



Hyperbolically embedded subgroups and quasi-isometries of pairs

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Abstract. We give technical conditions for a quasi-isometry of pairs to preserve a subgroup being hyperbolically embedded. We consider applications to the quasi-isometry and commensurability invariance of acylindrical hyperbolicity of finitely generated groups.

1 Introduction

A group G is *acylindrically hyperbolic* if it admits a nonelementary, acylindrical action on a hyperbolic space. An alternative characterization is that G is acylindrically hyperbolic if and only if G contains a *hyperbolically embedded subgroup* H , denoted $H \hookrightarrow_h G$, and we will give a characterization from [13] in Proposition 3.1.

The class of acylindrically hyperbolic groups generalizes the classes of nonelementary hyperbolic and relatively hyperbolic groups while sharing many similar properties [17]. In spite of this, there are still foundational questions that remain open, for instance, it is known that a group being hyperbolic or relatively hyperbolic is invariant under quasi-isometry [8, 9], but the corresponding question for acylindrical hyperbolicity is still open.

Question 1.1 [17, Question 2.20(a)] Is the class of finitely generated acylindrically hyperbolic groups closed under quasi-isometry?

Some partial results are known, for instance, acylindrical hyperbolicity passes to finite-index subgroups and is preserved by quotienting out a finite normal subgroup [15]. If the group is \mathcal{AH} -accessible, then acylindrical hyperbolicity can be passed to finite extensions [16]. The property of being \mathcal{AH} -accessible also passes to finite-index overgroups [3]. However, not every finitely presented acylindrically hyperbolic group

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is \mathcal{AH} -accessible [1, Theorem 2.18]. Some experts in the field do not expect a complete positive answer to Question 1.1.

This article relies on the notion of quasi-isometry of pairs, and our results provide technical conditions to ensure that a quasi-isometry of pairs carries the property of being a hyperbolically embedded subgroup.

Definition 1.1 (Quasi-isometry of pairs) Let X and Y be metric spaces, and let \mathcal{A} and \mathcal{B} be collections of subspaces of X and Y , respectively. A quasi-isometry $q: X \rightarrow Y$ is a *quasi-isometry of pairs* $q: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ if there is $M > 0$:

- (1) For any $A \in \mathcal{A}$, the set $\{B \in \mathcal{B}: \text{hdist}_Y(q(A), B) < M\}$ is nonempty.
- (2) For any $B \in \mathcal{B}$, the set $\{A \in \mathcal{A}: \text{hdist}_Y(q(A), B) < M\}$ is nonempty.

In this case, if $q: X \rightarrow Y$ is an (L, C) -quasi-isometry, then $q: (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ is called an (L, C, M) -*quasi-isometry*. If there is a quasi-isometry of pairs $(X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$, we say that (X, \mathcal{A}) and (Y, \mathcal{B}) are *quasi-isometric pairs*.

We specialize the previous definition to the case of finitely generated groups with finite collections of subgroups as follows.

Definition 1.2 (Quasi-isometry of group pairs) Consider two pairs (G, \mathcal{P}) and (H, \mathcal{Q}) where G and H are finitely generated groups with chosen word metrics dist_G and dist_H . Denote the Hausdorff distance between subsets of H by hdist_H . An (L, C) -quasi-isometry $q: G \rightarrow H$ is an (L, C, M) -*quasi-isometry of pairs* $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ if the relation

$$\dot{q} = \{(A, B) \in G/\mathcal{P} \times H/\mathcal{Q}: \text{hdist}_H(q(A), B) < M\}$$

satisfies that the projections into G/\mathcal{P} and H/\mathcal{Q} are surjective.

Example 1.1 (Quasi-isometry of pairs and finite extensions) Let H be a finite index normal subgroup of finitely generated group G , and let \mathcal{Q} be a finite collection of subgroups of H . Then the inclusion $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q})$ is a quasi-isometry of pairs if the collection $\{hQh^{-1}: h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G (see Proposition 4.1).

Recall that the *commensurator* of a subgroup P of a group G is the subgroup

$$\text{Comm}_G(P) = \{g \in G: P \cap gPg^{-1} \text{ is a finite index subgroup of } P \text{ and } gPg^{-1}\}.$$

Definition 1.3 (Refinements) Let \mathcal{P} be a collection of subgroups of group G . A *refinement* \mathcal{P}^* of \mathcal{P} is a set of representatives of conjugacy classes of the collection of subgroups

$$\{\text{Comm}_G(gPg^{-1}): P \in \mathcal{P} \text{ and } g \in G\}.$$

Example 1.2 (Refinements and qi of pairs) Let \mathcal{Q} be a finite collection of subgroups of a finitely generated group H , and let \mathcal{Q}^* be a refinement. If each $Q \in \mathcal{Q}$ is finite index in $\text{Comm}_H(Q)$, then the identity map on G is a quasi-isometry of pairs $(H, \mathcal{Q}) \rightarrow (H, \mathcal{Q}^*)$.

Example 1.3 (Refinements and finite extensions) Let A be a group, let \mathcal{H} be an almost malnormal collection of infinite subgroups, and let $F \leq \text{Aut}(A)$ be a finite subgroup. If F acts freely on \mathcal{H} and \mathcal{H}_F is a collection of representatives of F -orbits in \mathcal{H} , then a refinement of \mathcal{H} in $A \rtimes F$ is \mathcal{H}_F .

Definition 1.4 (Reduced collections) A collection of subgroups \mathcal{P} of a group G is *reduced* if for any $P, Q \in \mathcal{P}$ and $g \in G$, if P and gQg^{-1} are commensurable, then $P = Q$ and $g \in P$.

Our first result, Theorem A, describes a strategy to obtain positive results to Question 1.1. For a group G with a generating set S , let $\Gamma(G, S)$ denote the corresponding Cayley graph (see Definition 2.3).

Theorem A (Theorem 3.1) Let $q: G \rightarrow H$ be a quasi-isometry of finitely generated groups, let \mathcal{P} and \mathcal{Q} be finite collections of subgroups of G and H , respectively, and let S and T be (not necessarily finite) generating sets of G and H , respectively. Suppose that:

- (1) $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs and
- (2) $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry.

The following statements hold:

- (1) If \mathcal{P} and \mathcal{Q} are reduced collections in G and H , respectively, then $\mathcal{P} \hookrightarrow_h (G, S)$ if and only if $\mathcal{Q} \hookrightarrow_h (H, T)$.
- (2) If \mathcal{Q} contains only infinite subgroups and $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P}^* \hookrightarrow_h (G, S)$.

