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INELASTIC BOLTZMANN EQUATION UNDER SHEAR HEATING

José A. Carrillo[1], Kam Fai Chan[2], Renjun Duan[*2] and Zongguang Li[3]

[1]Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

[2]Department of Mathematics,
The Chinese University of Hong Kong, Shatin, Hong Kong, China

[3]Department of Applied Mathematics,
The Hong Kong Polytechnic University, Hong Kong, China

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Abstract. In this paper, we study the spatially homogeneous inelastic Boltzmann equation for angular cutoff pseudo-Maxwell molecules with an additional term of linear deformation. We establish the existence of non-Maxwellian self-similar profiles under the assumption of small deformation in the nearly elastic regime, and also obtain weak convergence to these self-similar profiles for global-in-time solutions with initial data that have finite mass and finite p-th order moment for any $2 < p \leq 4$. Our results confirm the competition between shear heating and inelastic cooling that governs the long-time behavior of temperature. Specifically, temperature increases to infinity if shear heating dominates, decreases to zero if inelastic cooling prevails, and converges to a positive constant if the two effects are balanced. In the balanced scenario, the corresponding self-similar profile aligns with the steady solution.

1. **Introduction.** The initial value problem of the spatially homogeneous inelastic Boltzmann equation under the effect of shear heating in $\mathbb{R}^d$ modeled by the matrix $A \in \mathbb{R}^{d\times d}$ with $d \geq 2$ is given by

$$\partial_t f - \nabla_v \cdot (Avf) = Q_e(f,f), \quad f(0,v) = f_0(v). \tag{1}$$

Here, the unknown function $f = f(t,v) \geq 0$ is the one-particle density distribution of rarefied gas particles with velocity $v \in \mathbb{R}^d$ at time $t \geq 0$, and the initial data $f(0,v) = f_0(v) \geq 0$ is given. The inelastic Boltzmann collision operator Q_e acting only on the velocity variable involves a parameter $e_{\mathrm{res}} \in (0,1]$ called the coefficient of restitution, which we assume to be constant through the paper. In the weak form, $Q_e(f,f)$ is defined as

$$\int_{\mathbb{R}^d} \psi(v) Q_e(f,f)(v)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \mathrm{B}(v-v_*,\sigma) f(v) f(v_*)$$

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*Corresponding author: Renjun Duan.

$$\times\left(\psi(v')+\psi(v'_*)-\psi(v)-\psi(v_*)\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v, \quad (2)$$

where the pre-collision velocity pair (v,v_*) and the post-collision velocity pair (v',v'_*) satisfy

$$\begin{cases} v'=\dfrac{v+v_*}{2}+\dfrac{1-z}{2}(v-v_*)+\dfrac{z}{2}|v-v_*|\sigma, \\ v'_*=\dfrac{v+v_*}{2}-\dfrac{1-z}{2}(v-v_*)-\dfrac{z}{2}|v-v_*|\sigma, \end{cases}$$

with $\sigma\in\mathbb{S}^{d-1}$ and

$$z:=\frac{1}{2}(1+e_{\mathrm{res}})\in(\frac{1}{2},1]. \quad (3)$$

When $e_{\mathrm{res}}=1$, which corresponds to $z=1$, the collision operator reverts back to the classical elastic case.

The Boltzmann collision kernel $\mathrm{B}(v-v_*,\sigma)$ in (2) takes the form

$$\mathrm{B}(v-v_*,\sigma)=|v-v_*|^{\gamma_0}b(\cos\theta),\ -d<\boldsymbol{\gamma}\leq 1, \quad (4)$$

where

$$\cos\theta=\frac{v-v_*}{|v-v_*|}\cdot\sigma,\ 0<\theta\leq\pi/2.$$

The cases $-d<\boldsymbol{\gamma}<0$, $\boldsymbol{\gamma}=0$, and $0<\boldsymbol{\gamma}\leq 1$ are called soft potentials, pseudo-Maxwell molecules, and hard potentials, respectively. Throughout the paper, we only focus on the case of pseudo-Maxwell molecules with $\boldsymbol{\gamma}=0$. The angular part of the collision kernel $b\geq 0$ is assumed to be a continuous function and satisfy the Grad cutoff assumption

$$b_n=\int_{\mathbb{S}^{d-1}}b(\hat{e}\cdot\sigma)(\hat{e}\cdot\sigma)^n\,\mathrm{d}\sigma<\infty,$$

for all unit vectors $\hat{e}\in\mathbb{S}^{d-1}$ and all natural numbers $n\in\mathbb{N}$. We further assume that the kernel is non-degenerate in the sense that

$$b_0>b_2>0,\qquad b_0>b_1. \quad (5)$$

We remark that the assumptions on Maxwell molecules and also on the homogeneous regime for granular media via the inelastic Boltzmann equation might be seen as an artificial ansatz, cf. [47] and [3]. Moreover, the assumption that the restitution coefficient is a fixed constant could be not seen as very physical as it may cause a problem to define the dynamics of the microscopic particle system (inelastic collapse), cf. [2] and [15].

