



On the mass of the world-sheet ‘axion’ in $SU(N)$ gauge theories in $3 + 1$ dimensions



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ABSTRACT

There is numerical evidence that the world sheet action of the confining flux tube in $D = 3 + 1$ $SU(N)$ gauge theories contains a massive excitation with 0^- quantum numbers whose mass shows some decrease as one goes from $SU(3)$ to $SU(5)$. Moreover it has been shown that the natural coupling of this pseudoscalar has a topological interpretation making it natural to call it the world-sheet ‘axion’. Recently it has been pointed out that if the mass of this ‘axion’ vanishes as $N \rightarrow \infty$ then it becomes possible for the world sheet theory to be integrable in the planar limit. In this paper we perform lattice calculations of this ‘axion’ mass from $SU(2)$ to $SU(12)$, which allows us to make a controlled extrapolation to $N = \infty$ and so test this interesting possibility. We find that the ‘axion’ does not in fact become massless as $N \rightarrow \infty$. So if the theory is to possess planar integrability then it must be some other world sheet excitation that becomes massless in the planar limit.

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1. Introduction

The spectrum and world sheet action of confining flux tubes in $D = 3 + 1$ $SU(N)$ gauge theories is now known to be, for the most part, remarkably simple. That is to say, the spectrum is very close to the spectrum of the light cone quantisation of the bosonic string theory [1], which is only consistent in $D = 26$ and $D = 3$, and which we shall refer to as the Nambu–Goto spectrum. (We restrict ourselves here to flux tubes that wind around one of the spatial tori, so that there is no Coulomb interaction, and no extra boundary terms to the world sheet action.) The remarkable simplicity obtained in lattice calculations (see [2] and references therein) is now well understood theoretically. Long flux tubes can be very accurately described by the established series of universal terms in the world sheet action [3] (see also [4]) while shorter flux tubes can be well understood from the near-integrability of the world-sheet theory in this limit [5–7]. The latter framework provides a powerful way to translate the observed energies into world sheet S-matrix elements and, where appropriate, into extra fields and in-

teractions in the world sheet action. In particular one ground state in [2] showed large deviations from the simple Nambu–Goto spectrum, in a way that suggested that it might consist of a massive pseudoscalar world-sheet particle on the background flux tube. The analysis in [6] using the formalism of [5] shows quite convincingly that this is indeed the case, and that the mass can be read off from the excitation energy above the absolute ground state to a good approximation. Moreover the natural coupling of this pseudoscalar has a topological interpretation [5,6], making it natural to call it the world-sheet ‘axion’. Recently [7] these authors have pointed out that the $N = \infty$ world-sheet theory might be integrable, but only if it possesses at least one massless mode in addition to the usual massless ‘phonons’ associated with the string’s spontaneous breaking of the bulk translation symmetry. They also noted that the observed decrease in the lattice estimates of the axion mass when one goes from $SU(3)$ to $SU(5)$ [2] raises the very interesting possibility that the axion mass might decrease to zero as $N \rightarrow \infty$ thus providing the extra massless mode needed for integrability. Locating a place for integrability in the planar limit of $SU(N)$ gauge theories has the potential to provide some analytic control over these theories and so is an exciting possibility. This motivates the present paper in which we perform lattice calculations of the axion mass for larger N , so as to see whether this mass vanishes or not in the planar limit.

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2. Lattice calculation

2.1. Lattice setup

The lattice calculations in this paper are essentially a direct continuation of the lattice calculations in [2]. Here we briefly recall that our lattice field variables are $SU(N)$ matrices, U_l , residing on the links l of a periodic $L_x \times L_y \times L_z \times L_t$ lattice, with lattice spacing a . The Euclidean path integral is

$$Z = \int \mathcal{D}U \exp\{-\beta S[U]\}, \quad (1)$$

where $\mathcal{D}U$ is the Haar measure and we use the standard plaquette action,

$$\beta S = \beta \sum_p \left\{ 1 - \frac{1}{N} \text{ReTr} U_p \right\} \quad ; \quad \beta = \frac{2N}{g^2}. \quad (2)$$

Here U_p is the ordered product of link matrices around the plaquette p . We write $\beta = 2N/g^2$, where g^2 becomes the continuum coupling when $a \rightarrow 0$. Monte Carlo calculations are performed using a standard Cabibbo–Marinari heat bath [8] plus over-relaxation algorithm.

2.2. Calculating flux tube energies

We calculate the energy of a flux tube that winds around the periodic space–time in the x direction and which has length $l = aL_x$. The details of the calculation are exactly the same as described at some length in [2]. Here it is useful to recall that such a flux tube has the following relevant quantum numbers: spin, J , around the axis; a parity, $P_{||}$, arising from $x \rightarrow -x$ reflections supplemented by charge conjugation; the $D = 2 + 1$ parity, P_{\perp} , arising from $(y, z) \rightarrow (-y, z)$. We set the momenta along and transverse to the flux tube to zero in this paper.

