# Motivic Donaldson–Thomas invariants of some quantized threefolds

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*In memory of Kentaro Nagao*

**Abstract**

This paper is motivated by the question of how motivic Donaldson–Thomas invariants behave in families. We compute the invariants for some simple families of noncommutative Calabi–Yau threefolds, defined by quivers with homogeneous potentials. These families give deformation quantizations of affine three-space, the resolved conifold, and the resolution of the transversal $A_n$-singularity. It turns out that their invariants are generically constant, but jump at special values of the deformation parameter, such as roots of unity. The corresponding generating series are written in closed form, as plethystic exponentials of simple rational functions. While our results are limited by the standard dimensional reduction techniques that we employ, they nevertheless allow us to conjecture formulae for more interesting cases, such as the elliptic Sklyanin algebras.

*Keywords:* Donaldson–Thomas theory, motivic vanishing cycle, Calabi–Yau algebra, quiver representation, dimensional reduction

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# 1 Introduction

Donaldson–Thomas (DT) invariants were introduced by Donaldson and Thomas in [19, 33] to give a numerical count of sheaves on three-dimensional projective Calabi–Yau varieties. One of the fundamental results of [33] was the statement that these integer quantities are indeed invariants, in the sense that they are unchanged when the underlying Calabi–Yau threefold moves in a connected projective family. It was later realized that DT-like invariants can be defined by counting objects in more general 3-Calabi–Yau categories, such as categories defined by a quiver with potential [32]. In a different direction, building on work of Behrend [5], DT-like invariants taking values in more general rings and not just in $\mathbb{Z}$, such as rings of (naive) motives [6, 24], were defined.

In this paper, we are interested in how motivic DT invariants behave in families. In the projective case, this problem seems difficult to study, since it is hard to compute motivic DT invariants of projective Calabi–Yau varieties in all but a handful of cases. Here, we instead study deformation properties of motivic DT invariants for some families of *noncommutative* Calabi–Yau threefolds, defined by quivers with homogeneous potentials. The motivic invariants we look at are attached to moduli spaces of finite-dimensional representations of the Jacobian algebra associated to the quiver with potential; this should be seen as the noncommutative analogue of studying moduli of finite sets of points on commutative threefolds. Moduli spaces of homogeneous potentials were studied recently in [31] in a specific example, where the question of the behaviour of motivic DT invariants was also raised. We can regard a family of *graded* deformations of a homogeneous potential as an analogue of a projective family in the local noncommutative situation.

The results we are reporting on here are rather limited. In particular, we do not introduce any new techniques for computing DT invariants in this paper; rather we use the dimensional reduction technique already used in [6], and systematized in [27], to see what we can learn about the structure of the invariants and their deformation properties. Thus, the cases in which we have been able to compute the generating series for the motivic DT invariants are the cases in which the potential is linear with respect to one of the generators of the algebra.

We explore only the simplest deformations of potentials for some well-studied quivers of geometric origin: the three-loop quiver underlying the Hilbert scheme of points on threefolds; the conifold quiver; and, generalizing the first, the cyclic quiver with loops. In the case of undeformed potentials, corresponding to commutative Calabi–Yau threefolds, the motivic invariants in these examples were computed in [6], [28] and [9, 27], respectively. We study some simple perturbations of these potentials, corresponding to certain deformation quantizations of the commutative threefolds, and compute or conjecture the corresponding motivic DT invariants. One reason for focusing on the three-loop quiver, in particular, is that such homogeneous deformations of the potential correspond to marginal deformations of $N = 4$ super Yang–Mills theory [7]; our formulae therefore correspond to refined BPS counts for these deformed theories [18].

The motivic DT invariants for our families are certainly not deformation-invariant. However, we find that they behave in rather well-controlled ways, having constant generic value, but jumping at special values of parameters (such as roots of unity) for which the quantized algebras become finite modules over their centres. At least for our simple quivers, the generating series of motivic DT invariants have a surprisingly simple form: they are motivic "Exponentials" of rather simple rational functions of the vertex variables. The fact that generating functions of DT invariants are Exponentials is now well known and underlies rationality (also known as BPS) conjectures and theorems [23, 24]. However, the fact that inside the Exponential we often have simple rational functions seems not to have been observed before in this generality.

The rational functions, including their coefficients, appear to be governed by *simple* finite-dimensional representations of the corresponding Jacobi algebra. In the case of the three-loop quiver, based on our results (Theorems 3.1-3.2) we are able to articulate a somewhat more precise conjectural formula (7), and use it to predict the answer in some cases which we cannot access via dimensional reduction—the homogenized Weyl algebra and the elliptic Sklyanin algebras (Conjectures 3.3-3.4). In multi-vertex cases, our results (Theorems 3.5-3.6) do not lead to any precise conjecture. Nevertheless, the coefficients in the rational functions still have intriguing geometric interpretations: they seem to correspond to the degeneracy loci of the Poisson structures that are quantized in order to produce the given families of noncommutative algebras. We thus uncover an intriguing landscape, which we leave for further study.

In a very recent paper [25], Le Bruyn has introduced a new technique to compute motivic Donaldson–Thomas invariants of Jacobi algebras associated to homogeneous potentials inductively via computing motives of Brauer-Severi schemes of Cayley-smooth algebras. In cases where the potential has a cut, he finds perfect agreement with our results. In more complicated cases, he finds that if one disregards monodromy, our conjectures do not hold. In these cases however, all our formulae involve nontrivial monodromy actions; repeating the computations of [25] taking full account of the monodromy appears to be an interesting challenge which we leave for future work.

The paper is organized as follows. In Section 2, we give a brief review of the necessary preliminaries about quiver representations; the ring of motivic classes; and the definition of motivic DT invariants and their generating series. In Section 3, we summarize our computations and conjectures for the generating series in the examples of interest, and we discuss their geometric interpretations. We finish in Section 4 with the proofs.

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## 2 Preliminaries

### 2.1 Quivers and their representations

Let $Q$ be a finite quiver, with vertex set $V(Q)$ and arrow set $E(Q)$. For an arrow $a \in E(Q)$, denote by $s(a) \in V(Q)$, respectively $t(a) \in V(Q)$, the vertex at which $a$ starts, respectively ends. The Euler-Ringel form $\chi$ on $\mathbb{Z}^{V(Q)}$ is

$$\chi(\alpha,\beta) = \sum_{i\in V(Q)} \alpha_i\beta_i - \sum_{a\in E(Q)} \alpha_{s(a)}\beta_{t(a)}, \qquad \alpha,\beta \in \mathbb{Z}^{V(Q)}.$$

Given a $Q$-representation $M$, its dimension vector $\underline{\dim}\, M \in \mathbb{N}^{V(Q)}$ is defined by $\underline{\dim}\, M = (\dim M_i)_{i\in V(Q)}$.

Let $\alpha \in \mathbb{N}^{V(Q)}$ be a dimension vector and let $V_i = \mathbb{C}^{\alpha_i}$, $i \in V(Q)$. Let

$$R(Q,\alpha) = \bigoplus_{a\in E(Q)} \mathrm{Hom}(V_{s(a)}, V_{t(a)})$$

and

$$G_\alpha = \prod_{i\in V(Q)} \mathrm{GL}(V_i).$$

Then $G_\alpha$ naturally acts on $R(Q,\alpha)$, and the quotient stack

$$\mathfrak{M}(Q,\alpha) = [R(Q,\alpha)/G_\alpha]$$

gives the moduli stack of representations of $Q$ with dimension vector $\alpha$.

Let $W$ be a potential on $Q$, a finite linear combination of cyclic paths in $Q$. Denote by $J_{Q,W}$ the Jacobian algebra, the quotient of the path algebra $\mathbb{C}Q$ by the two-sided ideal generated by formal partial derivatives of the potential $W$. Let

$$f_\alpha : R(Q,\alpha) \to \mathbb{C}$$

be the $G_\alpha$-invariant function defined by taking the trace of the map associated to the potential $W$. A point in the critical locus $\mathrm{crit}(f_\alpha)$ corresponds to a $J_{Q,W}$-module. The quotient stack

$$\mathfrak{M}(J_{Q,W},\alpha) = \left[\mathrm{crit}(f_\alpha)/G_\alpha\right]$$

gives the moduli stack of $J_{Q,W}$-modules with dimension vector $\alpha$.