The shearing matrix $A\in\mathbb{R}^{d\times d}$ is a constant real matrix with the norm $\|A\|=\sup_{|v|=1}|Av|$, which includes the special case of the simple uniform shear flow (USF)

$$A=\alpha E_{12},\qquad (E_{12})_{ij}=\delta_{i1}\delta_{2j} \quad (6)$$

and shearing parameter $\alpha\geq 0$, where δ_{ij} denotes the Kronecker delta.

Throughout the paper, we also assume that the initial value $f_0=f_0(v)$ satisfies

$$\int_{\mathbb{R}^d}f_0(v)\,\mathrm{d}v=1,\quad \int_{\mathbb{R}^d}vf_0(v)\,\mathrm{d}v=0.$$

By the fact that 1 *and* v are collision invariants of the inelastic collision operator and vanish on the shearing term after integration, it is well-known that the mass and momentum of the solution to (1) are conserved. However, the energy is not necessarily conserved since inelastic collision introduces a loss of energy, and the shearing term is not guaranteed to balance this effect. In fact, there is a competition between shear heating and inelastic cooling that governs the long-time behavior of temperature. To further study this phenomenon, one needs to define the natural cooling rate $\zeta \geq 0$ as

$$\zeta = z(1-z)(b_0 - b_1) = 2z(1-z)\int_{\mathbb{S}^{d-1}} b(\cos\theta)\sin^2\frac{\theta}{2}\,\mathrm{d}\sigma, \tag{7}$$

such that

$$-\zeta|v-v_*|^2 = \int_{\mathbb{S}^{d-1}} b(\cos\theta)(|v'|^2 + |v_*'|^2 - |v|^2 - |v_*|^2)\,\mathrm{d}\sigma,$$

which implies, in the case of no-shearing $A = 0$, that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|v|^2 f(t,v)\,\mathrm{d}v\\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\cos\theta)f(t,v)f(t,v_*)(|v'|^2+|v_*'|^2-|v|^2-|v_*|^2)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\
&= -\frac{1}{2}\zeta\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,v)f(t,v_*)|v-v_*|^2\,\mathrm{d}v_*\mathrm{d}v\\
&= -\zeta\int_{\mathbb{R}^d}|v|^2 f(t,v)\,\mathrm{d}v.
\end{aligned}$$

It is also necessary to denote the constant

$$c_{11} = \frac{b_0 - b_2}{d-1} = \frac{1}{d-1}\int_{\mathbb{S}^{d-1}} b(\cos\theta)\sin^2\theta\,\mathrm{d}\sigma \geq 0, \tag{8}$$

such that, by a direct calculation,

$$\int_{\mathbb{S}^{d-1}} b(\sigma\cdot\hat{e})\,\mathrm{d}\sigma = b_0, \tag{9}$$

$$\int_{\mathbb{S}^{d-1}} b(\sigma\cdot\hat{e})\sigma\,\mathrm{d}\sigma = b_1\hat{e}, \tag{10}$$

$$\int_{\mathbb{S}^{d-1}} b(\sigma\cdot\hat{e})\sigma^{\otimes 2}\,\mathrm{d}\sigma = (b_0 - dc_{11})\hat{e}^{\otimes 2} + c_{11}I, \tag{11}$$

for any unit vector $\hat{e} \in \mathbb{S}^{d-1}$, with the notation $v^{\otimes 2} = v\otimes v$ and $v \otimes w = vw^{\mathsf{T}}$, where w^{T} is the transpose of the column vector w.

In this paper, we aim at studying the long-time behavior of solutions to (1) under the competition of shear heating and inelastic cooling. Such competitive phenomenon was observed and discussed in [21, 32] and our goal is to make a rigorous mathematical analysis. Note that the two effects have been extensively investigated by [33, 35, 7] and [17] in their individual frameworks. In particular, solutions in both situations behave self-similarly over long times. Thus, a unified framework is developed to study the competition of two effects on the long-time behavior solutions. For this purpose, we shall consider the equation under the self-similar scaling

$$\tilde{f}(t,\tilde{v}) = e^{d\beta t} f(t, e^{\beta t}\tilde{v})$$

with some suitable self-similar parameter $\beta \in \mathbb{R}$ to be determined. This gives

$$\partial_t\tilde{f} - \nabla_{\tilde{v}}\cdot((A+\beta I)\tilde{v}\tilde{f}) = Q_e(\tilde{f},\tilde{f}). \tag{12}$$

The resulting equation is very similar to (1), with A replaced by the altered shearing matrix

$$A_\beta \coloneqq A + \beta I. \tag{13}$$

Hence, for simplicity, we drop the tilde in (12) and still denote $\tilde{f}, \tilde{v}$ by f, v. It is then natural to see the stationary profile G to (12), which satisfies

$$-\nabla_v \cdot (A_\beta v G) = Q_e(G,G), \tag{14}$$

which describes the long-time behavior of solutions to (12), or equivalently (1). Note that in the special case $\beta = 0$, (14) just corresponds to the steady state of (1).