1.1 Qi-characteristic collections

The first numbered hypothesis of Theorem A raises the following general problem: Given a finite collection of subgroups \mathcal{Q} of a group H and a quasi-isometry $q: G \rightarrow H$ of finitely generated groups, is there a collection \mathcal{P} of subgroups of G such that $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs?

This problem was studied in [14] where the notion of qi-characteristic collection is introduced and it is proved that if the collection \mathcal{Q} is qi-characteristic in H , then any quasi-isometry of finitely generated groups induces a collection \mathcal{P} .

Definition 1.5 (Qi-characteristic [14]) A collection of subgroups \mathcal{P} of a finitely generated group G is *quasi-isometrically characteristic* (or shorter *qi-characteristic*) if \mathcal{P} is finite; each $P \in \mathcal{P}$ has finite index in its commensurator; and for every $L \geq 1$ and $C \geq 0$, there is $M = M(G, \mathcal{P}, L, C) \geq 0$ such that every (L, C) -quasi-isometry $q: G \rightarrow G$ is an (L, C, M) -quasi-isometry of pairs $q: (G, \mathcal{P}) \rightarrow (G, \mathcal{P})$.

Example 1.4. The argument by Behrstock, Druţu, and Mosher proving quasi-isometric rigidity of relative hyperbolicity with respect to nonrelatively hyperbolic groups (NRH groups) shows that if H is hyperbolic group relative to a collection \mathcal{Q} of NRH subgroups, then \mathcal{Q} is qi-characteristic [4, Theorems 4.1 and 4.8]. Another example is provided by mapping class groups. Ruling out a few surfaces of low complexity, any self-quasi-isometry of the mapping class group is at uniform distance

from left multiplication by an element of the group (see the work of Behrstock, Kleiner, Minsky, and Mosher [6, Theorem 1.1]). As a consequence, the hyperbolically embedded (virtually cyclic) subgroup generated by a pseudo-Anosov is qi -characteristic. More generally, any finite collection of subgroups of such mapping class groups are qi -characteristic.

Corollary B *Let G and H be finitely generated groups, let T be a generating set of H , let \mathcal{Q} be a finite collection of subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H, T)$, and let $q: G \rightarrow H$ be a quasi-isometry. If:*

- (1) \mathcal{Q} is a qi -characteristic collection of subgroups of H and
- (2) *there is a generating set $S \subset G$ such that $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry,*

then there is a finite collection \mathcal{P} of subgroups of G such that $\mathcal{P} \hookrightarrow_h (G, S)$ and $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs.

Proof Without loss of generality, assume that all subgroups in \mathcal{Q} are proper infinite subgroups. Note that removing finite subgroups from \mathcal{Q} preserves being qi -characteristic and that $\mathcal{Q} \hookrightarrow_h (H, T)$. On the other hand, if \mathcal{Q} contains H , then the theorem is trivial by taking \mathcal{P} the collection that contains only G and S any finite generating set of G . Since \mathcal{Q} is qi -characteristic, the quasi-isometry $q: G \rightarrow H$ induces a finite collection \mathcal{P} such that $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs, and this is precisely [14, Theorem 1.1]. Then the second statement of Theorem A and $\mathcal{Q} \hookrightarrow_h (H, T)$ imply that $\mathcal{P}^* \hookrightarrow_h (G, S)$. ■

1.2 Uniform quasi-actions

The second numbered hypothesis of Theorem A raises the problem: Given a group H with a generating set T and a quasi-isometry $q: G \rightarrow H$ of finitely generated groups, is there a generating set $S \subset G$ such that $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry of Cayley graphs?

We show that a positive answer to this question is equivalent to asking that the quasi-action of G on H induced by q is T -uniform in the following sense (see Proposition C).

Definition 1.6 (Uniform induced quasi-action) *Let G and H be finitely generated groups, and let $q: G \rightarrow H$ be a quasi-isometry with quasi-inverse \bar{q} . Let $T \subset H$ be a generating set (possibly infinite). We say that the quasi-action of G on H induced by q is uniform with respect to T if there are constants $L \geq 1$, $C \geq 0$ such that, for each $g \in G$, the function $q_g: H \rightarrow H$ given by $q_g(h) = q(g \cdot \bar{q}(h))$ is an (L, C) -quasi-isometry $q_g: \Gamma(H, T) \rightarrow \Gamma(H, T)$.*

Example 1.5 (Uniform quasi-action and finite extensions) *Let H be a finite index normal subgroup of finitely generated group G , and let T be a generating set of H invariant under conjugation by G . The G -action by conjugation on H preserves the word metric induced by T . On the other hand, any transversal R of H in G induces*

a quasi-isometry $q: G \rightarrow H$ given by $q(hg) = h$ for $h \in H$ and $g \in R$. In this case, the quasi-action of G on H induced by q is uniform with respect to T (see Lemma 2.1).

Proposition C (Proposition 2.1) *Let G and H be groups with finite generating sets S_0 and T_0 , and let $q: \Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$ be a quasi-isometry. Let $T \subset H$ containing T_0 . The following statements are equivalent:*

- (1) *The quasi-action of G on H induced by q is uniform with respect to T .*
- (2) *There is $S \subset G$ containing S_0 such that $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry.*

Corollary D *Let G and H be finitely generated groups with finite collections of infinite subgroups \mathcal{P} and \mathcal{Q} , respectively. Suppose that $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs inducing a T -uniform quasi-action of G on H . If $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P}^* \hookrightarrow_h G$.*

Proof Since the quasi-action of G on H induced by q is T -uniform, Proposition C implies that there is a generating set S of G such that $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry. Then the second statement of Theorem A and $\mathcal{Q} \hookrightarrow_h (H, T)$ imply that $\mathcal{P}^* \hookrightarrow_h (G, S)$. ■

Let us remark that for this last corollary, in the case that T is finite, then there is a finite $S \subset G$ such that $\mathcal{P} \hookrightarrow_h (G, S)$; this case is implied by the results on quasi-isometric rigidity of relative hyperbolicity in [4].

1.3 Finite extensions

The following application is a particular instance of Theorem 4.1 in the main body of the article.

Theorem E (Theorem 4.1) *Let H be a finite index normal subgroup of a finitely generated group G , and let \mathcal{Q} be a finite collection of infinite subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H, T)$. Suppose that:*

- (1) *The set T is invariant under conjugation by G .*
- (2) *The collection $\{hQh^{-1}: h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G .*

If \mathcal{Q}^ is a refinement of \mathcal{Q} in G , then $\mathcal{Q}^* \hookrightarrow_h G$.*

Example 1.6. Let $G = \langle a, b, t: tat^{-1} = b, t^2 = 1 \rangle \cong F_2 \rtimes \mathbb{Z}_2$, let $H = \langle a, b \rangle$, and let $\mathcal{Q} = \{\langle a \rangle, \langle b \rangle\}$. Note that $\mathcal{Q} \hookrightarrow_h H$, and, for instance, one can take $\mathcal{Q}^* = \{\langle a \rangle\}$ and observe that $\mathcal{Q}^* \hookrightarrow_h G$. In contrast, for $\mathcal{Q}_0 = \{\langle a \rangle\} \hookrightarrow_h H$, the theorem does not apply since the conjugates of $\langle a \rangle$ in H are not invariant under conjugation by elements of G .