As described in detail in [2] our operators $\phi(t)$ are essentially Wilson lines that wind around the x -torus and are summed over y and z , as well as the starting point in x , so as to have zero momentum. These loops are decorated with various deviations from the direct path so as to allow us to produce operators $\phi_q(t)$ with various non-trivial quantum numbers q . We extract ground states from the asymptotic decay of the correlators of such operators, $\langle \phi_q^\dagger(t) \phi_q(0) \rangle \propto \exp\{-E_q(l)t\}$ as $t \rightarrow \infty$ where $E_q(l)$ is the ground state energy of the flux tube with the quantum numbers q . An important constraint is that the statistical error is roughly independent of t , so the exponentially decreasing ‘signal’ can rapidly disappear into the ‘noise’. That is to say we need $aE_q(l)$ to be small for a reliable calculation.

In this paper we shall focus on the quantum numbers $J^{P_{||}P_{\perp}} = 0^{++}$ and $J^{P_{||}P_{\perp}} = 0^{--}$. From the 0^{++} ground state we extract the string tension, while from the difference between the two ground state energies we can extract the ‘axion’ mass using

$$M_A \simeq E_{0^{--}}(l) - E_{0^{++}}(l) \quad (3)$$

as long as l is not too large. (If l is large, then the lightest 0^{--} state will be the one with massless phonons rather than with a massive axion [2].) This is, of course, an approximation but in practice it is a rather good one: using eqn. (3) with the spectra in [2] would give $M_A/\sqrt{\sigma} \sim 1.9 - 2.0$, while the correct value [5,6] is $M_A/\sqrt{\sigma} \sim 1.85(3)$.

2.3. Computational strategy

Ideally we would wish to repeat our calculation in [2] for a range of values of N that is large enough for us to have confidence

in our $N \rightarrow \infty$ extrapolation. Since the cost of pure gauge calculations increases roughly as $\propto N^3$, this would require substantially larger computer resources than employed in [2] where our main calculations were for $SU(3)$, with only some for $SU(5)$. So we shall follow a more limited strategy here which entails the risk of extra systematic errors, which we will need to address in detail later on in this paper.

We will perform calculations for $N \leq 12$ which should be an adequate range, given that the leading large- N correction should be $\propto 1/N^2$. However to reduce the computational cost we will work at a larger lattice spacing than in [2] since this means that we can use lattices that are smaller in lattice units. We will of course need to check (in Section 3) that the ensuing lattice spacing corrections are in fact negligible. Of course if a is larger, then so is the energy $aE(l)$ in lattice units, which means it will be harder to extract the energy reliably from the correlators, as discussed in Section 2.2. To compensate for this we perform the calculation for small values of l where the value of $aE(l)$ will be modest for those states where $E(l)$ decreases with decreasing l when l is small. This is the case for the absolute ground state but is in general not the case for excitations of this ground state because the ‘phonon’ momenta that usually provide the excitation of the flux tube increase roughly as $\propto 1/l$ as l decreases. Fortunately, the 0^{--} excited state we are interested in has no phonons and is a counterexample to this. So, as we shall see below, we are able to calculate the ground state energies of the 0^{++} and 0^{--} flux tubes with reasonable accuracy. (Although we will, unfortunately, not be able to calculate the energies of other excited states.) Of course we need to use a value of l where the extraction of the axion mass from the difference between the 0^{++} and 0^{--} flux tube energies is justified, and this we demonstrate in Section 3, where we shall also see that it is enough to perform calculations for a single value of l rather than a range of values. All these restrictions on our calculations will allow us to perform a reasonably accurate calculation even for this large range of N .

In addition to the above caveats there is a more general problem with lattice calculations at large N . This is the rapid loss of ergodicity as N increases in exploring fields with different topological charge Q . On a periodic lattice a change in Q requires a fluctuation that starts as a zero action gauge singularity around some hypercube, then under the action of the Monte Carlo update deforms into a dislocation, then grows into an instanton with a small core and then gradually grows into an instanton with a typically sized core. With such local Monte Carlo changes the instanton has to pass through a stage where it is small on physical length scales albeit not small on lattice scales. However at physically small scales one can do a semiclassical estimate and, as is well known [9], the probability of small instantons is exponentially suppressed in N . (Albeit with some caveats [10].) Thus the probability of changing Q in a Monte Carlo is also suppressed exponentially with N . This has long been known [11–13]. The question, then, is whether such a freezing of Q has damaging implications for the calculations in this paper. This is a particularly relevant question in our context because our ‘axion’ arises from a topological interaction in the world-sheet action which can be induced by a θ term in the bulk $SU(N)$ gauge theory [14]. This is the issue we address in detail in Section 4.

3. Results

We perform calculations in $SU(N)$ lattice gauge theories with $N = 2, 3, 4, 5, 6, 7, 8, 10, 12$. These calculations are performed at a lattice spacing that is (nearly) constant in physical units with $a\sqrt{\sigma} \simeq 0.300 \pm 0.001$, as we see from Table 1. We also keep the

Table 1

Ground state energies of flux tubes of length $l = aL_x$ with $J^{P_1 P_2} = 0^{++}$ and 0^{--} quantum numbers, and also the string tension. For lattices, groups and couplings shown.

