Generating Functional Analysis of a Model Economy with Heterogeneous Adaptive Consumers



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Trinity Term 2005

Thesis submitted for the degree of
Master of Science in Mathematical Modelling & Scientific Computing

Acknowledgements

I would like to thank my supervisor Dr Damien Challet and Dr Isaac Pérez Castillo not only for their encouragement and guidance during this project, but also for sharing with me their enthusiasm for the field of minority games. I am grateful to the 'Studienstiftung des deutschen Volkes', Booz Allen Hamilton and EPSRC for funding.
Special thanks go to my family for their support in every respect.

Abstract

This thesis examines the dynamics of a market with heterogeneous and adaptive consumers who make their purchase decisions repeatedly. The main mathematical tool applied is the generating functional formalism. This tool allows for forming an average over the randomly assigned strategies of the consumers while simultaneously retaining all macroscopic information. The focus lies on the analysis of the stationary state of the ergodic region.

First, the problem and the appropriate model is described, the generating functional analysis is then applied and finally the analytical results are interpreted and compared to numerical computer simulations.

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Chapter 1

Introduction

> *"Economics is changing currently from an emphasis on equilibrium and homogeneity to an emphasis on the formation of pattern and heterogeneity."*
> *W. Brian Arthur (taken from [1])*

This thesis describes a market with a large number of consumers who try to optimise their individual levels of satisfaction by purchasing preferable commodity bundles from sellers. 'Satisfaction' can have several characteristics, depending on the market examined, for instance the price or queue time. The microscopic objects of the market, the consumers, have different preferences according to their commodity choice, in other words, they are heterogeneous. As sophisticated persons, consumers are able to learn from *past* experiences and try to use this information for their *future* advantage.

The aim of this thesis is to describe the market by finding expressions of important macroscopic observables, such as the volatility or the consumers' sensitivity to perturbations. The mathematical tool applied to accomplish these calculations is the generating functional formalism; originating from theoretical physics. Without confining the range of applications, let the example of a daily vegetable market be our leading example throughout this thesis. This demonstrates the practicability of the methods used and eases understanding of the model and the conclusions.

The thesis starts with a brief historical outline and general introduction into the

novel research area of minority game.
In the second chapter, the problem situation is described and an appropriate model is given. The analysis is then fulfilled by applying the generating functional formalism. Exact calculations of macroscopic observables from a microscopic dynamic market system require us to carry out averages over the paths and the heterogeneity (which corresponds to the market disorder) of the consumers. For the sake of generality, we consider a general probability distribution to accomplish the disorder average. After calculating the correlation and response functions, we simplify our market dynamics to an effective single-consumer process. In the stationary state persistent order parameters, an approximation of the volatility and a measure for the efficiency of the market are obtained.

In the third chapter, the model results are applied to different probability distributions. Graphical visualisations, numerical computer simulations and interpretations conclude this chapter.

Often used mathematical techniques and identities are explained and proven in appendix A and all detailed calculations can be found in appendix B. Lengthy calculations are not introduced into the main text.

1.1 Minority Game and its Historical Development

There are many situations in everyday life in which people compete for limited resources. Car drivers during rush hour, agents in financial markets or consumers of certain products all have something in common: Every competitor tries to obtain a favourable position. Since certainly individual goals will be conflicting, it is hardly possible to find a globally efficient situation in which every individual goal is fulfilled.

Economic methods such as general equilibrium theory, often applied in similar contexts, require people to act strategically, logically and deductively. This kind of rational behaviour seems unrealistic for the examples from above, since beyond a certain complexity of information it is impossible to cope with information and therefore an individually optimal strategy cannot be found. This necessitates a different methodical approach to study those kind of problems. Minority game,

which recently has attracted lots of academic interest, offers an adequate alternative.

W. Brian Arthur's 1994 paper [2] gave the initial impulse for minority game theory. He considered a problem observed in the El Farol bar in Santa Fe: On Thursday evenings the bar plays Irish music but this is only enjoyable if less than 60 guests are present. Every week, each of the for instance 120 potential guests has to decide independently whether to attend or not to attend the event. If more than 60 people turn out to be at the pub it would have been better for them to stay at home, and vice versa.

The decision-making process of the people in the *El Farol Bar Problem* is characterised by limited rationale and lack of information; it is based solely on the numbers of attendance in the past. Additionally, the customers may learn from their experiences. With the invention of the minority game by Challet and Zhang [3] in 1997, this problem can be modelled satisfactorily for the first time.

Let us briefly describe the properties of the standard minority game. We assume that each of N (assumed to be odd) so-called agents has two possible choices A and B. After the decision of each agent was taken independently from the others' choices, the agents who find themselves in the minority make profit, whereas those in the majority reduce their level of satisfaction. In the first time step a random action is chosen, in the following repetitions each agent considers the past outcomes, called history, while making a decision, using a strategy from a predefined set of strategies. A strategy is a function on the set of history configurations and returns a choice. The strategy is chosen according to certain virtual success-points which are allocated to each strategy after each time step. If two or more strategies have the same highest number of virtual points, then one strategy out of these is chosen randomly. The history contains M entries, where each entry represents the choice (A or B) leading to the minority in this time step. For M binary entries there are 2^M possible configurations. Hence, for two choices there are 2^{2^M} possible strategies, which is a very high number, even for small M. Since no one can handle 2^{2^M} strategies[1], each agent is randomly given a moderate number of S strategies.

After this brief introduction to minority games, let us turn our focus to a model

[1] For example $M = 8$ yields the immense number $2^{2^M} = 1.15792 \cdot 10^{77}$!

economy with heterogeneous adaptive consumers.

Chapter 2

Economy with Heterogeneous Adaptive Consumers

Frequently, decisions on what goods to buy are made. For many goods like clothing or luxury articles individual preferences, cultural imprints, cost-performance ratios or vogues are the main influences in the individual decision-making process. For comestible goods, especially perishable agricultural goods like fresh vegetables, the purchase decision follows in general a different pattern. Vegetables are largely independent of temporary fashion or price fluctuations. Furthermore, we assume that only a very little amount of the consumer's income is spent on vegetables and hence small price differences of different vegetables are negligible. Individual preferences will result in different equally favoured recipes which necessitate certain commodities. Hence, vegetables are chosen differently to luxury articles. Here a reasonable influence in the decision-making process is for instance the queue time. Since vegetables are supposed to be eaten freshly, one usually purchases vegetables several times a week and therefore it is more important to save as much time as possible to buy these goods.

Let us concretise these ideas for a vegetable market, where heterogeneous adaptive consumers make their purchase decision daily in the next section.

Papers in the field of minority games, published in physical journals, focus mainly on financial problems. A good introduction for applications of minority games in finance and the most important papers in this area are given in the book by Challet et. al. (2005) [1]. Our situation, as a classical problem in economics,

exhibits many similarities to certain (but not all!) financial problems studied by means of minority games. In both cases the agents and consumers respectively want to be in a 'minority'. In the case of a vegetable market the consumers would like to find only a few people buying at the same seller, in the case of a financial market it is profitable for traders to be in the minority since not all traders can make profit. Despite numerous publications in the area of finance, those focussing on general economic problems are very limited. First ideas for a model economy with heterogeneous adaptive customers are presented in the recent work by De Martino and Marsili (2005) [4]. Building on a similar modelling approach as presented in [4] we will extend the analysis of the model by considering a general probability distribution rather than a particular one.

2.1 The Problem

We consider N heterogeneous consumers. 'Heterogeneous' means that each consumer has different preferences with respect to the available commodities, i. e. one consumer for instance does not like salad, the other one prefers certain vegetable combinations. The heterogeneity results in different possible shopping lists, the bundles of the consumers. It is assumed that all bundles of one consumer are equivalent for that consumer. Put another way: The consumer prefers no possible shopping list to another one. Each day, N consumers buy on a market with P stalls. For simplicity let us assume that each stall sells exactly one kind of vegetable and no two stands sell the same commodity. That means the consumers can choose from a variety of P possible vegetables, which are assumed to have the same price. The consumers try to minimise their waiting time and therefore choose the shopping-bundle which generated the smallest aggregated queue time in the past[1], i. e. they learn from experience. Each customer chooses therefore first a most promising bundle (out of all possible bundles) and afterwards it turns

[1] A spontaneous decision for the stand with the shortest queue (independent of past experiences) is unrealistic for two reasons: First, it is difficult to overlook the whole market, especially for a large number of sellers, i. e. it is necessary to spend a lot of unmeant time in searching for the shortest queue. Secondly, for practical reasons it is reasonable to assume that the recipe and therefore the choice of vegetables are usually determined in advance, since all ingredients have to be available on time.

out if the bundle was good (no or short queues) or not (long queues). All consumers are assumed to have the same perception for 'long queue' or 'short queue'. *A posteriori* the advantageous situation is spotted. Under the assumption that each market stand needs the same time to serve one customer, the waiting time is proportional to the demand at the market stall.

Having described the situation in detail, let us turn to the modelling process of the problem.

2.2 Modelling

As described in the introduction, it is practically very complex to consider all possible bundles for each consumer. Furthermore, in reference to the heterogeneity of the consumer it is reasonable to consider only certain shopping lists. Hence, we randomly assign S bundles, denoted by $\mathbf{q}_{i,g} = (q_{i,g}^{\mu})_{\mu=1,\dots,P}$ with $q_{i,g}^{\mu} \in \mathbb{R}_0^+$, to each consumer and keep them fixed for all repetitions of the market interaction game. $\mathbf{q}_{i,g}$ is therefore a fixed random vector. The index $i = 1, \dots, N$ describes the consumers, $g = 1, \dots, S$ labels the different bundles and $\mu = 1, \dots, P$ denotes the sellers. In the analysis and calculations of macroscopic observables (e. g. the volatility) of our model, we will first assume the case of a general probability distribution $P(\mathbf{q}_{i,g})$ with finite variance. We will see that, as an extension to De Martino and Marsili (2005) [4], the exact solution can be obtained without specifying the distribution, even for the general case, and only the first two moments of the distribution suffice.

The normalised demand for seller μ at time t and therefore a measure for the queue time is described by

$$D^{\mu}(t) = \frac{1}{N} \sum_{i=1}^{N} q_{i,g_i(t)}^{\mu}.$$

Each customer $i = 1, \dots, N$ chooses each day $t \geq 0$ the most promising bundle, denoted by $g_i(t)$. To establish what is meant by a 'promising bundle', let us define the *utility* $U_{i,g}(t)$. The utility $U_{i,g}(t)$ states the satisfaction of customer i at time t, when choosing bundle g. The *learning dynamics* are given by

$$U_{i,g}(t+1) = U_{i,g}(t) - \sum_{\mu=1}^{P} q_{i,g}^{\mu} [D^{\mu}(t) - k], \tag{2.1}$$

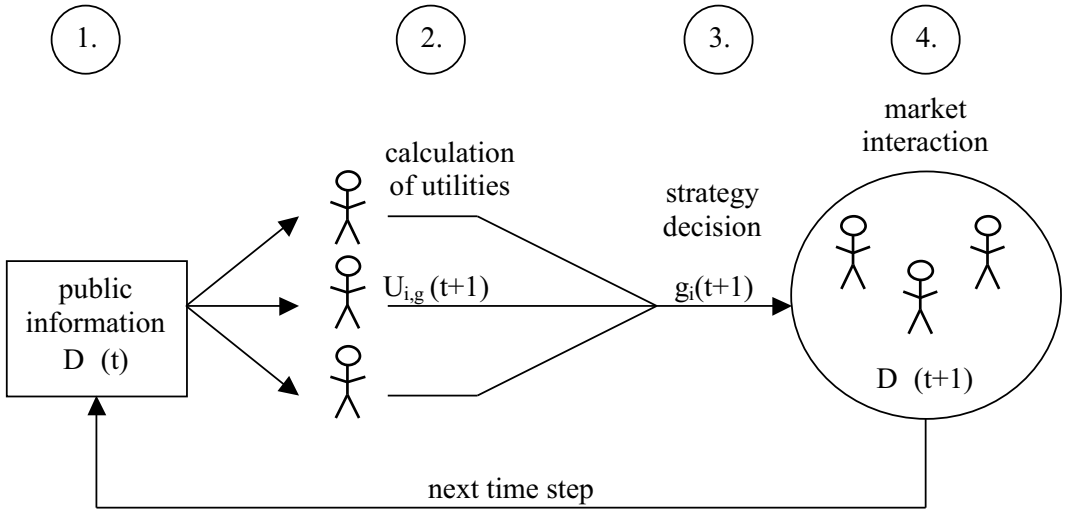

Figure 2.1: Model dynamics. The consumers $i = 1, \ldots, N$ calculate the utility (see equation (2.1)) for time step $t + 1$ for each of their strategies $g = 1, \ldots, S$ with the help of the public information $D^{\mu}(t)$ (demand for seller $\mu = 1, \ldots, P$ at time step t) from the previous time step. Having calculated the utility for each strategy, the strategy $g_i(t + 1)$ with the highest expected utility is chosen. The market interaction then results in certain demands $D^{\mu}(t + 1)$ which are available as public information in the next time step $t + 2$, etc.

for $t \geq 0$. If the normalised demand $D^{\mu}(t)$ is higher than a threshold k, which for simplicity is assumed to be equal for all customers, then the utility for bundle g decreases. Conversely, for a normalised demand lower than k, the queue time is acceptable and the utility for the associated bundle increases. At time $t = 0$ we choose randomly the initial utility $U_{i,g}(0)$. The most promising bundle $g_i(t)$ at time t for customer i can therefore be calculated by

$$g_i(t) = \arg\max_g U_{i,g}(t).$$

This completes the model for the consumer-seller market interaction.

In some aspects our model is very similar to the standard minority game by Challet and Zhang (1997) [3] described in the introduction. More precisely, it is equivalent to the simpler so-called *batch version* of the minority game, in which the success-points in the standard minority game are not re-evaluated at every time step but after a large number of market decisions.

2.3 Analysis

Our model describes the microscopic dynamics of the problem, consisting of NS utility equations per time step. Especially for large N or S, the system becomes hard to analyse. Furthermore, it is not the microscopic behaviour of the system which is of interest, but some important macroscopic observables, such as the amplitude of fluctuations. Analysis of macroscopic observables enables us to predict future behaviour of the market, i. e. it is of prime economic importance. Due to the similarity of our model to the classical minority game, presented in Challet and Zhang (1997) [3], which was analysed in many papers like Coolen and Heimel (2001) [5], the application of similar physical methods to our model is suggested (similar to De Martino and Marsili (2005) in [4]). The underlying mathematical theory is well described in the book by Coolen (2005) [6].

2.3.1 Techniques in Statistical Mechanics

The technique of the generating functional formalism allows us to examine the model dynamics analytically for an infinitely large market system. Such an infinitely large system behaves independently of the realisation of the disorder,

i. e. for different assigned sets of strategies and we say the system is *self-averaging*. The generating functional formalism was introduced by Martin et. al. in 1973 [7]. In 1978, De Dominicis [8] was the first to apply this formalism to systems with fixed disorder, like for instance spin glasses. Another physical or biological area of application is a system of neurons. Neurons interact with each other as well as consumers: In the case of neurons the interaction takes place by dendrites and synapses in form of electrical impulses, whereas the consumers' interaction is indirectly determined by the length of the queue. Further analysis of the connection between artificial neurons and interacting agents is also given in Heimel (2001) [9]. However, formally both systems exhibit strong similarities and hence the same mathematical method, viz. the generating functional formalism, can be applied to study the dynamic systems analytically. Usual tricks in this formalism are the introduction of identities in form of the integrated Dirac delta function, application of the steepest descent integration (see A.1) and Gaussian integrals (see A.3). Further information can be found in appendix A.

2.3.2 Definitions

Let us apply the generating functional analysis to the consumer market in detail. A short sketch of a similar problem can also be found in De Martino and Marsili (2005) [4]. It was shown in 1998 by Challet and Zhang [10] that for all numbers of strategies $S > 1$ the qualitative behaviour of the fluctuations stays almost the same. Let us therefore consider the simplest case $S = 2$. Especially interesting results are obtained for the case $P = \alpha N$, which we will analyse in the following. Note that a consumer limit $N \to \infty$, used throughout this thesis, therefore also implies a seller limit $P \to \infty$.

For technical reasons let us introduce auxiliary variables for $i = 1, \ldots, N$ and $t \geq 0$ of the form:

$$p_i(t) = U_{i,1}(t) - U_{i,2}(t),\ \boldsymbol{\omega}_i = \frac{\mathbf{q}_{i,1} + \mathbf{q}_{i,2}}{2},\ \boldsymbol{\xi}_i = \frac{\mathbf{q}_{i,1} - \mathbf{q}_{i,2}}{2},\ s_i(t) = \operatorname{sgn}[p_i(t)],$$

where $p_i(t)$ is called the *preference* of agent i at time t, since $p_i(t) > 0$ in the case of preferred strategy 1 and $p_i(t) < 0$ for preferred strategy 2. In the case that $p_i(t) = 0$ both strategies are perceived as equally appropriate. Since the sign of $p_i(t)$ is of importance to calculate the preferred strategy choice each time step, we defined the microscopic dynamic variable $s_i(t)$.

The actually selected strategy at time t, viz. $\mathbf{q}_{i,g_i(t)} = (q^1_{i,g_i(t)}, q^2_{i,g_i(t)}, \ldots, q^P_{i,g_i(t)})$ can be written explicitly as a function of $s_i(t)$:

$$\mathbf{q}_{i,g_i(t)} = \boldsymbol{\omega}_i + s_i(t)\boldsymbol{\xi}_i.$$

The time evolution of the preferences $p_i(t)$ is the following:

$$\begin{aligned} p_i(t+1) - p_i(t) &= U_{i,1}(t+1) - U_{i,1}(t) - U_{i,2}(t+1) + U_{i,2}(t) \\ &= -2\sum_{\mu=1}^{P} \xi_i^\mu \left(D^\mu(t) - k\right) \\ &= -2\sum_{\mu=1}^{P} \xi_i^\mu \left[\frac{1}{N}\sum_{j=1}^{N}\left(\omega_j^\mu + s_j(t)\xi_j^\mu\right) - k\right]. \end{aligned}$$

The system is now characterised by N Markovian equations with fixed disorder. The physical expression 'fixed disorder' (also 'quenched disorder') describes in our model the initially randomly assigned S strategies of the consumers, which are kept fixed in the repetitions of the game.

In the following we will examine the time evolution with an introduced auxiliary field $\theta_i(t)$ which probes the behaviour of the system under perturbations:

$$p_i(t+1) - p_i(t) = -2\sum_{\mu=1}^{P} \xi_i^\mu \left[\frac{1}{N}\sum_{j=1}^{N}\left(\omega_j^\mu + s_j(t)\xi_j^\mu\right) - k\right] + \theta_i(t). \tag{2.2}$$

2.3.3 The Generating Functional

The Markovian process (2.2) can be translated into the language of microscopic state probabilities $\rho_t(\mathbf{p})$, where $\mathbf{p} = (p_1(t), \ldots, p_N(t))$ denotes the preference vector at time t for all consumers $i = 1, \ldots, N$. The resulting equations involve a positive microscopic transition probability density, denoted by $W(\mathbf{p}|\mathbf{p}')$. It gives the probability of finding the microscopic state vector $\mathbf{p}$ at time step t, given that at the previous iteration step $t-1$ the state vector took the value $\mathbf{p}' = (p_1(t-1), \ldots, p_N(t-1))$. The Chapman-Kolmogorov equation for calculating $\rho_{t+1}(\mathbf{p})$ is given by:

$$\rho_{t+1}(\mathbf{p}) = \int W(\mathbf{p}|\mathbf{p}')\rho_t(\mathbf{p}')d\mathbf{p}'. \tag{2.3}$$

Note that unless otherwise stated, in this thesis we use integrals over the whole real axis.

The transition probability density is given by

$$\begin{aligned} W(\mathbf{p}|\mathbf{p}') &= \prod_{i=1}^{N} \delta \left[p_i - p_i' + 2 \sum_{\mu=1}^{P} \xi_i^\mu \left[\frac{1}{N} \sum_{j=1}^{N} \left(\omega_j^\mu + s_j' \xi_j^\mu \right) - k \right] - \theta_i' \right] \\ &= \int \exp \left[\mathrm{i} \sum_{i=1}^{N} \hat{p}_i \left(p_i - p_i' + 2 \sum_{\mu=1}^{P} \xi_i^\mu \left(\frac{1}{N} \sum_{j=1}^{N} \left(\omega_j^\mu + s_j' \xi_j^\mu \right) - k \right) \right. \right. \\ &\quad \left. \left. - \theta_i' \right) \right] \prod_{i=1}^{N} \frac{d\hat{p}_i}{2\pi}, \end{aligned} \tag{2.4}$$

with $s_j' = \mathrm{sgn}[p_j']$ and by using the integral representation of Dirac's delta $\delta(x) = \frac{1}{2\pi} \int \exp(\mathrm{i}\hat{x}x)\, d\hat{x}$. A conjugate variable, introduced in Dirac's delta integral representation is denoted by a hat. Note that for the sake of readability we use, where appropriate, only one integral sign $\int$ instead of multiple $\int \cdots \int$.

For the process (2.3) the probability of finding a certain path $\mathbf{p}(0) \to \mathbf{p}(1) \to \ldots \to \mathbf{p}(t)$ is given by the product of particular transition probability densities:

$$\mathrm{Prob}\left(\mathbf{p}(0), \mathbf{p}(1), \ldots, \mathbf{p}(t_{max})\right) = \rho_0(\mathbf{p}(0)) \prod_{t=0}^{t_{max}-1} W(\mathbf{p}(t+1)|\mathbf{p}(t)). \tag{2.5}$$

We are now in a position to define the *moment-generating functional* for our stochastic process, which is the Fourier transform of the probability distribution of the paths (cf. for instance page 70, 71 Coolen (2005) [6]). As we will see later, all necessary macroscopic information can be calculated with the help of the generating functional.