The next result illustrates concrete examples where Theorem E applies.

Theorem F (Theorem 5.1) *Let A be a finitely generated group with a (not necessarily finite) generating set T , and let \mathcal{H} be a finite collection of infinite subgroups such that $\mathcal{H} \hookrightarrow_h (A, T)$. If $F \leq \text{Aut}(A)$ is finite, T and \mathcal{H} are F -invariant, and the F -action on \mathcal{H} is free, then $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$ where \mathcal{H}_F is collection of representatives of F -orbits in \mathcal{H} .*

Example 1.7. Let $A = \bigstar_{i=1}^n B_i$ with each B_i isomorphic to a fixed finitely generated group B . Let $F = \mathbb{Z}_n$ act on A by cyclically permuting the copies of B . Consider the generating set of A given by $T = \bigcup_{i=1}^n B_i \setminus \{1\}$, then T is F -invariant. Now, the collection $\mathcal{H} = \{B_1, \dots, B_n\}$ is hyperbolically embedded into (A, T) and F acts freely by conjugation on \mathcal{H} . All of the hypotheses of the previous theorem have been verified, so we conclude that $B_1 \hookrightarrow_h (A \rtimes F, T \cup F)$.

1.4 Organization

The rest of the article is divided into five sections. Section 2 is on quasi-actions; it contains the proof of Proposition C as well as some corollaries. The proof of Theorem A is the content of Section 3. Then Sections 4 and 5 contain the proofs of Theorems E and F, respectively. Finally, Section 6 contains some questions and discussion about related to the results in this article and the definition of a quasi-isometry of pairs.

2 Uniform quasi-actions

Definition 2.1 (Uniform quasi-action) Let G be a group, and let X be a metric space. Let $\text{QI}(X)$ denote the set of quasi-isometries $X \rightarrow X$. A function $G \rightarrow \text{QI}(X)$, $g \mapsto f_g$, is a *quasi-action* if there is $K \geq 0$ such that for any $g_1, g_2 \in G$:

- (1) the map $f_{g_1 g_2}$ is at distance at most K from the map $f_{g_1} \circ f_{g_2}$ in the L_∞ -distance and
- (2) the map $f_{g_1} \circ f_{g_1^{-1}}$ is at distance at most K from the identity.

The quasi-action $G \rightarrow \text{QI}(X)$ is *uniform* if there are constants $L \geq 1$ and $C \geq 0$ such that, for any $g \in G$, the map f_g is an (L, C) -quasi-isometry.

It is well known that a quasi-isometry $q: G \rightarrow H$ of finitely generated groups induces a uniform quasi-action of G on H :

Definition 2.2 (Uniform quasi-action induced by a quasi-isometry) Let G be a group with a word metric induced by a finite generating set, let X be a metric space, and let $q: G \rightarrow X$ and $\bar{q}: X \rightarrow G$ be (L_0, C_0) -quasi-isometries such that $q \circ \bar{q}$ and $\bar{q} \circ q$ are at distance less than C_0 from the identity maps on X and G , respectively. For $g \in G$, let

$$L_g: G \rightarrow G, \quad x \mapsto gx,$$

and let

$$q_g: X \rightarrow X, \quad q_g = q \circ \mathbf{g} \circ \bar{q}.$$

It is an exercise to verify that there are constants $L \geq 1$ and $C \geq 0$ such that:

- For $g \in G$, $q_g: X \rightarrow X$ is an (L, C) -quasi-isometry.
- (G quasi-acts on X) For $g_1, g_2 \in G$, the map $q_{g_1 g_2}$ is at distance at most C from the map $q_{g_1} \circ q_{g_2}$, and the map $q_{g_1} \circ q_{g_1^{-1}}$ is at distance at most C from the identity.

- (G acts C_0 -transitively on X) For every $x, y \in X$, there is $g \in G$ such that $\text{dist}_G(x, q_g(y)) \leq C$.

The map $G \rightarrow \text{Ql}(X)$ given by $g \mapsto q_g$ is called the *uniform quasi-action* of G on X induced by q and \bar{q} .

Remark 2.1 (Equivalence of Definitions 1.6 and 2.2) In the context of Definition 1.6, if the induced quasi-action of G on H is uniform with respect to T , then $G \rightarrow \text{Ql}(\Gamma(H, T))$ given by $g \mapsto q_g$ is a uniform quasi-action in the sense of Definition 2.2. Indeed, since T contains a finite generating set of H , there is $M > 0$ such that $\text{dist}_{(H, T)} \leq M \text{dist}_{(H, T_0)}$. Hence, if two functions $H \rightarrow H$ are at finite L_∞ -distance with respect to $\text{dist}_{(H, T_0)}$, then the same holds for $\text{dist}_{(H, T)}$.

Definition 2.3 (Cayley graph) Let G be a group with a generating set S . The *Cayley graph* $\Gamma(G, S)$ of G with respect to S is the G -graph with vertex set G and edge set $\{\{g, gs\}: g \in G, s \in S\}$.

Proposition 2.1 (Proposition C) Let G and H be groups with finite generating sets S_0 and T_0 , and let $q: \Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$ be a quasi-isometry. Let $T \subset H$ containing T_0 . The following statements are equivalent:

- (1) The quasi-action of G on H induced by q is uniform with respect to T .
- (2) There is $S \subset G$ containing S_0 such that $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry.

Proof That the second statement implies the first one is immediate. Conversely, suppose that q and \bar{q} are (L_0, C_0) -quasi-isometries $\Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$ and $\Gamma(H, T_0) \rightarrow \Gamma(G, S_0)$, respectively. Without loss of generality, assume that $q(e) = e$ and $\bar{q}(e) = e$ where e denotes the identity in each corresponding group.

Let $K_0 = L_0 + C_0 + 1$ and define

$$S = \{f^{-1}g \in G: \text{there are } h \in H \text{ and } t \in T \text{ such that} \\ \text{dist}_{(H, T_0)}(q(f), h) \leq K_0 \text{ and } \text{dist}_{(H, T_0)}(q(g), ht) \leq K_0\}.$$

Note that $S_0 \subset S$ since $q: \Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$ is an (L_0, C_0) -quasi-isometry. In particular, S is a generating set of G .

