On feasible sets for coalitional MPC

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***Abstract*— This paper presents novel results and observations on the analysis of tube-based controllers for Large-Scale Systems. These systems group several, possibly heterogeneous, interconnected subsystems into a single entity. Different subgroupings of subsystems lead to different coupling structures that have an impact on the design of robust decentralised controllers. We study the nesting properties of feasible regions and robust positive invariant sets for different partitions of the large-scale system in terms of partial order relations induced by the power set of the elements composing it.**

I. Introduction

Model Predictive Control (MPC) has reached a mature state in its theoretical development, and has gained popularity in many industrial applications [1] because it excels in situations where off-line control laws are difficult to compute, *i.e.* in the presence of hard constraints. However, the new challenges for control point towards larger and more interconnected phenomena where the system can have a large number of states and/or be geographically spread [2]. In this context, *centralised* control techniques have limited applicability because the controllers would have to model, sense and control the whole plant. One of the fundamental assumptions for controlling Large-Scale System (LSS) lies in the partition of the system itself. Most of the existing approaches for LSS control consider a constant partition for the purposes of distributed controller design, see for example [3].

Depending on the partition chosen for the LSS, the degree of interconnectedness of the distributed controller changes; the more elements the partition has, and the stronger the interconnection between these elements, the more effort is required for the controllers to coordinate their control actions. In [4], the authors explore the effects of using different partitions to control a chemical system using a Distributed Model Predictive Control (DMPC) approach, and, not surprisingly, have found that using different partitions may boost aspects of system performance such as transient behaviour. There are several papers in the literature on DMPC attempt to exploit different partitions of the LSS at different stages of its time evolution. For example, the problem of switching between partitions is seen as a coalition formation game in [5]. When viewed as a game, several new metrics can be taken into account to assess each partition for the LSS, such as the Shapley value in [6]. Another approach for DMPC for this problem is [7], which proposes a cooperative DMPC strategy in order to promote different types of cooperation between subsystems within an interconnection graph.

In this paper we investigate properties of feasible reagions associated with tube MPC controllers for various possible partitions, thus providing an analysis which, surprisingly, is missing in the approaches described lines above. The possible partitions of the LSS have a power set structure which equips them with an order relation in terms of set inclusion. For example, a LSS composed of three elements $\mathcal{M} = \{1, 2, 3\}$ admits 5 possible partitions given by the power set $2^{\mathcal{M}}$. The number of elements of each partition determines the number of independent controllers the LSS has. For example, the partition $\{\{1, 2\}, \{3\}\}$ has two elements, within which subsystems 1 and 2 are joint in a single entity or *coalition*. The number of coalitions determines the coupling properties of the LSS, which is closely related to the Robust Positive Invariant (RPI) sets used by tube-based controllers. We study the nesting properties of these invariant sets and feasible regions with respect to the possible partitions of the LSS.

The organisation and contributions of the paper are as follows: Section II states the problem, challenges, and the basic definitions and the first controllability results for coalitions. Section III states the decentralised MPC problem. In Section IV, we study the feasible regions for tube MPC controllers; we state our main results regarding the nesting of the feasible regions and RPI sets for the LSS with respect to the partial order relation induced by the set of partitions.

Notation: The set of non-negative integers is $\mathbb{N}$, and a finite subset is $\mathbb{N}_{a:b}$ where a is the lowest value, and b the highest. For a square matrix $A \in \mathbb{R}^{n\times n}$, the spectral radius (*i.e.* the largest absolute value of any of its eigenvalues) is denoted $\rho(A)$. A *C-set* is a compact and convex set; a *PC-set* is a *C-set* that contains the origin in its non-empty interior. The set of all PC-sets in $\mathbb{R}^n$ is $\mathcal{K}^n$. The Minkowski sum of sets $\mathcal{A}$, $\mathcal{B}$ is $\mathcal{A} \oplus \mathcal{B} = \{a + b\colon\ a \in \mathcal{A},\ b \in \mathcal{B}\}$.

Definition 1 (RPI sets): The set $\mathcal{R} \subseteq \mathbb{R}^n$ is an RPI set for the uncertain dynamics $x^+ = Ax + w$ with constraints $(\mathbb{X}, \mathbb{W})$ if for any $x \in \mathcal{R}$ implies that $x^+ \in \mathcal{R}$ for all $w \in \mathbb{W}$.

II. Problem setup and basic formulation

A. System description and control objective

The LSS under consideration is discrete-time linear time-invariant:

$$x^+ = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and control input, and x^+ is the state at the next instant of time. This system admits a

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basic *decomposition* or *partitioning* of (1) into M subsystems. Each subsystem $i \in \mathcal{M} \triangleq \{1, \dots, M\}$ has dynamics

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i \text{ where } w_i \triangleq \sum_{j\in\mathcal{N}_i} A_{ij}x_j + B_{ij}u_j,$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ are the state and input of each subsystem, such that $x = (x_1, \dots, x_M)$, $u = (u_1, \dots, u_M)$, $\sum_{i\in\mathcal{M}} n_i = n$, and $\sum_{i\in\mathcal{M}} m_i = m$. That is, the subsystems are non-overlapping, which means they share no states or inputs but are interconnected whenever A or B contains nonzero off-diagonal block matrices. The resulting dynamic and input coupling is accounted for by the exogenous term w_i, and we define the set of *neighbours* of subsystem i as

$$\mathcal{M}_i \triangleq \big\{j \in \mathcal{M} \setminus \{i\} \colon [A_{ij}\ B_{ij}] \neq 0\big\}.$$

Assumption 1 (Controllability): For each $i \in \mathcal{M}$ the pair (A_{ii}, B_{ii}) is controllable.
The LSS is constrained via local, independent constraints on the states and inputs of each subsystem. For subsystem i,

$$x_i \in \mathbb{X}_i \qquad\qquad u_i \in \mathbb{U}_i.$$

Assumption 2 (Constraint sets): The sets $\mathbb{X}_i \subset \mathbb{R}^{n_i}$ and $\mathbb{U}_i \subset \mathbb{R}^{m_i}$ are polytopic PC-sets.

