1 . This means that \vec{H}'/\sim is obtained from \vec{H}/\sim by contracting to S_1 each W that is reachable from S_1 and from which V can be reached; this includes of course S_1 and V . The resulting vertex \hat{S} is a source of \vec{H}'/\sim and, clearly, any fractional cover ψ of \vec{H}/\sim becomes a fractional cover ψ' of \vec{H}'/\sim by setting $\psi'(\hat{S}) = \psi(S_1)$ and $\psi'(S_i) = \psi(S_i)$ for $i \geq 2$. Thus $\rho^*(\vec{H}') \leq \rho^*(\vec{H})$.

Case 2: There are sources S_2, \dots, S_t in \vec{H}'/\sim different from S_1 from which V can be reached; specifically, assume that S_2, \dots, S_t are all sources with this property. The identification of u and v in \vec{H} then corresponds to the following operation in \vec{H}'/\sim :

- (a) Similarly as in Case 1, we obtain a new vertex, called \hat{W} , by contracting all vertices in \vec{H}'/\sim that are reachable from S_1 and from which V can be reached.
- (b) In contrast to Case 1, \hat{W} is not a source, since there will be arcs from S_2, \dots, S_t to \hat{W} . However, \hat{W} is not reachable from any source S_i with $t < i \leq k$.

Observe that the following holds for all vertices $W \in \mathbf{W}$ of \vec{H}/\sim that were not contracted to \hat{W} in (a): If W is reachable from S_1 in \vec{H}/\sim , then W is reachable from all sources S_2, \dots, S_t in \vec{H}'/\sim .

Noting that $\mathbf{S}' = \mathbf{S} \setminus \{S_1\} = \{S_2, \dots, S_t, \dots, S_k\}$, we define ψ' as follows:

$$\psi'(S') := \begin{cases} \psi(S') & S' \in \{S_{t+1}, \dots, S_k\} \\ \psi(S') + \psi(S_1)/(t-1) & S' \in \{S_2, \dots, S_t\} \end{cases}$$

First we observe that, clearly,

$$\sum_{S' \in \mathbf{S}'} \psi'(S') = \sum_{S \in \mathbf{S}} \psi(S) = \rho^*(\vec{H}).$$

Hence it remains to show that ψ' is a fractional cover. To this end, let $W \in \mathbf{W}'$. Assume first that $W = \hat{W}$ and note that \hat{W} is reachable from S_2, \dots, S_t in \vec{H}'/\sim : If it would be reachable from S_i with $i > t$, then V would have been reachable in \vec{H}/\sim

from S_i , contradicting our choice of the S_2, \dots, S_t . Note further that V is reachable from (precisely) S_1, \dots, S_t in \vec{H} . Since ψ is a fractional cover of \vec{H}/\sim , we have

$$\sum_{i=1}^t \psi(S_i) \geq 1.$$

Hence, we have that

$$\sum_{\substack{S' \in \mathbf{S}' \\ W \in R'(S')}} \psi'(S') = \sum_{i=2}^t \psi'(S_i) = \sum_{i=1}^t \psi(S_i) \geq 1.$$

Next, assume that $W \neq \hat{W}$. For each $i \geq 2$, if W is reachable from S_i in \vec{H}'/\sim then W is also reachable from S_i in \vec{H}/\sim . Thus, if W is not reachable from S_1 in \vec{H}/\sim , then

$$\sum_{\substack{S' \in \mathbf{S}' \\ W \in R'(S')}} \psi'(S') = \sum_{\substack{S \in \mathbf{S} \\ W \in R(S)}} \psi'(S) \geq 1.$$

Finally, if W is reachable from S_1 in \vec{H}/\sim , we recall that W must be reachable from all S_2, \dots, S_t in \vec{H}'/\sim . Since our definition of ψ' adds to the value of those sources $\psi(S_1)/(t-1)$, we have

$$\sum_{\substack{S' \in \mathbf{S}' \\ W \in R'(S')}} \psi'(S') = (t-1) \cdot \psi(S_1)/(t-1) + \sum_{\substack{S \in \mathbf{S} \setminus \{S_1\} \\ W \in R(S)}} \psi(S) = \sum_{\substack{S \in \mathbf{S} \\ W \in R(S)}} \psi(S) \geq 1.$$

This concludes Case 2.

$(u \notin \mathcal{S}, v \in \mathcal{S})$ Symmetric to the previous case.

$(u \notin \mathcal{S}, v \notin \mathcal{S})$ Let U and V be the strongly connected components of \vec{H} containing u and v , respectively. Then \vec{H}'/\sim is obtained from \vec{H}/\sim by contracting U and V , and all vertices between them, to a single vertex; here, a vertex X is “between” U and V if there is a directed path from U to V (or vice versa) that contains X . Let us call the resulting vertex \hat{W} . Note that $\mathbf{S} = \mathbf{S}'$, that is, \vec{H}/\sim and \vec{H}'/\sim have the same sources. Note further that, for every $i \in \{1, \dots, k\}$ and non-source $W \neq \hat{W}$, if W is reachable from S_i in \vec{H}/\sim , then it is also reachable from S_i in \vec{H}'/\sim . Furthermore, \hat{W} is reachable from a source S_i in \vec{H}'/\sim if one of the vertices that was contracted to \hat{W} was reachable from S_i in \vec{H}/\sim . Thus, every fractional cover of \vec{H}/\sim must also be a fractional cover of \vec{H}'/\sim , concluding this case.

With all cases resolved, the proof is complete. \square

Lemma 62. *Let \vec{H} be a digraph. Then $\text{fhtw}(\mathcal{R}(\vec{H})) \leq \rho^*(\vec{H})$.*

Proof. Let S_1, \dots, S_k be the sources of \vec{H}/\sim , and let R_1, \dots, R_k be the hyperedges of $\mathcal{R}(\vec{H})$, that is, for each $i \in [k]$ the hyperedge R_i includes all vertices in \vec{H} that can be reached from S_i .

Consider the following tree decomposition $(\mathcal{T}, \mathcal{B})$ of $\mathcal{R}(\vec{H})$:

- We add one *center* bag $B := V(\vec{H}) \setminus \left(\bigcup_{i=1}^k S_i\right)$, that is, B contains all vertices of \vec{H} not included in strongly connected components that become sources in \vec{H}/\sim .
- For each $i \in [k]$ we add a bag $B_i := R_i$, which is made adjacent to the center bag B .

Note first that this yields indeed a tree decomposition. Clearly, each vertex in $V(\mathcal{R}(\vec{H})) = V(\vec{H})$ is contained in a bag, including isolated vertices of \vec{H} (since those will become isolated sources

in \vec{H}/\sim). By definition, each hyperedge R_i is fully contained in at least one bag. Finally, for every vertex $v \in V(\mathcal{R}(\vec{H}))$, the subtree $\mathcal{T}_v = \mathcal{T}[B \in \mathcal{B} \mid v \in B]$ is connected: If v is contained in S_i for some $i \in [k]$, then \mathcal{T}_v only consists of $B_i = R_i$. Otherwise, v is contained in the center bag B . Hence \mathcal{T}_v cannot be disconnected.

Now let us prove that each bag has a fractional edge cover of weight at most $\rho^*(\vec{H})$: For the center bag B , any fractional cover of \vec{H} yields, by definition, a fractional edge cover of B with the same weight (recall that the sources of \vec{H}/\sim correspond to the hyperedges of $\mathcal{R}(\vec{H})$). Hence, the fractional edge cover number of B is bounded by $\rho^*(\vec{H})$. Finally, each bag $B_i = R_i$ can clearly be covered by one hyperedge. Thus the fractional edge cover number of B_i is 1. \square

Theorem 63. *There is a computable function f such that the following is true. Let \vec{H} and \vec{G} be digraphs, let d be the maximum outdegree of \vec{G} , and let r be the fractional cover number of \vec{H} . We can compute $\#\text{Sub}(\vec{H} \rightarrow \vec{G})$ in time*

$$f(|\vec{H}|, d) \cdot |\vec{G}|^{r+O(1)}.$$

Moreover, let \vec{C} be a class of digraphs. Then $\#\text{DIRSUB}_d(\vec{C})$ is fixed-parameter tractable if \vec{C} has bounded fractional cover number.