Let $L_1 \geq 1$ and $C_1 \geq 0$ be such that the G -action on H induced by q is (L_1, C_1) -uniform with respect to T . In particular, for every $g \in G$, the function $q_g: H \rightarrow H$ is an (L_1, C_1) -quasi-isometry $\Gamma(H, T) \rightarrow \Gamma(H, T)$.

Now, we prove that if the induced quasi-action of G on H is uniform with respect to T , then $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry. Observe that every vertex of $\Gamma(H, T)$ is at distance at most C_0 from $q(G)$ with respect to $\text{dist}_{(H, T_0)}$ and hence with respect to $\text{dist}_{(H, T)}$. Below, we prove inequalities (2.1) and (2.2), which will conclude proof.

Claim There is constant \bar{L} such that

$$(2.1) \quad \text{dist}_{(H, T)}(q(a), q(b)) \leq \bar{L} \text{dist}_{(G, S)}(a, b),$$

for any $a, b \in G$.

Proof of claim Let $s \in S$. Then there are $f, g \in G$, $h \in H$, and $t \in T$ such that $s = f^{-1}g$ and

$$\text{dist}_{(H, T_0)}(q(f), h) \leq K_0, \quad \text{dist}_{(H, T_0)}(q(g), ht) \leq K_0.$$

It follows that

$$\text{dist}_{(H, T)}(q_f(e), q_g(e)) = \text{dist}_{(H, T)}(q(f), q(g)) \leq 2K_0 + 1.$$

Since the quasi-action of G on $\Gamma(H, T)$ is (L_1, C_1) -uniform, the previous inequality implies that

$$\begin{aligned} \text{dist}_{(H, T)}(e, q(s)) &= \text{dist}_{(H, T)}(q_e(e), q_{f^{-1}g}(e)) \\ &\leq L_1 \text{dist}_{(H, T)}(q_f \circ q_e(e), q_f \circ q_{f^{-1}g}(e)) + C_1 \\ &\leq L_1 \text{dist}_{(H, T)}(q_f(e), q_g(e)) + 3C_1 \\ &\leq L_1(2K_0 + 1) + 3C_1 =: \tilde{L}_0. \end{aligned}$$

For any $g \in G$ and $s \in S$, we have that

$$\begin{aligned} \text{dist}_{(H, T)}(q(g), q(gs)) &= \text{dist}_{(H, T)}(q_g(e), q_{gs}(e)) \\ &\leq L_1 \text{dist}_{(H, T)}(q_{g^{-1}} \circ q_g(e), q_{g^{-1}} \circ q_{gs}(e)) + C_1 \\ &\leq L_1 \text{dist}_{(H, T)}(e, q_{g^{-1}gs}(e)) + 3C_1 \\ &\leq L_1 \text{dist}_{(H, T)}(q(e), q(s)) + 3C_1, \end{aligned}$$

and hence

$$\text{dist}_{(H, T)}(q(g), q(gs)) \leq \text{dist}_{(H, T_0)}(q(g), q(gs)) \leq \tilde{L},$$

where $\tilde{L} = L_1(\tilde{L}_0) + 3C_1$. If $a, b \in G$ and $[u_0, \dots, u_\ell]$ is a geodesic in $\Gamma(G, S)$ from a to b , then the triangle inequality implies inequality (2.1). ■

Claim For any $a, b \in G$, we have

$$(2.2) \quad \text{dist}_{(G, S)}(a, b) \leq \text{dist}_{(H, T)}(q(a), q(b)).$$

Proof of claim Suppose that $[h_0, \dots, h_\ell]$ is a geodesic in $\Gamma(H, T)$ from $q(a)$ to $q(b)$. Since $q: \Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$ is an (L_0, C_0) -quasi-isometry, for each i , there is $g_i \in G$ such that $\text{dist}_{(H, T_0)}(q(g_i), h_i) \leq C_0$. Let $g_0 = a$ and $g_\ell = b$. Observe that $g_i^{-1}g_{i+1} \in S$ for $0 \leq i < \ell$, and hence $\text{dist}_{(G, S)}(g_i, g_{i+1}) \leq 1$. Now, $[g_0, \dots, g_\ell]$ is a path in $\Gamma(G, S)$ from a to b and therefore $\text{dist}_{(G, S)}(a, b) \leq \text{dist}_{(H, T)}(q(a), q(b))$ proving inequality (2.2). ■

Corollary 2.1 Let G and H be groups with finite generating sets S_0 and T_0 . Let $q: G \rightarrow H$ be a group homomorphism which is also an (L_0, C_0) -quasi-isometry $q: \Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$. If $T \subset H$ contains T_0 , then there is $S \subset G$ containing S_0 such that $q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry.

Proof Let $\tilde{q}: H \rightarrow G$ be a quasi-inverse of q and, by increasing L_0 and C_0 if necessary, assume that $\tilde{q}: \Gamma(H, T_0) \rightarrow \Gamma(G, S_0)$ is an (L_0, C_0) -quasi-isometry. Moreover,

suppose $q \circ \bar{q}$ and $\bar{q} \circ q$ are at distance at most C_0 from the corresponding identity maps with respect to $\text{dist}_{(H, T_0)}$ and $\text{dist}_{(G, S_0)}$. Note that for any $g \in G$,

$$q_g(h) = q(g \cdot \bar{q}(h)) = q(g) \cdot q(\bar{q}(h)).$$

Hence, q_g is a $(1, C_0)$ -quasi-isometry since it is the composition of $q \circ \bar{q}$ followed by the isometry given by multiplication on the left by $q(g)$. Then the proof concludes by invoking Proposition 2.1. ■

The following result is the particular case of Corollary 2.1 in which H is a finite index subgroup of G . In this case, one can give a more algebraic description of the generating set S . The proof follows the same lines as the previous argument modulo Lemma 2.1.

Proposition 2.2 *Let H be a finite index normal subgroup of a finitely generated group G . Let T be a generating set of H , let R be a right transversal of H in G , and let $S = T \cup R$. If the G -action by conjugation on H is a uniform quasi-action on $\Gamma(H, T)$, then the inclusion $\Gamma(H, T) \hookrightarrow \Gamma(G, S)$ is a quasi-isometry.*

We divert the proof of the proposition after the following lemma.