The definition of the moment-generating functional $Z[\boldsymbol{\psi}]$ is given by:

$$Z[\boldsymbol{\psi}] = \left\langle \exp \left(\mathrm{i} \sum_{t=0}^{t_{max}} \sum_{i=1}^{N} \psi_i(t) s_i(t) \right) \right\rangle, \tag{2.6}$$

where $\boldsymbol{\psi}$ denotes a set of generating fields $\{\psi_i\}$ and $\langle \cdot \rangle$ gives the average over all possible paths. By using the path probabilities (2.5) and the transition probabil-

ity density (2.4) we obtain

$$\begin{aligned} Z[\boldsymbol{\psi}] &= \int \rho_0(\mathbf{p}(0)) \prod_{t=0}^{t_{max}-1} [W(\mathbf{p}(t+1)|\mathbf{p}(t))] \\ &\quad \times \exp\left(\mathrm{i} \sum_{t=0}^{t_{max}} \sum_{i=1}^{N} \psi_i(t) s_i(t) \right) \prod_{t=0}^{t_{max}} \prod_{i=1}^{N} dp_i(t) \\ &= \int \exp\left[\mathrm{i} \sum_{t=0}^{t_{max}-1} \sum_{i=1}^{N} \hat{p}_i(t) \left(p_i(t+1) - p_i(t) \right.\right. \\ &\quad \left.\left. + 2 \sum_{\mu=1}^{P} \xi_i^\mu \left(\frac{1}{N} \sum_{j=1}^{N} \left(\omega_j^\mu + s_j(t) \xi_j^\mu \right) - k \right) - \theta_i(t) \right) \right] \mathcal{D}\hat{\mathbf{p}} \\ &\quad \times \rho_0(\mathbf{p}(0)) \exp\left(\mathrm{i} \sum_{t=0}^{t_{max}} \sum_{i=1}^{N} \psi_i(t) s_i(t) \right) \mathcal{D}\mathbf{p} \prod_{i=1}^{N} dp_i(t_{max}), \end{aligned}$$

where $\mathcal{D}\mathbf{p} = \prod_{t=0}^{t_{max}-1} \prod_{i=1}^{N} \frac{dp_i(t)}{\sqrt{2\pi}}$, $\mathcal{D}\hat{\mathbf{p}} = \prod_{t=0}^{t_{max}-1} \prod_{i=1}^{N} \frac{d\hat{p}_i(t)}{\sqrt{2\pi}}$. As the name implies, the moment-generating functional can generate all averages, correlations and responses of the random variables $\{s_i(t)\}$ at arbitrary times by taking derivatives with respect to the generating variables $\{\psi_i(t)\}$ and/or with respect to the perturbation variables $\{\theta_i(t)\}$. For instance we can get:

$$\langle s_i(t) \rangle = -\mathrm{i} \lim_{\boldsymbol{\psi} \to \mathbf{0}} \frac{\partial Z[\boldsymbol{\psi}]}{\partial \psi_i(t)}, \tag{2.7}$$

$$\langle s_i(t) s_i(t') \rangle = - \lim_{\boldsymbol{\psi} \to \mathbf{0}} \frac{\partial^2 Z[\boldsymbol{\psi}]}{\partial \psi_i(t) \partial \psi_i(t')}, \tag{2.8}$$

$$\frac{\partial \langle s_i(t) \rangle}{\partial \theta_i(t')} = -\mathrm{i} \lim_{\boldsymbol{\psi} \to \mathbf{0}} \frac{\partial^2 Z[\boldsymbol{\psi}]}{\partial \psi_i(t) \partial \theta_i(t')}. \tag{2.9}$$

We will see that the correlation between the choices of consumer i at times t and t', given by (2.8), and the response of consumer i at time t to a perturbation at time t', given by (2.9), is of prime importance. By taking the disorder average, being defined by $\overline{[\cdot]} := \int \prod_{i,g,\mu} P(q_{i,g}^\mu)[\cdot] \prod_{i,g,\mu} dq_{i,g}^\mu$, which is an average over the strategies $\mathbf{q}_{i,g}$ for all i, g, and taking the consumer limit $N \to \infty$, we obtain from equation (2.8) and (2.9) the macroscopic correlation and response functions $C(t,t')$ and $G(t,t')$ respectively:

$$C(t,t') = - \lim_{N \to \infty} \lim_{\boldsymbol{\psi} \to \mathbf{0}} \frac{1}{N} \sum_{i=1}^{N} \frac{\partial^2 \overline{Z}[\boldsymbol{\psi}]}{\partial \psi_i(t) \partial \psi_i(t')}, \tag{2.10}$$

$$G(t,t') = -\mathrm{i} \lim_{N \to \infty} \lim_{\boldsymbol{\psi} \to \mathbf{0}} \frac{1}{N} \sum_{i=1}^{N} \frac{\partial^2 \overline{Z}[\boldsymbol{\psi}]}{\partial \psi_i(t) \partial \theta_i(t')}. \tag{2.11}$$

Mathematically, $G(t,t')$ is called response function since it is for $N \to \infty$ the response of $\frac{1}{N}\sum_{i=1}^{N}\overline{\langle s_i(t)\rangle}$ to an infinitesimally small perturbation of the external field $\theta_i(t')$ at time t'.

To calculate the correlation and response functions, our next step will be to calculate the disorder average of the generating functional. To do so, it is beneficial to decouple the disorder variables $\{\omega_i^\mu, \xi_i^\mu\}$ by introducing the following identities in the form of delta functions

$$1 = \int \delta\left[w^\mu(t) - \sqrt{\frac{2}{N}}\sum_{i=1}^{N}\hat{p}_i(t)\xi_i^\mu\right] dw^\mu(t) \;\; \forall\mu,t,$$
$$1 = \int \delta\left[x^\mu(t) - \sqrt{\frac{2}{N}}\sum_{i=1}^{N}s_i(t)\xi_i^\mu\right] dx^\mu(t) \;\; \forall\mu,t.$$

Then the relevant term of the generating functional, i. e.

$$\exp\left[2\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{i=1}^{N}\hat{p}_i(t)\sum_{\mu=1}^{P}\xi_i^\mu\left(\frac{1}{N}\sum_{j=1}^{N}\left(\omega_j^\mu + s_j(t)\xi_j^\mu\right) - k\right)\right]$$

takes the decoupled form (cf. detailed calculations in appendix B)

$$\int \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\hat{w}^\mu(t)w^\mu(t) + \hat{x}^\mu(t)x^\mu(t) + x^\mu(t)w^\mu(t) - k\sqrt{2N}w^\mu(t)\right)\right]$$
$$\times \exp\left[\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\left(\sum_{j=1}^{N}\omega_j^\mu\right)w^\mu(t) - \sum_{i=1}^{N}\hat{w}^\mu(t)\hat{p}_i(t)\xi_i^\mu\right.\right.$$
$$\left.\left. - \sum_{i=1}^{N}\hat{x}^\mu(t)s_i(t)\xi_i^\mu\right)\right]\mathcal{D}\mathbf{w}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\hat{\mathbf{x}},$$

with shorthand notation $\mathcal{D}\mathbf{w} = \prod_{t=0}^{t_{max}-1}\prod_{\mu=1}^{P}\frac{dw^\mu(t)}{\sqrt{2\pi}}$, $\mathcal{D}\mathbf{x} = \prod_{t=0}^{t_{max}-1}\prod_{\mu=1}^{P}\frac{dx^\mu(t)}{\sqrt{2\pi}}$ for w^μ and x^μ and $\mathcal{D}\hat{\mathbf{w}} = \prod_{t=0}^{t_{max}-1}\prod_{\mu=1}^{P}\frac{d\hat{w}^\mu(t)}{\sqrt{2\pi}}$, $\mathcal{D}\hat{\mathbf{x}} = \prod_{t=0}^{t_{max}-1}\prod_{\mu=1}^{P}\frac{d\hat{x}^\mu(t)}{\sqrt{2\pi}}$ for the conjugate variables (introduced by the integral representation of Dirac's delta functions) $\hat{w}^\mu$ and $\hat{x}^\mu$.

2.3.4 The Disorder Average

The generating functional formalism allows for the macroscopic analysis of the consumer-seller market. Through macroscopic quantities like the volatility we

can find out more about the behaviour of the whole system rather than individual consumers. To obtain these macroscopic quantities from our microscopic dynamics we have to examine an average over the fixed disorder, which necessitates the knowledge of the strategy distribution $P(\mathbf{q}_{i,g})$. We will in the following carry out the calculations for a general distribution and the results can finally be obtained for different probability distributions without having to calculate all quite complex and time-intensive steps again.

The disorder average over the generating functional only has to be taken over the terms involving $q^{\mu}_{i,1}$ and $q^{\mu}_{i,2}$, i. e.

$$\begin{aligned}
\overline{Z}[\boldsymbol{\psi}] \;=\; & \int \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{i=1}^{N}\hat{p}_i(t)\left(p_i(t+1)-p_i(t)-\theta_i(t)\right)\right] \\
& \left[\int \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\hat{w}^{\mu}(t)w^{\mu}(t)+\hat{x}^{\mu}(t)x^{\mu}(t)+x^{\mu}(t)w^{\mu}(t)-k\sqrt{2N}w^{\mu}(t)\right)\right]\right. \\
& \times\exp\left[\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\left(\sum_{j=1}^{N}\omega_j^{\mu}\right)w^{\mu}(t)-\sum_{i=1}^{N}\hat{w}^{\mu}(t)\hat{p}_i(t)\xi_i^{\mu}\right.\right. \\
& \left.\left.\overline{\left.-\sum_{i=1}^{N}\hat{x}^{\mu}(t)s_i(t)\xi_i^{\mu}\right)\right]}\,\mathcal{D}\mathbf{w}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\hat{\mathbf{x}}\right]\mathcal{D}\hat{\mathbf{p}} \\
& \times\rho_0(\mathbf{p}(0))\exp\left(\mathrm{i}\sum_{t=0}^{t_{max}}\sum_{i=1}^{N}\psi_i(t)s_i(t)\right)\mathcal{D}\mathbf{p}\prod_{i=1}^{N}dp_i(t_{max}). \qquad (2.12)
\end{aligned}$$

Hence, let us consider the overlined part of the generating functional: (for detailed calculations see appendix B)

$$\begin{aligned}
& \overline{\exp\left[\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\left(\sum_{j=1}^{N}\omega_j^{\mu}\right)w^{\mu}(t)-\sum_{i=1}^{N}\hat{w}^{\mu}(t)\hat{p}_i(t)\xi_i^{\mu}-\sum_{i=1}^{N}\hat{x}^{\mu}(t)s_i(t)\xi_i^{\mu}\right)\right]} \\
= \;& \prod_{\mu=1}^{P}\prod_{i=1}^{N}\left[\int P(q_{i,1}^{\mu})\left[\exp\left(\frac{\mathrm{i}q_{i,1}^{\mu}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left(w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right)\right)\right]dq_{i,1}^{\mu}\right. \\
& \left.\int P(q_{i,2}^{\mu})\left[\exp\left(\frac{\mathrm{i}q_{i,2}^{\mu}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left(w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right)\right)\right]dq_{i,2}^{\mu}\right].
\end{aligned}$$

Using the series $\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ for $x\in\mathbb{R}$ and the definition for the first and second moment of the probability distribution $m_1=\int P(q_{i,g}^{\mu})q_{i,g}^{\mu}dq_{i,g}^{\mu}$ and

$m_2 = \int P(q^\mu_{i,g})(q^\mu_{i,g})^2 dq^\mu_{i,g}$ respectively gives

$$\prod_{\mu=1}^{P}\prod_{i=1}^{N}\Bigg[\Bigg(1+\frac{\mathrm{i}\,m_1}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^\mu(t)-\hat{w}^\mu(t)\hat{p}_i(t)-\hat{x}^\mu(t)s_i(t)\right]$$
$$-\frac{m_2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^\mu(t)-\hat{w}^\mu(t)\hat{p}_i(t)-\hat{x}^\mu(t)s_i(t)\right]\left[w^\mu(t')-\hat{w}^\mu(t')\hat{p}_i(t')\right.$$
$$\left.-\hat{x}^\mu(t')s_i(t')\right]\Bigg)$$
$$\times\Bigg(1+\frac{\mathrm{i}\,m_1}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^\mu(t)+\hat{w}^\mu(t)\hat{p}_i(t)+\hat{x}^\mu(t)s_i(t)\right]$$
$$-\frac{m_2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^\mu(t)+\hat{w}^\mu(t)\hat{p}_i(t)+\hat{x}^\mu(t)s_i(t)\right]\left[w^\mu(t')+\hat{w}^\mu(t')\hat{p}_i(t')\right.$$
$$\left.+\hat{x}^\mu(t')s_i(t')\right]\Bigg)\Bigg]$$

for $N\to\infty$. Note that it is sufficient to consider only the first terms of the exponential series, since all other terms vanish in the consumer limit $N\to\infty$. In the following we will use the series $\log(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$ for $|x|\le 1$ and the identity $1+x=\exp(\log(1+x))$. Let us henceforth label the variance as follows: $\Delta_2 := m_2 - m_1^2$. For the sake of clarity let us introduce the following abbreviations:

$$C(t,t')=\frac{1}{N}\sum_{i=1}^{N}s_i(t)s_i(t'),\ K(t,t')=\frac{1}{N}\sum_{i=1}^{N}s_i(t)\hat{p}_i(t'),\ L(t,t')=\frac{1}{N}\sum_{i=1}^{N}\hat{p}_i(t)\hat{p}_i(t').$$

Note that we again use the notation $C(t,t')$, since this function will turn out to be equal to the macroscopic correlation function, defined in (2.10).

However, using these definitions, the relevant part of the disorder average takes in the limit $N\to\infty$ the form

$$\exp\left(\frac{\mathrm{i}\sqrt{2}m_1}{\sqrt{N}}\sum_{\mu=1}^{P}\sum_{i=1}^{N}\sum_{t=0}^{t_{max}-1}w^\mu(t)\right)$$
$$\times\exp\Bigg[-\frac{m_2-m_1^2}{2}\sum_{\mu=1}^{P}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left(w^\mu(t)w^\mu(t')+\hat{w}^\mu(t)L(t,t')\hat{w}^\mu(t')\right.$$
$$\left.+2\hat{w}^\mu(t')K(t,t')\hat{x}^\mu(t)+\hat{x}^\mu(t)C(t,t')\hat{x}^\mu(t')\right)\Bigg].$$

We can isolate the two-time functions $C(t,t')$, $K(t,t')$ and $L(t,t')$ via the insertion of appropriate Dirac's delta functions. For simplicity let us define the shorthand notation $\mathcal{D}C := \prod_{t=0}^{t_{max}-1}\prod_{t'=0}^{t_{max}-1}\frac{dC(t,t')}{\sqrt{2\pi}}$, $\mathcal{D}K := \prod_{t=0}^{t_{max}-1}\prod_{t'=0}^{t_{max}-1}\frac{dK(t,t')}{\sqrt{2\pi}}$ and $\mathcal{D}L := \prod_{t=0}^{t_{max}-1}\prod_{t'=0}^{t_{max}-1}\frac{dL(t,t')}{\sqrt{2\pi}}$ and similar definitions containing the generated conjugate integration variables $\mathcal{D}\hat{C}$, $\mathcal{D}\hat{K}$ and $\mathcal{D}\hat{L}$.

Appropriate delta function identities, which can be obtained from the integral representation of the delta function and a change of variables, are of the following form:

$$N^{t_{max}^2}\int \exp\left[\mathrm{i}N\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\hat{C}(t,t')\left(C(t,t')-\frac{1}{N}\sum_{i=1}^{N}s_i(t)s_i(t')\right)\right]\mathcal{D}C\mathcal{D}\hat{C}=1, \tag{2.13}$$

$$N^{t_{max}^2}\int \exp\left[\mathrm{i}N\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\hat{K}(t,t')\left(K(t,t')-\frac{1}{N}\sum_{i=1}^{N}s_i(t)\hat{p}_i(t')\right)\right]\mathcal{D}K\mathcal{D}\hat{K}=1, \tag{2.14}$$

$$N^{t_{max}^2}\int \exp\left[\mathrm{i}N\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\hat{L}(t,t')\left(L(t,t')-\frac{1}{N}\sum_{i=1}^{N}\hat{p}_i(t)\hat{p}_i(t')\right)\right]\mathcal{D}L\mathcal{D}\hat{L}=1. \tag{2.15}$$

Note that C, K and L have become integration variables.

In the following we will use the value $k = m_1$ for the threshold-parameter k. This choice of k is reasonable, since many consumers will perceive the mean of the probability distribution as acceptable, whereas a demand higher than that seems not to be approveable any more. Mathematically, for a large number of randomly assigned strategies S the consumers will after some time automatically pick those strategies which have together an effective average of m_1, i. e. $\langle q_{i,g}^{\mu}\rangle_{used} = m_1$. That means $k = m_1$ is a natural choice. In the calculations the terms $\exp\left(\frac{\mathrm{i}\sqrt{2}m_1}{\sqrt{N}}\sum_{\mu=1}^{P}\sum_{i=1}^{N}\sum_{t=0}^{t_{max}-1}w^{\mu}(t)\right)$ and $\exp\left(-\mathrm{i}k\sqrt{2N}\sum_{\mu=1}^{P}\sum_{t=0}^{t_{max}-1}w^{\mu}(t)\right)$ will cancel. Note that in principle it is possible to accomplish the calculations for more general cases of k and m_1, as long as one of the following conditions is fulfilled:

- $k = m_1$
- $k \sim \frac{1}{\sqrt{N}}$ and $m_1 \sim \frac{1}{\sqrt{N}}$

- $k = m_1 + \frac{a}{\sqrt{N}}$, where $a \in \mathbb{R}$.

The mean m_1 and the parameter k cannot be completely general, since the term $\exp\left(\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}} \sum_{\mu=1}^{P} \sum_{i=1}^{N} \sum_{t=0}^{t_{max}-1} w^{\mu}(t)\right)$ would diverge for $N \to \infty$.

By substitution of the identities (2.13), (2.14) and (2.15) into $\overline{Z}[\boldsymbol{\psi}]$ we get:

$$\begin{aligned}
\overline{Z}[\boldsymbol{\psi}] &= \int \exp\left[\mathrm{i} \sum_{t=0}^{t_{max}-1} \sum_{i=1}^{N} \hat{p}_i(t)\,(p_i(t+1) - p_i(t) - \theta_i(t))\right] \\
&\quad \left[\int \exp\left[\mathrm{i} \sum_{t=0}^{t_{max}-1} \sum_{\mu=1}^{P} (\hat{w}^{\mu}(t) w^{\mu}(t) + \hat{x}^{\mu}(t) x^{\mu}(t) + x^{\mu}(t) w^{\mu}(t))\right]\right. \\
&\quad \times \exp\left[-\frac{\Delta_2}{2} \sum_{\mu=1}^{P} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} (w^{\mu}(t) w^{\mu}(t') + \hat{w}^{\mu}(t) L(t,t') \hat{w}^{\mu}(t')\right. \\
&\quad \left.\left. + 2\hat{w}^{\mu}(t') K(t,t') \hat{x}^{\mu}(t) + \hat{x}^{\mu}(t) C(t,t') \hat{x}^{\mu}(t'))\right] \mathcal{D}\mathbf{w} \mathcal{D}\mathbf{x} \mathcal{D}\hat{\mathbf{w}} \mathcal{D}\hat{\mathbf{x}}\right] \mathcal{D}\hat{\mathbf{p}} \\
&\quad \times \rho_0(\mathbf{p}(0)) \exp\left(\mathrm{i} \sum_{t=0}^{t_{max}} \sum_{i=1}^{N} \psi_i(t) s_i(t)\right) \mathcal{D}\mathbf{p} \prod_{i=1}^{N} dp_i(t_{max}) \\
&= \int \exp\left[N(\Psi + \Phi + \Omega) + \mathcal{O}(t_{max}^2 \log(N))\right] [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}],
\end{aligned}$$

where the following definitions were used:

$$\Psi := \mathrm{i} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} [\hat{C}(t,t') C(t,t') + \hat{L}(t,t') L(t,t') + \hat{K}(t,t') K(t,t')], \quad (2.16)$$

$$\begin{aligned}
\Phi &:= \alpha \log\left[\int \exp\left[\mathrm{i} \sum_{t=0}^{t_{max}-1} (\hat{w}(t) w(t) + \hat{x}(t) x(t) + x(t) w(t))\right]\right. \\
&\quad \times \exp\left[-\frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} (w(t) w(t') + \hat{w}(t) L(t,t') \hat{w}(t')\right. \qquad (2.17) \\
&\quad \left.\left. + 2\hat{w}(t') K(t,t') \hat{x}(t) + \hat{x}(t) C(t,t') \hat{x}(t'))\right] \mathcal{D}w \mathcal{D}x \mathcal{D}\hat{w} \mathcal{D}\hat{x}\right],
\end{aligned}$$

$$\begin{aligned}
\Omega &:= \frac{1}{N} \sum_{i=1}^{N} \log\left[\int \rho_0(p(0)) \exp\left[\mathrm{i} \sum_{t=0}^{t_{max}-1} \hat{p}(t)(p(t+1) - p(t) - \theta_i(t))\right]\right. \\
&\quad \times \exp\left[\mathrm{i} \sum_{t=0}^{t_{max}} \psi_i(t) s(t) - \mathrm{i} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \left(s(t) \hat{C}(t,t') s(t')\right.\right. \qquad (2.18) \\
&\quad \left.\left.\left. + s(t) \hat{K}(t,t') \hat{p}(t') + \hat{p}(t) \hat{L}(t,t') \hat{p}(t')\right)\right] \mathcal{D}p\, dp(t_{max}) \mathcal{D}\hat{p}\right],
\end{aligned}$$

with $s(t) := \mathrm{sgn}(p(t))$, $\mathcal{D}p := \prod_{t=0}^{t_{max}-1} \frac{dp(t)}{\sqrt{2\pi}}$, $\mathcal{D}w := \prod_{t=0}^{t_{max}-1} \frac{dw(t)}{\sqrt{2\pi}}$ and $\mathcal{D}x := \prod_{t=0}^{t_{max}-1} \frac{dx(t)}{\sqrt{2\pi}}$ and similar definitions for the conjugate integration variables $\mathcal{D}\hat{p}$, $\mathcal{D}\hat{w}$ and $\mathcal{D}\hat{x}$.

Hence we have found a term for $\overline{Z}[\boldsymbol{\psi}]$ in the limit $N \to \infty$ upon assuming an initial condition of the form $\rho_0(\mathbf{p}) = \prod_{i=1}^{N} \rho_0(p_i)$. Since the microscopic quantities have been integrated out, all quantities except the generating and perturbation fields $\{\psi_i(t)\}$ and $\{\theta_i(t)\}$ are macroscopic, i. e. independent of $i = 1, \ldots, N$.