B. Coalitions of subsystems and partitions of the system

The setting of the paper is to consider how subsystems may be grouped together into *coalitions*.

Definition 2 (Coalition of subsystems): A *coalition* of subsystems is a non-empty subset of $\mathcal{M}$.
The idea is that each coalition of subsystems operates and is controlled as a single entity. A coalitional controller, which is assumed to have access to the states and control inputs of subsystems within its coalition, replaces local subsystem controllers and may achieve better performance, albeit at a higher cost of complexity and communication. The grouping of subsystems into different coalitions induces alternative partitions of the LSS.

Definition 3 (Partition of the LSS): A *partition* of the LSS is an arrangement of the M subsystems into $C \leq M$ coalitions: formally, a partition of $\mathcal{M} = \{1, \dots, M\}$ is a set $\mathcal{C} = \{1, \dots, C\}$, satisfying the following properties:

1) Coalition $c \in \mathcal{C}$ contains subsystems $\mathcal{C}_c \subseteq \mathcal{M}$, and the cardinality of $\mathcal{C}_c$ is M_c.
2) Coalitions are non-overlapping: $\mathcal{C}_c \cap \mathcal{C}_d = \emptyset$ for all $c \neq d$.
3) Coalitions cover the set of subsystems: $\bigcup_{c\in\mathcal{C}} \mathcal{C}_c = \mathcal{M}$.

These definitions include the trivial cases of (i) a single, grand coalition of all subsystems ($C = 1$, $\mathcal{C}_1 = \mathcal{M}$) (the *centralized partition*) and (ii) the basic partitioning of the system, in which each subsystem is a coalition ($C = M$, $\mathcal{C} = \mathcal{M}$, $\mathcal{C}_i = \{i\}$ for each $i \in \mathcal{M}$) (the *decentralized partition*). More generally, the set of all possible partitions is

$$\Pi_{\mathcal{M}} \triangleq \{\mathcal{C} \colon \mathcal{C} \text{ is a partition of } \mathcal{M}\}$$

Given a partition $\mathcal{C}$, the state and input of coalition c are, respectively, $x_c = (x_i)_{i\in\mathcal{C}_c}$ and $u_c = (u_i)_{i\in\mathcal{C}_c}$ [1]. The dynamics of coalition c are

$$x_c^+ = A_{cc}x_c + B_{cc}u_c + w_c, \tag{2}$$

where $x_c \in \mathbb{R}^{n_c}$, $u \in \mathbb{R}^{n_c}$, and where the matrices A_{cc} and B_{cc} comprise the relevant matrices of the subsystems within the coalition: $A_{cc} = [A_{ij}]_{i,j\in\mathcal{C}_c}$, $B_{cc} = [B_{ij}]_{i,j\in\mathcal{C}_c}$. Similar to the basic decomposition of the systems into subsystems, the coalitions remain coupled via their dynamics: coalition c is coupled with coalition d via the matrices A_{cd} and B_{cd}, defined as

$$A_{cd} = \big[A_{ij}\big]_{i\in\mathcal{C}_c, j\in\mathcal{C}_d, d\neq c} \quad B_{cd} = \big[B_{ij}\big]_{i\in\mathcal{C}_c, j\in\mathcal{C}_d, d\neq c}$$

so that

$$w_c \triangleq \sum_{d\in\mathcal{M}_c} A_{cd}x_d + B_{cd}u_d,\ \ \mathcal{M}_c \triangleq \big\{d \in \mathcal{C} : [A_{cd}\ B_{cd}] \neq 0\big\}.$$

The disturbance set $\mathbb{W}_c \subseteq \mathbb{R}^{n_c}$, by Assumption 2, is a ***C-set*** satisfying $\mathbb{W}_c = \bigoplus_{d\in\mathcal{M}_c} A_{cd}\mathbb{X}_d \oplus B_{cd}\mathbb{U}_d$. The arrangement into coalitions, and subsequent control of each coalition as a single entity, affects the control performance. Between the extremes of the decentralised and centralised partitions, there exists a trade-off between the size of the coalitions (and the resulting dimension or complexity of the control law as well as the communication network required to support it) and control performance. The LSS satisfies a weak coupling assumption for the decentralised partition in $\Pi_{\mathcal{M}}$,

Assumption 3: For all $i \in \mathcal{M}$, the disturbance sets containing the interactions satisfy $\mathbb{W}_i \subset \mathbb{X}_i$.
Assumption 3, which implies weak interactions among subsystems and is a standard assumption in DMPC approaches, is a key ingredient to guarantee the existence of local controllers for the decentralised partition of the LSS as proved in [8].

Lemma 1: Suppose Assumptions 1–3 hold. If the decentralised partition $\mathcal{M} \in \Pi_{\mathcal{M}}$ admits for each $i \in \mathcal{M}$ a feedback gain K_i such that $\rho(A_{ii} + B_{ii}K_i) < 1$, and there exists constraint admissible RPI sets $\mathcal{R}_i \subset \mathbb{X}_i$ and $K_i\mathcal{R}_i \subset \mathbb{U}_i$; then, the diagonal feedback gain $\mathbf{K} = \text{diag}(K_i)$ satisfies $\rho(A + B\mathbf{K}) < 1$.
The following proposition, the proof of which is in the appendix, asserts that elements of $\Pi_{\mathcal{M}}$ inherit a weaker version of controllability from the basic partition.

Proposition 1 (Stabilizability for coalitions): Suppose Assumptions 1–3 together with Lemma 1 hypotheses hold. Then, the pair (A_{cc}, B_{cc}) is stabilizable for all $c \in \mathcal{C}$ and $\mathcal{C} \in \Pi_{\mathcal{M}}$.

III. Tube MPC formulation

Considering the interaction free dynamics arising from (2) for a coalition $c \in \mathcal{C}$

$$\bar{x}_c^+ = A_{cc}\bar{x}_c + B_{cc}\bar{u}_c \tag{3}$$

[1] Our intention is to make the notation as simple as possible by employing a single subscript to denote both a variable of a subsystem and a variable of a coalition; typically i for the former and c for the latter. Although there is potential for confusion, the meaning will be clear from the context.