### 2.2 The ring of motivic classes

Let $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ be the ring of isomorphism classes of reduced varieties over $\mathbb{C}$, equipped with a good action of a finite group of roots of unity, respecting the scissor relation and the relation $[\mathbb{A}^n, \mu_k] = [\mathbb{A}^n]$ for a linear representation of the group $\mu_k$ on affine space $\mathbb{A}^n$, as in [26]. Here $\mu_k$ denotes the group of $k$th roots of unity.

Let $\mathbb{L} = [\mathbb{A}^1] \in K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$ be the class of the affine line with trivial action. Then it is known that $\mathbb{L}$ admits a square root $\mathbb{L}^{\frac{1}{2}} \in K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$. We work in the motivic ring

$$\mathcal{M} = \left(K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})/\mathrm{Ann}(\mathbb{L})\right)[\mathbb{L}^{-\frac{1}{2}}, (1-\mathbb{L}^n)^{-1} \colon n \geq 1],$$

where $\mathrm{Ann}(\mathbb{L})$ denotes the annihilator [8] of $\mathbb{L}$ in $K^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$. We have the Euler characteristic specialization defined on classes of varieties by $[X] \mapsto \chi(X)$ (with or without compact support, which agree), and $\mathbb{L}^{\frac{1}{2}} \mapsto -1$; this is of course well-defined only on a subset of elements of $\mathcal{M}$ which we call *motives without denominators*.

For a regular function $f \colon X \to \mathbb{C}$ on a smooth variety $X$, Denef and Loeser [17, 26] define the motivic nearby cycle $[\psi_f] \in \mathcal{M}$ and the motivic vanishing cycle $[\varphi_f] = [\psi_f] - [f^{-1}(0)] \in \mathcal{M}$ of $f$ (at the value $0 \in \mathbb{C}$, the only case that will be relevant to us since all our functions $f$ will be homogeneous). For $f = 0$, we have $[\varphi_0] = -[X]$. Given a global critical locus $Z = \{df = 0\} \subset X$ for $f \colon X \to \mathbb{C}$ on a smooth complex variety $X$, define the virtual motive of $Z$ by

$$[Z]_{\mathrm{vir}} = -(-\mathbb{L}^{\frac{1}{2}})^{-\dim X}[\varphi_f] \in \mathcal{M}. \tag{1}$$

Thus for a smooth variety $X$ with $f = 0$, we have

$$[X]_{\mathrm{vir}} = (-\mathbb{L}^{\frac{1}{2}})^{-\dim X} \cdot [X].$$

The ring $\mathcal{M}$ is a so-called pre-$\lambda$-ring, with operations $\sigma_n : \mathcal{M} \to \mathcal{M}$ for $n \geq 0$ satisfying certain natural compatibilities [21, 14]. For classes $[X] \in \mathcal{M}$ represented by quasiprojective varieties $X$, we have $\sigma_n([X]) = [\mathrm{Sym}^n(X)]$. We therefore have $\sigma_n(\mathbb{L}) = \mathbb{L}^n$ and $\sigma_n(-\mathbb{L}^{\frac{1}{2}}) = (-\mathbb{L}^{\frac{1}{2}})^n$. There is an induced pre-$\lambda$-ring structure on the power series ring $\mathcal{M}[[t_1, \ldots, t_k]]$ defined by $\sigma_n(rt_j^i) = \sigma_n(r)t_j^{ni}$. Denoting finally by $\mathcal{M}[[t_1, \ldots, t_k]]_+ \subset \mathcal{M}[[t_1, \ldots, t_k]]$ the ideal generated by $t_1, \ldots, t_k$, we define the plethystic exponential

$$\mathrm{Exp} \colon \mathcal{M}[[t_1, \ldots, t_k]]_+ \to 1 + \mathcal{M}[[t_1, \ldots, t_k]]_+$$

by the formula

$$\mathrm{Exp}(r) = \sum_{n \geq 0} \sigma_n(r).$$

See [6, Section 2.5] as well as [13, Section 2] for a more leisurely introduction to aspects of this formalism and some more explicit formulae.

**Remark 2.1.** We remind the reader that the main motivation for definition (1) comes from the work of Behrend [5], who showed that Donaldson–Thomas type invariants can always be computed by integrating a certain specific constructible function on the relevant moduli scheme (or stack). He further proved that if the moduli scheme can (locally) be written as a critical locus $Z$ of some regular function $f$ on a smooth variety $X$, then the value of this constructible function at a point of $Z$ can be evaluated in terms of the Euler characteristic of the Milnor fibre of $f$ at this point. Taking the motivic vanishing cycle of $f$ is the right lift of Behrend's constuction to the ring of motivic classes. The prefactor $(-\mathbb{L}^{\frac{1}{2}})^{-\dim X}$ shifts this motive in the appropriate way to be self-dual with respect to Verdier duality. This makes the motive $[Z]_{\mathrm{vir}}$ behave like that of a smooth 0-dimensional variety with respect to duality, which is one expression of the fact that these invariants come from a moduli problem of virtual dimension 0.

### 2.3 Statement of the problem

Given a quiver with potential $(Q, W)$, we define motivic Donaldson-Thomas invariants

$$[\mathfrak{M}(J_{Q,W}, \alpha)]_{\mathrm{vir}} = \frac{[\mathrm{crit}(f_\alpha)]_{\mathrm{vir}}}{[G_\alpha]_{\mathrm{vir}}}, \tag{2}$$

where $[G_\alpha]_{\mathrm{vir}}$ refers to the virtual motive of the pair $(G_\alpha, 0)$. We package these invariants into a generating series by introducing a set $t = (t_i \colon i \in V(Q))$ of auxiliary variables, and setting

$$U_{Q,W}(t) = \sum_{\alpha \in \mathbb{N}^{V(Q)}} [\mathfrak{M}(J_{Q,W}, \alpha)]_{\mathrm{vir}} \cdot t^\alpha,$$

where we use multi-index notation for monomials $t^\alpha$. Our aim is to compute the series $U_{Q,W}(t)$ in closed form for some interesting classes of pairs $(Q, W)$. We will particularly be interested in how $U_{Q,W}(t)$ changes for a fixed $Q$ under homogeneous deformations of the potential $W$.

The series $U_{Q,W}$ is called the universal DT series in [28]. Generating series of framed invariants are related to $U_{Q,W}$ by wall crossing [23, 24, 29, 30].

**Remark 2.2.** Our definition (2), associating motivic DT invariants $[\mathfrak{M}(J_{Q,W}, \alpha)]_{\mathrm{vir}}$ to moduli spaces $\mathfrak{M}(J_{Q,W}, \alpha)$ of representations of the associative algebra $J_{Q,W}$ appears to depend on a particular presentation of this algebra as a quiver algebra with relations. In fact, our definition of the motivic invariant is *almost* independent from this presentation. Namely, in the cases we study, $J_{Q,W}$ is a 3-Calabi–Yau algebra. As discussed by [24], moduli spaces of representations of 3-Calabi–Yau algebras can always be described locally as critical loci, in a way that is intrinsic to the algebra only. After fixing a $\mathbb{Z}/2$-ambiguity on the moduli spaces, called *orientation data* by [24], motivic DT invariants can be defined. The choice of orientation data is implicitly fixed by the quiver description, and with this choice, our DT invariants agree with the invariants of [24], as proved by [11, Thm.7.1.3]. See [12] for more details and a concrete example.

## 3 Results and interpretations

### 3.1 Deformations of affine three-space

Let $Q_1$ be the quiver corresponding to affine three-space [6], with $V(Q_1)$ containing a single vertex and $E(Q_1) = \{x, y, z\}$ consisting of three loops based at the vertex as shown in Figure 1.

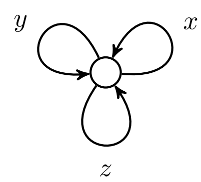


Figure 1: The quiver $Q_1$.

