Quantum Differentiability of Essentially Bounded Functions on Euclidean Space

Steven Lord, Edward McDonald, Fedor Sukochev, Dmitry Zanin

Abstract

We investigate the properties of the singular values of the quantised derivatives of essentially bounded functions on $\mathbb{R}^d$ with $d > 1$. The commutator $i[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]$ of an essentially bounded function f on $\mathbb{R}^d$ acting by pointwise multiplication on $L^2(\mathbb{R}^d)$ and the sign of the Dirac operator $\mathcal{D}$ acting on $\mathbb{C}^{2^{\lfloor d/2 \rfloor}} \otimes L^2(\mathbb{R}^d)$ is called the quantised derivative of f. We prove the condition that the function $x \mapsto \|(\nabla f)(x)\|_2^d := ((\partial_1 f)(x)^2 + \ldots + (\partial_d f)(x)^2)^{d/2}$, $x \in \mathbb{R}^d$, being integrable is necessary and sufficient for the quantised derivative of f to belong to the weak Schatten d-class. This problem has been previously studied by Rochberg and Semmes, and is also explored in a paper of Connes, Sullivan and Telemann. Here we give new and complete proofs using the methods of double operator integrals. Furthermore, we prove a formula for the Dixmier trace of the d-th power of the absolute value of the quantised derivative. For real valued f, when $x \mapsto \|(\nabla f)(x)\|_2^d$ is integrable, there exists a constant $c_d > 0$ such that for every continuous normalised trace φ on the weak trace class $\mathcal{L}_{1,\infty}$ we have $\varphi(|[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]|^d) = c_d \int_{\mathbb{R}^d} \|(\nabla f)(x)\|_2^d dx$.

Keywords: Quantum derivative; quantised calculus; essentially bounded function; trace formula; noncommutative geometry

---

1. Introduction

Let $d > 1$ be an integer, and let $x_1, x_2, \ldots, x_d$ be the coordinates of $\mathbb{R}^d$. Given a separable Hilbert space H, we denote the algebra of all bounded linear operators on H by $\mathcal{L}_\infty(H)$. The singular value function of a bounded operator $A \in \mathcal{L}_\infty(H)$ is defined by

$$\mu(t, A) = \inf\{\|A(1 - P)\| : P \text{ is a finite rank projection}, \mathrm{Tr}(P) \leq t\}, t \geq 0.$$

Preprint submitted to Journal of Functional Analysis *April 14, 2016*

The sequence $\{\mu(n,A)\}_{n=0}^{\infty}$ is called the sequence of singular values of the operator A. When A is a compact operator then $\mu(n,A)$, $n \geq 0$, is the $(n+1)$-th eigenvalue of the absolute value $|A|$ when the sequence of eigenvalues is arranged in decreasing order. We define the Schatten-Von Neumann space $\mathcal{L}_p(H)$, $p \in (0,\infty]$, as the subspace of operators in $\mathcal{L}_\infty(H)$ with a sequence of singular values in ℓ^p. Similarly the Schatten-Lorentz space $\mathcal{L}_{p,q}(H)$ is defined as the operators with singular values in $\ell^{p,q}$, for $p, q \in (0,\infty]$. When $p \neq \infty$ an operator $A \in \mathcal{L}_{p,q}(H)$ is compact. See [1, Chapter 4] for details on these spaces. We will suppress the dependence on H and write $\mathcal{L}_{p,q}$ when the Hilbert space is clear from context.

Given $A \in \mathcal{L}_{p,q}$ with a sequence of singular values $\{\mu(n,A)\}_{n=0}^{\infty}$, the quasinorm $\|A\|_{p,q}$ is defined to be the $\ell^{p,q}$ norm of $\{\mu(n,A)\}_{n=0}^{\infty}$.

For $j = 1,\ldots,d$, we define D_j to be the derivative in the direction x_j,

$$D_j = \frac{1}{i}\frac{\partial}{\partial x_j} = -i\partial_j.$$

When $f \in L^\infty(\mathbb{R}^d)$ is not a smooth function then $D_j f$ denotes the distributional derivative of f. We also consider D_j as a self-adjoint operator on $L^2(\mathbb{R}^d)$ with its standard domain of square integrable functions with a square integrable weak derivative in the direction x_j. This is equivalent to the closure of the symmetric operator D_j restricted to Schwartz functions. We use the notation $\nabla f = i(D_1 f, D_2 f, \ldots, D_d f)$ for an essentially bounded function $f \in L^\infty(\mathbb{R}^d)$. For a square integrable function f with a square integrable derivative in each direction we consider ∇ as an unbounded operator from $L^2(\mathbb{R}^d)$ to the Bochner space $L^2(\mathbb{R}^d, \mathbb{C}^d)$.

Let $N = 2^{\lfloor d/2 \rfloor}$. We use d-dimensional Euclidean gamma matrices, which are $N \times N$ self-adjoint complex matrices $\gamma_1, \ldots, \gamma_d$ satisfying the anticommutation relation,

$$\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}. \ \ 1 \leq j,k \leq d,$$

where δ is the Kronecker delta. The precise choice of matrices satisfying this relation is unimportant so we assume that a choice is fixed for the rest of this paper.

Using this choice of gamma matrices, we can define the d-dimensional Dirac operator,

$$\mathcal{D} = \sum_{j=1}^{d} \gamma_j \otimes D_j.$$

This is a linear operator on the Hilbert space $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$ initially defined with dense domain $\mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of functions on $\mathbb{R}^d$. It is easily seen that $\mathcal{D}$ is symmetric on this domain. Taking the closure we obtain a self-adjoint operator which we also denote $\mathcal{D}$.

We then define the sign of $\mathcal{D}$,

$$\operatorname{sgn}(\mathcal{D}) := \sum_{j=1}^{d} \gamma_j \otimes \frac{D_j}{\sqrt{D_1^2 + D_2^2 + \cdots + D_d^2}}.$$

This is defined through the joint Borel functional calculus [2, Theorem 13.24]. Specifically, the operator $D_j/\sqrt{D_1^2 + D_2^2 + \cdots + D_d^2}$ is the result of applying the function $x \mapsto x_j/\|x\|$, $x \in \mathbb{R}^d$, to $\mathcal{D}$. Consequently $\operatorname{sgn}(\mathcal{D})$ extends to a bounded operator on $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$.

Given $f \in L^\infty(\mathbb{R}^d)$, denote by M_f the operator of pointwise multiplication by f on the Hilbert space $L^2(\mathbb{R}^d)$. The operator $1 \otimes M_f$ is a bounded linear operator on $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$, where 1 denotes the identity operator on $\mathbb{C}^N$. The commutator,

$${\mathchar'26\mkern-12mu d}f := i[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f],$$

denotes the quantised derivative of Alain Connes introduced in [1, IV]. It is of particular interest in the quantised calculus to determine conditions on f such that ${\mathchar'26\mkern-12mu d}f \in \mathcal{L}_{d,\infty}(\mathbb{C}^N \otimes L^2(\mathbb{R}^d))$. The asymptotic behaviour of the singular values of the quantised derivative denote the dimension of the infinitesimal in the quantised calculus. That the sequence of singular values belongs to the weak space $\ell^{d,\infty}$ when the dimension of the Euclidean space is d indicates analogous behaviour between quantum derivatives and differential forms. Specifically, a product of d derivatives lies in the space $\mathcal{L}_{1,\infty}$, which is the only weak space admitting a non-trivial trace that acts as the integral.

In one dimension, necessary and sufficient conditions on $f \in L^\infty(\mathbb{R})$ such that $[\operatorname{sgn}(-id/dx), M_f] \in \mathcal{L}_{p,q}$ where $p, q \in (0, \infty]$ are provided by Peller in [3, Chapter 4, Theorem 4.4].

Janson and Wolfe [4], and Connes, Sullivan and Teleman [5] have studied necessary and sufficient conditions for ${\mathchar'26\mkern-12mu d}f \in \mathcal{L}_{p,q}$ with $p, q \in (0, \infty]$ in the higher dimensional case $d > 1$.

The case of $p = q$ was studied by Janson and Wolfe in their paper [4]. They proved that when $p > d$ a necessary and sufficient condition for ${\mathchar'26\mkern-12mu d}f$ to be in $\mathcal{L}_p$ is that f is in the Besov space $B^{d/p}_{pp}(\mathbb{R}^d)$. They also show that if $p \leq d$, then ${\mathchar'26\mkern-12mu d}f \in \mathcal{L}_p$ if and only if f is a constant.

The case of $p \neq q$ with $p \in [1,\infty)$ and $q \in [1,\infty]$ was answered by Rochberg and Semmes in [6, Corollary 2.8, Theorem 3.4]. Necessary and sufficient conditions on $f \in L^\infty(\mathbb{R}^d)$ are given so that $\,\text{đ}f \in \mathcal{L}_{p,q}$. These conditions are given in terms of the mean oscillation of f, and it is not obvious whether an equivalent condition could be given in terms of more familiar function spaces. In the Appendix of Connes, Sullivan and Teleman's paper [5, p. 679], it is proved that necessary and sufficient conditions for $\text{đ}f \in \mathcal{L}_{d,\infty}$ are that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla f \in L^d(\mathbb{R}^d,\mathbb{C}^d)$. The proof given in that Appendix is only a sketch.

We give a complete and different proof of this result under the assumption that $f \in L^\infty(\mathbb{R}^d)$ using double operator integrals. Our method allows us to give sharp bounds on the quasinorm $\|\text{đ}f\|_{d,\infty}$. When we write $\nabla f \in L^d(\mathbb{R}^d,\mathbb{C}^d)$ we implicitly assume that the essentially bounded function f has weak partial derivatives and that the Bochner norm of ∇f in $L^d(\mathbb{R}^d,\mathbb{C}^d)$,

$$\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} = \left(\int_{\mathbb{R}^d} \|(\nabla f)(x)\|_d^d\,dx\right)^{1/d} = \left(\int_{\mathbb{R}^d} \sum_{j=1}^d |D_j f(x)|^d dx\right)^{1/d},$$

is finite.

Theorem 1. *Let $d > 1$ and $f \in L^\infty(\mathbb{R}^d)$. Then, for $\text{đ}f \in \mathcal{L}_{d,\infty}(\mathbb{C}^N \otimes L^2(\mathbb{R}^d))$ it is necessary and sufficient that $\nabla f \in L^d(\mathbb{R}^d,\mathbb{C}^d)$. Further, there exist positive constants c and C depending only on d such that,*

$$c\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \leq \|\text{đ}f\|_{d,\infty} \leq C\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

In proving Theorem 1, we shall call the inequality

$$c\left(\int_{\mathbb{R}^d} \|(\nabla f)(x)\|_d^d\,dx\right)^{1/d} \leq \|\text{đ}f\|_{d,\infty}$$

in Theorem 1 the necessary direction. We call

$$\|\text{đ}f\|_{d,\infty} \leq C\left(\int_{\mathbb{R}^d} \|(\nabla f)(x)\|_d^d\,dx\right)^{1/d}$$

the sufficient direction.

Since all ℓ^p-norms on $\mathbb{C}^d$ are equivalent, there exist constants $c_{2,d}$ and $C_{2,d}$ such that for almost every $x \in \mathbb{R}^d$,

$$c_{2,d}\|\nabla f(x)\|_2^d \leq \|\nabla f(x)\|_d^d \leq C_{2,d}\|\nabla f(x)\|_2^d. \tag{1}$$

We note therefore that the norm equivalence in Theorem 1 can be written as the existence of positive constants c and C such that

$$c\left(\int_{\mathbb{R}^d}\|(\nabla f)(x)\|_2^d\,dx\right)^{1/d}\leq\|đf\|_{d,\infty}\leq C\left(\int_{\mathbb{R}^d}\|(\nabla f)(x)\|_2^d\,dx\right)^{1/d}.$$

In [7], Connes proves a trace formula relevant to quantised derivatives. This trace formula is defined for functions on a spin manifold; for background on this topic see [8, Chapter 2]. The setting of [7] is as follows. The manifold M is a compact spin^c Riemannian manifold of dimension $d>1$ with spinor bundle S and D is a Dirac-type operator defined on sections of S. Given a real valued $f\in C^\infty(M)$, let M_f be the operator of pointwise multiplication by f on sections of S. Then $\operatorname{sgn}(D)$ is defined by the functional calculus, and $đf$ is defined to be the commutator of $\operatorname{sgn}(D)$ with pointwise multiplication by f and multiplied by i, considered as an operator on square integrable sections of S.

Given an orientation of M, there is a Hodge $\star$ operator which maps 1-forms on M to $d-1$ forms. Given $f\in C^\infty(M)$, we have that $df\wedge\star df$ is a d-form on M, and thus $|df\wedge\star df|^{d/2}$ is a density on M.

Theorem 3(3) of [7] then gives a formula for the Dixmier trace tr_ω of $|đf|^d$. For details on Dixmier traces on the ideal $\mathcal{L}_{1,\infty}$, see [1, Chapter 4, Section 2β] and [9, Chapter 10]. The formula for a smooth function $f\in C^\infty(M)$ given is,

$$\operatorname{tr}_\omega(|đf|^d)=\lambda_d\int_M|df\wedge\star df|^{d/2}dx \tag{2}$$

where $\lambda_d>0$ is a constant depending on d that is independent of M and f.

We find a new formula that is analogous to (2). In our case, we deal with functions on $\mathbb{R}^d$. Since this space is not compact [7, Theorem 3(3)] does not apply and new techniques are needed. Our result is described as follows.

