

ON THE INVESTIGATION OF UTILITY FUNCTIONS ON OPTIMAL SENSOR LOCATIONS

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Abstract

Structural Health Monitoring uses data collected from sensors placed on structures to determine their operating condition and whether maintenance is required. Often, optimal sensor placement strategies are used to find the optimal locations for the identification of their modal properties, structural parameters and/or abnormal behaviours under the influence of model and measurement uncertainty. An approach that has been frequently used to solve the problem of sensor placement is the Bayesian experimental design. This approach chooses the locations using the data measured by the sensors to reduce the prior uncertainty of the parameters that are being inferred. The Bayesian experimental design minimizes the uncertainty of the parameters to be inferred through the use of metrics called utility functions. Most of these metrics are based on functions of the posterior distribution. In this paper, the use of three utility functions (Bayesian D-posterior precision, Bayesian A-posterior precision, and Expected Information Gain) is investigated for the problem of sensor placement.

The case study chosen consists of a beam with translational and rotational springs connected to the ground subject to an impulsive load. The goal of the analysis is to select the most informative position of a sensor in order to update the distribution of two uncertain physical parameters of the beam based on natural frequencies extracted using the Eigensystem Realization Algorithm. It is shown that for the case investigated, the three utility functions yield the same optimal sensor location.

Keywords: Optimal Sensor Placement, Uncertainty Quantification, Structural Health Monitoring.

1 INTRODUCTION

Structural Health Monitoring (SHM) often focuses on non-intrusive structure damage detection [1]. It can be used to provide early warnings on the health status of engineering systems. The equipment required for implementing SHM, includes sensors and data acquisition systems. The real time information obtained from the sensors has to be post-processed and statistical procedures are implemented to detect anomalies and suggest preventive actions [1,2]. Technological advances in sensor monitoring allow the development of optimal sensor strategies that have made SHM cost effective and easier to implement.

Structural parameters are usually inferred from the sensorial data (such as velocity or acceleration measurements), especially the modal parameters (natural frequency, damping ratio and mode shape). The inferred parameters are then used to assess the acceptability of the models of structures and to evaluate the structures' condition. For some cases, local forms of damage may be identified by a shift in the modal properties [3]. The position of the sensor can strongly influence the inference on the structural parameters. This has led to the widespread development of optimal sensor strategy techniques [4]. Broadly speaking, the sensor placement framework methods can be split into methods based on information theory or non-information methods [4]. The non-information-based methods are not discussed in this paper, however, more information on these techniques can be found in [4]. Work based on information theory heavily relies on the application and development of the general Bayesian framework [5]. This framework was proposed for system identification in [5,6] and it has been consequently extended to the problem of sensor placement [7–10]. The main objective is the selection of the location and number of sensors that maximises the information needed to estimate the uncertain parameters [8]. The research challenges linked to these approaches are the definition of the metric to be used to assess the different configurations of sensors (number and location of the sensors) and the choice of the most adequate optimisation technique [11,12].

In this paper a Bayesian experimental design framework [13] is used to solve the sensor placement problem. Within this framework, the number and locations of the sensors are chosen by using the data obtained from the sensors to reduce the prior uncertainty on the parameters to be inferred. Therefore, the framework's focus is the minimization of the uncertainty of certain physical parameters of interest to the practitioner by comparing different metrics, the so-called utility functions. Two physical parameters of a beam attached to ground using translational and rotational springs subject to an impulsive load have to be inferred by using a single sensor. The beam is investigated by building a Finite Element model. Numerical simulations of the dynamic response (velocity signal) at different locations are used to obtain numerical 'measurements' of possible sensor locations. An intermediate step requires the post-processing of the numerical 'measurements' to obtain the modal parameters that are subsequently used as the data used to reduce the model parametric uncertainty via Bayesian model updating. This is achieved by using the Eigensystem Realization Algorithm. Model updating is then used to obtain the posterior probability density function of the parameters to be inferred, having assigned a uniform prior distribution and applying Monte Carlo sampling-based strategies. The obtained posterior is used to evaluate a utility function that is then used to select the optimal sensor location. Three utility functions are investigated.