Lemma 2.1 *Let H be a finite index normal subgroup of a finitely generated group G . Let T be a generating set of H containing a finite generating set T_0 , let R be transversal of H in G , let S_0 be a finite generating set of G , and let $q: \Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$ be the quasi-isometry defined by $q(hg) = h$ for $h \in H$ and $g \in R$. The following statements are equivalent:*

- (1) *The G -action by conjugation on H is a uniform quasi-action on $\Gamma(H, T)$.*
- (2) *The quasi-action of G on H induced by q is uniform with respect to T .*

Proof Take as the quasi-inverse of q the inclusion $H \hookrightarrow G$. For $h \in H$, let $L_h: H \rightarrow H$ be given by $L_h(x) = hx$, i.e., multiplication on the left. Note that $L_h: \Gamma(H, T) \rightarrow \Gamma(H, T)$ is an isometry for every $h \in H$.

Let $g \in G$ and suppose that $g = h_* g_*$ where $h_* \in H$ and $g_* \in R$. Then

$$q_g(h) = q(gh) = q(ghg^{-1}h_*g_*) = ghg^{-1}h_* = h_*g_*hg^{-1}h_*^{-1}h_* = h_*g_*hg_*^{-1}$$

and hence

$$q_g = L_{h_*} \circ \text{Ad}(g_*),$$

where $\text{Ad}(g_*)$ is conjugation by g_* . It follows $q_g: \Gamma(H, T) \rightarrow \Gamma(H, T)$ is an (L, C) -quasi-isometry for all $g \in G$ if and only if $\text{Ad}(g_*): \Gamma(H, T) \rightarrow \Gamma(H, T)$ is an (L, C) -quasi-isometry for all $g_* \in R$. In particular, the first statement implies the second by Remark 2.1, and the second statement implies the first since the constants L and C hold for all conjugations. ■

Proof [Proof of Proposition 2.2.] Let $T_0 \subset T$ be a finite generating set of H , and let $S_0 = T_0 \cup R$. Note that S_0 is a finite generating set of G . Then $q: \Gamma(G, S_0) \rightarrow \Gamma(H, T_0)$

is an (L_0, C_0) quasi-isometry for some $L_0 \geq 1$ and $C_0 \geq 0$, and the quasi-inverse \bar{q} can be taken as the inclusion $\Gamma(H, T_0) \hookrightarrow \Gamma(G, S_0)$.

Observe that, in $\Gamma(G, S)$, the vertices $g = hr$ and $q(g) = h$ are adjacent since $r \in S$. Therefore, if $[v_0, \dots, v_\ell]$ is a geodesic path in $\Gamma(H, T)$ from $q(a)$ to $q(b)$, then $[a, v_0, \dots, v_\ell, b]$ is a path in $\Gamma(G, S)$ from a to b , and hence

$$\text{dist}_{(G,S)}(a, b) \leq \text{dist}_{(H,T)}(q(a), q(b)) + 2.$$

We now prove the other inequality. Since the G -action on H by conjugation is a uniform quasi-action on $\Gamma(H, T)$, Lemma 2.1 implies that the quasi-action of G on H induced by q is (L_1, C_1) -uniform with respect to T , for some $L_1 \geq 1$ and $C_1 \geq 0$.

Let $K_0 = L_0 + C_0 + 1$. Observe that

$$S \subseteq \{f^{-1}g \in G: \text{there are } h \in H \text{ and } t \in T \text{ such that} \\ \text{dist}_{(H,T_0)}(q(f), h) \leq K_0 \text{ and } \text{dist}_{(H,T_0)}(q(g), ht) \leq K_0\}.$$

Indeed, let $s \in S = T \cup R$, and there are two cases. First, if $s \in T$, let $f = h = e$ and $g = t = s$, and second, if $s \in R$, let $f = h = e$, $g = s$, and t be any element of T_0 . Then, exactly as in the first claim in the proof of Proposition 2.1, one defines a constant $\bar{L} = \bar{L}(L_1, C_1, K_0)$ and deduces the inequality

$$(2.3) \quad \text{dist}_{(H,T)}(q(a), q(b)) \leq \bar{L} \text{dist}_{(G,S)}(a, b).$$

It remains to show that

$$(2.4) \quad \text{dist}_{(G,S)}(a, b) \leq \text{dist}_{(H,T)}(q(a), q(b)) + 2,$$

for any $a, b \in G$, concluding the proof. This is clear since $\Gamma(H, T)$ is a subgraph of $\Gamma(G, T)$ and $\text{dist}_{G,S}(g, q(g)) \leq 1$ for any $g \in G$. ■

The following example by Minasyan and Osin illustrates the need for the hypothesis relating to the conjugation action in Corollary 2.2.

Example 2.1 [16] Let $H = \langle a, b \rangle$ be the free group of rank 2, let $G = \langle a, b, t: tat^{-1} = b, t^2 = e \rangle$, let $T = \{b, a, a^{-1}, a^2, a^{-2}, \dots\}$, and let $S = T \cup \{t\}$. The inclusion $\Gamma(H, T) \rightarrow \Gamma(G, S)$ is not a quasi-isometry. Indeed, in G , we have $ta^n t^{-1} = b^n$, and hence $\text{dist}_{(G,S)}(e, b^n) = 3$, but $\text{dist}_{(H,T)}(e, b^n) = n$ for every n . In particular, the map $\Gamma(H, T) \rightarrow \Gamma(H, T)$ given by $h \mapsto tht^{-1}$ is not a quasi-isometry, and hence the G -action on H by conjugation is not an action by quasi-isometries.

3 Quasi-isometries and hyperbolically embedded subgroups

In this section, we will prove Theorem A. The theorem is obtained by putting together a simple characterization of hyperbolically embedded subgroups in terms of coned-off Cayley graphs which appeared in the work of Rashid and the second author (see [13, Propositions 1.5 and 5.8]), some results about quasi-isometries of pairs from [11], and some basic facts about hyperbolically embedded subgroups from [7]. Below, we state these results and then we discuss the proof of Theorem 3.1.

Definition 3.1 (Reduced collections) A collection of subgroups \mathcal{Q} of a group H is *reduced* if for any $P, Q \in \mathcal{Q}$ and $g \in H$, if P and gQg^{-1} are commensurable subgroups, then $P = Q$ and $g \in P$.

Remark 3.1. An almost malnormal collection is reduced.

Definition 3.2 (Fine) Let Γ be a graph, and let v be a vertex of Γ . Let

$$T_v\Gamma = \{w \in V(\Gamma) \mid \{v, w\} \in E(\Gamma)\}$$

denote the set of the vertices adjacent to v . For $x, y \in T_v\Gamma$, the *angle metric* $\angle_v(x, y)$ is the length of the shortest path in the graph $\Gamma \setminus \{v\}$ between x and y , with $\angle_v(x, y) = \infty$ if there is no such path. The graph Γ is *fine at v* if $(T_v\Gamma, \angle_v)$ is a locally finite metric space. The graph Γ is *fine at $C \subseteq V(\Gamma)$* if Γ is fine at v for all $v \in C$.