Recall that a trace on $\mathcal{L}_{1,\infty}$ is a linear functional $\varphi:\mathcal{L}_{1,\infty}\to\mathbb{C}$ such that $\varphi([A,B])=0$ for all bounded operators A and for all $B\in\mathcal{L}_{1,\infty}$. The trace φ is called continuous when it is continuous with respect to the $\mathcal{L}_{1,\infty}$ quasinorm. Given an orthonormal basis $\{e_n\}_{n=0}^\infty$ of H, we define the operator $T:=\operatorname{diag}\left\{\frac{1}{n+1}\right\}_{n=0}^\infty$ by $\langle e_n,Te_m\rangle=\delta_{n,m}\frac{1}{n+1}$.

We say that φ is normalised when

$$\varphi\left(\operatorname{diag}\left\{\frac{1}{n+1}\right\}_{n=0}^\infty\right)=1.$$

The property that φ is normalised is independent of the choice of orthonormal basis, since for all unitary operators U and all bounded operators B we have $\varphi(UBU^*) = \varphi(B)$.

Theorem 2. *Let $f \in L^\infty(\mathbb{R}^d)$ be real valued and such that $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$. Then there is a constant $c_d > 0$ such that for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$ we have*

$$\varphi(|{\rlap{\,/}d} f|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d \, dx.$$

Theorem 2 is the analogue of [7, Theorem 3(3)] for functions on the non-compact manifold $\mathbb{R}^d$. Theorem 2 is also stated for a larger class of functions than [7, Theorem 3(3)] which is proved for smooth functions.

Our proof of Theorem 1 will proceed as follows. In Section 3, we will prove the sufficiency direction of Theorem 1. In Section 4 we will prove Theorem 2. Finally we will prove the necessary direction of Theorem 1 in Section 5.

2. Preliminaries

2.1. Quantised differentials

For the remainder of the text, $d > 1$ and $N := 2^{\lfloor d/2 \rfloor}$. The functions $x_1, x_2, \ldots, x_d$ are orthogonal coordinates for $\mathbb{R}^d$. Given a distribution f on $\mathbb{R}^d$ the j-th partial distributional derivative is denoted $\partial_j f$, $j = 1, \ldots, d$, and we define

$$D_j := -i\partial_j, \quad j = 1, \ldots, d.$$

The linear operator D_j restricts to a self-adjoint operator on functions within $L^2(\mathbb{R}^d)$ with square integrable weak derivatives and to the normal partial derivative on Schwartz functions $\mathcal{S}(\mathbb{R}^d)$. The family $\{\gamma_j\}_{j=1}^d$ denotes d-dimensional gamma matrices, which are fixed $N \times N$ complex matrices satisfying $\gamma_j^2 = 1$, $\gamma_j^* = \gamma_j$ and $\gamma_j \gamma_k = -\gamma_k \gamma_j$ for $j \neq k$. Define

$$\mathcal{D} := \sum_{j=1}^d \gamma_j \otimes D_j$$

and define the Laplacian

$$\Delta := -\sum_{j=1}^d D_j^2.$$

The Laplacian satisfies $\mathcal{D}^2 = -1 \otimes \Delta$, where 1 is the identity on $\mathbb{C}^N$. Furthermore we define

$$\operatorname{sgn}(\mathcal{D}) := \sum_{j=1}^{d} \gamma_j \otimes \frac{D_j}{\sqrt{-\Delta}}.$$

This defines a bounded self adjoint operator on $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$, since it is the result of applying the bounded function $t \mapsto \operatorname{sgn}(t)$, $t \in \mathbb{R}$, to $\mathcal{D}$.

Given an essentially bounded function $f \in L^\infty(\mathbb{R}^d)$, M_f denotes the operator on $L^2(\mathbb{R}^d)$ given by pointwise multiplication by f. We define the "quantised derivative" of f,

$$đf := i[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f]$$

which is a bounded operator on $\mathbb{C}^N \otimes L^d(\mathbb{R}^d)$.

2.2. Sobolev Spaces

Given a distribution f on $\mathbb{R}^d$ define $\nabla f = i(D_1 f, D_2 f, \ldots, D_d f)$ for those distributions with weak partial derivatives. We will frequently refer to the d-Sobolev seminorm,

$$\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} := \left(\int_{\mathbb{R}^d} \|(\nabla f)(x)\|_d^d \, dx\right)^{1/d} = \left(\int_{\mathbb{R}^d} \sum_{j=1}^{d} |D_j f(x)|^d dx\right)^{1/d}.$$

The class of locally integrable functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ with weak partial derivatives such that $\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$ is finite is the homogeneous Sobolev space $\dot{W}^{1,d}(\mathbb{R}^d)$, as equivalently defined in [10, Definition 6.2.5]. The class of f such that $f \in \dot{W}^{1,d}(\mathbb{R}^d)$ and $f \in L^d(\mathbb{R}^d)$ is the Sobolev space $W^{1,d}(\mathbb{R}^d)$. The d-Sobolev norm of $f \in W^{1,d}(\mathbb{R}^d)$ is defined by $\|f\|_d + \|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$.

The following theorem is well known, and a proof can be found in [11, Theorem 2.1].

Theorem 3. *Equip $\dot{W}^{1,d}(\mathbb{R}^d)$ with the topology generated by the seminorm $f \mapsto \|\nabla f\|_{L^p(\mathbb{R}^d,\mathbb{C}^d)}$. Then $C_c^\infty(\mathbb{R}^d)$ is dense in $\dot{W}^{1,d}(\mathbb{R}^d)$. Moreover it is possible to find a sequence $\{f_n\}_{n=0}^\infty$ of smooth compactly supported functions such that ∇f_n converges to ∇f in the $L^d(\mathbb{R}^d, \mathbb{C}^d)$ sense and $\{f_n\}_{n=0}^\infty$ converges uniformly on compact subsets to f, modulo constant functions.*

2.3. Double operator integrals

We shall use the technique of double operator integrals. The specific definition that we use follows [12]. We restate it here in the specific case of bounded operator on a Hilbert space for convenience.

Definition 1. *Let H be a complex separable Hilbert space. Let D_0 and D_1 be self-adjoint (potentially unbounded) operators on H and let E^0 and E^1 be the associated spectral measures.*

For all $x, y \in \mathcal{L}_2(H)$, the measure

$$(\lambda, \mu) \mapsto \mathrm{Tr}(x dE^0(\lambda) y dE^1(\mu))$$

is a countably additive complex valued measure on $\mathbb{R}^2$.

Let $\phi \in L^\infty(\mathbb{R}^2)$. We say that ϕ is $E^0 \otimes E^1$ integrable if there exists an operator $T_\phi^{D_0,D_1} \in \mathcal{L}_\infty(H)$ such that for all $x, y \in \mathcal{L}_2(H)$,

$$\mathrm{Tr}(x T_\phi^{D_0,D_1} y) = \int_{\mathbb{R}^2} \phi(\lambda, \mu) \ \mathrm{Tr}(x dE^0(\lambda) y dE^1(\mu)).$$

The operator $T_\phi^{D_0,D_1}$ is called the transformer, and we then define for $A \in \mathcal{L}_\infty(\mathcal{H})$,

$$\int_{\mathbb{R}^d} \phi(\lambda, \mu) \, dE^0(\lambda) A dE^1(\mu) := T_\phi^{D_0,D_1}(A).$$

This is called a double operator integral.

2.4. Pseudodifferential operators

A multi-index of order $|\beta|$ is

$$\beta = (\beta_1, \ldots, \beta_d) \in \mathbb{Z}_+^d$$

such that

$$|\beta| := \beta_1 + \ldots + \beta_d.$$

For partial derivatives we use the notation

$$\partial_x^\beta := \frac{\partial^{|\beta|}}{\partial_{x_1}^{\beta_1} \ldots \partial_{x_d}^{\beta_d}}, \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Let $M_N(\mathbb{C})$ denote the $N \times N$ complex valued matrices.

Definition 2. *If $p \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, M_N(\mathbb{C}))$ and there is a fixed value $m \in \mathbb{R}$ and a constant $C > 0$ independent of x and y such that, for each multi-index α, β,*

$$\|(\partial_x^\alpha \partial_y^\beta p)(x, y)\|_2 \leq C(1 + \|y\|_2^2)^{(m-|\beta|)/2}, \qquad x, y \in \mathbb{R}^d, \tag{3}$$

we say that p is a symbol of order m.

In general terms we have introduced the uniform symbols of Hörmander type (1,0). The following are standard results, see e.g. [13, Chapter 2].

Proposition 4. *If $p \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d, M_N(\mathbb{C}))$ is a symbol order m, then the operator defined by*

$$(A\, v \otimes f)(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot y} p(x, y) v \hat{f}(y)\, dy, \qquad v \in \mathbb{C}^N, f \in \mathcal{S}(\mathbb{R}^d) \tag{4}$$

is a continuous linear operator

$$A : \mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d).$$

Here $\hat{f}$ denotes the Fourier transform of a Schwartz class function.

Definition 3. *The integral operator A in (4) is called a pseudodifferential operator of order $m \in \mathbb{R}$. The symbol of a pseudodifferential operator A we often denote by p_A.*

A pseudodifferential operator A is said to be compactly based when there exists $\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $(1 \otimes M_\psi)A = A$.

A pseudodifferential operator A of order m is called a classical pseudodifferential operator if its symbol has an asymptotic expansion of the form

$$p_A(x, y) \sim \sum_{j=0}^{\infty} p_{A,m-j}(x, y)$$

where each $p_{A,m-j}$ is symbol of order $m - j$ that is a homogeneous matrix valued function of order $m - j$ in y for $|y| \geq 1$, and $p_A - \sum_{j=0}^{k} p_{A,m-j}$ is a symbol of order $m - k - 1$ for every $k \geq 1$. The matrix valued smooth function $p_{A,m}$ is called the principal symbol of A.

2.5. Cwikel-type estimates

Definition 4. *Suppose* $f \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$*, and let* Δ_α *be the cube with centre* $\alpha \in \mathbb{Z}^d$ *of side length* 1*. If* χ_{Δ_α} *is the characteristic function of* Δ_α*, then we say that* $f \in \ell^p(L^2)$ *when*

$$\{\|f\chi_{\Delta_\alpha}\|_2\}_{\alpha\in\mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d).$$

Similarly $f \in \ell^{p,\infty}(L^2)$ *when the above sequence is in* $\ell^{p,\infty}(\mathbb{Z}^d)$*.*

In the following, we will frequently require the following Cwikel-type estimate.

Lemma 5. *Let* $\psi \in L^\infty(\mathbb{R}^d)$ *be a function of rapid decay. That is, for all multi-indices* α*, the expressions,*

$$|x_1|^{\alpha_1}|x_2|^{\alpha_2}\cdots|x_d|^{\alpha_d}\psi(x), \qquad x = (x_1,\ldots,x_d) \in \mathbb{R}^d$$

are essentially bounded. If $\beta > 0$*, then*

$$(1\otimes M_\psi)(1+\mathcal{D}^2)^{-\beta/2} \in \mathcal{L}_{d/\beta,\infty}.$$

In the case where $d/\beta > 2$*, we have the following estimate for the* $(d/\beta,\infty)$ *quasinorm,*

$$\|(1\otimes M_\psi)(1+\mathcal{D}^2)^{-\beta/2}\|_{d/\beta,\infty} \leq C\|\psi\|_{d/\beta}$$

where the constant $C > 0$ *depends only on* d *and* β*.*

Proof. Without loss of generality, ψ is non-negative, since

$$\|(1\otimes M_\psi)(1+\mathcal{D}^2)^{-\beta/2}\|_{d/\beta,\infty} = \||(1\otimes M_\psi)|(1+\mathcal{D}^2)^{-\beta/2,\infty}\|_{d/\beta,\infty}.$$

Further, since $\mathcal{D}^2 = -1\otimes\Delta$, we have that $\|(1\otimes M_\psi)(1+\mathcal{D}^2)^{-\beta/2}\|_{d/\beta,\infty}$ is equivalent to $\|M_\psi(1-\Delta)^{-\beta/2}\|_{d/\beta,\infty}$.

We shall deal with the cases $d/\beta \in (0,2)$, $d/\beta = 2$ and $d/\beta > 2$ separately.

First let $d/\beta > 2$. We refer to [14, Theorem 4.2] which states that if $p > 2$, $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p,\infty}(\mathbb{R}^d)$, then,

$$\|M_f g(-i\nabla)\|_{p,\infty} \leq c_{p,d}\|f\|_p\|g\|_{p,\infty}$$

where $c_{p,d} > 0$ depends only on p and d. We apply [14, Theorem 4.2] with $f = \psi$, $p = d/\beta$ and $g(x) = (1+\|x\|^2)^{-\beta/2}$. Note that,

$$|g(x)|^{d/\beta} = (1+\|x\|^2)^{-d/2}.$$

When $\|x\| > 1$, we have $|g(x)|^{d/\beta} \le \|x\|^{-d}$. For $\|x\| \le 1$, $|g(x)|^{d/\beta}$ is bounded above by 1. Hence, $|g|^{d/\beta} \in L^{1,\infty}(\mathbb{R}^d)$ and so $g \in L^{d/\beta,\infty}(\mathbb{R}^d)$. Since ψ is of rapid decay by assumption, we have $\psi \in L^{d/\beta}(\mathbb{R}^d)$. Hence,

$$\|M_\psi(1-\Delta)^{-\beta/2}\|_{d/\beta,\infty} \le c_{d/\beta,d}\|\psi\|_{d/\beta}\|g\|_{d/\beta,\infty}$$

and the required assertion is proved for $d/\beta > 2$.

Now consider the case $0 < d/\beta < 2$. In this case we apply [15, Theorem 5.7]. This theorem states that for $0 < p < 2$, if $f \in \ell^p(L^2)$ and $g \in \ell^{p,\infty}(L^2)$ then $M_f g(-i\nabla) \in \mathcal{L}_{p,\infty}$.