The DMPC formulation follows a conventional tube approach, where the nominal model (3) is used to predict the future behaviour of the system up to a horizon N. The corresponding open-loop optimal control problem for each coalition $c \in \mathcal{C}$ can be expressed as the optimization problem

$$\bar{\mathbb{P}}_c(\bar{x}_c)\colon \min_{\bar{\mathbf{u}}_c} \{ J_c^N(\bar{x}_c, \bar{\mathbf{u}}_c) : \bar{\mathbf{u}}_c \in \bar{\mathcal{U}}_c^N(\bar{x}_c) \} \tag{4}$$

where J_c^N is the finite-horizon regulation cost

$$J_c^N(\bar{x}_c, \bar{\mathbf{u}}_c) = V_f(\bar{x}_c) + \sum_{k=0}^{N-1} \bar{x}_c^\top(k) Q_c \bar{x}_c(k) + \bar{u}_c^\top(k) R_c \bar{u}_c(k),$$

and $\bar{\mathcal{U}}_c^N(\bar{x}_c)$ is defined by the following constraints for $k \in \mathbb{N}_{0:N-1}$:

$$\bar{x}_c(0) = \bar{x}_c, \tag{5a}$$
$$\bar{x}_c(k+1) = A_{cc}\bar{x}_c(k) + B_c \bar{u}_c(k), \tag{5b}$$
$$\bar{x}_c(k) \in \bar{\mathbb{X}}_c, \tag{5c}$$
$$\bar{u}_c(k) \in \bar{\mathbb{U}}_c, \tag{5d}$$
$$\bar{x}_c(N) \in \bar{\mathbb{X}}_c^f. \tag{5e}$$

The solution of this problem is a sequence of nominal control actions $\bar{\mathbf{u}}_c^0(\bar{x}_c) \triangleq \{\bar{u}_c^0(0; \bar{x}_c), \ldots, \bar{u}_c^0(N-1; \bar{x}_c)\}$.

This formulation, however, does not take into account the interactions between the coalitions associated with the partition $\mathcal{C} \in \Pi_{\mathcal{M}}$. To account for the input and dynamic coupling, tube based methods rely on the concept of invariant sets $\mathcal{R}_c(\mathbb{W}_c) \subset \mathbb{R}^{n_c}$. Traditional tube MPC methods use an RPI set with a linear control law $\hat{\kappa}_c(x_c) = K_c x_c$ to reject the disturbances arising from interactions. A generalisation of the RPI-based tube MPC is the homothetic tube approach of [9], where the invariant sets are scaled and translated accordingly to the disturbance action; on the other hand [10] uses predictive laws and Robust Control Invariant (RCI) sets to handle disturbances. These invariant sets are used to appropriately tighten the constraints sets to ensure constraint satisfaction, *i.e.* $\bar{\mathbb{X}}_c = \mathbb{X}_c \ominus \mathcal{R}_c(\mathbb{W}_c) \subseteq \mathbb{X}_c \ \forall c \in \mathcal{C}$.

Remark 1: If subsystem i is to be controlled by its own controller, then the interactions $w_i = \sum_{j \in \mathcal{N}_i} A_{ij} x_j + B_{ij} u_j$ need to be accounted for in a robust way. However, when a subsystem i is brought into a coalition with c, then the new coalition $\mathcal{C}_d = \mathcal{C}_c \cup \{i\}$ includes in its dynamics the interaction information given by the respective off-diagonal blocks of A and B. This implies that this information is available for feedback purposes.

Remark 2: The choice of a robust framework favours flexibility in design, since the controllers rely on local information to synthesise their control laws. This flexibility is missing in classical decoupling methods relying on relative gain and direct Nyquist arrays. Another benefit of a robust approach is that process noise can be incorporated into the framework seamlessly. However, the price to pay for this flexibility is conservatism.

IV. Controllability sets and feasible regions

In this section, we study the properties of the underlying partition set, and its relation to the LSS invariance properties.

A. Feasibility regions

The region of feasibility for optimisation problem (4) can be characterised by the set-valued controllability map

$$\mathfrak{Q}(\mathcal{A}) \triangleq \{ z_c \in \bar{\mathbb{X}}_c \colon \exists \bar{u}_c \in \bar{\mathbb{U}}_c \ A_{cc} z_c + B_{cc} \bar{u}_c \in \mathcal{A} \} \tag{6}$$

for any $\mathcal{A} \subset \mathbb{R}^{n_c}$, and the recursion $\bar{\mathcal{X}}_c^{k+1} = \mathfrak{Q}(\bar{\mathcal{X}}_c^k)$ with $\bar{\mathcal{X}}_c^0 = \bar{\mathbb{X}}_c^f$. For every $k \in \mathbb{N}$, the set $\bar{\mathcal{X}}_c^k$ characterises the points that can be steered to $\bar{\mathbb{X}}_c^f$ in k steps or less. These controllability sets represent the domain of the set-valued map $\bar{\mathcal{U}}_c^N \colon \bar{\mathcal{X}}_c^N \rightrightarrows \mathbb{U}_c^N$, defined by the constraints of problem (4), the graph of which is a polyhedron in $\mathbb{R}^{n_c + N m_c}$ and is defined as

$$\begin{aligned} \bar{\mathcal{Z}}_c^N = \{ (\bar{x}_c, \bar{\mathbf{u}}_c) \colon & A^k \bar{x}_c + \tilde{B}_c^k \bar{\mathbf{u}}_c \in \bar{\mathbb{X}}_c, k \in \mathbb{N}_{0:N-1} \\ & A^N \bar{x}_c + \tilde{B}_c^N \bar{\mathbf{u}}_c \in \bar{\mathbb{X}}_c^f, \ \bar{\mathbf{u}}_c \in \bar{\mathbb{U}}_c^N \}, \end{aligned} \tag{7}$$

where $\tilde{B}_c^k$ denotes the k-step controllability matrix. Let $z = (\bar{x}_c, \bar{\mathbf{u}}_c)$, then the H-representation $\bar{\mathcal{Z}}_c^N = \{ z : P_{z,c} z \leq q_{z,c} \}$ is available as a direct consequence of (7), taking the product of each $\bar{\mathcal{Z}}_c^N$ defines the set for the overall partition $\bar{\mathcal{Z}}^N(\mathcal{C}) := \prod_{c \in \mathcal{C}} \bar{\mathcal{Z}}_c^N$. This, in turn, defines a map between the set of LSS partitions $\Pi_{\mathcal{M}}$ and the compact sets $\bar{\mathcal{Z}}^N(\mathcal{C})$, and we likewise define the map $\bar{\mathcal{X}}^N(\mathcal{C}) := \prod_{c \in \mathcal{C}} \bar{\mathcal{X}}_c^N$.