After having accomplished the disorder average of the generating functional $\overline{Z}[\boldsymbol{\psi}]$, we now turn to the calculations of the macroscopic correlation $C(t,t')$ and response functions $G(t,t')$. Upon taking derivatives of $\overline{Z}[\boldsymbol{\psi}]$ with respect to the generating and perturbation fields $\{\psi_i(t)\}$ and $\{\theta_i(t)\}$ we find the following terms:

$$\begin{aligned}
\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial \overline{Z}[\boldsymbol{\psi}]}{\partial \psi_i(t)} &= \mathrm{i}\overline{\langle s_i(t)\rangle} \qquad (2.19)\\
&= \lim_{\boldsymbol{\psi}\to\mathbf{0}} \int \exp\left[N(\Psi+\Phi+\Omega) + \mathcal{O}(t_{max}^2 \log(N))\right]\\
&\quad \times N \frac{\partial \Omega}{\partial \psi_i(t)} [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}],\\
\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2 \overline{Z}[\boldsymbol{\psi}]}{\partial \psi_i(t)\partial \psi_j(t')} &= -\overline{\langle s_j(t') s_i(t)\rangle} \qquad (2.20)\\
&= \lim_{\boldsymbol{\psi}\to\mathbf{0}} \int \exp\left[N(\Psi+\Phi+\Omega) + \mathcal{O}(t_{max}^2 \log(N))\right]\\
&\quad \times \left(N^2 \frac{\partial \Omega}{\partial \psi_i(t)} \frac{\partial \Omega}{\partial \psi_j(t')} + N \frac{\partial^2 \Omega}{\partial \psi_i(t) \partial \psi_j(t')} \right) [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}],\\
\lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\partial^2 \overline{Z}[\boldsymbol{\psi}]}{\partial \psi_i(t)\partial \theta_j(t')} &= \mathrm{i}\frac{\partial \overline{\langle s_i(t)\rangle}}{\partial \theta_j(t')} \qquad (2.21)\\
&= \lim_{\boldsymbol{\psi}\to\mathbf{0}} \int \exp\left[N(\Psi+\Phi+\Omega) + \mathcal{O}(t_{max}^2 \log(N))\right]\\
&\quad \times \left(N^2 \frac{\partial \Omega}{\partial \psi_i(t)} \frac{\partial \Omega}{\partial \theta_j(t')} + N \frac{\partial^2 \Omega}{\partial \psi_i(t) \partial \theta_j(t')} \right) [\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}].
\end{aligned}$$

Since $N \to \infty$, we are able to apply the *steepest descent integration* to an integral of the form $\int \exp(Nf(x))\, g(x) dx$. For an exact derivation of this method and conditions for the functions $f(x)$ and $g(x)$ see appendix A.1. Steepest descent integration gives:

$$\lim_{N\to\infty} \frac{\int \exp(Nf(x)) g(x) dx}{\int \exp(Nf(x)) dx} = g(x_0),$$

where x_0 denotes the saddle point.

Therefore equation (2.19) becomes[2] in the consumer limit $N \to \infty$ by using $\overline{Z}[\mathbf{0}] = 1$:

$$\begin{aligned} & \lim_{\boldsymbol{\psi}\to\mathbf{0}} \int \exp\left[N(\Psi+\Phi+\Omega)+\mathcal{O}(t_{max}^2\log(N))\right] N\frac{\partial\Omega}{\partial\psi_i(t)}[\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}] \\ = & \lim_{\boldsymbol{\psi}\to\mathbf{0}} \frac{\int \exp\left[N(\Psi+\Phi+\Omega)+\mathcal{O}(t_{max}^2\log(N))\right] N\frac{\partial\Omega}{\partial\psi_i(t)}[\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]}{\int \exp\left[N(\Psi+\Phi+\Omega)+\mathcal{O}(t_{max}^2\log(N))\right][\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]} \\ = & N\frac{\partial\Omega_0}{\partial\psi_i(t)}, \end{aligned}$$

where the index '0' denotes the evaluation of the respective function at the saddle point. Calculating the derivative $\frac{\partial\Omega_0}{\partial\psi_i(t)}$ we get

$$\overline{\langle s_i(t)\rangle} = \langle s(t)\rangle_{*,i},$$

where the average over a function depending on p and $\hat{p}$ is given by

$$\langle f[\{q,\hat{q}\}]\rangle_{*,i} := \frac{\int M_i[\{p,\hat{p}\}]f[\{p,\hat{p}\}]\mathcal{D}p\,dp(t_{max})\mathcal{D}\hat{p}}{\int M_i[\{p,\hat{p}\}]\mathcal{D}p\,dp(t_{max})\mathcal{D}\hat{p}},$$

with the probability measure[3]

$$\begin{aligned} M_i[\{p,\hat{p}\}] := & \rho_0(p(0))\exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\hat{p}(t)(p(t+1)-p(t)-\theta_i(t))\right] \\ & \times\exp\left[-\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\Big(s(t)\hat{C}_0(t,t')s(t')+s(t)\hat{K}_0(t,t')\hat{p}(t') \right. \\ & \left. +\hat{p}(t)\hat{L}_0(t,t')\hat{p}(t')\Big)\right]. \end{aligned} \tag{2.22}$$

Note that we still have a dependence on 'i' in the definition of $\langle\cdot\rangle_{*,i}$ and $M_i[\{p,\hat{p}\}]$, since the function Ω contains the site-dependent perturbation fields $\{\theta_i(t)\}$.

[2]Since the steepest descent integration is only applicable for the consumer limit, we consider henceforth the limit $N\to\infty$. Therefore, in all equations $\lim_{N\to\infty}$ should be present, which we left out for the sake of clarity.

[3]Note that the limit of the generating field $\boldsymbol{\psi}\to\mathbf{0}$ was taken and therefore the probability measure $M_i[\{p,\hat{p}\}]$ is independent of $\{\psi_i(t)\}$.

Similarly, for equation (2.20) and (2.21) we get

$$\begin{aligned}
\overline{\langle s_j(t')s_i(t)\rangle} &= \langle s(t)\rangle_{*,i}\langle s(t')\rangle_*^j + \delta_{ij}\left(\langle s(t)s(t')\rangle_{*,i} - \langle s(t)\rangle_{*,i}\langle s(t')\rangle_*^j\right) \quad (2.23)\\
\mathrm{i}\frac{\partial\overline{\langle s_i(t)\rangle}}{\partial\theta_j(t')} &= \langle s(t)\rangle_{*,i}\langle\hat{p}(t')\rangle_*^j + \delta_{ij}\left(\langle s(t)\hat{p}(t')\rangle_{*,i} - \langle s(t)\rangle_{*,i}\langle\hat{p}(t')\rangle_*^j\right)\\
&= \langle s(t)\hat{p}(t')\rangle_{*,i}, \quad (2.24)
\end{aligned}$$

where $\delta_{ij} = \begin{cases} 1 & : \quad i = j \\ 0 & : \quad i \neq j \end{cases}$. Note that $\langle\hat{p}(t)\rangle_{*,i} = 0$ and $\langle\hat{p}(t)\hat{p}(t')\rangle_{*,i} = 0$, which can be easily shown by taking the first and second derivative of the averaged moment-generating functional $\overline{Z}[\boldsymbol{\psi}]$ with respect to $\theta_i(t)$ and $\theta_j(t')$ respectively (see appendix A.2).

We know that the partial derivative of $(\Psi+\Phi+\Omega)$ with respect to $\{C, \hat{C}, K, \hat{K}, L, \hat{L}\}$ is equal to zero at the saddle point. Hence, by taking these derivatives we obtain six *maximal-point equations*:

$$\begin{aligned}
0 = \left.\frac{\partial(\Psi+\Omega)}{\partial\hat{C}(t,t')}\right|_{s.p.} &= \mathrm{i}C_0(t,t') - \mathrm{i}\frac{1}{N}\sum_{i=1}^{N}\langle s(t)s(t')\rangle_{*,i}\\
\Longleftrightarrow C_0(t,t') &= \frac{1}{N}\sum_{i=1}^{N}\langle s(t)s(t')\rangle_{*,i}, \quad (2.25)\\
0 = \left.\frac{\partial(\Psi+\Omega)}{\partial\hat{K}(t,t')}\right|_{s.p.} &= \mathrm{i}K_0(t,t') - \mathrm{i}\frac{1}{N}\sum_{i=1}^{N}\langle s(t)\hat{p}(t')\rangle_{*,i}\\
\Longleftrightarrow K_0(t,t') &= \frac{1}{N}\sum_{i=1}^{N}\langle s(t)\hat{p}(t')\rangle_{*,i}, \quad (2.26)\\
0 = \left.\frac{\partial(\Psi+\Omega)}{\partial\hat{L}(t,t')}\right|_{s.p.} &= \mathrm{i}L_0(t,t') - \mathrm{i}\frac{1}{N}\sum_{i=1}^{N}\langle\hat{p}(t)\hat{p}(t')\rangle_{*,i}\\
\Longleftrightarrow L_0(t,t') &= \frac{1}{N}\sum_{i=1}^{N}\langle\hat{p}(t)\hat{p}(t')\rangle_{*,i} = 0, \quad (2.27)
\end{aligned}$$

where 's.p.' denotes the evaluation at the saddle point.

Similarly, for the conjugate variables $\hat{C}_0(t,t')$, $\hat{K}_0(t,t')$ and $\hat{L}_0(t,t')$ we find the

following results

$$0 = \left.\frac{\partial(\Psi+\Phi)}{\partial C(t,t')}\right|_{s.p.} = \mathrm{i}\hat{C}_0(t,t') + \left.\frac{\partial\Phi}{\partial C(t,t')}\right|_{s.p.} \Longleftrightarrow \hat{C}_0(t,t') = \mathrm{i}\left.\frac{\partial\Phi}{\partial C(t,t')}\right|_{s.p.},$$

$$0 = \left.\frac{\partial(\Psi+\Phi)}{\partial K(t,t')}\right|_{s.p.} = \mathrm{i}\hat{K}_0(t,t') + \left.\frac{\partial\Phi}{\partial K(t,t')}\right|_{s.p.} \Longleftrightarrow \hat{K}_0(t,t') = \mathrm{i}\left.\frac{\partial\Phi}{\partial K(t,t')}\right|_{s.p.},$$

$$0 = \left.\frac{\partial(\Psi+\Phi)}{\partial L(t,t')}\right|_{s.p.} = \mathrm{i}\hat{L}_0(t,t') + \left.\frac{\partial\Phi}{\partial L(t,t')}\right|_{s.p.} \Longleftrightarrow \hat{L}_0(t,t') = \mathrm{i}\left.\frac{\partial\Phi}{\partial L(t,t')}\right|_{s.p.}.$$

For the sake of clarity we will in the following drop the index '0'.

Let us use equation (2.23) and (2.24) to formulate the above results for $C(t,t')$ and $K(t,t')$ in a different way, which allows an easier interpretation:

$$C(t,t') = \frac{1}{N}\sum_{i=1}^{N}\langle s(t)s(t')\rangle_{*,i} = \frac{1}{N}\sum_{i=1}^{N}\overline{\langle s_i(t)s_i(t')\rangle},$$

$$K(t,t') = \frac{1}{N}\sum_{i=1}^{N}\langle s(t)\hat{p}(t')\rangle_{*,i} = \frac{1}{N}\sum_{i=1}^{N}\mathrm{i}\frac{\partial\overline{\langle s_i(t)\rangle}}{\partial\theta_i(t')},$$

and hence $K(t,t') = \mathrm{i}G(t,t')$. For an infinitely large system with site-independent perturbation field $\{\theta(t)\}$ it follows for $C(t,t')$ and $G(t,t')$ that

$$C(t,t') = \langle s(t)s(t')\rangle_*, \tag{2.28}$$

$$G(t,t') = -\mathrm{i}\langle s(t)\hat{p}(t')\rangle_*. \tag{2.29}$$

Since the perturbation field is site-independent, the definition of $M_i[\{p,\hat{p}\}]$ is henceforth site-independent as well. To evaluate equations (2.28) and (2.29) we need $\hat{C}$, $\hat{K}$ and $\hat{L}$, because they appear in the probability measure $M[\{p,\hat{p}\}]$. However, this requires the evaluation of Φ.

2.3.5 Evaluation of Φ

Our aim is to find an easier expression for Φ which does not involve integrals. This means we have to integrate over $\{x(t)\}$, $\{\hat{x}(t)\}$, $\{w(t)\}$ and $\{\hat{w}(t)\}$. Taking a closer look at equation (2.17) one finds that the introduction of suitable delta functions and the application of Gaussian integrals (see appendix A.3) succeeds. First of all we take the $\{x(t)\}$ integrals and receive a delta function of the form

$\delta(\hat{x}(t)+w(t))$. Taking afterwards the $\{\hat{x}(t)\}$ integrals changes the expression only slightly (replacing all $\hat{x}$ by $-w$):

$$\begin{aligned}\Phi &= \alpha \log \Bigg[\int \exp\left(\mathrm{i} \sum_{t=0}^{t_{max}-1} \hat{w}(t) w(t) \right) \\ &\quad \times \exp\Bigg[-\frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \left(w(t)w(t') + \hat{w}(t) L(t,t') \hat{w}(t') \right. \\ &\quad \left. -2\hat{w}(t') K(t,t') w(t) + w(t) C(t,t') w(t') \right) \Bigg] \mathcal{D}w \mathcal{D}\hat{w} \Bigg].\end{aligned}$$

A Gaussian integral allows us to take the integrals over $\{w(t)\}$. Denoting the identity matrix by I and defining

$$\begin{aligned} G'(t,t') &:= \left(I + \Delta_2 G\right)(t,t'), \\ D(t,t') &:= 1 + C(t,t'), \end{aligned}$$

we can rewrite the function Φ as follows

$$\begin{aligned}\Phi &= \alpha \log \Bigg[\int \exp\Bigg[-\frac{1}{2\Delta_2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \hat{w}(t) \left[(G')^T D^{-1} G' \right](t,t') \hat{w}(t') \Bigg] \\ &\quad \times \exp\left(-\frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \hat{w}(t) L(t,t') \hat{w}(t') \right) \mathcal{D}\hat{w} \Bigg] \\ &\quad -\frac{1}{2} \alpha \log \Big[\det\left(\Delta_2 D\right) \Big].\end{aligned}$$

It remains to take the $\{\hat{w}(t)\}$ integrals. For this purpose we use the exponential series for L small[4] for the second exponential term, leave the first one unchanged and calculate a Gaussian integral, obtaining

$$\begin{aligned}\Phi &= \alpha \log \left[1 - \frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} L(t,t') \left[\frac{1}{\Delta_2} (G')^T D^{-1} G' \right]^{-1} (t,t') \right] \\ &\quad -\frac{1}{2} \alpha \log \left[\det\left((G')^T G' \right) \right] + \mathcal{O}(L^2).\end{aligned}$$

Finally, we use the following identity $\log[\det(A)] = \mathrm{tr}[\log(A)]$ for $A \in \mathbb{R}^{m \times m}$ (cf. page 79 in Coolen's book [6]) and the series $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ for

[4] Note that we use equation (2.27) at this stage.

$|x| \leq 1$ and get the result

$$\begin{aligned}\Phi &= -\alpha\frac{\Delta_2^2}{2}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1} L(t,t')\left[(G')^T D^{-1} G'\right]^{-1}(t,t') \\ &\quad -\alpha\,\mathrm{tr}\left[\log\left(G'\right)\right] + \mathcal{O}(L^2).\end{aligned}$$

Detailed calculations for the evaluation of function Φ can be found in appendix B. For the sake of clarity we will henceforth use the following definitions:

$$\begin{aligned}R &:= -\alpha\Delta_2 (G')^{-1}, \\ \Lambda &:= \Delta_2 (G')^{-1} D((G')^T)^{-1}.\end{aligned}$$

As we will find out later, matrix Λ will turn out to be the covariance-matrix of a random variable in an effective single-consumer process, which can be derived mathematically exactly from our market dynamics (2.1).

The conjugate order parameters are found to be

$$\begin{aligned}\hat{C}(t,t') &= \mathrm{i}\frac{\partial\Phi}{\partial C(t,t')}\bigg|_{s.p.} = 0, \\ \hat{K}(t,t') &= \mathrm{i}\frac{\partial\Phi}{\partial K(t,t')}\bigg|_{s.p.} = R(t',t), \\ \hat{L}(t,t') &= \mathrm{i}\frac{\partial\Phi}{\partial L(t,t')}\bigg|_{s.p.} = -\mathrm{i}\alpha\frac{\Delta_2}{2}\Lambda(t,t'),\end{aligned}$$

where we used the following identity for a matrix $U \in \mathbb{R}^{m\times m}$ (cf. page 79 Coolen's book [6]): $\frac{\partial\,\mathrm{tr}[\log(U)]}{\partial U(t,t')} = (U^{-1})(t',t)$. With these new definitions the effective measure $M[\{p,\hat{p}\}]$ reads:

$$\begin{aligned}M[\{p,\hat{p}\}] &= \rho_0(p(0))\exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\hat{p}(t)(p(t+1)-p(t)-\theta(t))\right] \\ &\quad\times\exp\left[-\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\hat{p}(t)R(t,t')s(t')\right] \\ &\quad\times\exp\left[-\alpha\frac{\Delta_2}{2}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\hat{p}(t)\Lambda(t,t')\hat{p}(t')\right].\end{aligned} \tag{2.30}$$

2.3.6 The Effective Single-Consumer Equation

The very complicated system, initially described by the numerous equations (2.1), will turn out to be equivalent to one single equation. This so-called *effective single-consumer equation* allows for an enormous simplification of the dynamics.

First of all, we have to introduce the concept of *causality*. Causality makes sure that the response functions $G(t,t')$ are equal to zero for the case $t \leq t'$. This is reasonable insofar that a perturbation at time t' of the system can only affect the future but not the past. Assigned to equation (2.30), this means in particular that only values of $p(t')$ with $t' \leq t$ are involved.

Causality allows for calculating the denominator of

$$\langle f[\{p,\hat{p}\}]\rangle_* = \frac{\int M[\{p,\hat{p}\}] f[\{p,\hat{p}\}] \mathcal{D}p\, dp(t_{max}) \mathcal{D}\hat{p}}{\int M[\{p,\hat{p}\}] \mathcal{D}p\, dp(t_{max}) \mathcal{D}\hat{p}},$$

viz. we calculate

$$\begin{aligned}
&\int \rho_0(p(0)) \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1} \hat{p}(t)(p(t+1)-p(t)-\theta(t))\right] \\
&\times \exp\left[-\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1} \hat{p}(t) R(t,t') s(t')\right] \\
&\times \exp\left[-\alpha\frac{\Delta_2}{2}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1} \hat{p}(t)\Lambda(t,t')\hat{p}(t')\right] \mathcal{D}p\, dp(t_{max}) \mathcal{D}\hat{p}.
\end{aligned}$$

We integrate iteratively over $\{p(t)\}$ and $\{\hat{p}(t)\}$. Note that $\{dp(t)\}$ contains $t_{max}+1$ factors, whereas $\{d\hat{p}(t)\}$ contains only t_{max} factors.

First, we integrate over $p(t_{max})$ which introduces a delta function:

$$\begin{aligned}
&\int \rho_0(p(0)) \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-2} \hat{p}(t)(p(t+1)-p(t)-\theta(t))\right] \delta\left(\hat{p}(t_{max}-1)\right) \\
&\times \exp\left[\mathrm{i}\hat{p}(t_{max}-1)(-p(t_{max}-1)-\theta(t_{max}-1))\right] \\
&\times \exp\left[-\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1} \hat{p}(t) R(t,t') s(t')\right] \\
&\times \exp\left[-\alpha\frac{\Delta_2}{2}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1} \hat{p}(t)\Lambda(t,t')\hat{p}(t')\right] \\
&\prod_{t=0}^{t_{max}-1} [dp(t)d\hat{p}(t)] \prod_{t=0}^{t_{max}-2} \frac{1}{2\pi}.
\end{aligned}$$

Integrating over $\hat{p}(t_{max}-1)$ makes the delta function disappear:

$$\int \rho_0(p(0)) \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-2}\hat{p}(t)(p(t+1)-p(t)-\theta(t))\right]$$
$$\times \exp\left[-\mathrm{i}\sum_{t=0}^{t_{max}-2}\sum_{t'=0}^{t_{max}-2}\hat{p}(t)R(t,t')s(t')\right]$$
$$\times \exp\left[-\alpha\frac{\Delta_2}{2}\sum_{t=0}^{t_{max}-2}\sum_{t'=0}^{t_{max}-2}\hat{p}(t)\Lambda(t,t')\hat{p}(t')\right]$$
$$\prod_{t=0}^{t_{max}-1} dp(t) \prod_{t=0}^{t_{max}-2}\frac{d\hat{p}(t)}{2\pi}.$$

Then we integrate over $p(t_{max}-1)$ and $\hat{p}(t_{max}-2)$ and so on. After the last iterative step we obtain:

$$\int \rho_0(p(0))dp(0) \;=\; 1.$$

The definition for $\langle\cdot\rangle_*$ has now been simplified to an expression without fraction, viz.

$$\langle f[\{p,\hat{p}\}]\rangle_* = \int M[\{p,\hat{p}\}]f[\{p,\hat{p}\}]\mathcal{D}p\,dp(t_{max})\mathcal{D}\hat{p}.$$

In the numerator we calculate $\int[\cdot]\prod_{t=0}^{t_{max}-1}\frac{d\hat{p}(t)}{2\pi}$ and introduce a suitable delta function. The integral over the $M[\{p,\hat{p}\}]$ takes the following form

$$\begin{aligned} M[\{p\}] \;=\; & \rho_0(p(0))\int \exp\left(\mathrm{i}\sqrt{\alpha\Delta_2}\sum_{t=0}^{t_{max}-1}\hat{p}(t)\eta(t)\right) \\ & \times \prod_{t=0}^{t_{max}-1}\delta\left[\eta(t)-\frac{p(t+1)-p(t)-\theta(t)-\sum_{t'=0}^{t_{max}-1}R(t,t')s(t')}{\sqrt{\alpha\Delta_2}}\right] \\ & \times \exp\left[-\alpha\frac{\Delta_2}{2}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\hat{p}(t)\Lambda(t,t')\hat{p}(t')\right]\prod_{t=0}^{t_{max}-1}\left[\frac{d\hat{p}(t)d\eta(t)}{2\pi}\right]. \end{aligned}$$

Using a Gaussian integral we obtain

$$\begin{aligned} M[\{p\}] \;=\; & \rho_0(p(0))\int \frac{\exp\left[-\frac{1}{2}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\eta(t)\Lambda^{-1}(t,t')\eta(t')\right]}{\sqrt{\det(\Lambda)}} \\ & \times \prod_{t=0}^{t_{max}-1}\delta\left[p(t+1)-p(t)-\theta(t)-\sum_{t'=0}^{t_{max}-1}R(t,t')s(t')\right. \\ & \left.-\sqrt{\alpha\Delta_2}\eta(t)\right]\prod_{t=0}^{t_{max}-1}\left[\frac{d\eta(t)}{\sqrt{2\pi}}\right]. \end{aligned}$$

We deduce from this that our system can be expressed as an effective single-consumer process of the form

$$p(t+1) - p(t) \;=\; \sum_{t'=0}^{t} R(t,t')s(t') + \sqrt{\alpha\Delta_2}\eta(t) + \theta(t), \tag{2.31}$$

where $\eta(t)$ is a Gaussian noise (random variable) with zero mean and with correlation $\Lambda(t,t') = \langle \eta(t)\eta(t') \rangle$ given by

$$\Lambda = \Delta_2 \left(I + \Delta_2 G\right)^{-1} D \left(I + \Delta_2 G^T\right)^{-1}. \tag{2.32}$$

Note that we used causality by replacing the sum $\sum_{t'=0}^{t_{max}-1}$ by $\sum_{t'=0}^{t}$ in the effective single-consumer equation (2.31).