Gauge Group	β	$L_x \times L_y \times L_z \times L_t$	$a\sqrt{\sigma}$	$aE_{0^{++}}$	$aE_{0^{--}}$
$SU(2)$	2.360	$8 \times 12 \times 12 \times 16$	0.29944(77)	0.5716(38)	1.290(14)
$SU(3)$	5.825	$8 \times 12 \times 12 \times 16$	0.29733(67)	0.5613(33)	1.157(25)
		$10 \times 12 \times 12 \times 16$	0.2987(13)	0.7806(81)	1.357(52)
$SU(4)$	10.70	$8 \times 12 \times 12 \times 16$	0.30007(75)	0.5747(37)	1.159(21)
$SU(5)$	17.00	$8 \times 12 \times 12 \times 16$	0.29951(57)	0.5720(28)	1.138(29)
$SU(6)$	24.71	$8 \times 12 \times 12 \times 16$	0.29809(72)	0.5650(35)	1.089(26)
$SU(7)$	33.825	$8 \times 12 \times 12 \times 16$	0.29935(63)	0.5712(31)	1.094(17)
$SU(8)$	44.355	$8 \times 12 \times 12 \times 16$	0.29915(98)	0.5702(48)	1.103(27)
		$8 \times 16 \times 16 \times 20$	0.30047(83)	0.5767(41)	1.080(23)
		$8 \times 18 \times 18 \times 24$	0.30007(89)	0.5747(44)	1.105(20)
		$8 \times 24 \times 24 \times 32$	0.30031(51)	0.5759(25)	1.090(21)
$SU(10)$	69.617	$8 \times 12 \times 12 \times 16$	0.30078(55)	0.5782(27)	1.121(27)
		$8 \times 16 \times 16 \times 20$	0.30037(77)	0.5762(38)	1.110(23)
		$8 \times 18 \times 18 \times 24$	0.30049(43)	0.5768(21)	1.097(11)
		$8 \times 24 \times 24 \times 32$	0.30086(77)	0.5786(38)	1.109(20)
$SU(12)$	100.50	$8 \times 12 \times 12 \times 16$	0.30090(67)	0.5788(33)	1.120(26)
		$8 \times 16 \times 16 \times 20$	0.30045(59)	0.5766(29)	1.088(23)
		$8 \times 18 \times 18 \times 24$	0.30082(53)	0.5784(26)	1.063(18)
		$8 \times 24 \times 24 \times 32$	0.30112(59)	0.5799(29)	1.109(27)

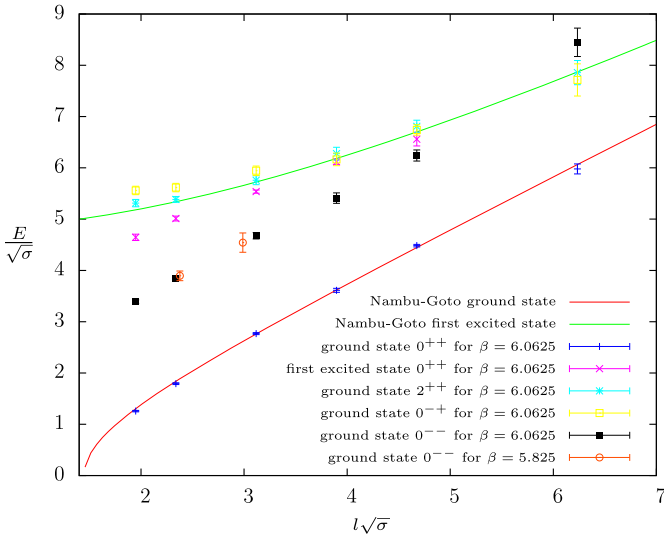


Fig. 1. Energies of the low-lying flux tube states with the quantum numbers shown, versus the length of the flux tube. Results for $SU(3)$ at $\beta = 6.0635$ taken from [2], compared to our new results for the $J^{P_1 P_2} = 0^{--}$ flux tube at $\beta = 5.825$ (○). (For interpretation of the references to colour in this figure, the reader is referred to the web version of this article.)

same the length $l = 8a$ of the winding flux tube, i.e. $l\sqrt{\sigma} \simeq 2.40$ in physical units.

We first need to check that using this relatively coarse a does not lead to significant lattice spacing corrections. To do that we focus on $SU(3)$, where we have an additional calculation using $l = 10a$, and we compare our results to those we obtained in [2] at a smaller value of a , corresponding to $a\sqrt{\sigma} \simeq 0.195$ (and which was itself checked in [2] against results obtained on an even finer lattice with $a\sqrt{\sigma} \simeq 0.129$). The comparison is displayed in Fig. 1 and we see that our values of the 0^{--} flux tube energy are perfectly consistent with those at the smaller lattice spacing. Given that the leading lattice spacing corrections should be $\propto a^2\sigma$, which decreases by a factor of ~ 2.4 between these two calculations, it

appears that any such corrections must be negligible in our calculation.

The second thing we learn from Fig. 1 is that our choice of $l\sqrt{\sigma} \simeq 2.40$ for the calculation of the energy gap $\Delta E(l) = E_{0^{--}}(l) - E_{0^{++}}(l)$ is in the range where $\Delta E(l)$ is approximately independent of l , and so provides a good estimate of the ‘axion’ mass using $M_A \simeq \Delta E(l)$.

We will assume that our above checks for $SU(3)$ carry over to our calculations at other values of N , which is certainly reasonable for $N > 3$.