Proposition 2: Suppose Assumptions 1 and 2 hold. The mapping $\bar{\mathcal{Z}}^N : \Pi_{\mathcal{M}} \to \mathcal{K}^{n+mN}$ and $\bar{\mathcal{X}}^N : \Pi_{\mathcal{M}} \to \mathcal{K}^n$ are well defined.

Lemma 2: Suppose Assumption 2 holds. If the constraint sets $\bar{\mathbb{X}}_c$ and $\bar{\mathbb{U}}_c$ are symmetric, then the sets $\bar{\mathcal{X}}_c^N$ and $\bar{\mathcal{Z}}_c^N$ are symmetric for all $c \in \mathcal{C}$, $\mathcal{C} \in \Pi_{\mathcal{M}}$.

B. Robust invariant sets for coalitions

The characterisation of the region of attraction requires the notion of an invariant set bounding the effects of the couplings between subsystems and coalitions. Given a partition of the LSS $\mathcal{C} \in \Pi_{\mathcal{M}}$ and its corresponding uncertain dynamics (2), the error $e_c = x_c - \bar{x}_c$ and the tube linear control law $u_c = \bar{u}_c + \kappa_c(x_c - \bar{x}_c)$ yields the error dynamics

$$e_c^+ = A_{cc} e_c + B_{cc} \kappa(e_c) + w_c. \tag{8}$$

As mentioned in the previous section, the structure of the control law determines the type of invariant set used by the tube controller. For clarity of exposition, we use linear feedback laws and the minimal RPI set which is computed via the infinite Minkowski sum[2]

$$\mathcal{R}_c(\mathbb{W}_c) = \bigoplus_{k=0}^{\infty} (A_{cc} + B_{cc} K_c)^k \mathbb{W}_c. \tag{9}$$

[2]The minimal RPI set is only computable for a nilpotent choice of the feedback gain K_c, which yields a finite computation of the set. This, however, has severe drawbacks in terms of performance of the closed-loop system. Outer invariant approximations yielding finite time computation and polytopic structures are used to circumvent this issue; we refer the reader to [11] and [12] for methods for computing ε-outer approximations and fixed complexity approximations respectively.

The gain K_c is chosen such $\rho(A_{cc} + B_{cc}K_c) < 1$ which, by virtue of Proposition 1, is always possible. Following [13], forcing the error to satisfy $e_c(0) \in \mathcal{R}_c(\mathbb{W}_c)$, or equivalently $x_c(0) \in \bar{x}_c \oplus \mathcal{R}_c(\mathbb{W}_c)$, and replacing (5a) ensures that the feasible region for partition $\mathcal{C}$ is given by

$$\mathcal{X}^N(\mathcal{C}) = \prod_{c \in \mathcal{C}} \big(\bar{\mathcal{X}}_c^N \oplus \mathcal{R}_c(\mathbb{W}_c)\big). \tag{10}$$

We are interested in the way the different regions of attraction $\mathcal{X}^N(\mathcal{C})$ relate to each other. Each of these sets characterises the initial states that can be steered to a neighbourhood of the origin in N steps, specifically the neighbourhood given by the product of invariant sets $\mathcal{R}(\mathcal{C}) = \prod_{c \in \mathcal{C}} \mathcal{R}_c(\mathbb{W}_c)$.

Remark 3: The overall invariant set for a partition $\mathcal{C}$, namely $\mathcal{R}(\mathcal{C}) \subset \mathbb{R}^n$, is the product of the invariant sets $\mathcal{R}_c(\mathbb{W}_c)$ for each coalition $c \in \mathcal{C}$. Similarly, the set $\Pi_{\mathcal{M}}$ generates a family of invariant sets for the LSS.

C. Structure of the partition set

The set of partitions $\Pi_{\mathcal{M}}$ is a *poset* (in fact, a *lattice*) when equipped with the following *refinement* order relation, denoted by '$\preceq$': given $\mathcal{C}, \mathcal{C}' \in \Pi_{\mathcal{M}}$, the partition $\mathcal{C} \preceq \mathcal{C}'$ ($\mathcal{C}$ *refines* $\mathcal{C}'$, or $\mathcal{C}'$ *coarsens* $\mathcal{C}$) if every member of $\mathcal{C}$ is contained in some member of $\mathcal{C}'$. The Hasse diagram in Figure 1 illustrates the poset of partitions for a system with four subsystems, *i.e.* $\mathcal{M} = \{1, 2, 3, 4\}$. There are 15 possible partitions.

Because the ordering is partial, however, not all pairs of partitions are comparable in this way: a *chain* of $\Pi_{\mathcal{M}}$ is a set of comparable elements (partitions of $\mathcal{M}$) and an *anti-chain* of $\Pi_{\mathcal{M}}$ is a set of incomparable elements. The set of chains of $\Pi_{\mathcal{M}}$ is $\mathbb{C}$ and the set of chains involving partition $\mathcal{C} \in \Pi_{\mathcal{M}}$ is $\mathbb{C}(\mathcal{C}) \subset \mathbb{C}$; Figure 1 illustrates chains with solid lines and anti-chains with dashed lines. The ordering of the set Π_{M} satisfies the following Lemma.