2 BAYESIAN OPTIMAL DESIGN FRAMEWORK

Bayesian optimal design [13] allows the designation of resources required to obtain information for reduction of systematic error, inference of unknown parameters (i.e., reduction of prior uncertainty), obtaining future predictions and the comparison of models chosen to represent a

system [13]. The framework's objective is the maximisation of the information obtained from a set of measurements for the inference of the unknown parameters of the model used to describe the physical system [13]. The choice of the optimal design improves parameter inference and reduces the experimental costs.

Lindley proposed a unifying theory of Bayesian optimal design in [14]. The definition of the best possible design given a set of objectives and restrictions is described by a utility function. The maximization of this function is used to choose the possible design that measures how well the set of objectives and restrictions are obeyed [15].

One of the major challenges in Bayesian optimal design methods is the reduction of their high computational cost incurred in the calculation of their utility functions [13]. This is because the utility functions require the knowledge of the posterior distribution of the parameters to be inferred. These distributions are dependent on the set of measurements available and therefore are different when different designs are considered.

2.1 Bayesian Framework

Probability density functions are used to model the uncertain model parameters in the Bayesian inference framework [5,16]. The prior knowledge on the uncertain parameters before any measurements or data is obtained, is described by the prior density function $P(\boldsymbol{\theta})$. The likelihood function $p(\mathbf{y} | \boldsymbol{\theta})$, is normally assumed to follow a specific distribution (e.g., Gaussian). The $p(\mathbf{y} | \boldsymbol{\theta})$ measures the degree of suitability of the model to justify the obtained measurements. The denominator $p(\mathbf{y})$ of eq.(1) below is the evidence pdf and normalizes the pdf of the posterior. If the above described pdfs are known, the eq.(1) can be used to calculate the so-called posterior distribution $p(\boldsymbol{\theta} | \mathbf{y})$:

$$p(\boldsymbol{\theta} | \mathbf{y}) = \frac{p(\boldsymbol{\theta})p(\mathbf{y} | \boldsymbol{\theta})}{p(\mathbf{y})} \quad (1)$$

The posterior distribution obtained can then be used to determine the utility function. Hence, it is important to obtain accurate estimations of both location (median or mean) and scale (interquartile range or standard deviation) of the posterior [13]. Most frequently, it is not possible to express the posterior distributions with a closed form, so computational methodologies are used to obtain samples from the posterior or to approximate it [16–21]. In this work, the sampling-based model updating techniques are used. Specifically, the Sequential Monte Carlo (SMC) [16] sampling and the Transitional Markov Chain Monte Carlo (TMCMC) [20] are chosen to infer the two physical parameters of the case study investigated.

2.2 Bayesian Utility Functions

Many different utility functions have been developed for inferring parameters of a model [13]. Metrics that quantify the performance of experiments are obtained by using a set of utility functions that are maximized (or minimized) with the objective of identifying the optimal experiment [13]. Three utility functions are reviewed in what follows.

A well-known utility function is expressed as the inverse of the determinant of the posterior covariance matrix [13]. This utility function also known as the Bayesian D-posterior precision maximises the posterior precision of the model parameters to be inferred and it is given by [13]:

$$U_D(\mathbf{d}, \mathbf{y}) = \frac{1}{\det(\text{cov}(\boldsymbol{\theta}|\mathbf{d}, \mathbf{y}))} \quad (2)$$

Where \mathbf{d} is the vector that represents the experimental design to be optimized (e.g. the sensor positions for a given number of sensors).