A further check one needs to perform is whether our transverse lattice size is large enough. To check this we show in Fig. 2 how the ground state energies of the 0^{++} and 0^{--} flux tubes vary with the volume, using the values for $N = 8, 10, 12$ in Table 1. We see that there is no visible volume dependence, demonstrating that our initial choice of a $8 \times 12 \times 12 \times 16$ lattice is entirely adequate, at least for these larger values of N .

An important final issue that still needs to be addressed has to do with the rapid loss of topological ergodicity as one increases N . This will be dealt with in detail in Section 4 where we will argue that this poses no obstacle to our calculations.

We can now turn to our estimates of the ‘axion’ mass, M_A , using eqn. (3). This is shown as a function of N in Fig. 3, where we have averaged the values obtained on the four different volumes for $N = 8, 10, 12$. We fit these values with a linear function of $1/N^2$, which is the expected leading large- N correction [15], giving

$$\frac{M_A}{\sqrt{\sigma}} = 1.713(14) + \frac{2.74(7)}{N^2} \quad ; \quad \chi^2/n_{dof} = 1.12 \quad N \in [2, 12]. \quad (4)$$

As shown, the fit has an entirely acceptable χ^2 per degree of freedom, so we do not need to include any higher order terms in $1/N^2$. (No doubt we would need to do so if our calculations had much smaller statistical errors.) This fit is shown in Fig. 3, and while one sees that $M_A/\sqrt{\sigma}$ does indeed decrease with increasing N , it is equally clear that the ‘axion’ mass does not vanish as $N \rightarrow \infty$.

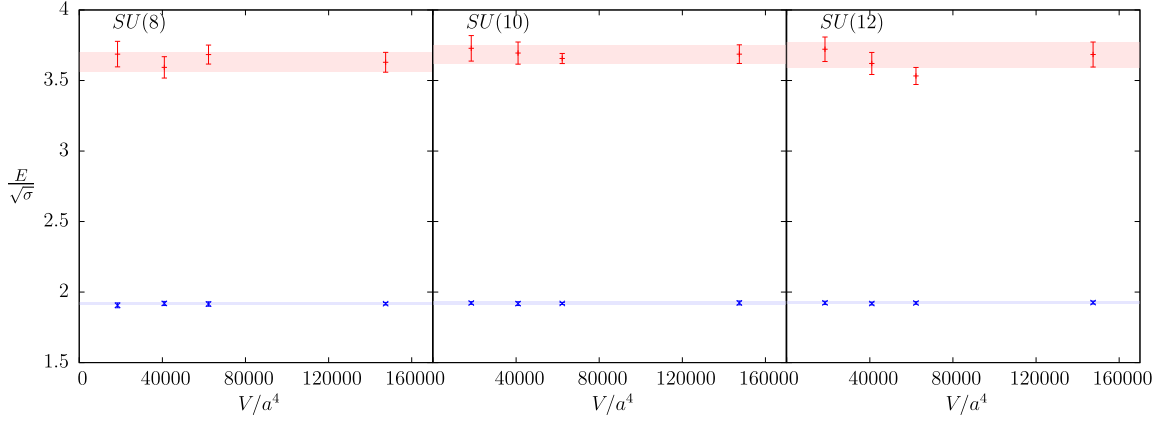


Fig. 2. The volume dependence of the energies of the $J^{P_u P_\perp} = 0^{++}$ (\times , blue) and 0^{--} ($+$, red) flux tube ground states. The bands correspond to the energy levels obtained with the largest volume i.e. $8 \times 24 \times 24 \times 32$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 2

Various energy scales associated with the bulk gauge theory, together with their large- N extrapolation. Glueball quantum number labels are J^{PC} .

Group	$M_{0^{++}}/\sqrt{\sigma} _{\beta_a}$	$M_{0^{++}}/\sqrt{\sigma} _{cont}$	$(M_{0^{-+}} - M_{0^{++}})/\sqrt{\sigma}$	$(M_{0^{++}} - M_{0^{++}})/\sqrt{\sigma}$
$SU(2)$	3.46(6)	3.78(7)		
$SU(3)$	2.97(6)	3.55(7)	2.40(10)	2.06(7)
$SU(4)$	2.82(6)	3.36(6)	2.22(9)	2.20(8)
$SU(5)$	2.81(7)	3.38(16)	2.15(10)	2.22(9)
$SU(6)$	2.67(7)	3.25(9)	2.26(10)	2.18(8)
$SU(8)$	2.68(8)	3.55(12)	2.14(10)	2.16(10)
$SU(\infty)$	2.62(5)	3.29(6)	2.11(9)	2.23(8)