We again apply this to $f = \psi$ and $g = (1+\|x\|^2)^{-\beta/2}$ and $p = d/\beta$. Since ψ by assumption is bounded and of rapid decrease we have that $\psi \in \ell^{d/\beta}(L^2)$. What we require to verify is that $g \in \ell^{d/\beta,\infty}(L^2)$.

Let $\alpha \in \mathbb{Z}^d$. It is required to estimate $\|g\chi_\alpha\|_2$,

$$\begin{aligned}\|g\chi_\alpha\|_2^2 &= \int_{\Delta_\alpha} \frac{1}{(1+\|x\|^2)^\beta}\,dx \\ &= \int_{[-1/2,1/2]^d} \frac{1}{(1+\|x+\alpha\|^2)^\beta}\,dx \\ &\le \max_{x\in[-1/2,1/2]^d} \frac{1}{(1+\|x+\alpha\|^2)^\beta}.\end{aligned}$$

The maximum of the function $x \mapsto (1+\|x+\alpha\|^2)^{-\beta}$ on $[-1/2,1/2]^d$ is obtained when $\|x+\alpha\|$ is minimised. Since the cube $[-1/2,1/2]^d$ is contained in the closed ball of radius $(d/2)^{1/2}$ at the origin, this minimum can be no less than $\|\alpha\| - (d/2)^{1/2}$. Hence,

$$\|g\chi_\alpha\|_2 \le \left(1 + (\|\alpha\| - \sqrt{d/2})^2\right)^{-\beta/2}.$$

To show that this defines a sequence indexed by α in $\ell^{d/\beta,\infty}(\mathbb{Z}^d)$, we take the d/β power, hence all that is required to show is that

$$\left\{\left(1 + (\|\alpha\| - \sqrt{d/2})^2\right)^{-d/2}\right\}_{\alpha\in\mathbb{Z}^d} \in \ell^{1,\infty}(\mathbb{Z}^d),$$

which is easily verified.

Finally, when $d/\beta = 2$, we have $\beta = d/2$ and so it is required to estimate,

$$\|M_\psi(1-\Delta)^{-d/4}\|_{2,\infty}.$$

This is exactly equal to to,

$$\|M_\psi(1-\Delta)^{-d/2}M_{\overline{\psi}}\|_{1,\infty}^{1/2}.$$

However by our previous argument for $0 < d/\beta < 2$, we have that $M_\psi(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$ and hence $M_\psi(1-\Delta)^{-d/4} \in \mathcal{L}_{2,\infty}$ as required.

□

The Cwikel estimate can be applied to compactly based pseudo-differential operators of negative order.

Lemma 6. *Let A be a compactly based pseudodifferential operator of order $-\beta$, $\beta > 0$. Then $A \in \mathcal{L}_{d/\beta,\infty}$.*

Proof. Since A is compactly based then there exists a smooth function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $A = (1 \otimes M_\psi)A$. Since A is a uniform pseudodifferential operator of order $-\beta$ then there exists an order 0 pseudodifferential operator W such that $A = (1+\mathcal{D}^2)^{-\beta/2}W$, [9, Proposition 10.2.13(c)]. The operator W has a bounded extension to $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$, [9, Proposition 10.1.15(b)]. Hence

$$A = (1 \otimes M_\psi)A = (1 \otimes M_\psi)(1+\mathcal{D}^2)^{-\beta/2}W \in \mathcal{L}_{d/\beta,\infty}$$

using Lemma 5. □

Within a commutator the bounded operator $\mathrm{sgn}(\mathcal{D})$ can be approximated by the order 0 pseudodifferential operator $\mathcal{D}(1+\mathcal{D}^2)^{-1/2}$ modulo $\mathcal{L}_p$ spaces.

Lemma 7. *Suppose that $p > d/2$ and $f \in \mathcal{S}(\mathbb{R}^d)$. If $p \geq 2$, then there exists a constant $C_{p,d} > 0$ such that*

$$\left\|\left[\mathrm{sgn}(\mathcal{D}) - \frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}, 1 \otimes M_f\right]\right\|_p \leq C_{p,d}\|f\|_p.$$

If $1 \leq p < 2$ then there exists a constant $B_{p,d} > 0$ such that,

$$\left\|\left[\mathrm{sgn}(\mathcal{D}) - \frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}, 1 \otimes M_f\right]\right\|_p \leq B_{p,d}\|f\|_{p;2}.$$

Here $\|f\|_{p;2}$ denotes the norm on $\ell^p(L^2)$ given by

$$\|f\|_{p;2} := \left(\sum_{\alpha \in \mathbb{Z}^d} \|f\chi_{\Delta_\alpha}\|_2^p\right)^{1/p}.$$

Proof. Let $1 \leq j \leq d$ and for $x \in \mathbb{R}^d$ define,

$$h_j(x) := \frac{x_j}{\|x\|} - \frac{x_j}{(1+\|x\|^2)^{1/2}} = \frac{x_j}{(\|x\| + (1+\|x\|^2)^{1/2})\|x\|(1+\|x\|^2)^{1/2}}.$$

It is clear that $|h_j(x)| \leq 1$ for all x. For $\|x\| > 1$, we have $|h_j(x)| \leq \|x\|^{-2}$ and hence $|h_j(x)|^p \leq \|x\|^{-d}$ as $p > d/2$. Thus $h_j \in L^p(\mathbb{R}^d)$.

Expanding out the commutator,

$$\begin{aligned}\left[\operatorname{sgn}(\mathcal{D}) - \frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}, 1 \otimes M_f\right] &= \left[\sum_{j=1}^{d} \gamma_j \otimes h_j(-i\nabla), 1 \otimes M_f\right] \\ &= \sum_{j=1}^{d} \gamma_j \otimes [h_j(-i\nabla), M_f].\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\left[\operatorname{sgn}(\mathcal{D}) - \frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}, 1 \otimes M_f\right]\right\|_p &\leq d \max_{1\leq j\leq d} \|[h_j(-i\nabla), M_f]\|_p \\ &\leq d \max_{1\leq j\leq d} (\|h_j(-i\nabla)M_f\|_p + \|M_f h_j(-i\nabla)\|_p) \\ &\leq d \max_{1\leq j\leq d} (\|h_j(-i\nabla)M_f\|_p + \|h_j(-i\nabla)M_{\overline{f}}\|_p).\end{aligned}$$

Suppose $p \geq 2$. We can apply Theorem 4.1 of [14], which states that, for measurable functions a, b on $\mathbb{R}^d$ and $p \geq 2$, $\|a(-i\nabla)M_b\|_p \leq (2\pi)^{-d/p}\|a\|_p\|b\|_p$. In our case, we take $b = f$ or $b = \overline{f}$ and $a = h_j$. By assumption, $\|f\|_p$ is finite, and we have shown that $\|h_j\|_p$ is finite. Hence, if we let $C_{p,d} = 2d(2\pi)^{-d/p}\max_{1\leq j\leq d}\|h_j\|_p$, we have the desired result.

Suppose $1 \leq p < 2$. Define h_j as above. For $\|x\| > 1$, we have $|h_j(x)| \leq \|x\|^{-2}$ and for $\|\alpha\| \geq 0$ we have

$$\left(\int_{\Delta_\alpha} |h_j(x)|^2 \, dx\right)^{1/2} \leq K\|\alpha\|^{-2}$$

for a constant $K > 0$. Then

$$\sum_{\alpha\in\mathbb{Z}^d} \|h_j \chi_{\Delta_\alpha}\|_2^p \leq K^p \sum_{\alpha\in\mathbb{Z}^d} \|\alpha\|^{-2p} \leq K^p L \int_0^\infty r^{-2p+d-1} dr = \frac{K^p L}{2p-d}$$

for a constant $L > 0$. The last integral can be performed because of the condition $-2p+d < 0$. Thus, $\|h_j\|_{2;p}$ is finite. Theorem 4.5 of [14] states that if $1 \leq p < 2$, then there is a positive constant D_p such that $\|a(-i\nabla)M_b\|_p \leq D_p\|a\|_{2;p}\|b\|_{2;p}$. Repeating the argument above verbatim using [14, Theorem 4.5] instead of [14, Theorem 4.1], we obtain the required result. □

3. Sufficiency

In this section we will prove that for all integers $d > 1$ there is a constant $C_d > 0$ depending only on d such that the inequality,

$$\|[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} \leq C_d\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$$

holds for all $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and thus we conclude that $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is sufficient for $đf \in \mathcal{L}_{d,\infty}$.

The following proof uses the technique of double operator integrals according to Definition 1, and is modelled on proofs in [12] and results from [16]. The definition used in [16] for the double operator integral is visually different to ours, however the discussion in [17] under equation (2.12) of that paper demonstrates the equivalence of Definition 1 with the definition used in [16].

The following lemma is similar to [18]. We include the proof here for completeness.

Lemma 8. *Define the function $\psi : \mathbb{R}^2 \to \mathbb{R}$ by the formula,*

$$\psi(\lambda, \mu) := \frac{(1+\lambda^2)^{1/4}(1+\mu^2)^{1/4}}{(1+\lambda^2)^{1/2}+(1+\mu^2)^{1/2}}, \quad \lambda, \mu \in \mathbb{R}.$$

Then, for all self-adjoint potentially unbounded operators A on a complex separable Hilbert space H, we have,

$$\|T_\psi^{A,A}\|_{\mathcal{L}_\infty \to \mathcal{L}_\infty} < \infty, \ \|T_\psi^{A,A}\|_{\mathcal{L}_1 \to \mathcal{L}_1} < \infty.$$

Proof. Let $B = \frac{1}{4}\log(1+A^2)$ and define $\psi_0 : \mathbb{R}^2 \to \mathbb{R}$ by the formula

$$\psi_0(\lambda, \mu) := \frac{e^\lambda e^\mu}{e^{2\lambda}+e^{2\mu}} = \frac{1}{2\cosh(\lambda-\mu)}.$$

Denote $h(x) = \frac{1}{2\cosh(x)}$ so that $\psi_0(\lambda, \mu) = h(\lambda-\mu)$. Note that $h \in \mathcal{S}(\mathbb{R})$, and therefore $\widehat{h} \in \mathcal{S}(\mathbb{R})$.

Let $s, t \in \mathbb{R}$, $s' = \frac{1}{4}\log(1+s^2)$ and $t' = \frac{1}{4}\log(1+t^2)$. Then,

$$\begin{aligned}\psi_0(s', t') &= \frac{(1+s^2)^{1/4}(1+t^2)^{1/4}}{(1+s^2)^{1/4}+(1+t^2)^{1/4}} \\ &= \psi(s,t).\end{aligned}$$

Thus, by [12, Lemma 8], we have $T_{\psi_0}^{B,B} = T_{\psi}^{A,A}$.

Now [12, Lemma 6] states that if f is a function on $\mathbb{R}$ such that $\widehat{f}$ is integrable, and D_0 and D_1 are self-adjoint potentially unbounded operators then, $T_{\phi}^{D_0,D_1}$ is bounded from $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$, where $\phi(\lambda,\mu) = f(\lambda-\mu)$. Thus, since $\widehat{h}$ is integrable,

$$\|T_{\psi}^{A,A}\|_{\mathcal{L}_\infty\to\mathcal{L}_\infty} = \|T_{\psi_0}^{B,B}\|_{\mathcal{L}_\infty\to\mathcal{L}_\infty} < \infty.$$

By an identical argument to that of [12, Theorem 1], we conclude similarly that $\|T_{\psi}^{A,A}\|_{\mathcal{L}_1\to\mathcal{L}_1} < \infty$. □

Given a function $f : \mathbb{R} \to \mathbb{R}$, define

$$f^{[1]}(\lambda,\mu) := \frac{f(\lambda)-f(\mu)}{\lambda-\mu}, \quad \lambda,\mu\in\mathbb{R}, \lambda\neq\mu.$$

Lemma 9. *Let $g(t) = t(1+t^2)^{-1/2}$, $t \in \mathbb{R}$. Then $g^{[1]}(\lambda,\mu) = (\psi_1\psi_2\psi_3)(\lambda,\mu)$ for all $\lambda,\mu\in\mathbb{R}, \lambda\neq\mu$, where*

$$\begin{aligned}\psi_1(\lambda,\mu) &:= 1 + \frac{1-\lambda\mu}{(1+\lambda^2)^{\frac{1}{2}}(1+\mu^2)^{\frac{1}{2}}}, \\ \psi_2(\lambda,\mu) &:= \frac{(1+\lambda^2)^{\frac{1}{4}}(1+\mu^2)^{\frac{1}{4}}}{(1+\lambda^2)^{\frac{1}{2}}+(1+\mu^2)^{\frac{1}{2}}}, \\ \psi_3(\lambda,\mu) &:= \frac{1}{(1+\lambda^2)^{\frac{1}{4}}(1+\mu^2)^{\frac{1}{4}}}.\end{aligned}$$

Proof. Let $\langle\alpha\rangle := (1+\alpha^2)^{1/2}$. Then we have

$$\begin{aligned}g^{[1]}(\lambda,\mu) &= \frac{\lambda\langle\mu\rangle - \mu\langle\lambda\rangle}{(\lambda-\mu)\langle\lambda\rangle\langle\mu\rangle} \\ &= \frac{\lambda^2\langle\mu\rangle - \lambda\mu\langle\lambda\rangle + \lambda\mu\langle\mu\rangle - \mu^2\langle\lambda\rangle}{(\lambda^2-\mu^2)\langle\lambda\rangle\langle\mu\rangle}.\end{aligned}$$