Lemma 3 (Maximal elements): The partition set $\Pi_{\mathcal{M}}$ together with the refinement relation $\preceq$ admits unique maximal (minimal) element $\mathcal{C} \preceq \mathcal{C}^{\max}$ ($\mathcal{C}^{\min} \preceq \mathcal{C}$) for all $\mathcal{C} \in \Pi_{\mathcal{M}}$.

Intuition may argue that the relationship between the family of feasible regions $\{\mathcal{X}^N(\mathcal{C})\}_{\mathcal{C} \in \Pi_{\mathcal{M}}}$ follows a similar ordering to that of $\Pi_{\mathcal{M}}$. The line of thought is as follows: consider a chain in $\mathbb{C}(\mathcal{C}) \subset \Pi_{\mathcal{M}}$, then the maximal element $\mathcal{C}^{\max}$ of that chain contains a larger region of the state space than any other partition of that chain. This implies a nesting $\mathcal{X}^N(\mathcal{D}) \subseteq \mathcal{X}^N(\mathcal{C}^{\max})$ for all $\mathcal{D} \in \mathbb{C}(\mathcal{C})$, which follows from the fact that "$\subseteq$" is an ordering in $\mathcal{K}^n$. However, this nesting does not hold in general; the following result states the counter-nesting of the invariant sets.

Proposition 3 (Counter-nesting of RPI sets): Suppose Assumptions 1 and 2 hold. Consider a partition $\mathcal{C} \in \Pi_{\mathcal{M}}$ and a chain $\mathbb{C}(\mathcal{C})$ containing it. If $\mathcal{D} \in \mathbb{C}(\mathcal{C})$ and $\mathcal{D} \succeq \mathcal{C}$, then $\mathcal{R}(\mathcal{D}) \subseteq \mathcal{R}(\mathcal{C})$.

Proof: The proof consists of two parts: for the first part we prove the counter-nesting occurs for the disturbance sets; and the second part derives the desired result for RPI sets. For an element of $\mathcal{C} \in \Pi_{\mathcal{M}}$, and a chain $\mathbb{C}(\mathcal{C})$ containing it, there exists an element, following Lemma 3, such that

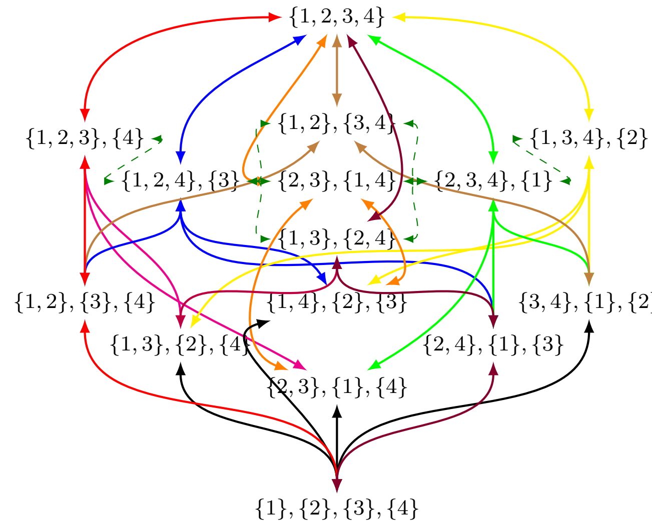

Fig. 1. Refinement relation defined over $\Pi_{\{1,2,3,4\}}$. Each path of different colour points towards the elements that are related to each other, through refinement. Solid lines represent elements of chains whereas dashed lines represent members of anti-chains.

$\mathcal{C}^{\max}$ is maximal in $\mathbb{C}(\mathcal{C})$, and $\mathcal{C}$ refines $\mathcal{C}^{\max}$. The refinement definition implies that for any $d \in \mathcal{C}^{\max}$, there exists $c \in \mathcal{C}$ such that $\mathcal{C}_c \subseteq \mathcal{C}_d^{\max}$, which in turn implies that the cardinality of the elements of $\mathcal{C}_c$ is smaller than that of the elements of $\mathcal{C}_d^{\max}$, *i.e.* $|\mathcal{C}_c| \leq |\mathcal{C}_d^{\max}|$. Now pick $d \in \mathcal{C}^{\max}$ so that its set of dynamic neighbours has a cardinality $|\mathcal{M}_d|$, and suppose (without loss of generality) that the set $\{i, j\}$ is a subset of $\mathcal{C}_d^{\max}$ with $\{i\} \subset \mathcal{C}_c$, and $\{j\} \not\subset \mathcal{C}_c$. The number of dynamic neighbours for coalition c is therefore $|\mathcal{M}_c| \geq |\mathcal{M}_d| + 1$. The change in coupling structure when coarsening $\mathcal{C}$ to $\mathcal{C}^{\max}$ is reflected in the disturbance set size and structure $\mathbb{W}_d = (\bigoplus_{e \in \mathcal{M}_d} A_{de}\mathbb{X}_e \oplus B_{de}\mathbb{U}_e) \subset \mathbb{R}^{n_d}$. The two new coalitions $c_1 = \{j\}$ and $c = \mathcal{C}_d^{\max} \setminus \{j\}$ have disturbance sets satisfying $\mathbb{W}_{c_1} = (\bigoplus_{e \in \mathcal{M}_d \cup \{c\}} A_{c_1 e}\mathbb{X}_e \oplus B_{c_1 e}\mathbb{U}_e) \subset \mathbb{R}^{n_{c_1}}$ and $\mathbb{W}_c = (\bigoplus_{e \in \mathcal{M}_d \cup \{c_1\}} A_{ce}\mathbb{X}_e \oplus B_{ce}\mathbb{U}_e) \subset \mathbb{R}^{n_c}$ where $n_d = n_c + n_{c_1}$. Following the properties of the Minkowski sum and the cartesian product, we obtain

$$\begin{aligned}
\tilde{\mathbb{W}}_d &= \mathbb{W}_c \times \mathbb{W}_{c_1} \\
&= \big(\bigoplus_{e \in \mathcal{M}_d} A_{ce}\mathbb{X}_e \oplus B_{ce}\mathbb{U}_e\big) \times \big(\bigoplus_{e \in \mathcal{M}_d} A_{c_1 e}\mathbb{X}_e \oplus B_{c_1 e}\mathbb{U}_e\big) \\
&\quad \oplus (A_{cc_1}\mathbb{X}_{c_1} \oplus B_{cc_1}\mathbb{U}_{c_1}) \times (A_{c_1 c}\mathbb{X}_c \oplus B_{c_1 c}\mathbb{U}_c) \\
&= \mathbb{W}_d \oplus \mathbb{W}_{cc_1} \supseteq \mathbb{W}_d.
\end{aligned}$$

Therefore, the disturbance set for the finer coalition has extra terms of the form $\mathbb{W}_{cc_1}$ for each of the missing interconnections, and these result in the desired nesting $\prod_{d \in \mathcal{C}^{\max}} \mathbb{W}_d \subset \prod_{c \in \mathcal{C}} \mathbb{W}_c$.