Definition 3.3 (Coned-off Cayley graph) Let G be a group, let \mathcal{P} be an arbitrary collection of subgroups of G , and let S be a subset of G . Denote by G/\mathcal{P} the set of all cosets gP with $g \in G$ and $P \in \mathcal{P}$. The *coned-off Cayley graph of G with respect to \mathcal{P}* is the graph $\hat{\Gamma}(G, \mathcal{P}, S)$ with vertex set $G \cup G/\mathcal{P}$ and edges are of the following type:

- $\{g, gs\}$ for $s \in S$ and
- $\{x, gP\}$ for $g \in G$, $P \in \mathcal{P}$, and $x \in gP$.

We call vertices of the form gP *cone points*.

Proposition 3.1 [13] Let \mathcal{P} be a collection of infinite subgroups of G , and let S be a subset of G . Then $\mathcal{P} \hookrightarrow_h (G, S)$ if and only if the Coned-off Cayley graph $\hat{\Gamma}(G, \mathcal{P}, S)$ is a connected hyperbolic graph which is fine at every cone vertex.

Proposition 3.2 [11, Proposition 5.6] Let G and H be groups, let $S \subset G$ and $T \subset H$, and let $S_0 \subset S$ and $T_0 \subset T$ be finite generating sets of G and H , respectively. Consider collections \mathcal{P} and \mathcal{Q} of subgroups of G and H , respectively. Let $q: G \rightarrow H$ be a function.

Suppose that q is a quasi-isometry $\Gamma(G, S) \rightarrow \Gamma(H, T)$, is a quasi-isometry of pairs $(G, \mathcal{P}, S_0) \rightarrow (H, \mathcal{Q}, T_0)$, and \hat{q} is a bijection $G/\mathcal{P} \rightarrow H/\mathcal{Q}$.

- (1) Let $\hat{q} = q \cup \hat{q}$, then \hat{q} is a quasi-isometry $\hat{\Gamma}(G, \mathcal{P}, S) \rightarrow \hat{\Gamma}(H, \mathcal{Q}, T)$.
- (2) If $\hat{\Gamma}(H, \mathcal{Q}, T)$ is fine at cone vertices, then $\hat{\Gamma}(G, \mathcal{P}, S)$ is fine at cone vertices.
- (3) If $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P} \hookrightarrow_h (G, S)$.

Items (1) and (2) of Proposition 3.2 are taken from [11, Proposition 5.6], and the last item is a direct consequence of Proposition 3.1.

Proposition 3.3 [11, Proposition 5.12] Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be an (L, C, M) -quasi-isometry of pairs. Then:

- (1) \hat{q} is a surjective function $G/\mathcal{P} \rightarrow H/\mathcal{Q}$ if \mathcal{Q} is reduced.
- (2) \hat{q} is a bijection $G/\mathcal{P} \rightarrow H/\mathcal{Q}$ if \mathcal{P} and \mathcal{Q} are reduced.

Proposition 3.4 [11, Proposition 6.2] Let \mathcal{P}^* be a refinement of a finite collection of subgroups \mathcal{P} of a finitely generated group G . If P is a finite index subgroup of $\text{Comm}_G(P)$

for every $P \in \mathcal{P}$, then (G, \mathcal{P}) and (G, \mathcal{P}^*) are quasi-isometric pairs via the identity map on G .

Proposition 3.5 [11, Proposition 6.7] *Let $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ be a quasi-isometry of pairs. If \mathcal{Q} is an almost malnormal finite collection of infinite subgroups and \mathcal{P} is a finite collection, then any refinement \mathcal{P}^* of \mathcal{P} is almost malnormal.*

Proposition 3.6 [7, Proposition 4.33] *Let \mathcal{P} be a collection of subgroups of a group G . If $\mathcal{P} \hookrightarrow_h G$, then \mathcal{P} is an almost malnormal collection.*

We are now ready to prove Theorem A.

Theorem 3.1 (Theorem A) *Let $q: G \rightarrow H$ be a quasi-isometry of finitely generated groups, let \mathcal{P} and \mathcal{Q} be finite collections of subgroups of G and H , respectively, and let S and T be (not necessarily finite) generating sets of G and H , respectively. Suppose that:*

- (1) *$q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs and*
- (2) *$q: \Gamma(G, S) \rightarrow \Gamma(H, T)$ is a quasi-isometry.*

The following statements hold:

- (1) *If \mathcal{P} and \mathcal{Q} are reduced collections in G and H , respectively, then $\mathcal{P} \hookrightarrow_h (G, S)$ if and only if $\mathcal{Q} \hookrightarrow_h (H, T)$.*
- (2) *If \mathcal{Q} contains only infinite subgroups and $\mathcal{Q} \hookrightarrow_h (H, T)$, then $\mathcal{P}^* \hookrightarrow_h (G, S)$.*

Proof For the first statement, since \mathcal{P} and \mathcal{Q} are reduced, Proposition 3.3 implies that $\hat{q}: G/\mathcal{P} \rightarrow H/\mathcal{Q}$ is a bijection. Then Proposition 3.2 implies that $\hat{\Gamma}(G, \mathcal{P}, S)$ is hyperbolic and fine at cone vertices if and only if $\hat{\Gamma}(H, \mathcal{Q}, T)$ is hyperbolic and fine at cone vertices. Then Proposition 3.1 concludes the proof of the first statement.

The second statement is a consequence of the first statement as follows. That $\mathcal{Q} \hookrightarrow_h H$ implies that \mathcal{Q} is an almost malnormal collection of subgroups in H (see Proposition 3.6). It follows that \mathcal{Q} is reduced in H . Then, since \mathcal{Q} contains only infinite subgroups, Proposition 3.5 implies that \mathcal{P}^* is reduced. By Proposition 3.4, $q: (G, \mathcal{P}^*) \rightarrow (H, \mathcal{Q})$ is a quasi-isometry of pairs. Then $\mathcal{Q} \hookrightarrow_h H$ and the first statement of the proposition imply that $\mathcal{P}^* \hookrightarrow (G, S)$. ■

4 Hyperbolically embedded subgroups and commensurability

In this section, we prove Theorem E. The argument uses the following proposition, which is a strengthening of [14, Proposition 2.15]. It essentially follows from the proof in the cited article, but we have included the proof for the convenience of the reader.