Our definition Λ now turns out, as mentioned before, to have a mathematical interpretation, viz. it is the covariance-matrix of the random variable $\eta(t)$.

Finally, the definition of $\langle\cdot\rangle_*$ was simplified to

$$\langle f[\{p\}] \rangle_* = \int M[\{p\}] f[\{p\}] \prod_{t=0}^{t_{max}} p(t). \tag{2.33}$$

Therefore the expressions

$$C(t,t') = \langle s(t)s(t') \rangle_*, \qquad G(t,t') = \frac{\partial \langle s(t) \rangle_*}{\partial \theta(t')} \tag{2.34}$$

only involve the simpler expression (2.33).

Our market with interaction between N consumers and P sellers can now be expressed as a market with *retarded self-interaction* of an effective single consumer. Mathematically, both dynamics are equivalent for $N \to \infty$. It simplifies the analysis enormously to consider the effective single-consumer dynamics (2.31) instead of the dynamics of the whole market (2.1). The effective single-consumer equation misleads in that point that one could think we simplified the problem to a single equation. This is not completely correct, since the noise η is a random variable with a long memory and as such has infinitely many realisations. Thus, strictly speaking, we substituted our (in the limit) infinitely large problem by another one. Indeed, the original process was Markovian, viz. its dependence was only from the directly previous time step, whereas now the single consumer acquires an *effective memory*. We therefore obtained a non-Markovian stochastic process. Nevertheless, as we will see in the next section, the effective single-consumer equation with Gaussian noise η generates useful equations in the stationary state and is therefore of main importance for our further analysis of the model dynamics.

2.3.7 The Stationary State

The effective single-consumer process (2.31) reaches in a long time limit a stationary state[5], i. e. a state in which the probability of any quantity (e. g. variance) does not change in time. In this situation the correlation and response functions satisfy the following three properties:

(i) time-translation invariance

(ii) no anomalous response

(iii) no long-term memory.

In a time-translation invariant stationary state the correlation and response functions $C(t+\tau,t)$ and $G(t+\tau,t)$ respectively only depend on the time difference τ. We therefore have the following properties for the stationary state:

$$\lim_{t\to\infty} C(t+\tau,t) = C(\tau), \qquad \lim_{t\to\infty} G(t+\tau,t) = G(\tau).$$

Moreover, with equation (2.32) we conclude that $\lim_{t\to\infty}\Lambda(t+\tau,t) = \Lambda(\tau)$. In the stationary state without anomalous response, short-term perturbations decay sufficiently fast, viz. the so-called integrated response χ is finite and non-negative

$$0 \leq \chi := \lim_{\tau\to\infty} \sum_{t=0}^{\tau-1} G(t) < \infty. \tag{2.35}$$

The last property of no long-term memory means that there is no response in the far future of the system to a temporary perturbation at time t

$$\lim_{\tau\to\infty} G(t+\tau,t) = 0 \;\forall t < \infty.$$

In the stationary state we can find two types of consumers: The first type are consumers, who choose at every time step of the game the same strategy and are therefore called *frozen consumers*. This kind of consumer is in the lucky position that at each step his/her own choice was the optimal one corresponding to a low queue time and there is no reason for changing the strategy in the future. This in

[5]For a justification that such a stationary state is actually reached see chapter 7 in Coolen (2005) [6].

turn means that the difference of the utility of both available strategies, i. e. the absolute value of the preference p_i increases linearly with time.

The second group consisting of *fickle consumers* changes the strategy sometimes. They do this because of dissatisfaction with the past queue time and hope by changing the strategy frequently to improve the situation in the future. This means that both possible strategies are nearly equally promising and therefore p_i fluctuates around zero.

In the effective single-consumer process (2.31) the original complex system is simplified to a self-interacting consumer. This means that the fraction of all N consumers being frozen denoted by ϕ and fickle labelled with $\bar{\phi}$ can now be interpreted as the fraction of the single consumer trajectories being frozen and fickle respectively. It is practically difficult to separate both fraction types, since for frozen consumers the preference p_i is not finite in the time limit $t \to \infty$. Therefore, we will not consider the preferences themselves, but the velocity $\tilde{p}(t) = \frac{p(t)}{t}$ with which the preferences increase.

Let us now turn to the effective single-consumer process (2.31). The perturbation field $\{\theta(t)\}$ is no longer required and is in the following set to zero:

$$p(t+1) - p(t) = \sum_{t'=0}^{t} R(t,t')s(t') + \sqrt{\alpha\Delta_2}\eta(t)$$

$$\Leftrightarrow (t+1)\tilde{p}(t+1) - t\tilde{p}(t) = \sum_{t'=0}^{t} R(t,t')\tilde{s}(t') + \sqrt{\alpha\Delta_2}\eta(t),$$

where $\tilde{s}(t') := \mathrm{sgn}(t'\tilde{p}(t'))$. Taking the time average and the time limit leads to

$$\begin{aligned} & \lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} [(t'+1)\tilde{p}(t'+1) - t'\tilde{p}(t')] \\ = & \lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} \left[\sum_{t''=0}^{t'} R(t',t'')\tilde{s}(t'') + \sqrt{\alpha\Delta_2}\eta(t') \right]. \end{aligned}$$

In the telescoping sum on the left hand side all terms except the first and the last one cancel. The first term is multiplied by zero and therefore only the last term remains:

$$\lim_{t\to\infty} \tilde{p}(t) = \lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} \left[\sum_{t''=0}^{t'} R(t',t'')\tilde{s}(t'') + \sqrt{\alpha\Delta_2}\eta(t') \right].$$

By defining $\tilde{p} := \lim_{t\to\infty} \tilde{p}(t)$ and the random variable[6] $\bar{\eta} := \lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} \eta(t')$ we can simplify this process to:

$$\tilde{p} = \lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} \sum_{t''=0}^{t'} R(t', t'')\tilde{s}(t'') + \sqrt{\alpha\Delta_2}\bar{\eta}.$$

The following lemmata 1 and 2 as consequences of the absence of anomalous response, mentioned in Heimel & Coolen (2001) [5], will be required in the calculations of the re-scaled effective single-consumer dynamics. Proofs for both lemmata can be found in appendix A.4.

Lemma 1 *Let $\{A_t\}$ and $\{b_t\}$ be two bounded sequences of real numbers. Assume a and b exist with $b = \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} b_t$ and $a = \lim_{\tau\to\infty} \sum_{t=0}^{\tau-1} A_t$. Then*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \sum_{t'=0}^{t} A_{t-t'} b_{t'} = ab.$$

Lemma 2 *Let $G(t, t') = G(t - t') \in \mathbb{R}$, where $G(t, t') = 0$ for all $t \leq t'$ and define $\chi := \lim_{\tau\to\infty} \sum_{t=0}^{\tau-1} G(t)$ and $s := \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} s(t)$. Then for all $n \in \mathbb{N}_0$*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \sum_{t'=0}^{t} G^n(t, t')s(t') = \chi^n s.$$

Using the following result deduced from the power series of the logarithm $\log(1 + x) = \sum_{n\geq 1} \frac{(-1)^{n-1}x^n}{n}$ for $|x| \leq 1$

$$(G')^{-1}(t, t') = (I + \Delta_2 G)^{-1}(t, t') = \sum_{n\geq 0} (-1)^n \Delta_2^n G^n(t, t'), \tag{2.36}$$

[6] Note that $\bar{\eta}$ depends on the realisation of the random variable η.

lemma 2 and the definition $\bar{s} = \lim_{t\to\infty} \frac{1}{t}\sum_{t'=0}^{t-1} \tilde{s}(t')$ we finally find

$$\begin{aligned}\tilde{p} &= \lim_{t\to\infty} -\alpha\Delta_2 \frac{1}{t}\sum_{t'=0}^{t-1}\sum_{t''=0}^{t'}\sum_{n\geq 0}(-1)^n \Delta_2^n G^n(t',t'')\tilde{s}(t'') + \sqrt{\alpha\Delta_2}\bar{\eta} \\ &= -\sum_{n\geq 0}\alpha(-1)^n\Delta_2^{n+1}\chi^n\bar{s} + \sqrt{\alpha\Delta_2}\bar{\eta}.\end{aligned}$$

Using again the power series trick results in

$$\tilde{p} = -\Delta_2\frac{\alpha}{1+\Delta_2\chi}\bar{s} + \sqrt{\alpha\Delta_2}\bar{\eta}. \tag{2.37}$$

Let us now examine the cases 'fickle' and 'frozen' separately.

Fickle: $\tilde{p} = 0$. In the case of a fickle effective consumer we can deduce from equation (2.37) that

$$\bar{s} = \sqrt{\frac{1}{\alpha\Delta_2}}(1+\Delta_2\chi)\bar{\eta}, \tag{2.38}$$

for one realisation of the random variable $\bar{\eta}$. Furthermore, from the definition of $\bar{s}$ we derive that

$$\begin{aligned}|\bar{s}| &= \left|\lim_{t\to\infty}\frac{1}{t}\sum_{t'=0}^{t-1}\mathrm{sgn}(t'\tilde{p}(t'))\right| \\ &\leq 1,\end{aligned} \tag{2.39}$$

where we define $\mathrm{sgn}(0) = 0$. From equations (2.38) and (2.39) we conclude the condition for a fickle effective consumer:

$$|\bar{\eta}| \leq \frac{\sqrt{\alpha\Delta_2}}{1+\Delta_2\chi}, \tag{2.40}$$

where we assumed that $1+\Delta_2\chi > 0$. This is fulfilled since one condition of the integrated response (2.35) was its non-negativity.

Frozen: $\tilde{p} \neq 0$. Again, we will use the simplified single-consumer process (2.37):

$$\tilde{p} = -\Delta_2\frac{\alpha}{1+\Delta_2\chi}\bar{s} + \sqrt{\alpha\Delta_2}\bar{\eta}. \tag{2.41}$$

As defined, a frozen consumer always chooses the same strategy, i. e. $\tilde{p}(t')$ has permanently the same sign in the stationary state, therefore

$$\begin{aligned} \bar{s} &= \lim_{t\to\infty} \frac{1}{t} \sum_{t'=0}^{t-1} \operatorname{sgn}(t'\tilde{p}(t')) \\ &= \operatorname{sgn}(\tilde{p}). \end{aligned} \tag{2.42}$$

From equations (2.41) and (2.42) we deduce the condition for a frozen effective consumer:

$$\operatorname{sgn}(\tilde{p}) = (1+\Delta_2\chi)\left(\sqrt{\frac{1}{\alpha\Delta_2}}\bar{\eta} - \frac{1}{\alpha\Delta_2}\tilde{p}\right).$$

In turn, we can conclude from this equation that

$$\operatorname{sgn}(\tilde{p}) = \operatorname{sgn}(\bar{\eta})$$

and also

$$\begin{aligned} 1 &= (1+\Delta_2\chi)\left|\sqrt{\frac{1}{\alpha\Delta_2}}\bar{\eta} - \frac{1}{\alpha\Delta_2}\tilde{p}\right| \\ &< (1+\Delta_2\chi)\sqrt{\frac{1}{\alpha\Delta_2}}|\bar{\eta}| \end{aligned}$$

$$\iff |\bar{\eta}| > \frac{\sqrt{\alpha\Delta_2}}{1+\Delta_2\chi}. \tag{2.43}$$

Equations (2.40) and (2.43) yield an important result – the conditions for a fickle and frozen effective consumer respectively are complementary. It is therefore sufficient to calculate the frozen fraction ϕ and deduce from this the fickle fraction $\bar{\phi} = 1-\phi$. Let us emphasise once more that for one certain realisation of the noise η the consumer can either be fickle or frozen, as indicated by the complementary equations (2.40) and (2.43). $\bar{\eta}$ then gives the time average over one possible realisation of η. $\langle\cdot\rangle_{\bar{\eta}}$ denotes henceforth the average over all realisations of the time-averaged noise $\bar{\eta}$, i. e. this expression is not either fickle or frozen any more, it exhibits both in a certain ratio. As we will use shortly, the average with measure M is equal to the average over the noise, i. e. $\langle\cdot\rangle_* = \langle\cdot\rangle_{\bar{\eta}}$ (shown in appendix B).

Our aim is to describe the stationary state completely by using equations (2.40) and (2.43). The following observables will be of main interest:

- persistent correlation c
- frozen fraction ϕ
- susceptibility χ.

In the calculations of c, ϕ and χ the probability distribution $P(\bar{\eta})$ is required. Since η is Gaussian with zero mean, then $\bar{\eta}$ is Gaussian and exhibits zero mean, viz. $\langle\bar{\eta}\rangle_{\bar{\eta}} = 0$. The probability distribution is therefore of the form:

$$P(\bar{\eta}) = \frac{1}{\sqrt{2\pi\sigma_{\bar{\eta}}^2}} \exp\left[-\frac{1}{2\sigma_{\bar{\eta}}^2}\bar{\eta}^2\right], \tag{2.44}$$

where $\sigma_{\bar{\eta}}^2$ denotes the variance of $\bar{\eta}$. The only hitherto unknown value in equation (2.44) is $\sigma_{\bar{\eta}}^2$. The variance is defined as $\sigma_{\bar{\eta}}^2 = \langle\bar{\eta}^2\rangle_{\bar{\eta}} - \langle\bar{\eta}\rangle_{\bar{\eta}}^2 = \langle\bar{\eta}^2\rangle_{\bar{\eta}}$. Thus, with equation (2.32) we find the variance to be

$$\begin{aligned}\langle\bar{\eta}^2\rangle_{\bar{\eta}} &= \Delta_2 \lim_{\tau\to\infty} \frac{1}{\tau^2} \sum_{t,t'=0}^{\tau-1} \sum_{s=0}^{t} \sum_{s'=0}^{t'} \Big[\left(I + \Delta_2 G\right)^{-1}(t,s) \\ &\quad \times D(s,s') \left(I + \Delta_2 G^T\right)^{-1}(s',t')\Big].\end{aligned}$$

Using the power series of the logarithm and lemma 2 we get

$$\begin{aligned}\langle\bar{\eta}^2\rangle_{\bar{\eta}} &= \frac{\Delta_2}{\left(1+\Delta_2\chi\right)^2}\left[1 + \lim_{\tau\to\infty}\frac{1}{\tau^2}\sum_{t,t'=0}^{\tau-1} C(t,t')\right] \\ &= \frac{\Delta_2(1+c)}{\left(1+\Delta_2\chi\right)^2},\end{aligned} \tag{2.45}$$

where the persistent correlation $c = \lim_{\tau\to\infty}\frac{1}{\tau^2}\sum_{t,t'=0}^{\tau-1} C(t,t')$ is defined as $c := \lim_{\tau\to\infty} C(\tau)$ (both expressions are shown to be equal in appendix A.2).

By using equation (2.28), we can rewrite $C(t,t')$ in terms of $s(t)$: $C(t,t') = \langle s(t)s(t')\rangle_*$, i. e. $c = \langle\bar{s}^2\rangle_*$. In the stationary state the average with measure M is an average over the noise $\bar{\eta}$ (see appendix B for more details). Note that both possibilities, fickle and frozen, have to be taken into account. Making a change

of variables $z = \frac{\bar{\eta}}{\sigma_\eta}$, we find the persistent correlation to be

$$\begin{aligned} c &= \int P(\bar{\eta}) \left[\mathcal{H}\left(\frac{\sqrt{\alpha\Delta_2}}{(1+\Delta_2\chi)} - |\bar{\eta}| \right) \frac{\bar{\eta}^2 \left(1+\Delta_2\chi\right)^2}{\alpha\Delta_2} \right. \\ &\quad \left. + \mathcal{H}\left(|\bar{\eta}| - \frac{\sqrt{\alpha\Delta_2}}{(1+\Delta_2\chi)} \right) \right] d\bar{\eta} \\ &= 2\int_0^\infty \exp\left[-\frac{z^2}{2}\right] \left[\mathcal{H}\left(\frac{\sqrt{\alpha}}{\sqrt{1+c}} - z \right) \frac{(1+c)z^2}{\alpha} + \mathcal{H}\left(z - \frac{\sqrt{\alpha}}{\sqrt{1+c}} \right) \right] \frac{dz}{\sqrt{2\pi}}, \end{aligned}$$

where $\mathcal{H}(x) := \begin{cases} 1 & : \quad x > 0 \\ 0 & : \quad x \leq 0 \end{cases}$ is the Heaviside function. This is simply

$$c = \frac{2(1+c)}{\alpha} \int_0^{\sqrt{\frac{\alpha}{1+c}}} z \left[z \exp\left(-\frac{z^2}{2}\right) \right] \frac{dz}{\sqrt{2\pi}} + 2\int_{\sqrt{\frac{\alpha}{1+c}}}^\infty \exp\left(-\frac{z^2}{2}\right) \frac{dz}{\sqrt{2\pi}}.$$

By using integration by parts, the persistent correlation can be simplified to

$$\begin{aligned} c &= \frac{1+c}{\alpha} \left[\operatorname{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}} \right) - \sqrt{\frac{2\alpha}{\pi(1+c)}} \exp\left(-\frac{\alpha}{2(1+c)} \right) \right] \\ &\quad + \left[1 - \operatorname{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}} \right) \right], \end{aligned} \tag{2.46}$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\left[-t^2\right] dt$ is the error function for $x \in \mathbb{R}$.

Similarly, we obtain the fraction of frozen consumers ϕ (and therefore also the fraction of fickle consumers $\bar{\phi} = 1 - \phi$):

$$\begin{aligned} \phi &= \int P(\bar{\eta}) \mathcal{H}\left(|\bar{\eta}| - \frac{\sqrt{\alpha\Delta_2}}{(1+\Delta_2\chi)} \right) d\bar{\eta} \\ &= 1 - \operatorname{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}} \right). \end{aligned} \tag{2.47}$$

Finally, we calculate the susceptibility or integrated response $\chi = \lim_{\tau\to\infty} \sum_{t=0}^{\tau-1} G(t)$ (cf. datailed calculations in appendix B). By using equation (2.34) and taking the fact that both $\theta(t)$ and $\eta(t)$ occur linearly in the effective single-consumer equation (2.31) into consideration, we find $G(t,t') = \frac{\partial\langle s(t)\rangle_*}{\partial\theta(t')} = \sqrt{\frac{1}{\alpha\Delta_2}} \langle \frac{\partial s(t)}{\partial\eta(t')} \rangle_*$, i. e. $\chi = \sqrt{\frac{1}{\alpha\Delta_2}} \langle \frac{\partial\bar{s}}{\partial\bar{\eta}} \rangle_*$. In the stationary state the perturbation field $\{\theta(t)\}$ is independent of the time t, that means we consider $\theta(t) = \theta$. Using integration by

parts and proceeding similarly to the calculations for c and ϕ, we obtain

$$\begin{aligned}\chi &= \sqrt{\frac{1}{\alpha\Delta_2}}\int P(\bar{\eta})\frac{\partial\bar{s}}{\partial\bar{\eta}}d\bar{\eta} \\ &= \frac{(1+\Delta_2\chi)}{\alpha\Delta_2}\,\mathrm{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right).\end{aligned}$$

Since χ also appears on the right hand side, let us make a last modification to get the following explicit expression

$$\chi = \frac{\mathrm{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)}{\Delta_2\left(\alpha-\mathrm{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)\right)}. \tag{2.48}$$

2.4 Results

Taking equations (2.46), (2.47) and (2.48) together we finally obtain the results:

$$\begin{aligned}c &= \frac{1+c}{\alpha}\left[1-\phi-\sqrt{\frac{2\alpha}{\pi(1+c)}}\exp\left(-\frac{\alpha}{2(1+c)}\right)\right]+\phi \\ \phi &= 1-\mathrm{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right) \\ \chi &= \frac{\mathrm{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)}{\Delta_2\left(\alpha-\mathrm{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)\right)}.\end{aligned}$$

Note that the persistent correlation c and the frozen fraction ϕ are both independent of the variance Δ_2. As a condition for the stationary state, the integrated response has to be finite and non-negative as expressed by equation (2.35). The result for χ satisfies these conditions for $\alpha > \alpha_c$, where α_c is the point at which χ diverges. Altogether, this means we have found exact solutions for the case $\alpha > \alpha_c$. It is still an unsolved problem in the research of minority games to analyse the region $\alpha < \alpha_c$ completely. Numerical simulations show that the stationary state is not unique and turns out to be non-ergodic, i. e. dependent on the

initial conditions (cf. Heimel and Coolen (2001) [5] or Coolen (2005) [6]). This means for relatively few sellers (in comparison to the number of consumers) on the daily market the consumers' choices in all repetitions of the game depend strongly on the initial strategy valuation. However, in this thesis we will focus on the analytical solution and therefore on the region $\alpha > \alpha_c$.

By using the above results the critical value for α is given by

$$\alpha_c = \operatorname{erf}(x),$$

where x is a solution of the following equation

$$\operatorname{erf}(x) = 2 - \frac{1}{x\sqrt{\pi}} \exp\left(-x^2\right).$$

The approximate solution is $\alpha_c \approx 0.337400189$ (with $x \approx 0.308529472$). Note that the critical value α_c is independent of the variance Δ_2 and therefore of the probability distribution choice.

The equation for c cannot be solved in an explicit way. To visualise the result for c numerically we use fixed point iteration (figure 2.2). The analytical results for c and ϕ are independent of the probability distribution choice. The time-averaged asymptotic correlation c decreases for higher α, i. e. the larger is the number of sellers in comparison to the number of consumers. The same behaviour can be observed for the frozen fraction ϕ – the more sellers exist in comparison to the consumers on the market, the smaller the fraction of frozen consumers and the larger the fraction of fickle consumers. Economically this can be explained as follows: For many vegetable options it is likely that the consumers find many equally good options and the difference in profitability of the strategies is small. This leads to frequent changes in strategies. This suggests that the fluctuations represented by the volatility also increase with α. We will turn to the volatility in the next section.