To study (1) in the frequency space, we take the Fourier transform $\varphi(k) = (\mathcal{F}f)(k) = \int_{\mathbb{R}^d} e^{-ik\cdot v} f(v)\,\mathrm{d}v$ in (1). Recall $\gamma = 0$. We then formally obtain the equation for φ as

$$\partial_t \varphi + A^{\mathsf{T}} k \cdot \nabla_k \varphi = \widehat{Q_e}(\varphi, \varphi). \tag{15}$$

Here, A^{T} is the transpose of A, and the collision operator $\widehat{Q_e}$ as in [9] can be split as

$$\widehat{Q_e}(\varphi,\varphi)(k) = \int_{\mathbb{S}^{d-1}} b\big(\sigma \cdot \frac{k}{|k|}\big)\big(\varphi(k^+)\varphi(k^-) - \varphi(0)\varphi(k)\big)\,\mathrm{d}\sigma = \widehat{Q_e^+}(\varphi,\varphi)(k) - b_0\varphi(k), \tag{16}$$

for Maxwell molecules, where the negative term is obtained in terms of the mass conservation,

$$\widehat{Q_e^+}(\varphi,\psi)(k) = \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)\varphi(k^+)\psi(k^-)\,\mathrm{d}\sigma$$

is the bilinear gain part operator, $\cos\vartheta = \frac{k}{|k|}\cdot\sigma$, and

$$\begin{aligned} k^- &= \frac{z}{2}(k - |k|\sigma), \\ k^+ &= k - k^- = (1 - \frac{z}{2})k + \frac{z}{2}|k|\sigma, \end{aligned}$$

are the corresponding Fourier variables.

The linearized gain part operator $\mathcal{L}_e$, linearized around the constant function 1, is defined as

$$\mathcal{L}_e\varphi = \widehat{Q_e^+}(\varphi,1) + \widehat{Q_e^+}(1,\varphi) = \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)(\varphi(k^+) + \varphi(k^-))\,\mathrm{d}\sigma. \tag{17}$$

Similar to how we obtain (15), after applying a Fourier transform to (12), one gets

$$\partial_t \varphi + A_\beta^{\mathsf{T}} k \cdot \nabla_k \varphi = \widehat{Q_e}(\varphi,\varphi), \tag{18}$$

which has a similar form to (15) but with a replaced shearing matrix.

Our motivation to study (1) also originates from the recent study of homo-energetic solutions of the elastic Boltzmann equation, as introduced by [45] and [30], where the second- and the third-order moments are computed. The behavior of these moments are later investigated in numerous works, including [43, 42, 46, 26, 44, 41, 1, 13]. The initial value problem, as shown by Cercignani in [20], has a global-in-time L^1 solution for a class of cutoff hard or pseudo-Maxwell collision kernels and for suitable initial data (cf. [33]). More recently, [35] considered the Radon measure-valued solution and existence of self-similar profiles, [7] considered the equation via Fourier transform to establish the self-similar asymptotics of weak solutions in large time, [24, 25] obtained the regularity and structure of shear dependency of the solution, and [38, 37] generalized the results to non-cutoff, pseudo-Maxwellian, and hard potentials.

On the other hand, the study of the inelastic Boltzmann equation, which models the behavior of granular gases, is a quite related topic. In [8], the Boltzmann equation for inelastic pseudo-Maxwell potential is derived, together with the existence and uniqueness of the solution with initial data in L^1_2. The behavior of the solution was studied in [9, 11, 5, 6, 4], justifying the behavior conjectured in [27]. In [14], convergence of solutions with pseudo-Maxwell molecule in 2-Wasserstein metric was tackled. A survey of the use of Wassertein and Toscani metrics in this framework was performed in [19]. In [40, 39], global well-posedness of the initial value problem for a generic class of collision kernels was settled, and Haff's law and the self-similar structure for the solution was justified for the hard sphere model, which seems more difficult to treat. Concerning the interaction between inelastic collision and shearing, an explicit stationary state for the second moments was given in [21], and the result was further discussed in [22, 32]. See also a recent work [23] for the linear equation.