Since $\langle\alpha\rangle^2 = \alpha^2 + 1$, this is

$$\begin{aligned}
&= \frac{\lambda^2\langle\mu\rangle - \mu^2\langle\lambda\rangle - \lambda\mu(\langle\lambda\rangle - \langle\mu\rangle)}{(\langle\lambda\rangle^2 - \langle\mu\rangle^2)\langle\lambda\rangle\langle\mu\rangle} \\
&= \frac{\lambda^2\langle\mu\rangle - \mu^2\langle\lambda\rangle - \lambda\mu(\langle\lambda\rangle - \langle\mu\rangle)}{(\langle\lambda\rangle - \langle\mu\rangle)\langle\lambda\rangle\langle\mu\rangle} \cdot \frac{1}{\langle\lambda\rangle + \langle\mu\rangle} \\
&= \left[\frac{\lambda^2\langle\mu\rangle - \mu^2\langle\lambda\rangle}{(\langle\lambda\rangle - \langle\mu\rangle)\langle\lambda\rangle\langle\mu\rangle} - \frac{\lambda\mu}{\langle\lambda\rangle\langle\mu\rangle}\right] \cdot \frac{1}{\langle\lambda\rangle + \langle\mu\rangle} \\
&= \left[\frac{\lambda^2\langle\mu\rangle - \mu^2\langle\lambda\rangle}{(\langle\lambda\rangle - \langle\mu\rangle)\langle\lambda\rangle\langle\mu\rangle} - \frac{1}{\langle\lambda\rangle\langle\mu\rangle} + \frac{1 - \lambda\mu}{\langle\lambda\rangle\langle\mu\rangle}\right] \cdot \frac{1}{\langle\mu\rangle + \langle\lambda\rangle} \\
&= \left[\frac{\lambda^2\langle\mu\rangle - \mu^2\langle\lambda\rangle - \langle\lambda\rangle + \langle\mu\rangle}{(\langle\lambda\rangle - \langle\mu\rangle)\langle\lambda\rangle\langle\mu\rangle} + \frac{1 - \lambda\mu}{\langle\lambda\rangle\langle\mu\rangle}\right] \cdot \frac{1}{\langle\lambda\rangle + \langle\mu\rangle} \\
&= \left[\frac{(\lambda^2 + 1)\langle\mu\rangle - (\mu^2 + 1)\langle\lambda\rangle}{(\lambda^2 + 1)\langle\mu\rangle - (\mu^2 + 1)\langle\lambda\rangle} + \frac{1 - \lambda\mu}{\langle\lambda\rangle\langle\mu\rangle}\right] \cdot \frac{1}{\langle\lambda\rangle + \langle\mu\rangle} \\
&= \psi_1(\lambda, \mu)\frac{\langle\lambda\rangle^{1/2}\langle\mu\rangle^{1/2}}{\langle\lambda\rangle + \langle\mu\rangle} \cdot \frac{1}{\langle\lambda\rangle^{1/2}\langle\mu\rangle^{1/2}} \\
&= \psi_1(\lambda, \mu)\psi_2(\lambda, \mu)\psi_3(\lambda, \mu).
\end{aligned}$$

□

Lemma 10. *Let* $f \in \mathcal{S}(\mathbb{R}^d)$. *Then,*

$$\left\| \left[\frac{\mathcal{D}}{\sqrt{1 + \mathcal{D}^2}}, 1 \otimes M_f \right] \right\|_{d,\infty} \leq B_d \|\nabla f\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)} \tag{5}$$

where $B_d > 0$ *depends only on* d.

Proof. Set $g(t) := t(1 + t^2)^{-1/2}$ with $t \in \mathbb{R}$. By [16, Theorem 4.1],

$$[g(\mathcal{D}), 1 \otimes M_f] = T^{\mathcal{D},\mathcal{D}}_{g^{[1]}}([\mathcal{D}, 1 \otimes M_f]). \tag{6}$$

By Lemma 9, we have $g^{[1]} = \psi_1\psi_2\psi_3$, using the functions ψ_1, ψ_2 and ψ_3 in that lemma. We express the transformer $T^{\mathcal{D},\mathcal{D}}_{g^{[1]}}$ as a product by [12, Theorem 1],

$$T^{\mathcal{D},\mathcal{D}}_{g^{[1]}} = T^{\mathcal{D},\mathcal{D}}_{\psi_1} T^{\mathcal{D},\mathcal{D}}_{\psi_2} T^{\mathcal{D},\mathcal{D}}_{\psi_3}. \tag{7}$$

We claim that the transformers $T^{\mathcal{D},\mathcal{D}}_{\psi_k}$, with $k = 1, 2, 3$ are bounded linear transformations from $\mathcal{L}_{d,\infty}$ to $\mathcal{L}_{d,\infty}$.

We start by showing that the transformers are bounded from $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$ and $\mathcal{L}_1$ to $\mathcal{L}_1$ and then use interpolation. For $k = 2$, this is precisely the statement of Lemma 8. For the cases $k = 1$ and $k = 3$, note that $\psi_k(\lambda, \mu)$, $k = 1, 3$, can be written as a linear combination of terms of the form $a(\lambda)b(\mu)$ for some bounded functions a and b on $\mathbb{R}$. By [12, Corollary 2], the transformer $T^{\mathcal{D},\mathcal{D}}_{\psi_k}$ with $k = 1, 3$ is bounded from $\mathcal{L}_\infty$ to $\mathcal{L}_\infty$, and by an identical argument we conclude that it is bounded from $\mathcal{L}_1$ to $\mathcal{L}_1$. As $T^{\mathcal{D},\mathcal{D}}_{\psi_k}$, $k = 1, 2, 3$, is bounded on $\mathcal{L}_1$ and $\mathcal{L}_\infty$ we find that $T^{\mathcal{D},\mathcal{D}}_{\psi_k}$ is bounded from $\mathcal{L}_{d,\infty}$ to $\mathcal{L}_{d,\infty}$ by real interpolation on the pair $(\mathcal{L}_1, \mathcal{L}_\infty)$, [19].

We now exploit the identity in (6) and the product of terms in (7), noticing that

$$\begin{aligned}\|[g(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} &\leq \|T^{\mathcal{D},\mathcal{D}}_{\psi_1}\|_{\mathcal{L}_{d,\infty}\to\mathcal{L}_{d,\infty}} \\ &\qquad \times \|T^{\mathcal{D},\mathcal{D}}_{\psi_2}\|_{\mathcal{L}_{d,\infty}\to\mathcal{L}_{d,\infty}} \\ &\qquad\qquad \times \|T^{\mathcal{D},\mathcal{D}}_{\psi_3}([\mathcal{D}, 1 \otimes M_f])\|_{d,\infty} \\ &\leq C_d \|T^{\mathcal{D},\mathcal{D}}_{\psi_3}([\mathcal{D}, 1 \otimes M_f])\|_{d,\infty}\end{aligned}$$

for a constant $C_d > 0$ independent of f. Since $\psi_3(\lambda, \mu) = (1 + \lambda^2)^{-1/4}(1 + \mu^2)^{-1/4}$, then

$$T^{\mathcal{D},\mathcal{D}}_{\psi_3}([\mathcal{D}, 1 \otimes M_f]) = (1 + \mathcal{D}^2)^{-1/4}[\mathcal{D}, 1 \otimes M_f](1 + \mathcal{D}^2)^{-1/4}.$$

Hence,

$$\|[g(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} \leq C_d \|(1 + \mathcal{D}^2)^{-1/4}[\mathcal{D}, 1 \otimes M_f](1 + \mathcal{D}^2)^{-1/4}\|_{d,\infty}.$$

Expanding out $\mathcal{D}$, and using the quasi-triangle inequality for $\mathcal{L}_{d,\infty}$ we have

$$\begin{aligned}&\|(1 + \mathcal{D}^2)^{-1/4}[\mathcal{D}, 1 \otimes M_f](1 + \mathcal{D}^2)^{-1/4}\|_{d,\infty} \\ &\leq K_d \sum_{j=1}^{d} \|(1 + \mathcal{D}^2)^{-1/4}[\gamma_j \otimes D_j, 1 \otimes M_f](1 + \mathcal{D}^2)^{-1/4}\|_{d,\infty}.\end{aligned}$$

Here $K_d > 0$ depends only on d. Now using the fact that $[\gamma_j \otimes D_j, 1 \otimes M_f] = -i\gamma_j \otimes M_{\partial_j f}$ it follows that,

$$\|(1 + \mathcal{D}^2)^{-1/4}\gamma_j \otimes M_{\partial_j f}(1 + \mathcal{D}^2)^{-1/4}\|_{d,\infty} = \|(1 - \Delta)^{-1/4} M_{\partial_j f}(1 - \Delta)^{-1/4}\|_{d,\infty}.$$

Let $1 \leq j \leq d$. Let $\operatorname{sgn}(\partial_j f)(x) = (\partial_j)f(x)/|(\partial_j f)(x)|$ for those $x \in \mathbb{R}^d$ such that $\partial_j f(x) \neq 0$ and equal to 1 otherwise. Then,

$$\partial_j f = |\partial_j f|^{1/2} \operatorname{sgn}(\partial_j f)|\partial_j f|^{1/2}.$$

Using the fact that $M_{\operatorname{sgn}(\partial_j f)}$ is unitary and the Hölder inequality,

$$\|(1-\Delta)^{-1/4} M_{\partial_j f}(1-\Delta)^{-1/4}\|_{d,\infty} \leq \|(1-\Delta)^{-1/4} M_{|\partial_j f|^{1/2}}\|_{2d,\infty} \|M_{|\partial_j f|^{1/2}}(1-\Delta)^{-1/4}\|_{2d,\infty}$$

By Lemma 5,

$$\|(1-\Delta)^{-1/4} M_{|\partial_j f|^{1/4}}\|_{2d,\infty} \leq Q_d \||\partial_j f|^{1/2}\|_{2d}$$

for some constant Q_d. Taking $B_d > dQ_d^2 C_d K_d$,

$$\begin{aligned} \|[g(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} &\leq B_d \sum_{j=1}^d \||\partial_j f|^{1/2}\|_{2d}^2 = B_d \sum_{j=1}^d \|\partial_j f\|_d \\ &\leq B_d \left(\sum_{j=1}^d \int_{\mathbb{R}^d} \|\partial_j f(x)\|^d dx \right)^{1/d}. \end{aligned}$$

The last inequality follows from Jensen's inequality applied to the concave function $t \mapsto t^{1/d}$, $t > 0$. Thus the required bound is satisfied. □

Now we can complete the proof of the sufficiency direction of Theorem 1.

Theorem 11. *Let $d > 1$. If $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then*

$$\|[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} = \|{\mathchar'26\mkern-12mu d} f\|_{d,\infty} \leq C_d \|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$$

where $C_d > 0$ depends only on d.

Proof. Initially, let $f \in \mathcal{S}(\mathbb{R}^d)$. Combining Lemmas 7 and 10, we find that

$$\|[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} \leq C_{d,d}\|f\|_d + B_d \|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

We can remove the dependence on $\|f\|_d$ on the right hand side by a dilation argument as follows. Let $t > 0$ and let σ_t be the dilation action on $\mathcal{S}(\mathbb{R}^d)$, defined by $(\sigma_t f)(x) = f(t^{-1}x)$. Then ${\mathchar'26\mkern-12mu d}(\sigma_t f) = {\mathchar'26\mkern-12mu d}f$ as the symbol of $\mathrm{sgn}(\mathcal{D})$ as a Fourier multiplier is dilation invariant. Thus,

$$\begin{aligned}
\|[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} &= \|[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_{\sigma_t f}]\|_{d,\infty} \\
&\leq C_{d,d}\|\sigma_t f\|_d + B_d\|\sigma_t \nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \\
&= C_{d,d} t\|f\|_d + B_d\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.
\end{aligned}$$

Taking $t \to 0$, we obtain

$$\|[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} \leq B_d\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \tag{8}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$.

To complete the proof, let $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Let $\{f_n\}_{n=0}^\infty \subset \mathcal{S}(\mathbb{R}^d)$ be the sequence defined in Theorem 3. Then ∇f_n converges to ∇f in the L^d sense, and f_n, modulo constants, converges to f uniformly on bounded sets.

Let $n, m \geq 0$. Applying equation (8) to $f_n - f_m \in \mathcal{S}(\mathbb{R}^d)$,

$$\|{\mathchar'26\mkern-12mu d}f_n - {\mathchar'26\mkern-12mu d}f_m\|_{d,\infty} \leq B_d\|\nabla f_n - \nabla f_m\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

Since by definition $\{\nabla f_n\}_{n=0}^\infty$ is Cauchy in the L^d norm, the sequence $\{{\mathchar'26\mkern-12mu d}f_n\}_{n=0}^\infty$ is Cauchy in $\mathcal{L}_{d,\infty}$. Thus $\{{\mathchar'26\mkern-12mu d}f_n\}_{n=0}^\infty$ converges to some limit L in the $\mathcal{L}_{d,\infty}$ quasinorm.

Since $f_n \to f$ uniformly on bounded sets modulo constants, the sequence $\{M_{f_n}\}_{n=0}^\infty$ converges strongly to M_f modulo constants. Thus ${\mathchar'26\mkern-12mu d}f_n$ converges in the strong operator topology to ${\mathchar'26\mkern-12mu d}f$ and so $L = {\mathchar'26\mkern-12mu d}f$.

Hence ${\mathchar'26\mkern-12mu d}f \in \mathcal{L}_{d,\infty}$, and using the quasinorm triangle inequality there exists $K > 0$ such that

$$\begin{aligned}
\|{\mathchar'26\mkern-12mu d}f\|_{d,\infty} &\leq K\|{\mathchar'26\mkern-12mu d}f - {\mathchar'26\mkern-12mu d}f_n\|_{d,\infty} + K\|{\mathchar'26\mkern-12mu d}f_n\|_{d,\infty} \\
&\leq K\|{\mathchar'26\mkern-12mu d}f - {\mathchar'26\mkern-12mu d}f_n\|_{d,\infty} + KB_d\|\nabla f_n\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.
\end{aligned}$$

Taking the limit as $n \to \infty$ and setting $C_d = KB_d$ we obtain the desired inequality.