Another useful utility function similar to the Bayesian D-posterior precision is given by the inverse of the trace of the posterior covariance matrix [15]. This utility function also known as the ‘Bayesian A-posterior precision’ maximises the marginal posterior precision of the model parameters to be inferred and it is given by [15]:

$$U_A(\mathbf{d}, \mathbf{y}) = \frac{1}{\text{trace}(\text{cov}(\boldsymbol{\theta}|\mathbf{d}, \mathbf{y}))} \quad (3)$$

Alternatively, the utility function may be expressed as the expected Kullback–Leibler (KL) divergence from the posterior distribution to the prior distribution [14]. The expected KL divergence utility function is also known as Expected Information Gain (EIG) over the parameters to be inferred [22], and it is expressed as:

$$U_{EIG}(\mathbf{d}) = E_{\mathbf{y}} \left[D_{KL} \left(p_{\boldsymbol{\theta}|\mathbf{y}} \parallel p_{\boldsymbol{\theta}} \right) \right] = \int_{\mathcal{Y}} \int_{\mathcal{Q}} p(\boldsymbol{\theta}|\mathbf{y}) \log \left(\frac{p(\boldsymbol{\theta}|\mathbf{y})}{p(\boldsymbol{\theta})} \right) p(\mathbf{y}) d\mathbf{y} d\boldsymbol{\theta} \quad (4)$$

Where $E_{\mathbf{y}}$ is the expectation with respect to the measurements \mathbf{y} , $D_{KL} \left(p_{\boldsymbol{\theta}|\mathbf{y}} \parallel p_{\boldsymbol{\theta}} \right)$ is the KL-divergence from the posterior distribution to the prior distribution, \mathcal{Y} and \mathcal{Q} are the support of the measurements \mathbf{y} and the parameters to be inferred $\boldsymbol{\theta}$ respectively.

The EIG can be interpreted as a non-linear generalization of the Bayesian D-optimal utility function [23]. It has been found [24] that this metric can be approximated using a Monte Carlo approach:

$$\tilde{U}(\mathbf{d}) = \frac{1}{N_{out}} \sum_{i=1}^{N_{out}} \left\{ \ln \left[p(\mathbf{y}^i | \boldsymbol{\theta}^i, \mathbf{d}) \right] - \ln \left[p(\mathbf{y}^i | \mathbf{d}) \right] \right\} \quad (5)$$

$$p(\mathbf{y}^i | \mathbf{d}) \approx \frac{1}{N_{in}} \sum_{j=1}^{N_{in}} p(\mathbf{y}^i | \boldsymbol{\theta}^j, \mathbf{d}) \quad (6)$$

Where N_{out} is the number of samples used in the outer loop and N_{in} is the number of samples used in the inner loop of the Monte Carlo approximations. The samples are obtained from the prior distribution and the likelihood is evaluated for these samples.

The number of likelihood function evaluations can be reduced if $N = N_{in} = N_{out}$, so that [24,25]:

$$\hat{U}(\mathbf{d}) = \frac{1}{N} \sum_{i=1}^N \left\{ \ln \left[p(\mathbf{y}^i | \boldsymbol{\theta}^i, \mathbf{d}) \right] - \ln \left(\frac{1}{N} \sum_{j=1}^N p(\mathbf{y}^i | \boldsymbol{\theta}^j, \mathbf{d}) \right) \right\} \quad (7)$$

However, this result is a biased estimate of the EIG [24,25]. A large number of samples may be required if the prior assumed has a large support at regions of low probability density, as this results in arithmetic underflow [26].

Another way to calculate the EIG is by calculating the difference between the differential entropy of the prior $h(\boldsymbol{\theta})$ and the differential entropy of the posterior $h(\boldsymbol{\theta} | \mathbf{y}, \mathbf{d})$ [15]:

$$U_{EIG,2}(\mathbf{d}) = I(\boldsymbol{\theta}, \mathbf{y}) = h(\boldsymbol{\theta}) - h(\boldsymbol{\theta} | \mathbf{y}, \mathbf{d}) \quad (8)$$

The calculation of the differential entropy using samples from the posterior distribution can be approximated using the recursive copula splitting approach given in [27].

