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On the Group Rings of Abelian Minimax Groups

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INTRODUCTION

An abelian group G is called *minimax* if it contains a finitely generated subgroup H such that G/H satisfies the minimal condition for subgroups (which I shall abbreviate to min). In this case, we may choose H to be free abelian (by making it smaller if necessary), or we may choose G/H to be divisible (by making H bigger if necessary). Recall that the divisible abelian groups with min are direct products of finitely many quasicyclic groups (groups of type C_{p^∞}, for various primes p), and that an abelian group with min is the direct product of a divisible one with a finite group.

There are two reasons for being interested in the group rings of such groups, "external" and "internal." The external reason, and original motivation for the present work, comes from the theory of infinite soluble groups. It has become apparent in recent years that among finitely generated groups, the *soluble groups of finite rank* are singled out by quite a varied range of finiteness conditions, such as polynomial subgroup growth or finite upper rank [LMS, MS]. In proving results of this nature, some sort of induction argument may well reduce the problem to the consideration of a group Γ with an abelian normal subgroup A such that Γ/A is soluble of finite rank; to conclude that Γ itself has finite rank one then has to understand the nature of A as a module for the group ring $\mathbb{Z}(\Gamma/A)$. The analogous situation in the case where Γ/A was *polycyclic* was studied by P. Hall in the 1950s and J. E. Roseblade in the 1970s; the key idea of their beautiful and highly successful theory was to pick an abelian normal subgroup G in Γ/A, and consider A as a module for the group ring $\mathbb{Z}G$, with Γ/A as a group of operators. In their situation, $\mathbb{Z}G$ was a finitely generated commutative ring, and the whole arsenal of Noetherian ring

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theory was available for exploitation. In our case, we are only allowed to suppose that G is a *minimax* group (as is every abelian subgroup of a finitely generated soluble group of finite rank): to have any hope of emulating Hall and Roseblade we therefore need to develop some analogous theory for rings like $\mathbb{Z}G$. That is what this paper sets out to do. That the results obtained go some way towards fulfilling the stated purpose will be demonstrated elsewhere [S]; a sample of the methodology is provided by the proof of the final theorem in this paper, 8.5 below.

The internal reason for pursuing this topic is that in fact a rather satisfying theory emerges. At first glance, one might expect this to be little more than a routine exercise in commutative algebra: after all, $\mathbb{Z}G$ is an integral extension of its subring $\mathbb{Z}H$, and $\mathbb{Z}H$ is a finitely generated commutative ring. However, the main results seem to depend critically on the interplay between the ideals of the ring and the structure of the multiplicative subgroup G; the resulting theory is specific to such group rings.

Before summarising the main results let us establish some terminology. Throughout the paper, G will denote an abelian minimax group. The set of primes p such that G has an element of order p is denoted $\pi(G)$.

A subgroup H is *dense* in G if G/H is periodic (this is equivalent to G/H satisfying min); and H is *co-divisible* if G/H is divisible. Note that every finitely generated subgroup of G is contained in a finitely generated dense subgroup of G. We write $\text{spec}(G) = \pi(G/H)$, where H is a finitely generated dense co-divisible subgroup of G; this is the set of primes p such that C_{p^∞} is a section of G.

G is said to be *reduced* if the torsion subgroup of G is finite; this is equivalent to G being residually finite.

We work over a finitely generated commutative ring k. A kG-module M is *non-singular* if

$$\pi(M) \cap \text{spec}(G) = \varnothing.$$

An ideal I of kG is *regular* if the module kG/I is non-singular; in particular, a prime ideal P is regular if and only if $\text{char}(kG/P) \notin \text{spec}(G)$.

For an ideal I of kG we write $I^\dagger = (I + 1) \cap G$, and say that I is *faithful* if $I^\dagger = 1$.

The first main result is

1.2 COROLLARY. *Every faithful regular prime ideal of kG is finitely generated.*

This is fundamental for the whole theory. The second important result shows that, under some circumstances, the group ring kG shares another key property of finitely generated rings; the proof, however, depends on both Hilbert's irreducibility theorem and Chebotarev's theorem:

4.2 THEOREM. *If G is reduced then every faithful regular prime ideal of kG is an intersection of maximal ideals of finite index in kG.*

The last three sections develop a kind of primary decomposition, for modules which satisfy the following

DEFINITION. A kG-module M is *quasi-residually finite*, or qrf, if $G/\mathrm{C}_G(a)$ is reduced for every $a \in M$.

This property is a generalisation of residual finiteness, but has the advantage of passing more readily to quotient modules. It is a curious, though not particularly useful, fact that for non-singular modules, being qrf is equivalent to being "poly-(locally-(poly-(residually finite)))." We show that if a non-zero non-singular module M is qrf then the set $\mathscr{P}(M)$ of associated primes of M is non-empty (this is the set of prime ideals of kG of the form $\mathrm{ann}_{kG}(a)$ for some $a \in M$). I shall say that M is *unmixed* if, additionally, $\mathscr{P}(M)$ consists of *maximal* annihilators, and *primary* if $\mathscr{P}(M)$ is a singleton. What then emerges from the theory is that

- every non-singular qrf module has a finite filtration in which each factor is unmixed;
- every unmixed module has a natural decomposition as a subdirect sum of primary modules.

This decomposition serves to reduce the study of general (non-singular qrf) modules to that of *prime* modules, which may be more tractable; the point is that it is available in situations that are generally far from Noetherian. This is illustrated in Section 8, where we consider certain finitely generated $k\Gamma$-modules where Γ is a group in which G sits as a normal subgroup, with Γ/G polycyclic.

1. PRIME MODULES

In this section, F denotes a field, p denotes a prime, and ζ_m denotes a primitive mth root of unity when m is odd and a primitive $2m$th root of unity when m is even. Note that $\zeta_m \in F$ entails $\mathrm{char}(F) \nmid m$.

The group of units of a ring R is denoted R^*. The field of fractions of a domain R is denoted $\mathrm{ffr}(R)$.

An R-module M is said to be *prime* if $\mathrm{ann}_R(M)$ is a prime ideal and M is torsion-free when considered as an $R/\mathrm{ann}_R(M)$-module. A kG-module

M is *induced* from a subgroup H of G if there is a kH-submodule U of M such that the natural mapping

$$U \bigotimes_{kH} kG \to M$$

is an isomorphism. In this case we write

$$M = U \uparrow_H^G .$$

Note that in this case (i) if the kG-module M has one of the properties *finitely generated*, *prime*, or *torsion-free* then the kH-module U has the same property; (ii) M is induced from H_1 for every subgroup H_1 of G containing H.

The main result of this section shows how a prime module is built from a module over a Noetherian subring of kG:

1.1 THEOREM. *Let M be a finitely generated non-singular prime kG-module. Then M is induced from a dense subgroup H of G containing $\mathrm{C}_G(M)$ such that $H/\mathrm{C}_G(M)$ is finitely generated.*

Before embarking on the proof we deduce

1.2 COROLLARY. *Let P be a faithful regular prime ideal of kG. Then*

(i) *G contains a finitely generated dense subgroup H such that*

$$P = (P \cap kH)\uparrow_H^G \qquad \text{and} \qquad kG/P = ((kH + P)/P)\uparrow_H^G .$$

(ii) *P is finitely generated.*

Proof. Put $M = kG/P$. Let H be as in 1.1; since P is faithful, $\mathrm{C}_G(M) = 1$, so H is finitely generated. Now let $\{g_i\}$ be a transversal to the cosets of H in G. Then $kG = \oplus kHg_i$, and $M = \oplus Ug_i$ for some kH-submodule U of M. It follows that

$$P = \mathrm{ann}_{kG}(M) = \mathrm{ann}_{kG}(U) = \bigoplus \mathrm{ann}_{kH}(U)g_i = \bigoplus (P \cap kH)g_i,$$

which implies that

$$kG/P = \sum (kHg_i + P)/P = \bigoplus (kHg_i + P)/P.$$

This gives (i). Since H is finitely generated, kH is a Noetherian ring. Hence $P \cap kH$ is a finitely generated ideal of kH, and (ii) follows. ∎

We build up to the proof of Theorem 1.1 with a series of lemmas.

1.3 LEMMA. *Let* $F = F(y_0) \subseteq F(y_1) \subseteq \cdots \subseteq F(y_n)$ *be a chain of fields, where* $y_i^p = y_{i-1}$ *for* $i = 1, \ldots, n$. *If* F *contains* ζ_p *and* $F(y_n) = F(y_{n-1})$ *then* $F(y_n) = F$.

Proof. This follows from [Ka, Theorem 51]. ∎

1.4 LEMMA. *Let* S *be an integrally closed domain with field of fractions* F, *and let* $E = F(C)$ *be an extension field of* F, *where* C *is a subgroup of* E^*. *Suppose that* $C/(C \cap S^*)$ *is a* π*-group, where* π *is a finite set of primes such that* $\zeta_p \in S$ *for all* $p \in \pi$. *Then the* π*-torsion subgroup of* E^*/S^* *is* CS^*/S^*.

Proof. We may clearly suppose that $C \geq S^*$ and that C/S^* is finite. Now we argue by induction on $|C : S^*|$. It will suffice to show that if $x \in E^*$ and $x^p \in C$, where $p \in \pi$, then $x \in C$. Replacing x by a suitable power of x, we may assume also that $x^{p^m} \in S^*$ for some m.

Case 1. Suppose $C = S^*\langle x^p \rangle$. Then $x \in F(x^p)$. Since $x^{p^m} \in F$, Lemma 1.3 shows that $x \in F$. As S is integrally closed, it follows that $x \in S$, and hence that $x \in S^*$ since $x^{p^m} \in S^*$.

Case 2. Suppose $C > S^*\langle x^p \rangle$. Then $S^*\langle x^p \rangle \leq D < C$ where $C = D\langle y \rangle$ and $y^q \in D$ for some $q \in \pi$. If $x \in F(D)$ we are done by inductive hypothesis. Suppose $x \notin F(D)$. Then $F(D) < F(D)(x) \subseteq E = F(D)(y)$; as $(F(D)(y) : F(D)) = q$ it follows that $F(D)(x) = F(D)(y)$ and that $q = p$. Hence there exists $z \in F(D)$ such that $x^p = y^{hp} z^p$ for some integer h (Kummer theory). Now $z^p \in D$, and so by inductive hypothesis $z \in D$. Consequently $x = y^h z \zeta_p^j \in C$ for some j. ∎

1.5 LEMMA. *Let* $R = k[A]$ *be a domain, where* A *is a subgroup of* R^*. *Let* B *be a subgroup of* A *with* $A/B \cong C_{p^\infty}$, *where* $p \neq \operatorname{char}(R)$. *Then one of the following holds*:

(i) A/B *has a finite subgroup* C/B *such that* $R \subseteq \operatorname{ffr}(k[C])$, *or*

(ii) A/B *has a finite subgroup* C/B *such that* $R = k[C]\uparrow_C^A$.

Proof. A is the ascending union of a chain of subgroups B_i, with $B_0 = B$ and B_i/B cyclic of order p^i for each $i \geq 1$. Put $F = \operatorname{ffr}(k[B])$ and $F' = F(\zeta_p)$.