Proof. Given an instance (\vec{H}, \vec{G}) we first cast the problem as a linear combination of homomorphism counts. Concretely, using Lemma 29, we have

$$\#\text{Sub}(\vec{H} \rightarrow \vec{G}) = \sum_{\vec{F}} \text{sub}_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \vec{G}), \quad (19)$$

where the sum is over all (isomorphism classes of) digraphs \vec{F} and the coefficients $\text{sub}_{\vec{H}}(\vec{F})$ only depend on \vec{H} and are non-zero if and only if \vec{F} is a quotient graph of \vec{H} . Thus, we can proceed by computing all $\text{sub}_{\vec{H}}(\vec{F})$ in time only depending on \vec{H} , and all terms $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ with a non-zero coefficient using our algorithm for counting homomorphisms (Theorem 42): By Lemmas 61 and 62, we have that, for each quotient \vec{F} of \vec{H} ,

$$\text{fhtw}(\mathcal{R}(\vec{F})) \leq \rho^*(\vec{F}) \leq \rho^*(\vec{H}) = r.$$

Additionally, $|\vec{F}| \leq |\vec{H}|$. Thus, the computation of $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ takes time $g(|\vec{H}|, d) \cdot |\vec{G}|^{r+O(1)}$, for some computable function g , by the running time bound given in Theorem 42, concluding the proof. \square

7.2 Lower Bounds

Recall that α and α^* denote respectively the independence number and the fractional independence number. Let \vec{C} be a class of digraphs, and let $\Gamma(\vec{C})$ the class of all contours of digraphs in \vec{C} . We show that $\#\text{DIRSUB}_d(\vec{C})$ is intractable when $\alpha^*(\Gamma(\vec{C})) = \infty$. Together with the upper bounds of Section 7.1, this yields a complete characterization of the tractability of $\#\text{DIRSUB}_d(\vec{C})$.

To prove that $\#\text{DIRSUB}_d(\vec{C})$ is hard when $\alpha^*(\Gamma(\vec{C})) = \infty$, we first look at the integral independence number $\alpha(\Gamma(\vec{C}))$. We show that, if $\alpha(\Gamma(\vec{C})) = \infty$, then $\#\text{DIRSUB}_d(\vec{C})$ is hard because \vec{C} contains “hard” quotients. If instead $\alpha(\Gamma(\vec{C})) < \infty$ then we can show a reduction from $\#\text{DIRHOM}_d(\vec{C}')$ where \vec{C}' is a class of canonical DAGs with $\text{aw}(\vec{C}') = \infty$, which implies hardness by Lemma 58.

First, using our interpolation result based on Dedekind’s Theorem (Lemma 33), we establish a hardness result in two steps.

Lemma 64. *Let \vec{C} be a recursively enumerable class of digraphs and let \vec{Q} be a class of quotient graphs of \vec{C} . Then*

$$\#\text{DIRHOM}_d(\vec{Q}) \leq_{\text{fpt}} \#\text{DIRSUB}_d(\vec{C}).$$

Proof. Let \vec{H}' and \vec{G}' be an input instance of $\#\text{DIRHOM}_d(\vec{Q})$, and let d be the outdegree of \vec{G}' . Search for a graph $\vec{H} \in \vec{C}$ such that \vec{H}' is a quotient of \vec{H} - note that this takes time only depending on \vec{H}' . By Lemma 29, we have that

$$\#\text{Sub}(\vec{H} \rightarrow \star) = \sum_{\vec{F}} \text{sub}_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \star).$$

Moreover, we have that $\text{sub}_{\vec{H}}(\vec{H}') \neq 0$ since \vec{H}' is a quotient of \vec{H} . Let us set $\iota = \text{sub}_{\vec{H}}$. This allows us to invoke Lemma 33 since we can then simulate the oracle required by Lemma 33 using our own oracle for $\#\text{DIRSUB}_d(\vec{C})$. The algorithm \mathbb{A} in Lemma 33 then returns all pairs $(\vec{F}, \#\text{Hom}(\vec{F} \rightarrow \vec{G}'))$ with $\text{sub}_{\vec{F}} \neq 0$; this includes $(\vec{H}', \#\text{Hom}(\vec{H}' \rightarrow \vec{G}'))$. All oracle queries posed by \mathbb{A} have outdegree bounded by $f(|\iota|) \cdot d$, which guarantees that the parameter of each oracle call we forward to $\#\text{DIRSUB}_d(\vec{C})$ only depends on \vec{H}' (recall that the parameter is $|\vec{H}'| + d$). Moreover, the total running time is fixed-parameter tractable, concluding the proof. \square

Lemma 65. *Let \vec{C} be a recursively enumerable class of digraphs and let \vec{Q} be the class of all quotient graphs of \vec{C} . If the class of canonical DAGs that are MR minors of digraphs in \vec{Q} has unbounded adaptive width, then $\#\text{DIRSUB}_d(\vec{C})$ is not fixed-parameter tractable unless ETH fails.*

Proof. Assume ETH holds. By Lemma 58, the problem $\#\text{DIRHOM}_d(\vec{Q})$ is not fixed-parameter tractable. The claim thus follows by invoking Lemma 64. \square

7.2.1 The case of unbounded independence number

Let us introduce:

Definition 66 (Induced Matching Gadget). *Let \vec{H} be a DAG. An induced matching gadget of size k of \vec{H} is a set of arcs $(s_1, w_1), \dots, (s_k, w_k) \in E(\vec{H})$ such that no two distinct w_i, w_j are reachable from a single source of \vec{H} .*

We denote by $\text{img}(\vec{H})$ the maximum size of an induced matching gadget in \vec{H} . Given a directed (not necessarily acyclic) graph \vec{H} , we set $\text{img}(\vec{H}) := \text{img}(\vec{H}/\sim)$.

Lemma 67. *Every digraph \vec{H} satisfies $\alpha(\Gamma(\vec{H}/\sim)) = \text{img}(\vec{H})$.*

Proof. First, we prove that $\text{img}(\vec{H}) \geq \alpha(\Gamma(\vec{H}/\sim))$. Let $\{v_1, \dots, v_k\}$ be an independent set of $\Gamma(\vec{H}/\sim)$, and let s_1, \dots, s_k be sources of \vec{H}/\sim such that v_i is reachable from s_i for each $i \in [k]$. Note that those sources must exist: if v_i is not reachable by any source, then v_i is a source itself and thus not in $V(\Gamma(\vec{H}/\sim))$. Furthermore, those sources must be distinct: if $s_i = s_j$ for some $i \neq j$, then v_i and v_j can both be reached from $s_i = s_j$ and thus they are contained in a common edge of $\Gamma(\vec{H}/\sim)$, contradicting the fact that $\{v_1, \dots, v_k\}$ is independent.

For each $i \in [k]$ let P_i be a shortest directed path from s_i to v_i , and let w_i be the successor of s_i in P_i (if $(s_i, v_i) \in E(\vec{H}/\sim)$, then $w_i = v_i$). Note that w_1, \dots, w_k are pairwise distinct and form an independent set in $\Gamma(\vec{H}/\sim)$. Indeed, if $w_i = w_j$ for some $i \neq j$, then v_i and v_j can both be reached from s_i and s_j and are thus contained in a common hyperedge in $\Gamma(\vec{H}/\sim)$, a contradiction; if instead $\{w_1, \dots, w_k\}$ is not an independent set in $\Gamma(\vec{H}/\sim)$, then there is a source s of \vec{H}/\sim from which both w_i and w_j , and thus both v_i and v_j , can be reached, which implies v_i and v_j are contained in a common edge of $\Gamma(\vec{H}/\sim)$, yielding again a contradiction. Finally, observe that $(s_1, w_1), \dots, (s_k, w_k)$ is an induced matching gadget since $\{w_1, \dots, w_k\}$ is an independent set in $\Gamma(\vec{H}/\sim)$.

Now we prove that $\alpha(\Gamma(\vec{H}/\sim)) \geq \text{img}(\vec{H})$. Let $(s_1, w_1), \dots, (s_k, w_k)$ be an induced matching gadget of \vec{H} , and for each $i \in [k]$ let S_i and W_i be the classes of \sim containing respectively s_i and w_i . Note that $S_i = \{s_i\}$ since s_i is a source, so $W_i \neq S_i$; and $(S_i, W_i) \in E(\vec{H}/\sim)$ since $(s_i, w_i) \in E(\vec{H})$, hence W_i is not a source of \vec{H}/\sim . The definition of $V(\Gamma(\vec{H}))$ then implies

$W_i \subseteq V(\Gamma(\vec{H}))$ for all $i \in [k]$, and so $\{w_1, \dots, w_k\} \subseteq V(\Gamma(\vec{H}))$. Now observe that $\{w_1, \dots, w_k\}$ is an independent set in $\Gamma(\vec{H}/\sim)$; if this was not the case, then w_i would be reachable in \vec{H} from w_j , and thus from s_j , for some $j \neq i$, contradicting the definition of induced matching gadget. \square

Next we show that large induced matching gadgets yield as quotients the directed splits of arbitrary graphs.