The integrated response is dependent on the probability distribution, i. e. more precisely the variance of the distribution Δ_2. Nevertheless, by using the following expression of χ we are able to visualise graphically the result for a general probability distribution (figure 2.3):

$$\tilde{\chi} := \Delta_2 \chi = \frac{\operatorname{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)}{\alpha - \operatorname{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)}. \tag{2.49}$$

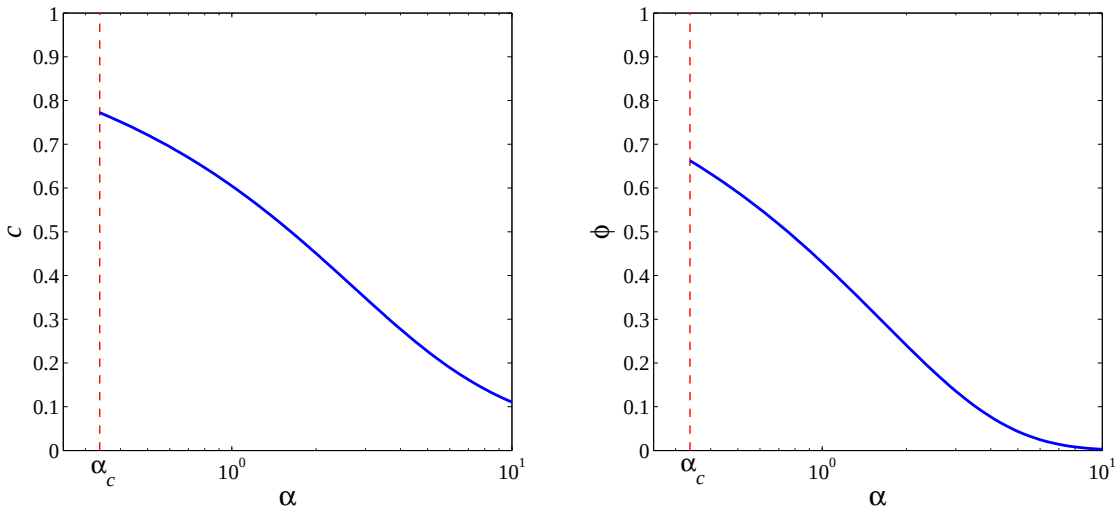

Figure 2.2: Left: Persistent correlation c (2.46) versus the ratio of the number of sellers to the number of consumers $\alpha = \frac{P}{N}$. Right: Frozen fraction ϕ (2.47) versus α. The dashed lines represent the critical value α_c. For $\alpha > \alpha_c$ (solid lines) the predicted solutions for c and ϕ respectively are shown.

The point α_c at which χ diverges can be observed as the discontinuity in figure 2.3. Near α_c the susceptibility of the consumers to perturbations is very strong. For $\alpha \gg \alpha_c$ the behaviour of the integrated response $\Delta_2\chi$ changes only slightly with α. In this region the accumulated response to perturbations seems not to be significantly dependent on the number of sellers in comparison to the number of consumers. Nevertheless, it asymptotically approaches zero.

2.4.1 Volatility & Solutions

The *volatility* in our problem situation is a measure for the demand fluctuations of the whole underlying market, i. e. it measures the risk of the consumers: The higher the volatility the lower the certainty of demands from one day to another. The disorder-averaged volatility matrix for the consumer limit $N \to \infty$ has the

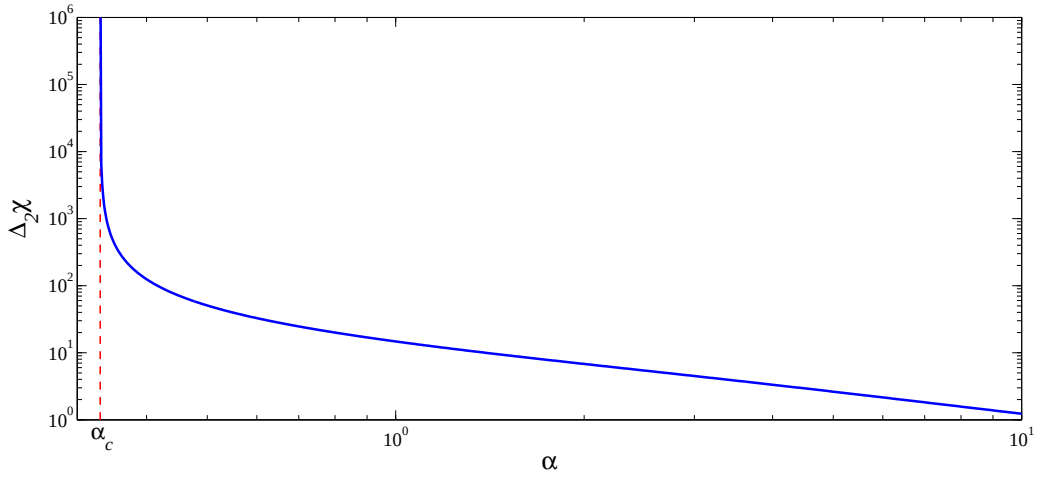

Figure 2.3: Susceptibility $\Delta_2\chi$ (2.49) versus the ratio of the number of sellers to the number of consumers $\alpha = \frac{P}{N}$. The dashed line represents the critical value α_c, at which χ diverges. For $\alpha > \alpha_c$ (solid line) the predicted solution for $\Delta_2\chi$ is shown.

form

$$\begin{aligned}\overline{\Xi}(t,t') &= \lim_{N\to\infty}\frac{1}{P}\sum_{\mu=1}^{P}\overline{\langle N(D^\mu(t)-k)(D^\mu(t')-k)\rangle} \\ &- \overline{\left[\frac{1}{P}\sum_{\mu=1}^{P}\left\langle\sqrt{N}(D^\mu(t)-k)\right\rangle\right]}\,\overline{\left[\frac{1}{P}\sum_{\nu=1}^{P}\left\langle\sqrt{N}(D^\nu(t')-k)\right\rangle\right]},\end{aligned}$$

where $\sigma_v^2 := \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau-1}\overline{\Xi}(t,t)$ is the volatility itself. We will see that it is cumbersome to calculate the exact solution for the volatility, hence we will search for an approximation. But we are able to calculate exactly a measure for the distribution of the consumers over the sellers for the limits $N\to\infty$ and $\tau\to\infty$:

$$\begin{aligned}H := \lim_{\tau\to\infty}\overline{\Xi}(t+\tau,t) &= \lim_{N\to\infty}\lim_{\tau\to\infty}\left[\frac{1}{P}\sum_{\mu=1}^{P}\overline{\langle N(D^\mu(t+\tau)-k)(D^\mu(t)-k)\rangle}\right. \\ &- \overline{\left[\frac{1}{P}\sum_{\mu=1}^{P}\left\langle\sqrt{N}(D^\mu(t+\tau)-k)\right\rangle\right]} \\ &\times \left.\overline{\left[\frac{1}{P}\sum_{\nu=1}^{P}\left\langle\sqrt{N}D^\nu(t)-k)\right\rangle\right]}\right].\end{aligned}$$

H is also of economic importance since it gives information about the efficiency of the considered market. This interpretation can be obtained from the equivalent expression for H, viz. $H = \lim_{N\to\infty} \frac{1}{P} \sum_{\mu=1}^{P} \overline{\left\langle \sqrt{N}(D^\mu(t) - k) \right\rangle}^2$. For $H = 0$ the consumers are evenly distributed over the sellers, giving an overall optimal situation. In the other case, $H > 0$, the consumers are not evenly distributed over the sellers which is obviously not optimal from the collective point of view. Hence, a market with $H = 0$ is called *efficient*, whereas for $H > 0$ the market is *inefficient*.

Our first step is to express the volatility in terms of the correlation and response functions. To do so, Coolen (2005) [6] suggests the generating functional formalism. The calculations presented in the analysis from section 2.3 will only be changed slightly. The main differences to the previous calculations are given in appendix B. The considered moment-generating functional is the following:

$$Z[\boldsymbol{\varphi}] = \left\langle \exp\left(\mathrm{i}\sqrt{2} \sum_{t=0}^{t_{max}} \sum_{\mu=1}^{P} \varphi_\mu(t) \sqrt{N}(D^\mu(t) - k) \right) \right\rangle .$$

The aim is to calculate the averages given by

$$\begin{aligned} \lim_{N\to\infty} \overline{\left\langle \sqrt{N}(D^\mu(t) - k) \right\rangle} &= -\frac{\mathrm{i}}{\sqrt{2}} \lim_{N\to\infty} \lim_{\boldsymbol{\varphi}\to\mathbf{0}} \frac{\partial \overline{Z}[\boldsymbol{\varphi}]}{\partial \varphi_\mu(t)}, \\ \lim_{N\to\infty} \overline{\langle N(D^\mu(t) - k)(D^\mu(t') - k) \rangle} &= -\frac{1}{2} \lim_{N\to\infty} \lim_{\boldsymbol{\varphi}\to\mathbf{0}} \frac{\partial^2 \overline{Z}[\boldsymbol{\varphi}]}{\partial \varphi_\mu(t) \partial \varphi_\mu(t')}. \end{aligned}$$

Calculating the disorder average and the derivatives of the averaged generating functional $\overline{Z}[\boldsymbol{\varphi}]$ by following the lines of the previous calculations, leads to the following result for the volatility for $N \to \infty$ in the stationary state[7]:

$$\sigma_v^2 = \lim_{\tau\to\infty} \frac{\Delta_2}{2\tau} \sum_{t=0}^{\tau-1} \left[(I + \Delta_2 G)^{-1} D \left(I + \Delta_2 G^T \right)^{-1} \right] (t,t). \tag{2.50}$$

This result (2.50) gives not only an easier expression for the volatility, but it also proves that the noise η in the effective single-consumer equation (2.31) really represents the market fluctuations, since $\sigma_v^2 = \lim_{\tau\to\infty} \frac{1}{2\tau} \sum_{t=0}^{\tau-1} \Lambda(t,t)$.

[7] Recall that the limit $N \to \infty$ was required to apply the steepest descent integration (see A.1).

Let us simplify equation (2.50) a bit further:

$$\sigma_v^2 = \lim_{\tau\to\infty} \frac{\Delta_2}{2\tau} \sum_{t=0}^{\tau-1} \sum_{t',t''=0}^{t} \left(I + \Delta_2 G\right)^{-1}(t,t') \left[1 + C(t',t'')\right] \left(I + \Delta_2 G\right)^{-1}(t,t'').$$

We would like to calculate the volatility in terms of the already known macroscopic observables c, ϕ and χ. It is hard to find an exact solution to this, since the volatility involves not only long-term fluctuations but also short-term ones. Hence, it is already obvious at this early point of the calculations that our solution for the volatility will be an approximation. The book by Coolen (2005) [6] (pages 100-104) and the paper by Heimel & Coolen (2001) [5] show volatility approximations for a different model. Let us use a similar approach.

Similarly to the calculations for the variance of the probability distribution (2.45) we obtain for the volatility, still in an exact form,

$$\begin{aligned} \sigma_v^2 \;=\; & \frac{\Delta_2(1+c)}{2\left(1+\tilde{\chi}\right)^2} + \lim_{\tau\to\infty} \frac{\Delta_2}{2\tau} \sum_{t=0}^{\tau-1} \sum_{t',t''=0}^{t} \left(I + \Delta_2 G\right)^{-1}(t,t') \\ & \times \tilde{C}(t',t'') \left(I + \Delta_2 G\right)^{-1}(t,t''), \end{aligned} \tag{2.51}$$

where we separated the correlations $C(t',t'')$ into the persistent c and non-persistent $\tilde{C}(t',t'')$ part[8]. Since we know for the non-persistent part of the correlations that the condition $\lim_{\tau\to\infty} \tilde{C}(t'+\tau,t') = 0$ must be satisfied, we find the following exact market efficiency measure H to be only dependent on the persistent order parameters $\{c,\phi,\chi\}$:

$$H = \lim_{\tau\to\infty} \overline{\Xi}(t+\tau,t) = \frac{\Delta_2(1+c)}{2\left(1+\tilde{\chi}\right)^2}. \tag{2.52}$$

Since C and G emerge for all times in equation (2.51), the exact calculations for the volatility are very extensive. Hence, let us in the following try to find an approximation for the volatility in the stationary state involving only the persistent order parameters $\{c,\phi,\chi\}$. In the time-translation invariant stationary

[8] Note that there is no persistent response $G(t,t')$, i. e. $G(t,t') = \tilde{G}(t,t')$ for the region $\alpha > \alpha_c$ (cf. Heimel & Coolen (2001) [5]).

state, we can express, involving equation (2.34), the correlations $C(t', t'') = C(t' - t'')$ in a form involving fickle and frozen consumers seperately:

$$C(t' - t'') = \phi + (1 - \phi)\langle s(t')s(t'')\rangle_{fickle},$$

where $\langle\cdot\rangle_{fickle}$ denotes the average over the effective single-consumer process (2.31), but now only for the fickle fraction. We used the fact that $\langle s(t')s(t'')\rangle_{frozen} = 1$ (for a similar definition of $\langle\cdot\rangle_{frozen}$ as for $\langle\cdot\rangle_{fickle}$), which follows from the definition of a frozen consumer. We can write the following identities concerning the definitions of the brackets: $\langle\cdot\rangle_* = \langle\cdot\rangle_{\bar{\eta}} = \langle\cdot\rangle_{fickle} + \langle\cdot\rangle_{frozen}$. Since the correlations consist of persistent and non-persistent parts we can write the non-persistent part in the following form:

$$\tilde{C}(t' - t'') = \phi + (1 - \phi)\langle s(t')s(t'')\rangle_{fickle} - c.$$

By inserting this into equation (2.51) we obtain

$$\begin{aligned}\sigma_v^2 &= \frac{\Delta_2(1+\phi)}{2\left(1+\tilde{\chi}\right)^2} \\ &+ (1-\phi)\lim_{\tau\to\infty}\frac{\Delta_2}{2\tau}\sum_{t=0}^{\tau-1}\left\langle\left[\sum_{t'=0}^{t}\left(I+\Delta_2 G\right)^{-1}(t,t')s(t')\right]^2\right\rangle_{fickle}.\end{aligned}$$

The term $\sum_{t'=0}^{t}\left(I+\Delta_2 G\right)^{-1}(t,t')s(t')$ describes the retarded self-interaction term from the effective single-consumer process (2.31). We can now use the fact that this self-interaction term has a small magnitude (or even none) for the case of a fickle consumer (and can have a large magnitude for a frozen consumer). The reason for this is that the preference $\tilde{p}$ (see page 31) is for fickle consumers is equal to zero. Hence, only the term $\left[\left(I+\Delta_2 G\right)^{-1}(t,t)s(t)\right]^2 = 1$ remains and the approximation for the volatility takes the form

$$\sigma_v^2 = \frac{\Delta_2(1+\phi)}{2\left(1+\tilde{\chi}\right)^2} + \frac{\Delta_2}{2}(1-\phi). \tag{2.53}$$

We will see in short that the approximation for the volatility turns out to be quite accurate.

Chapter 3

Applications

In the previous chapter, we studied the case of a general probability distribution and obtained exact expressions for the persistent order parameters $\{c, \phi, \chi\}$, the measure for the efficiency of the market H and an approximation for the volatility σ_v^2. In this chapter we will consider concrete examples by applying certain probability distributions.

Before doing so let us lay emphasis on the following important observation: Examining the equations for the efficiency measure H (see (2.52)) and the volatility σ_v^2 (see (2.53)) the fact is striking that for two different probability distribution functions with the same variance the same results for H, σ_v^2 and χ will be found. The variance of a probability distribution is a measure for how widely spread the values of the random variable (here $q_{i,g}^{\mu}$) are. The larger the variance, the more scattered are the values. From equations (2.52) and (2.53) we deduce that for an increasing variance Δ_2, σ_v^2 and H increase, i. e. the fluctuations increase and the distribution of the consumers over the sellers is not uniform. Precisely, σ_v^2 and H depend linearly on Δ_2. At first it is surprising that the magnitude of the fluctuations and the magnitude of how evenly the consumers are distributed over the sellers are only dependent on the variance of the probability distribution, and not on the probability distribution itself, since in spite of two probability distributions having the same variance their shape can be largely dissimilar. To visualise this, the exponential distribution compared with another distribution, for instance the continuous uniform distribution, is convenient, since the variance of the exponential distribution can be chosen. The exponential distribution has

the following form

$$P(q_{i,g}^{\mu}) = \begin{cases} \lambda \exp\left(-\lambda q_{i,g}^{\mu}\right) & : \quad q_{i,g}^{\mu} \geq 0,\ \lambda > 0 \\ 0 & : \quad q_{i,g}^{\mu} < 0 \end{cases},$$

with mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$. The continous uniform distribution, viz.

$$P(q_{i,g}^{\mu}) = \begin{cases} \frac{1}{q_{max}} & : \quad 0 \leq q_{i,g}^{\mu} \leq q_{max} \\ 0 & : \quad q_{i,g}^{\mu} < 0 \text{ or } q_{i,g}^{\mu} > q_{max} \end{cases},$$

has the following variance

$$\Delta_2 = \frac{q_{max}^2}{12}.$$

Choosing $\frac{1}{\lambda^2}$ equal to $\frac{q_{max}^2}{12}$, viz. $\lambda = \frac{\sqrt{12}}{q_{max}}$, the probability density functions take the well known dissimilar forms, whereas σ_v^2 and H are equal for both distributions.

An explanation for this phenomenon is the following: In the limit $N \to \infty$, only the first two moments of $P(q_{i,g}^{\mu})$ matter as long as the variance of P is finite by virtue of the central limit theorem.

3.1 Discrete Probability Distribution

In the previous chapter we assumed that $q_{i,g}^{\mu} \in \mathbb{R}_0^+$ and we considered a general probability distribution. Let us now apply our results from the previous chapter to a particular probability distribution. In a first step we assume that in our model we only regard the fact whether consumer $i \in \{1, \ldots, N\}$ purchases from seller $\mu \in \{1, \ldots, P\}$ (without specifying the exact amount of goods) or not. This means $q_{i,g}^{\mu} \in \{0, 1\}$, where $q_{i,g}^{\mu} = 0$ implies 'buying' and $q_{i,g}^{\mu} = 1$ 'not buying'. Hence, it is reasonable to assume the following probability distribution suggested by De Martino and Marsili (2005) in [4]:

$$P(q_{i,g}^{\mu}) = (1 - q)\delta(1 - q_{i,g}^{\mu}) + q\delta(q_{i,g}^{\mu}), \tag{3.1}$$

where $0 < 1 - q < 1$ is the probability that any good is part of a bundle. As indicated above, it is sufficient to calculate the variance of the probability distribution to obtain the required macroscopic observables. For the given distribution

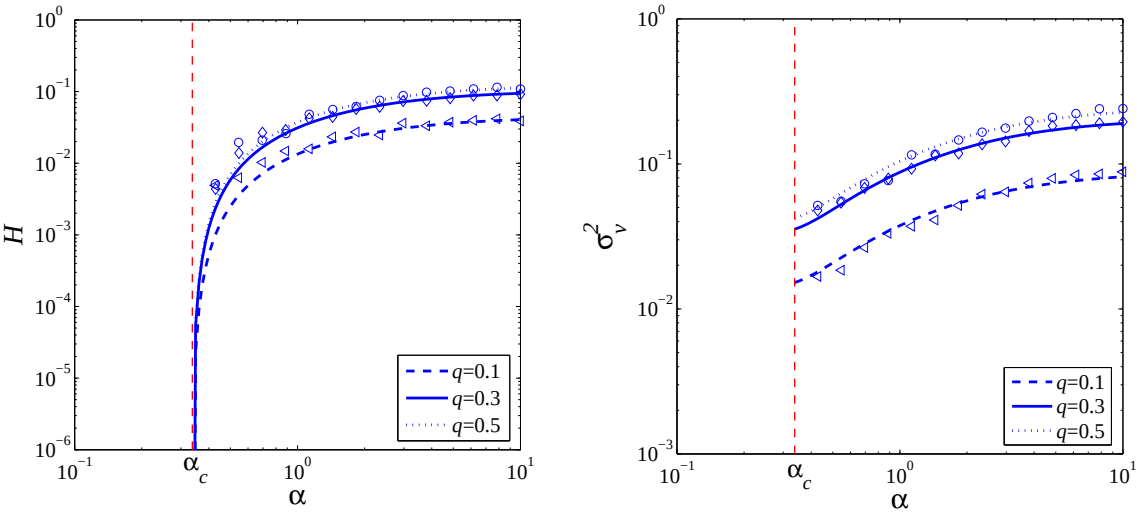

Figure 3.1: Left: Market efficiency measure in the stationary state for $N \to \infty$, viz. H from equation (3.2) versus the ratio of the number of sellers to the number of consumers $\alpha = \frac{P}{N}$ for different q. Right: Approximation for the volatility in the stationary state for $N \to \infty$, viz. σ_v^2 from equation (3.3) versus α for different q. The vertical lines represent the critical value α_c. For $\alpha > \alpha_c$ the predicted solution for H and the approximate solutions for σ_v^2 respectively are shown. The markers correspond to numerical simulations of the model equations (see section 2.2) for the fixed product $NP = 250,000$.

we find $\Delta_2 = \int P(q_{i,g}^{\mu})(q_{i,g}^{\mu})^2 dq_{i,g}^{\mu} - \left(\int P(q_{i,g}^{\mu}) q_{i,g}^{\mu} dq_{i,g}^{\mu}\right)^2 = q(1-q)$. Hence we obtain for H and the approximate volatility the following equations:

$$H = \frac{q(1-q)(1+c)}{2\left(1+\tilde{\chi}\right)^2} \tag{3.2}$$

$$\sigma_v^2 = \frac{q(1-q)(1+\phi)}{2\left(1+\tilde{\chi}\right)^2} + \frac{q(1-q)}{2}(1-\phi), \tag{3.3}$$

where $\tilde{\chi} = q(1-q)\chi$.

From the economic point of view situations where the fluctuations are small and the market is efficient, i. e. σ_v^2 small and $H = 0$, are desirable, since this would mean that the market is predictable and an external consumer is not able to exploit uneven consumer distributions over the sellers. Figure 3.1 shows the approximate volatility σ_v^2 and the measure for the distribution of the consumers over the sellers H respectively versus the ratio between the sellers and the consumers. The graphs for different values of q are given, where $1 - q$ represents

the expected demand. Obviously, since q only appears in form of the variance in both measures σ_v^2 and H respectively and the variance $\Delta_2 = q(1-q)$ is invariant to a transformation of the form $q \to 1-q$, only values $q \leq 0.5$ are chosen. The higher the variance (highest possible value for $q = 0.5$) of the probability distribution, the higher σ_v^2 and H. To show that the approximation for the volatility σ_v^2 is quite accurate and the analytical solution for H is calculated correctly, we plotted numerical simulations of the model equations (see section 2.2), displayed as the markers in figure 3.1. Due to computational constraints[1] the markers do not exactly lie on the solid lines. The volatility curves for different values of q are parallel translations in the logarithmic plot. The smallest volatility, i. e. the best market predictability is given for α near α_c. We find our assumption from the last chapter confirmed, since the volatility increases with α. σ_v^2 and H converge for $\alpha \to \infty$ to $\Delta_2 = q(1-q)$ and $\frac{\Delta_2}{2} = \frac{q(1-q)}{2}$ respectively. This means that the consumers act in the limit $\alpha \to \infty$ completely randomly. They act as if they would toss a coin to make their goods decisions. In the market there are so many sellers that the immense information cannot be used for a collectively positive behaviour. The more consumers enter the market for this fixed number of sellers the more the fluctuations decrease, i. e. the consumers are able to improve their behaviour to a situation better than for random behaviour.