Case 1. Where $F'(B_n) = F'(B_{n-1})$ for infinitely many values of $n \geq 0$. Suppose we have $F' < F'(B_m)$ for some m. Let $n > m$, and choose y such that $B_n = B\langle y \rangle$; putting $y_j = y^{p^{n-j}}$ for $j = 0, \ldots, n$, we have $F'(B_j) = F'(y_j)$, and Lemma 1.3 shows that $F'(B_n) > F'(B_{n-1})$. It follows that in fact $F' = F'(B_m)$ for all m. Since $(F' : F)$ is finite, there exists m such that $F(B_n) = F(B_m)$ for all $n \geq m$. Put $C = B_m$. Then

$$R \subseteq \bigcup_{n \geq 0} F(B_n) = F(C) = \mathrm{ffr}(k[C]),$$

and (i) holds.

Case 2. There exists m such that $F'(B_n) > F'(B_{n-1})$ for all $n > m$. Let $j \geq 0$, and choose y such that $B_{m+j} = B\langle y \rangle$. Then $(F'(B_{m+j}) : F'(B_m)) = p^j$, so the first p^j powers of y are linearly independent over $F'(B_m)$, and *a fortiori* over $k[B_m]$. As this holds for each j, we see that (ii) holds with $C = B_m$. ∎

1.6 LEMMA. *Let* A, B, *and* R *be as in* 1.5, *and let* U *be a finitely generated torsion-free* R*-module. Assume that* $\mathrm{Jac}(R) = 0$. *Suppose that* U *is not induced from any proper subgroup of* A *containing* B. *Then there exists a finite subgroup* C/B *of* A/B *such that* $R \subseteq \mathrm{ffr}(k[C])$.

Proof. The R-module U contains a maximal R-linearly independent set S, say (Zorn's lemma); then S generates a free R-submodule W, and U/W is a torsion module for R. As U is finitely generated it follows that $\mathrm{ann}_R(U/W) \neq 0$, and as $R = k[A]$ is integral over $k[B]$ (because A/B is a periodic group) this implies that $\mathrm{ann}_{k[B]}(U/W) \neq 0$. Choose $\lambda \in \mathrm{ann}_{k[B]}(U/W) \setminus \{0\}$.

Since $\mathrm{Jac}(R) = 0$, there is a maximal ideal L of R with $\lambda \notin L$. Then $R = L + \lambda R$, so $U = UL + U\lambda = UL + W$ and $UL \cap W = WL$. Therefore $W/WL \cong U/UL$ is a finitely generated R/L-module. As W is a free R-module it follows that W is finitely generated; say

$$W = \bigoplus_{i=1}^{d} w_i R.$$

Now suppose that the lemma is false. Then 1.5(ii) holds. Say $U = \sum_{i=1}^{r} u_i R$, and put $V = \sum_{i=1}^{r} u_i k[C]$ where C is as in 1.5(ii). Then λV is a finitely generated $k[C]$-submodule of W; since $R = k[A]$ and A/C is periodic, it follows that A/C contains a finite subgroup C_1/C such that $\lambda V \leq \oplus_{i=1}^{d} w_i k[C_1]$. Now let T be a transversal to the cosets of C_1 in A. Then

$$R = \bigoplus_{t \in T} k[C_1]t.$$

We can therefore write

$$W = \bigoplus_{t \in T} \left(\bigoplus_{i=1}^{d} w_i k[C_1] \right) t. \tag{1.1}$$

Now

$$U = Vk[A] = \sum_{t \in T} Vk[C_1]t. \tag{1.2}$$

Suppose $\Sigma v_j t_j = 0$ where each $v_j \in Vk[C_1]$ and the t_j are distinct elements of T. Then $\Sigma \lambda v_j t_j = 0$; but $\lambda v_j \in \oplus_{i=1}^{d} w_i k[C_1]$ for each j, so (1.1) gives $\lambda v_j = 0$ for each j. Since U is torsion-free it follows that $v_j = 0$ for each j. Thus the sum in (1.2) is direct, and so U is induced from C_1. This contradicts the hypothesis, and so completes the proof. ∎

1.7 LEMMA. *Let S_0 be a finitely generated domain with field of fractions F_0, let F be a finite extension field of F_0, and let S be the integral closure of S_0 in F. Then S^* is a finitely generated group.*

Proof. By the generalised units theorem [La, p. 39], it will suffice to show that S is contained in some finitely generated subring of F. If char $S_0 \neq 0$ then S_0 is a finitely generated algebra over its prime field P, and [ZS, Chap. V, Theorem 9] shows that S is a finitely generated algebra over P. Suppose now that char $S_0 = 0$. It follows from Noether's normalisation lemma that for some integer $m \neq 0$, the ring $S_0[\frac{1}{m}]$ is integral over a polynomial subring of the form $\mathbb{Z}[\frac{1}{m}][T_1, \ldots, T_r] = S_1$, say. Then F is still a finite extension of ffr(S_1), and S is contained in the integral closure S_2, say, of S_1 in F. In this case, [ZS, Chap. V, Cor. 1 to Theorem 7] shows that S_2 is finitely generated as a module over S_1, and the result follows. ∎

We are now ready for the

Proof of Theorem 1.1. M is a finitely generated prime kG-module, with $\pi(M) \cap \text{spec}(G) = \varnothing$. We may assume without loss of generality that $\text{ann}_k(M) = 0$ and $\text{C}_G(M) = 1$. Now G has a co-divisible finitely generated dense subgroup G_0; then G/G_0 satisfies min, so we may choose a subgroup A/G_0 of G/G_0 minimal subject to M being induced from A. Thus $M = U \uparrow_A^G$, where U is a finitely generated prime kA-module. It will suffice to prove that A is finitely generated. Put $\pi = \pi(A/G_0)$, so $\pi \subseteq \text{spec}(G)$.

Assume, then, that A is not finitely generated. Then A has a subgroup $B \geq G_0$ such that $A/B \cong C_{p^\infty}$ for some $p \in \pi$. Let R be the image of kA in the endomorphism ring of U. Since (we have assumed that) M is faithful for both k and G, we see that U is faithful for both k and A; so we may identify k and A with their images in R, and we can write $R = k[A]$ (for any subgroup A_1 of A, $k[A_1]$ will then denote the subring of R generated over k by A_1). Since U is a prime kA-module, R is a domain and U is torsion-free as an R-module, and char$(R) \notin \pi$. Since k is a Hilbert ring and G_0 is finitely generated, kG_0 is again a Hilbert ring, so $\text{Jac}(k[G_0]) = 0$, and as R is integral over $k[G_0]$ it follows that $\mathrm{Jac}(R) = 0$. The minimal choice of A ensures that U is not induced from any proper subgroup of A containing B.

Thus we are in the situation of Lemma 1.6. Hence there exists a finite subgroup C/B of A/B such that

$$R \subseteq \mathrm{ffr}(k[C]) = E_0,$$

say.

Now put $S_0 = k[G_0]$ and $F_0 = \mathrm{ffr}(S_0)$. Then $F_0(C) = E_0$. So if $Z = \{\zeta_l \mid l \in \pi\}$ and $E = E_0(Z)$, $F = F_0(Z)$, we have $F(C) = E$. Let S be the integral closure of S_0 in F. Then $F = \mathrm{ffr}(S)$, since $Z \subseteq S$, and since $S^* \geq G_0$, both $C/(C \cap S^*)$ and AS^*/S^* are π-groups. It follows by Lemma 1.4 that $A \leq CS^*$. But S^* is a finitely generated group, by Lemma 1.7; this implies that A/C is finitely generated, a contradiction since $A/C \cong C_{p^\infty}$.

This completes the proof. ∎

Corollary 1.2 says that regular faithful prime ideals are finitely generated. It is easy to see that a non-faithful prime ideal need not be finitely generated; the next result shows that, in a sense, a regular prime ideal is not far from being finitely generated:

1.8 PROPOSITION. *Let P be a regular prime ideal of kG. Then P contains a finitely generated ideal $P^\natural$ such that for every positive integer m, the additive group $P/(P^\natural + P^m)$ is a divisible π-group, where $\pi = \mathrm{spec}(G)$.*

Proof. Put $K = P^\dagger$. By Corollary 1.2, there is a finitely generated subgroup H/K of G/K such that

$$P = \mathfrak{k}kG + (P \cap kH)kG,$$

where $\mathfrak{k} = (K - 1)kK$ is the augmentation ideal of K. Now K has a finitely generated subgroup F such that K/F is a divisible π-group; then $\bar{\mathfrak{f}} = (F - 1)kK$ is a finitely generated ideal of kK. As $k(H/K)$ is Noetherian we have

$$P \cap kH = \mathfrak{k}kH + XkH$$

for some finite subset X of $P \cap kH$. We define

$$P^\natural = XkG + \bar{\mathfrak{f}}kG.$$

As $P = \mathfrak{k}kG + XkG$, we see that $P/(P^\natural + P^m)$ is an image of $\mathfrak{k}kG/(\bar{\mathfrak{f}}kG + \mathfrak{k}^m kG)$; as $\mathfrak{k}kG$ is additively generated by G-translates of $\mathfrak{k}$, it will be enough to show that $\mathfrak{k}/(\bar{\mathfrak{f}} + \mathfrak{k}^m)$ is a divisible π-group.

Write $\tilde{K} = k \otimes_{\mathbb{Z}} K$, and $\tilde{F}$ for the image of $k \otimes_{\mathbb{Z}} F$ in $\tilde{K}$. One verifies easily that $\tilde{K}/\tilde{F}$ is a divisible π-group. The group epimorphism $\tilde{K} \to \mathfrak{k}/\mathfrak{k}^2$ given by $\lambda \otimes x \mapsto \lambda(x-1) + \mathfrak{k}^2$ maps $\tilde{F}$ onto $(\bar{\mathfrak{f}} + \mathfrak{k}^2)/\mathfrak{k}^2$, so $\mathfrak{k}/(\bar{\mathfrak{f}} + \mathfrak{k}^2)$ is an image of $\tilde{K}/\tilde{F}$. For each $i > 1$, $(\bar{\mathfrak{f}} + \mathfrak{k}^i)/(\bar{\mathfrak{f}} + \mathfrak{k}^{i+1})$ is an image of $\mathfrak{k}^{i-1} \otimes_{\mathbb{Z}} \mathfrak{k}/(\bar{\mathfrak{f}} + \mathfrak{k}^2)$. It follows that $\mathfrak{k}/(\bar{\mathfrak{f}} + \mathfrak{k}^m)$ is a divisible π-group as required. ∎

However, the hypothesis that P be regular is definitely necessary in 1.2:

1.9 EXAMPLE. Let k be an integral domain of characteristic p, and let G be the additive group of $\mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}[\frac{1}{p}]$, written multiplicatively. Then kG contains a faithful prime ideal which is not finitely generated.

Proof. Write $G = X \times Y \times Z$, where

$$X = \langle x_n \mid n \in \mathbb{Z} \rangle, \qquad Y = \langle y_n \mid n \in \mathbb{Z} \rangle, \qquad Z = \langle z_n \mid n \in \mathbb{Z} \rangle$$

with $x_n^p = x_{n-1}$, $y_n^p = y_{n-1}$, and $z_n^p = z_{n-1}$ for each n. Let S be the group ring $k(XY)$, and for each n put $t_n = x_n + y_n$. Then $t_n^p = t_{n-1}$ for each n, so the set $\{t_n \mid n \in \mathbb{Z}\}$ generates inside the unit group of $S[t_0^{-1}]$ a group isomorphic to $\mathbb{Z}[\frac{1}{p}]$. We may therefore define a homomorphism θ: $kG \to S[t_0^{-1}]$ by setting $z_n\theta = t_n$ for all n, $s\theta = s$ for all $s \in S$.