In this paper, we shall follow the approaches of [35] and [7] and consider (1) both from the physical space and from the frequency space. More precisely, we will show that the results in [35, 7] for the elastic Boltzmann equation, taken as an evolution equation on the space of probability measures, generalize naturally to (1) with inelastic collisions. In particular, we obtain the following result concerning the existence of a stationary profile for inelastic collision.

Theorem 1.1. *Assume that $e_{\rm res} \in (0,1]$ for $e_{\rm res}$ given in (3), and the collision kernel (4) is cutoff pseudo-Maxwellian. For sufficiently small $\epsilon > 0$ ensuring small shear, $\|A\| < \epsilon$, and small inelasticity, $1 - e_{\rm res} < \epsilon$, there exists $\beta \in \mathbb{R}$ such that with this self-similar parameter β, (14) has a nonnegative Radon-measure solution $G \in \mathcal{M}_+$ with finite mass, zero momentum, and finite energy. Furthermore, G has finite p-moment for some $p > 2$.*

Here we refer to Section 2 for the definition of solutions. Theorem 1.1 will be shown with a fixed-point argument using a uniform-in-time p-moment bound on the solutions of (12). For technical reasons, instead of $\mathbb{R}^d$, we will consider the Radon measures on $\mathbb{R}^d_c$ and the compatification of $\mathbb{R}^d$. The selection of an appropriate self-similar parameter β will be made by a perturbation argument. As noted in [7], in the case of elastic collision, only a limited selection of β yields a meaningful solution of (12). We will see shortly that the same restriction applies to our model of inelastic interactions.

We note that the correct choice of self-similar parameter $\beta \in \mathbb{R}$ that gives the self-similar profile depends on both the shearing matrix A and the restitution coefficient $e_{\rm res}$, and contrary to the case of elastic collision, it is not necessarily positive in the case of uniform shear flow. In fact, β is chosen such that the inelastic cooling effect is balanced by the shear heating effect that comes from the altered shearing matrix. If inelastic cooling is the dominant effect, it would be necessary to choose a negative β, which corresponds to an anti-drift force instead of the friction force that corresponds to a positive β (cf. [47]).

For the simple uniform shear flow, we can determine the sign of β; its proof will be given in Section 2.2.

Theorem 1.2. *Assume $e_{\rm res} \in (0,1)$. Let $A = \alpha E_{12}$ be the uniform shear flow matrix with shearing parameter $\alpha \geq 0$ and $\beta \in \mathbb{R}$ be the self-similar parameter from Theorem 1.1 in the small inelasticity regime. Then, there exists a computable strictly positive constant $\alpha_0 > 0$ depending only on $e_{\rm res}, d,$ and c_{11} such that $\alpha_0 \to 0$ as $e_{\rm res} \to 1-$ and the following holds:*

1. *$\beta > 0$ if $\alpha > \alpha_0$;*
2. *$\beta < 0$ if $\alpha < \alpha_0$;*

3. $\beta = 0$ *if* $\alpha = \alpha_0$.

Therefore, α_0 *is a critical shearing parameter where the shear heating effect and the inelastic cooling effect are balanced; in particular, for the balanced case* $\alpha = \alpha_0$*, the self-similar profile determined by* (14) *is reduced to the steady solution of* (1)*.*

Note that for the Cauchy problem (1), the usual fixed-point approach can be adopted to prove the global existence of a class of measure-valued solutions even for any constant shearing matrix A and any restitution coefficient $e_{\rm res} \in (0, 1]$; see Theorem 2.5 later on. With this in hand, we focus on the long-time convergence of solutions to those self-similar profiles obtained in Theorem 1.1. In fact, we have the following result.

Theorem 1.3. *Assume that* $e_{\rm res} \in (0, 1]$ *and the collision kernel* (4) *is cutoff pseudo-Maxwellian. Then, for all* $p \in (2, 4]$*, there exists* $\epsilon > 0$ *such that if* $\|A\| < \epsilon$ *and* $1 - e_{\rm res} < \epsilon$*, then the following holds: Let* $f \in C^0([0,\infty), \mathcal{M}_+)$ *be a measure-valued solution to* (1) *with initial data* $f_0 \in \mathcal{M}_+$ *being a nonnegative Radon measure with finite mass, zero momentum, and finite* p*-moment. Then, there exist* $\beta \in \mathbb{R}$*,* $\lambda > 0$*, and a nonnegative probability measure* $G \in \mathcal{M}_+$ *with zero momentum and finite* p*-moment such that* $e^{d\beta t} f(t, e^{\beta t} v) \to \lambda^{-d} G(\lambda^{-1} v)$ *on* $t \to \infty$ *in the weak topology of* $\mathcal{M}_+$ *with a computable convergence rate in* 2*-Toscani metric.*

Here, the Toscani metric is introduced in (69) later on, see [19] for a review of this topic. Theorem 1.3 will be shown by studying the equation in the frequency space via Fourier transform. We will first show that the same results on Radon measures on $\mathbb{R}^d_c$ have analogs on the characteristic functions of probability measures on $\mathbb{R}^d$, with additional convergence results in Toscani metrics. The desired result then follows by reverting the Fourier transform with an appropriate scaling. Moreover, regarding the stationary self-similar profile G, we can obtain its existence in both Theorem 2.8 and Theorem 3.9 using the different approaches. In the meantime, Theorem 3.9 also gives the uniqueness of G in an appropriate function space in the Fourier framework.