□

4. Trace Formula

Let φ be a continuous normalised trace on $\mathcal{L}_{1,\infty}$. In this section, we will prove for all real valued $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ that there is a positive constant c_d independent of f and φ such that

$$\varphi(|{\mathchar'26\mkern-12mu d}f|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d \, dx. \tag{9}$$

Initially we work with the assumption that $f \in C_c^\infty(\mathbb{R}^d)$, and then we show that (9) holds for any $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

To prove this we will require some preliminary lemmas. The following is a variant of Connes' trace theorem, [9, Theorem 11.5.1]. We remark that a uniform pseudodifferential operator $A : \mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d)$ of order 0 as it is defined in Section 2.4 has a bounded extension, [9, Proposition 10.1.15(b)]. Unless the context is ambiguous we also denote the extension by $A : \mathbb{C}^N \otimes L^2(\mathbb{R}^d) \to \mathbb{C}^N \otimes L^2(\mathbb{R}^d)$. For example, it is implicit in the next lemma that we apply the continuous functional calculus to the bounded and self-adjoint extension of an order 0 symmetric pseudodifferential operator A.

Lemma 12. *Let A be a compactly based classical pseudodifferential operator of order 0 with self-adjoint extension, and let $h \in C(\mathbb{R})$ be such that $h(0) = 0$. Then for every normalised trace φ on $M_N(\mathbb{C}) \otimes \mathcal{L}_{1,\infty}$ we have,*

$$\varphi(h(A)(1+\mathcal{D}^2)^{-d/2}) = c_d \int_{S^{d-1}} \int_{\mathbb{R}^d} \operatorname{tr}(h(p_{0,A}(x,s)))\, dxds.$$

where $p_{0,A}$ is the principal symbol of A.

Proof. First we prove the result for $h(A) = A$, then extend it to all polynomial h, then we use an approximation argument to conclude the result for all continuous h.

In the first case, this is an application of Connes' trace formula, specifically [9, Theorem 11.5.1]. This follows since the only nontrivial trace on $M_N(\mathbb{C})$ is the classical trace tr, and any trace on the tensor product $M_N(\mathbb{C}) \otimes \mathcal{L}_{1,\infty}$ splits as $\operatorname{tr} \otimes \varphi'$ for a trace φ' on $\mathcal{L}_{1,\infty}$. Moreover, the compactly based classical pseudodifferential operator $A(1+\mathcal{D}^2)^{-d/2}$ has principal symbol $p_{0,A}(x,y)\|y\|^{-d}$, $x, y \in \mathbb{R}^d$, $\|y\| \geq 1$, since the principal symbol of a product is the product of principal symbols, by [9, Proposition 10.2.12].

If $h(A) = A^n$, for $n \geq 1$, then it is easy to see that $h(A)$ is a compactly based pseudodifferential operator, and has principal symbol $h(p_{0,A})$ again by [9, Proposition 10.2.12]. By linearity the result then holds for all polynomials h with $h(0) = 0$.

Now suppose that $h \in C(\mathbb{R})$ is such that $h(0) = 0$. Choose a sequence $\{h_n\}_{n=0}^\infty$ of polynomials such that h_n uniformly approximates h on $[-\|A\|, \|A\|]$ and for each n, $h_n(0) = 0$. Hence $\|h_n(A) - h(A)\| \to 0$ and $h(A)$ is compactly based. If $\psi \in C_c^\infty(\mathbb{R}^d)$ is such that $(1 \otimes M_\psi)A = A = A(1 \otimes M_\psi)$, then $(1 \otimes M_\psi)h(A) = h(A) = h(A)(1 \otimes M_\psi)$. Due to the continuity of φ we have,

$$
\begin{aligned}
|\varphi((h(A)-h_n(A))(1+\mathcal{D}^2)^{-d/2})| &= |\varphi((h(A)-h_n(A))(1\otimes M_\psi)(1+\mathcal{D}^2)^{-d/2})| \\
&\leq \|h(A)-h_n(A)\|\varphi((1\otimes M_\psi)(1+\mathcal{D}^2)^{-d/2}).
\end{aligned}
$$

Thus, $\varphi(h_n(A)(1+\mathcal{D}^2)^{-d/2})$ converges to $\varphi(h(A)(1+\mathcal{D}^2)^{-d/2})$.

Now we establish convergence on the right hand side. As $A(1\otimes M_\psi) = A$, we can apply the principal symbol map to obtain $p_{0,A}(1\otimes\psi) = p_{0,A}$. It follows that $h_n(p_{0,A}(1\otimes\psi)) = h_n(p_{0,A})(1\otimes\psi)$ and hence $h(p_{0,A}(1\otimes\psi)) = h(p_{0,A})(1\otimes\psi)$. Thus,

$$
\begin{aligned}
&\left|\int_{S^{d-1}}\int_{\mathbb{R}^d}\operatorname{tr}(h(p_{0,A}(x,s))) - \operatorname{tr}(h_n(p_{0,A}(x,s)))dxds\right| \\
&\leq \int_{S^{d-1}}\int_{\mathbb{R}^d}|\operatorname{tr}((h-h_n)(p_{0,A}(x,s)(1\otimes\psi(x)))|\,dxds \\
&\leq \int_{S^{d-1}}\int_{\mathbb{R}^d}|\operatorname{tr}((h_n-h)(p_{0,A}(x,s)))||\psi(x)|\,dxds \\
&\leq \|h_n-h\|_\infty\int_{S^{d-1}}\int_{\mathbb{R}^d}|\psi(x)|\,dxds = C\|h_n-h\|_\infty
\end{aligned}
$$

for a constant $C>0$. Hence the convergence of both sides is proved. □

Lemma 13. *Suppose $d>1$. Let A be a compactly based classical pseudodifferential operator of order 0 with self-adjoint extension. Then for all $0<\alpha<d-1$ and $\beta\geq 0$,*

$$
[(1+\mathcal{D}^2)^{-\alpha/2},|A|](1+\mathcal{D}^2)^{-\beta/2}\in\mathcal{L}_{d/(\alpha+\beta+1),\infty}.
$$

Proof. Let $K := (1+\mathcal{D}^2)^{-1/2}$. First we prove that for $\psi\in C_c^\infty(\mathbb{R}^d)$ and $\alpha,\beta\geq 0$

$$
[K^\alpha, 1\otimes M_\psi]K^\beta\in\mathcal{L}_{d/(\alpha+\beta+1),\infty}. \tag{10}
$$

Choose $0<\phi\in C_c^\infty(\mathbb{R}^d)$ such that $\psi\phi=\psi$. Then,

$$
[K^\alpha,1\otimes M_\psi]K^\beta = (1\otimes M_\psi)[K^\alpha,1\otimes M_\phi]K^\beta + [K^\alpha,1\otimes M_\psi](1\otimes M_\phi)K^\beta
$$

By [9, Proposition 10.2.13(c)], the operator $(1\otimes M_\psi)[K^\alpha,1\otimes M_\phi]K^\beta$ is a compactly based pseudodifferential operator of order $(-\alpha-\beta-1)$. By Lemma 6

$$
(1\otimes M_\psi)[K^\alpha,1\otimes M_\phi]K^\beta\in\mathcal{L}_{d/(\alpha+\beta+1)}.
$$

Similarly, $[K^\alpha, 1\otimes M_\psi]((1\otimes M_{\phi^{1/2}})$ is the adjoint of a compactly based pseudodifferential operator of order $(-\alpha-1)$, and $(1\otimes M_{\phi^{1/2}})K^\beta$ is a compactly based pseudodifferential operator of order $-\beta$. Hence, by Lemma 6 and the Hölder inequality

$$[K^\alpha, 1\otimes M_\psi](1\otimes M_\phi)K^\beta \in \mathcal{L}_{d/(\alpha+1)}\mathcal{L}_{d/\beta} \subseteq \mathcal{L}_{d/(\alpha+\beta+1)}.$$

We have proven equation (10).

Now let A be a compactly based classical pseudodifferential operator of order 0 with self-adjoint extension, and let $\psi \in C_c^\infty(\mathbb{R}^d)$ be such that $(1\otimes M_\psi)A = A$. Then

$$[K^\alpha, A] = (1\otimes M_\psi)[K^\alpha, A] + [K^\alpha, 1\otimes M_\psi]A.$$

Using equation (10) with $\beta = 0$,

$$[K^\alpha, A] \in (1\otimes M_\psi)[K^\alpha, A] + \mathcal{L}_{d/(\alpha+1),\infty}.$$

By [9, Proposition 10.2.13(c)], the operator $(1\otimes M_\psi)[K^\alpha, A]$ is a compactly based pseudodifferential operator of order $(-\alpha-1)$. By Lemma 6, we conclude that $[K^\alpha, A] \in \mathcal{L}_{d/(\alpha+1),\infty}$.

Let $h(t) = |t|$. Since h is Lipschitz and $d/(\alpha+1) > 1$, from [20], the transformer $T^{A,A}_{h^{[1]}}$ maps $\mathcal{L}_{d/(\alpha+1),\infty}$ to $\mathcal{L}_{d/(\alpha+1),\infty}$. Hence,

$$[K^\alpha, |A|] \in \mathcal{L}_{d/(\alpha+1),\infty}.$$

We note that, using for example polar decomposition on A, that $|A|(1\otimes M_\psi) = |A|$. Then,

$$[K^\alpha, |A|]K^\beta = |A|[K^\alpha, 1\otimes M_\psi]K^\beta + [K^\alpha, |A|](1\otimes M_\psi)K^\beta.$$

We have shown in equation (10) that $[K^\alpha, 1\otimes M_\psi]K^\beta \in \mathcal{L}_{d/(\alpha+\beta+1),\infty}$. From above $[K^\alpha, |A|] \in \mathcal{L}_{d/(\alpha+1),\infty}$ and from Lemma 6, $(1\otimes M_\psi)K^\beta \in \mathcal{L}_{d/\beta,\infty}$, so the result follows from the Hölder inequality. □

Lemma 14. *Let A be a compactly based classical pseudodifferential operator of order 0 with self-adjoint extension, and let $T \in \mathcal{L}_\infty$ be any bounded operator such that*

$$T \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{2d/3,\infty}.$$

Then $|T|^d \in \mathcal{L}_{1,\infty}$ and for any continuous normalised trace on $\mathcal{L}_{1,\infty}$, we have

$$\varphi(|T|^d) = \varphi(|A|^d(1+\mathcal{D}^2)^{-d/2}).$$

Proof. Let $K = (1+\mathcal{D}^2)^{-1/2}$. Since A is compactly based, there is a smooth function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $|A|K = |A|(1 \otimes M_\psi)K$. By Lemma 5, $|A|K \in \mathcal{L}_{d,\infty}$. By assumption, $T - |A|K \in \mathcal{L}_{2d/3,\infty}$, hence $T \in \mathcal{L}_{d,\infty}$.

Taking the adjoint of the condition,

$$T - AK \in \mathcal{L}_{2d/3,\infty}, \tag{11}$$

we have

$$T^* - KA \in \mathcal{L}_{2d/3,\infty}. \tag{12}$$

Multiplying the expressions in (11) and (12) together,

$$(T^* - KA)(T - AK) \in \mathcal{L}_{2d/3,\infty} \cdot \mathcal{L}_{2d/3,\infty}.$$

By Hölder's inequality, $\mathcal{L}_{2d/3,\infty} \cdot \mathcal{L}_{2d/3,\infty} \subseteq \mathcal{L}_{d/3,\infty}$, so we have

$$T^*T - KAT - T^*AK + KA^2K \in \mathcal{L}_{d/3,\infty}.$$

Rearranging,

$$|T|^2 - K|A|^2K \in KAT + T^*AK - 2KA^2K + \mathcal{L}_{d/3,\infty},$$

and

$$|T|^2 - K|A|^2K \in KA(T - AK) + (T^* - KA)AK + \mathcal{L}_{d/3,\infty}.$$

Since AK and KA are in $\mathcal{L}_{d,\infty}$ and $T - AK$ and $T^* - KA$ are each in $\mathcal{L}_{2d/3,\infty}$ from (11) and (12), we conclude from the Hölder inequality that $KA(T-AK)$ and $(T^* - KA)AK$ are in $\mathcal{L}_{2d/5,\infty}$. Thus,

$$|T|^2 - K|A|^2K \in \mathcal{L}_{2d/5,\infty}. \tag{13}$$

We now complete the proof for $d > 2$ and $d = 2$ separately. Assume $d = 2$. Since $\mathcal{L}_{4/5,\infty} \subset \mathcal{L}_1$, we apply a continuous normalised trace φ on $\mathcal{L}_{1,\infty}$ to (13) and we obtain

$$\varphi(|T|^2 - |A|^2K^2) = 0.$$

Since $|A|^2K^2 = |A|^2(1 \otimes M_\psi)K^2$, it follows from Lemma 5 that $|A|^2K^2 \in \mathcal{L}_{1,\infty}$, and from (13) we also have $|T|^2 \in \mathcal{L}_{1,\infty}$. Hence,

$$\varphi(|T|^2) = \varphi(|A|^2K^2)$$

which is the required assertion in the case $d = 2$.

Now to complete the proof, assume $d > 2$.