Put $P = \ker\theta$. This is evidently a prime ideal of kG, and it is faithful, since if $(x_i^a y_j^b z_l^c)\theta = 1$ then $x_i^a y_j^b t_l^c = 1$ in $S[t_0^{-1}]$, which occurs only for $a = b = c = 0$. On the other hand, P cannot be finitely generated. For if $P = (P \cap S\langle z_m \rangle)kG$, then

$$S\left[t_0^{-1}\right] = kG\theta = S\langle z_m \rangle\theta \uparrow_{\langle z_m \rangle}^{Z} = S\left[t_0^{-1}\right] \uparrow_{\langle z_m \rangle}^{Z},$$

which is plainly impossible. ∎

2. INDUCED MODULES

In this section, we consider the relationship between the submodules of finite index in a kG-module M and those in a kH-submodule U, when H is a subgroup of G and $M = U \uparrow_H^G$. We write $D(G)$ to denote the maximal divisible subgroup of G.

A subgroup X of H is said to be *open* in H if $X = H \cap K$ for some subgroup K of finite index in G. Note that if H is co-divisible in G, then X is open in H if and only if G/X splits over H/X; in this case, $G/X = H/X \times D_X/X$, where $D_X/X = D(G/X)$ is the unique complement to H/X in G/X.

In the following proposition, G can be any abelian group, and k can be any ring with identity.

2.1 PROPOSITION. *Let M be a kG-module and let H be a co-divisible subgroup of G, and suppose that $M = UkG$ where U is a kH-submodule of M. Put*

$$\mathscr{N} = \{N \leq_{kG} M \mid |G : \mathrm{C}_G(M/N)| \text{ is finite}\},$$
$$\mathscr{V} = \{V \leq_{kH} U \mid \mathrm{C}_H(U/V) \text{ is open in } H\}.$$

There exists a mapping $\Phi: \mathscr{N} \to \mathscr{V}$ given by $\Phi(N) = N \cap U$, and the following hold:

(i) *Φ is a bijection from $\mathscr{N}$ onto $\Phi(\mathscr{N})$, with inverse Ψ given by*

$$\begin{aligned}\Psi(V) &= VkG + M(D_X - 1) \\ &= V + U(D_X - 1), \end{aligned} \tag{2.1}$$

where $X = \mathrm{C}_H(U/V)$.

(ii) *If $M = U \uparrow_H^G$ then $\Phi(\mathscr{N}) = \mathscr{V}$.*

Both Φ and Ψ preserve inclusions. Moreover, if $N \in \mathscr{N}$ then $U/\Phi(N)$ and M/N are isomorphic kH-modules, and if $X = \mathrm{C}_H(U/\Phi(N))$ then $D_X = \mathrm{C}_G(M/N)$.

Proof. Let $N \in \mathscr{N}$ and put $Y = \mathrm{C}_G(M/N)$, $V = N \cap U$, $X = \mathrm{C}_H(U/V)$. Then $HY = G$ since G/H has no proper finite quotients. Thus

$$M = UkG = U(kH + (Y - 1)kH) \subseteq U + N, \tag{2.2}$$

so $M = U + N$; hence $M/N \cong U/(N \cap U) = U/V$ as kH-modules. It follows that $X = H \cap Y$, so X is open in H and $V \in \mathscr{V}$. Since now $G/X = H/X \times Y/X$, we also have $Y = D_X$, and this implies that

$$VkG + M(D_X - 1) \leq N. \tag{2.3}$$

Moreover, $G = HY$ implies $kG(Y - 1) = kH(Y - 1)$, so

$$M(D_X - 1) = UkG(Y - 1) = U(Y - 1). \tag{2.4}$$

On the other hand, suppose $w \in N$. Then (as in (2.2)) we can write $w = u_0 + \sum_{i \geq 1} u_i(y_i - 1)$ with each $u_i \in U$ and $y_i \in Y$; the sum on the right lies in $M(Y - 1) \leq N$, so $u_0 \in N \cap U = V$. Thus $N \leq V + M(Y - 1)$. With (2.3) and (2.4) this establishes the second equality in (2.1), and shows that $\Psi(V) = N$.

It is obvious that Φ preserves inclusions. That Ψ does so follows from (2.1), and the easily verified fact that if $X_1 \leq X_2$ are open subgroups of H then $D_{X_1} \leq D_{X_2}$.

We have established everything except (ii). Suppose now that $M = U\uparrow_H^G$; we have to show that Φ maps $\mathscr{N}$ onto $\mathscr{V}$. So let $V \in \mathscr{V}$, and put $X = \mathrm{C}_H(U/V)$, $Y = D_X$, and $N = V + M(Y-1)$. I claim that $N \in \mathscr{N}$ and that $N \cap U = V$. Now N is certainly a kG-submodule of M, because $G = HY$ and V is a kH-submodule of M. Since Y has finite index in G and $\mathrm{C}_G(M/N) \geq Y$, it follows that $N \in \mathscr{N}$. Finally, let T be a transversal to the cosets of X in Y, with $1 \in T$; as $G = HY$, we have

$$kG(Y-1) = \sum_{t\in T} kH(X-1)t + \sum_{t\in T} kH(t-1).$$

Since $G/X = H/X \times Y/X$, T is also a transversal to the cosets of H in G, so

$$M = \bigoplus_{t\in T} Ut.$$

Now

$$\begin{aligned} M(Y-1) &= \sum_{t\in T} UkH(X-1)t + \sum_{t\in T} UkH(t-1) \\ &\leq \sum_{t\in T} Vt + \sum_{t\in T} U(t-1); \end{aligned}$$

so

$$M(Y-1) \cap U \leq V$$

(for if $u = \sum v_t t + \sum u_t(t-1) \in U$ then $v_t + u_t = 0$ for each $t \neq 1$, giving $u = v_1 + \sum_{t\neq 1} v_t \in V$). Therefore $N \cap U = V + (M(Y-1) \cap U) = V$, as required. ▮

2.2 COROLLARY. (i) *Every finite kG-module image of M is kH-isomorphic to some kH-module image of U.* (ii) *Suppose that G/H is a π-group and that $M = U\uparrow_H^G$. If $\overline{U}$ is a finite kH-module image of U and $H/\mathrm{C}_H(\overline{U})$ is a π'-group, then $\overline{U}$ is kH-isomorphic to some kG-module image of M.*

We shall mostly apply Proposition 2.1 with $M = kG/P$ where P is a prime ideal of kG. In this case we can say a little more (we revert to the standing assumption that G is a minimax group):

2.3 PROPOSITION. *Let P be an ideal of kG and let H be a co-divisible subgroup of G. Put $S = kG/P$ and $R = (kH+P)/P$, and assume that $S = R\uparrow_H^G$. Write $\mathscr{N}_f$ for the set of all ideals of finite index in S and $\mathscr{V}_f$ for the set of all ideals J of finite index in R such that $J^\dagger$ is open in H. Then*

(i) *the mapping $\Phi: \mathscr{N}_f \to \mathscr{V}_f$ given by $I \mapsto I \cap R$ is a bijection, and $R/\Phi(I) \cong S/I$ for each $I \in \mathscr{N}_f$;*

(ii) *let* $I \in \mathcal{N}_f$ *and* $n \in \mathbb{N}$; *then* $I^n \in \mathcal{N}_f$, *and if* $|S : I|$ *is coprime to* $\pi(G/H)$ *then* $\Phi(I^n) = \Phi(I)^n$.

Proof. (i) We apply Proposition 2.1 with $M = S$ and $U = R$. Since $C_G(S/I) = I^\dagger$ for each ideal I of S, it is clear that $\mathcal{N}_f \subseteq \mathcal{N}$. As $R/\Phi(I) \cong S/I$ for each such I, it follows that $\Phi(\mathcal{N}_f) \subseteq \mathcal{V}_f$. If $J \in \mathcal{V}_f$ then, similarly, $J \in \mathcal{V}$, so we have $\Psi(J) \in \mathcal{N}$ and $S/\Psi(J) \cong R/J$; thus $S/\Psi(J)$ is finite and $\Psi(J) \in \mathcal{N}_f$.

(ii) Put $q = |S : I|$ and $G_n = (I^n)^\dagger$. Then $I^\dagger/G_n$ has exponent dividing q^{n-1}, and $G/I^\dagger$ is finite, so G/G_n is finite. Also $k/(k \cap I^n) = k_n$, say, is a finite ring. It follows that the group ring $k_n(G/G_n)$ is finite, and hence so is its image S/I^n. Thus $I^n \in \mathcal{N}_f$.

Suppose now that q is coprime to $\pi(G/H)$. Put $J = \Phi(I)$. Arguing as above, with H for G and J for I, we see that J^n has finite index in R and that $J^\dagger/(J^n)^\dagger$ is a finite group of exponent dividing q^{n-1}. As $J^\dagger$ is open in H, it now follows that $(J^n)^\dagger$ is open in H. Thus $J^n \in \mathcal{V}_f$. Put $I_n = \Psi(J^n)$. Then $I_n \leq I$, since Ψ is inclusion-preserving, and $I_n + R = S$, by (2.2) above. Therefore

$$I = I_n + (I \cap R) = I_n + J,$$

so $I^n \leq I_n + J^n = I_n$ and so $I^n \cap R \leq I_n \cap R = J^n$. The reverse inclusion being obvious, we deduce that $\Phi(I^n) = I^n \cap R = J^n = \Phi(I)^n$. ∎

3. AN INTERSECTION THEOREM

Here we establish

3.1 THEOREM. *Let* M *be a non-singular, finitely generated prime* kG*-module, and let* $I/\operatorname{ann}_{kG} M$ *be an ideal of finite index* $q > 1$ *in* $kG/\operatorname{ann}_{kG} M$. *If* q *is coprime to* $\operatorname{spec}(G)$ *then*

$$\bigcap_{n=1}^{\infty} MI^n = 0.$$

Proof. Put $S = kG/\operatorname{ann}_{kG} M$, so M is a finitely generated torsion-free S-module, S is an integral domain, and $\operatorname{char}(S) \notin \operatorname{spec}(G)$. As in the proof of Lemma 1.6, we see that $M\lambda \leq W \leq M$ for some free S-module W and $0 \neq \lambda \in S$. Now put $L = I/\operatorname{ann}_{kG} M$. It will suffice to prove that

$$\bigcap_{n=1}^{\infty} L^n = 0, \tag{3.1}$$

for if this holds then also $\bigcap_{n=1}^{\infty} WL^n = 0$, giving the result since $M \cong M\lambda \le W$.

Assume without loss of generality that M is faithful for G. Corollary 1.2 shows that G has a finitely generated dense subgroup H such that $S = R\uparrow_H^G$, where R denotes the image of kH in S; we can choose H to be co-divisible in G. Proposition 2.3(ii) shows that then

$$L^n \cap R = (L \cap R)^n \tag{3.2}$$

for each n. Since R is a finitely generated domain and $L \cap R$ is a proper ideal of R, Krull's intersection theorem tells us that $\bigcap_{n=1}^{\infty}(L \cap R)^n = 0$. Then (3.2) shows that $R \cap \bigcap_{n=1}^{\infty} L^n = 0$, and (3.1) follows since S is integral over R.