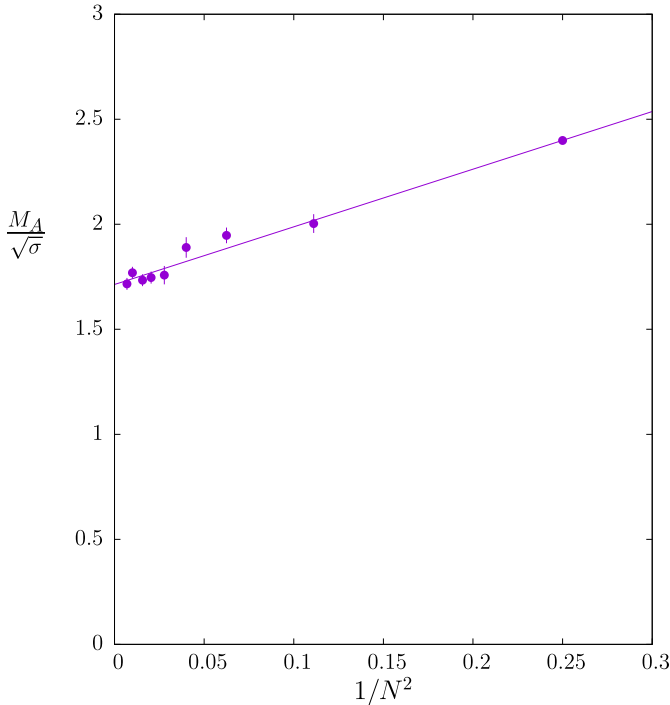


Fig. 3. The ‘axion’ mass, using eqn. (3), in units of the string tension, versus $1/N^2$ together with a linear extrapolation in $1/N^2$ to $N = \infty$.

As an aside, it is interesting to compare our above values for M_A to other characteristic energy scales in $SU(N)$ gauge theories, particularly as $N \rightarrow \infty$. We do this in Table 2. Note that our labelling of the quantum numbers of a glueball state is the conventional J^{PC} . The first column gives estimates of the mass of

the lightest scalar glueball, $M_{0^{++}}$, in units of the string tension as calculated at the same values $\beta = \beta_a$ as our ‘axion’ mass. (The scalar mass is the mass gap and is from [16] for $N = 2, 3, 4, 6, 8$ and from [11] for $N = 5$. In some cases we need to interpolate from neighbouring β values.) The second column in Table 2 lists the continuum values of the scalar mass [16,11]. In a naive constituent model for the lightest scalar glueball, half its mass provides a rough estimate for the mass of a ‘constituent’ gluon (rather like half the ρ mass providing a rough estimate for the mass of a ‘constituent’ quark). The comparison with the ‘axion’ mass is complicated by the large lattice spacing corrections to the mass gap visible at $\beta = \beta_a$ in Table 2 while, as we have seen in Fig. 1, the ‘axion’ mass does not show significant lattice corrections. If we compare M_A to half the continuum mass gap we see that the values are quite similar at $N = \infty$, but much less so if we use the mass gap at $\beta = \beta_a$. In the third column of Table 2 we list the value of the gap between the lightest scalar and pseudoscalar glueballs.¹ (The masses are from [17], and have been obtained at values of β that are slightly larger than our β_a values. We expect that the gap would increase slightly if one were to extrapolate to β_a .) In a model for glueballs in which they are composed of closed loops of (fundamental) flux, the pseudoscalar could be due to the excitation of an ‘axion’ mode on the flux loop of the scalar glueball, so that $M_{0^{-+}} - M_{0^{++}} \simeq M_A$. Of course a 0^{-+} can also arise from the naive oscillations and rotations of the flux tube. However in a simple model [18] one finds in the case of $SU(2)$ (see Fig. 3b of [19]) that if one chooses the short distance cut-off parameter f of the model [19] at a natural value of $f \sim 1$ then while that reproduces the $M_{0^{-+}}$ and $M_{0^{++}}$ masses it is at the cost of losing the observed splitting between the 2^{++} and 2^{-+} . If we want to split the latter states then we require a larger value of f that leads to a mass

¹ We thank the referee for encouraging us to look at this quantity.

for the 0^{-+} that is too large. So in this latter case the idea that the relatively light observed 0^{-+} is in fact the lightest 0^{++} with an ‘axion’ excitation is not implausible in principle. Although the $N \rightarrow \infty$ value of the splitting listed in Table 2 is somewhat larger than the value of M_A , given the rough approximations involved, it is surely consistent with this general possibility. Finally in the fifth column we list the gap between the ground and first excited scalar glueballs. (Again from [17].) This is again a typical low-lying excitation energy of the bulk theory, but one which has no obvious link to the world-sheet ‘axion’. The fact that it is very similar to the gap between the 0^{-+} and the 0^{++} suggests that the most that one can claim firmly from these comparisons is that the ‘axion’ mass is similar to typical excitation energies of the low-lying glueball spectrum.

4. Topology and non-ergodicity

As we explained earlier the usual local Monte Carlo algorithms rapidly lose the ability to change the topological charge Q of a lattice gauge field as N is increased. This suppression is related to the suppression of small instantons. If ρ is the size of the instanton and $\lambda(\rho) = g^2(\rho)N$ is the ‘t Hooft coupling on the scale of ρ , then the density of instantons is $D(\rho) \propto \exp\{-8\pi^2 N/\lambda(\rho)\}$ once ρ is small enough for this semi-classical calculation to be accurate [9]. (There are significant qualifications [10] that we do not enter into here.) Now, a change in Q is accompanied by an instanton growing from $\rho \sim a$ to $\rho \sim 1/\Lambda$ where $\Lambda \sim \sqrt{\sigma}$ is the physical length scale of the gauge theory. (Or the reverse process.) So this change will be suppressed by some factor $\propto \exp\{-8\pi^2 N/\lambda(\rho)\}$ with $\rho \sim O(a)$. Since $\lambda(\rho \sim a)$ decreases as a decreases, one way to delay (in N) this suppression is to work at a value of a that is not very small. In fact this is what we have done in this paper, in part for other reasons. However this only delays the onset of the problem, and eventually one needs to confront it.