The consumers are more evenly distributed over the sellers the more we approach α_c. This leads to the following connection between H, σ_v^2 and $\bar{\phi}$ – an uneven distribution of the consumers over the sellers leads to dissatisfaction for most consumers, precisely those who have to wait in a longer queue. The dissatisfaction causes strategy changes which in turn increases the fluctuations and the fraction of fickle consumers.

For α near α_c the market achieves in the regime $\alpha > \alpha_c$ the most desirable state from the collective point of view – the fluctuations are small and the distribution of the consumers over the sellers is almost uniform. Admittedly, in this situation the integrated response diverges, i. e. the consumers react very strongly to perturbations. This in turn indicates that this optimal situation is very sensitive. A certain distance to α_c is therefore advisable.

[1] The stationary state is not yet reached.

3.2 Continuous Probability Distribution

The model with discrete values for $q_{i,g}^{\mu}$ can be easily extended to a more realistic model involving continuous values for $q_{i,g}^{\mu}$ just by changing the probability distribution.

Let us assume that the consumers are able to determine an exact amount of goods they are going to purchase, i. e. $q_{i,g}^{\mu} \in [0, q_{max}]$. q_{max} denotes the maximum amount of one kind of good, which can be bought per consumer each day. A reasonable choice for the probability distribution is for instance to weight the whole interval uniformly. Another possibility would be to constitute that small amounts are more likely to be bought than large ones. Although many different distributions are reasonable let us focus on the continuous uniform distribution. As we have seen before the results for H and σ_v^2 in the case of a continuous uniform distribution are equal to those obtained for the case of an exponential distribution with $\frac{1}{\lambda^2} = \frac{q_{max}^2}{12}$. Hence, the following solutions are also valid for the case of the exponential distribution with appropriate choice of λ.

Using the variance of the continuous uniform distribution, we obtain figure 3.2. Figure 3.2 shows, similar to figure 3.1, the measure for the distributions of the consumers over the sellers H and the approximate volatility σ_v^2 respectively versus $\alpha = \frac{P}{N}$. Dependent on the choice of the variance, i. e. $\frac{q_{max}^2}{12}$, the curves take a different quantitative behaviour. The qualitative behaviour is not affected by different variance choices.

Just by choosing any reasonable probability distribution and calculating its variance, we are able to adapt our solutions to a wide range of models.

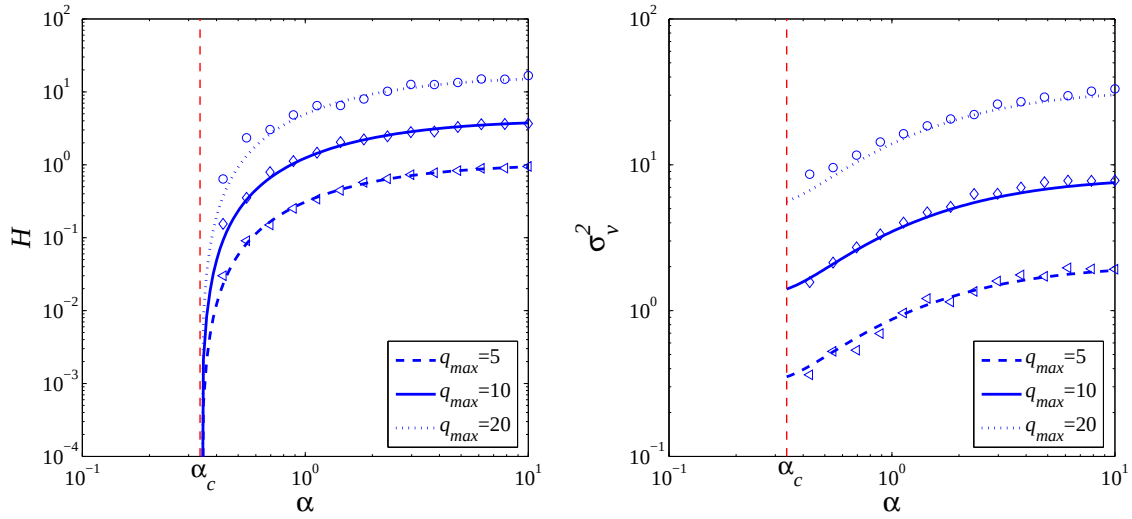

Figure 3.2: Left: Market efficiency measure in the stationary state for $N \to \infty$, viz. H versus the ratio of the number of sellers to the number of consumers $\alpha = \frac{P}{N}$ for different q_{max}. Right: Approximation for the volatility in the stationary state for $N \to \infty$, viz. σ_v^2 versus α for different q_{max}. The vertical lines represent the critical value α_c. For $\alpha > \alpha_c$ the predicted solution for H and the approximate solutions for σ_v^2 respectively are shown. The markers correspond to numerical simulations of the model equations (see section 2.2) for the fixed product $NP = 250,000$.

Chapter 4

Conclusions

> *"Modern physics can offer much to economics. Not just different [mathematical] tools and different methods of analysis, but different concepts [...] Economics needs this."*
> *W. Brian Arthur (taken from [1])*

In chapter 1, we started with a brief introduction to minority games in general and the historical development of this novel field of research. It was shown that minority games and mathematical tools such as generating functional formalism often applied in this context enable mathematicians, physicists and economists to analyse an economic problem, which cannot be examined by the means of general equilibrium theory. Even though mostly applied to financial problems, minority game has a far-ranging application area. For instance seller-consumer markets can be well modelled with the help of minority games and analysed with the means of generating functional formalism.

In chapter 2, we modelled the dynamics of a market with heterogeneous adaptive consumers and discussed some associated ideas for the example of a daily vegetable market. Afterwards, the generating functional formalism is introduced and then applied to our model situation. After having accomplished an average over the strategies of the generating functional for a general probability distribution, we found expressions for the macroscopic correlation and response functions. Further calculations led us to the effective single-consumer equation, which described exactly the same dynamics as our original model but simplified the analysis enormously. In the stationary state we obtained macroscopic persistent observables

(correlations, fraction of frozen consumers, susceptibility) which described the model dynamics exactly. Calculations for a measure of how evenly consumers are distributed over the sellers and the approximate volatility were accomplished for the region $\alpha > \alpha_c$. All our results were dependent only on the variance of the probability distribution and not on the distribution itself.

In chapter 3, we applied our results from chapter 2 to different model assumptions by choosing appropriate probability distributions and calculating their variance. For α near α_c the market, independent of the chosen probability distribution, achieved the most desirable state from the collective point of view in the regime $\alpha > \alpha_c$. The fluctuations are small and the distribution of the consumers over the sellers is almost uniform. This observation suggests, at least when we only consider the region $\alpha > \alpha_c$, that (almost) efficient markets have a certain consumer-seller ratio, viz. near α_c. For these efficient markets the integrated response increases very rapidly. Hence, to avoid a sensitive market situation it would be advisable to keep a certain distance to α_c.

Altogether, one has to find a compromise between both, the efficiency of the market and the non-susceptibility of the consumers to perturbations.

4.1 Outlook

Our considered model allows applications for a variety of different economies, as indicated in the introduction. As well as a vegetable market, markets which satisfy the model assumption can be examined by this thesis' means. Another prime example could for instance be a flea market. The satisfaction level of each consumer is determined in this example by the price of the goods. The higher the demand, the larger the prices will be. Therefore the consumers aim to find a seller with low prices, i. e. little demand.

Further study starting from the results of this thesis could include the application of more probability distributions. The solutions can be compared and conclusions drawn. Comparison with statistical data can then provide solutions to the problem of how realistic our model is in general or for different probability distributions. Also model extensions such as introducing the property of passive consumers, i. e. each consumer is given the trivial strategy not to purchase at all,

are reasonable. For these extensions we may suppose that an exact solution can be obtained.

The most challenging further research will certainly be the examination of the non-ergodic region $\alpha < \alpha_c$. In 2005, Coolen [6] already discovered that "for this region the system has the tendency to develop asymptotic solutions which are either fixed-points, or period-two oscillations, or randomly perturbed period-two oscillations". First steps in the direction of finding an analytical solution in the non-ergodic regime have been attempted, but many questions remain still unsolved. The generating functional formalism will certainly play an important role in answering these questions.

The conjunction between mathematical modelling and economics offers the best possibilities for the future to find methods, which are successful in explaining people's behaviour and indicate how to improve the overall situation.

Appendix

Appendix A

Mathematical Methods and Proofs

In this first part of the appendix we describe and prove some important methods and results used frequently in this thesis. For further reading especially Coolen's book [6] (2005) can be recommended.

A.1 Steepest Descent Integration

The steepest descent integration is a method to approximate integrals of the following form

$$\int \exp\left(Nf(x)\right) g(x) dx,$$

where $f(x)$ is twice-differentiable, $g(x)$ is continuous and $N \in \mathbb{R}$ is large. The method is, especially when applied to real-valued functions f, also referred to as Laplace's method. Let us prove the real case here and refer for the complex case to Coolen's book [6] (2005). In the following we will use, for the sake of clarity, the commonly used expressions $f'(x_0)$ and $f''(x_0)$ instead of $\frac{df}{dx}\Big|_{x=x_0}$ and $\frac{d^2f}{dx^2}\Big|_{x=x_0}$ respectively.

The method is subject to the following assumptions:

- $f(x)$ has a unique global maximum at x_0 with $f''(x_0) < 0$,

- x_0 is not an endpoint of the interval of integration (in our case fulfilled anyway, since the integral endpoints are $\pm\infty$),
- $f(x) \ll f(x_0)$ unless x is close to x_0.

Note that in principle the proof can also be accomplished for different assumptions. For more details see Coolen (2005) [6].

We can expand $f(x)$ around x_0 by using Taylor's theorem and find

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \mathcal{O}(|x - x_0|^3).$$

According to our assumptions, x_0 has to be a stationary point, i. e. $f'(x_0) = 0$. The function $f(x)$ for x close to x_0 can therefore be approximated by

$$f(x) \approx f(x_0) - \frac{1}{2}|f''(x_0)|(x - x_0)^2.$$

Using this to approximate the requested integral $\int \exp(Nf(x))\, g(x)dx$, we get

$$\int \exp(Nf(x))\, g(x)dx \approx \exp(Nf(x_0)) \int \exp\left(-\frac{N}{2}|f''(x_0)|(x - x_0)^2\right) g(x)dx.$$

Finally, making a change of variables $x = x_0 + \frac{y}{\sqrt{N}}$, we obtain

$$\int \exp(Nf(x))\, g(x)dx \approx \frac{\exp(Nf(x_0))}{\sqrt{N}} \int \exp\left(-\frac{1}{2}|f''(x_0)|y^2\right) g\left(x_0 + \frac{y}{\sqrt{N}}\right) dy.$$

From this we can deduce by using a Gaussian integral (appendix A.3) the following identity for $N \to \infty$:

$$\begin{aligned}\lim_{N\to\infty} \frac{\int \exp(Nf(x))g(x)dx}{\int \exp(Nf(x))dx} &= \frac{g(x_0)\sqrt{\frac{2\pi}{|f''(x_0)|}}}{\sqrt{\frac{2\pi}{|f''(x_0)|}}} \\ &= g(x_0) \\ &= g(\arg\max_{x\in\mathbb{R}} f(x)).\end{aligned}$$

A.2 Proofs of Results Used in Chapter 2

Proposition 1 is used on page 21.

Proposition 1 $\langle\hat{p}(t)\rangle_{*,i} = 0$ *and* $\langle\hat{p}(t)\hat{p}(t')\rangle_{*,i} = 0$.

Proof. First part of the proof: Since the argument of $\overline{Z}[\boldsymbol{\psi}]$, $\exp\left(\mathrm{i}\sum_{t=0}^{t_{max}}\sum_{i=1}^{N}\psi_i(t)s_i(t)\right)$, is independent of $\theta_i(t)$ we find

$$\begin{aligned}\lim_{\boldsymbol{\psi}\to\mathbf{0}}\frac{\partial\overline{Z}[\boldsymbol{\psi}]}{\partial\theta_i(t)} &= 0\\ &= \lim_{\boldsymbol{\psi}\to\mathbf{0}}\int\exp\left[N(\Psi+\Phi+\Omega)+\mathcal{O}(t_{max}^2\log(N))\right]\\ &\quad\times N\frac{\partial\Omega}{\partial\theta_i(t)}[\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\\ &= -\mathrm{i}\langle\hat{p}_i(t)\rangle_{*,i}\end{aligned}$$

$$\Longrightarrow \langle\hat{p}_i(t)\rangle_{*,i} = 0.$$

Second part of the proof: By using proposition 1 we get

$$\begin{aligned}\lim_{\boldsymbol{\psi}\to\mathbf{0}}\frac{\partial^2\overline{Z}[\boldsymbol{\psi}]}{\partial\theta_i(t)\partial\theta_j(t')} &= 0\\ &= \lim_{\boldsymbol{\psi}\to\mathbf{0}}\int\exp\left[N(\Psi+\Phi+\Omega)+\mathcal{O}(t_{max}^2\log(N))\right]\\ &\quad\times\left(N^2\frac{\partial\Omega}{\partial\theta_i(t)}\frac{\partial\Omega}{\partial\theta_j(t')}+N\frac{\partial^2\Omega}{\partial\theta_i(t)\partial\theta_j(t')}\right)[\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]\\ &= -\langle\hat{p}(t)\rangle_{*,i}\langle\hat{p}(t')\rangle_*^j+\delta_{ij}\left(\langle\hat{p}(t)\rangle_{*,i}\langle\hat{p}(t')\rangle_*^j-\langle\hat{p}(t)\hat{p}(t')\rangle_{*,i}\right)\\ &= \langle\hat{p}(t)\hat{p}(t')\rangle_{*,i}\end{aligned}$$

$$\Longrightarrow \langle\hat{p}(t)\hat{p}(t')\rangle_{*,i} = 0.$$

□

Proposition 2 is used on page 33.

Proposition 2 $\lim_{\tau\to\infty}\frac{1}{\tau^2}\sum_{t,t'=0}^{\tau-1}C(t,t') = \lim_{\tau\to\infty}C(\tau)$.

Proof. We assume that the limit $c := \lim_{\tau\to\infty}C(\tau)$ exists. In the time-translation invariant stationary state we have the condition $C(t,t') = C(t-t')$. From the

definition of the correlation we deduce that $C(-t) = C(t)$. After a change of variables one finds

$$\begin{aligned}\frac{1}{\tau^2}\sum_{t,t'=0}^{\tau-1} C(t-t') &= \frac{1}{\tau^2}\sum_{s=1-\tau}^{\tau-1} C(s)|\tau - s| \\ &= \frac{2}{\tau^2}\sum_{s=0}^{\tau-1} C(s)(\tau - s) - \frac{C(0)}{\tau}\end{aligned}$$

$$\stackrel{\tau\to\infty}{\longrightarrow} \lim_{\tau\to\infty}\frac{2}{\tau^2}\sum_{s=0}^{\tau-1} C(s)(\tau - s).$$

We define $E := \lim_{\tau\to\infty}\frac{2}{\tau^2}\sum_{s=0}^{\tau-1} C(s)(\tau - s)$. Since $C(s) \stackrel{s\to\infty}{\longrightarrow} c$, $\forall\epsilon > 0\ \exists\ S\ \forall s > S : |C(S) - c| < \epsilon$. Hence,

$$\begin{aligned}E &= \lim_{\tau\to\infty}\frac{2}{\tau^2}\left[\sum_{s=0}^{S} C(s)(\tau - s) + \sum_{s=S+1}^{\tau-1} C(s)(\tau - s)\right] \\ &= \lim_{\tau\to\infty}\left[\frac{2}{\tau^2}\sum_{s=0}^{S} C(s)(\tau - s)\right] + \lim_{\tau\to\infty}\left[\frac{2}{\tau^2}\sum_{s=S+1}^{\tau-1} C(s)(\tau - s)\right] \\ &= \lim_{\tau\to\infty}\left[\frac{2}{\tau^2}\sum_{s=S+1}^{\tau-1} C(s)(\tau - s)\right] \\ &=: D.\end{aligned}$$

We know that

$$\begin{aligned}
D &\leq \lim_{\tau\to\infty}\left[\frac{2}{\tau^2}\sum_{s=S+1}^{\tau-1}[c+\epsilon](\tau-s)\right] \\
&= \lim_{\tau\to\infty}\frac{2}{\tau^2}[c+\epsilon]\sum_{s=S+1}^{\tau-1}(\tau-s) \\
&= \lim_{\tau\to\infty}\frac{2}{\tau^2}[c+\epsilon]\sum_{s=S+1}^{\tau-1}(\tau-s) + \underbrace{\lim_{\tau\to\infty}\frac{2}{\tau^2}[c+\epsilon]\sum_{s=0}^{S}(\tau-s)}_{=0} \\
&= \lim_{\tau\to\infty}\frac{2}{\tau^2}[c+\epsilon]\sum_{s=0}^{\tau-1}(\tau-s) \\
&= \lim_{\tau\to\infty}\frac{2}{\tau^2}[c+\epsilon]\sum_{t=1}^{\tau}t \\
&= \lim_{\tau\to\infty}\frac{2}{\tau^2}[c+\epsilon]\frac{\tau(\tau+1)}{2} \\
&= c+\epsilon.
\end{aligned}$$

Analogously, $D \geq c-\epsilon$.
Since both hold for $\forall\epsilon>0$, $D=c$ and therefore $E=c$. □

A.3 Gaussian Integration

The following Gaussian integrals are frequently used in this thesis. For proofs see Coolen [6] (2005).

$$\int \exp\left(-\frac{1}{2}x^2\right)dx = \sqrt{2\pi} \quad \text{for} \quad x\in\mathbb{R}.$$

$$\int\cdots\int \exp\left(-\frac{1}{2}\mathbf{x^T A x}+\mathbf{b^T x}\right)\prod_{\ell=1}^{N}\frac{dx_\ell}{\sqrt{2\pi}} = \frac{\exp\left(\frac{1}{2}\mathbf{b^T A^{-1} b}\right)}{\sqrt{\det(\mathbf{A})}}$$

for $\mathbf{x}\in\mathbb{R}^N$ and $\mathbf{A}\in\mathbb{R}^{N\times N}$ symmetric positive definite.

$$\int\cdots\int x_i x_j \exp\left(-\frac{1}{2}\mathbf{x^T A x}\right)\prod_{\ell=1}^{N}\frac{dx_\ell}{\sqrt{2\pi}} = \frac{(\mathbf{A}^{-1})_{i,j}}{\sqrt{\det(\mathbf{A})}}$$

for $i,j\in\{1,2,\ldots,N\}$, $\mathbf{x}\in\mathbb{R}^N$ and $\mathbf{A}\in\mathbb{R}^{N\times N}$ symmetric positive definite.

A.4 Proofs of Lemmata 1 and 2

Proof of lemma 1 (page 30). Define $a_\tau = \sum_{t=0}^{\tau-1} A_t$. Let us first substitute $t - t'$ by s and t' by s'.

$$\frac{1}{\tau}\sum_{t=0}^{\tau-1}\sum_{t'=0}^{t} A_{t-t'}b_{t'} = \frac{1}{\tau}\sum_{s'=0}^{\tau-1} b_{s'} \sum_{s=0}^{\tau-1-s'} A_s = \frac{1}{\tau}\sum_{s'=0}^{\tau-1} a_{\tau-1-s'}b_{s'}$$

Since the sequences $\{a_t\}$ and $\{b_t\}$ are bounded[1], real numbers B_a and B_b exist with $|a_t| < B_a$ and $|b_t| < B_b$ for all $t \geq 0$. Furthermore $\lim_{t\to\infty} a_t = a$, i. e. for any ϵ there exists a number K such that for all $t > K$: $|a_t - a| < \frac{\epsilon}{3B_b}$. We choose a number M such that for all $\tau > M$: $\left|\frac{1}{\tau}\sum_{t=0}^{\tau-1} b_t - b\right| < \frac{\epsilon}{3|a|}$ and $\frac{KB_aB_b}{\tau} < \frac{\epsilon}{3}$. Then we get for all $\tau > M$ by using the triangle inequality

$$\begin{aligned}
& \left|\frac{1}{\tau}\sum_{s'=0}^{\tau-1} a_{\tau-1-s'}b_{s'} - ab\right| \\
= & \left|\frac{1}{\tau}\sum_{s'=0}^{\tau-K-1} a_{\tau-1-s'}b_{s'} + \frac{1}{\tau}\sum_{s'=\tau-K}^{\tau-1} a_{\tau-1-s'}b_{s'} - ab\right| \\
\leq & \left|\frac{1}{\tau}\sum_{s'=0}^{\tau-K-1} (a_{\tau-1-s'} - a)\, b_{s'} - a\left(b - \frac{1}{\tau}\sum_{s'=0}^{\tau-K-1} b_{s'}\right)\right| + \left|\frac{1}{\tau}\sum_{s'=\tau-K}^{\tau-1} a_{\tau-1-s'}b_{s'}\right| \\
\leq & \left|\frac{1}{\tau}\sum_{s'=0}^{\tau-K-1} (a_{\tau-1-s'} - a)\, b_{s'}\right| + |a|\left|b - \frac{1}{\tau}\sum_{s'=0}^{\tau-K-1} b_{s'}\right| + \frac{KB_aB_b}{\tau} \\
\leq & \epsilon
\end{aligned}$$

and the proposition is proved. □

Proof by induction of lemma 2 (page 30). For $n = 0$ the proposition is trivially true. Assume the following hypothesis is true for all $0 \leq n \leq k$: $\lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=0}^{\tau-1}\sum_{t'=0}^{t} G^k(t,t')s(t') = \chi^k s$. Then

$$\begin{aligned}
& \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=0}^{\tau-1}\sum_{t'=0}^{t} G^{k+1}(t,t')s(t') \\
= & \lim_{\tau\to\infty} \frac{1}{\tau}\sum_{t=0}^{\tau-1}\sum_{t''=0}^{t-1} G(t,t'')\sum_{t'=0}^{t''} G^k(t'',t')s(t').
\end{aligned}$$

[1] The first is true, because $\{A_t\}$ is bounded and $a_\tau = \sum_{t=0}^{\tau-1} A_t$.