4. IDEALS OF FINITE INDEX

We begin with a result about finitely generated rings:

4.1 PROPOSITION. *Let R be a finitely generated integral domain, and let H be a torsion-free subgroup of R^*. Let π be a finite set of primes. Then the maximal ideals M of finite index in R satisfying*

$$|H : (1 + M) \cap H| \text{ is a } \pi'\text{-number}$$

intersect in zero. If $\mathrm{char}(R) = 0$, *then the same holds of the maximal ideals of prime index in R.*

Proof. If $\mathrm{char}(R) = l \neq 0$, we shall assume that $l \notin \pi$; this will not affect the conclusion, since if M is a maximal ideal of finite index in R then $|H : (1 + M) \cap H|$ is a divisor of $|(R/M)^*|$, which is coprime to l.

We start with some reductions. By [GS1, Theorem A], there exist homomorphisms θ from R into global fields of characteristic l such that $\theta_{|H}$ is injective, and the kernels of all such θ intersect in zero. Replacing R by its image under such a θ, we may therefore assume that R is a subring of a global field k. We may assume that $\zeta_p \in k$ for each $p \in \pi$. As R is finitely generated, there is a finite set S of primes of k such that R consists of S-integers; enlarging R if necessary, we may suppose that R is the full ring of S-integers in k. Now R^* is finitely generated, so we have $R^*/H = G/H \times T/H$, where G/H is free abelian and T/H is finite, of exponent $\prod p^{e(p)}$, say. Let e be the maximum of all the $e(p)$, and denote by $\mu(p)$ the group of p-power roots of unity in k. Since G is torsion-free we then have

$$k^{*p^e} \cap G\mu(p) \le G \tag{4.1}$$

for each prime p. Note also that, since H is torsion-free and $\mu(p)$ is cyclic, $|\mu(p)| \mid p^e$ for each prime p.

Put $m = \prod_{p \in \pi} p^e$, for each $p \in \pi$ put $\eta_p = \zeta_{p^{e+1}}$, and put

$$K = k\left(\zeta_m, G^{1/m}\right), \qquad L = K\left(\eta_p \mid p \in \pi\right), \qquad E_p = k(\eta_p) \qquad \text{for } p \in \pi.$$

Below I shall prove that

$$\eta_p \notin K\left(\eta_q \mid q \in \pi \setminus \{p\}\right) = L_p, \tag{4.2}$$

say, for each $p \in \pi$. This implies that $\mathrm{Gal}(L/K)$ is the direct product of its subgroups $\mathrm{Gal}(L/L_p)$, each of which has order p; it follows that $\mathrm{Gal}(L/K)$ is not contained in $\bigcup_{p \in \pi} \mathrm{Gal}(L/E_p)$. Let

$$\sigma \in \mathrm{Gal}(L/K) \setminus \bigcup_{p \in \pi} \mathrm{Gal}\left(L/E_p\right).$$

Chebotarev's theorem [FJ, Chap. 5] tells us that there exist infinitely many primes $\mathfrak{P}$ of L such that

$$\mathfrak{P} \cap k = \mathfrak{p} \notin S, \qquad \mathfrak{p} \text{ is unramified in } L,$$

$$\text{and} \qquad \left(\frac{L/k}{\mathfrak{P}}\right) = \sigma;$$

in the number-field case, when $\mathrm{char}(k) = 0$, infinitely many of these $\mathfrak{p}$ have absolute residue-class degree 1. Now choose such a $\mathfrak{P}$, and put $\mathfrak{P}_0 = \mathfrak{P} \cap K$. Write $\theta\colon R \to R/\mathfrak{p}R = \mathbb{F}_q$, say, for the residue-class map, and $\overline{R}$ for the integral closure of R in K; note that $\mathfrak{p}R = R \cap \mathfrak{P}_0\overline{R}$. Now the choice of $\mathfrak{P}$ ensures that $\mathfrak{p}$ splits completely in K but does not split completely in E_p for any $p \in \pi$. The first statement implies that $R + \mathfrak{P}_0\overline{R} = \overline{R} \supseteq G^{1/m}$, and hence that $G\theta$ consists of mth powers in $(R\theta)^*$. The second statement implies that $q \not\equiv 1 \pmod{p^{e+1}}$ for each $p \in \pi$, and hence that the π-part of $|(R\theta)^*|$ divides m. It follows that $G\theta$ is a π'-group.

Now we take $M = \mathfrak{p}R$. Then $|H : (1 + M) \cap H|$ divides $|G\theta|$, a π'-number, and $R/M = \mathbb{F}_q$. (When $\mathrm{char}(k) = 0$, we also have that $q = |R : \mathfrak{p}R|$ is prime.) As there are infinitely many different possible choices for $\mathfrak{P}$, the intersection of all such M is zero. This completes the proof, modulo the

Proof of (4.2). Fix $p \in \pi$, and put $E = k(\zeta_m, \eta_q \mid q \in \pi \setminus \{p\})$, so we have $L_p = E(G^{1/m})$. Suppose that $\eta_p \in L_p$. Then

$$E(G^{1/m}) = E\left(\left(G\langle \zeta_p \rangle\right)^{1/m}\right),$$

since η_p^m is a primitive pth root of unity if p is odd and a primitive fourth root of unity if $p = 2$. It follows by Kummer theory that $\zeta_p \in E^{*m} \cdot G$; thus $\zeta_p = y^{p^e} g$ for some $y \in E^*$ and $g \in G$.

Now $E^{*p^e} \cap k^* \subseteq k^{*p^e}\mu(p)$, by [GS2, Lemmas 5.2 and 5.3]. Since ζ_p and g are in k^* we therefore have $y^{p^e} = x^{p^e}\lambda$, where $x \in k^*$ and $\lambda \in \mu(p)$; then (4.1) gives

$$x^{p^e} = y^{p^e}\lambda^{-1} = g^{-1}\zeta_p\lambda^{-1} \in k^{*p^e} \cap G\mu(p) \subseteq G.$$

As G is torsion-free this implies that $g = x^{-p^e}$ and hence that $\zeta_p = (yx^{-1})^{p^e}$. It follows that $\eta_p \in E$. Putting $E_0 = k(\eta_q \mid q \in \pi \setminus \{p\})$, and noting that $E = E_0(\zeta_{p^e})$, we deduce from Lemma 1.3 that in fact $E = E_0$. As $(E_0 : k)$ is prime to p and $\zeta_p \in k$ this implies that $\eta_p \in k$, contradicting our observation that $|\mu(p)| \mid p^e$. This contradiction completes the proof. ∎

Now we state the main result of this section:

4.2 THEOREM. *Let P be a regular prime ideal of kG such that $G/P^\dagger$ is reduced. Let $S = kG/P$.*

(i) *The maximal ideals of finite index in S intersect in* 0. *If* $\mathrm{char}(S) = 0$ *then the same holds for the maximal ideals of prime index in S.*

(ii) *Assume that S is infinite. Then for infinitely many maximal ideals L of finite index in S we have $\bigcap_{n=1}^{\infty} L^n = 0$; if* $\mathrm{char}(S) = 0$ *then the same holds for infinitely many maximal ideals of prime index in S.*

Proof. We may clearly assume that $P^\dagger = 1$. Then G has a torsion-free subgroup G_1 of finite index, and, by Corollary 1.2, G has a finitely generated dense subgroup H such that $P = (P \cap kH)kG$; we choose H so that G/H is divisible, and write $H_1 = H \cap G_1$. Then $HG_1 = G$ and $G_1/H_1 \cong G/H$. So putting $R = (kH + P)/P \cong kH/(kH \cap P)$ we have

$$S = R\uparrow_H^G = R\uparrow_{H_1}^{G_1}.$$

Now put $\pi = \pi(G/H)$. Suppose M is a maximal ideal of finite index in R such that $|H_1 : (1 + M) \cap H_1|$ is a π'-number. Then Corollary 2.2 shows that S has a kG_1-submodule $\tilde{M}$ such that $R + \tilde{M} = S$ and $R \cap \tilde{M} = M$, and it is clear from the construction given in Proposition 2.1 that $\tilde{M}$ is in fact an R-submodule of S. Since $S = R[G_1]$ it follows that $\tilde{M}$ is an ideal of S, and $S/\tilde{M} \cong R/M$ is a finite field.

Proposition 4.1 says that the intersection of all ideals M as above is zero (counting only those of prime index in the case where $\mathrm{char}(S) = 0$). It follows that if I is the intersection of all the corresponding ideals $\tilde{M}$ of S then $I \cap R = 0$. But the integral domain S is an integral extension of its subring R, since G/H is periodic. Consequently $I = 0$. Part (i) of the theorem follows.

Part (ii) follows from (i) and Theorem 3.1. ∎

Remark. The hypothesis that $G/P^{\dagger}$ be reduced is necessary in Theorem 4.2. If G is not finitely generated, then G has C_{p^∞} as a homomorphic image, for some prime p; provided only that $\text{char}(k) \neq p$, it is then easy to see that kG can be mapped onto an infinite field of prime characteristic. The kernel P of such a map will be a maximal ideal of infinite index in kG.

An application of Theorem 4.2 is the following:

4.3 COROLLARY. *Let M be a finitely generated non-singular prime kG-module. Then the following are equivalent:*

(a) *$G/\mathrm{C}_G(M)$ is reduced;*

(b) *M is residually finite as kG-module;*

(c) *there exist a subgroup K of finite index in G and a maximal ideal J of finite index in k such that*

$$\bigcap_{n=1}^{\infty} M(JkG + (K-1)kG)^n = 0. \tag{4.3}$$

If $\text{char}(M) = 0$, *then these conditions are equivalent to*

(d) *for infinitely many primes p, there exist a subgroup K of finite index in G and a maximal ideal J of index p in k such that* (4.3) *holds.*

Proof. If (4.3) holds, for some J and K of finite index, then M is residually finite, by Proposition 2.3(ii), so (d) ⇒ (c) ⇒ (b). It is easy to see that (b) implies (a). Suppose now that $G/\mathrm{C}_G(M)$ is reduced, and put $P = \text{ann}_{kG}(M)$. Then P is a regular prime ideal and $G/P^{\dagger}$ is reduced, so Theorem 4.2(i) shows that kG/P has a maximal ideal L/P of finite index. Theorem 3.1 shows that $\bigcap_{n=1}^{\infty} ML^n = 0$, provided that $\text{char}(kG/L) \notin \text{spec}(G)$. Taking $K = L^{\dagger}$ and $J = L \cap k$, we see that (4.3) holds. If $\text{char}(kG/P) = l \neq 0$ then also $\text{char}(kG/L) = l \notin \text{spec}(G)$, and (c) follows. If $\text{char}(M) = 0$, we may choose L to have prime index p in kG, and clearly there are infinitely many choices for p. As $\text{spec}(G)$ is finite, this gives (d). ∎

5. MINIMAL PRIMES

Since kG is integral over a finitely generated subring kH, where H is a finitely generated dense subgroup of G, we see that kG has finite Krull dimension: in fact $\text{Dim}(kG) = \text{Dim}(k) + \mathrm{h}(H)$, where $\mathrm{h}(H)$ is the Hirsch length (torsion-free rank) of H. It follows that if I is a proper ideal of kG, then the set of prime ideals containing I has minimal members. I shall call these the *minimal primes of I*.