Lemma 68. *Let F be an undirected graph with ℓ edges, and let \vec{H} be a digraph with an induced matching gadget of size 2ℓ . Then \vec{F}^2 is an MR minor of a quotient graph of \vec{H} .*

Proof. First we claim that we can assume w.l.o.g. that \vec{H} is a DAG: In the very first step, we take the quotient of \vec{H} corresponding to \sim . Note that the resulting graph is equal to \vec{H}/\sim except for possibly having loops. Note also that this operation does not change the size of a maximum induced matching gadget. Since all loops can in the end be deleted by the loop deletion operation of MR minors, to avoid notational clutter, we can assume that there are none.

Now, for each edge $e = \{u, v\}$ of F , we choose two arcs of the induced matching gadget that will correspond to e in the construction of our quotient. We will denote those arcs by (s_e^u, w_e^u) and (s_e^v, w_e^v) . Let us now define the partition σ of $V(\vec{H})$ which will yield our quotient graph.

1. For each vertex $v \in V(F)$, we add a block $B_v = \{w_e^v \mid e \in E(F)\}$.
2. For each edge $e = \{u, v\} \in E(F)$, we add a block $B_e = \{s_e^u, s_e^v\}$.
3. Each vertex of \vec{H} not contained in any of the B_v or B_e becomes a singleton block.

Let $\vec{H}' = \vec{H}/\sigma$, that is, \vec{H}' is the quotient graph obtained from \vec{H} by contracting each B_v and each B_e to a single vertex — it will be convenient to also call those vertices B_v and B_e . Observe that the subgraph of \vec{H}' induced by the vertices B_v for $v \in V(F)$ and B_e for $e \in E(F)$ is isomorphic to \vec{F}^2 : By construction of σ , it is clear that \vec{F}^2 is a *subgraph* of the subgraph of \vec{H}' induced by the B_v and B_e . For isomorphism, we have to argue that there are no additional arcs: First, there cannot be any arc between B_e and $B_{e'}$ for $e \neq e'$ since the s_e^u and $s_{e'}^v$ have been sources. Furthermore, there cannot be an arc between $B_{\{u,v\}}$ and B_x for a vertex $x \notin \{u, v\}$ since this would only be possible if either s_e^u or s_e^v has an arc to some $w_{e'}^x$ for some $e' \neq \{u, v\}$. However, in that case $w_{e'}^x$ and one of w_e^u or w_e^v would be reachable from the same source, contradicting the definition of an induced matching gadget. A similar argument shows that there cannot be an arc between B_v and B_u for two distinct vertices $u, v \in V(F)$. Also, we observe that the B_e must be sources of \vec{H}' .

Now perform the following operations on \vec{H}' until none of them can be applied anymore:

- Delete a sink that is not one of the B_v .
- Let s be a source of \vec{H}' not among the B_e , and let y be a descendant of s (y might be one of the B_v). Contract the arc (s, y) .

This procedure stops if the only vertices remaining are the B_e and the B_v . Crucially, the contraction of arcs from sources (not among the B_e) can never create additional arcs between the B_e and the B_v since, by definition of induced matching gadgets, no distinct pair of the w_e^u is reachable from a common source. \square

We are now able to establish hardness for the case of unbounded independence number of the contours.

Lemma 69. *Let \vec{C} be a recursively enumerable class of digraphs, and let $\Gamma(\vec{C})$ be the contours of \vec{C} . If the independence number of $\Gamma(\vec{C})$ is unbounded, then $\#\text{DIRSUB}_d(\vec{C})$ is not fixed-parameter tractable, unless ETH fails.*

Proof. By Lemma 25 and Lemma 67, the class \vec{C} contains induced matching gadgets of unbounded size. Let K be the family of all complete (undirected) graphs; clearly, the treewidth of K is

unbounded. Let furthermore $\vec{K}^2 := \{\vec{F}^2 \mid F \in K\}$ be the set of all directed splits of complete graphs. Observe that \vec{K}^2 is a class of canonical DAGs, and observe further that $\Gamma(\vec{K}^2) = K$: Given $\vec{F}^2 \in \vec{K}^2$, each source s of \vec{F}^2 corresponds to an edge $\{u, v\}$ of F , and the only two vertices reachable from s in \vec{F}^2 are precisely u and v . Hence, the reachability hypergraph is 3-uniform and contains the hyperedges $\{s, u, v\}$. In the contour, we delete the former sources from each hyperedge, which then yields F again (thinking of a graph as a 2-uniform hypergraph).

Now, it is well-known that adaptive width and tree-width are equivalent for graphs. Hence the adaptive width of \vec{K}^2 must be unbounded (recall that the adaptive width of a digraph is defined to be the adaptive width of its contour).

Finally, by Lemma 68, and using that \vec{C} has induced matching gadgets of unbounded size, we obtain that the set of quotient graphs of digraphs in \vec{C} admits as MR minors the canonical DAGs in \vec{K}^2 . Since the adaptive width of the latter is unbounded, we can conclude the proof by applying Lemma 65. \square

7.2.2 The case of bounded independence number

Let \mathcal{H} be a nonempty hypergraph. Without loss of generality we may assume $V(\mathcal{H}) = \cup E(\mathcal{H})$. Let (T, B) be a tree decomposition of \mathcal{H} . For every $\{r, u\} \in E(T)$ let T_u^r be the connected component of $T \setminus e$ containing u but not r , and define:

$$V_u^r = (\cup_{x \in V(T_u^r)} B_x) \setminus (B_r \cap B_u) \quad (20)$$

Note that $V(\mathcal{H}) = V_u^r \dot{\cup} V_r^u \dot{\cup} (B_r \cap B_u)$.

The next result bounds the integrality gap of $\alpha^*(\mathcal{H})$ through $\text{aw}(\mathcal{H})$.

Lemma 70. $\alpha(\mathcal{H}) \geq \frac{1}{2} + \frac{\alpha^*(\mathcal{H})}{4 \text{aw}(\mathcal{H})}$.

Proof. We use induction on $|E(\mathcal{H})|$. If $|E(\mathcal{H})| = 1$ then one can see that $\alpha(\mathcal{H}) = \alpha^*(\mathcal{H}) = \text{aw}(\mathcal{H})$, so the claim holds. Now suppose $|E(\mathcal{H})| > 1$, and assume the claim holds for every hypergraph with less than $|E(\mathcal{H})|$ edges. Let $\mu : V(\mathcal{H}) \rightarrow \mathbb{R}_{\geq 0}$ be a fractional independent set for \mathcal{H} with $\mu(V(\mathcal{H})) = \alpha^*(\mathcal{H})$, and let (T, \mathcal{B}) be a tree decomposition for \mathcal{H} of smallest order (i.e., that minimizes $|V(T)|$) such that $\mu\text{-width}(T, \mathcal{B}) \leq \text{aw}(\mathcal{H})$. Choose any $\{r, u\} \in E(T)$ and let $S = B_r \cap B_u$. Finally, let $\mathcal{C}(S)$ be the set of connected components of $\mathcal{H} \setminus S$. Note that no $e \in E(\mathcal{H})$ intersects two distinct elements of $\mathcal{C}(S)$: indeed, by the properties of tree decompositions S separates V_u^r and V_r^u , and any such e would intersect both V_u^r and V_r^u , a contradiction.

We can now deduce the following facts. First, $\alpha(\mathcal{H}) \geq \sum_{C \in \mathcal{C}(S)} \alpha(\mathcal{H}[C])$, since no $e \in E(\mathcal{H})$ intersects more than one element of $\mathcal{C}(S)$. Second, $|\mathcal{C}(S)| \geq 2$; indeed, if $|\mathcal{C}(S)| \leq 1$, then $B_r \subseteq B_u$ (or vice versa) and thus we could replace $\{B_r, B_u\}$ with B_u (or with B_r) without increasing $\mu\text{-width}(T, \mathcal{B})$, contradicting the minimality of $|V(T)|$. Third, $|E(\mathcal{H}[C])| < |E(\mathcal{H})|$ for all $C \in \mathcal{C}(S)$; indeed, every $C \in \mathcal{C}(S)$ is intersected by some $e \in E(\mathcal{H})$, and as noted above $|\mathcal{C}(S)| \geq 2$ and no $e \in E(\mathcal{H})$ intersects more than one element of $\mathcal{C}(S)$. Using these facts and the inductive hypothesis on each $\mathcal{H}[C]$, we obtain:

$$\alpha(\mathcal{H}) \geq \sum_{C \in \mathcal{C}(S)} \alpha(\mathcal{H}[C]) \quad (21)$$

$$\geq \sum_{C \in \mathcal{C}(S)} \left(\frac{1}{2} + \frac{\alpha^*(\mathcal{H}[C])}{4 \text{aw}(\mathcal{H}[C])} \right) \quad (22)$$

$$(23)$$

$$\geq \sum_{C \in \mathcal{C}(S)} \left(\frac{1}{2} + \frac{\alpha^*(\mathcal{H}[C])}{4 \text{aw}(\mathcal{H})} \right) \quad (24)$$

$$\geq 1 + \frac{1}{4 \text{aw}(\mathcal{H})} \sum_{C \in \mathcal{C}(S)} \alpha^*(\mathcal{H}[C]) \quad (25)$$