When the potential on $Q_1$ is $W = xyz - xzy$, the Jacobian algebra $J_{Q_1,W}$ is simply the polynomial ring $\mathbb{C}[x, y, z]$, corresponding to the simplest commutative Calabi–Yau threefold: affine three space $\mathbb{C}^3$. This perspective was used in [6] to compute the motivic DT invariants of Hilbert schemes of threefolds.

We will consider homogeneous cubic deformations of the potential $W$, resulting in flat deformations of the polynomial ring as a graded Calabi–Yau algebra. These algebras correspond to quantizations (i.e. noncommutative deformations) of $\mathbb{C}^3$ and its projectivization $\mathbb{P}^2$. Such deformations have received substantial attention in the noncommutative geometry literature, starting with [1, 4]. In particular, considerable work has been done on their representation theory [2, 15, 16, 35].

#### 3.1.1 Quantum affine three-space

Start with the potential

$$W_q = xyz - qxzy$$

on $Q_1$, where $q \in \mathbb{C}^*$ is a constant. The corresponding Jacobian algebra $J_{Q_1,W_q}$ is the coordinate ring of quantum affine three-space

$$J_{Q_1,W_q} = \mathbb{C}\langle x, y, z\rangle \,/\, ([x, y]_q, [y, z]_q, [z, x]_q)$$

with $[a, b]_q = ab - qba$. The requirement that $q$ be nonzero is important, as it ensures that the algebra is Calabi–Yau.

Since the quiver has only one vertex, a dimension vector is just a single number, indicating the dimension of the representation, so that the universal series is a function of a single variable $t$. Corresponding to the fact that $W_q$ is linear in the generator $z$, the algebra $J_{Q_1,W_q}$ has an extra $\mathbb{C}^*$-symmetry, given by rescaling $z$. In Section 4.2.1, we exploit this symmetry to prove the following

**Theorem 3.1.** *If $q \in \mathbb{C}^*$ is a primitive $r$th root of unity, then*

$$U_{Q_1,W_q}(t) = \mathrm{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t} + (\mathbb{L}-1)\frac{t^r}{1-t^r}\right).$$

*Otherwise,*

$$U_{Q_1,W_q}(t) = \mathrm{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t}\right).$$

### 3.1.2 The Jordan deformation

Up to isomorphism, the only other deformation of $W$ that is linear in one of the generators is given by

$$W_J = xyz - xzy - zy^2.$$

One of the relations of the Jacobian algebra $J_{Q_1,W_J}$ is $[x,y] = y^2$, which is precisely the relation between the generators $x, y$ of the non-commutative affine plane commonly known as the Jordan plane. In Section 4.2.2, we prove:

**Theorem 3.2.** *For the Jordan deformation, we have*

$$U_{Q_1,W_J}(t) = \operatorname{Exp}\left(\frac{\mathbb{L}}{\mathbb{L}-1}\frac{t}{1-t}\right).$$

### 3.1.3 The homogenized Weyl deformation

Consider now the potential

$$W_{hW} = xyz - xzy - \frac{1}{3}z^3$$

on the three-loop quiver. One of the relations of the Jacobian algebra $J_{Q_1,W_W}$ is $[x,y] = z^2$, the homogenization of the Weyl algebra relation $[x,y] = 1$. In Section 3.4 below, we will explain the following

**Conjecture 3.3.** *We have*

$$U_{Q_1,W_{hW}}(t) = \operatorname{Exp}\left(\frac{\mathbb{L}(1-[\mu_3])}{\mathbb{L}-1}\frac{t}{1-t}\right),$$

*where by slight abuse of notation we denote by $[\mu_3]$ the equivariant motivic class of $\{z^3 = 1\} \subset \mathbb{C}$ carrying the canonical action of $\mu_3$.*

### 3.1.4 Sklyanin deformations

Consider finally the family of Sklyanin deformations

$$W_{a,b,c} = a\,xyz + b\,xzy + \frac{c}{3}(x^3 + y^3 + z^3),$$

for $[a:b:c] \in \mathbb{P}^2$. We assume that $abc \neq 0$ and $(3abc)^3 \neq (a^3+b^3+c^3)^3$. The relations in the Jacobian algebra $J_{Q_1,W_{a,b,c}}$ are given by

$$axy + byx + cz^2 = ayz + bzy + cx^2 = azx + bxz + cy^2 = 0.$$

As explained in [1, p. 38], to each such algebra is associated a pair of smooth cubic curves in $\mathbb{P}^2$. The first is the *point scheme* $E_{pt}$, which parametrizes isomorphism classes of graded modules with Hilbert series $(1-t)^{-1}$; it is given explicitly by

$$E_{\text{pt}} = \left\{(a^3+b^3+c^3)XYZ - abc(X^3+Y^3+Z^3) = 0\right\} \subset \mathbb{P}^2,$$

using homogeneous coordinates $[X : Y : Z]$ on $\mathbb{P}^2$. The functor which shifts the grading of a module induces a translation $\sigma : E_{\mathrm{pt}} \to E_{\mathrm{pt}}$ in the group law of the cubic curve. The second cubic curve, which typically has a different $j$-invariant, is the curve $E_{\mathrm{DT}}$ defined by the vanishing of the potential $f : \mathbb{C}^3 \to \mathbb{C}$ for one-dimensional representations:

$$E_{\mathrm{DT}} = \left\{(a+b)XYZ + \tfrac{c}{3}(X^3+Y^3+Z^3) = 0\right\} \subset \mathbb{P}^2.$$

Two Sklyanin algebras determine the same pair $(E_{\mathrm{pt}}, E_{\mathrm{DT}})$ if and only if they are either isomorphic or opposite as graded algebras. In Section 3.4, we explain the following conjecture.

**Conjecture 3.4.** *Let $J_{Q_1,W_{a,b,c}}$ be a Sklyanin algebra as above, let $(E_{\mathrm{pt}}, E_{\mathrm{DT}})$ be the associated elliptic curves, and let $\sigma : E_{\mathrm{pt}} \to E_{\mathrm{pt}}$ be the induced automorphism. Let $S_{\mathrm{DT}}$ be the affine cubic surface*

$$S_{\mathrm{DT}} = \left\{(a+b)xyz + \tfrac{c}{3}(x^3+y^3+z^3) = 1\right\} \subset \mathbb{A}^3,$$

*the universal cover of $\mathbb{P}^2 \backslash E_{\mathrm{DT}}$, carrying the canonical action of $\mu_3 \cong \pi_1(\mathbb{P}^2 \backslash E_{\mathrm{DT}})$. Define the virtual motive*

$$M_1 = \mathbb{L}^{-\frac{3}{2}}\left([S_{\mathrm{DT}}, \mu_3] - [E_{\mathrm{DT}}](\mathbb{L}-1) - 1\right).$$

*Then we have the following conjectural formulae for the universal series:*

1. *If $|\sigma| = \infty$, then*

$$U_{Q_1,W_{a,b,c}}(t) = \mathrm{Exp}\left(-\frac{M_1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t}\right).$$

2. *If $|\sigma|$ is finite but not a multiple of three, then*

$$U_{Q_1,W_{a,b,c}}(t) = \mathrm{Exp}\left(-\frac{M_1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t} - \frac{M_\sigma}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t^{|\sigma|}}{1-t^{|\sigma|}}\right),$$

   *where the virtual motive $M_\sigma = \mathbb{L}^{\frac{1}{2}}([\mathbb{P}^2] - [E_{\mathrm{pt}}/\sigma])$ involves the elliptic curve $E_{\mathrm{pt}}/\sigma$ isogeneous to $E_{\mathrm{pt}}$.*

3. *If $|\sigma|$ is a multiple of three, then*

$$U_{Q_1,W_{a,b,c}}(t) = \mathrm{Exp}\left(-\frac{M_1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t} - \sum_r \frac{M_r}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}\frac{t^r}{1-t^r}\right).$$

   *where the summation index $r$ ranges over a subset of $\{|\sigma|/3, \ldots, |\sigma|\}$ and $M_r$ are virtual motives without denominators.*

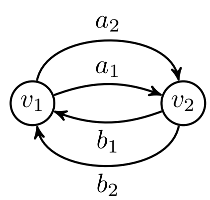


Figure 2: The quiver $Q_{\mathrm{con}}$.