5.1 PROPOSITION. *Let I be a proper ideal of kG and let P be a minimal prime of I.*

(i) *G has a finitely generated dense subgroup H_0 such that $P \cap kH$ is a minimal prime of $I \cap kH$ for every finitely generated subgroup H of G containing H_0.*

(ii) *For each such H, there exists $r \in kH$ such that $P \cap kH = \mathrm{ann}_{kH}(r \bmod I)$.*

(iii) *If I is regular then P is regular.*

(iv) *If P is regular and $P^\dagger/I^\dagger$ is finitely generated, then there exists $r \in kG$ such that $P = \mathrm{ann}_{kG}(r \bmod I)$.*

Proof. We assume without loss of generality that $I^\dagger = 1$. Put

$$\mathscr{S} = \{kH \mid H \text{ is a finitely generated dense subgroup of } G\}.$$

For $S \in \mathscr{S}$, define

$$\mu(S) = \min\{\mathrm{ht}(Q) \mid Q \text{ a prime ideal of } S \text{ with } I \cap S \subseteq Q \subseteq P \cap S\}.$$

Now if $S \subseteq T \in \mathscr{S}$ then

$$\mu(S) \le \mu(T) \le \mathrm{Dim}(S) = \mathrm{Dim}(kG);$$

so if we choose $S_0 = kH_0$ so that $\mu(S_0)$ is maximal and write $\mathscr{S}_0 = \{S \in \mathscr{S} \mid S \supseteq S_0\}$ we have

$$S \in \mathscr{S}_0 \quad \Rightarrow \quad \mu(S) = \mu(S_0) = \mu,$$

say. For each $S \in \mathscr{S}_0$ let

$$\mathfrak{P}(S) = \{Q \mid Q \text{ a prime ideal of } S \text{ with } I \cap S \subseteq Q \subseteq P \cap S \text{ and } \mathrm{ht}(Q) = \mu\}.$$

Then $\mathfrak{P}(S)$ is non-empty, it consists of minimal primes of $I \cap S$, and hence is finite since S is Noetherian. If $T \supseteq S \in \mathscr{S}_0$ then $Q \mapsto Q \cap S$ maps $\mathfrak{P}(T)$ into $\mathfrak{P}(S)$; the family $\{\mathfrak{P}(S) \mid S \in \mathscr{S}_0\}$ with these maps then forms an inverse system whose inverse limit is non-empty. Let $(P_S)_{S \in \mathscr{S}}$ be an element of this inverse limit, and put

$$\tilde{P} = \bigcup_{S \in \mathscr{S}_0} P_S.$$

As kG is the union of its subrings $S \in \mathscr{S}_0$, it is clear that $I \subseteq \tilde{P} \subseteq P$ and that $\tilde{P}$ is a prime ideal of kG. It follows that $P \cap S = P_S$ is a minimal prime of $I \cap S$ for each $S \in \mathscr{S}_0$. This proves (i).

Claim (ii) follows from the fact that S is Noetherian, and (iii) follows from (ii).

Finally, suppose that P is regular and that $P^\dagger$ is finitely generated. Corollary 1.2 shows that then $P = (P \cap S)kG$ for some $S \in \mathscr{S}_0$. There exists $r \in S$ such that $P \cap S = \operatorname{ann}_S(r \bmod I)$. Now put $K = \operatorname{ann}_{kG}(r \bmod I)$. Then $K \cap S = P \cap S$, and so $K \supseteq (P \cap S)kG = P$. Since kG is integral over S this implies that $K = P$, and (iv) follows. ∎

Remark. We shall see in the next section that when kG/I is *residually finite*, the minimal primes of I behave just as in the Noetherian situation (Corollary 6.6).

6. QUASI-RESIDUALLY FINITE MODULES

A subgroup C of G is *closed* if G/C is reduced (i.e., if C is closed in the profinite topology on G). A kG-module M is qrf (*quasi-residually finite*) if $\mathrm{C}_G(a)$ is closed in G for every element $a \in M$. Since the intersection of any family of closed subgroups is closed, we see that M is qrf if and only if $\mathrm{C}_G(S)$ is closed in G for *every* subset S of M. It is easy to see that any module which is locally residually finite for kG is necessarily qrf. As we shall see, the converse is not far from being true also; however, the qrf hypothesis is more convenient to handle.

For an ideal X of kG, we write ${}^*X = \{a \in M \mid aX = 0\}$. For a non-empty set $\mathscr{X}$ of ideals, we write $\langle \mathscr{X} \rangle$ to denote the set of all finite products of ideals in $\mathscr{X}$, and

$$M(\mathscr{X}) = \bigcup \{ {}^*X \mid X \in \langle \mathscr{X} \rangle \}.$$

We also put $M(\varnothing) = 0$. For a subset S of M, we put $S^* = \operatorname{ann}_{kG} S$. We write

$$\mathscr{P}(M) = \{ a^* \mid 0 \neq a \in M \text{ and } a^* \text{ is prime} \},$$

and denote by $\mathscr{M}(M)$ the set of all maximal members of $\mathscr{P}(M)$. It is well known, and easy to see, that $\mathscr{M}(M)$ contains all maximal annihilators of non-empty, non-zero subsets of M; it will follow from 6.3, below, that when M is non-singular and qrf then $\mathscr{M}(M)$ consists of such maximal annihilators. We shall also be interested in the set $\mathscr{Q}(M)$ of all *minimal* members of $\mathscr{P}(M)$.

We put $\pi = \operatorname{spec}(G)$. I shall say that a module N has *reduced π-torsion* if the π-torsion subgroup $\tau_\pi(N)$ of N is reduced as an abelian group.

We start with some elementary observations; here M is an arbitrary kG-module.

6.1 LEMMA. *The set of all closed subgroups of G satisfies the ascending chain condition.*

Proof. If C is a closed subgroup of G then G/C is finite-by-torsion free; if $C_1 < C_2$ then either $\mathrm{h}(G/C_2) < \mathrm{h}(G/C_1)$ or $|\tau(G/C_2)| < |\tau(G/C_1)|$. ▮

6.2 LEMMA. (i) *Let (M_α) be a chain of submodules of M with union U. If M/M_α is* qrf *for each α then M/U is* qrf.

(ii) *Let (M_α) be a family of submodules of M with intersection V. If M/M_α is* qrf *for each α then M/V is* qrf.

(iii) *Let N be a submodule of M which has reduced π-torsion. If N and M/N are* qrf *then M is* qrf.

(iv) *If M is* qrf *then so are $M/{}^*X$, for any ideal X, and $M/\tau_\sigma(M)$, for any set of primes σ.*

Proof. (i) Let $a \in M$, and put $C_\alpha = \mathrm{C}_G(a \bmod M_\alpha)$ for each α. Then $\mathrm{C}_G(a \bmod U) = \bigcup_\alpha C_\alpha$. Now the C_α form a chain of closed subgroups in G; it follows from 6.1 that $\bigcup_\alpha C_\alpha = C_\beta$ for some β.

(ii) Using the notation above, we have $\mathrm{C}_G(a \bmod V) = \bigcap C_\alpha$, which is closed.

(iii) Let $a \in M$ and put $A = \mathrm{C}_G(a \bmod N)$, $B = \mathrm{C}_G(akG \cap N)$. Then A and B are closed, so $G/(A \cap B)$ is reduced. The mapping $x \mapsto a(x-1)$ is a homomorphism from $A \cap B$ into N with kernel equal to $\mathrm{C}_G(a)$, so $(A \cap B)/\mathrm{C}_G(a)$ is a section of G with reduced π-torsion. It follows that $G/\mathrm{C}_G(a)$ is reduced.

(iv) If $a \in M$ then $\mathrm{C}_G(a \bmod ({}^*X)) = \mathrm{C}_G(aX)$, which is closed in G. Now $\tau_\sigma(M)$ is the ascending union of a family of submodules of the form ${}^*n$, where n ranges over a sequence of σ-numbers, each dividing the next; the second claim therefore follows by (i). ▮

Remark. The hypothesis that N have reduced π-torsion is necessary in (iii). Without that hypothesis, it becomes false: consider $G = C_{p^\infty}$ acting on $M = C_{p^\infty} \oplus \mathbb{Z}$ via $x \mapsto \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$.

From now on, we shall assume that *M is a non-singular qrf kG-module*. Note that every member of $\mathscr{P}(M)$ is then a regular prime ideal.

The next, key, lemma shows the usefulness of the qrf hypothesis: it implies that every annihilator is contained in a maximal one.

6.3 LEMMA. *If $S \not\subseteq \{0\}$ is a subset of M then $S^* \subseteq P$ for some $P \in \mathscr{M}(M)$.*

Proof. Let $X = \{v \in M \setminus \{0\} \mid v^* \supseteq S^*\}$, and let $Y = \{v \in X \mid \mathrm{C}_G(v)$ is maximal for $v \in X\}$. Then Y is non-empty by 6.1. Let $v \in Y$, put $I = v^*$, and let P be a minimal prime of I. By Proposition 5.1, there is a finitely generated subgroup H_0 of G such that $P \cap kH$ is a minimal prime of $I \cap kH$ whenever $H_0 \leq H \leq G$ and H is finitely generated; and for any such H, there exists $r \in kH$ such that $P \cap kH = \mathrm{ann}_{kH}(r \bmod I) = \mathrm{ann}_{kH}(vr)$. This implies, in particular, that P is regular.

Let H and r be such a pair. Then $vr \in X$ and $\mathrm{C}_G(vr) \geq \mathrm{C}_G(v)$, so

$$P^{\dagger} \cap H = \mathrm{C}_H(vr) \leq \mathrm{C}_G(vr) = \mathrm{C}_G(v) = I^{\dagger}.$$

As G is covered by subgroups like H it follows that $P^{\dagger} = I^{\dagger}$. The last part of Proposition 5.1 now shows that there exists $r \in kG$ such that

$$P = \mathrm{ann}_{kG}(r \bmod I) = (vr)^*,$$

and $vr \in Y$ since now $\mathrm{C}_G(vr) = \mathrm{C}_G(v)$. Thus the set

$$Z = \{w \in Y \mid w^* \text{ is prime}\}$$

is non-empty.

Now kG has the ascending chain condition on prime ideals since $\mathrm{Dim}(kG)$ is finite, so the set $\{w^* \mid w \in Z\}$ has a maximal member Q. Since $Z \subseteq X$ we have $S^* \subseteq Q$. I claim that $Q \in \mathscr{M}(M)$. To see this, suppose $Q \subseteq v^*$ for some $v \neq 0$. Then $v \in X$ and $\mathrm{C}_G(v) \geq \mathrm{C}_G(w)$ where $w \in Y$ with $Q = w^*$, so $v \in Y$. Let P be a minimal prime of v^*. By the preceding paragraphs we see that $P = w_1^*$ for some $w_1 \in Z$; so from $Q \subseteq v^* \subseteq P$ it follows that $Q = P = v^*$. This establishes the claim and completes the proof. ∎

6.4 Corollary. *If $M \neq 0$ then $\mathscr{P}(M)$ is non-empty, and*

$$\begin{aligned}\pi(M) &= \{\mathrm{char}(kG/P) \mid P \in \mathscr{P}(M)\} \setminus \{0\} \\ &= \{\mathrm{char}(kG/P) \mid P \in \mathscr{M}(M)\} \setminus \{0\}.\end{aligned}$$

6.5 Lemma. *Let $\mathscr{X}$ be a set of prime ideals of kG, and put $N = M(\mathscr{X})$. Then*

(i) *M/N is* qrf, *and $\pi(M/N) \subseteq \pi(M)$ (so in particular M/N is non-singular).*

(ii) *$\mathscr{P}(M/N) \subseteq \mathscr{P}(M) \setminus \mathscr{X}$ and $\mathscr{X} \cap \mathscr{P}(M) \subseteq \mathscr{P}(N)$.*

(iii) *Suppose that $\mathscr{X} \subseteq \mathscr{P}(M)$; if $\mathscr{P}(M) \setminus \mathscr{X} \subseteq \mathscr{Q}(M)$ then $\mathscr{P}(M/N) = \mathscr{P}(M) \setminus \mathscr{X}$ and $\mathscr{P}(N) = \mathscr{X}$.*

(iv) $M = M(\mathscr{P}(M)) = M(\mathscr{Q}(M))$.