In the case of simple USF (6), we note from Theorem 1.2 that for a given $e_{\rm res}$ in the nearly elastic regime, if the shear parameter α is exactly α_0, then Theorem 1.3 above gives the weak convergence of the solution to the initial value problem on (1) with appropriate initial data towards a non-degenerate stationary profile defined via (14) with self-similar parameter $\beta = 0$, which is just a stationary solution of (1). Moreover, the temperature of solutions increases to infinity for $\beta > 0$ corresponding to the shear-heating dominated case and decreases to zero for $\beta < 0$ corresponding to the inelastic-cooling dominated case, both exponentially fast as $t \to \infty$.

We divide the rest of this paper into two parts. In the first part of Section 2, we treat (1) in the framework of nonnegative Radon measures. We first establish the existence and uniqueness of global in time solutions as well as propagation of moment bounds. To obtain the existence of a non-degenerate stationary profile, we analyze the behavior of the second moment of Radon solutions in Section 2.1, together with the special case of USF in Section 2.2. The stationary Radon profile will be obtained in Section 2.3. In Section 3, we treat (15) in frequency space for analogous results on existence of both global-in-time solutions and self-similar profiles in Section 3.1, in particular obtaining the weak convergence of global solutions to self-similar profiles.

2. **Radon measure-valued solution.** In this part, we consider the mild solutions to (1) that are Radon measure-valued, see [29, Chapter 7]. First, we give the formulation for mild solutions in the inelastic case, which is consistent with [36] for the classical case. Denote by $\mathcal{M}_+(\mathbb{R}^d_c)$ the space of nonnegative Radon measures on $\mathbb{R}^d_c$, which is the one-point compactification of $\mathbb{R}^d$ appending the point ∞, equipped with the total variation distance

$$\|f\|_{\mathcal{M}} = \int_{\mathbb{R}^d_c} |f|(\mathrm{d}v). \tag{19}$$

For $s > 0$, let the s-moment norm of $f \in \mathcal{M}_+$ be

$$\|f\|_s = \int_{\mathbb{R}^d_c} (1 + |v|^s)|f|(\mathrm{d}v). \tag{20}$$

For simplicity of notation, we write $f\mathrm{d}v$ as if f is a function. Equation (1) should now be understood in the sense of measure. More precisely, we will focus on the following notion of solutions to (1).

Definition 2.1 (Weak solution)**.** We say that $f \in C([0,\infty), \mathcal{M}_+)$ is a weak solution of (1) with initial condition $f(0,v) = f_0(v) \in \mathcal{M}_+$ if for any $T > 0$ and for any test function $\psi \in C^1([0,T], C^1(\mathbb{R}^d_c))$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \psi(T,v) f(T,\mathrm{d}v) - \int_{\mathbb{R}^d} \psi(0,v) f_0(\mathrm{d}v) \\
&= - \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \partial_t \psi\, f(t,\mathrm{d}v) - \int_0^T dt \int_{\mathbb{R}^d} (Av \cdot \nabla_v \psi) f(t,\mathrm{d}v) \\
&\quad + \frac{1}{2} \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathrm{d}\sigma\, b\big(\sigma \cdot \frac{v - v_*}{|v - v_*|}\big) f(t,\mathrm{d}v) f(t,\mathrm{d}v_*) \\
&\qquad\qquad (\psi(t,v') + \psi(t,v'_*) - \psi(t,v) - \psi(t,v_*)).
\end{aligned} \tag{21}$$