Lemma 13, setting $\alpha = \beta = \frac{1}{2}$, states that $K|A| \in K^{1/2}|A|K^{1/2} + \mathcal{L}_{d/2,\infty}$. Taking the adjoint we also have $|A|K \in K^{1/2}|A|K^{1/2} + \mathcal{L}_{d/2,\infty}$. Multiplying these identities together we arrive at

$$K|A|^2K \in (K^{1/2}|A|K^{1/2})^2 + K^{1/2}|A|K^{1/2}\mathcal{L}_{d/2,\infty} + \mathcal{L}_{d/2,\infty}K^{1/2}|A|K^{1/2} + \mathcal{L}_{d/2,\infty}\mathcal{L}_{d/2,\infty}.$$

By Lemma 13, $K^{1/2}|A|K^{1/2} \in |A|K + \mathcal{L}_{d/2,\infty}$. Therefore,

$$K|A|^2K \in (K^{1/2}|A|K^{1/2})^2 + \mathcal{L}_{d,\infty}\mathcal{L}_{d/2,\infty} + \mathcal{L}_{d/2,\infty}\mathcal{L}_{d/2,\infty},$$

and using the Hölder inequality,

$$K|A|^2K \in (K^{1/2}|A|K^{1/2})^2 + \mathcal{L}_{d/3,\infty}. \tag{14}$$

Then, adding together (13) and (14),

$$|T|^2 \in (K^{1/2}|A|K^{1/2})^2 + \mathcal{L}_{2d/5,\infty}.$$

We now reduce this to an estimate for $|T|$ using a formula of Birman, Koplienko and Solomyak, [21], which implies that for non-negative self-adjoint operators X and Y with $X - Y \in \mathcal{L}_{2d/5,\infty}$ we have $X^{1/2} - Y^{1/2} \in \mathcal{L}_{4d/5,\infty}$. Applying this to $X = |T|^2$ and $Y = (K^{1/2}|A|K^{1/2})^2$,

$$|T| \in K^{1/2}|A|K^{1/2} + \mathcal{L}_{4d/5,\infty}.$$

Using Lemma 13 again with $\alpha = \beta = 1/2$, and the inclusion $\mathcal{L}_{d/2,\infty} \subset \mathcal{L}_{4d/5,\infty}$ we obtain

$$|T| \in |A|K + \mathcal{L}_{4d/5,\infty}.$$

We now claim that

$$|T|^d \in (|A|K)^d + \mathcal{L}_1. \tag{15}$$

This follows from using the expansion,

$$|T|^d - (|A|K)^d = \sum_{j=0}^{d-1} |T|^j(|T| - |A|K)(A|K|)^{d-j-1},$$

where we can apply the Hölder inequality to the j-th summand,

$$|T|^j(|T| - |A|K)(A|K|)^{d-j-1} \in \mathcal{L}_{d/j,\infty} \cdot \mathcal{L}_{4d/5,\infty} \cdot \mathcal{L}_{d/(d-j-1),\infty} \subseteq \mathcal{L}_{4d/(4d+1),\infty} \subset \mathcal{L}_1.$$

To prove the required statement of the lemma it will be sufficient to show that

$$(|A|K)^d \imath 0n|A|^{d-2}K^d|A|^2+\mathcal{L}_1. \tag{16}$$

We will prove (16) via the following claim: for each integer, $0 \leq j \leq d-2$,

$$(|A|K)^d \in |A|^j K^j (|A|K)^{d-j}+\mathcal{L}_1. \tag{17}$$

We will prove equation (17) by induction on j, with the initial case $j=0$ being trivial. Suppose that (17) is true for $0 \leq j \leq k$, where $0 \leq k < d-2$. Then using the induction hypothesis

$$\begin{aligned}(|A|K)^d &\in |A|^k K^k |A| K (|A|K)^{d-k-1}+\mathcal{L}_1\\ &= |A|^k K^k ([|A|,K]+K|A|)(|A|K)^{d-k-1}+\mathcal{L}_1\\ &= |A|^k K^k [|A|,K](|A|K)^{d-k-1}+|A|^k K^{k+1}|A|(|A|K)^{d-k-1}+\mathcal{L}_1.\end{aligned}$$

Lemma 13 indicates that $[|A|,K] \in \mathcal{L}_{d/2,\infty}$ by setting $\alpha=1$, $\beta=0$ and recalling that $d>2$.

Applying Lemma 5, $|A|^k K^k = |A|^k (1 \otimes M_\psi) K^k \in \mathcal{L}_{d/k,\infty}$ and $(|A|K)^{d-k-1} \in \mathcal{L}_{d/(d-k-1),\infty}$. By the Hölder inequality then,

$$|A|^k K^k [|A|,K](|A|K)^{d-k-1} \in \mathcal{L}_{d/k,\infty}\cdot\mathcal{L}_{d/2,\infty}\cdot\mathcal{L}_{d/(d-k-1),\infty} \subseteq \mathcal{L}_{d/(d+1),\infty} \subset \mathcal{L}_1.$$

Thus,

$$\begin{aligned}(|A|K)^d &\in |A|^k K^{k+1}|A|(|A|K)^{d-k-1}+\mathcal{L}_1\\ &= |A|^k([K^{k+1},|A|]+|A|K^{k+1})(|A|K)^{d-k-1}+\mathcal{L}_1\\ &= |A|^{k+1}K^{k+1}(|A|K)^{d-k-1}+|A|^k[K^{k+1},|A|](|A|K)^{d-k-1}+\mathcal{L}_1.\end{aligned}$$

Now, by Lemma 13, $[K^{k+1},|A|] \in \mathcal{L}_{d/(k+2),\infty}$ using $\alpha=k+1$, $\beta=0$ and recalling that $k<d-2$. Hence $[K^{k+1},|A|](|A|K)^{d-k-1} \in \mathcal{L}_{d/(k+2),\infty}\cdot\mathcal{L}_{d/(d-k-1),\infty}$. So, by Hölder's inequality,

$$\begin{aligned}(|A|K)^d &\in |A|^{k+1}K^{k+1}(|A|K)^{d-k-1}+\mathcal{L}_{d/(d+1),\infty}+\mathcal{L}_1\\ &= |A|^{k+1}K^{k+1}(|A|K)^{d-k-1}+\mathcal{L}_1.\end{aligned}$$

Thus (17) is proved for $j=k+1$. By induction, we have proved (16).

A rearrangement indicates that,

$$\begin{aligned}(|A|K)^d &\in |A|^{d-2}K^{d-2}(|A|K)^2 + \mathcal{L}_1\\ &= |A|^{d-2}K^{d-2}|A|K|A|K + \mathcal{L}_1\\ &= |A|^{d-2}K^{d-2}[|A|,K]|A|K + |A|^{d-2}K^{d-1}|A|^2K + \mathcal{L}_1\\ &= |A|^{d-2}K^{d-2}[|A|,K]|A|K + |A|^{d-2}K^{d-1}[|A|^2,K] + |A|^{d-2}K^d|A|^2 + \mathcal{L}_1.\end{aligned}$$

From the Hölder inequality we have

$$|A|^{d-2}K^{d-2}[|A|,K]|A|K + |A|^{d-2}K^{d-1}[|A|^2,K] \in \mathcal{L}_1.$$

Hence, from (15)

$$|T|^d \in |A|^{d-2}K^d|A|^2 + \mathcal{L}_1.$$

Applying the trace φ,

$$\varphi(|T|^d - |A|^{d-2}K^d|A|^2) = 0.$$

However, since $|A|^{d-2}K^d|A|^2 = |A|^{d-2}(1 \otimes M_\psi)K^d|A|^2$ we conclude from Lemma 5 that $|A|^{d-2}K^d|A|^2 \in \mathcal{L}_{1,\infty}$. Hence $|T|^d \in \mathcal{L}_{1,\infty}$ and

$$\varphi(|T|^d) = \varphi(|A|^{d-2}K^d|A|^2).$$

Due to the cyclicity of φ, the required assertion follows. □

Lemma 15. *Let $f \in C_c^\infty(\mathbb{R}^d)$ be real valued and let $\psi \in C_c^\infty(\mathbb{R}^d)$ be real valued such $f\psi = f$. Define the operator $A_k : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, for $k = 1, \dots, d$ by setting,*

$$A_k := M_{\partial_k f} - \frac{1}{2}\sum_{j=1}^{d}\left(M_{\partial_j f}\frac{D_jD_k}{1-\Delta}M_\psi + M_\psi\frac{D_jD_k}{1-\Delta}M_{\partial_j f}\right).$$

Then $A : \mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d)$ defined by

$$A := \sum_{k=1}^{d} \gamma_k \otimes A_k,$$

is a compactly based classical pseudodifferential operator of order 0 with self-adjoint extension such that

$$i\left[\frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}, 1 \otimes M_f\right] \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{d/2,\infty}.$$

Proof. We note the relations between f and ψ and their derivatives

$$f = \psi f \ , \ \partial_j f = \psi \partial_j f \ , \ f \partial_j \psi = 0 \ , \quad j = 1, \ldots, d. \tag{18}$$

Set $g(\mathcal{D}) := \mathcal{D}(1 + \mathcal{D}^2)^{-1/2}$. We claim that

$$(1 \otimes M_\psi) i[g(\mathcal{D}), 1 \otimes M_f] \in A(1 + \mathcal{D}^2)^{-1/2} + \mathcal{L}_{d/2,\infty} \tag{19}$$

and

$$i[g(\mathcal{D}), 1 \otimes M_\psi](1 \otimes M_f) \in \mathcal{L}_{d/2,\infty}. \tag{20}$$

If we can prove these claims then

$$\begin{aligned} i[g(\mathcal{D}), 1 \otimes M_f] &= i[g(\mathcal{D}), 1 \otimes M_\psi M_f] \\ &= (1 \otimes M_\psi) i[g(\mathcal{D}), 1 \otimes M_f] + i[g(\mathcal{D}), 1 \otimes M_\psi](1 \otimes M_f) \\ &= A(1 + \mathcal{D}^2)^{-1/2} + \mathcal{L}_{d/2,\infty} \end{aligned}$$

and the result would be shown.

We prove the assertion (19). The operator $A(1 + \mathcal{D}^2)^{-1/2}$ has principal symbol, for $x, y \in \mathbb{R}^d$, $\|y\| \geq 1$,

$$p_{-1}(x, y) = \sum_{k=1}^{d} \gamma_k \otimes \frac{1}{\|y\|} \left((\partial_k f)(x) - \frac{y_k}{\|y\|^2} \sum_{j=1}^{d} y_j (\partial_j f)(x) \right).$$

The order 0 pseudodifferential operators $g(\mathcal{D})$ and $1 \otimes M_f$ are both classical pseudodifferential operators. It follows that $i[g(\mathcal{D}), 1 \otimes M_f]$ is a classical pseudodifferential operator of order -1. The principal symbol of the commutator is given by the Poisson bracket, [22, Formula (1.45), p. 13],

$$\begin{aligned} \left\{ p_{0,g(\mathcal{D})}, p_{0,1 \otimes M_f} \right\} &= \sum_{j,k=1}^{d} \gamma_k \otimes \left(\frac{\partial f(x)}{\partial x_j} \frac{\partial (y_k \|y\|^{-1})}{\partial y_j} - \frac{\partial f(x)}{\partial y_j} \frac{\partial (y_k \|y\|^{-1})}{\partial x_j} \right) \\ &= \sum_{j,k=1}^{d} \gamma_k \otimes (\partial_j f)(x) \partial_j (y_k \|y\|^{-1}) \\ &= \sum_{j,k=1}^{d} \gamma_k \otimes (\partial_j f)(x) (\delta_{j,k} \|y\|^{-1} - y_k y_j \|y\|^{-3}) \\ &= \sum_{k=1}^{d} \gamma_k \otimes \frac{1}{\|y\|} \left((\partial_k f)(x) - \frac{y_k}{\|y\|^2} \sum_{j=1}^{d} y_j (\partial_j f)(x) \right). \end{aligned}$$

Hence $i[g(\mathcal{D}), 1 \otimes M_f]$ and $A(1+\mathcal{D}^2)^{-1/2}$ have the same principal symbol. Using equation (18) it is evident that $(1 \otimes M_\psi) i[g(\mathcal{D}), 1 \otimes M_f]$ and $A(1+\mathcal{D}^2)^{-1/2}$ also have the same principal symbol. Hence

$$(1 \otimes M_\psi) i[g(\mathcal{D}), 1 \otimes M_f] - A(1+\mathcal{D}^2)^{-1/2}$$

is a compactly based pseudodifferential operator of order -2. By Lemma 6 it belongs to $\mathcal{L}_{d/2,\infty}$. The claim in formula (19) is proven.

Let B be the operator defined by

$$B := \sum_{k=1}^{d} \gamma_k \otimes \left(M_{\partial_k \psi} - \sum_{j=1}^{d} M_f \frac{D_j D_k}{1-\Delta} M_{\partial_j \psi} \right).$$

By similar calculations, $i[g(\mathcal{D}), 1 \otimes M_\psi]$ and $(1+\mathcal{D}^2)^{-1/2} B$ have the same principal symbol and

$$i[g(\mathcal{D}), 1 \otimes M_\psi] - (1+\mathcal{D}^2)^{-1/2} B$$

is a pseudodifferential operator of order -2. By equation (18)

$$B(1 \otimes M_f) = 0.$$

Hence

$$i[g(\mathcal{D}), 1 \otimes M_\psi](1 \otimes M_f) = (i[g(\mathcal{D}), 1 \otimes M_f] - (1+\mathcal{D}^2)^{-1/2} B)(1 \otimes M_f)$$

is the adjoint of a compactly based pseudodifferential operator of order -2. By Lemma 6 it belongs to $\mathcal{L}_{d/2,\infty}$ and the claim in formula (20) is proven. $\square$

Now we can prove the trace formula for the case where f is smooth and compactly supported.