Proof. (i) Let $w \in M \setminus N$. By 6.1 we may choose $D_0 \in \langle \mathscr{X} \rangle$ so as to maximise $C_G(wD_0)$. I claim that then $C_G(wD_0) = C_G(w \bmod N)$. Indeed,

$$\begin{aligned} x \in C_G(wD_0) \quad &\Rightarrow \quad w(x-1)D_0 = wD_0(x-1) = 0 \\ &\Rightarrow \quad w(x-1) \in N \\ &\Rightarrow \quad x \in C_G(w \bmod N). \end{aligned}$$

On the other hand, if $x \in C_G(w \bmod N)$ then $w(x-1)D = 0$ for some $D \in \langle \mathscr{X} \rangle$, so $x \in C_G(wD) \leq C_G(wDD_0) = C_G(wD_0)$ by the maximal choice of the latter.

It follows that $C_G(w \bmod N) = C_G(wD_0)$ is closed; thus M/N is qrf.

Suppose now that $qw \in N$ where q is a prime number. Then $qwD = 0$ for some $D \in \langle \mathscr{X} \rangle$, and $wD \neq 0$ since $w \notin N$. Hence $q \in \pi(M)$. This shows that $\pi(M/N) \subseteq \pi(M)$.

(ii) Suppose now that $Q = \operatorname{ann}(w \bmod N)$ is prime, where $w \in M \setminus N$ as above; it follows from (i) that Q is regular. Putting $I = \operatorname{ann}(wD_0)$, we have $Q^\dagger = C_G(w \bmod N) = C_G(wD_0) = I^\dagger$. It follows by Corollary 1.2 that Q/I is finitely generated, so we have $Q = I + J$ for some finitely genrated ideal J. Then $wJD_1 = 0$ for some $D_1 \in \langle \mathscr{X} \rangle$, since now wJ is a finitely generated submodule of N. Since $wD_0 I = 0$ by definition it follows that $wQD_1D_0 = 0$. Thus putting $D = D_1D_0$ we have $D \in \langle \mathscr{X} \rangle$ and $Q \subseteq \operatorname{ann}(wD)$.

Now $wD \nsubseteq N$; for if $wD \subseteq N$ then $D \subseteq Q$ and then $wD^2 = 0$, which is false since $w \notin N$. Hence there exists $x \in D$ such that $wx \notin N$. Put $Y = (wx)^*$. Then $xY \subseteq Q$ since $wxY = 0$, but $x \notin Q$ since $wx \notin N$; consequently $Y \subseteq Q$. Thus

$$Q \subseteq (wD)^* \subseteq (wx)^* = Y \subseteq Q,$$

showing that $Q = Y \in \mathscr{P}(M)$. On the other hand, $Q \notin \mathscr{X}$ since $wDQ = 0$ and $w \notin N$.

To complete the proof of (ii), note that if $X \in \mathscr{X} \cap \mathscr{P}(M)$ then $X = a^*$ for some $x \in M$, and then $a \in M(\{X\}) \leq N$, so in fact $X \in \mathscr{P}(N)$.

(iii) Assume now that $\mathscr{X} \subseteq \mathscr{P}(M)$ and that $\mathscr{P}(M) \setminus \mathscr{X} \subseteq \mathscr{Q}(M)$. Let $P \in \mathscr{P}(M) \setminus \mathscr{X}$. Then $P = a^*$ for some $a \in M$. If $a \in N$ then $a^* \geq D$ for some $D \in \langle \mathscr{X} \rangle$. But as P is prime and minimal in $\mathscr{P}(M)$ this would force P to be equal to some member of $\mathscr{X}$, so $a \notin N$, and hence, by Lemma 6.3, $P \leq Q$ for some $Q \in \mathscr{M}(M/N) \subseteq \mathscr{P}(M) \setminus \mathscr{X} \subseteq \mathscr{Q}(M)$. Then $P = Q$, and the first claim of (iii) follows. For the second claim, suppose that $Y \in \mathscr{P}(N)$. Then $Y \geq D$ for some $D \in \langle \mathscr{X} \rangle$, so $Y \geq X$ for some $X \in \mathscr{X}$; if $Y \notin \mathscr{X}$ then $Y \in \mathscr{Q}(M)$, forcing $Y = X \in \mathscr{X}$, a contradiction, so we must have $Y \in \mathscr{X}$. Thus $\mathscr{P}(N) \subseteq \mathscr{X}$, and the result follows from (ii).

Finally, putting $\mathscr{X} = \mathscr{P}(M)$ and $N = M(\mathscr{X})$, we see from (ii) that $\mathscr{P}(M/N) = \emptyset$, and hence by 6.4 that $M = M(\mathscr{P}(M))$. Since $\mathrm{Dim}(KG)$ is finite, each member of $\mathscr{P}(M)$ contains a member of $\mathscr{Q}(M)$, so (iv) follows. ∎

6.6 COROLLARY. *Let I be a non-singular ideal of kG, and suppose that kG/I is* qrf *as a kG-module. Then I has only finitely many minimal primes, $P_1, \dots, P_m$, say, and there exist natural numbers n_i such that*

$$\prod_{i=1}^{m} P_i^{n_i} \subseteq I.$$

For each i there exists $x_i \in kG$ such that $P_i = \mathrm{ann}_{kG}(x_i \bmod I)$.

Proof. Applying 6.5(iv) to the module $M = kG/I$, we see that the element $1 + I$ is annihilated by a product $P_1^{n_1} \dots P_m^{n_m}$, with each $P_i \in \mathscr{Q}(M)$. It follows that $P_1^{n_1} \dots P_m^{n_m} \subseteq I \subseteq P_i$ for each i, and this implies that $\{P_1, \dots, P_m\}$ is exactly the set of minimal primes of I. The final sentence follows from the definition of $\mathscr{Q}(M)$. ∎

Recall that M is said to be *unmixed* if $\mathscr{P}(M) = \mathscr{M}(M)$. We show next that an arbitrary non-singular qrf module has a finite filtration, of bounded length, with each factor unmixed. First we introduce some notation. For $X \in \mathscr{P}(M)$, let $h_*(X)$ denote the maximal length of a chain

$$X = X_0 > X_1 > \cdots > X_n$$

with $X_i \in \mathscr{P}(M)$ for each i. We define $\lambda(M)$ to be *the maximal length of any chain in* $\mathscr{P}(M)$; note that

$$\lambda(M) = \sup_{X \in \mathscr{P}(M)} h_*(X) \leq \mathrm{Dim}(kG/M^*),$$

and that $X \in \mathscr{Q}(M)$ if and only if $X \in \mathscr{P}(M)$ and $h_*(X) = 0$.

6.7 PROPOSITION. *For $0 \leq i \leq \lambda(M)$, let*

$$\mathscr{X}_i = \{X \in \mathscr{P}(M) \mid h_*(X) < i\}, \qquad M_i = M(\mathscr{P}(M) \setminus \mathscr{X}_i).$$

Then

(i) $M = M_0 > M_1 > \cdots > M_k = 0$, *where* $k = \lambda(M) + 1$;

(ii) *for each i, we have* $\mathscr{P}(M_{i-1}/M_i) = \mathscr{M}(M_{i-1}/M_i) = \mathscr{X}_i \setminus \mathscr{X}_{i-1}$.

Proof. Note to begin with that $M_0 = M$ by 6.5(iv), because $\mathscr{X}_0$ is empty, and that $M_k = 0$ because $\mathscr{P}(M) \setminus \mathscr{X}_k$ is empty. Next, we prove (ii). Soince $\mathscr{X}_1 = \mathscr{Q}(M)$, we see from 6.5(iii) that $\mathscr{P}(M/M_1) = \mathscr{X}_1 = \mathscr{X}_1 \setminus \mathscr{X}_0$; thus (ii) holds for $i = 1$, as the members of $\mathscr{X}_1$ are pairwise incomparable.

The second part of 6.5(iii) shows that $\mathscr{P}(M_1) = \mathscr{P}(M) \setminus \mathscr{X}_1$; that (ii) holds for $i > 1$ now follows by induction on $\lambda(M)$, on replacing M by M_1.

To complete the proof of (i), note now that if $1 \leq i \leq k$ then $\mathscr{X}_i \setminus \mathscr{X}_{i-1}$ is non-empty, so Corollary 6.4 shows that $M_{i-1}/M_i \neq 0$. ∎

6.8 COROLLARY. *A non-singular kG-module is* qrf *if and only if it is PLPL-residually finite for kG.*

PLPL means "poly locally-(poly locally-)." We shall see in Section 8 that in fact "PLPL" can be replaced by "PLP."

Proof. Let M be a non-singular qrf kG-module. In the notation of Proposition 6.7, it will suffice to show that if U is a finitely generated submodule of M_{i-1}/M_i, for some $i \leq k$, then U is poly locally-(residually finite). Now 6.7(ii) implies that $\mathscr{P}(U) = \mathscr{M}(U)$, and as U is finitely generated, it follows from 6.5(iv) that $UP_1 \dots P_s = 0$ for some $P_1, \dots, P_s \in \mathscr{M}(U)$. Taking s to be minimal, put $U_0 = 0$ and $U_i = {}^*(P_1 \dots P_i)$ for $i = 1, \dots, s$. For each i, U_i/U_{i-1} is easily seen to be prime kG-module with annihilator P_i; so if W is a non-zero finitely generated submodule of U_i/U_{i-1} then W is prime and $\mathrm{C}_G(W) = P_i^\dagger$ is closed in G. It follows by Corollary 4.3 that W is residually finite. Thus each U_i/U_{i-1} is locally residually finite for kG.

The converse follows from Lemma 6.2(iii), since every residually finite kG-module is qrf, and every locally qrf module is qrf. Indeed, this argument shows, more generally, that every module with reduced π-torsion which is $(\mathrm{PL})^n$-(residually finite) for kG for some n is a qrf module. ∎

7. DECOMPOSITION OF UNMIXED MODULES

With a view to applications, we consider in this section kG-modules with operators. We shall suppose that G is contained as a normal subgroup in another group Γ, and that M is a $k\Gamma$-module; adjectives such as "qrf" applied to M will always refer to the kG-structure of M.

It is clear that Γ permutes the sets $\mathscr{P}(M)$ and $\mathscr{M}(M)$. If also M is non-singular and qrf, then the subsets $\mathscr{X}_i$ of $\mathscr{P}(M)$ and the submodules M_i of M given in Proposition 6.7 are all Γ-invariant.

Throughout this section, the $k\Gamma$-module M is supposed to be non-singular and qrf as a kG-module. We assume also that M is *unmixed*, which means that $\lambda(M) = 0$; that is,

$$\mathscr{P}(M) = \mathscr{M}(M) = \mathscr{Q}(M).$$

The main result, 7.3 below, shows that if M is finitely generated as a $k\Gamma$-module, then M can be embedded in a *finite direct sum* of modules that are induced from *primary kG-modules*.