3 NUMERICAL RESULTS

A beam connected to the ground via two sets of translational and rotational spring positioned along the length of the beam, as shown on fig.1, is investigated in this paper. This simple case study has been chosen as it can represent a variety of practical situations where a component is attached to some fixtures, but there are uncertainties that may due to its assembly, boundary conditions and/or manufacture. In particular, in this case, the location of the first set of springs and the magnitude of the rotational spring are investigated. A prior distribution is assigned to each of these two parameters. The goal of the analysis is to select the most informative position of a sensor in order to update the distribution of these two uncertain parameters. The utility functions defined in section 2 are used to assess the optimal position of the sensor.

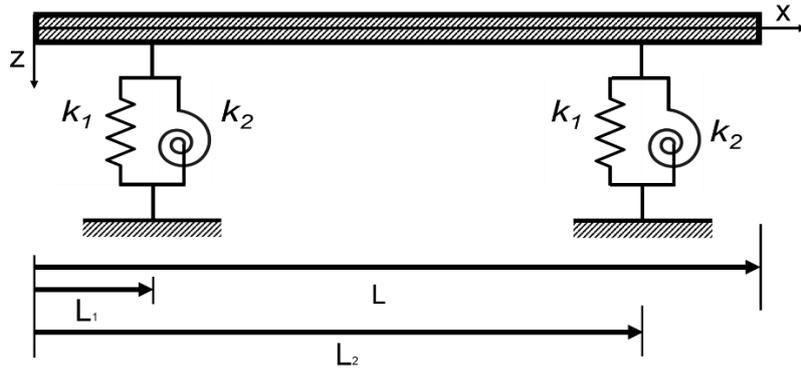


Figure 1: Beam attached to ground by translational and rotational springs.

The following geometric and material properties were used: L (length)=0.6m; b (base)=0.04m; h (height)=0.003m; ρ (density) =8000kg/m³; E (Young's modulus) =100GPa; k_1 (translational spring stiffness) = 1kN/m; k_2 (rotational spring stiffness) = 101.7Nm/rad; L_1 (length to springs) = 0.181m; L_2 (length to springs) = 0.4m. Modal damping was introduced into the system ($\eta=0.01$ for all modes). A force F (triangular pulse of length 10ms and maximum amplitude of 50N) applied at length L is used to excite the beam. The parameters to be inferred are the stiffness k_2 of both rotational springs and the location L_1 of the first rotational spring.

A Finite Element (FE) model is used to calculate the transversal velocity signals of the beam at several locations, to investigate the effects of the position of a single sensor on the utility functions. In particular, a 2-dimensional Euler-Bernoulli beam model is considered. This is discretized uniformly using 200 Euler-Bernoulli beam FEs with 2 degrees of freedom per node. Moreover, to simulate experimental conditions, for each transversal velocity signal measure at each node point, ten different realizations are created contaminating each signal using a white Gaussian noise with a noise to signal ratio (rms) of 5%.

The numerically contaminated velocity signals obtained at each possible sensor location (where the locations available are the ones at each node of the FE system) are post-processed using the Eigensystem Realization Algorithm (ERA) [28] to calculate the modal properties. Therefore, it was required to apply ERA 200 times to cover all the possible sensor locations in the system. These modal properties are then used as the data observed in the likelihood function. The likelihood function is then approximated by using the kernel smoothing function (ksdensity function of MATLAB [29]) on the set of modal properties obtained from ERA using the 10 different realizations of the contaminated velocity signals for each possible sensor location. Uniform priors were used for both the stiffness (100Nm/rad to 103 Nm/rad) of the rotational springs and location (0.17m to 0.19m) of the rotational spring. The joint posterior distribution of k_2 and L_1 is calculated using two Bayesian model updating techniques [16]: Sequential Monte Carlo (SMC) sampling and the Transitional Markov Chain Monte Carlo (TMCMC). In the SMC sampling approach [16] the samples obtained from the prior were re-used in all possible sensor locations to reduce the amount of forward simulations needed and to investigate how the bias resulting from this approach could affect the calculation of the utility functions. The results obtained with this implementation of SMC were compared with the result obtained using the unbiased TMCMC [20] that required new simulations each time a possible sensor location was considered. While SMC required 20,000 forward simulations to obtain acceptable estimations of the posterior distribution, the TMCMC required only 6,000 simulations but each time a new sensor position was considered the forward simulations could not be reused.