For the second part of the proof, consider linear feedback gains K_d and $\tilde{K}_d = \mathrm{diag}(K_c, K_{c_1})$ that stabilise the pairs (A_{dd}, B_{dd}) and

$$(\tilde{A}_d, \tilde{B}_d) = (\mathrm{diag}(A_{cc}, A_{c_1 c_1}), \mathrm{diag}(B_{cc}, B_{c_1 c_1}))$$

respectively. What remains to be proven is that $\mathcal{R}_d(\mathbb{W}_d) \subseteq \tilde{\mathcal{R}}_d(\tilde{\mathbb{W}}_d)$, where $\tilde{\mathcal{R}}_d(\tilde{\mathbb{W}}_d) = \mathcal{R}_c(\mathbb{W}_c) \times \mathcal{R}_{c_1}(\mathbb{W}_{c_1})$ and $\mathcal{C}_d^{\max} =$

$\mathcal{C}_c \cup \{j\}$. Let $x \in \mathcal{R}_d(\mathbb{W}_d)$, then

$$A_{dd}x + B_{dd}u_d + w_d \in \mathcal{R}_d(\mathbb{W}_d)$$

so that

$$A_{dd}x + B_{dd}u_d + w_d + w_{cc_1} - w_{cc_1} \in \mathcal{R}_d(\mathbb{W}_d)$$

and hence

$$\tilde{A}_d x + \tilde{B}_d u_d + \tilde{w}_d \in \mathcal{R}_d(\mathbb{W}_d)$$

where $\tilde{w}_d = w_d + w_{cc_1} \in \tilde{\mathbb{W}}_d$ and $u_d = K_d x$. Therefore, if the control law changes to $\tilde{u}_d = \tilde{K}_d x$, we must then obtain $\tilde{A}_d x + \tilde{B}_d \tilde{K}_d x + \tilde{w}_d \in \tilde{\mathcal{R}}_d(\tilde{\mathbb{W}}_d)$ which yields the desired result. ∎

The following Theorem provides a counter-example to the intuitive argument that it is possible to obtain a nesting of the feasible regions respect to the coarsening and refinement operations defined on $\Pi_{\mathcal{M}}$.

Theorem 4 (Non Nesting of the feasible regions): Suppose Assumptions 1 and 2 hold. The family of feasible regions $\{\mathcal{X}^N(\mathcal{C})\}_{\mathcal{C}\in\Pi_{\mathcal{M}}}$ is not partially ordered with respect to the inclusion relation in $\mathcal{K}^n$

Remark 4: The nesting and counter-nesting properties of the invariant sets and feasible regions are important in controllers capable of switching between these partitions. A rearrangement of subsystems into fewer, larger coalitions may lead to a loss of feasibility. This fact has not been observed in the literature in coalitional control.

In order to prove the theorem it is enough to find an example that does not exhibit the nesting property. We now consider two examples illustrating the results of this section. For a system composed of two subsystem $M = 2$, the induced partition set is $\Pi_{\mathcal{M}} = \{\mathcal{D}, \mathcal{C}\}$, where $\mathcal{D}$ represents the decentralised partition and $\mathcal{C}$ the centralised partition. The centralised dynamics are characterised by the matrices

$$A_{\text{ex1}} = \begin{bmatrix} 1.1 & 0.1 \\ 0.1 & 1.1 \end{bmatrix} \quad B_{\text{ex1}} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

with constraint sets $\mathbb{X}_i = \{x_i \in \mathbb{R} \colon |x_i| \leq 2\}$, and $\mathbb{U}_i = \{u_i \in \mathbb{R} \colon |u_i| \leq 0.5\}$, and for simplicity we choose the terminal set to be the origin $\bar{\mathbb{X}}_i^f = \{0\}$.

For a prediction horizon of $N = 2$, Figure 2 shows that the feasible region of the decentralised system is contained within the feasible set for the centralised one, satisfying the property $\mathcal{X}^2(\mathcal{C}) \supseteq \mathcal{X}^2(\mathcal{D})$. The feasible region for the decentralised partition has a rectangular shape which characterises the absence of coupling between both subsystems. On the other hand, the shape of the centralised feasible region shows how the coupling can be constructive or disruptive. The flattened regions correspond to those states for which the coupling pushes the state towards instability; whereas the other two corners have a beneficial coupling that enlarges the feasible region. However, for a different system with centralised dynamics given by

$$A_{\text{ex2}} = \begin{bmatrix} 0.6 & 0.1 \\ 0.1 & 0.6 \end{bmatrix} \quad B_{\text{ex2}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

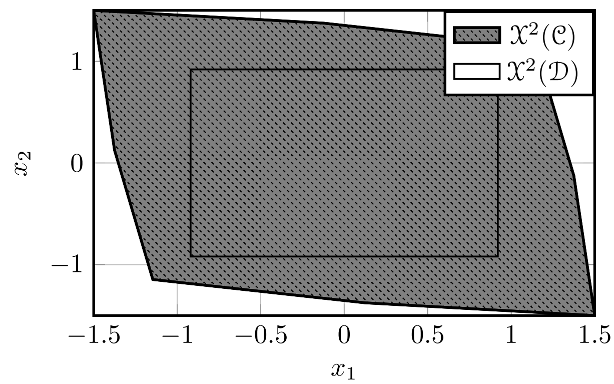

Fig. 2. Feasible regions for $\mathcal{M} = \{1, 2\}$, $\Pi_{\mathcal{M}} = \{\mathcal{D}, \mathcal{C}\}$ and centralised dynamics A_{ex1} and B_{ex1}. (▩) represents the feasible region for $\mathcal{C}$ and (▭) represents the feasible region for $\mathcal{D}$.