7.1 LEMMA. *Let* $\mathscr{X}_i$ $(i \in \mathscr{I})$ *be a family of pairwise disjoint subsets of* $\mathscr{P}(M)$, *and put* $\mathscr{X} = \bigcup_{i \in \mathscr{I}} \mathscr{X}_i$,

$$M_i = M(\mathscr{X}_i), \qquad M^i = M(\mathscr{X} \backslash \mathscr{X}_i)$$

for each i. *Then*

(i) $\sum_{i \in \mathscr{I}} M_i = \bigoplus_{i \in \mathscr{I}} M_i$;

(ii) *if* $\mathscr{X} = \mathscr{P}(M)$ *then the natural map* $M \to \prod_{i \in \mathscr{I}} M/M^i$ *is injective and maps* M *into* $\bigoplus_{i \in \mathscr{I}} M/M^i$.

Proof. (i) Suppose $0 \neq a \in M_1 \cap (M_2 + \cdots + M_n)$, where $1, 2, \ldots, n$ are distinct indices in $\mathscr{I}$. Then there exist $X \in \langle \mathscr{X}_1 \rangle$ and $Y \in \langle \mathscr{X}_2 \cup \cdots \cup \mathscr{X}_n \rangle$ such that $a(X + Y) = 0$. Then $X + Y \subseteq P$ for some $P \in \mathscr{M}(M)$, forcing $P = X_1 = Y_j$ for some $X_1 \in \mathscr{X}_1$ and $Y_j \in \mathscr{X}_j$, where $j \geq 2$. This is impossible since $\mathscr{X}_1 \cap \mathscr{X}_j = \varnothing$.

(ii) Let $0 \neq a \in M$, and put $I = a^*$. Then $kG/I \cong akG$ is non-singular and qrf; by Corollary 6.6, I has finitely many minimal primes $P_1, \ldots, P_m$, say, and these satisfy $\prod_{j=1}^{m} P_j^{n_j} \subseteq I$. Also each P_j is the annihilator of some element of akG, and since $M = M(\mathscr{X})$, by 6.5(iv), each P_j contains some member of $\mathscr{X}$. It follows that $P_j \in \mathscr{X}$ for each j. Now $a \in M^i$ if and only if I contains some member of $\langle \mathscr{X} \backslash \mathscr{X}_i \rangle$; this holds whenever

$$\mathscr{X}_i \cap \{P_1, \ldots, P_m\} = \varnothing,$$

which is true for all but finitely many values of i. It follows that the image of a in $\prod M/M^i$ lies in $\oplus M/M^i$, as claimed. On the other hand, if $P_j \in \mathscr{X}_{i(j)}$ then $a \notin M^{i(j)}$, for $j = 1, \ldots, m$: for if I contains some member of $\langle \mathscr{X} \backslash \mathscr{X}_{i(j)} \rangle$ then $P_j \supseteq Q$ for some $Q \in \mathscr{X} \backslash \mathscr{X}_{i(j)}$, forcing $P_j = Q \in \mathscr{X} \backslash \mathscr{X}_{i(j)}$. Thus the image of a in $\prod M/M^i$ is not zero, showing that the map $M \to \prod M/M^i$ is injective. ∎

7.2 COROLLARY. *Suppose that* $\mathscr{X} = \mathscr{M}(M)$, *and that the action of* Γ *on* $\mathscr{X}$ *is transitive. Fix* $P \in \mathscr{X}$, *put* $\Delta = \mathrm{N}_{\Gamma}(P)$, *and let*

$$U = M/M(\mathscr{X} \backslash \{P\}).$$

(i) *M has a natural embedding into the $k\Gamma$-module* $U \uparrow_{\Delta}^{\Gamma}$.

(ii) *If M is finitely generated as a $k\Gamma$-module then U is finitely generated as a $k\Delta$-module.*

Proof. (i) Let T be a transversal to the right cosets of Δ in Γ. Then $\mathscr{X} = \{P^t \mid t \in T\}$. We apply 7.1 with $\mathscr{I} = T$, $\mathscr{X}_t = \{P^t\}$, and $M^t = M(\mathscr{X} \backslash \{P^t\})$ for each t. Supposing that $1 \in T$, we thus have $U = M/M^1$, a

$k\Delta$-module since clearly M^1 is Δ-invariant, and $M^t = M^1 t$ for each $t \in T$. The operation of $t \in T$ therefore induces a k-module isomorphism θ_t: $U = M/M^1 \to M/M^t$. Writing $U\uparrow_\Delta^\Gamma = \oplus_{t \in T} U \otimes t$, we may define a k-module isomorphism θ: $U\uparrow_\Delta^\Gamma \to \oplus_{t\in T} M/M^t$ by putting $(u \otimes t)\theta = u\theta_t$. Now let

$$\Psi = \Pi \circ \theta^{-1}: M \to U\uparrow_\Delta^\Gamma,$$

where Π: $M \to \oplus_{t \in T} M/M^t$ is the natural injective mapping given by 7.1(ii). As $G \leq \Gamma$, we only have to verify that Ψ respects the action of Γ. So let $\gamma \in \Gamma$, and suppose that $t\gamma = \alpha_t s_t$, where $\alpha_t \in \Delta$ and $s_t \in T$, for each $t \in T$. Write —: $M \to U$ for the quotient mapping. Then for $a \in M$ we have

$$\begin{aligned}(a\Psi)\gamma &= \left(\sum \overline{at^{-1}} \otimes t\right)\gamma \\ &= \sum \overline{at^{-1}\alpha_t} \otimes s_t \\ &= \sum \overline{a\gamma s_t^{-1}} \otimes s_t = (a\gamma)\Psi,\end{aligned}$$

as required.

(ii) Let S be a finite subset of M. Then $S\Psi$ is contained in $V\uparrow_\Delta^\Gamma$ for some finitely generated $k\Delta$-submodule V of U. Writing π_1 for the projection $U\uparrow_\Delta^\Gamma \to U \otimes 1 = U$, we have

$$Sk\Gamma\Psi\pi_1 = S\Psi k\Gamma\pi_1 \leq \left(V\uparrow_\Delta^\Gamma\right)\pi_1 = V.$$

Hence if $Sk\Gamma = M$ then $U = M\Psi\pi_1 \leq V$. Thus $U = V$ and the result follows. ∎

7.3 PROPOSITION. *Let $\{P_j \mid j \in \mathscr{J}\}$ be a set of representatives for the orbits of Γ in $\mathscr{X} = \mathscr{M}(M)$. For each j let $\Delta_j = \mathrm{N}_\Gamma(P_j)$ and put $U_j = M/M(\mathscr{X} \setminus \{P_j\})$. Then $\mathscr{P}(U_j) = \{P_j\}$ for each j, and M has a natural embedding as a $k\Gamma$-module into*

$$\bigoplus_{j \in \mathscr{J}} U_j \uparrow_{\Delta_j}^\Gamma .$$

If M is finitely generated as a $k\Gamma$-module, then $\mathscr{J}$ is finite, and for each $j \in \mathscr{J}$, U_j is finitely generated as a $k\Delta_j$-module and satisfies $U_j P_j^{e_j} = 0$ for some $e_j \in \mathbb{N}$.

Proof. Denote by $\mathscr{X}_j$ the Γ-orbit of P_j, and put $M^j = M(\mathscr{X} \setminus \mathscr{X}_j)$. Lemma 7.1 shows that M embeds as a subdirect sum in $\oplus_{j \in \mathscr{J}} M/M^j$; each M^j is Γ-invariant, and it is clear that this embedding is a $k\Gamma$-homomorphism. Now fix $j \in \mathscr{J}$. By Lemma 6.5(iii) we have $\mathscr{P}(M/M^j) = \mathscr{X}_j$; it follows that $\mathscr{M}(M/M^j) = \mathscr{P}(M/M^j) = \mathscr{X}_j$ is permuted transitively by Γ. This also shows, by 6.5(iv), that $M/M^j = (M/M^j)(\mathscr{X}_j)$, and by 6.4 that $M \neq M_j$.

That $\mathscr{P}(U_j) = \{P_j\}$ also follows from 6.5.

Put $M^{j\sharp}/M^j = (M/M^j)(\mathscr{X}_j \setminus \{P_j\})$. The preceding corollary gives a natural embedding of $k\Gamma$-modules

$$M/M^j \to \left(M/M^{j\sharp}\right)\uparrow_{\Delta_j}^{\Gamma}.$$

So to establish the first part it suffices now to show that $M^{j\sharp} = M(\mathscr{X} \setminus \{P_j\})$. It is clear that $M^j \leq M(\mathscr{X} \setminus \{P_j\}) \leq M^{j\sharp}$; consequently, by 6.5(iii),

$$\mathscr{P}\Big(M^{j\sharp}/M\big(\mathscr{X} \setminus \{P_j\}\big)\Big) \subseteq \{P_j\} \cap \big(\mathscr{X}_j \setminus \{P_j\}\big) = \varnothing.$$

Therefore $M^{j\sharp} = M(\mathscr{X} \setminus \{P_j\})$ as required.

Suppose now that M is finitely generated as a $k\Gamma$-module. Then so is M/M^j, for each j; and 7.2(ii) shows that $U_j = M/M^{j\sharp}$ is then finitely generated as a $k\Delta_j$-module. Since $U_j = U(\{P_j\})$, by 6.5, and P_j is Δ_j-invariant, it follows that some finite power of P_j annihilates U_j. Now let S be a finite generating set for M as a $k\Gamma$-module. Lemma 7.1 shows that $S \subseteq M^j$ for all but finitely many indices $j \in \mathscr{I}$. As each M^j is Γ-invariant, it follows that $M = M^j$ for all but finitely many indices $j \in \mathscr{I}$. But we have seen above that $M \neq M^j$ for all $j \in \mathscr{I}$, so $\mathscr{I}$ must in fact be finite. ∎

8. A FILTRATION

As in Section 7, G is supposed to be a normal subgroup of a group Γ. We saw in Corollary 4.3 that if M is a finitely generated non-singular prime kG-module and M is residually finite, then there is a subgroup of finite index in G that acts "residually nilpotently" on M. We are now in a position to generalise this to the case where M is finitely generated as a $k\Gamma$-module, provided that Γ/G is a polycyclic group.

Throughout this section, P will denote a Γ-invariant regular prime ideal of kG. We write

$$l = \operatorname{char}(kG/P), \qquad \pi = \operatorname{spec}(G).$$

8.1 LEMMA. *Let M be a finitely generated $k\Gamma$-module such that $MP^m = 0$ for some m. Then MP contains a finitely generated $k\Gamma$-submodule N such that the additive group MP/N is a divisible π-group. If $l \neq 0$ then MP is finitely generated as a $k\Gamma$-module.*

Proof. By Proposition 1.8, P contains a finitely generated ideal $P^\natural$ such that $P/(P^\natural + P^m)$ is a divisible π-group. Let U be a finitely generated kG-submodule of M such that $M = U\Gamma$, and put $N = UP^\natural k\Gamma$. Now MP/N is additively generated by Γ-translates of its subgroups $(uP + N)/N$, with $u \in U$. Since $uP \cap N \geq u(P^\natural + P^m)$, each of these subgroups is an image of $P/(P^\natural + P^m)$, and hence a divisible π-group. It follows that MP/N is likewise a divisible π-group.