Proposition 4.1 *Let H be a finite index subgroup of a finitely generated group G , and let \mathcal{Q} be a finite collection of subgroups of H . The following statements are equivalent:*

- (1) *The inclusion $H \hookrightarrow G$ is a quasi-isometry of pairs $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q})$.*
- (2) *For any $Q \in \mathcal{Q}$ and $g \in G$, there is $Q' \in \mathcal{Q}$ and $h \in H$ such that $\text{hdist}_G(gQ, hQ') < \infty$.*

Proof That (1) implies (2) is trivial. Assume statement (2). Since H is a finite index subgroup of the finitely generated group G , assume that $H \hookrightarrow G$ is an (L, C) quasi-isometry. Since H is finite index in G , and \mathcal{Q} is a finite collection, the H -action on G/\mathcal{Q} has finitely many orbits. For $gQ \in G/\mathcal{Q}$, let

$$\text{hdist}_G(gQ, H/\mathcal{Q}) := \min \{ \text{hdist}_G(gQ, hQ') : hQ' \in H/\mathcal{Q} \}.$$

Let \mathcal{R} be a finite collection of orbit representatives of the H -action on G/\mathcal{Q} . By hypothesis, for $gQ \in \mathcal{R}$, there is $hQ' \in H/\mathcal{Q}$ such that $\text{hdist}(gQ, hQ') < \infty$ and therefore

$$M = \max \{ \text{hdist}_G(gQ, H/\mathcal{Q}) : gQ \in \mathcal{R} \} < \infty$$

is a well-defined integer since \mathcal{R} is a finite set. Since the subset H/\mathcal{Q} of G/\mathcal{Q} is H -invariant,

$$\text{hdist}_G(gQ, H/\mathcal{Q}) = \text{hdist}_G(hgQ, H/\mathcal{Q})$$

for every $gQ \in \mathcal{R}$ and $h \in H$. Since \mathcal{R} is a collection of representatives of orbits of G/\mathcal{Q} ,

$$\text{hdist}_G(gQ, H/\mathcal{Q}) \leq M$$

for every $gQ \in G/\mathcal{Q}$. Hence, $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q})$ is an (L, C, M) quasi-isometry of pairs. ■

Remark 4.1. Let G be a group, and let T and S generating sets with finite symmetric difference. Then the identity map on G is a quasi-isometry $\Gamma(G, T) \rightarrow \Gamma(G, S)$.

Theorem 4.1 (Theorem E) *Let H be a finite index normal subgroup of a finitely generated group G , and let \mathcal{Q} be a finite collection of infinite subgroups of H such that $\mathcal{Q} \hookrightarrow_h (H, T)$. Suppose that:*

- (1) *The G -action by conjugation on H is a uniform quasi-action on $\Gamma(H, T)$.*
- (2) *The collection $\{hQh^{-1} : h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by G .*

If \mathcal{Q}^ is a refinement of \mathcal{Q} in G and R is a transversal of H in G , then $\mathcal{Q}^* \hookrightarrow_h (G, T \cup R)$.*

Proof Since H is finitely generated, by adding a finitely many elements, we can assume that T generates H . Note that this preserves $\mathcal{Q} \hookrightarrow_h (H, T)$ by [7, Corollary 4.27], and the quasi-isometry type of $\Gamma(H, T)$ by Remark 4.1. Under this assumption, the conclusion will follow from the second statement of Theorem 3.1 applied to the quasi-isometry of finitely generated groups given by the inclusion $H \hookrightarrow G$.

Since $\mathcal{Q} \hookrightarrow_h (H, T)$, \mathcal{Q} is an almost malnormal collection (see Proposition 3.6). The assumption that \mathcal{Q} consists only of infinite subgroups implies that for any $Q \in \mathcal{Q}$,

$$Q = \text{Comm}_H(Q) = \text{Comm}_G(Q) \cap H.$$

Since H is finite index in G , we have that Q is finite index in $\text{Comm}_G(Q)$. Then Proposition 3.4 implies that the identity map on G is a quasi-isometry of pairs $(G, \mathcal{Q}) \rightarrow (G, \mathcal{Q}^*)$. On the other hand, since the collection $\{hQh^{-1} : h \in H \text{ and } Q \in \mathcal{Q}\}$ is invariant under conjugation by elements of G , we have for any $g \in G$ and $Q \in \mathcal{Q}$ there is $h \in H$ such that $gQg^{-1} = hQ'h^{-1}$ and hence

$$\text{hdist}_G(gQ, hQ') \leq \text{hdist}_G(gQ, Q^g) + \text{hdist}_G(Q^g, (Q')^h) + \text{hdist}((Q')^h, hQ') < \infty.$$

Proposition 4.1 implies that $H \hookrightarrow G$ is a quasi-isometry of pairs $(H, \mathcal{Q}) \rightarrow (G, \mathcal{Q})$. It follows that $H \hookrightarrow G$ is a quasi-isometry of pairs $(H, \mathcal{Q}) \rightarrow (G, \mathcal{Q}^*)$ as it is the composition $(H, \mathcal{Q}) \hookrightarrow (G, \mathcal{Q}) \rightarrow (G, \mathcal{Q}^*)$. Let R be a transversal of H in G , and let $S = T \cup R$. Since the G -action by conjugation on H is uniform on $\Gamma(H, T)$, Proposition 2.2 implies that $H \hookrightarrow G$ is a quasi-isometry $\Gamma(H, T) \rightarrow (G, S)$. The hypothesis of Theorem 3.1 has been verified, and therefore $\mathcal{Q} \hookrightarrow_h (H, T)$ implies $\mathcal{Q}^* \hookrightarrow_h (G, S)$. ■

5 Semidirect products and hyperbolically embedded subgroups

In this section, we will prove Theorem F about semidirect products. The hypothesis of the following proposition and theorem reflects the issues posed by the example of Minasyan and Osin (Example 2.1).

Proposition 5.1 *Let A be a group with (not necessarily finite) generating set T , let \mathcal{H} be a collection of subgroups, and let $F \leq \text{Aut}(A)$ be a finite subgroup. Suppose that T and \mathcal{H} are F -invariant and that the F -action on \mathcal{H} is free. Let \mathcal{H}_F be a collection of representatives of F -orbits in \mathcal{H} . Then the inclusion $A \hookrightarrow A \rtimes F$ induces:*

- (1) *a quasi-isometry $\Gamma(A, T) \rightarrow \Gamma(A \rtimes F, T \cup F)$ and*
- (2) *if A is finitely generated, a quasi-isometry of pairs $(A, \mathcal{H}) \rightarrow (A \rtimes F, \mathcal{H}_F)$.*