Note that $\alpha^*(\mathcal{H}[C]) \geq \mu(C)$ since the restriction of μ to C is a fractional independent set for $\mathcal{H}[C]$. Moreover $\sum_{C \in \mathcal{C}(S)} \mu(C) = \mu(V(\mathcal{H})) - \mu(S)$, and $\mu(S) \leq \text{aw}(\mathcal{H})$ by the choice of (T, \mathcal{B}) . Therefore:

$$\alpha(\mathcal{H}) \geq 1 + \frac{1}{4 \text{aw}(\mathcal{H})} \sum_{C \in \mathcal{C}(S)} \mu(C) \quad (26)$$

$$= 1 + \frac{1}{4 \text{aw}(\mathcal{H})} (\mu(V(\mathcal{H})) - \mu(S)) \quad (27)$$

$$\geq 1 + \frac{\alpha^*(\mathcal{H}) - \text{aw}(\mathcal{H})}{4 \text{aw}(\mathcal{H})} \quad (28)$$

$$> \frac{1}{2} + \frac{\alpha^*(\mathcal{H})}{4 \text{aw}(\mathcal{H})} \quad (29)$$

which concludes the proof. \square

A construction of Canonical DAGs that preserves α^* In what follows, recall that the vertices of the DAG \vec{H}/\sim are the strongly connected components of \vec{H} ; we use capital letters to denote the vertices of \vec{H}/\sim .

Lemma 71. *Let \vec{H} be a digraph and $(U, V) \in E(\vec{H}/\sim)$ where U is not a source of \vec{H}/\sim . Then there is a fractional independent set $\hat{\mu}$ for $\Gamma(\vec{H})$ of maximum weight such that $\hat{\mu}(v) = 0$ for every $v \in V$.*

Proof. Let $\hat{\mu}^* : V(\Gamma(\vec{H})) \rightarrow \mathbb{R}_{\geq 0}$ be any fractional independent set for $\Gamma(\vec{H})$ of maximum weight. By Lemma 25, there exists a maximum fractional independent set $\mu^* : V(\Gamma(\vec{H}/\sim)) \rightarrow \mathbb{R}_{\geq 0}$, the weight of which equal to the weight of $\hat{\mu}^*$.

We define a fractional independent set μ of $\Gamma(\vec{H}/\sim)$ as follows:

$$\mu(X) = \begin{cases} \mu^*(X) & X \notin \{U, V\} \\ \mu^*(U) + \mu^*(V) & X = U \\ 0 & X = V \end{cases} \quad (30)$$

Clearly μ and μ^* have the same weight. Now let $e \in E(\Gamma(\vec{H}/\sim))$. If $U \notin e$ then $\mu(e) \leq \mu^*(e)$. Otherwise $\{U, V\} \subseteq e$, and since $\mu(U) + \mu(V) = \mu^*(U) + \mu^*(V)$, then again $\mu(e) \leq \mu^*(e)$. Therefore μ is a fractional independent set for $\Gamma(\vec{H}/\sim)$. Note that μ must also be of maximum weight since otherwise, μ^* would not have been of maximum weight.

Now define $\hat{\mu} : V(\Gamma(\vec{H})) \rightarrow \mathbb{R}_{\geq 0}$ as follows: For any strongly connected component $X = x_1, \dots, x_k$ of \vec{H} we set $\hat{\mu}(x_1) = \mu(X)$ and $\hat{\mu}(x_i) = 0$ for all $i \in \{2, \dots, k\}$. Clearly, $\hat{\mu}$ has the same total weight as μ . Furthermore, note that the hyperedges of $\Gamma(\vec{H}/\sim)$ are obtained from the hyperedges of $\Gamma(\vec{H})$ by contracting each vertex set X corresponding to a strongly connected component in \vec{H} into a single vertex. For each such set X and hyperedge $e \in E(\Gamma(\vec{H}))$ we have that either $X \subseteq e$ or $e \cap X = \emptyset$ — this follows from the definition of reachability hypergraphs and of the contour. Thus $\hat{\mu}$ is a fractional independent set of $\Gamma(\vec{H})$. Furthermore, since $\hat{\mu}$ has the same weight as μ and since μ is a fractional independent set of $\Gamma(\vec{H}/\sim)$ of maximum weight, we have by Lemma 25 that $\hat{\mu}$ is of maximum weight as well. Finally, $\hat{\mu}(v) = 0$ for each $v \in V$ since $\mu(V) = 0$, concluding the proof. \square

Lemma 72. *For every digraph \vec{H} there is a digraph \vec{F} such that the following conditions are satisfied:*

1. \vec{F} can be obtained from \vec{H} via a sequence of sink deletions,
2. $\alpha^*(\Gamma(\vec{F})) \geq \alpha^*(\Gamma(\vec{H}))$, and
3. \vec{F}/\sim is a canonical DAG.

Proof. If \vec{H}/\sim is a canonical DAG then we set $\vec{F} = \vec{H}$.

If \vec{H}/\sim is not a canonical DAG, then there is an arc $(U, V) \in E(\vec{H}/\sim)$ such that both U and V are not sources. Moreover we can assume V is a sink (otherwise replace U with V , and V with one of its children). By Lemma 71, there is a fractional independent set $\hat{\mu}$ for $\Gamma(\vec{H})$ of maximum weight such that $\hat{\mu}(v) = 0$ for every $v \in V$. Since V is a sink of \vec{H}/\sim , we can set $\vec{H}' = \vec{H} \setminus V$, that is, we perform a sink deletion. Let $\hat{\mu}'$ be the restriction of $\hat{\mu}$ to $V(\Gamma(\vec{H}'))$. Note that $V(\Gamma(\vec{H}')) = V(\Gamma(\vec{H})) \setminus V$; since $\hat{\mu}(v) = 0$ for all $v \in V$, this implies that $\hat{\mu}'$ has the same weight as $\hat{\mu}$.

Moreover $\hat{\mu}'$ is clearly a fractional independent set for $\Gamma(\vec{H}')$, since for every $e' \in E(\Gamma(\vec{H}'))$ there is $e \in E(\Gamma(\vec{H}))$ such that $e' \subseteq e$, and $\hat{\mu}(e) = \hat{\mu}'(e')$. We conclude that $\alpha^*(\Gamma(\vec{H}')) \geq \alpha^*(\Gamma(\vec{H}))$. If \vec{H}'/\sim is a canonical DAG, then setting $\vec{F} = \vec{H}'$ concludes the proof; otherwise just repeat the argument on \vec{H}' . \square

We are now able to prove the intractability part of our classification.

Lemma 73. *Let \vec{C} be a recursively enumerable class of digraphs of unbounded fractional cover number. Then $\#\text{DIRSUB}_d(\vec{C})$ is not fixed-parameter tractable, unless ETH fails.*

Proof. Let $\Gamma(\vec{C})$ be the class of contours of digraphs in \vec{C} . By Observation 60 and Lemma 25 $\Gamma(\vec{C})$ has unbounded fractional edge cover number and thus, by Fact 22, $\Gamma(\vec{C})$ has unbounded fractional independence number. The proof now considers two cases.

Case 1: The independence number of $\Gamma(\vec{C})$ is unbounded. Then the claim follows from the construction based on induced matching gadgets (Lemma 69).

Case 2: The independence number of $\Gamma(\vec{C})$ is bounded. We show that $\#\text{DIRHOM}_d(\vec{C})$ is not fixed-parameter tractable, unless ETH fails. The claim then follows since, by Lemma 64 and the trivial fact that each digraph is a quotient graph of itself, we have

$$\#\text{DIRHOM}_d(\vec{C}) \leq_{\text{fpt}} \#\text{DIRSUB}_d(\vec{C}).$$

Let \vec{C}' be the class of all digraphs \vec{H}' such that

1. \vec{H}' can be obtained by a sequence of sink deletions from a graph $\vec{H} \in \vec{C}$, disallowing deletions of sinks that are also sources, and
2. \vec{H}'/\sim is a canonical DAG.

Now note that a sink deletion in a digraph \vec{H} corresponds to vertex-deletions in the contour. More precisely, let \vec{H}' be obtained from \vec{H} by deleting the sink T of \vec{H}/\sim . Then $\Gamma(\vec{H}') = \Gamma(\vec{H})[V(\vec{H}) \setminus T]$. Thus, clearly, the independence number of $\Gamma(\vec{H}')$ is upper bounded by the independence number of $\Gamma(\vec{H})$. Thus, the independence number of the class $\Gamma(\vec{C}')$ of the contours of digraphs in \vec{C}' is bounded (since the independence number of $\Gamma(\vec{C})$ is bounded by the assumption of this case.). Next, by Lemma 72, we have that the fractional independence number of $\Gamma(\vec{C}')$ is still unbounded. In combination with Lemma 70, this is only possible if the adaptive width of $\Gamma(\vec{C}')$ is unbounded.