Figures 2, 3 show the precision values obtained for the Bayesian D-posterior precision and Bayesian A-posterior precision utility functions as a function of a sensor location along the length of the beam when using SMC and TMCMC.

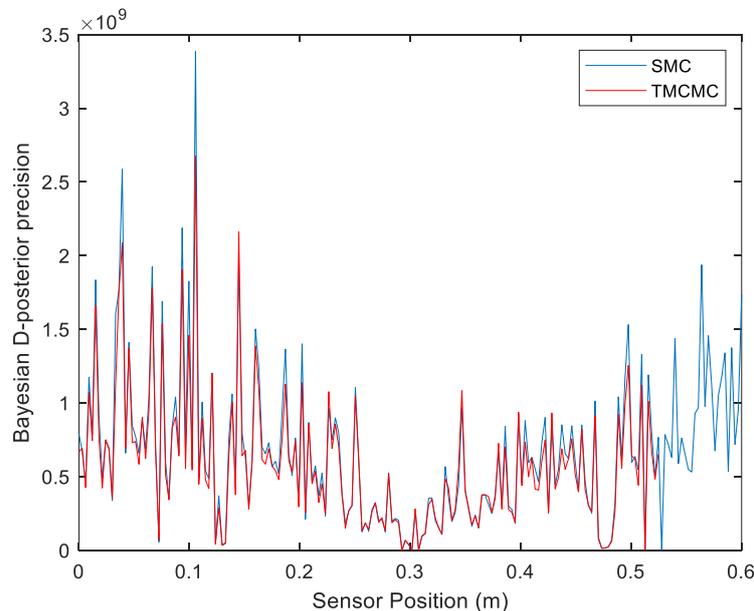


Figure 2: Bayesian D-posterior precision values vs sensor location.

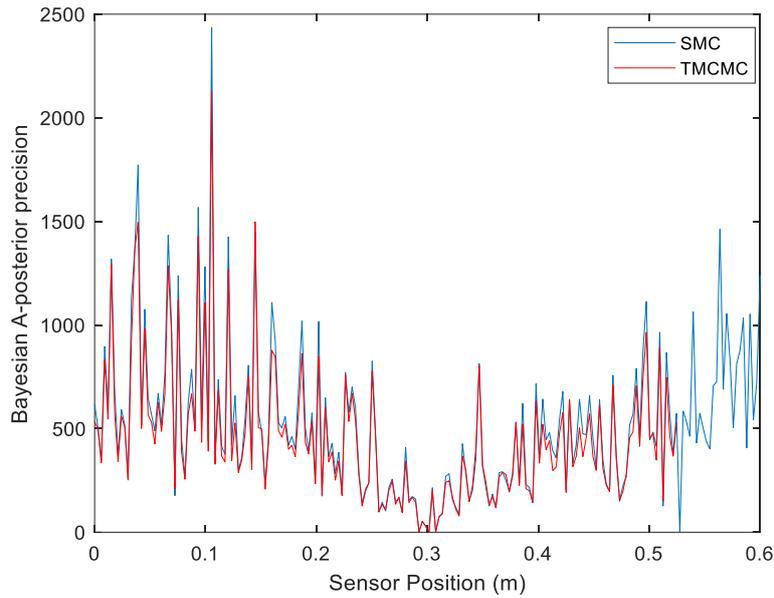


Figure 3: Bayesian A-posterior precision values vs sensor location.