Lemma 16. *Let $f \in C_c^\infty(\mathbb{R}^d)$ be real valued, and let φ be a continuous normalised trace on $\mathcal{L}_{1,\infty}$. Then $|đf|^d \in \mathcal{L}_{1,\infty}$ and there is a positive constant c_d depending only on d such that*

$$\varphi(|đf|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d \, dx.$$

Proof. Setting $p = 2d/3$ in Lemma 7 indicates that

$$đf \in i\left[\frac{\mathcal{D}}{\sqrt{1+\mathcal{D}^2}}, 1 \otimes M_f\right] + \mathcal{L}_{2d/3,\infty}$$

since $\|f\|_{2d/3}$ is finite when $d > 2$ and $\|f\|_{2;2d/3}$ is finite in the case $d = 2$. We have also used the inclusion $\mathcal{L}_{2d/3} \subseteq \mathcal{L}_{2d/3,\infty}$.

Lemma 15 implies that

$$đf \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{2d/3,\infty}$$

where A is the compactly based classical pseudodifferential operator of order 0 with self-adjoint extension defined in Lemma 15.

By Lemma 14, $|đf|^d \in \mathcal{L}_{1,\infty}$ and for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$,

$$\varphi(|đf|^d) = \varphi(|A|^d(1+\mathcal{D}^2)^{-d/2}).$$

We now complete the proof by applying Lemma 12 with $h(t) = |t|^d$ as follows. Restricted to the unit sphere, the principal symbol of A is

$$p_{0,A}(x,s) = \sum_{k=1}^{d} \gamma_k \otimes i\left((\partial_k f)(x) - s_k \sum_{j=1}^{d} s_j(\partial_j f)(x)\right),\ x \in \mathbb{R}^d, s \in S^{d-1}.$$

Simplified further we write

$$p_{0,A}(x,s) = \sum_{k=1}^{d} \gamma_k \otimes i\sigma(x,s)_k,\ x \in \mathbb{R}^d, s \in S^{d-1}$$

where

$$\sigma(x,s) := (\nabla f)(x) - s(s, (\nabla f)(x)),\ x \in \mathbb{R}^d, s \in S^{d-1}.$$

Using the properties of the gamma matrices γ_j, $j = 1, \ldots, d$,

$$\begin{aligned} |p_{0,A}(x,s)|^2 &= \sum_{k',k=1}^{d} \gamma_{k'}^* \gamma_k \otimes \overline{\sigma(x,s)_{k'}}\sigma(x,s)_k = 1 \otimes \sum_{k=1}^{d} |\sigma(x,s)_k|^2 \\ &= 1 \otimes \|(\nabla f)(x) - s(s,(\nabla f)(x))\|_2^2. \end{aligned}$$

Applying Lemma 12,

$$\begin{aligned} \varphi(|đf|^d) &= k_d \int_{\mathbb{R}^d} \int_{S^{d-1}} \mathrm{tr}(|p_{0,A}(x,s)|^d)\, ds dx \\ &= k_d \int_{\mathbb{R}^d} \int_{S^{d-1}} \|(\nabla f)(x) - s(s,(\nabla f)(x))\|_2^d\, ds dx. \end{aligned}$$

By rotation invariance, we can choose orthonormal coordinates $(e_1, e_2, \dots, e_d)$ for the inner integral such that $\nabla f = \|\nabla f\|_2 e_1$. Thus,

$$\varphi(|đf|^d) = k_d \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d dx \int_{S^{d-1}} \|e_1 - s(s, e_1)\|_2^d \, ds.$$

We let $c_d = k_d \int_{S^{d-1}} \|e_1 - s(s, e_1)\|_2^d \, ds$ and this completes the proof. □

Finally we can prove the trace formula in full generality.

Theorem 17. *Let $d > 1$ and $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be real valued. Then $|đf|^d \in \mathcal{L}_{1,\infty}$ and there is a constant c_d depending on d only such that for every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$ we have*

$$\varphi(|đf|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d dx.$$

Proof. The formula is proven for $f \in C_c^\infty(\mathbb{R}^d)$ by Lemma 16. Let $f \in \dot{W}^{1,d}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. From Theorem 11 we have $đf \in \mathcal{L}_{d,\infty}$ and hence $|đf|^d \in \mathcal{L}_{1,\infty}$. By Theorem 3, we can choose a sequence $\{f_n\}_{n=0}^\infty \subset C^\infty(\mathbb{R}^d)$ such that $\|\nabla f - \nabla f_n\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)} \to 0$ as $n \to \infty$. The result of the theorem is shown if we show convergence of $\int_{\mathbb{R}^d} \|\nabla f_n(x)\|_2^d dx$ to $\int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d dx$ and the convergence of $\varphi(|đf_n|^d)$ to $\varphi(|đf|^d)$.

We show that $\int_{\mathbb{R}^d} \|\nabla f_n(x)\|_2^d dx$ converges to $\int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d dx$.

For brevity set $h_n(x) := \|\nabla f_n(x)\|_2$, $n \geq 0$, and $h(x) := \|\nabla f(x)\|_2$. By the assumption on f and using the equivalence in equation (1), $\|h\|_d < \infty$. Notice also that $\|\nabla f - \nabla f_n\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)} \to 0$ implies the uniform boundedness of $\|\nabla f_n\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)}$. Hence $\sup_{n \geq 0} \|h_n\|_d < \infty$. We have

$$\left| \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d - \|\nabla f_n(x)\|_2^d dx \right| \leq \|h^d - h_n^d\|_1, \quad n \geq 0.$$

Using the identity,

$$x^d - y^d = \sum_{j=0}^{d-1} x^j (x - y) y^{d-j-1}, \quad x, y \in \mathbb{C}$$

and Hölder's inequality

$$\begin{aligned}\|h^d - h_n^d\|_1 &\leq \sum_{j=0}^{d-1} \|h^j\|_{d/j}\|h-h_n\|_d\|h_n^{d-j-1}\|_{d/(d-j-1)} \\ &\leq \sum_{j=0}^{d-1} \|h\|_d^j\|h-h_n\|_d\|h_n\|_d^{d-j-1} \\ &\leq K_1\|h-h_n\|_d\end{aligned}$$

where K_1 is a constant independent of n. Now

$$\begin{aligned}\|h-h_n\|_d^d &= \int_{\mathbb{R}^d} |\|\nabla f(x)\|_2 - \|\nabla f_n(x)\|_2|^d dx \\ &\leq \int_{\mathbb{R}^d} \|\nabla f(x) - \nabla f_n(x)\|_2^d dx \\ &\leq K_2\|\nabla f - \nabla f_n\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}^d\end{aligned}$$

where K_2 is a constant independent of n obtained from the equivalence in equation (1). We have shown that

$$\left|\int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d - \|\nabla f_n(x)\|_2^d dx\right| \leq K_3\|\nabla f - \nabla f_n\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}, \quad n \geq 0$$

for a constant K_3 independent of n. The required convergence follows.

We show that $\varphi(|\text{đ}f_n|^d)$ converges to $\varphi(|\text{đ}f|^d)$.

As φ is a continuous normalised trace on $\mathcal{L}_{1,\infty}$,

$$|\varphi(|\text{đ}f|^d - |\text{đ}f_n|^d)| \leq \||\text{đ}f|^d - |\text{đ}f_n|^d\|_{1,\infty}.$$

Let $d > 1$ and let X and Y be self-adjoint operators in $\mathcal{L}_{d,\infty}$. Then

$$|X|^d - |Y|^d = \sum_{j=0}^{d-1} |X|^j(|X|-|Y|)|Y|^{d-j-1}.$$

The Hölder inequality, applied to the j-th summand, gives

$$\begin{aligned}\||X|^j(|X|-|Y|)|Y|^{d-j-1}\|_{1,\infty} &\leq \|X^j\|_{d/j,\infty}\||X|-|Y|\|_{d,\infty}\|Y^{d-j-1}\|_{d/(d-j-1),\infty} \\ &\leq \|X\|_{d,\infty}^j\||X|-|Y|\|_{d,\infty}\|Y\|_{d,\infty}^{d-j-1}.\end{aligned}$$

Since $d > 1$, by [20] we have a constant $K_4 > 0$ such that

$$\||X| - |Y|\|_{d,\infty} \leq K_4 \|X - Y\|_{d,\infty}.$$

Hence, using the quasinorm triangle inequality, there is a constant $K_5 > 0$ such that

$$\||X|^d - |Y|^d\|_{1,\infty} \leq K_5 \sum_{j=0}^{d} \|X\|_{d,\infty}^{j} \||X - Y\|_{d,\infty} \|Y\|_{d,\infty}^{d-j-1}. \tag{21}$$

From Theorem 11, $\|đf - đf_n\|_{d,\infty} \leq B_d \|\nabla f - \nabla f_n\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$. Hence, $\|đf - đf_n\|_{d,\infty} \to 0$ as $n \to \infty$ and $\sup_{n\geq 0} \|đf_n\|_{d,\infty} < \infty$. Substituting into equation (21) we have the inequality,

$$|\varphi(|đf|^d - |đf_n|^d)| \leq \||đf|^d - |đf_n|^d\|_{1,\infty} \leq K_6 \|đf - đf_n\|_{d,\infty}$$

for a constant $K_6 > 0$ independent of n. We have the required convergence.

This completes the proof. □

Remark 18. *The constant c_d in Theorem 17 has no dependence on the continuous normalised trace φ. This implies that the operator $|đf|^d \in \mathcal{L}_{1,\infty}$ is a measurable operator in the sense of Connes' quantised calculus, [9, Chapter 10]. It also implies convergence properties of the ordered eigenvalues of $|đf|^d$, see [23].*

5. Necessity

In this section we will complete the proof of Theorem 1. That is, if $f \in L^\infty(\mathbb{R}^d)$ is such that $đf \in \mathcal{L}_{d,\infty}$, then $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$ and

$$c_d \|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \leq \|đf\|_{d,\infty}$$

for a constant $c_d > 0$.

The following two lemmas are routine. The next result follows from [24, Proposition 2.5.18] and interpolation. We provide a short proof for completeness.

Lemma 19. *Let ν be a finite signed Borel measure on $\mathbb{R}$. Let $t \to V_t$ be a strongly continuous family of unitary operators, and let A be a bounded operator on a separable Hilbert space $\mathcal{H}$.*

Define the integral $\int_{\mathbb{R}} V_t^{-1} A V_t d\nu(t)$ *as the unique operator* $T(A) \in \mathcal{L}_\infty(\mathcal{H})$ *such that for all* $x, y \in \mathcal{H}$*,*

$$(x, T(A)y) = \int_{\mathbb{R}} (x, V_t^{-1} A V_t y)\, d\nu(t).$$

If $A \in \mathcal{L}_{d,\infty}$ *then* $\int_{\mathbb{R}} V_t^{-1} A V_t d\nu(t) \in \mathcal{L}_{d,\infty}$*, and*

$$\left\| \int_{\mathbb{R}} V_t^{-1} A V_t d\nu(t) \right\|_{d,\infty} \leq \|A\|_{d,\infty} \|\nu\|.$$

Proof. Let $\mathcal{E}_N := \{E_n^N = [a_n, b_n]\}_{n=0}^{J(N)}$, $N \geq 0$, be finite sets of intervals of $\mathbb{R}$ such that $\sum_{n=0}^{J(N)} f(a_n)\nu(E_n^N)$ converges to $\int_{\mathbb{R}} f(x) d\nu(x)$ as $N \to \infty$ for every continuous function $f \in C(\mathbb{R})$. The sequence of bounded operators

$$T_N(A) := \sum_{n=0}^{J(N)} V_{a_n}^{-1} A V_{a_n} \nu(E_n^N),\ N \geq 0$$

converges weakly to $T(A)$. Using [9, Theorem 3.3.3] and properties of the singular value function [9, Lemma 2.3.12], for each $N \geq 0$ the operators $T_N(A)$ are submajorised by $A \cdot \|\nu\|$,

$$\mu(t, T_N(A)) \prec\prec \sum_{n=0}^{J(N)} \mu(t, V_{a_n}^{-1} A V_{a_n} \nu(E_n^N)) \leq \mu(t, A) \cdot \|\nu\|, \quad t \geq 0. \tag{22}$$

Suppose $A \in \mathcal{L}_1$. Equation (22) implies that $T_N(A) \in \mathcal{L}_1$ and $\|T_N(A)\|_1 \leq \|A\|_1 \|\nu\|$ for each $N \geq 0$. By the noncommutative Fatou lemma, [14, Theorem 2.7(d)], $T(A) \in \mathcal{L}_1$ and $\|T(A)\|_1 \leq \|A\|_1 \|\nu\|$. The same argument verbatim shows that if $A \in \mathcal{L}_\infty$, then $T(A) \in \mathcal{L}_\infty$ and $\|T(A)\| \leq \|A\| \|\nu\|$. Hence T is a linear operator such that $\|T\|_{\mathcal{L}_1 \to \mathcal{L}_1} \leq \|\nu\|$ and $\|T\|_{\mathcal{L}_\infty \to \mathcal{L}_\infty} \leq \|\nu\|$. The result of the lemma follows from real interpolation on the pair $(\mathcal{L}_1, \mathcal{L}_\infty)$, [19]. □

Lemma 20. *Let* $1 < p < \infty$*. Let* $\psi \in C_c^\infty(\mathbb{R}^d)$ *be radially symmetric and equal to* 1 *in a neighbourhood of the origin. Define* $\psi_n(x) := (\sigma_n \psi)(x) = \psi(x/n)$ *for* $n \geq 1$*. If* $f \in L^\infty(\mathbb{R}^d)$ *satisfies*

$$\sup_{n \geq 1} \|\nabla(\psi_n (f * \widehat{\psi_n}))\|_{L^p(\mathbb{R}^d, \mathbb{C}^d)} < \infty \tag{23}$$

then $\nabla f \in L^p(\mathbb{R}^d, \mathbb{C}^d)$*.*

Proof. By assumption, the sequence $\{\nabla(\psi_n(f*\widehat{\psi}_n))\}_{n=1}^{\infty}$ is uniformly bounded in $L^p(\mathbb{R}^d,\mathbb{C}^d)$. Since $1<p<\infty$, $L^p(\mathbb{R}^d,\mathbb{C}^d)$ is reflexive and there is a subsequence of $\{\nabla(\psi_n(f*\widehat{\psi}_n))\}_{n=1}^{\infty}$ which converges weakly to some $g\in L^p(\mathbb{R}^d,\mathbb{C}^d)$.