Next, let \vec{C}'/\sim be the class of all DAGs \vec{H}'/\sim with $\vec{H}' \in \vec{C}'$. By definition of \vec{C}' , each element of \vec{C}'/\sim must be a canonical DAG. Now note that the contours of the canonical DAGs in \vec{C}'/\sim can be obtained from the contours of digraphs in \vec{C}' by a sequence of contractions of vertices u and v such that u and v are contained in precisely the same hyperedges (since we contract strongly connected components into single vertices). Those contractions cannot decrease the adaptive width as shown in Lemma 25.

As a consequence, we have established the following three facts:

- (A) The elements in \vec{C}'/\sim are MR minors of the digraphs in \vec{C} : \vec{C}' is obtained by sink-deletions, and \vec{C}'/\sim is obtained by arc contractions — each strongly connected component can be contracted into a single vertex using only arc contractions.
- (B) The adaptive width of \vec{C}'/\sim is unbounded.

(C) Each element of \vec{C}'/\sim is a canonical DAG.

In combination, (A), (B), and (C) allow for the application of Lemma 58 which yields intractability as desired, concluding the proof. \square

7.3 Proof of the Classification

We are now able to combine our upper and lower bounds and prove a complete and explicit classification:

Theorem 74. *Let \vec{C} be a recursively enumerable class of digraphs and assume that ETH holds. Then the problem $\#\text{DIRSUB}_d(\vec{C})$ is fixed-parameter tractable if and only if the fractional cover number of \vec{C} is bounded.*

Proof. The “if” direction is Theorem 63, and the “only if” direction is Lemma 73. \square

Furthermore, we note that all of our hardness results apply also in case of digraphs without loops, and even for DAGs; the reason for this is two-fold: First the intractability boils down to counting homomorphisms from canonical DAGs to DAGs of small outdegree via our reduction from $\#\text{CSP}$ (see Lemma 57 and Lemma 51). Second, each intermediate step in our reduction, including the interpolation method based on Dedekind’s Theorem (Lemma 33), does not create any additional cycles (including loops) in the host, provided our pattern is without loops and cycles. Concretely, we obtain the following variations of Theorem 74.

Theorem 75. *Let \vec{C} be a recursively enumerable class of digraphs without loops and assume that ETH holds. Then the problem $\#\text{DIRSUB}_d(\vec{C})$, restricted on host graphs without loops, is fixed-parameter tractable if and only if the fractional cover number of \vec{C} is bounded.*

Theorem 76. *Let \vec{C} be a recursively enumerable class of DAGs and assume that ETH holds. Then the problem $\#\text{DIRSUB}_d(\vec{C})$, restricted on acyclic host graphs, is fixed-parameter tractable if and only if the fractional cover number of \vec{C} is bounded.*

8 Counting Induced Subgraphs

We will establish boundedness of the following invariant as sufficient and necessary condition for the fixed-parameter tractability of $\#\text{DIRSUB}_d$.

Definition 77 ($\alpha_s(\vec{H})$). *Given a digraph \vec{H} , we denote by $\alpha_s(\vec{H})$ the number of sources of the DAG \vec{H}/\sim .*

8.1 Implications of Upper and Lower Bounds on $\alpha_s(\vec{H})$

Lemma 78 (Lower Bound). *Let F be an undirected graph with k vertices and ℓ edges, and let \vec{H} be a digraph with $\alpha_s(\vec{H}) \geq k + \ell$. Then there exists an arc supergraph \vec{H}' of \vec{H} satisfying that \vec{F}^2 is an MR minor of \vec{H}' .*

Proof. We will first construct \vec{H}' . To this end, observe that \vec{F}^2 has precisely $k + \ell$ vertices, and we assume w.l.o.g. that the vertex set of \vec{F}^2 is $[k + \ell]$. Since $\alpha_s(\vec{H}) \geq k + \ell$ there exists set of sources $S_1, \dots, S_{k+\ell}$ of \vec{H}/\sim . We emphasize that a set of sources must always also be an independent set, since there cannot be arcs between the sources. Recall that, by definition of \vec{H}/\sim , the S_i are strongly connected components of \vec{H} . Now pick $s_i \in S_i$ arbitrarily for each $i \in [k + \ell]$. We obtain the graph \vec{H}' from \vec{H} as follows: Whenever (i, j) is an arc of \vec{F}^2 , we add an arc (s_i, s_j) to \vec{H} . Observe that we do not create loops in this construction since \vec{F}^2 does not contain loops. For this proof, it will also be convenient to consider the digraph \vec{G} obtained from \vec{H}/\sim by adding an arc (S_i, S_j) whenever (i, j) is an arc of \vec{F}^2 . We will see that $\vec{G} = \vec{H}'/\sim$.

Observe that \vec{G} is still acyclic: Assuming otherwise, there must be a (not necessarily simple) directed cycle in \vec{G} . Since \vec{H}/\sim is acyclic, we have that at least on arc of the cycle must be one of the freshly added arcs (S_i, S_j) . Additionally, there must be a directed path P from S_j to S_i in \vec{G} . Consider two cases: If P contains any vertex V which is not a source of \vec{H}/\sim , then we obtain a contradiction immediately, since it is not possible to reach any source from V by a directed path — recall that we only added arcs between sources in the construction of \vec{G} . Otherwise, all vertices of P are sources. However, in this case we created a cycle in \vec{G} only consisting of sources, and thus, this cycle must correspond to a cycle in \vec{F}^2 by construction of \vec{G} . Since \vec{F}^2 is acyclic by definition, we obtain the contradiction.

Next we claim that the compositions into strongly connected components of \vec{H} and \vec{H}' are the same, that is, the relation \sim has the same equivalence classes in both $V(\vec{H})$ and $V(\vec{H}')$. Assume for contradiction that this is not the case. Since adding arcs can only merge strongly connected components, there must be vertices x and y which are not in the same strongly connected component of \vec{H} , but they are in the same strongly connected component in \vec{H}' , that is, there is a (not necessarily simple) directed cycle in \vec{H}' containing both x and y . Let $X \neq Y$ be the connected components of \vec{H} containing x and y , respectively. Now identify all vertices in this cycle that are in the same connected component of \vec{H} and observe that this creates a cycle in \vec{G} (note that we needed the assumption that $X \neq Y$ to make sure that the entire cycle does not collapse to a single vertex in \vec{G}). Since \vec{G} is acyclic, we obtain the contradiction.

As a consequence, we infer that \vec{G} is indeed the graph \vec{H}'/\sim . Using this fact, we are able to show that \vec{F}^2 is an MR minor of \vec{H}' : First, contract each strongly connected component of \vec{H}' into a single vertex. By definition this yields precisely $\vec{H}'/\sim = \vec{G}$. Next we iteratively delete sinks until only the vertices $S_1, \dots, S_{k+\ell}$ remain and claim that the resulting graph is \vec{F}^2 as desired. To see this, observe that there is no vertex in $V(\vec{G}) \setminus \{S_1, \dots, S_{k+\ell}\}$ from which one can reach any of the S_1, \dots, S_k ; this is true since the S_i have been sources in \vec{H}/\sim . Therefore, in combination with the fact that \vec{G} is acyclic, we have that there must always be a sink outside of the $S_1, \dots, S_{k+\ell}$ as long as there are still vertices outside of the $S_1, \dots, S_{k+\ell}$ remaining. Note that the latter property is invariant under the deletion of sinks outside of $S_1, \dots, S_{k+\ell}$ — for clarification we note that even former sources of \vec{H}/\sim not included in $\{S_1, \dots, S_{k+\ell}\}$ will be deleted at some point, since they become sinks if all of their descendants have been deleted in previous iterations. Hence, we can conclude that at the end of the process of iteratively deleting sinks, we obtain the (induced) subgraph of \vec{G} only consisting of the $S_1, \dots, S_{k+\ell}$ which is equal to \vec{F}^2 (recall that we added an arc between S_i and S_j if and only if there is an arc (i, j) in \vec{F}^2 , and since the $S_1, \dots, S_{k+\ell}$ have been sources in \vec{H}/\sim there are no further arcs between them.) \square

Next we invoke our reduction based on Dedekind's Theorem to the case of counting induced subgraphs:

Lemma 79. *Let \vec{C} be a recursively enumerable class of digraphs and let \vec{A} be a class of arc supergraphs of digraphs in \vec{C} . Then*

$$\#\text{DIRHOM}_d(\vec{A}) \leq_{\text{T}}^{\text{fpt}} \#\text{DIRINDSUB}_d(\vec{C}).$$