## 3.2 A deformation of the conifold algebra

Let $Q_{\mathrm{con}}$ be the conifold quiver, with two vertices $V(Q_{\mathrm{con}}) = \{v_1, v_2\}$, and four arrows $E(Q_{\mathrm{con}}) = \{a_1, a_2, b_1, b_2\}$ with head and tail as indicated in Figure 2.

The standard quartic potential of the conifold quiver

$$W = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$$

gives a Calabi–Yau algebra $J_{Q_{\mathrm{con}},W}$ whose centre is given by

$$Z = \mathbb{C}[x, y, z, t]/(xt - yz).$$

In this way, $J_{Q_{\mathrm{con}},W}$ is a noncommutative crepant resolution of the conifold singularity $\operatorname{Spec} Z$ in the sense of [34]. In particular, $J_{Q_{\mathrm{con}},W}$ is derived equivalent to a standard (commutative) crepant resolution of $Z$.

We may therefore think of the one-parameter deformation

$$W_q = a_1 b_1 a_2 b_2 - q a_1 b_2 a_2 b_1$$

with $q \in \mathbb{C}^*$ as a quantization of the resolved conifold. The condition $q \neq 0$ once again corresponding to the Calabi–Yau property for the Jacobian algebra.

**Theorem 3.5.** *If $q \in \mathbb{C}^*$ is not a root of unity, then*

$$U_{Q_{\mathrm{con}},W_q}(t_0, t_1) = \operatorname{Exp}\left( \frac{3\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{t_0 t_1}{1 - t_0 t_1} - \frac{1}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{t_0 + t_1}{1 - t_0 t_1} \right).$$

*If $q$ is a primitive $r$-th root of unity, then the above expression is multiplied by a further factor*

$$\operatorname{Exp}\left( (\mathbb{L} - 1) \frac{t_0^r t_1^r}{1 - t_0^r t_1^r} \right).$$

More general deformations of the conifold potential are studied in [10, 31]. We leave the investigation of motivic DT invariants for these deformations for future work.

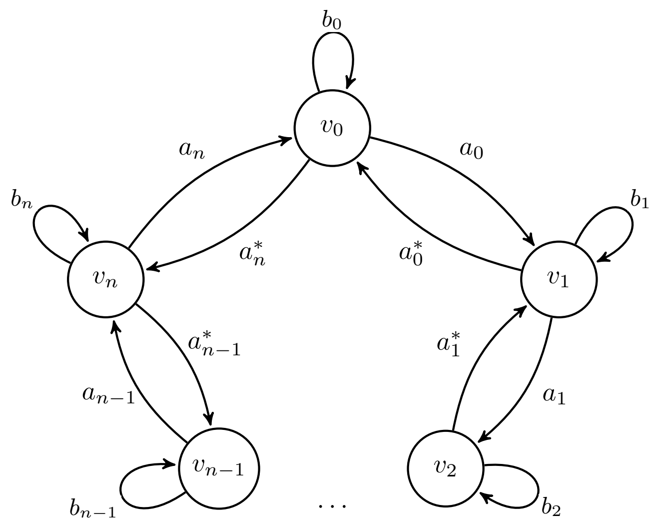


Figure 3: The $A_n$-quiver $Q_{n+1}$.

## 3.3 A deformation of the cyclic quiver

Finally let $n \geq 1$ and consider the quiver $Q_{n+1}$ with $V(Q_{n+1}) = \{v_0, \ldots, v_n\}$ and arrows depicted as in Figure 3.

As is well known, $Q_{n+1}$ is the McKay quiver for the embedding $\mu_{n+1} < \mathrm{SL}(3, \mathbb{C})$ with weights $(1, -1, 0)$. We are interested in deformations of the potential

$$W_1 = \sum_{i=0}^{n} \left(b_{i+1} a_i^* a_i - b_i a_i a_i^*\right), \tag{3}$$

where the index labels are to be understood modulo $n+1$. Once again, the Jacobi algebra $J_{Q_{n+1}, W_1}$ is a non-commutative crepant resolution of its center

$$Z \cong \mathbb{C}[x, y, z, t]/(xy - z^{n+1}).$$

The threefold $\operatorname{Spec} Z$ has transverse $A_n$ singularities along a line.

We concentrate on the family of homogeneous deformations of the potential $W_1$ given by

$$W_{\underline{q}} = \sum_{i=0}^{n-1} \left(q_i' b_{i+1} a_i^* a_i - q_i b_i a_i a_i^*\right), \tag{4}$$

where we assume $\prod_j q_j' q_j \neq 0$. It is straightforward to verify that any such potential is equivalent (by rescaling variables) to a member of the one-parameter family

$$W_q = \sum_{i=0}^{n-1} \left(b_{i+1} a_i^* a_i - q b_i a_i a_i^*\right),$$

parametrized by $q \in \mathbb{C}^*$. The condition $q \neq 0$ is still equivalent to the Jacobian algebra being Calabi–Yau.

Let $t = (t_0, \ldots, t_n)$ be the variables in the universal DT series; we continue to use multi-index notation for monomials in these variables. Let $\delta_i$ be dimension vector which takes the value 1 on $i$th vertex and 0 otherwise. Define

$$\Delta = \{\delta_i + \ldots + \delta_{i+k} \colon i \in \{0, \ldots, n\}, k \in \{0, \ldots, n-1\}\},$$

where we read the indices modulo $n+1$. Thus, for example, the vector $\delta_n + \delta_0$ lies in $\Delta$. Finally, we let $\delta = \sum_{i=0}^{n} \delta_i$.

**Theorem 3.6.** *If $q \in \mathbb{C}^*$ is not a root of unity, then*

$$U_{Q_{n+1}, W_q}(t) = \operatorname{Exp}\left( \frac{(n+1)\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{t^\delta}{1 - t^\delta} + \sum_{\alpha \in \Delta} \frac{\mathbb{L}^{\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{t^\alpha}{1 - t^\delta} \right).$$

*If $q$ is a primitive $r$-th root of unity, then the above expression is multiplied by a further factor*

$$\operatorname{Exp}\left( (\mathbb{L} - 1) \frac{t^{r\delta}}{1 - t^{r\delta}} \right).$$

## 3.4 Interpretations

In this section, we analyze the results stated above. The pattern that will emerge is that, at least in our examples, the universal series has the shape of an Exponentiated rational function in the vertex variables with coefficient motives closely related to motives of spaces of simple representations of the corresponding algebra.

### 3.4.1 Interpreting the results on the three-loop quiver

Let us start by interpreting our results on the quiver $Q_1$ that leads to variants of affine three-space. For the potential $W_q = xyz - qxzy$ at $q = 1$, we recover the result of [6] for the commutative case:

$$U_{Q_1, W_1}(t) = \operatorname{Exp}\left( -\frac{-\mathbb{L}^{3/2}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{t}{1 - t} \right).$$

Here the motive appearing in the numerator, $-\mathbb{L}^{3/2}$, is the virtual motive of affine three-space $\mathbb{C}^3$, the moduli space of simple one-dimensional modules of the algebra $U_{Q_1, W_1}$. These are clearly the only simple modules.

For generic quantum three-space, with parameter $q \in \mathbb{C}^*$ not a root of unity, by Theorem 3.1 above we have

$$U_{Q_1, W_q}(t) = \operatorname{Exp}\left( -\frac{-2\mathbb{L}^{\frac{1}{2}} + \mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{t}{1 - t} \right). \tag{5}$$

The only simple modules are still the one-dimensional modules by Lemma 3.8 below; these are parametrized by the subscheme $\{df = 0\} \subset \mathbb{C}^3$ where $f \colon \mathbb{C}^3 \to \mathbb{C}$

is given by $f(x,y,z) = xyz$. The motivic vanishing cycle of $f$ is $2\mathbb{L}^2 - \mathbb{L}$, so its critical locus has virtual motive $-2\mathbb{L}^{\frac{1}{2}} + \mathbb{L}^{-\frac{1}{2}}$, which is the expression appearing in the numerator of (5).