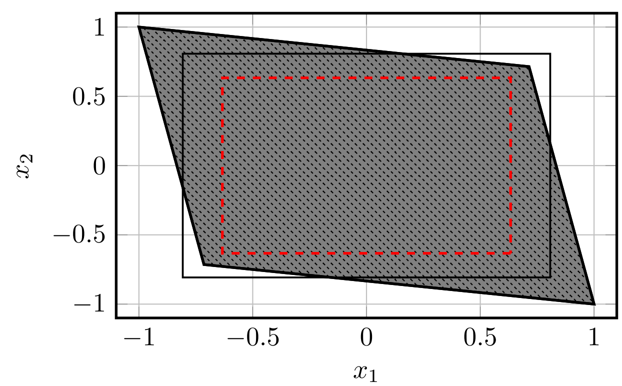

Fig. 3. Feasible regions for $\mathcal{M} = \{1, 2\}$, $\Pi_{\mathcal{M}} = \{\Lambda_D, \Lambda_C\}$ and dynamics given by A_{ex2} and B_{ex2}.(▩) represents the feasible region for Λ_C, (▭) represents the feasible region for Λ_D for an LQR feedback gain, and (⬚) is the feasible set for a nilpotent feedback gain.

The feasible regions are not nested, *i.e.* $\mathcal{X}^2(\mathcal{C}) \not\supseteq \mathcal{X}^2(\mathcal{D})$ and $\mathcal{X}^2(\mathcal{D}) \not\supseteq \mathcal{X}^2(\mathcal{C})$, as is evident in Figure 3. The difference between these two systems can be understood from the spectral radius: for the first system $\rho(A_{\text{ex1}}) = 1.2$ while for the second $\rho(A_{\text{ex2}}) = 0.7$. In the second case, the couplings have a stronger effect on the evolution of the state, which is in part the reason why the nesting is lost. Including the effect of couplings, as invariant sets and constraint tightenings, into the feasible region, following (10), does not guarantee the nesting property either. Figure 3 shows how the choice of invariant set affect the size of the resulting region of attraction: for the nilpotent feedback gain the resulting feasible region is the smallest, whereas for the LQR gain the feasible region is the largest. The remaining cases lie in between these two sets.

V. Concluding remarks

In this paper we analyse feasibility regions for an LSS subject to different subsystem partitions; the process of partition refinement induces an order relation on the invariant sets for the LSS but not on the feasible regions. We show that a controllability assumption on the finest partition implies stabilizability for coarser ones. These properties, both controllability and nesting of invariant sets, may be useful for designing controllers capable of switching between partitions. Finding conditions that allow feasible switching between partitions is part of our future work.

Appendix

A. Proof of Proposition 1

To achieve this, we use a preliminary result on the controllability of systems subject to bounded disturbances developed in [14] and [15].[3]

Lemma 5 (Controllability under disturbances): The system $x^+ = Ax + Bu + w$ with bounded control input $u \in \mathbb{U}$ and disturbance $w \in \mathbb{W}$, where $\mathbb{U}$ and $\mathbb{W}$ are C-sets, is controllable to the target set $\mathcal{X}_T \subseteq \mathbb{R}^n$ if and only if for some $k \in \mathbb{N}$ the disturbance-free system $y^+ = Ay + Bu$, $u \in \mathbb{U}$, is controllable to

$$\mathcal{Y}_T(k) = \{y \in \mathbb{R}^n \colon a^\top y \leq b(a,k),\ \forall a \in \mathbb{R}^n \text{ with } \|a\| = 1\}$$

where $b(a,k) = \max_{x \in \mathcal{X}_T} a^\top x - \sum_{j=0}^{k-1} \max_{w(j) \in \mathbb{W}} a^\top A^{k-1-j} w(j)$.□

The proof of Proposition 1 will be given for the case of two subsystems $i, j \in \mathcal{M}$ that form a coalition $\mathcal{C}_c = \{i, j\}$. This case can easily be extended using induction on the number of subsystems forming a coalition. By Assumption 1, controllability follows trivially for the system

$$y_c^+ = A_{cc}^{\text{diag}} y_c + B_{cc}^{\text{diag}} u_c, \tag{11}$$

with $A_{cc}^{\text{diag}} = \text{diag}(A_i, A_j)$ and $B_{cc}^{\text{diag}} = \text{diag}(B_i, B_j)$. However, the coalition dynamics are composed of block matrices that do not, in general, have a diagonal structure but can instead be written in terms of their diagonal components:

$$x_c^+ = A_{cc}^{\text{diag}} x_c + B_{cc}^{\text{diag}} u_c + w_c,$$

with $w_c = A_c^{\text{c}} x_c + B_c^{\text{c}} u_c$, $A_c^{\text{c}} = A_{cc} - A_{cc}^{\text{diag}}$ and $B_c^{\text{c}} = B_{cc} - B_{cc}^{\text{diag}}$. The controllability of (11) implies the existence of a linear feedback gain K_c^{diag} such that the spectral radius $\rho(A_{cc}^{\text{diag}} + B_{cc}^{\text{diag}} K_c^{\text{diag}}) < 1$, the minimal RPI set corresponding to K_c satisfy $\mathcal{R}_c \subset \mathbb{X}_c$ and $K_c \mathcal{R}_c \subset \mathbb{U}_c$, and such that $P_c^{\text{diag}} = P_c^{\text{diag}\top} \succ 0$ satisfies the Lyapunov equation

$$P_c^{\text{diag}} - (A_{cc}^{\text{diag}} + B_{cc}^{\text{diag}} K_c^{\text{diag}})^\top P_c^{\text{diag}} (A_{cc}^{\text{diag}} + B_{cc}^{\text{diag}} K_c^{\text{diag}}) = Q_c$$