Under the cutoff assumption, we can write the corresponding strong form of the inelastic collision operator as

$$\begin{aligned}
Q_e(f,f) &= Q_e^+(f,f) - Q_e^-(f,f), \\
Q_e^+(f,f)(v) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b^+(\cos\theta) {e_{\mathrm{res}}}^{-1} f(v'') f(v''_*)\, \mathrm{d}\sigma \mathrm{d}v_*, \\
Q_e^-(f,f)(v) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} b(\cos\theta) f(v) f(v_*)\, \mathrm{d}\sigma \mathrm{d}v_* = b_0 f(v) \int_{\mathbb{R}^d} f(v_*)\, \mathrm{d}v_*,
\end{aligned} \tag{22}$$

where

$$\begin{aligned}
v'' &= \frac{v + v_*}{2} + \frac{1 - z''}{2}(v - v_*) + \frac{z''}{2}|v - v_*|\sigma, \\
v''_* &= \frac{v + v_*}{2} - \frac{1 - z''}{2}(v - v_*) - \frac{z''}{2}|v - v_*|\sigma,
\end{aligned} \tag{23}$$

are images of the inverse of the scattering mapping, $z'' = \frac{1}{2}(1 + {e_{\mathrm{res}}}^{-1}) = z/e_{\mathrm{res}}$, and

$$b^+(c) = b\Big(\frac{(1 + {e_{\mathrm{res}}}^2)c - (1 - {e_{\mathrm{res}}}^2)}{(1 + {e_{\mathrm{res}}}^2) - (1 - {e_{\mathrm{res}}}^2)c}\Big) \sqrt{\frac{2}{(1 + {e_{\mathrm{res}}}^2) - (1 - {e_{\mathrm{res}}}^2)c}},$$

for $-1 \le c \le 1$. The derivation of the strong form (22) from the weak form (2) is based strongly on the inelastic reflection map method as in the appendix of [17], where the ω representation of the post-collision velocities plays a crucial role. If one defines $u := v - v_*$ and $\omega := \frac{u-|u|\sigma}{|u-|u|\sigma|}$ such that

$$v'' = v - \frac{1+e_{\rm res}}{2e_{\rm res}}((v-v_*)\cdot\omega)\omega, \quad v''_* = v_* + \frac{1+e_{\rm res}}{2e_{\rm res}}((v-v_*)\cdot\omega)\omega,$$

and also denotes $\tilde{b}$ in the sense that $b(c) = \tilde{b}(\sqrt{(1-c)/2})(2\sqrt{(1-c)/2})^{-1}$, then it holds that

$$b(\cos\theta)\mathrm{d}\sigma = 2\tilde{b}(\omega\cdot\frac{v-v_*}{|v-v_*|})\mathrm{d}\omega, \tag{24}$$

and

$$\begin{aligned}&\tilde{b}(\omega\cdot\frac{v''-v''_*}{|v''-v''_*|})\mathrm{d}\omega\\ &\quad= \frac{1}{2}b(\frac{(1+{e_{\rm res}}^2)\cos\theta-(1-{e_{\rm res}}^2)}{(1+{e_{\rm res}}^2)-(1-{e_{\rm res}}^2)\cos\theta})\sqrt{\frac{2}{(1+{e_{\rm res}}^2)-(1-{e_{\rm res}}^2)\cos\theta}}\mathrm{d}\sigma. \end{aligned}\tag{25}$$

One may refer to [17, A.3] for a detailed derivation, see also [31, 18]. With the help of the strong form, we give another notion of solutions.

Definition 2.2 (Mild solution)**.** Let $L^1([0,T],\mathcal{M}_+)$ denote the space of all real-valued functions $t\to\|f(t)\|_{\mathcal{M}}$ that are integrable on $[0,T]$. For $g,h_0\in L^1([0,T],\mathcal{M}_+)$ and $0\le t_0\le t\le T<\infty$, we define

$$S_g(t,t_0)h_0(v) := e^{(t-t_0)\mathrm{tr}(A)}e^{-b_0\int_{t_0}^t\|g(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau}h_0(e^{(t-t_0)A}v), \tag{26}$$

such that $h(t,v) = S_g(t,t_0)h_0(v)$ solves

$$\partial_t h - \nabla_v\cdot(Avh) = -b_0 h\int_{\mathbb{R}^d} g(t,v)\,\mathrm{d}v = -b_0 h\,\|g(t)\|_{\mathcal{M}},$$

in the sense of measure on $t\in[t_0,T]$ with initial condition $h(t_0,v) = h_0(v)$. Then, we say that $f\in C([0,\infty),\mathcal{M}_+(\mathbb{R}^d_c))$ is a mild solution of (1) with initial condition $f_0\in\mathcal{M}_+(\mathbb{R}^d_c)$ if

$$f(t,v) = S_f(t,0)f_0(v) + \int_0^t S_f(t,\tau)Q_e^+(f,f)(\tau,v)\,\mathrm{d}\tau.$$

Remark 2.3. We point out that $Q_e^\pm(f,f)$ defines measures in $\mathcal{M}_+$ by the continuity and integrability of $b(\cos\theta)$. Here, the sense of measure is understood by integrating by parts and passing the derivative to the test function. Note that if f is a mild solution, then (1) holds in the sense of measure, which implies that (21) is satisfied on appropriate test functions, and thus f is also a weak solution (see also [35]).