By applying lemma 1 for the bounded and real number sequence $b_{t''} = \sum_{t'=0}^{t''} G^k(t'', t')s(t')$, we get

$$\lim_{\tau\to\infty} \frac{1}{\tau} \sum_{t=0}^{\tau-1} \sum_{t'=0}^{t} G^{k+1}(t, t')s(t') = \chi \cdot \chi^k s = \chi^{k+1} s.$$

Hence the proposition is true for all $n \in \mathbb{N}_0$. □

Appendix B

Detailed Calculations

The generating functional formalism allows to find analytical solutions for our problem. This advantage requires extensive calculations, which would go beyond the scope of this thesis. Therefore detailed calculations are introduced in this part of the appendix. All definitions used are defined in the main text of the thesis.

Calculations for the disorder average from page 14 and 15:

$$
\begin{aligned}
& \exp\left[2\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{i=1}^{N}\hat{p}_i(t)\sum_{\mu=1}^{P}\xi_i^\mu\left(\frac{1}{N}\sum_{j=1}^{N}\left(\omega_j^\mu + s_j(t)\xi_j^\mu\right) - k\right)\right] \\
= \; & \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{i=1}^{N}\hat{p}_i(t)\left(\frac{2}{N}\sum_{\mu=1}^{P}\sum_{j=1}^{N}\xi_i^\mu\omega_j^\mu + \frac{2}{N}\sum_{\mu=1}^{P}\sum_{j=1}^{N}\xi_i^\mu\xi_j^\mu s_j(t) - 2\sum_{\mu=1}^{P}\xi_i^\mu k\right)\right] \\
= \; & \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left[\left(\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\omega_j^\mu\right)\left(\sqrt{\frac{2}{N}}\sum_{i=1}^{N}\hat{p}_i(t)\xi_i^\mu\right)\right.\right. \\
& \left.\left. + \left(\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\xi_j^\mu s_j(t)\right)\left(\sqrt{\frac{2}{N}}\sum_{i=1}^{N}\hat{p}_i(t)\xi_i^\mu\right) - \left(2k\sum_{i=1}^{N}\hat{p}_i(t)\xi_i^\mu\right)\right]\right] \\
= \; & \prod_{t=0}^{t_{max}-1}\prod_{\mu=1}^{P}\int\delta\left[w^\mu(t) - \sqrt{\frac{2}{N}}\sum_{i=1}^{N}\hat{p}_i(t)\xi_i^\mu\right]\delta\left[x^\mu(t) - \sqrt{\frac{2}{N}}\sum_{i=1}^{N}s_i(t)\xi_i^\mu\right] \\
& \times\exp\left[\mathrm{i}\left(\left(\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\omega_j^\mu\right)w^\mu(t) + x(t)^\mu w^\mu(t) - k\sqrt{2N}w^\mu(t)\right)\right]dw^\mu(t)dx^\mu(t)
\end{aligned}
$$

$$
\begin{aligned}
= & \int \exp\left[\mathrm{i}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\hat{w}^{\mu}(t)w^{\mu}(t)+\hat{x}^{\mu}(t)x^{\mu}(t)+x^{\mu}(t)w^{\mu}(t)-k\sqrt{2N}w^{\mu}(t)\right)\right] \\
& \times \exp\left[\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\left(\sum_{j=1}^{N}\omega_j^{\mu}\right)w^{\mu}(t)-\sum_{i=1}^{N}\hat{w}^{\mu}(t)\hat{p}_i(t)\xi_i^{\mu}\right.\right. \\
& \left.\left.-\sum_{i=1}^{N}\hat{x}^{\mu}(t)s_i(t)\xi_i^{\mu}\right)\right]\mathcal{D}\mathbf{w}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\hat{\mathbf{x}}.
\end{aligned}
$$

$$
\begin{aligned}
& \overline{\exp\left[\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{t=0}^{t_{max}-1}\sum_{\mu=1}^{P}\left(\left(\sum_{j=1}^{N}\omega_j^{\mu}\right)w^{\mu}(t)-\sum_{i=1}^{N}\hat{w}^{\mu}(t)\hat{p}_i(t)\xi_i^{\mu}-\sum_{i=1}^{N}\hat{x}^{\mu}(t)s_i(t)\xi_i^{\mu}\right)\right]} \\
= & \prod_{\mu=1}^{P}\prod_{i=1}^{N}\overline{\exp\left[\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{t=0}^{t_{max}-1}\left(\omega_i^{\mu}w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)\xi_i^{\mu}-\hat{x}^{\mu}(t)s_i(t)\xi_i^{\mu}\right)\right]} \\
= & \prod_{\mu=1}^{P}\prod_{i=1}^{N}\left[\int\prod_{a=1,2}P(q_{i,a}^{\mu})\left[\exp\left(\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_{t=0}^{t_{max}-1}\left(\omega_i^{\mu}w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)\xi_i^{\mu}-\right.\right.\right.\right. \\
& \left.\left.\left.\left.\hat{x}^{\mu}(t)s_i(t)\xi_i^{\mu}\right)\right)\right]\prod_{a=1,2}dq_{i,a}^{\mu}\right] \\
= & \prod_{\mu=1}^{P}\prod_{i=1}^{N}\left[\int P(q_{i,1}^{\mu})\left[\exp\left(\frac{\mathrm{i}q_{i,1}^{\mu}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left(w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right)\right)\right]dq_{i,1}^{\mu}\right. \\
& \left.\int P(q_{i,2}^{\mu})\left[\exp\left(\frac{\mathrm{i}q_{i,2}^{\mu}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left(w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right)\right)\right]dq_{i,2}^{\mu}\right] \\
= & \prod_{\mu=1}^{P}\prod_{i=1}^{N}\left[\int P(q_{i,1}^{\mu})\left[1+\frac{\mathrm{i}q_{i,1}^{\mu}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right]\right.\right. \\
& -\frac{(q_{i,1}^{\mu})^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right]\left[w^{\mu}(t')-\hat{w}^{\mu}(t')\hat{p}_i(t')\right. \\
& \left.\left.-\hat{x}^{\mu}(t')s_i(t')\right]\right]dq_{i,1}^{\mu} \\
& \times\int P(q_{i,2}^{\mu})\left[1+\frac{\mathrm{i}q_{i,2}^{\mu}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(q^{\mu}_{i,2})^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right]\left[w^{\mu}(t')+\hat{w}^{\mu}(t')\hat{p}_i(t')\right.\\
&\left.+\hat{x}^{\mu}(t')s_i(t')\right]\Bigg]dq^{\mu}_{i,2}\Bigg]\\
=\;&\prod_{\mu=1}^{P}\prod_{i=1}^{N}\Bigg[\Bigg(1+\int P(q^{\mu}_{i,1})\frac{\mathrm{i}q^{\mu}_{i,1}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right]dq^{\mu}_{i,1}\\
&-\int P(q^{\mu}_{i,1})\frac{(q^{\mu}_{i,1})^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right]\left[w^{\mu}(t')\right.\\
&\left.-\hat{w}^{\mu}(t')\hat{p}_i(t')-\hat{x}^{\mu}(t')s_i(t')\right]dq^{\mu}_{i,1}\Bigg)\\
&\times\Bigg(1+\int P(q^{\mu}_{i,2})\frac{\mathrm{i}q^{\mu}_{i,2}}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right]dq^{\mu}_{i,2}\\
&-\int P(q^{\mu}_{i,2})\frac{(q^{\mu}_{i,2})^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right]\left[w^{\mu}(t')\right.\\
&\left.+\hat{w}^{\mu}(t')\hat{p}_i(t')+\hat{x}^{\mu}(t')s_i(t')\right]dq^{\mu}_{i,2}\Bigg)\Bigg]\\
=\;&\prod_{\mu=1}^{P}\prod_{i=1}^{N}\Bigg[\Bigg(1+\frac{\mathrm{i}\,m_1}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right]\\
&-\frac{m_2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^{\mu}(t)-\hat{w}^{\mu}(t)\hat{p}_i(t)-\hat{x}^{\mu}(t)s_i(t)\right]\left[w^{\mu}(t')-\hat{w}^{\mu}(t')\hat{p}_i(t')\right.\\
&\left.-\hat{x}^{\mu}(t')s_i(t')\right]\Bigg)\\
&\times\Bigg(1+\frac{\mathrm{i}\,m_1}{\sqrt{2N}}\sum_{t=0}^{t_{max}-1}\left[w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right]\\
&-\frac{m_2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}\left[w^{\mu}(t)+\hat{w}^{\mu}(t)\hat{p}_i(t)+\hat{x}^{\mu}(t)s_i(t)\right]\left[w^{\mu}(t')+\hat{w}^{\mu}(t')\hat{p}_i(t')\right.\\
&\left.+\hat{x}^{\mu}(t')s_i(t')\right]\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \prod_{\mu=1}^{P}\prod_{i=1}^{N}\Bigg[& \exp\Bigg[\frac{\mathrm{i}\sqrt{2}m_1}{\sqrt{N}}\sum_{t=0}^{t_{max}-1} w^\mu(t) - \frac{m_2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}(w^\mu(t) \\
& -\hat{w}^\mu(t)\hat{p}_i(t) - \hat{x}^\mu(t)s_i(t))\,(w^\mu(t') - \hat{w}^\mu(t')\hat{p}_i(t') - \hat{x}^\mu(t')s_i(t'))\Bigg] \\
& \times \exp\Bigg[-\frac{m_2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}(w^\mu(t) + \hat{w}^\mu(t)\hat{p}_i(t) + \hat{x}^\mu(t)s_i(t))\,(w^\mu(t') \\
& +\hat{w}^\mu(t')\hat{p}_i(t') + \hat{x}^\mu(t')s_i(t'))\Bigg] \\
& \times \exp\Bigg[\frac{m_1^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}(w^\mu(t) - \hat{w}^\mu(t)\hat{p}_i(t) - \hat{x}^\mu(t)s_i(t))\,(w^\mu(t') \\
& -\hat{w}^\mu(t')\hat{p}_i(t') - \hat{x}^\mu(t')s_i(t'))\Bigg] \\
& \times \exp\Bigg[\frac{m_1^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}(w^\mu(t) + \hat{w}^\mu(t)\hat{p}_i(t) + \hat{x}^\mu(t)s_i(t))\,(w^\mu(t') \\
& +\hat{w}^\mu(t')\hat{p}_i(t') + \hat{x}^\mu(t')s_i(t'))\Bigg]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \prod_{\mu=1}^{P}\prod_{i=1}^{N}\Bigg[& \exp\Bigg[\frac{\mathrm{i}\sqrt{2}m_1}{\sqrt{N}}\sum_{t=0}^{t_{max}-1} w^\mu(t) - \frac{m_2-m_1^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}(w^\mu(t) \\
& -\hat{w}^\mu(t)\hat{p}_i(t) - \hat{x}^\mu(t)s_i(t))\,(w^\mu(t') - \hat{w}^\mu(t')\hat{p}_i(t') - \hat{x}^\mu(t')s_i(t'))\Bigg] \\
& \times \exp\Bigg[-\frac{m_2-m_1^2}{4N}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}(w^\mu(t) + \hat{w}^\mu(t)\hat{p}_i(t) + \hat{x}^\mu(t)s_i(t))\,(w^\mu(t') \\
& +\hat{w}^\mu(t')\hat{p}_i(t') + \hat{x}^\mu(t')s_i(t'))\Bigg]\Bigg] \\
= \exp & \left(\frac{\mathrm{i}\sqrt{2}m_1}{\sqrt{N}}\sum_{\mu=1}^{P}\sum_{i=1}^{N}\sum_{t=0}^{t_{max}-1} w^\mu(t)\right) \\
& \times \exp\Bigg[-\frac{m_2-m_1^2}{2}\sum_{\mu=1}^{P}\sum_{t=0}^{t_{max}-1}\sum_{t'=0}^{t_{max}-1}(w^\mu(t)w^\mu(t') + \hat{w}^\mu(t)L(t,t')\hat{w}^\mu(t') \\
& +2\hat{w}^\mu(t')K(t,t')\hat{x}^\mu(t) + \hat{x}^\mu(t)C(t,t')\hat{x}^\mu(t'))\Bigg],
\end{aligned}
$$

Evaluation of Φ from page 23:

$$
\begin{aligned}
\Phi &= \alpha \log \Bigg[\int \exp \Bigg[\mathrm{i} \sum_{t=0}^{t_{max}-1} (\hat{w}(t)w(t) + \hat{x}(t)x(t) + x(t)w(t)) \Bigg] \\
&\quad \times \exp \Bigg[-\frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} (w(t)w(t') + \hat{w}(t)L(t,t')\hat{w}(t') \\
&\quad +2\hat{w}(t')K(t,t')\hat{x}(t) + \hat{x}(t)C(t,t')\hat{x}(t')) \Bigg] \mathcal{D}w\mathcal{D}x\mathcal{D}\hat{w}\mathcal{D}\hat{x} \Bigg] \\
&= \alpha \log \Bigg[\int \exp \left(\mathrm{i} \sum_{t=0}^{t_{max}-1} \hat{w}(t)w(t) \right) \prod_{t=0}^{t_{max}-1} \delta(\hat{x}(t) + w(t)) \\
&\quad \times \exp \Bigg[-\frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} (w(t)w(t') + \hat{w}(t)L(t,t')\hat{w}(t') \\
&\quad +2\hat{w}(t')K(t,t')\hat{x}(t) + \hat{x}(t)C(t,t')\hat{x}(t')) \Bigg] \mathcal{D}w\mathcal{D}\hat{w} \prod_t d\hat{x}(t) \Bigg] \\
&= \alpha \log \Bigg[\int \exp \left(\mathrm{i} \sum_{t=0}^{t_{max}-1} \hat{w}(t)w(t) \right) \\
&\quad \times \exp \Bigg[-\frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} (w(t)w(t') + \hat{w}(t)L(t,t')\hat{w}(t') \\
&\quad -2\hat{w}(t')K(t,t')w(t) + w(t)C(t,t')w(t')) \Bigg] \mathcal{D}w\mathcal{D}\hat{w} \Bigg] \\
&= \alpha \log \Bigg[\int \exp \Bigg[-\frac{1}{2\Delta_2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \hat{w}(t) \left[(G')^T D^{-1} G' \right](t,t')\hat{w}(t') \Bigg] \\
&\quad \times \exp \left(-\frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \hat{w}(t)L(t,t')\hat{w}(t') \right) \mathcal{D}\hat{w} \Bigg] \\
&\quad -\frac{1}{2}\alpha \log \Big[\det(\Delta_2 D) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha \log \Bigg[\int \exp \Bigg[-\frac{1}{2\Delta_2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \hat{w}(t) \left((G')^T D^{-1} G' \right)(t,t') \hat{w}(t') \Bigg] \\
&\quad \times \frac{1 - \frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \hat{w}(t) L(t,t') \hat{w}(t') + \mathcal{O}(L^2)}{\left[\det \left(\frac{1}{\Delta_2} (G')^T D^{-1} G' \right) \right]^{-\frac{1}{2}}} \mathcal{D}\hat{w} \Bigg] \\
&\quad -\frac{1}{2} \alpha \log \left[\det \left(\Delta_2 D \right) \right] - \frac{1}{2} \alpha \log \left[\det \left(\frac{1}{\Delta_2} (G')^T D^{-1} G' \right) \right] \\
&= \alpha \log \Bigg[1 - \frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} L(t,t') \sqrt{\det \left(\frac{1}{\Delta_2} (G')^T D^{-1} G' \right)} \\
&\quad \int \hat{w}(t) \hat{w}(t') \exp \Big[-\frac{1}{2\Delta_2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} \hat{w}(t) \left((G')^T D^{-1} G' \right)(t,t') \hat{w}(t') \Big] \mathcal{D}\hat{w} \Bigg] \\
&\quad -\frac{1}{2} \alpha \log \left[\det \left((G')^T G' \right) \right] + \mathcal{O}(L^2) \\
&= \alpha \log \Bigg[1 - \frac{\Delta_2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} L(t,t') \left[\frac{1}{\Delta_2} (G')^T D^{-1} G' \right]^{-1} (t,t') \Bigg] \\
&\quad -\frac{1}{2} \alpha \log \left[\det \left((G')^T G' \right) \right] + \mathcal{O}(L^2) \\
&= -\alpha \frac{\Delta_2^2}{2} \sum_{t=0}^{t_{max}-1} \sum_{t'=0}^{t_{max}-1} L(t,t') \left[(G')^T D^{-1} G' \right]^{-1} (t,t') \\
&\quad -\alpha \operatorname{tr} \left[\log \left(G' \right) \right] + \mathcal{O}(L^2)
\end{aligned}
$$

Calculation of the persistent correlations from page 34:

$$
\begin{aligned}
c &= \lim_{\tau\to\infty}\frac{1}{\tau^2}\sum_{t,t'=0}^{\tau-1}\langle s(t)s(t')\rangle_* \\
&= \langle \bar{s}^2\rangle_* \\
&= \int \rho_0(p(0))\int \frac{\exp\left[-\frac{1}{2}\sum_{t,t'=0}^{\infty}\eta(t)\Lambda^{-1}(t,t')\eta(t')\right]}{\sqrt{\det(\Lambda)}} \\
&\quad \times\prod_{t=0}^{\infty}\delta\left[p(t+1)-p(t)-\theta(t)-\sum_{t'=0}^{\infty}R(t,t')s(t')\right. \\
&\quad \left.-\sqrt{\alpha\Delta_2}\eta(t)\right]\prod_{t=0}^{\infty}\left[\frac{d\eta(t)}{\sqrt{2\pi}}\right]\bar{s}^2\prod_{t=0}^{\infty}dp(t) \\
&= \lim_{\tau\to\infty}\int \frac{\exp\left[-\frac{1}{2}\sum_{t,t'=0}^{\tau-1}\eta(t)\Lambda^{-1}(t,t')\eta(t')\right]}{\sqrt{\det(\Lambda)}}\prod_{t=0}^{\tau-1}\left[\frac{d\eta(t)}{\sqrt{2\pi}}\right]\left[\frac{1}{\tau}\sum_{t=0}^{\tau-1}s(t)\right]^2 \\
&= \lim_{\tau\to\infty}\int \delta\left[\bar{\eta}-\frac{1}{\tau}\sum_{t=0}^{\tau-1}\eta(t)\right]\frac{\exp\left[-\frac{1}{2}\sum_{t,t'=0}^{\tau-1}\eta(t)\Lambda^{-1}(t,t')\eta(t')\right]}{\sqrt{\det(\Lambda)}}\prod_{t=0}^{\tau-1}\left[\frac{d\eta(t)}{\sqrt{2\pi}}\right] \\
&\quad \left[\mathcal{H}\left(\frac{\sqrt{\alpha\Delta_2}}{1+\Delta_2\chi}-|\bar{\eta}|\right)\frac{\bar{\eta}^2\left(1+\Delta_2\chi\right)^2}{\alpha\Delta_2}\right. \\
&\quad \left.+\mathcal{H}\left(|\bar{\eta}|-\frac{\sqrt{\alpha\Delta_2}}{1+\Delta_2\chi}\right)\right]d\bar{\eta} \\
&= \int P(\bar{\eta})\left[\mathcal{H}\left(\frac{\sqrt{\alpha\Delta_2}}{1+\Delta_2\chi}-|\bar{\eta}|\right)\frac{\bar{\eta}^2\left(1+\Delta_2\chi\right)^2}{\alpha\Delta_2}\right. \\
&\quad \left.+\mathcal{H}\left(|\bar{\eta}|-\frac{\sqrt{\alpha\Delta_2}}{1+\Delta_2\chi}\right)\right]d\bar{\eta} \\
&= 2\int_0^{\infty}\exp\left[-\frac{z^2}{2}\right]\left[\mathcal{H}\left(\frac{\sqrt{\alpha}}{\sqrt{1+c}}-z\right)\frac{(1+c)z^2}{\alpha}+\mathcal{H}\left(z-\frac{\sqrt{\alpha}}{\sqrt{1+c}}\right)\right]\frac{dz}{\sqrt{2\pi}} \\
&= \frac{2(1+c)}{\alpha}\int_0^{\sqrt{\frac{\alpha}{1+c}}}z\left[z\exp\left(-\frac{z^2}{2}\right)\right]\frac{dz}{\sqrt{2\pi}}+2\int_{\sqrt{\frac{\alpha}{1+c}}}^{\infty}\exp\left(-\frac{z^2}{2}\right)\frac{dz}{\sqrt{2\pi}} \\
&= \frac{1+c}{\alpha}\left[\operatorname{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)-\sqrt{\frac{2\alpha}{\pi(1+c)}}\exp\left(-\frac{\alpha}{2(1+c)}\right)\right] \\
&\quad +\left[1-\operatorname{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)\right]
\end{aligned}
$$

We used the fact that

$$\begin{aligned}
& \lim_{\tau\to\infty}\int \delta\left[\bar{\eta}-\frac{1}{\tau}\sum_{t=0}^{\tau-1}\eta(t)\right]\frac{\exp\left[-\frac{1}{2}\sum_{t,t'=0}^{\tau-1}\eta(t)\Lambda^{-1}(t,t')\eta(t')\right]}{\sqrt{\det\left(\Lambda\right)}}\prod_{t=0}^{\tau-1}\left[\frac{d\eta(t)}{\sqrt{2\pi}}\right] \\
= & \lim_{\tau\to\infty}\int \exp\left[\mathrm{i}\hat{\bar{\eta}}\left(\bar{\eta}-\frac{1}{\tau}\sum_{t=0}^{\tau-1}\eta(t)\right)\right]\frac{1}{\sqrt{\det\left(\Lambda\right)}} \\
& \exp\left[-\frac{1}{2}\sum_{t,t'=0}^{\tau-1}\eta(t)\Lambda^{-1}(t,t')\eta(t')\right]\prod_{t=0}^{\tau-1}\left[\frac{d\eta(t)}{\sqrt{2\pi}}\right]\frac{d\hat{\bar{\eta}}}{2\pi} \\
= & \lim_{\tau\to\infty}\int \exp\left[\mathrm{i}\hat{\bar{\eta}}\bar{\eta}\right]\exp\left[-\frac{1}{2}\hat{\bar{\eta}}^2\frac{1}{\tau^2}\sum_{t,t'=0}^{\tau-1}\Lambda(t,t')\right]\frac{d\hat{\bar{\eta}}}{2\pi} \\
= & \int \exp\left[\mathrm{i}\hat{\bar{\eta}}\bar{\eta}\right]\exp\left[-\frac{1}{2}\hat{\bar{\eta}}^2\sigma_{\bar{\eta}}^2\right]\frac{d\hat{\bar{\eta}}}{2\pi} \\
= & \frac{1}{\sqrt{2\pi\sigma_{\bar{\eta}}^2}}\exp\left[-\frac{1}{2\sigma_{\bar{\eta}}^2}\bar{\eta}^2\right]=P(\bar{\eta})
\end{aligned}$$