Let us now turn to the Jordan deformation, with

$$J_{Q_1,W_J} = \mathbb{C}\langle x,y,z\rangle/([x,y]-y^2,[y,z],[z,x]-2yz).$$

Here we have, by Theorem 3.2,

$$U_{Q_1,W_J}(t) = \mathrm{Exp}\left(-\frac{-\mathbb{L}^{\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t}\right). \tag{6}$$

On the other hand, we still only have one-dimensional simple representations.

**Lemma 3.7.** *Every finite-dimensional simple module for $J_{Q_1,W_J}$ has dimension one.*

*Proof.* Let $V$ be a finite-dimensional simple module. We have the relation

$$y^{n+2} = y^n[x,y] = [y^n x, y],$$

which implies that every positive power of $y^2$ is a commutator, and therefore acts with trace zero on $V$. Hence $y^2$ acts nilpotently on $V$, and so the action of $y$ on $V$ must have a nontrivial kernel $K \subset V$. But the relations for $J_{Q_1,W_J}$ then imply that $K$ is actually a $J_{Q_1,W_J}$-submodule. Since $V$ is simple, we must therefore have $K = V$. Hence $y$ acts trivially on $V$, so that $V$ descends to a representation of the quotient

$$J_{Q_1,W_J}/(y) \cong \mathbb{C}[x,z],$$

which is commutative, and therefore only has one-dimensional simple modules. □

One-dimensional modules are parametrized by the critical locus $\{df_J = 0\} \subset \mathbb{C}^3$, where $f_J\colon \mathbb{C}^3 \to \mathbb{C}$ is given by $f(x,y,z) = zy^2$. The motivic vanishing cycle of this function is $\mathbb{L}^2$, so the corresponding virtual motive is $(-\mathbb{L})^{-\frac{3}{2}}\mathbb{L}^2 = -\mathbb{L}^{\frac{1}{2}}$, the numerator of (6).

In the cases discussed in this section so far, the Jacobian algebra has only one-dimensional simple representations. Quantized affine three-space at roots of unity behaves differently (see [7, 15]):

**Lemma 3.8.** *The algebra*

$$J_{Q_1,W_q} = \mathbb{C}\langle x,y,z\rangle/([x,y]_q,[y,z]_q,[z,x]_q)$$

*corresponding to the potential $W = xyz - qxzy$ on $Q_1$ has simple modules of dimension $r > 1$ if and only if $q$ is a primitive $r$-th root of unity. Moreover, the space of one-dimensional representations is independent of $q$ provided $q \neq 1$.*

Note that the formula in Theorem 3.1 has terms in precise correspondence with the possible dimensions of simple representations. Indeed, when $q$ is a primitive $r$th root of unity, we have

$$\begin{aligned} U_{Q,W}(t) &= \operatorname{Exp}\left(\frac{2\mathbb{L}-1}{\mathbb{L}-1}\frac{t}{1-t} + (\mathbb{L}-1)\frac{t^r}{1-t^r}\right) \\ &= \operatorname{Exp}\left(-\frac{-2\mathbb{L}^{\frac{1}{2}}+\mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t}{1-t} - \frac{-\mathbb{L}^{-\frac{3}{2}}\cdot\mathbb{L}(\mathbb{L}-1)^2}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t^r}{1-t^r}\right). \end{aligned}$$

The first summand is again given by the virtual motive of one-dimensional representations, which are, of course, all simple. But there are also simple representations of dimension $r$, and they contribute the rational function $\frac{t^r}{1-t^r}$ to the generating series. Our aim now is to provide a geometric interpretation of this contribution.

The coefficient of $\frac{t^r}{1-t^r}$ has a factor $(\mathbb{L}-1)^2$. In our calculation in Section 4.2.1, this factor will be obtained by a dimensional reduction procedure, in which the problem is reduced to studying representations of the simpler algebra $\mathbb{C}\langle x,y\rangle/(xy-qyx)$. The simple $r$-dimensional modules of the latter are parametrized by a copy of $(\mathbb{C}^*)^2$, which is naturally obtained by removing the coordinate axes in $\mathbb{A}^2$. So in that sense, the motive is explained geometrically by dimensional reduction. But it is unclear how to generalize this interpretation to potentials for which dimensional reduction is not applicable. We therefore propose another interpretation that is better suited to generalizations, based on the noncommutative projective geometry approach of [2, 3, 15].

We observe that the moduli stack $\mathfrak{M}(J_{Q_1,W_q}, r)$ of $r$-dimensional representations has a Zariski-open substack $\mathfrak{M}_{\mathrm{inv}} \subset \mathfrak{M}(J_{Q_1,W_q}, r)$ consisting of simple representations on which the central element $g = xyz$ acts invertibly. Let us denote by $\mathbb{A}^3_q$ the noncommutative affine space defined by $J_{Q_1,W_q}$. Thus elements of $\mathfrak{M}_{\mathrm{inv}}$ correspond to skyscraper sheaves on the noncommutative variety $\mathbb{A}^3_q \setminus \{g = 0\}$, obtained by removing the coordinate planes.

According to [15], the stack $\mathfrak{M}_{\mathrm{inv}}$ has a coarse moduli space $X$ that is a smooth threefold, and the stabilizers at each point are the scalars $\mathbb{C}^*$. Moreover, the grading on $J_{Q_1,W_q}$ gives rise to a $\mathbb{C}^*$-action on representations, and this action makes $X$ into a principal $\mathbb{C}^*$-bundle over $\mathbb{P}^2 \setminus Y$, where $Y \subset \mathbb{P}^2$ is a triangle. This fibration has the following interpretation: starting with a skyscraper sheaf $\mathcal{F}$ on $\mathbb{A}^3_q \setminus \{g = 0\}$, we take its direct image $\pi_*\mathcal{F}$ along the quotient map $\pi : \mathbb{A}^3_q \setminus \{0\} \to \mathbb{P}^2_q$ to the corresponding noncommutative $\mathbb{P}^2$. The support of $\pi_*\mathcal{F}$ is a point in the "centre" of $\mathbb{P}^2_q \setminus \{g = 0\}$, which is the commutative variety $\mathbb{P}^2 \setminus Y$.

Assembling these facts, we can easily compute the virtual motive of this open substack:

$$[\mathfrak{M}_{\mathrm{inv}}]_{\mathrm{vir}} = \frac{[X]_{\mathrm{vir}}}{[\mathbb{C}^*]_{\mathrm{vir}}} = \frac{[\mathbb{A}^1 \setminus \{0\}]_{\mathrm{vir}}[\mathbb{P}^2 \setminus Y]_{\mathrm{vir}}}{[\mathbb{C}^*]_{\mathrm{vir}}}$$

On the other hand, the coefficient of $\frac{t^r}{1-t^r}$ in the exponential $U_{Q,W}(t)$ above is given by

$$\frac{-\mathbb{L}^{-\frac{3}{2}} \cdot \mathbb{L}(\mathbb{L}-1)^2}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} = \frac{[\mathbb{A}^1]_{\rm vir}[\mathbb{P}^2 \setminus Y]_{\rm vir}}{[\mathbb{C}^*]_{\rm vir}}.$$

Thus this coefficient is the virtual motive of a line bundle over $\mathbb{P}^2 \setminus Y$, rather than a $\mathbb{C}^*$-bundle. We expect that this discrepancy can be explained by considering the closure $\overline{\mathfrak{M}_{\rm inv}} \subset \mathfrak{M}(J_{Q_1,W_q}, r)$.

### 3.4.2 The conjectures

Notice that in all cases studied in the previous section, the answer had the general rational form

$$U_{Q_1,W}(t) = \operatorname{Exp}\left( -\sum_{i=1}^{k} \frac{M_i}{\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}} \frac{t^{m_i}}{1-t^{m_i}} \right), \tag{7}$$

where $m_1 = 1, \ldots, m_k \in \mathbb{N}$ are the dimensions in which there exist *simple* modules for the algebra $J_{Q_1,W}$, and $M_i \in \mathcal{M}$ are motivic expressions without denominators, with $M_1$ being the virtual motive of the scheme parametrizing one-dimensional simple modules.