An important point is that since our theory has a non-zero mass gap, the value of an observable will only depend on the topological fluctuations within a finite neighbourhood of that observable, i.e. within a distance $\sim 1/\Lambda$. So the effect on an observable of the total topological charge Q being frozen at some constant value can be made arbitrarily small by making the space-time volume sufficiently large. So a direct method to check whether an observable is affected by this freezing is simply to calculate it on a range of ever larger volumes to see if it changes. This is in fact the main reason that we calculated the flux tube energies on four different volumes for our largest three values of N in Table 1. The smallest volume of the four is the ‘standard’ volume we use for $2 \leq N \leq 7$. In physical units the space-time volume orthogonal to the flux tube is already a substantial $\sim \{3.6/\sqrt{\sigma}\}^2 \times 4.8/\sqrt{\sigma}$. The largest volume is a much larger $\sim \{7.2/\sqrt{\sigma}\}^2 \times 9.6/\sqrt{\sigma}$. Nonetheless as we see in Fig. 2 there is no sign of any change in the relevant flux tube energies as we increase the volume, strongly suggesting that the topological freezing is already unimportant on our ‘standard’ volume.

While such direct tests are the most convincing, it is interesting to see what is actually happening to the topological charge at these larger values of N . Since the method of calculation is fairly standard, we only briefly summarise it. We recall that the topological charge Q is the integral over Euclidean space-time of a topological charge density, $Q(x)$, which can be expressed in terms of the field strengths as $32\pi^2 Q(x) = \epsilon_{\mu\nu\rho\sigma} \text{Tr}\{F_{\mu\nu}(x)F_{\rho\sigma}(x)\}$. If on the lattice we replace $F_{ij}(x)$ by the plaquette $U_{ij}(x)$ then we obtain a lattice topological charge density $a^4 Q_L(x)$ such that $Q_L(x) \rightarrow Q(x)$ for smooth fields. At finite β this lattice measure receives both additive and multiplicative renormalisation, which can be removed

Table 3

Fluctuations of the total topological charge, Q , and the integral of the absolute value of the topological charge density, $|Q(x)|$, on various lattice volumes for the $N = 8, 10, 12$.

Gauge Group	Lattice	$\langle Q_L^2 \rangle$	$\langle \sum_x Q_L(x) \rangle$
$SU(8)$	$8 \times 12 \times 12 \times 16$	4.16(96)	2.55(13)
	$8 \times 16 \times 16 \times 20$	4.08(72)	5.23(15)
	$8 \times 18 \times 18 \times 24$	9.88(1.74)	8.48(17)
	$8 \times 24 \times 24 \times 32$	27.4(4.0)	20.08(19)
$SU(10)$	$8 \times 12 \times 12 \times 16$	1.09(21)	2.22(6)
	$8 \times 16 \times 16 \times 20$	2.61(73)	5.43(13)
	$8 \times 18 \times 18 \times 24$	7.86(1.96)	8.21(17)
	$8 \times 24 \times 24 \times 32$	7.66(1.10)	19.76(17)
$SU(12)$	$8 \times 12 \times 12 \times 16$	0.07(4)	2.09(6)
	$8 \times 16 \times 16 \times 20$	0.35(11)	5.22(12)
	$8 \times 18 \times 18 \times 24$	0.47(13)	8.23(10)
	$8 \times 24 \times 24 \times 32$	0.48(10)	19.47(15)

by smoothening the lattice fields in various ways. We shall employ ‘cooling’ [20] which consists of performing ‘Monte Carlo’-type sweeps with the difference that one locally minimises the action. Under this process Q will be quasi-stable, since instantons are minima of the continuum action. $Q(x)$ on the other hand will be gradually deformed as one performs more cooling sweeps. (For example, neighbouring instantons and anti-instantons can reduce their action by gradually annihilating each other.) We refer to [11, 13, 20] for much more detailed discussions.

In our calculation we perform 40 cooling sweeps on a sample of 40 or 80 lattice fields that should be mutually independent for standard physical observables. From experience we expect that this amount of cooling will leave the total charge Q unaffected, except for the possible disappearance of very small instantons. But at larger N there will be almost none of these. (That is, after all, why Q freezes in the Monte Carlo ensemble.) Each of these lattice fields was the last of a separate Monte Carlo sequence used for the calculation of the flux tube energies, and each started from a near-frozen starting lattice field with $Q = 0$. So if the ergodicity in Q were to be seriously suppressed, then we would expect to find the fluctuations of Q around $Q = 0$ to be suppressed. We list in Table 3 the values of $\langle Q^2 \rangle$ that we obtain. We recall that Q is the difference between the number of instantons, n_+ , and the number of anti-instantons, n_- , and since the correlation length is finite, and since $\langle n_+ \rangle = \langle n_- \rangle$, we will have $\langle Q^2 \rangle \propto V$ once the space-time volume V is large enough. Indeed in the dilute gas approximation one can readily show that $\langle Q^2 \rangle = \langle n_+ \rangle + \langle n_- \rangle = 2\langle n_+ \rangle$. So it is usual to define the topological susceptibility $\chi_t = \langle Q^2 \rangle/V$, and to express it in physical units, e.g. $\chi_t^{1/4}/\sqrt{\sigma}$. If we take our