Proof. Let \vec{H}' and \vec{G}' be an input instance of $\#\text{DIRHOM}_d(\vec{A})$, and let d be the outdegree of \vec{G}' . Search for a graph $\vec{H} \in \vec{C}$ such that \vec{H}' is an arc supergraph of \vec{H} — note that this takes time only depending on \vec{H}' . By Lemma 32, we have that

$$\#\text{IndSub}(\vec{H} \rightarrow \star) = \sum_{\vec{F}} \text{indsub}_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \star).$$

Moreover, we have that $\text{indsub}_{\vec{H}}(\vec{H}') \neq 0$ since \vec{H}' is an arc supergraph of \vec{H} (see condition 3. in Lemma 32). Let us set $\iota = \text{indsub}_{\vec{H}}$. This allows us to invoke Lemma 33 since we can

then simulate the oracle required by Lemma 33 using our own oracle for $\#\text{DIRINDSUB}_d(\vec{C})$. The algorithm \mathbb{A} in Lemma 33 then returns all pairs $(\vec{F}, \#\text{Hom}(\vec{F} \rightarrow \vec{G}'))$ with $\text{indsub}_{\vec{F}} \neq 0$; this includes $(\vec{H}', \#\text{Hom}(\vec{H}' \rightarrow \vec{G}'))$. All oracle queries posed by \mathbb{A} have outdegree bounded by $f(|\iota|) \cdot d$, which guarantees that the parameter of each oracle call we forward to $\#\text{DIRINDSUB}_d(\vec{C})$ only depends on \vec{H}' (recall that the parameter is $|\vec{H}'| + d$). Moreover, the total running time is fixed-parameter tractable, concluding the proof. \square

Now, relying on our reduction chain based on Dedekind's Theorem and on our hardness result for $\#\text{DIRHOM}_d$, we can prove the following intractability result.

Lemma 80. *Let \vec{C} be a recursively enumerable class of digraphs and assume that ETH holds. If $\alpha_s(\vec{C})$ is unbounded then $\#\text{DIRINDSUB}_d(\vec{C})$ is not fixed-parameter tractable.*

Proof. The setup is similar to the proof of Lemma 69: Let K be the family of all complete (undirected) graphs. The treewidth of K is unbounded. Let furthermore $\vec{K}^2 := \{\vec{F}^2 \mid F \in K\}$ be the set of all directed splits of complete graphs. Observe that \vec{K}^2 is a class of canonical DAGs, and observe further that $\Gamma(\vec{K}^2) = K$. Since adaptive width and treewidth are equivalent for graphs, the adaptive width of \vec{K}^2 must be unbounded (recall that the adaptive width of a digraph is defined to be the adaptive width of its contour).

Finally, by Lemma 78, and using that $\alpha_s(\vec{C})$ is unbounded, we obtain that the set of arc supergraphs of digraphs in \vec{C} admits as MR minors the canonical DAGs in \vec{K}^2 . Since the adaptive width of the latter is unbounded, we can conclude the proof by applying Lemma 79 and Lemma 58. \square

Lemma 81 (Upper Bound). *Let \vec{H} be a digraph with $\alpha_s(\vec{H}) \leq c$, and let \vec{H}' be a quotient of an arc supergraph of \vec{H} . Then $|E(\mathcal{R}(\vec{H}'))| \leq c$.*

Proof. Observe that neither of the operations of adding arcs to or identifying vertices of a digraph \vec{F} can increase the number of sources of \vec{F}/\sim . Thus $\alpha_s(\vec{H}') \leq c$ as well. By definition of reachability hypergraphs, we can immediately conclude that $|E(\mathcal{R}(\vec{H}'))| \leq c$ since we create one hyperedge for each source of \vec{H}'/\sim . \square

We obtain the following algorithm.

Theorem 82. *There is a computable function f such that the following is true. Let \vec{H} and \vec{G} be digraphs, let d be the maximum outdegree of \vec{G} , and let $r = \alpha_s(\vec{H})$. We can compute $\#\text{IndSub}(\vec{H} \rightarrow \vec{G})$ in time*

$$f(|\vec{H}|, d) \cdot |\vec{G}|^{r+O(1)}.$$

Moreover, let \vec{C} be a class of digraphs. Then $\#\text{DIRINDSUB}_d(\vec{C})$ is fixed-parameter tractable if $\alpha_s(\vec{C})$ is bounded.

Proof. By Lemma 32, we have

$$\#\text{IndSub}(\vec{H} \rightarrow \vec{G}) = \sum_{\vec{F}} \text{indsub}_{\vec{H}}(\vec{F}) \cdot \#\text{Hom}(\vec{F} \rightarrow \vec{G}), \quad (31)$$

such that the following conditions are satisfied:

1. $\text{indsub}_{\vec{H}}$ has finite support and only depends on \vec{H} .
2. If $\text{indsub}_{\vec{H}}(\vec{F}) \neq 0$ then \vec{F} is a quotient of an arc supergraph of \vec{H} .

By Lemma 81, the second condition implies that the only terms $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ surviving with a non-zero coefficient satisfy $|E(\mathcal{R}(\vec{F}))| \leq r$. Clearly, this also implies that $\text{fhtw}(\mathcal{R}(\vec{F})) \leq r$. Note further that the size of any quotient of any arc supergraph of \vec{H} is bounded by $|\vec{H}|^2$. Hence, using the algorithm for counting homomorphisms in Theorem 42 for each term $\#\text{Hom}(\vec{F} \rightarrow \vec{G})$ with a non-zero coefficient we can evaluate the linear combination in time

$$f(|\vec{H}|, d) \cdot |\vec{G}|^{r+O(1)}$$

for some computable function f . This concludes the proof. \square

8.2 Proof of the Classification

We are now able to combine our upper and lower bounds and prove a complete and explicit classification:

Theorem 83. *Let \vec{C} be a recursively enumerable class of digraphs and assume that ETH holds. Then the problem $\#\text{DIRINDSUB}_d(\vec{C})$ is fixed-parameter tractable if and only if $\alpha_s(\vec{C})$ is bounded.*

Proof. The “if” direction is Theorem 82, and the “only if” direction is Lemma 80. \square

Finally, similarly to the case of counting subgraphs, our proofs readily classify also the cases of digraphs without loops and DAGs:

Theorem 84. *Let \vec{C} be a recursively enumerable class of digraphs without loops and assume that ETH holds. Then the problem $\#\text{DIRINDSUB}_d(\vec{C})$, restricted on host graphs without loops, is fixed-parameter tractable if and only if $\alpha_s(\vec{C})$ is bounded.*

Theorem 85. *Let \vec{C} be a recursively enumerable class of DAGs and assume that ETH holds. Then the problem $\#\text{DIRINDSUB}_d(\vec{C})$, restricted on acyclic host graphs, is fixed-parameter tractable if and only if $\alpha_s(\vec{C})$ is bounded.*

References

- [1] N. K. Ahmed, J. Neville, R. A. Rossi, and N. G. Duffield. Efficient graphlet counting for large networks. In C. C. Aggarwal, Z. Zhou, A. Tuzhilin, H. Xiong, and X. Wu, editors, *2015 IEEE International Conference on Data Mining, ICDM 2015, Atlantic City, NJ, USA, November 14-17, 2015*, pages 1–10. IEEE Computer Society, 2015. doi:10.1109/ICDM.2015.141.
- [2] R. Albert and A.-L. Barabási. Statistical mechanics of complex networks. *Rev. Mod. Phys.*, 74:47–97, Jan 2002. doi:10.1103/RevModPhys.74.47.
- [3] N. Alon, P. Dao, I. Hajirasouliha, F. Hormozdiari, and S. C. Sahinalp. Biomolecular network motif counting and discovery by color coding. *Bioinformatics*, 24(13):i241–i249, 07 2008. doi:10.1093/bioinformatics/btn163.
- [4] M. Arenas, L. A. Croquevielle, R. Jayaram, and C. Riveros. When is approximate counting for conjunctive queries tractable? In *Proc. of ACM STOC*, page 1015–1027, 2021. doi:10.1145/3406325.3451014.
- [5] E. Artin. *Galois Theory*, volume 2. Notre Dame Mathematical Lectures, 1948.
- [6] V. Arvind and V. Raman. Approximation Algorithms for Some Parameterized Counting Problems. In *Proc. of ISAAC*, pages 453–464, 2002. doi:10.1007/3-540-36136-7_40.
- [7] S. K. Bera, L. Gishboliner, Y. Levanzov, C. Seshadhri, and A. Shapira. Counting subgraphs in degenerate graphs. *J. ACM*, 69(3):23:1–23:21, 2022. doi:10.1145/3520240.