for any given $Q_c = Q_c^\top \succ 0$. Then $V_c^{\text{diag}}(x_c) = x_c^\top P_c^{\text{diag}} x_c$ is a Lyapunov function demonstrating (local) asymptotic stability of (11), and $V_c^{\text{diag}}(\cdot)$ is Lipschitz continuous on $\mathcal{X}_T$ with constant L_K. The descent property of the Lyapunov function implies

$$\begin{aligned} V_c^{\text{diag}}(x_c^+) - V_c^{\text{diag}}(x_c) &\leq -x_c^\top Q_c x_c \\ &\qquad + L_K \|A_c^{\text{c}} + B_c^{\text{c}} K_c^{\text{diag}}\| \|x_c\| \\ &\leq -\gamma x_c^\top Q_c x_c + \big(L_K \|A_c^{\text{c}} + B_c^{\text{c}} K_c^{\text{diag}}\| \\ &\qquad - (1-\gamma)\lambda_{\min}(Q_c)\|x_c\|\big)\|x_c\| \\ &\leq -\gamma x_c^\top Q_c x_c \end{aligned}$$

whenever $\|x_c\| \geq (1-\gamma)^{-1} \lambda_{\min}^{-1}(Q_c) L_K \|A_c^{\text{c}} + B_c^{\text{c}} K_c^{\text{diag}}\|$ and $\gamma \in (0,1]$ which implies that x_c converges asymptotically to a neighbourhood of the origin dependent on L_k (and hence on $\mathbb{X}_c := \mathbb{X}_i \times \mathbb{X}_j$) and on $\|A_c^{\text{c}} + B_c^{\text{c}} K_c^{\text{diag}}\|$. Note that the property that $V_c^{\text{diag}}(x_c)$ is decreasing outside of a compact set containing the origin also implies that for sufficiently small $\|x_c(0)\|$ the state and input constraints will be satisfied provided $L_K \|A_c^{\text{c}} + B_c^{\text{c}} K_c^{\text{diag}}\|$ is sufficiently small.

Using Assumption 2, the interactions lie in a bounded set $\mathbb{W}_c^{\text{c}}(k) = A_c^{\text{c}} \mathbb{X}_c \oplus B_c^{\text{c}} u_c(k)$ where $\mathbf{u}_c = \{u_c(0), \ldots, u_c(T-1)\}$ is a control sequence used to steer the diagonal system to the origin in finite time. Given the controllability of the diagonal dynamics, Lemma 5 implies that the coalition dynamics can be steered to

$$\mathcal{X}_T = \bigoplus_{j=0}^{T-1} (A_{cc}^{\text{diag}} + B_{cc} K_c)^{T-1-j} (A_c^{\text{c}} \mathbb{X}_c \oplus B_c^{\text{c}} u_c(j)),$$

which is a *C-set* by construction, and $\mathcal{X}_T \subset \mathcal{R}_c$. The controllability to the origin follows as a consequence of Lemma 1. ■

[3]The proof of Lemma 5 can be found in Theorem 1 of [14] which is based on Theorem 4.1 of [15].

References

[1] D. Mayne, "Model predictive control: Recent developments and future promise," *Automatica*, vol. 50, no. 12, pp. 2967–2986, 2014.

[2] D. D. Siljak, *Decentralized control of complex systems*. Dover Publications, 2012.

[3] J. Maestre and R. R. Negenborn, *Distributed Model Predictive Control Made Easy*, 2014, vol. 69.

[4] D. B. Pourkargar, A. Almansoori, and P. Daoutidis, "Distributed model predictive control of process networks: Impact of control architecture," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 12 452–12 457, jul 2017.

[5] J. Maestre, D. Muñoz de la Peña, and E. F. Camacho, "Distributed model predictive control based on a cooperative game," *Optim. Control Appl. Methods*, vol. 32, no. 2, pp. 153–176, mar 2011.

[6] F. Muros, J. Maestre, E. Algaba, T. Alamo, and E. Camacho, "An Iterative Design Method for Coalitional Control Networks with Constraints on the Shapley Value," *IFAC Proc. Vol.*, vol. 47, no. 3, pp. 1188–1193, 2014.

[7] P. Trodden and A. Richards, "Adaptive cooperation in robust distributed model predictive control," in *2009 IEEE Int. Conf. Control Appl.*, no. 1. IEEE, jul 2009, pp. 896–901.

[8] B. Hernandez, P. Baldivieso, and P. Trodden, "Distributed MPC: Guaranteeing Global Stability from Locally Designed Tubes," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 11 829–11 834, jul 2017.

[9] S. Raković, B. Kouvaritakis, R. Findeisen, and M. Cannon, "Homothetic tube model predictive control," *Automatica*, vol. 48, no. 8, pp. 1631–1638, 2012.

[10] P. R. Baldivieso Monasterios, B. Hernandez, and P. A. Trodden, "Nested distributed MPC," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 11 822–11 828, jul 2017.

[11] S. Raković, E. Kerrigan, K. Kouramas, and D. Mayne, "Invariant approximations of the minimal robust positively Invariant set," *IEEE Trans. Automat. Contr.*, vol. 50, no. 3, pp. 406–410, mar 2005.

[12] P. Trodden, "A One-Step Approach to Computing a Polytopic Robust Positively Invariant Set," *IEEE Trans. Automat. Contr.*, vol. 61, no. 12, pp. 4100–4105, dec 2016.

[13] D. Mayne, M. Seron, and S. Raković, "Robust model predictive control of constrained linear systems with bounded disturbances," *Automatica*, vol. 41, no. 2, pp. 219–224, feb 2005.

[14] R. Van Til and W. Schmitendorf, "Constrained controllability of discrete-time systems with additive disturbances," in *Poceedings 24th Conf. Decis. Control*, 1985, pp. 874–875.

[15] B. R. Barmish and W. E. Schmitendorf, "The associated disturbance-free system: A means for investigating the controllability of a disturbed system," *J. Optim. Theory Appl.*, vol. 38, no. 4, pp. 525–540, 1982.