Proof To prove the first statement, let $S = T \cup F$ and let dist_T and dist_S be the word metrics on A and $A \rtimes F$ induced by T and S , respectively. Let $q: A \rightarrow A \rtimes F$ be the inclusion, and let $\bar{q}: A \rtimes F \rightarrow A$ be such that for $a \in A$ and $f \in F$, $\bar{q}(af) = a$. Note that \bar{q} is a well-defined A -equivariant map since each element of $A \rtimes F$ can be expressed as a product af in a unique way. Observe that $\bar{q} \circ q$ is the identity on A , and $q \circ \bar{q}$ is at distance 1 from the identity map on $A \rtimes F$ with respect to dist_S . Since the Cayley graph $\Gamma(A, T)$ is a subgraph of $\Gamma(A \rtimes F, T \cup F)$, it is immediate that for any $u, v \in A$, $\text{dist}_S(q(u), q(v)) \leq \text{dist}_T(u, v)$. To conclude the proof of the statement, we show that for any $u, v \in A \rtimes F$, $\text{dist}_T(\bar{q}(u), \bar{q}(v)) \leq \text{dist}_S(u, v)$. Note that it is enough to consider the case that $\text{dist}_S(u, v) = 1$. Let $w_1, w_2 \in A \rtimes F$ such that $\text{dist}_S(w_1, w_2) = 1$. Then $w_1 = a_1 f_1$ and $w_2 = a_2 f_2$ and $\bar{q}(w_i) = a_i$. It follows that $g = (a_1 f_1)^{-1} a_2 f_2 \in T \cup F$. Observe that

$$g = f_1^{-1} a_1^{-1} a_2 f_2 = (a_1^{-1})^{f_1^{-1}} f_1^{-1} a_2 f_2 = (a_1^{-1})^{f_1^{-1}} a_2^{f_1^{-1}} f_1^{-1} f_2 = (a_1^{-1} a_2)^{f_1^{-1}} f_1^{-1} f_2 \in T \cup F.$$

There are two cases, either $g \in T$ or $g \in F$, since $T \cap F = \emptyset$. We regard $T \cup F$ and F as a subset and a subgroup of $A \rtimes F$, respectively. If $g \in T$, then $f_1 = f_2$ and hence $(a_1^{-1} a_2)^{f_1^{-1}} \in T$; since T is F -invariant, a_1 and a_2 are adjacent in Γ , and hence $\text{dist}_T(\bar{q}(w_1), \bar{q}(w_2)) = 1$. If $g \in F$, then $a_1 = a_2$ and hence $\text{dist}_T(\bar{q}(w_1), \bar{q}(w_2)) = 0$.

For the second statement, suppose that A is finitely generated and let dist denote word metric on $A \rtimes F$ induced by finite generating set, and let $\text{hdist}_{A \rtimes F}$ be the induced Hausdorff distance. Let $M = \max_{f \in F} \text{dist}(1, f)$. Since the inclusion $A \hookrightarrow A \rtimes F$ is a quasi-isometry of finitely generated groups and $\mathcal{H}_F \subset \mathcal{H}$, it is enough to prove that for any $H \in \mathcal{H}$, there is a left coset in $(A \rtimes F)/\mathcal{H}_F$ at Hausdorff distance at most M in $A \rtimes F$. Let $H \in \mathcal{H}$. Since the F -action on \mathcal{H} by conjugation is free, there is a unique

$f \in F$ and a unique $K \in \mathcal{H}_F$ such that $H = fKf^{-1}$. Observe that

$$\text{hdist}(H, fK) = \text{hdist}(fKf^{-1}, fK) \leq \text{dist}(1, f^{-1}) \leq M,$$

and this completes the proof. \blacksquare

Theorem 5.1 (Theorem F) *Let A be a finitely generated group with (not necessarily finite) generating set T , and let \mathcal{H} be a finite collection of infinite subgroups such that $\mathcal{H} \hookrightarrow_h (A, T)$. If $F \leq \text{Aut}(A)$ is finite, T and \mathcal{H} are F -invariant, and the F -action on \mathcal{H} is free, then $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$, where \mathcal{H}_F is collection of representatives of F -orbits in \mathcal{H} .*

Proof By Proposition 5.1, the inclusion $A \hookrightarrow A \rtimes F$ induces a quasi-isometry $\Gamma(A, T) \rightarrow \Gamma(A \rtimes F, T \cup F)$, and a quasi-isometry of pairs $(A, \mathcal{H}) \rightarrow (A \rtimes F, \mathcal{H}_F)$. Since $\mathcal{H} \hookrightarrow_h A$, the collection \mathcal{H} is almost malnormal in A ; then the assumption that F acts freely on \mathcal{H} implies that a refinement of \mathcal{H} in $A \rtimes F$ is \mathcal{H}_F , and this was observed in Example 1.3. Since \mathcal{H} contains only infinite subgroups and $\mathcal{H} \hookrightarrow_h A$, Theorem 3.1 implies that $\mathcal{H}_F \hookrightarrow_h (A \rtimes F, T \cup F)$. \blacksquare

6 Concluding remarks

A positive answer to the following question would allow us to drop the first hypothesis of Theorem A for the relevant groups.

Question 6.1. Let G be a finitely generated NRH acylindrically hyperbolic group. Does G contain a qi-characteristic collection of hyperbolically embedded subgroups?

It is possible that \mathcal{AH} -accessibility as defined in [1] may be necessary for a positive answer to Question 6.1. Note that this property does not always hold (see [2]).

It is tempting to weaken the definition of a quasi-isometry of pairs $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ to remove the uniform constant M bounding the Hausdorff distances on the cosets and instead ask the relation

$$\dot{q} = \{(A, B) \in G/\mathcal{P} \times H/\mathcal{Q} : \text{hdist}_H(q(A), B) < \infty\}$$

satisfies that the projections into G/\mathcal{P} and H/\mathcal{Q} are surjective. We shall call the map q in this modified definition an *almost quasi-isometry of pairs* following [10, Section 5].

Indeed, there is work of Margolis [12] where the main theorems do not require this additional hypothesis. However, Margolis shows that the hypotheses assumed in the main results of loc. cit. in fact imply that such a constant M exists (see [12, Theorem 4.1]). Note that our results in this article rely on the existence of a constant M —primarily due to the use of [11, Proposition 5.6]. Thus, we raise the following question.

Question 6.2. Let G and H be finitely generated groups with finite collections of subgroups \mathcal{P} and \mathcal{Q} , respectively. When is an (L, C) -almost quasi-isometry of pairs $q: (G, \mathcal{P}) \rightarrow (H, \mathcal{Q})$ an (L, C, M) -quasi-isometry of pairs?

Motivated by results of [5], the referee of the article suggested that it might be interesting to investigate other relaxations of the definition of a quasi-isometry of pairs (Defintion 1.1), for example, in the sense that the image of every element of the collection \mathcal{A} lies at uniform Hausdorff distance of the union of finitely many elements in the collection \mathcal{B} . Having a more general notion could allow a broader strategy toward tackling Question 1.1 based on the methods in this article.

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