- [8] S. K. Bera, N. Pashanasangi, and C. Seshadhri. Linear Time Subgraph Counting, Graph Degeneracy, and the Chasm at Size Six. In *Proc. of ITCS*, volume 151, pages 38:1–38:20, 2020. doi:10.4230/LIPIcs.ITCS.2020.38.
- [9] S. K. Bera, N. Pashanasangi, and C. Seshadhri. Near-linear time homomorphism counting in bounded degeneracy graphs: The barrier of long induced cycles. In *Proc. of ACM-SIAM SODA*, pages 2315–2332, 2021. doi:10.1137/1.9781611976465.138.
- [10] S. K. Bera and C. Seshadhri. How the degeneracy helps for triangle counting in graph streams. In D. Suciú, Y. Tao, and Z. Wei, editors, *Proceedings of the 39th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems, PODS 2020, Portland, OR, USA, June 14-19, 2020*, pages 457–467. ACM, 2020. doi:10.1145/3375395.3387665.
- [11] C. Brand, H. Dell, and T. Husfeldt. Extensor-coding. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018*, pages 151–164, 2018. doi:10.1145/3188745.3188902.
- [12] M. Bressan. Faster algorithms for counting subgraphs in sparse graphs. *Algorithmica*, 2021. doi:10.1007/s00453-021-00811-0.
- [13] M. Bressan and M. Roth. Exact and approximate pattern counting in degenerate graphs: New algorithms, hardness results, and complexity dichotomies. In *62nd IEEE Annual Symposium on Foundations of Computer Science, FOCS 2021, Denver, CO, USA, February 7-10, 2022*, pages 276–285. IEEE, 2021. doi:10.1109/FOCS52979.2021.00036.
- [14] H. Chen, G. Gottlob, M. Lanzinger, and R. Pichler. Semantic width and the fixed-parameter tractability of constraint satisfaction problems. In *Proc. of IJCAI*, pages 1726–1733. ijcai.org, 2020. doi:10.24963/ijcai.2020/239.
- [15] H. Chen and S. Mengel. Counting Answers to Existential Positive Queries: A Complexity Classification. In *Proc. of ACM PODS*, pages 315–326, 2016. doi:10.1145/2902251.2902279.
- [16] J. Chen, B. Chor, M. Fellows, X. Huang, D. W. Juedes, I. A. Kanj, and G. Xia. Tight lower bounds for certain parameterized NP-hard problems. *Inf. Comput.*, 201(2):216–231, 2005. doi:10.1016/j.ic.2005.05.001.
- [17] J. Chen, X. Huang, I. A. Kanj, and G. Xia. Strong computational lower bounds via parameterized complexity. *J. Comput. Syst. Sci.*, 72(8):1346–1367, 2006. doi:10.1016/j.jcss.2006.04.007.
- [18] Y. Chen, M. Thurley, and M. Weyer. Understanding the Complexity of Induced Subgraph Isomorphisms. In *Proc. of ICALP*, pages 587–596, 2008. doi:10.1007/978-3-540-70575-8_48.
- [19] Z. Chen, A. Galanis, L. A. Goldberg, W. Perkins, J. Stewart, and E. Vigoda. Fast Algorithms at Low Temperatures via Markov Chains. In D. Achlioptas and L. A. Végh, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2019, September 20-22, 2019, Massachusetts Institute of Technology, Cambridge, MA, USA*, volume 145 of *LIPIcs*, pages 41:1–41:14. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPIcs.APPROX-RANDOM.2019.41.
- [20] N. Chiba and T. Nishizeki. Arboricity and subgraph listing algorithms. *SIAM J. Comput.*, 14(1):210–223, 1985. doi:10.1137/0214017.
- [21] R. Curticapean. Counting matchings of size k is $\#W[1]$ -hard. In *Proc. of ICALP*, volume 7965, pages 352–363, 2013. doi:10.1007/978-3-642-39206-1_30.

- [22] R. Curticapean. *The simple, little and slow things count: On parameterized counting complexity*. PhD thesis, Saarland University, 2015. URL <http://scidok.sulb.uni-saarland.de/volltexte/2015/6217/>.
- [23] R. Curticapean, H. Dell, and D. Marx. Homomorphisms are a good basis for counting small subgraphs. In *Proc. of ACM STOC*, pages 210–223, 2017. doi:10.1145/3055399.3055502.
- [24] R. Curticapean and D. Marx. Complexity of Counting Subgraphs: Only the Boundedness of the Vertex-Cover Number Counts. In *Proc. of IEEE FOCS*, pages 130–139, 2014. doi:10.1109/FOCS.2014.22.
- [25] V. Dalmau and P. Jonsson. The complexity of counting homomorphisms seen from the other side. *Theor. Comput. Sci.*, 329(1-3):315–323, 2004. doi:10.1016/j.tcs.2004.08.008.
- [26] H. Dell, M. Roth, and P. Wellnitz. Counting Answers to Existential Questions. In *Proc. of ICALP*, volume 132, pages 113:1–113:15, 2019. doi:10.4230/LIPIcs.ICALP.2019.113.
- [27] A. Durand and S. Mengel. Structural Tractability of Counting of Solutions to Conjunctive Queries. *Theory Comput. Syst.*, 57(4):1202–1249, 2015. doi:10.1007/s00224-014-9543-y.
- [28] J. Flum and M. Grohe. The Parameterized Complexity of Counting Problems. *SIAM J. Comput.*, 33(4):892–922, 2004. doi:10.1137/S0097539703427203.
- [29] J. Flum and M. Grohe. *Parameterized Complexity Theory*. Springer, 2006. doi:10.1007/3-540-29953-X.
- [30] J. Focke, L. A. Goldberg, M. Roth, and S. Zivný. Approximately counting answers to conjunctive queries with disequalities and negations. In L. Libkin and P. Barceló, editors, *PODS ’22: International Conference on Management of Data, Philadelphia, PA, USA, June 12 - 17, 2022*, pages 315–324. ACM, 2022. doi:10.1145/3517804.3526231.
- [31] J. Focke and M. Roth. Counting small induced subgraphs with hereditary properties. In S. Leonardi and A. Gupta, editors, *STOC ’22: 54th Annual ACM SIGACT Symposium on Theory of Computing, Rome, Italy, June 20 - 24, 2022*, pages 1543–1551. ACM, 2022. doi:10.1145/3519935.3520008.
- [32] L. Gishboliner, Y. Levanzov, A. Shapira, and R. Yuster. Counting homomorphic cycles in degenerate graphs. In J. S. Naor and N. Buchbinder, editors, *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*, pages 417–430. SIAM, 2022. doi:10.1137/1.9781611977073.21.
- [33] M. Grohe and D. Marx. Constraint solving via fractional edge covers. *ACM Trans. Algorithms*, 11(1):4:1–4:20, 2014. doi:10.1145/2636918.
- [34] R. Impagliazzo and R. Paturi. On the Complexity of k-SAT. *J. Comput. Syst. Sci.*, 62(2):367–375, 2001. doi:10.1006/jcss.2000.1727.
- [35] M. Jerrum and K. Meeks. The parameterised complexity of counting connected subgraphs and graph motifs. *J. Comput. Syst. Sci.*, 81(4):702–716, 2015. doi:10.1016/j.jcss.2014.11.015.
- [36] O. Kuchaiev, T. Milenković, V. Memišević, W. Hayes, and N. Pržulj. Topological network alignment uncovers biological function and phylogeny. *Journal of the Royal Society Interface*, 7(50):1341–1354, 2010.
- [37] D. Lokshtanov, A. Björklund, S. Saurabh, and M. Zehavi. Approximate counting of k -paths: Simpler, deterministic, and in polynomial space. *ACM Trans. Algorithms*, 17(3):26:1–26:44, 2021. doi:10.1145/3461477.