For the homogenized Weyl case, we have

$$J_{Q_1,W_{hW}} = \mathbb{C}\langle x,y,z\rangle/([x,y]-z^2,[x,z],[y,z]).$$

**Lemma 3.9.** *Every finite-dimensional simple module for $J_{Q_1,W_{hW}}$ has dimension one.*

*Proof.* For $n \geq 0$, we have

$$z^{n+2} = z^n[x,y] = [z^n x, y] - [z^n, y]x = [z^n x, y]$$

in $J_{Q_1,W_{hW}}$. Thus $z^{n+2}$ is a commutator for $n \geq 2$. The rest of the proof follows that of Lemma 3.7 above. □

One-dimensional representations of $J_{Q_1,W_{hW}}$ are parametrized by the (scheme-theoretic) critical locus of the function $f_{hW} = z^3$ on $\mathbb{C}^3$, the double plane $\{z^2 = 0\}$. The motivic vanishing cycle of $f_{hW}$ is $\mathbb{L}^2(1-[\mu_3])$, and thus the corresponding virtual motive is $-\mathbb{L}^{\frac{1}{2}}(1-[\mu_3])$. Using Lemma 3.9, formula (7) turns into Conjecture 3.3 above.

Let us finally turn to the Sklyanin algebras, with potential

$$W_{a,b,c} = a\,xyz + b\,xzy + \frac{c}{3}(x^3+y^3+z^3).$$

The only one-dimensional representation of these algebras is the trivial one, but the moduli space is highly non-reduced; it is the critical locus of the function

$$f(x,y,z) = (a+b)xyz + \frac{c}{3}(x^3+y^3+z^3),$$

i.e. it is the scheme-theoretic singular locus of a simple elliptic surface singularity of type $\widetilde{E}_6$, and therefore has length eight.

One-dimensional representations have the following motivic DT invariant:

**Lemma 3.10.** *The virtual motive for the moduli space of one-dimensional representations is given by*

$$[\mathfrak{M}(J_{Q_1,W_{a,b,c}},1)]_{\mathrm{vir}} = \mathbb{L}^{-\frac{3}{2}}\left([S_{\mathrm{DT}},\mu_3] - [E_{\mathrm{DT}}](\mathbb{L}-1) - 1\right)$$

*where, as before, $S_{\mathrm{DT}}$ is the affine triple cover of $\mathbb{P}^2 \setminus E_{\mathrm{DT}}$ with its canonical action of $\mu_3$.*

*Proof.* The function $f$ has an isolated critical point at the origin, and $f^{-1}(0)$ is the cone over the elliptic curve $E_{\mathrm{DT}} \subset \mathbb{P}^2$. Blowing up the origin in $\mathbb{C}^3$ gives a normal crossings resolution of $f$ on $X = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^2}(-1))$, whose irreducible components are given by the zero section (with multiplicity three), and the total space of $\mathcal{O}_{E_{\mathrm{DT}}}(-1)$. The result is then an immediate consequence of Denef and Loeser's formula [17, Thm. 3.3]. □

To motivate Conjecture 3.4, we recall some basic facts about simple representations of the Sklyanin algebras [2, 3, 16, 35]. Let $r = |\sigma|$ be the order of the translation $\sigma : E_{\mathrm{pt}} \to E_{\mathrm{pt}}$. Then higher-dimensional simple representations exist if and only if $r < \infty$. In this case there, are explicit bounds on the dimensions [35], which depend on whether of not $r$ is divisible by three. Combining these bounds with (7), explains the rough shape of the expressions appearing in Conjecture 3.4.

When $r$ is not divisible by three, somewhat more information is available. In this case, all nontrivial simple representations have dimension $r$. As in the discussion following Lemma 3.10, we can consider the moduli stack $\mathfrak{M}_{\mathrm{inv}}$ of simple $r$-dimensional representations on which a certain cubic central element $g$ is invertible. Its coarse moduli space is a $\mathbb{C}^*$-bundle over the complement of the elliptic curve $E_{\mathrm{pt}}/\sigma \subset \mathbb{P}^2$. The analogy between $E_{\mathrm{pt}}/\sigma$ and the triangle $Y \subset \mathbb{P}^2$ in (**??**) explains the appearance of the isogenous curve in part (ii) of the conjecture.

### 3.4.3 Interpreting the results for the conifold quiver

For the undeformed conifold, the formula of Theorem 3.5 includes [28, Thm.2.1]:

$$\begin{aligned} U_{Q,W}(t_1,t_2) &= \mathrm{Exp}\left(\frac{(\mathbb{L}+\mathbb{L}^2)t_1t_2 - \mathbb{L}^{\frac{1}{2}}(t_1+t_2)}{\mathbb{L}-1}\sum_{n\geq 0}(t_1t_2)^n\right) \\ &= \mathrm{Exp}\left(-\frac{-(\mathbb{L}^{\frac{3}{2}}+\mathbb{L}^{\frac{1}{2}})}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t_1t_2}{1-t_1t_2}\right. \\ &\qquad \left. -\frac{1}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t_1}{1-t_1t_2} - \frac{1}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t_2}{1-t_1t_2}\right). \end{aligned}$$

In this case, there are two vertex simples, the representations with dimension vectors $(1,0)$ and $(0,1)$; there are also simples of dimension vector $(1,1)$. We are unable to give a systematic explanation of the denominators of these rational functions: for all three dimension vectors, we get the denominator $1 - t_1t_2$. The numerators are clear: they simply record the dimension vectors. As for the motivic coefficitents, for $(1,0)$ and $(0,1)$ the moduli space is a reduced point, in agreement with the coefficients above. On the other hand, it can be checked that a representation with dimension vector $(1,1)$ is simple if and only if there is a nonzero arrow in each direction, and thus the parameter space of simple representations is $(\mathbb{C}^2 \setminus \mathrm{pt})^2/\mathbb{C}^*$ which is the complement of the zero section in the resolved conifold. Thus its virtual motive is $-\mathbb{L}^{-\frac{3}{2}}(\mathbb{L}+1)^2(\mathbb{L}-1)$ which is close to, but not quite, the numerator above. Instead, the numerator $-(\mathbb{L}^{\frac{3}{2}}+\mathbb{L}^{\frac{1}{2}})$ is the virtual motive of the resolved (commutative) conifold $X$, the resolution $\pi\colon X \to Z$ of the singular conifold $Z = \operatorname{Spec}\mathbb{C}[x,y,z,t]/(xt-yz)$.

For the generic deformed conifold, we have

$$U_{Q_{\mathrm{con}},W_q} = \operatorname{Exp}\left(\frac{3\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t_0t_1}{1-t_0t_1} - \frac{1}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{t_0+t_1}{1-t_0t_1}\right).$$

Once again, the numerator of the first term is *not* the motive of the space of simples of dimension vector $(1,1)$. But it can be interpreted geometrically in the following way. We begin by considering the global function $f = xt = yz$ on the conifold singularity $Z$, and its pullback $g = \pi^*(f)$ along the resolution $\pi : X \to Z$. A straightforward calculation using the standard charts on the conifold shows that the virtual motive of $\operatorname{crit}(g)$ is the desired expression $-3\mathbb{L}^{\frac{1}{2}}+\mathbb{L}^{-\frac{1}{2}}$.

Now, since $X$ is a smooth Calabi–Yau threefold, a choice of global holomorphic volume form gives rise to an isomorphism $\Omega^1_X \cong \wedge^2 T_X$ between forms and bivectors. Under this isomorphism, the form $dg$ gives a Poisson structure $\pi \in H^0(X, \wedge^2 T_X)$. Via the derived equivalence $D(X) \cong D(J_{Q_{\mathrm{con}},W_q})$ for $q = 1$, the $q$-deformation of the Jacobi algebra corresponds to a noncommutative deformation of $X$ that we expect to be a deformation quantization of this Poisson structure. In particular, the critical points of $g$ are exactly the zero-dimensional symplectic leaves of $\pi$, which are precisely the points one expects to quantize to skyscraper sheaves on the noncommutative deformation.