Finally, if $l \neq 0$ then MP/N has finite exponent dividing l^{m-1}, as well as being divisible; so $MP = N$ and the last claim follows. ∎

Let $\lambda \in kG$. We say that a kG-module B is *$\langle \lambda^\Gamma \rangle$-torsion* if every element of B is annihilated by a product of the form $\lambda^{\gamma_1} \dots \lambda^{\gamma_n}$ with $\gamma_1, \dots, \gamma_n \in \Gamma$, that is, if $B = B(I^\Gamma)$, where $I = \lambda kG$ and $I^\Gamma = \{I^\gamma \mid \gamma \in \Gamma\}$. The pair $(kG/P, \Gamma)$ has the *Hall–Roseblade property* if for every finitely generated $k\Gamma$-module A satisfying $AP = 0$, there exists a free kG/P-submodule F of A such that A/F is $\langle \lambda^\Gamma \rangle$-torsion for some $\lambda \in kG \setminus P$. It is easy to see that this holds when $\Gamma = G$; a theorem of Brown [Bn] shows that it holds whenever Γ/G is polycyclic.

For any module M and ideal J, we write

$$MJ^\infty = \bigcap_{n=1}^{\infty} MJ^n.$$

8.2 Lemma. *Let $A \geq B$ be kG-modules and let I, J be ideals of kG. Suppose that $A/B = (A/B)(I^\Gamma)$ and that $J + I^\gamma = kG$ for every $\gamma \in \Gamma$. Then $AJ^\infty/BJ^\infty = (AJ^\infty/BJ^\infty)(I^\Gamma)$, and*

$$A/AJ^n \cong B/BJ^n$$

for each n.

Proof. Suppose $a = \sum a_j y_j \in AJ^n \cap B$, where each $a_j \in A$ and each $y_j \in J^n$. There exists $X \in \langle I^\Gamma \rangle$ such that $a_j X \subseteq B$ for each j. Then $J^n + X = kG$, so

$$a \in aJ^n + \sum a_j X y_j \subseteq BJ^n.$$

It follows that $AJ^n \cap B = BJ^n$, for each n. Since $A/(AJ^n + B)$ is both $\langle I^\Gamma \rangle$-torsion and $\langle J \rangle$-torsion, we also have $AJ^n + B = A$. The second claim follows. For the first claim, note that

$$AJ^\infty \cap B = \bigcap_n AJ^n \cap B = \bigcap_n BJ^n = BJ^\infty.$$

∎

Now we make an *ad hoc* definition: an ideal J of kG is I^Γ*-admissible* if

$$P \leq J < kG, \qquad |kG : J| \text{ is a finite } \pi'\text{-number}, \qquad \text{and}$$
$$J + I^\gamma = kG \qquad \text{for every } \gamma \in \Gamma$$

This depends on P, but the relevant P will always be clear from the context.

8.3 PROPOSITION. *Assume that the pair $(kG/P, \Gamma)$ has the Hall–Roseblade property. Let M be a finitely generated $k\Gamma$-module such that $MP^m = 0$, where $m \geq 1$. Then there exist an ideal I of kG, strictly containing P, and a chain*

$$M = M_0 \geq M_1 \geq \cdots \geq M_m$$

of $k\Gamma$-submodules in M such that

(i) *for $i = 0, \dots, m-1$, $(M_i/M_{i+1})J^\infty = 0$ for every I^Γ-admissible ideal J;*

(ii) $M_m \leq M(I^\Gamma)$.

M also contains a chain of finitely generated $k\Gamma$-submodules (N_i) such that

$$M_i/(M_iJ^n + M_{i+1}) \cong N_i/(N_iJ^n + N_{i+1})$$

for each i and J as in (i) *and all n.*

Proof. We start by constructing the chain (N_i). Put $N_0 = M_0 = M$. Now let $i \geq 0$. Having specified the finitely generated $k\Gamma$-submodule N_i of MP^i, we use Lemma 8.1 to find a finitely generated $k\Gamma$-submodule N_{i+1} of N_iP such that N_iP/N_{i+1} is a π-group. In this way we obtain a chain

$$M = N_0 \geq N_0P \geq N_1 \geq N_1P \geq N_2 \geq \cdots \geq N_{m-1} \geq N_{m-1}P = N_m = 0.$$

For each i, the Hall–Roseblade property ensures that N_i/N_iP contains a free kG/P-submodule F_i/N_iP such that N_i/F_i is $\langle \lambda_i^\Gamma \rangle$-torsion for some $\lambda_i \in kG \setminus P$. Put

$$\lambda = q\lambda_0 \dots \lambda_{m-1} \qquad \text{and} \qquad I = P + \lambda kG,$$

where $q = \prod_{p \in \pi} p$ (note that $q \notin P$ as P is regular). The chain (M_i) is now defined recursively by setting

$$M_{i+1}/N_{i+1} = (M_i/N_{i+1})(I^\Gamma)$$

for $i \geq 0$. Note that $N_iP \leq M_{i+1}$ and that M_i/F_i is $\langle I^\Gamma \rangle$-torsion, for each i.

Certainly (ii) holds, as $N_m = 0$. To establish (i), fix $i \in \{0, \ldots, m-1\}$ and let J be an I^Γ-admissible ideal. Write $\overline{}: M \to M/N_{i+1}$. Now Theorem 3.1 shows that $\overline{F}_i J^\infty \leq \overline{N}_i P \leq \overline{M_{i+1}}$, and Lemma 8.2 shows that $\overline{M}_i J^\infty / \overline{F}_i J^\infty$ is $\langle \lambda^\Gamma \rangle$-torsion. It follows that $\overline{M}_i J^\infty \leq \overline{M_{i+1}}$; but the $\langle \lambda^\Gamma \rangle$-torsion module $\overline{M_{i+1}}$ has no non-zero image annihilated by J, so in fact $\overline{M}_i J^\infty = \overline{M_{i+1}}$. This implies (i).

For the final claim, put $Y = M_{i+1} + N_i$ and $T = M_{i+1} \cap N_i$. Then $N_i J^n + N_{i+1} \geq T$, since T/N_{i+1} is $\langle I^\Gamma \rangle$-torsion; so we have

$$N_i/(N_i J^n + N_{i+1}) = N_i/(N_i J^n + T) \cong Y/(YJ^n + M_{i+1})$$
$$\cong M_i/(M_i J^n + M_{i+1})$$

by 8.2. ∎

8.4 Corollary. *Let A be a finitely generated non-singular* qrf *kG-module. If A is unmixed then A is poly-(residually finite) for kG.*

Proof. By Proposition 6.5(iv), each element of A is annihilated by some member of $\mathscr{M}(A)$. As A is finitely generated, it follows that $\mathscr{M}(A) = \{P_1, \ldots, P_t\}$, say, is finite, and that $AP_1^m \ldots P_t^m = 0$ for some m. Lemma 7.1 shows that A embeds in $\oplus_{i=1}^t A/A^i$, where now $A^i = A(\{P_j \mid j \neq i\})$. It will suffice to show that each A/A^i is poly-(residually finite). Fix $i \in \{1, \ldots, t\}$ and put $P = P_i$, $M = A/A^i$. Then M is finitely generated and $MP^m = 0$. It follows from 6.5(iii) that $\mathscr{M}(M) = \{P\}$.

We now apply 8.3, taking $\Gamma = G$. The existence of an I-admissible ideal J is assured by Theorem 4.2(i); note that our hypotheses imply that P is regular and that $G/P^\dagger$ is reduced. Proposition 2.3(ii) shows that $N_i/(N_i J^n + N_{i+1})$ is finite for each i and all n, so the same holds for $M_i/(M_i J^n + M_{i+1})$. Hence M_i/M_{i+1} is residually finite as a kG-module. Finally, $M_m = 0$ because $\mathscr{M}(M) = \{P\}$. Thus M is poly-(residually finite). ∎

Inserting Corollary 8.4 into the proof of Corollary 6.8, we see now that every non-singular qrf kG-module is poly-(locally-(poly-(residually finite))).

8.5 Theorem. *Suppose that Γ/G is polycyclic. Let M be a finitely generated $k\Gamma$-module which is unmixed as a kG-module. Then there exist an ideal J of finite index in kG and a finite chain*

$$M = M_0 \geq M_1 \geq \cdots \geq M_m = 0$$

of $k\Gamma$-submodules in M such that $(M_{i-1}/M_i)J^\infty = 0$ for $1 = 1, \ldots, m$.

In group-theoretic terms, this result shows that if E is any extension of the module M by G, then E has a subgroup E_1 of finite index (namely the inverse image of $J^\dagger$) such that the lower central series of E_1, continued transfinitely, terminates at the identity.

Proof. According to Proposition 7.3, we can embed M into a direct sum

$$\bigoplus_{j \in \mathscr{I}} U^j \uparrow_{\Delta_j}^{\Gamma};$$

here, $\mathscr{I}$ is a finite set of indices such that $\mathscr{P}(M) = \{P_j^\gamma \mid j \in \mathscr{I}, \gamma \in \Gamma\}$, and for each $j \in \mathscr{I}$, $\Delta_j = \mathrm{N}_\Gamma(P_j)$ and U^j is a finitely generated $k\Delta_j$-module which is P_j-primary as a kG-module. If the result holds for each of the finitely many summands, it holds for M; so, changing notation, we may as well suppose that $M = U \uparrow_\Delta^\Gamma$, where $U = U^j$ and $\Delta = \Delta_j$ for some j. Thus U is P-primary where $P = P_j$.

Now we apply Proposition 8.3 (with U in place of M and Δ in place of Γ). This provides an ideal I of kG, strictly containing P, and a chain

$$U = U_0 \geq U_1 \geq \cdots \geq U_m$$

of $k\Delta$-submodules in U such that

(i) for each i, $(U_{i-1}/U_i)J^\infty = 0$ for every I^Δ-admissible ideal J of kG, and

(ii) $U_m = U_m(I^\Delta)$.

As U is P-primary and P is Δ-invariant, it follows from (ii) that in fact $U_m = 0$.

Now one of the main theorems of [S] (an application of Theorem 4.2 above) shows that *the maximal ideals L/P of finite index in kG/P which satisfy $L + I^\gamma = kG$ for all $\gamma \in \Delta$ intersect in P*. If $\mathrm{char}(kG/P) = 0$ then $\mathrm{char}(kG/L)$ must take infinitely many distinct values as L ranges over all such maximal ideals, so we may choose one such L with $\mathrm{char}(kG/L) \notin \mathrm{spec}(G)$. If $\mathrm{char}(kG/P) = l \neq 0$, then $l \notin \mathrm{spec}(G)$, since M is non-singular and $P \in \mathscr{P}(M)$; so for any such ideal L we have $\mathrm{char}(kG/L) = l \notin \mathrm{spec}(G)$. Thus in either case we find an I^Δ-admissible ideal L of kG, containing lkG for some prime l. There exists $e \in \mathbb{N}$ such that $G^e \leq L^\dagger$, and we put

$$J = (L \cap k)kG + (G^e - 1)kG.$$

Then J is a Γ-invariant ideal of finite index in kG and $J \leq L$, so $(U_{i-1}/U_i)J^\infty = 0$ for each i. Putting $M_i = U_i \uparrow_\Delta^\Gamma$ for $i = 0, 1, \ldots, m$ we now get $(M_{i-1}/M_i)J^\infty = 0$, as required. ∎

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