Calculation for the integrated response from page 35:

$$\begin{aligned}
\chi &= \sqrt{\frac{1}{\alpha\Delta_2}}\int P(\bar{\eta})\frac{\partial\bar{s}}{\partial\bar{\eta}}d\bar{\eta} \\
&= \frac{1+\Delta_2\chi}{\Delta_2\sqrt{\alpha(1+c)}}\,2\int_0^\infty \exp\left[-\frac{z^2}{2}\right]z \\
&\quad \left[\mathcal{H}\left(\frac{\sqrt{\alpha}}{\sqrt{1+c}}-z\right)\frac{\sqrt{1+c}}{\sqrt{\alpha}}z+\mathcal{H}\left(z-\frac{\sqrt{\alpha}}{\sqrt{1+c}}\right)\right]\frac{dz}{\sqrt{2\pi}} \\
&= \frac{1+\Delta_2\chi}{\alpha\Delta_2}\,2\int_0^{\frac{\sqrt{\alpha}}{\sqrt{1+c}}}\exp\left[-\frac{z^2}{2}\right]z^2\frac{dz}{\sqrt{2\pi}} \\
&\quad +\frac{1+\Delta_2\chi}{\Delta_2\sqrt{\alpha(1+c)}}\,2\int_{\frac{\sqrt{\alpha}}{\sqrt{1+c}}}^\infty \exp\left[-\frac{z^2}{2}\right]z\frac{dz}{\sqrt{2\pi}} \\
&= \frac{1+\Delta_2\chi}{\alpha\Delta_2}\,\mathrm{erf}\left(\sqrt{\frac{\alpha}{2(1+c)}}\right)
\end{aligned}$$

Generating functional analysis for the volatility calculations (see page 39):
For further details see also Coolen (2005) [6]. The following moment-generating functional is considered:

$$Z[\boldsymbol{\varphi}] \;=\; \left\langle \exp\left(\mathrm{i}\sqrt{2}\sum_t \sum_\mu \varphi_\mu(t)\sqrt{N}(D^\mu(t)-k)\right)\right\rangle .$$

Note that the factor $\sqrt{2}$ was introduced for mathematical convenience. Let us rewrite the term inside the brackets as follows:

$$\begin{aligned}
& \exp\left[\mathrm{i}\sum_t\sum_\mu \varphi_\mu(t)\sqrt{2N}(D^\mu(t)-k)\right] \\
= \;& \exp\left[\mathrm{i}\sum_t\sum_\mu \varphi_\mu(t)\sqrt{2N}\left[\frac{1}{N}\sum_i(\omega_i^\mu + s_i(t)\xi_i^\mu)-k\right]\right] \\
= \;& \exp\left[\mathrm{i}\sum_t\sum_\mu \varphi_\mu(t)\left[\sqrt{\frac{2}{N}}\sum_i \omega_i^\mu + x^\mu(t) - k\sqrt{2N}\right]\right].
\end{aligned}$$

By making a coordinate transform, equation (2.12) without average takes the form:

$$\begin{aligned}
Z[\boldsymbol{\psi}] \;=\; & \int \exp\left[\mathrm{i}\sum_t\sum_i \hat{p}_i(t)\left(p_i(t+1)-p_i(t)-\theta_i(t)\right)\right] \\
& \left[\int \exp\left[\mathrm{i}\sum_t\sum_\mu\left(\hat{w}^\mu(t)(w^\mu(t)-\varphi_\mu(t)) + \hat{x}^\mu(t)x^\mu(t)\right.\right.\right. \\
& \left.\left.\left. + w^\mu(t)\left(x^\mu(t)+\sqrt{\frac{2}{N}}\sum_j \omega_j^\mu - k\sqrt{2N}\right)\right)\right] \\
& \times \exp\left[-\frac{\mathrm{i}\sqrt{2}}{\sqrt{N}}\sum_t\sum_\mu\left(\sum_i \hat{w}^\mu(t)\hat{p}_i(t)\xi_i^\mu\right.\right. \\
& \left.\left. + \sum_i \hat{x}^\mu(t)s_i(t)\xi_i^\mu\right)\right] \mathcal{D}\mathbf{w}\mathcal{D}\mathbf{x}\mathcal{D}\hat{\mathbf{w}}\mathcal{D}\hat{\mathbf{x}}\Big] \,\mathcal{D}\hat{\mathbf{p}} \\
& \times \rho_0(\mathbf{p}(0))\mathcal{D}\mathbf{p}.
\end{aligned}$$

The disorder average is not affected by our changes and remains as given in the main text. The disorder averaged generating functional is then given by the expression:

$$\overline{Z}[\boldsymbol{\psi}] \;=\; \int \exp\left[N(\Psi+\Phi+\Omega)+\mathcal{O}(t_{max}^2\log(N))\right][\mathcal{D}C\mathcal{D}\hat{C}][\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}],$$

where the following definitions were used:

$$
\begin{aligned}
\Psi &:= \mathrm{i}\sum_t\sum_{t'}[\hat{C}(t,t')C(t,t')+\hat{L}(t,t')L(t,t')+\hat{K}(t,t')K(t,t')],\\
\Phi &:= \frac{1}{N}\sum_\mu \log\left[\int \exp\left[\mathrm{i}\sum_t\left(\hat{w}(t)(w(t)-\varphi_\mu(t))+\hat{x}(t)x(t)+x(t)w(t)\right)\right]\right.\\
&\quad\times\exp\left[-\frac{\Delta_2}{2}\sum_t\sum_{t'}\left(w(t)w(t')+\hat{w}(t)L(t,t')\hat{w}(t')\right.\right.\\
&\quad\left.\left.+2\hat{w}(t')K(t,t')\hat{x}(t)+\hat{x}(t)C(t,t')\hat{x}(t')\right)\right]\mathcal{D}w\mathcal{D}x\mathcal{D}\hat{w}\mathcal{D}\hat{x}\Big],\\
\Omega &:= \frac{1}{N}\sum_i \log\left[\int \rho_0(p(0))\exp\left[\mathrm{i}\sum_t\hat{p}(t)(p(t+1)-p(t)-\theta_i(t))\right]\right.\\
&\quad\times\exp\left[-\mathrm{i}\sum_t\sum_{t'}\Big(s(t)\hat{C}(t,t')s(t')\right.\\
&\quad\left.\left.+s(t)\hat{K}(t,t')\hat{p}(t')+\hat{p}(t)\hat{L}(t,t')\hat{p}(t')\Big)\right]\mathcal{D}p\mathcal{D}\hat{p}\right].
\end{aligned}
$$

All three functions Ψ, Φ and Ω are almost equal to the ones in the main text (2.16), (2.17) and (2.18) respectively. Different is only the absence of the generating field in Ω and the presence of the generating field in Φ. At the saddle point we find the following maximal-point equations:

$$L=0,\quad K=\mathrm{i}G.$$

Our aim is to calculate the following quantities which are required for the volatility:

$$
\begin{aligned}
\lim_{N\to\infty}\overline{\left\langle\sqrt{N}(D^\mu(t)-k)\right\rangle} &= -\frac{\mathrm{i}}{\sqrt{2}}\lim_{N\to\infty}\lim_{\boldsymbol{\varphi}\to\mathbf{0}}\frac{\partial\overline{Z}[\boldsymbol{\varphi}]}{\partial\varphi_\mu(t)},\\
\lim_{N\to\infty}\overline{\langle N(D^\mu(t)-k)(D^\mu(t')-k)\rangle} &= -\frac{1}{2}\lim_{N\to\infty}\lim_{\boldsymbol{\varphi}\to\mathbf{0}}\frac{\partial^2\overline{Z}[\boldsymbol{\varphi}]}{\partial\varphi_\mu(t)\partial\varphi_\mu(t')}.
\end{aligned}
$$

Since the generating field $\{\varphi_\mu(t)\}$ is only present in function Φ, let us simplify Φ

at the saddle point as follows:

$$\begin{aligned}
\Phi &= \frac{1}{N}\sum_{\mu}\log\left[\int \exp\left[\mathrm{i}\sum_{t}\left(\hat{w}(t)(w(t)-\varphi_{\mu}(t))+\hat{x}(t)x(t)+x(t)w(t)\right)\right]\right. \\
&\quad \times \exp\left[-\frac{\Delta_2}{2}\sum_{t}\sum_{t'}\left(w(t)w(t')+2\mathrm{i}\hat{w}(t')G(t,t')\hat{x}(t)\right.\right. \\
&\quad \left.\left.+\hat{x}(t)C(t,t')\hat{x}(t')\right)\right]\mathcal{D}w\mathcal{D}x\mathcal{D}\hat{w}\mathcal{D}\hat{x}\Bigg] \\
&= \frac{1}{N}\sum_{\mu}\log\left[\int \exp\left(\mathrm{i}\sum_{t}\hat{w}(t)(w(t)-\varphi_{\mu}(t))\right)\prod_{t}\delta(\hat{x}(t)+w(t))\right. \\
&\quad \times \exp\left[-\frac{\Delta_2}{2}\sum_{t}\sum_{t'}\left(w(t)w(t')+2\mathrm{i}\hat{w}(t')G(t,t')\hat{x}(t)\right.\right. \\
&\quad \left.\left.+\hat{x}(t)C(t,t')\hat{x}(t')\right)\right]\mathcal{D}w\mathcal{D}\hat{w}\prod_{t}d\hat{x}(t)\Bigg] \\
&= \frac{1}{N}\sum_{\mu}\log\left[\int \exp\left[\mathrm{i}\sum_{t}\hat{w}(t)\left[\sum_{t'}\left(I+\Delta_2 G^T\right)(t,t')w(t')-\varphi_{\mu}(t)\right]\right]\right. \\
&\quad \left.\times \exp\left[-\frac{\Delta_2}{2}\sum_{t}\sum_{t'}w(t)D(t,t')w(t')\right]\mathcal{D}w\mathcal{D}\hat{w}\right].
\end{aligned}$$

By defining $u = (I+\Delta_2 G^T)w - \varphi_{\mu}$ and using $w = (I+\Delta_2 G^T)^{-1}(u+\varphi_{\mu})$ and $\frac{\partial u}{\partial w} = \det\left(I+\Delta_2 G^T\right)$ (determinant of Jacobian) we get:

$$\begin{aligned}
\Phi &= \frac{1}{N}\sum_{\mu}\log\left[\int \exp\left(\mathrm{i}\sum_{t}\hat{w}(t)u(t)\right)\right. \\
&\quad \times \exp\left[-\frac{\Delta_2}{2}\sum_{t}\sum_{t'}(u(t)+\varphi_{\mu}(t))((I+\Delta_2 G)^{-1}D(I+\Delta_2 G^T)^{-1})(t,t')\right. \\
&\quad \left.\left.\times(u(t')+\varphi_{\mu}(t'))\right]\frac{1}{\det(I+\Delta_2 G^T)}\prod_{t}\left[\frac{du(t)d\hat{w}(t)}{2\pi}\right]\right].
\end{aligned}$$

Using the integral representation of Dirac's delta function we obtain:

$$\begin{aligned}\Phi &= -\alpha \log\left[\det(I+\Delta_2 G^T)\right] + \frac{1}{N}\sum_\mu \log\left[\int \exp\left[-\frac{\Delta_2}{2}\sum_t\sum_{t'}(u(t)+\varphi_\mu(t))\right.\right.\\ &\quad \left.\left.\times((I+\Delta_2 G)^{-1}D(I+\Delta_2 G^T)^{-1})(t,t')(u(t')+\varphi_\mu(t'))\right]\prod_t[\delta(u(t))du(t)]\right]\\ &= -\alpha\log\left[\det(I+\Delta_2 G^T)\right] - \frac{\Delta_2}{2N}\sum_\mu\sum_t\sum_{t'}\varphi_\mu(t)\\ &\quad \times((I+\Delta_2 G)^{-1}D(I+\Delta_2 G^T)^{-1})(t,t')\varphi_\mu(t').\end{aligned}$$

By using steepest descent integration (see appendix A.1) in the consumer limit $N\to\infty$ we calculate:

$$\begin{aligned}&\lim_{N\to\infty}\overline{\left\langle\sqrt{N}(D^\mu(t)-k)\right\rangle}\\ &= -\frac{\mathrm{i}}{\sqrt{2}}\lim_{N\to\infty}\lim_{\boldsymbol{\varphi}\to\mathbf{0}}\frac{\partial\overline{Z}[\boldsymbol{\varphi}]}{\partial\varphi_\mu(t)}\\ &= -\frac{\mathrm{i}}{\sqrt{2}}\lim_{N\to\infty}\lim_{\boldsymbol{\varphi}\to\mathbf{0}}\frac{\int f(N,\Psi,\Phi,\Omega)N\frac{\partial\Phi}{\partial\varphi_\mu(t)}\{\mathcal{D}\}}{\int f(N,\Psi,\Phi,\Omega)\{\mathcal{D}\}}\\ &= 0,\end{aligned}$$

$$\begin{aligned}&\lim_{N\to\infty}\overline{\langle N(D^\mu(t)-k)(D^\nu(t')-k)\rangle}\\ &= -\frac{1}{2}\lim_{N\to\infty}\lim_{\boldsymbol{\varphi}\to\mathbf{0}}\frac{\partial^2\overline{Z}[\boldsymbol{\varphi}]}{\partial\varphi_\mu(t)\partial\varphi_\nu(t')}\\ &= -\frac{1}{2}\lim_{N\to\infty}\lim_{\boldsymbol{\varphi}\to\mathbf{0}}\frac{\int f(N,\Psi,\Phi,\Omega)\left(N^2\frac{\partial\Phi}{\partial\varphi_\mu(t)}\frac{\partial\Phi}{\partial\varphi_\nu(t')}+N\frac{\partial^2\Phi}{\partial\varphi_\mu(t)\partial\varphi_\nu(t')}\right)\{\mathcal{D}\}}{\int f(N,\Psi,\Phi,\Omega)\{\mathcal{D}\}}\\ &= \frac{\Delta_2}{2}\delta_{\mu,\nu}\left[(I+\Delta_2 G)^{-1}D(I+\Delta_2 G^T)^{-1}\right](t,t'),\end{aligned}$$

where $f(N,\Psi,\Phi,\Omega) := \exp\left[N(\Psi+\Phi+\Omega)+\mathcal{O}(t_{max}^2\log(N))\right]$ and $\{\mathcal{D}\} := [\mathcal{D}C\mathcal{D}\hat{C}]$ $[\mathcal{D}K\mathcal{D}\hat{K}][\mathcal{D}L\mathcal{D}\hat{L}]$. As in the main text, $\overline{Z}[\mathbf{0}]=1$ was used. Hence, since for $N\to\infty$ the term $\frac{1}{P}\sum_\mu\langle\sqrt{N}(D^\mu(t)-k)\rangle$ must become independent of the microscopic realisation of the disorder (self-averaging property), i. e.

$$\overline{\left[\frac{1}{P}\sum_\mu\langle\sqrt{N}(D^\mu(t)-k)\rangle\right]\left[\frac{1}{P}\sum_\mu\langle\sqrt{N}(D^\mu(t)-k)\rangle\right]}$$

$$\longrightarrow \overline{\left[\frac{1}{P}\sum_{\mu}\langle\sqrt{N}(D^{\mu}(t)-k)\rangle\right]}\,\overline{\left[\frac{1}{P}\sum_{\mu}\langle\sqrt{N}(D^{\mu}(t)-k)\rangle\right]},$$

we find the volatility matrix for $N \to \infty$ to be:

$$\begin{aligned}
\overline{\Xi}(t,t') &= \lim_{N\to\infty}\frac{1}{P}\sum_{\mu=1}^{P}\overline{\langle N(D^{\mu}(t)-k)(D^{\mu}(t')-k)\rangle} \\
&- \overline{\left[\frac{1}{P}\sum_{\mu=1}^{P}\left\langle\sqrt{N}(D^{\mu}(t)-k)\right\rangle\right]}\,\overline{\left[\frac{1}{P}\sum_{\nu=1}^{P}\left\langle\sqrt{N}(D^{\nu}(t')-k)\right\rangle\right]} \\
&= \frac{\Delta_2}{2}(I+\Delta_2 G)^{-1}D(I+\Delta_2 G^{T})^{-1}(t,t') \\
&= \frac{1}{2}\Lambda(t,t').
\end{aligned}$$

Appendix C

MATLAB Code

```
%f.m

%function used in fixed_point.m
function y = f(c, alpha)
    o = alpha/(1+c);
    phi = 1-erf(sqrt(o/2));
    p = sqrt(2*o/pi)*exp(-o/2);
    y = (1-phi-p)/o+phi;
end


%fixed_point.m

format long
%calculations for the critical value of alpha
x = erfinv(0.3374001886);
xold = 0;
while abs(x-xold) > 1e-15
    xold = x;
    x = (x*erf(x)+exp(-x^2)/sqrt(pi))/2;
end
alpha_critical = erf(x);
aux = [logspace(-2,log10(alpha_critical),100),logspace(log10(alpha_critical),1,1000)];
aux = [aux(1:100),alpha_critical+1e-6,aux(102:1100)];
%initialisation
q = 0.1;
qmax = 20;
Alpha1 = []; Alpha2 = [];
```

```
C1 = []; C2 = [];
Phi1 = []; Phi2 = [];
Chi1 = []; Chi2 = [];
H1 = []; H2 = [];
Sigma1 = []; Sigma2 = [];
for alpha = aux
    c = 0.5;
    cold = 0;
    while abs(c-cold) > 1e-14
        cold = c;
        c = f(c,alpha);
    end
    phi = 1-erf(sqrt(alpha/(2*(1+c))));
    chi = erf(sqrt(alpha/(2*(1+c))))/(q*(1-q)*(alpha-erf(sqrt(alpha/(2*(1+c))))));
    %chi = 12*erf(sqrt(alpha/(2*(1+c))))/(qmax^2*(alpha-erf(sqrt(alpha/(2*(1+c))))));
    h = q*(1-q)*(1+c)/2/(1+q*(1-q)*chi)^2;
    %h = qmax^2*(1+c)/24/(1+qmax^2/12*chi)^2;
    sigma = q*(1-q)*(1+phi)/2/(1+q*(1-q)*chi)^2+q*(1-q)/2*(1-phi);
    %sigma = qmax^2*(1+phi)/24/(1+qmax^2/12*chi)^2+qmax^2/24*(1-phi);
     %non-ergodic regime
     if alpha<=alpha_critical
        Alpha1 = [Alpha1,alpha];
        C1 = [C1,c];
        Phi1 = [Phi1,phi];
        Chi1 = [Chi1,chi];
        H1 = [H1,h];
        Sigma1 = [Sigma1, sigma];
     end
     %ergodic regime
     if alpha>=alpha_critical
        Alpha2 = [Alpha2,alpha];
        C2 = [C2,c];
        Phi2 = [Phi2,phi];
        Chi2 = [Chi2,chi];
        H2 = [H2,h];
        Sigma2 = [Sigma2, sigma];
     end
end

% plot results
alpha_c=alpha_critical*[1,1];
```

```
y = [0,1];

clf;
subplot(1,2,1)
semilogx(alpha_c,y,'r--','linewidth',1);
hold on
semilogx(Alpha2,C2,'b','linewidth',2);
semilogx(Alpha1,C1,'b:','linewidth',2);
xlabel('\alpha','fontsize',20)
ylabel('c','fontsize',20)
hold on

subplot(1,2,2)
semilogx(alpha_c,y,'r--','linewidth',1);
hold on
semilogx(Alpha2,Phi2,'b','linewidth',2);
semilogx(Alpha1,Phi1,'b:','linewidth',2);
xlabel('\alpha','fontsize',20)
ylabel('\phi','fontsize',20)
hold on
pause;
clf;

loglog(alpha_c,[1e0,1e7],'r--','linewidth',1);
hold on
loglog(Alpha2,(Chi2),'b','linewidth',2);
hold on
loglog(Alpha1,(Chi1),'b:','linewidth',2);
xlabel('\alpha','fontsize',20)
ylabel('\Delta_2\chi','fontsize',20)
pause;
clf;

subplot(1,2,1)
loglog(Alpha2,H2,'b:','linewidth',2);
hold on
loglog(alpha_c,[1e-6,1e4],'r--','linewidth',1);
loglog(Alpha1,H1,'b:','linewidth',2);
xlabel('\alpha','fontsize',20)
ylabel('H','fontsize',20)
```

```
subplot(1,2,2)
loglog(Alpha2,Sigma2,'b:','linewidth',2);
hold on
loglog(alpha_c,[1e-3,1e4],'r--','linewidth',1);
loglog(Alpha1,Sigma1,'b:','linewidth',2);
xlabel('\alpha','fontsize',20)
ylabel('\sigma_v^2','fontsize',20)

%simulation_volatility.m

%initialisation
PNmax = 400^2;
T = 10000;
T_mid = 7000;
q = 0.3;
qmax = 20;
k = 1-q;
% k = qmax/2;
alpha_c = 0.3374001886;
Delta = [];
DeltaH = [];
Alpha = [];

for alpha = logspace(-1,1,20)
    %initial values
    N = round(sqrt(PNmax/alpha));
    P = round(alpha*N);
    alpha_ = P/N;
    rand('state',sum(100*clock));
    % display('random number generation'); tic
    Q = (rand(N,2,P)>q);
    % Q = rand(N,2,P)*qmax;
    % toc
    U = 2*(rand(N,2)-.5);
    [aux1,G] = max(U,[],2);
    G = (G'-1)*N+(1:N);
    [aux1,aux2] = meshgrid(G,0:2*N:(P-1)*2*N);
    aux1 = aux1+aux2;
    D = sum(Q(aux1),2)/N;

   %calculations of volatility for each alpha
   for t = 1:T
```

```
        % display('meshgrid 3d'); tic
        [aux1,aux2,aux3] = meshgrid(1:2,1,D(:,t)-k);
        aux1 = zeros(N,2);
        for j = 1:N
            aux1(j,1:2) = sum(Q(j,1:2,:).*aux3,3);
        end
        U = U-aux1/P;
        % display('sum of product 3d');
        % toc
        [aux1,G] = max(U,[],2);
        G = (G'-1)*N+(1:N);
        % display('meshgrid 2d'); tic
        [aux1,aux2] = meshgrid(G,0:2*N:(P-1)*2*N);
        % toc
        aux1 = aux1+aux2;
        % display('referencing Q'); tic
        D = [D,sum(Q(aux1),2)/N];
        % toc
    end
    %measure for efficiency of the market H
    D_average_square = mean(sqrt(N)*(D(:,T_mid:T)-k),2).^2;
    %volatility
    D_average_2 = mean(N*(D(:,T_mid:T)-k).^2,2);
    delta = mean(D_average_2);
    deltaH = mean(D_average_square);
    if alpha_>=alpha_c
        Delta = [Delta,delta];
        DeltaH = [DeltaH,deltaH];
        Alpha = [Alpha,alpha_];
    end
end
hold on
loglog(Alpha,DeltaH,'o');
%loglog(Alpha,Delta,'s');
```

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