We first prove some useful properties of Q_e^+ and S.

Lemma 2.4. *If $f,g\in\mathcal{M}_+(\mathbb{R}^d_c)$, then*

$$\left\|Q_e^+(f,f)\right\|_{\mathcal{M}} = b_0\|f\|_{\mathcal{M}}^2, \tag{27}$$

$$\left\|Q_e^+(f,f)-Q_e^+(g,g)\right\|_{\mathcal{M}} \le b_0(\|f\|_{\mathcal{M}}+\|g\|_{\mathcal{M}})\|f-g\|_{\mathcal{M}}. \tag{28}$$

Also, if $h_1,h_2\in L^1([0,T],\mathcal{M}_+(\mathbb{R}^d_c))$, then for $T\ge t\ge t'\ge 0$, it holds that

$$\|S_{h_1}(t,t')f\|_{\mathcal{M}} = \exp(-b_0\int_{t'}^t\|h_1(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau)\int_{\mathbb{R}^d} f(v)\,\mathrm{d}v \le \|f\|_{\mathcal{M}}, \tag{29}$$

$$\|S_{h_1}(t,t')f - S_{h_2}(t,t')g\|_{\mathcal{M}} \le \|f-g\|_{\mathcal{M}}$$

$$+\frac{b_0}{2}(\|f\|_{\mathcal{M}}+\|g\|_{\mathcal{M}})\int_{t'}^{t}\|h_1(\tau)-h_2(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau. \quad (30)$$

Furthermore, if $\|f\|_s, \|g\|_s < \infty$ for some $s > 0$, then one has

$$\|Q_e^+(f,f)\|_s \le C\,\|f\|_{\mathcal{M}}\,\|f\|_s, \quad (31)$$

$$\|Q_e^+(f,f)-Q_e^+(g,g)\|_s \le C(\|f\|_s+\|g\|_s)\|f-g\|_s, \quad (32)$$

$$\|S_{h_1}(t,t')f\|_s \le C_T\|f\|_s, \quad (33)$$

and

$$\|S_{h_1}(t,t')f-S_{h_2}(t,t')g\|_s \le C\|f-g\|_s+C(\|f\|_s+\|g\|_s)\int_{t'}^{t}\|h_1(\tau)-h_2(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau, \quad (34)$$

where the constant C depends only on $A, s, T,$ and b_0.

Proof. Noticing that $Q_e^+(f,f)\ge 0$ and $S_g(t,t')f\ge 0$ by the definitions of Q_e^+ and S in (22) and (26), by the fact that 1 is a collision invariant, we have (27) by

$$\begin{aligned}\left\|Q_e^+(f,f)\right\|_{\mathcal{M}} &= \int_{\mathbb{R}^d} Q_e(f,f)\,\mathrm{d}v+\int_{\mathbb{R}^d} Q_e^-(f,f)\,\mathrm{d}v\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\cos\theta)f(v)f(v_*)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\ &= b_0\,\|f\|_{\mathcal{M}}^2.\end{aligned}$$

Next, it is direct to get

$$\begin{aligned}&\left\|Q_e^+(f,f)-Q_e^+(g,g)\right\|_{\mathcal{M}}\\ &\qquad\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b^+(\cos\theta){e_{\mathrm{res}}}^{-1}|f(v'')f(v_*'')-g(v'')g(v_*'')|\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v.\end{aligned}$$

From the definition of v''*and* v_*'' in (23), a straightforward calculation gives that the Jacobian matrix for the transformation $(v,v_*)\mapsto(v'',v_*'')$ satisfies

$$|J| = {e_{\mathrm{res}}}^{-1}.$$

Then, using a change of variables, (24), and (25), we get

$$\begin{aligned}&\left\|Q_e^+(f,f)-Q_e^+(g,g)\right\|_{\mathcal{M}}\\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\cos\theta)|f(v)f(v_*)-g(v)g(v_*)|\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\ &\le b_0\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(|f(v)||f(v_*)-g(v_*)|+|g(v_*)||f(v)-g(v)|)\,\mathrm{d}v_*\mathrm{d}v\\ &= b_0(\|f\|_{\mathcal{M}}+\|g\|_{\mathcal{M}})\,\|f-g\|_{\mathcal{M}},\end{aligned}$$

which implies (28). Moreover, (29) follows from

$$\begin{aligned}\|S_{h_1}(t,t')f\|_{\mathcal{M}} &= \int_{\mathbb{R}^d} S_{h_1}(t,t')f(v)\,\mathrm{d}v\\ &= \exp\big((t-t')\mathrm{tr}(A)-b_0\int_{t'}^{t}\|h_1(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau\big)\int_{\mathbb{R}^d} f(e^{(t-t')A}v)\,\mathrm{d}v\\ &= \exp\big(-b_0\int_{t'}^{t}\|h_1(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau\big)\int_{\mathbb{R}^d} f(v)\,\mathrm{d}v\\ &\le \|f\|_{\mathcal{M}}.\end{aligned}$$