Let $\phi\in\mathcal{S}(\mathbb{R}^d)$. Then,

$$(\psi_n(f*\widehat{\psi}_n),\phi)=(\widehat{\psi}_n*f,\psi_n\phi)=(f,\widehat{\psi}_n*(\psi_n\phi)).$$

The sequence $\{\widehat{\psi}_n*(\psi_n\phi)\}_{n=1}^{\infty}$ converges in the standard topology of $\mathcal{S}(\mathbb{R}^d)$ to ϕ, [25, Exercise 2.3.5]. Hence, $\{\psi_n(f*\widehat{\psi}_n)\}_{n=1}^{\infty}$ converges to f in the sense of tempered distributions, and so $\{\nabla(\psi_n(f*\widehat{\psi_n}))\}_{n=1}^{\infty}$ converges to the distributional derivative ∇f in the sense of tempered distributions.

Since weak convergence in $L^p(\mathbb{R}^d,\mathbb{C}^d)$ implies convergence in the sense of distributions, we conclude that $g=\nabla f$, and so $\nabla f\in L^p(\mathbb{R}^d,\mathbb{C}^d)$.

$\square$

Theorem 21. *Let* $f\in L^\infty(\mathbb{R}^d)$ *and* $d>1$. *If* ${\mathchar'26\mkern-12mu d}f\in\mathcal{L}_{d,\infty}$, *then* $f\in\dot{W}^{1,d}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ *and there is a constant* $c_d>0$ *depending only on* d *such that*

$$c_d\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}\le\|{\mathchar'26\mkern-12mu d}f\|_{d,\infty}.$$

Proof. Suppose $f\in L^\infty(\mathbb{R}^d)$ is real valued. If we establish that $\nabla f\in L^d(\mathbb{R}^d,\mathbb{C}^d)$ then the estimate follows from Theorem 17. Theorem 17 implies that there is a constant $c_d>0$ such that, for all continuous normalised traces φ on $\mathcal{L}_{1,\infty}$,

$$c_d\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}=\varphi(|{\mathchar'26\mkern-12mu d}f|^d)^{1/d}\le\|{\mathchar'26\mkern-12mu d}f\|_{d,\infty}. \tag{24}$$

In (24) we implicitly use the equivalence in equation (1).

We establish that $\nabla f\in L^d(\mathbb{R}^d,\mathbb{C}^d)$ when f is real valued. Suppose that $f\in L^\infty(\mathbb{R}^d)$ is such that ${\mathchar'26\mkern-12mu d}f\in\mathcal{L}_{d,\infty}$. Let $\{\psi_n\}_{n=1}^{\infty}$ be the sequence defined in Lemma 20. The Fourier transform $\widehat{\psi_n}$ satisfies

$$\widehat{\psi_n}(s)=n\widehat{\psi}(ns),\quad s\in\mathbb{R}^d.$$

Then

$$\|\widehat{\psi_n}\|_1=\int_{\mathbb{R}^d}|\widehat{\psi_n}(s)|ds=\int_{\mathbb{R}^d}|\widehat{\psi}(ns)|d(ns)=\|\widehat{\psi}\|_1$$

which is finite since $\widehat{\psi}$ is a Schwartz function. Since $\psi_n(f*\widehat{\psi_n})$ is smooth and compactly supported we can apply equation (24), and we have

$$c_d\|\nabla(\psi_n(f*\widehat{\psi_n}))\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}\le\|{\mathchar'26\mkern-12mu d}(\psi_n(f*\widehat{\psi_n}))\|_{d,\infty}.$$

Expanding out $đ(\psi_n(f * \widehat{\psi_n}))$,

$$\|đ(\psi_n(f * \widehat{\psi_n}))\|_{d,\infty} \leq K(\|(đ\psi_n)M_{f*\widehat{\psi_n}}\|_{d,\infty} + \|M_{\psi_n}đ(f * \widehat{\psi}_n)\|_{d,\infty})$$

where the constant $K > 0$ above is needed since $\|\cdot\|_{d,\infty}$ is a quasinorm. Expanding further the first term using Young's inequality

$$\begin{aligned}\|đ(\psi_n)M_{f*\widehat{\psi_n}}\|_{d,\infty} &\leq \|đ\psi_n\|_{d,\infty}\|f * \widehat{\psi_n}\|_\infty \leq \|đ\psi_n\|_{d,\infty}\|f\|_\infty\|\widehat{\psi_n}\|_1 \\ &\leq \|đ\psi_n\|_{d,\infty}\|f\|_\infty\|\widehat{\psi}\|_1.\end{aligned}$$

Looking at the second term, using $\|\psi_n\|_\infty = 1$,

$$\|M_{\psi_n}đ(f * \widehat{\psi_n})\|_{d,\infty} \leq \|đ(f * \widehat{\psi_n})\|_{d,\infty}.$$

Let $t \mapsto U_t$ be the group action of translations on $\mathbb{R}^d$. That is, for $t, s \in \mathbb{R}^d$, $(U_t f)(s) = f(s-t)$. By the definition of convolution,

$$f * \widehat{\psi}_n(s) = \int_{\mathbb{R}^d} (U_t f)(s)\widehat{\psi}_n(t)dt.$$

Hence, since $\mathcal{D}$ commutes with translations,

$$[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_{f*\widehat{\psi}_n}] = \int_{\mathbb{R}^d} (1 \otimes U_t^{-1})[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f](1 \otimes U_t)\widehat{\psi}_n(t)dt.$$

Applying Lemma 19 to the finite signed Borel measure $\widehat{\psi}_n(t)dt$,

$$\begin{aligned}\|[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_{f*\widehat{\psi}_n}]\|_{d,\infty} &\leq \|[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty}\|\widehat{\psi}_n\|_1 \\ &\leq \|[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty}\|\widehat{\psi}\|_1.\end{aligned}$$

We have proved that,

$$\|\nabla(\psi_n(f * \widehat{\psi}_n))\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \leq (c_1\|đ\psi_n\|_{d,\infty}\|f\|_\infty + c_2\|đf\|_{d,\infty})\|\widehat{\psi}\|_1.$$

for some $c_1, c_2 > 0$. As $\psi_n \in C_c^\infty(\mathbb{R}^d)$, we apply Theorem 11 to obtain,

$$\|đ\psi_n\|_{d,\infty} \leq C_d\|\nabla\psi_n\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

However $\psi_n := \sigma_n\psi$, and this implies that $\nabla\psi_n = \frac{1}{n}\sigma_n\nabla\psi$. Thus,

$$\|đ\psi_n\|_{d,\infty} \leq \frac{C_d}{n}\|\sigma_n\nabla\psi\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

Since $\|\sigma_n \nabla\psi\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} = n\|\nabla\psi\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$, we obtain

$$\sup_{n\geq 1} \|\nabla(\psi_n(f * \widehat{\psi}_n))\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \leq (c_1 C_d \|\nabla\psi\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} + c_2\|đf\|_{d,\infty})\|\widehat{\psi}\|_1.$$

By Lemma 20 we conclude that $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$.

If $f \in L^\infty(\mathbb{R}^d)$ is complex valued, write $f = f_1 + if_2$ where

$$f_1 = \frac{1}{2}(f + \overline{f})\ ,\ f_2 = \frac{1}{2i}(f - \overline{f})$$

are real valued. If $đf \in \mathcal{L}_{d,\infty}$ then $(đf)^* = đ\overline{f} \in \mathcal{L}_{d,\infty}$. Hence $đf_1, đf_2 \in \mathcal{L}_{d,\infty}$, which implies by the above results that $\nabla f_1, \nabla f_2 \in L^d(\mathbb{R}^d,\mathbb{C}^d)$. By linearity $\nabla f \in L^d(\mathbb{R}^d,\mathbb{C}^d)$.

The norm estimate in (24) applies to the real terms f_1 and f_2. Then

$$\begin{aligned}\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} &\leq \|\nabla f_1\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} + \|\nabla f_2\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \\ &\leq K_1(\|đf_1\|_{d,\infty} + \|đf_2\|_{d,\infty})\end{aligned}$$

for a constant $K_1 > 0$. Using the quasinorm triangle inequality, there is a constant $K_2 > 0$ such that

$$\begin{aligned}\|đf_1\|_{d,\infty} + \|đf_2\|_{d,\infty} &= \frac{1}{2}\|đ(f+\overline{f})\|_{d,\infty} + \frac{1}{2}\|đ(f-\overline{f})\|_{d,\infty} \\ &\leq K_2(\|đf\|_{d,\infty} + \|đ\overline{f}\|_{d,\infty}) \\ &= K_2(\|đf\|_{d,\infty} + \|(đf)^*\|_{d,\infty}) = 2K_2\|đf\|_{d,\infty}.\end{aligned}$$

Hence

$$\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \leq 2K_1K_2\|đf\|_{d,\infty}.$$

The result is proven by setting $c_d = (2K_1K_2)^{-1}$.

□

[1] A. Connes, Noncommutative geometry, Academic Press, Inc., San Diego, CA, 1994.

[2] W. Rudin, Functional analysis, 2nd Edition, International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York, 1991.

[3] V. V. Peller, Hankel operators and their applications, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[4] S. Janson, T. H. Wolff, Schatten classes and commutators of singular integral operators, Ark. Mat. 20 (2) (1982) 301–310.

[5] A. Connes, D. Sullivan, N. Teleman, Quasiconformal mappings, operators on Hilbert space, and local formulae for characteristic classes, Topology 33 (4) (1994) 663–681.

[6] R. Rochberg, S. Semmes, Nearly weakly orthonormal sequences, singular value estimates, and Calderon-Zygmund operators, J. Funct. Anal. 86 (2) (1989) 237–306.

[7] A. Connes, The action functional in noncommutative geometry, Comm. Math. Phys. 117 (4) (1988) 673–683.

[8] H. B. Lawson, Jr., M.-L. Michelsohn, Spin geometry, Vol. 38 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1989.

[9] S. Lord, F. Sukochev, D. Zanin, Singular traces: theory and applications, Vol. 46, Walter de Gruyter, 2012.

[10] L. Grafakos, Modern Fourier analysis, 3rd Edition, Vol. 250 of Graduate Texts in Mathematics, Springer, New York, 2014.

[11] C. Ortner, E. Süli, A note on linear elliptic systems on $\mathbb{R}^d$, arXiv preprint arXiv:1202.3970.

[12] D. Potapov, F. Sukochev, Unbounded Fredholm modules and double operator integrals, J. Reine Angew. Math. 626 (2009) 159–185.

[13] M. Ruzhansky, V. Turunen, Pseudo-differential operators and symmetries, Vol. 2 of Pseudo-Differential Operators. Theory and Applications, Birkhäuser Verlag, Basel, 2010, background analysis and advanced topics. doi:10.1007/978-3-7643-8514-9.
URL `http://dx.doi.org/10.1007/978-3-7643-8514-9`

[14] B. Simon, Trace ideals and their applications, Vol. 35 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge-New York, 1979.

[15] M. S. Birman, G. E. Karadzhov, M. Z. Solomyak, Boundedness conditions and spectrum estimates for the operators $b(X)a(D)$ and their analogs 7 (1991) 85–106.

[16] M. S. Birman, M. Z. Solomyak, Operator integration, perturbations and commutators, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 170 (Issled. Linein. Oper. Teorii Funktsii. 17) (1989) 34–66, 321.

[17] M. Š. Birman, M. Z. Solomjak, Double Stieltjes operator integrals, in: Probl. Math. Phys., No. I, Spectral Theory and Wave Processes (Russian), Izdat. Leningrad. Univ., Leningrad, 1966, pp. 33–67.

[18] D. Potapov, F. Sukochev, Double operator integrals and submajorization, Math. Model. Nat. Phenom. 5 (4) (2010) 317–339.

[19] P. G. Dodds, T. K. Dodds, B. de Pagter, Fully symmetric operator spaces, Integral Equations Operator Theory 15 (6) (1992) 942–972.

[20] D. Potapov, F. Sukochev, Operator-Lipschitz functions in Schatten-von Neumann classes, Acta Math. 207 (2) (2011) 375–389.

[21] M. S. Birman, M. Z. Solomyak, Estimates for the difference of fractional powers of selfadjoint operators under unbounded perturbations, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 178 (Issled. Linein. Oper. Teorii Funktsii. 18) (1989) 120–145, 185.

[22] M. Agranovich, M. Capinski, Y. Egorov, S. Ejdel'man, M. Shubin, R. Cooke, S. Levendorskij, B. Paneah, Partial Differential Equations VI: Elliptic and Parabolic Operators, Encyclopaedia of Mathematical Sciences, Springer Berlin Heidelberg, 2013.
URL `https://books.google.co.uk/books?id=Tqr9CAAAQBAJ`

[23] E. Semenov, F. Sukochev, A. Usachev, D. Zanin, Banach limits and traces on $\mathcal{L}_{1,\infty}$, Adv. Math. 285 (2015) 568–628.

[24] O. Bratteli, D. W. Robinson, Operator Algebras and Quantum Statistical Mechanics: Volume 1: C*-and W*-Algebras. Symmetry Groups. Decomposition of States, Springer Science & Business Media, 2012.

[25] L. Grafakos, Classical Fourier analysis, 3rd Edition, Vol. 249 of Graduate Texts in Mathematics, Springer, New York, 2014.