### 3.4.4 Interpreting results for the cyclic quiver

We will be brief, since the interpretations we can offer are analogous to the cases already studied. We refer for the details to [10]. As usual, at $q = 1$ the formula in Theorem 3.6 recovers the result of Bryan and Morrison [9, 27]

$$U_{Q_{n+1},W_1}(t) = \operatorname{Exp}\left(\frac{\mathbb{L}^{\frac{3}{2}}+(n-1)\mathbb{L}^{\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{y^\delta}{1-y^\delta} + \sum_{\alpha\in\Delta}\frac{\mathbb{L}^{\frac{1}{2}}}{\mathbb{L}^{\frac{1}{2}}-\mathbb{L}^{-\frac{1}{2}}}\frac{y^\alpha}{1-y^\delta}\right).$$

The first coefficient here, up to sign, is the motive of the (unique) crepant resolution of the quotient singularity $\operatorname{Spec} Z$. The other coefficients are all virtual motives of the affine line. For $\alpha = \delta_i$, it is indeed clear that the moduli space is simply the affine line, parameterized by the value of the loop arrow $b_i$. For generic $q$ on the other hand, the coefficient $(n+1)\mathbb{L}^{\frac{1}{2}} - \mathbb{L}^{-\frac{1}{2}}$ of the first term in Theorem 3.6 can again be interpreted, up to sign, as the virtual motive of the zero-set of a natural one-form on the crepant resolution.

# 4 Proofs

## 4.1 Quivers with cuts

A subset $I \subset E(Q)$ is called a cut of $(Q, W)$, if the potential $W$ is homogeneous of degree 1 for the arrow grading of $Q$ where arrows in $I$ have degree 1 and arrows not in $I$ have degree zero. Given a cut $I$ of $(Q, W)$, let $Q_I = (V(Q), E(Q)\backslash I)$, and let $J_{W,I}$ be the quotient of $\mathbb{C}Q_I$ by the ideal

$$(\partial_I W) = (\partial W/\partial a, a \in I).$$

Then by [28, Prop.1.7],

$$U_{Q,W}(t) = \sum_{\alpha\in\mathbb{N}^{V(Q)}} (-\mathbb{L}^{\frac{1}{2}})^{\chi(\alpha,\alpha)+2d_I(\alpha)} \frac{[R(J_{W,I},\alpha)]}{[G_\alpha]} t^\alpha, \tag{8}$$

where $d_I(\alpha) = \sum_{(a:i\to j)\in I} \alpha_i\alpha_j$ for any $\alpha \in \mathbb{Z}^{V(Q)}$.

## 4.2 Deformations of affine three-space

### 4.2.1 Quantum affine three-space

In this section, we prove Theorem 3.1. The proof will make heavy use of the assumption $q \neq 0$. As already observed, this is precisely the condition for the Jacobian algebra $J_{Q_1,W_q}$ to be 3-Calabi–Yau, although we will not use this fact directly.

We begin by applying the cut $I = \{z\}$ to $(Q_1, W_q)$, which reduces the problem to studying representations of the algebras

$$\mathbb{C}_q[x, y] = \mathbb{C}\langle x, y\rangle / (xy - qyx)$$

for $q \in \mathbb{C}^\times$. More precisely, using formula (8), we have

$$U_{Q_1,W_q}(t) = \sum_{n\geq 0} \frac{[R_q(n)]}{[\mathrm{GL}(n)]} t^n$$

where the variety

$$R_q(n) = \{(A, B) \in \mathrm{End}(V) \times \mathrm{End}(V) : AB = qBA\}$$

is the set of pairs of $q$-commuting $(n \times n)$-matrices, and $V = \mathbb{C}^n$ is a fixed $n$-dimensional vector space. We think of $\mathbb{C}_q[x, y]$ geometrically as functions on a quantum plane $\mathbb{A}^2_q$, so that finite-dimensional representations of $\mathbb{C}_q[x, y]$ correspond to torsion coherent sheaves on $\mathbb{A}^2_q$; compare [9].

For the calculation, it will be useful to consider the four subvarieties

$$R_q^{I,I}(n) = \{(A, B) \in R_q(n) \,|\, A, B \text{ are invertible}\}$$
$$R_q^{I,N}(n) = \{(A, B) \in R_q(n) \,|\, A \text{ is invertible and } B \text{ is nilpotent}\}$$
$$R_q^{N,I}(n) = \{(A, B) \in R_q(n) \,|\, A \text{ is nilpotent and } B \text{ is invertible}\}$$
$$R_q^{N,N}(n) = \{(A, B) \in R_q(n) \,|\, A, B \text{ are nilpotent}\}$$

The geometric interpretation of these subvarieties is as follows. The quantum plane $\mathbb{A}^2_q$ contains a privileged pair of commutative affine lines $L_x, L_y \subset \mathbb{A}^2_q$ corresponding to the two-sided ideals $(x), (y) \subset \mathbb{C}_q[x, y]$. These lines intersect in the point $0 \in \mathbb{A}^2_q$ corresponding to the ideal $(x, y)$. This gives a stratification of $\mathbb{A}^2_q$, and points of the varieties $R_q^{-,-}$ above correspond to sheaves on $\mathbb{A}^2_q$ whose supports are completely contained in one of the four strata.

Define the generating series

$$U_q^{I,I}(t) = \sum_{n \geq 0} \frac{[R_q^{I,I}(n)]}{[\mathrm{GL}(n)]} t^n,$$

and define $U_q^{N,I}, U_q^{N,N}$ and $U_q^{I,N}$ similarly. Then we have the following basic

**Lemma 4.1** (cf. [20, Lemma 1])**.** *For all $q \in \mathbb{C}^\times$, there is a factorisation*

$$U_{Q_1,W_q} = U_q^{I,I} \cdot U_q^{I,N} \cdot U_q^{N,I} \cdot U_q^{N,N}$$

*of the universal generating series.*

*Proof.* Suppose given a point $(A, B) \in R_q(n)$ corresponding to a representation of $\mathbb{C}_q[x, y]$ on the vector space $V = \mathbb{C}^n$. Then the kernel and image of $A^N$ for $N \gg 0$ decompose $V$ into direct summands on which $A$ acts nilpotently and invertibly, respectively. Since $q^N \neq 0$, the relation $A^N B = q^N B A^N$ implies that $B$ preserves the kernel and image of $A^N$. Decomposing further using the action of $B$, we obtain a canonical decomposition

$$V = V_{I,I} \oplus V_{I,N} \oplus V_{N,I} \oplus V_{N,N}$$

into subrepresentations, corresponding to sheaves whose supports lie on a single stratum in $\mathbb{A}^2_q$. The result now follows easily. □

**Lemma 4.2.** *The series $U_q^{N,N}$, $U_q^{N,I}$ and $U_q^{I,N}$ are independent of $q \in \mathbb{C}^\times$, namely*

$$U_q^{N,N}(t) = \mathrm{Exp}\left(\frac{1}{\mathbb{L} - 1} \frac{t}{1 - t}\right),$$

*and*

$$U_q^{N,I}(t) = U_q^{I,N}(t) = \mathrm{Exp}\left(\frac{t}{1 - t}\right)$$

*for all $q \in \mathbb{C}^\times$.*

*Proof.* The formulae in question for the case $q = 1$ are easily extracted from the results of [9], so it suffices to demonstrate that the series are independent of $q$. Moreover, since the equations defining $R_q(n) \subset \mathrm{End}(V) \times \mathrm{End}(V)$ are symmetric in the two factors, it is enough to show that the series $U_q^{N,I}$ and $U_q^{N,*} = U_q^{N,N} \cdot U_q^{N,I}$ are independent of $q$. Notice that, by the argument in Lemma 4.1, the series $U_q^{N,*}$ is the universal series for the sequence of varieties

$$R_q^{N,*}(n) = \{(A, B) \in C_q(n) \,|\, A \text{ is nilpotent}\}.$$

So, we must show that the motivic classes $[R_q^{N,*}(n)]$ and $[R_q^{N,I}(n)]$ are independent of $q$. To do so, we consider the maps from $R_q^{N,*}$ and $R_q^{N,I}$ to the nilpotent cone in $\mathrm{End}(V) \cong \mathfrak{gl}(n, \mathbb{C})$, given by projection on the first factor. We will show that, when restricted to a fixed nilpotent orbit, these maps are Zariski-locally trivial fibrations, and that their fibres are independent of $q$.