largest volume in Table 3 for $SU(8)$ we find $\chi_t^{1/4}/\sqrt{\sigma} \simeq 0.389(14)$ which is entirely consistent with the values at smaller N listed in Table 11 of [11] and Table 2 of [13]. That is to say, there is no visible suppression in the fluctuations of Q implying that we still have adequate ergodicity in Q in $SU(8)$ at $\beta = 44.355$. This is in fact no surprise since we chose this value of $a(\beta)$ for our calculations in the expectation, based on earlier calculations, that this would indeed be the case for $SU(8)$. More interesting is $SU(10)$ and $SU(12)$. Since $a\sqrt{\sigma}$ is the same as for $SU(8)$, we would expect that values of $\langle Q^2 \rangle$ to be very similar for $N = 8, 10, 12$ if there continues to be adequate ergodicity in Q . However what we observe in Table 3 is a dramatic suppression of $\langle Q^2 \rangle$ as we increase N to $N = 10$ and then $N = 12$. Clearly Q is indeed freezing for our highest values of N , despite our rather coarse value of $a(\beta)$.

As an important aside we remind the reader that while changing Q is necessarily associated with fields that contain small instantons (as described above) and so will be suppressed at large N ,

one can produce an instanton anti-instanton pair as a normal unsuppressed long-distance fluctuation. As one updates the field the separation between such pairs can grow so that even if $Q = 0$ for the total volume V , we expect that in any sufficiently large subvolume \tilde{V} with $V \gg \tilde{V}$ the integrated topological charge $\tilde{Q} = \sum_{x \in \tilde{V}} Q(x)$ will have a restricted susceptibility $\langle \tilde{Q}^2 \rangle / \tilde{V}$ that equals the true susceptibility $\langle Q^2 \rangle / V$, when the latter is calculated with ‘infinite’ statistics so as to overcome any partial non-ergodicity in Q .

So, as remarked earlier, the relevant question here is whether the topological fluctuations in a subvolume that is large enough to contain the flux tube physics of interest, are significantly suppressed by the observed freezing of Q on the total volume for $N = 10, 12$. Answering this question directly is certainly possible but would require calculations that take us beyond the scope of this paper. However we do perform a step in that direction. This is provided by our calculation of $Q_{abs} \equiv \sum_x |Q(x)|$, whose average values are listed in Table 3. In the dilute gas approximation one can easily show that $\langle Q_{abs} \rangle = \langle (n_+ + n_-) \rangle = \langle Q^2 \rangle$. Of course the cooling will tend to decrease Q_{abs} because of the gradual annihilation of nearby (anti)instanton pairs, and in any case in reality the ‘gas’ is surely not dilute. Nonetheless the approximate equality in Table 3 between $\langle Q_{abs} \rangle$ and $\langle Q^2 \rangle$ for $SU(8)$, suggests that this argument has an approximate validity, and that $\langle Q_{abs} \rangle$ does give us a measure of the local density of topological fluctuations. If we now compare to the values of $\langle Q_{abs} \rangle$ listed in Table 3 for $SU(10)$ and $SU(12)$, we see that they are almost exactly the same. We take this as some evidence that the observed onset of a serious non-ergodicity in Q at our largest values of N will not have a significant impact on the topological fluctuations in the relevant subvolume \tilde{V} as long as $\tilde{V} \ll V$. For our largest volume, $V \simeq \{2.4/\sqrt{\sigma}\} \times \{7.2/\sqrt{\sigma}\}^2 \times 9.6/\sqrt{\sigma}$, this inequality is presumably well satisfied.

5. Conclusions

In this paper we provided a calculation of the world-sheet ‘axion’ mass in $D = 3 + 1$ $SU(N)$ gauge theories, using the difference between the energies of the lightest $J^{P_1 P_2} = 0^{++}$ and 0^{--} flux tubes to estimate that mass. Although our calculations were at a fixed and rather coarse value of the lattice spacing, comparisons with earlier $SU(3)$ calculations reassured us that the estimates are reliable. Our calculations covered a much larger range of N than before, so as to allow a convincing large- N extrapolation. This required us to address the known rapid loss of lattice Monte Carlo ergodicity in the topological charge, when N becomes large. We addressed this directly by performing our calculations on a range of space-time volumes, and also by calculating the topological charge density, all of which strongly suggests that the clearly visible loss of this ergodicity does not have a significant impact on our results.

Our unambiguous conclusion is that the world sheet ‘axion’ has a finite non-zero mass at $N = \infty$, and that its mass is similar to typical excitation energies of the low-lying glueball spectrum. (Such as the gap between the lightest scalar and pseudoscalar, or the lightest two scalars, or indeed the ‘constituent’ gluon mass defined as half the mass gap.) That is to say, it cannot play the role of the extra massless world-sheet mode that is needed if the $N = \infty$ world sheet theory is to be integrable [7]. It needs to be stressed however that the currently available flux tube calculations [2] are incomplete and do not provide accurate calculations of all quantum numbers. Such calculations will require a basis of flux tube operators that is more extensive than that employed in [2] and until such a calculation is completed the possibility of some extra massless world-sheet modes at $N = \infty$ certainly cannot be excluded.