The results obtained with the EIG utility function are shown in figure 4. These results were obtained by using the recursive copula splitting approach from [27] as using the Monte Carlo approximation shown in eq.(7) resulted in evaluating likelihoods at supports of low probability density which lead to arithmetic underflow.

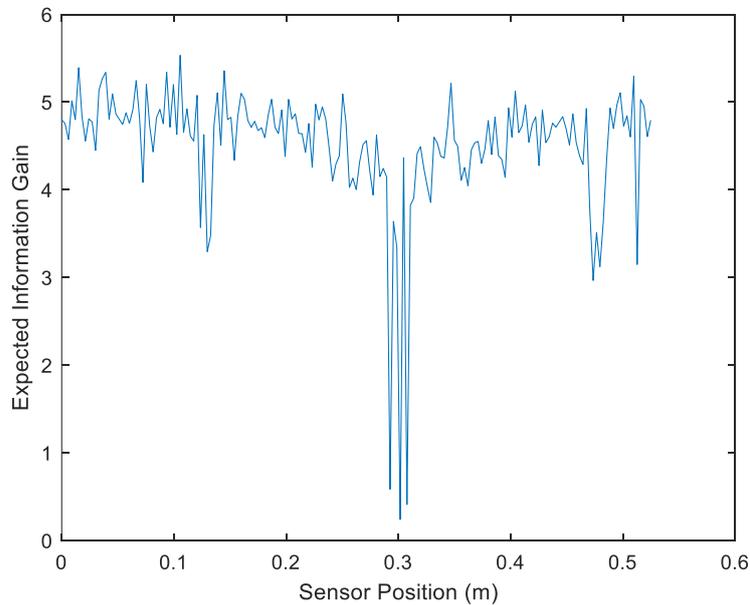


Figure 4: Expected information gain vs sensor location.

It can be observed that the three utility functions used have identified the same best sensor location - that is 0.106m. Locations where the utility values were low were close to nodal points and hence the modal properties resulting from ERA were less accurate. For this beam, the results obtained using the three different utility functions investigated, have been found to be

similar. However, it is expected that if a large number of physical parameters would have been inferred, the utility functions may have shown dissimilar results as the sensor location that maximises the joint posterior precision may have not been the same as the sensor location that maximises the marginal posterior precision.

It was also shown that for this case study the SMC and TMCMC provide similar results. The discrepancies found in the values of the utility function are largely due to the bias introduced by reusing samples from the prior and the choice of using a sequential importance sampling algorithm instead of a sequential importance resampling algorithm. If a sequential importance resampling algorithm had been chosen a lower bias would have been introduced in exchange for a higher computational cost.

4 CONCLUSIONS

The optimal sensor placement for the identification of two physical parameters of a beam attached to ground by translational and rotational springs has been investigated by considering three utility functions: Bayesian D-posterior precision and Bayesian A-posterior precision, and Expected Information Gain. It was shown that these utility functions led to the same best sensor location. As expected, poor values of the utility function were found for locations close to nodal points, as the modal properties estimated by ERA were less accurate. This result is expected as the measurements obtained at nodal points would not have as much information as other points along the beam system.

The utility functions chosen require the calculation of the posterior distribution. Therefore, the computational cost is reliant on the Bayesian inference technique being used. However, the choice of the inference technique usually shows a trade-off between computational cost and accuracy. Reusing samples, as in SMC techniques, may limit the amount of likelihood evaluations but this is at the risk of not evaluating samples close to regions of high probability densities. However, it was found that SMC and TMCMC lead to the same results for the case under investigation. The current challenge for the Bayesian optimal design approach would be the development of a fast inference technique that estimates the posterior at a limited computational cost.

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