To this end, choose a nilpotent orbit $\mathcal{O} \subset \mathrm{End}(V)$ and a matrix $A_0 \in \mathcal{O}$. Since $A_0$ is nilpotent and $q \neq 0$, we may choose an element $P \in \mathrm{GL}(V)$ such that $PA_0P^{-1} = q^{-1}A_0$. Let $H \subset \mathrm{GL}(V)$ be the stabilizer of $A_0$ under the conjugation action, and let $\mathfrak{h} \subset \mathrm{End}(V)$ be its Lie algebra. Acting by conjugation on $A_0$, we have a principal bundle $\mathrm{GL}(V) \to \mathcal{O}$ with structure group $H$, which is Zariski-locally trivial since $H$ is special. Given a local section $s : U \to \mathrm{GL}(V)$ over a subvariety $U \subset \mathcal{O}$, consider maps

$$U \times H \to R_q^{N,I}$$

and

$$U \times \mathfrak{h} \to R_q^{N,*},$$

both defined by the formula

$$(A, h) \mapsto (A, s(A)hPs(A)^{-1}).$$

One easily checks that these maps give local trivializations of $R_q^{N,I}$ and $R_q^{N,*}$ over $U$. Since the fibres are independent of $q$, the lemma follows. □

In order to complete the proof of Theorem 3.1, it remains to compute the series $U_q^{I,I}$. We remark that the varieties $R_q^{I,I}$ parametrize modules over the localized algebra $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$, which we can think of as functions on the quantum torus $T_q$ obtained by removing the lines $L_x, L_y \subset \mathbb{A}_q^2$. As is well known, the algebra $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ is isomorphic to the skew group ring $\mathbb{Z} * (\mathbb{C}[x, x^{-1}])$ associated with the $\mathbb{Z}$-action on the variety $\mathbb{C}^*$, where the generator $1 \in \mathbb{Z}$ acts by multiplication by $q$. If we denote by $[\mathbb{C}^*/\mathbb{Z}]_q$ the quotient stack, then finite-dimensional modules over $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ are equivalent to torsion coherent sheaves on $[\mathbb{C}^*/\mathbb{Z}]_q$.

When $q$ is not a root of unity, the orbits of the $\mathbb{Z}$-action are infinite, and hence there can be no nontrivial equivariant sheaves of finite length. We therefore have

$$U_q^{I,I}(t) = 1$$

if $q$ is not a root of unity.

On the other hand, if $q$ is a primitive $r$th root of unity, we have an isomorphism of stacks $[\mathbb{C}^*/\mathbb{Z}]_q \cong [\mathbb{C}^*/\mathbb{Z}]_1$ induced by the $r$th-power map $\mathbb{C}^* \to \mathbb{C}^*$ and the inclusion $\mathbb{Z} \cong r\mathbb{Z} \subset \mathbb{Z}$ of the stabilizer of the action. Thus, there is an equivalence between finite-length sheaves on $T_q$ and finite-length sheaves on the commutative torus $T_1 = \mathrm{Spec}\,\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$. Since pulling back along the $r$th power map multiplies the length of a sheaf on $\mathbb{C}^*$ by $r$, this equivalence takes $n$-dimensional representations of $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ to $rn$-dimensional representations of $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$. We therefore have

$$U_q^{I,I}(t) = U_1^{I,I}(t^r) = U_1^{N,N}(t^r)^{(\mathbb{L}-1)^2} = \mathrm{Exp}\left((\mathbb{L}-1)\frac{t^r}{1-t^r}\right),$$

where the last two identities use the power structure and the results of [9].

#### 4.2.2 The Jordan plane

We now prove Theorem 3.2. Applying a cut along $I = \{z\}$ again, we reduce to representations of the algebra

$$\mathbb{C}_J[x, y] = \mathbb{C}\langle x, y\rangle/(xy - yx - y^2),$$

the ring of functions on the Jordan plane. We define the representation varieties

$$R_J(n) = \left\{(A, B) \in \mathrm{End}(V) \times \mathrm{End}(V) \,\middle|\, [A, B] = B^2\right\}$$

for $n \geq 0$, where once again $V$ denotes a fixed $n$-dimensional vector space. Using (8), we have the equality

$$U_{Q_1,W_J}(t) = \sum_{n\geq 0} \frac{[R_J(n)]}{[\mathrm{GL}(n)]} t^n.$$

The series on the right can be easily computed using the results in the previous section.

Indeed, if $(A, B) \in R_J(n)$ then $B$ is nilpotent, as observed in Lemma 3.7; see also [22, Lemma 2.1]. Projection on the second factor therefore gives a map from $R_J(n)$ to the nilpotent cone in $\mathrm{End}(V)$. Over a fixed nilpotent orbit $\mathcal{O} \subset \mathfrak{gl}_n$ this map is an affine bundle for the vector bundle over $\mathcal{O}$ whose fibre at $B \in \mathcal{O}$ is the centralizer of $B$ in $\mathrm{End}(V)$. Hence $R_J(n)$ has the same motivic class as the variety of pairs of commuting matrices, the second of which is nilpotent. But these varieties are precisely those considered in the proof of Lemma 4.2 in the case $q = 1$. We therefore conclude that

$$U_{Q_1,W_J}(t) = U_q^{N,I}(t) \cdot U_q^{N,N}(t) = \mathrm{Exp}\left(\frac{\mathbb{L}}{\mathbb{L}-1}\frac{t}{1-t}\right),$$

proving Theorem 3.2.

### 4.3 The deformed conifold

In this section, we sketch the proof of Theorem 3.5, which follows that of Theorem 3.1 and [28, Sect. 2.2]. We refer the reader to [10] for full details.

Using $I = \{a_1\}$ as the cut, we are lead to considering representations of the quiver in Figure 4, with the single relation $b_1ab_2 = qb_2ab_1$, where $a = a_2$.

Thus, to compute the generating series, we must consider the varieties

$$\{(A, B_1, B_2) \in \mathrm{Hom}(V_1, V_2) \times \mathrm{Hom}(V_2, V_1)^{\times 2} \colon B_1AB_2 - qB_2AB_1 = 0\},$$

where $d = (d_1, d_2)$ is a dimension vector for the quiver $Q_2$, and $V_i$ are fixed vector spaces of dimension $d_i$. Given an element $(A, B_1, B_2)$ of such a variety, consider the linear map

$$A_2 \oplus B_2 \in \mathrm{Hom}(V_0, V_1) \oplus \mathrm{Hom}(V_1, V_0) \subset \mathrm{End}(V),$$

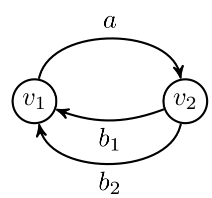


Figure 4: The cut of $Q_2$ along $a_1$

where $V = V_0 \oplus V_1$. As in Lemma 4.1, the relation implies that we can decompose $V = V^N \oplus V^I$ into subrepresentations on which $A \oplus B_2$ acts nilpotently and invertibly, respectively, so that the generating series factors

$$U_{Q_2,W_q} = U^N \cdot U^I$$

into nilpotent and invertible contributions.

Once again, one shows that the series $U^N$ is independent of $q$, while the computation of the series $U^I$ can be reduced to the study of the $q$-commuting varieties of Section 4.2.1. Combining that calculation with the formulae in [28, Sect. 2.2] for the undeformed conifold yields the result.

### 4.4 The cyclic quiver

The proof of Theorem 3.6 proceeds analogously to the conifold case, using dimensional reduction and appropriate splittings, reducing the calculation to the case $q = 1$ already done in [9, 27]. Once again, we refer for the details to [10].

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