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References

- [1] P. Goddard, J. Goldstone, C. Rebbi, C. Thorn, Quantum dynamics of a massless relativistic string, Nucl. Phys. B 56 (1973) 109; J. Arvis, The exact Q anti- Q potential in Nambu string theory, Phys. Lett. B 127 (1983) 106.
- [2] A. Athenodorou, B. Bringoltz, M. Teper, Closed flux tubes and their string description in $D=3+1$ $SU(N)$ gauge theories, JHEP 1102 (2011) 030, arXiv:1007.4720.
- [3] O. Aharony, Z. Komargodski, The effective theory of long strings, JHEP 1305 (2013) 118, arXiv:1302.6257; O. Aharony, E. Karzbrun, On the effective action of confining strings, JHEP 0906 (2009) 012, arXiv:0903.1927; S. Dubovsky, R. Flauger, V. Gorbenko, Effective string theory revisited, JHEP 1209 (2012) 044, arXiv:1203.1054.
- [4] M. Lüscher, Peter Weisz, String excitation energies in $SU(N)$ gauge theories beyond the free-string approximation, JHEP 0407 (2004) 014, arXiv:hep-th/0406205; J.M. Drummond, Universal subleading spectrum of effective string theory, arXiv:hep-th/0411017; M. Lüscher, Symmetry breaking aspects of the roughening transition in gauge theories, Nucl. Phys. B 180 (1981) 317; M. Lüscher, K. Symanzik, P. Weisz, Anomalies of the free loop wave equation in the WKB approximation, Nucl. Phys. B 173 (1980) 365.
- [5] S. Dubovsky, R. Flauger, V. Gorbenko, Flux tube spectra from approximate integrability at low energies, J. Exp. Theor. Phys. 120 (3) (2015) 399, arXiv:1404.0037; S. Dubovsky, R. Flauger, V. Gorbenko, Solving the simplest theory of quantum gravity, JHEP 1209 (2012) 133, arXiv:1205.6805.
- [6] S. Dubovsky, R. Flauger, V. Gorbenko, Evidence for a new particle on the world-sheet of the QCD flux tube, Phys. Rev. Lett. 111 (6) (2013) 062006, arXiv:1301.2325.
- [7] S. Dubovsky, V. Gorbenko, Towards a theory of the QCD string, JHEP 1602 (2016) 022, arXiv:1511.01908; P. Cooper, S. Dubovsky, V. Gorbenko, A. Mohsen, S. Storace, Looking for integrability on the worldsheet of confining strings, JHEP 1504 (2015) 127, arXiv:1411.0703.
- [8] N. Cabibbo, E. Marinari, A new method for updating $SU(N)$ matrices in computer simulations of gauge theories, Phys. Lett. B 119 (1982) 387.
- [9] E. Witten, Instantons, the quark model, and the $1/N$ expansion, Nucl. Phys. B 149 (1979) 285.
- [10] M. Teper, Instantons and the $1/N$ expansion, Z. Phys. C 5 (1980) 233.
- [11] B. Lucini, M. Teper, $SU(N)$ gauge theories in four dimensions: exploring the approach to $N = \infty$, JHEP 0106 (2001) 050, arXiv:hep-lat/0103027.
- [12] L. Del Debbio, H. Panagopoulos, P. Rossi, E. Vicari, Spectrum of confining strings in $SU(N)$ gauge theories, JHEP 0201 (2002) 009, arXiv:hep-th/0111090.
- [13] B. Lucini, M. Teper, U. Wenger, Topology of $SU(N)$ gauge theories at $T=0$ and $T=T_c$, Nucl. Phys. B 715 (2005) 461, arXiv:hep-lat/0401028.
- [14] P. Mazur, V. Nair, Strings in QCD and theta vacua, Nucl. Phys. B 284 (1987) 146.
- [15] G. 't Hooft, A planar diagram theory for strong interactions, Nucl. Phys. B 72 (1974) 461.
- [16] B. Lucini, M. Teper, U. Wenger, Glueballs and k-strings in $SU(N)$ gauge theories: calculations with improved operators, JHEP 0406 (2004) 012, arXiv:hep-lat/0404008.
- [17] B. Lucini, A. Rago, E. Rinaldi, Glueball masses in the large N limit, JHEP 1008 (2010) 119, arXiv:1007.3879.

- [18] N. Isgur, J. Paton, A flux tube model for hadrons in QCD, *Phys. Rev. D* 31 (1985) 2910.
- [19] T. Moretto, M. Teper, Glueball spectra of SU(2) gauge theories in three-dimensions and four-dimensions: a comparison with the Isgur–Paton flux tube model, [arXiv:hep-lat/9312035](https://arxiv.org/abs/hep-lat/9312035).
- [20] M. Teper, Instantons in the quantized SU(2) vacuum: a lattice Monte Carlo investigation, *Phys. Lett. B* 162 (1985) 357;
- M. Teper, The topological susceptibility in SU(2) lattice gauge theory: an exploratory study, *Phys. Lett. B* 171 (1986) 86;
- D. Smith, M. Teper, Topological structure of the SU(3) vacuum, *Phys. Rev. D* 58 (1998) 014505, [arXiv:hep-lat/9801008](https://arxiv.org/abs/hep-lat/9801008).