- [38] L. Lovász. *Large Networks and Graph Limits*, volume 60 of *Colloquium Publications*. American Mathematical Society, 2012.
- [39] D. Marx. Can You Beat Treewidth? *Theory Comput.*, 6(1):85–112, 2010. doi:10.4086/toc.2010.v006a005.
- [40] D. Marx. Tractable Hypergraph Properties for Constraint Satisfaction and Conjunctive Queries. *J. ACM*, 60(6):42:1–42:51, 2013. doi:10.1145/2535926.
- [41] D. Marx, P. D. Seymour, and P. Wollan. Rooted grid minors. *Journal of Combinatorial Theory, Series B*, 122:428–437, 2017. doi:10.1016/j.jctb.2016.07.003.
- [42] K. Meeks. The challenges of unbounded treewidth in parameterised subgraph counting problems. *Discrete Appl. Math.*, 198:170–194, 2016. doi:10.1016/j.dam.2015.06.019.
- [43] R. Milo, S. Itzkovitz, N. Kashtan, R. Levitt, S. Shen-Orr, I. Ayzenshtat, M. Sheffer, and U. Alon. Superfamilies of evolved and designed networks. *Science*, 303(5663):1538–1542, 2004. doi:10.1126/science.1089167.
- [44] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon. Network Motifs: Simple Building Blocks of Complex Networks. *Science*, 298(5594):824–827, 2002. doi:10.1126/science.298.5594.824.
- [45] M. Roth. *Counting Problems on Quantum Graphs: Parameterized and Exact Complexity Classifications*. PhD thesis, Saarland University, 2019. URL <https://scidok.sulb.uni-saarland.de/bitstream/20.500.11880/27575/1/thesis.pdf>.
- [46] M. Roth, J. Schmitt, and P. Wellnitz. Detecting and counting small subgraphs, and evaluating a parameterized tutte polynomial: Lower bounds via toroidal grids and cayley graph expanders. In N. Bansal, E. Merelli, and J. Worrell, editors, *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*, volume 198 of *LIPIcs*, pages 108:1–108:16. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.ICALP.2021.108.
- [47] E. Scheinerman and D. Ullman. *Fractional Graph Theory: A Rational Approach to the Theory of Graphs*. Dover Publications, Inc., 2011.
- [48] B. Schiller, S. Jager, K. Hamacher, and T. Strufe. StreaM - A Stream-Based Algorithm for Counting Motifs in Dynamic Graphs. In *Algorithms for Computational Biology*, pages 53–67, Cham, 2015.
- [49] S. Shen-Orr, R. Milo, S. Mangan, and U. Alon. Network motifs in the transcriptional regulation network of escherichia coli. *Nature Genetics*, 31(1):64–68, 2002. doi:10.1038/ng881.
- [50] N. H. Tran, K. P. Choi, and L. Zhang. Counting motifs in the human interactome. *Nature communications*, 4(1):1–8, 2013.
- [51] C. E. Tsourakakis, J. Pachocki, and M. Mitzenmacher. Scalable motif-aware graph clustering. In *Proc. of WWW*, page 1451–1460, 2017. doi:10.1145/3038912.3052653.
- [52] J. Ugander, L. Backstrom, and J. M. Kleinberg. Subgraph frequencies: mapping the empirical and extremal geography of large graph collections. In D. Schwabe, V. A. F. Almeida, H. Glaser, R. Baeza-Yates, and S. B. Moon, editors, *22nd International World Wide Web Conference, WWW '13, Rio de Janeiro, Brazil, May 13-17, 2013*, pages 1307–1318. International World Wide Web Conferences Steering Committee / ACM, 2013. doi:10.1145/2488388.2488502.

A Proof of Dedekind's Theorem

Theorem 86 (Theorem 36, restated). *Let $(G, *)$ be a semigroup. Let $(\varphi_i)_{i \in [k]}$ with $\varphi_i : G \rightarrow \mathbb{Q}$ be pairwise distinct semigroup homomorphisms of $(G, *)$ into (\mathbb{Q}, \cdot) , that is, $\varphi_i(g_1 * g_2) = \varphi_i(g_1) \cdot \varphi_i(g_2)$ for all $i \in [k]$ and $g_1, g_2 \in G$. Let $\phi : G \rightarrow \mathbb{Q}$ be a function*

$$\phi : g \mapsto \sum_{i=1}^k a_i \cdot \varphi_i(g), \quad (32)$$

where the a_i are rational numbers. Suppose furthermore that the following functions are computable:

1. The operation $*$.
2. The mapping $(i, g) \mapsto \varphi_i(g)$.
3. A mapping $i \mapsto g_i$ such that $\varphi_i(g_i) \neq 0$.
4. A mapping $(i, j) \mapsto g_{i,j}$ such that $\varphi_i(g_{i,j}) \neq \varphi_j(g_{i,j})$ whenever $i \neq j$.

Then there is a constant B only depending on the φ_i (and not on the a_i), and an algorithm $\hat{\mathbb{A}}$ such that the following conditions are satisfied:

- $\hat{\mathbb{A}}$ is equipped with oracle access to ϕ .
- $\hat{\mathbb{A}}$ computes a_1, \dots, a_k .
- Each oracle query \hat{g} only depends on the φ_i (and not on the a_i).
- The running time of $\hat{\mathbb{A}}$ is bounded by $O\left(B \cdot \sum_{i=1}^k \log a_i\right)$

Proof. Let g_i and $g_{i,j}$ be as in 3. and 4. in the statement of the theorem. The algorithm $\hat{\mathbb{A}}$ will perform recursion over k : If $k = 1$, then we just output

$$\varphi_1(g_1)^{-1} \cdot \phi(g_1) = a_1.$$

If $k > 1$, we consider the element $g_{1,k}$ and observe that for all $g \in G$ we have

$$\sum_{i=1}^k a_i \cdot \varphi_i(g_{1,k} * g) - \varphi_k(g_{1,k}) \cdot \sum_{i=1}^k a_i \cdot \varphi_i(g) = \sum_{i=1}^{k-1} a_i \cdot (\varphi_i(g_{1,k}) - \varphi_k(g_{1,k})) \cdot \varphi_i(g). \quad (33)$$

Now set $\hat{a}_i := a_i \cdot (\varphi_i(g_{1,k}) - \varphi_k(g_{1,k}))$ for all $0 < i < k$, and observe that (33) enables us to use our oracle to simulate an oracle for

$$g \mapsto \sum_{i=1}^{k-1} \hat{a}_i \cdot \varphi_i(g).$$

By recursion, we can thus obtain the value \hat{a}_1 . Since $\varphi_1(g_{1,k}) \neq \varphi_k(g_{1,k})$, we are able to compute $a_1 = \hat{a}_1 \cdot (\varphi_1(g_{1,k}) - \varphi_k(g_{1,k}))^{-1}$. Knowing a_1 , we can use our oracle to simulate an oracle for

$$g \mapsto \sum_{i=1}^k a_i \cdot \varphi_i(g) - a_1 \cdot \varphi_1(g) = \sum_{i=2}^k a_i \cdot \varphi_i(g).$$

Thus we can go into recursion again and compute a_2, \dots, a_k .

Note that all oracle queries only depend on the φ_i , and the same holds true for the recursion depth, and thus the number of arithmetic operations. This yields the desired running time; we emphasize that we need the factor of $\sum_{i=1}^k \log a_i$ in the running time to perform the arithmetic operations. \square

B Classifications for Unbounded Outdegrees

In this final section we quickly describe how the classifications for counting homomorphisms, subgraphs and induced subgraphs are easy consequences of the works of Dalmau and Jonsson [25], and of Curticapean, Dell and Marx [23], using our interpolation method via Dedekind's Theorem.

Recall that, given a class of digraphs \vec{C} , the problems $\#\text{DIRHOM}(\vec{C})$, $\#\text{DIRSUB}_d(\vec{C})$, and $\#\text{DIRINDSUB}(\vec{C})$ ask, respectively, given as input a pair $\vec{H} \in \vec{C}$ and \vec{G} , to compute $\#\text{Hom}(\vec{H} \rightarrow \vec{C})$, $\#\text{Sub}(\vec{H} \rightarrow \vec{C})$, and $\#\text{IndSub}(\vec{H} \rightarrow \vec{C})$. The crucial difference to the problems considered so far is that the parameter is just $|\vec{H}|$, rather than $|\vec{H}| + d(\vec{G})$, that is, we do not assume anymore that the host digraph has small outdegree.

Dalmau and Jonsson [25] proved their classification for counting homomorphisms not only for graphs, but in the more general setting of bounded arity relational structures. Since digraphs are precisely the relational structures over the signature containing one binary relation symbol, we can apply their main result and obtain:

Theorem 87. *Let \vec{C} be a recursively enumerable class of digraphs and assume ETH holds. Then $\#\text{DIRHOM}(\vec{C})$ is fixed-parameter tractable if and only if \vec{C} has bounded treewidth.*

Using Theorem 87 as the starting point, and relying on the transformation of subgraph and induced subgraph counts as a linear combination of homomorphism counts (see Lemma 29 and Lemma 32), we can mimic the proof of the classification theorems in the undirected setting due to Curticapean, Dell and Marx [23] almost verbatim. The only difference is the application of Dedekind's Theorem (Theorem 36) as an interpolation method for reducing the computation of a linear combination of directed homomorphism counts from the computation of its individual terms (Lemma 33). This yields the following results; for the first one, we define the vertex-cover number of a digraph as the vertex-cover number of its underlying undirected graph.

Theorem 88. *Let \vec{C} be a recursively enumerable class of digraphs and assume ETH holds. Then $\#\text{DIRSUB}(\vec{C})$ is fixed-parameter tractable if and only if \vec{C} has bounded vertex-cover number.*

Theorem 89. *Let \vec{C} be a recursively enumerable class of digraphs and assume ETH holds. Then $\#\text{DIRINDSUB}(\vec{C})$ is fixed-parameter tractable if and only if \vec{C} is finite.*