Using a change of variables and the fact that $|e^{-x_1}-e^{-x_2}|\le |x_1-x_2|$ on $x_1,x_2\ge 0$, we get (30) by

$$
\begin{aligned}
&\|S_{h_1}(t,t')f-S_{h_2}(t,t')g\|_{\mathcal{M}}\\
&=e^{(t-t')\mathrm{tr}(A)}\int_{\mathbb{R}^d}\Big|e^{-b_0\int_{t'}^t\|h_1(\tau)\|_{\mathcal{M}}\mathrm{d}\tau}f(e^{(t-t')A}v)-e^{-b_0\int_{t'}^t\|h_2(\tau)\|_{\mathcal{M}}\mathrm{d}\tau}g(e^{(t-t')A}v)\Big|\,\mathrm{d}v\\
&\le\int_{\mathbb{R}^d}\Big(\frac{f(v)+g(v)}{2}b_0\int_{t'}^t\big|\,\|h_1(\tau)\|_{\mathcal{M}}-\|h_2(\tau)\|_{\mathcal{M}}\big|\,\mathrm{d}\tau+|f(v)-g(v)|\Big)\,\mathrm{d}v\\
&\le\frac{b_0}{2}(\|f\|_{\mathcal{M}}+\|g\|_{\mathcal{M}})\int_{t'}^t\|h_1(\tau)-h_2(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau+\|f-g\|_{\mathcal{M}}.
\end{aligned}
$$

Noticing $|v'|^s\le(|v|+|v_*|)^s\le C(|v|^s+|v_*|^s)$ and $|v|^s\le C(|v''|^s+|v''_*|^s)$ for $s\ge 0$ and some constant C which depends only on s, by a change of variables, (24), and (25), (31) then holds from

$$
\begin{aligned}
\|Q_e^+(f,f)\|_s&\le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}(2+|v''|^s+|v''_*|^s)b^+(\cos\theta){e_{\mathrm{res}}}^{-1}f(v'')f(v''_*)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\
&=C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}(2+|v|^s+|v_*|^s)b(\cos\theta)f(v)f(v_*)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\
&\le C\,\|f\|_{\mathcal{M}}\,\|f\|_s.
\end{aligned}
$$

Similarly, we yield (32) by

$$
\begin{aligned}
&\|Q_e^+(f,f)-Q_e^+(g,g)\|_s\\
&\le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}(2+|v''|^s+|v''_*|^s)b^+(\cos\theta){e_{\mathrm{res}}}^{-1}|f(v'')f(v''_*)-g(v'')g(v''_*)|\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\
&=C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}(2+|v|^s+|v_*|^s)b(\cos\theta)|f(v)f(v_*)-g(v)g(v_*)|\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\
&\le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(2+|v|^s+|v_*|^s)(f(v)|f(v_*)-g(v_*)|+g(v_*)|f(v)-g(v)|)\,\mathrm{d}v_*\mathrm{d}v\\
&\le C(\|f\|_s+\|g\|_s)\|f-g\|_s.
\end{aligned}
$$

One can further prove (33) and (34) by

$$
\begin{aligned}
\|S_{h_1}(t,t')f\|_s&=\int_{\mathbb{R}^d}(1+|v|^s)e^{(t-t')\mathrm{tr}(A)}e^{-b_0\int_{t'}^t\|h_1(\tau)\|_{\mathcal{M}}\mathrm{d}\tau}f(e^{(t-t')A}v)\,\mathrm{d}v\\
&\le\int_{\mathbb{R}^d}(1+\big|e^{-(t-t')A}v\big|^s)f(v)\,\mathrm{d}v\\
&\le C_T\|f\|_s,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|S_{h_1}(t,t')f-S_{h_2}(t,t')g\|_s\\
&\le C\int_{\mathbb{R}^d}(1+|v|^s)\Big(\frac{f(v)+g(v)}{2}b_0\int_{t'}^t\big|\,\|h_1(\tau)\|_{\mathcal{M}}-\|h_2(\tau)\|_{\mathcal{M}}\big|\,\mathrm{d}\tau+|f(v)-g(v)|\Big)\,\mathrm{d}v\\
&\le C(\|f\|_s+\|g\|_s)\int_{t'}^t\|h_1(\tau)-h_2(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau+C\|f-g\|_s.
\end{aligned}
$$

This then completes the proof of Lemma 2.4. □

With all the above estimates, we now prove the existence of a mild solution by a fixed-point approach.

Theorem 2.5. *If $f_0 \in \mathcal{M}_+(\mathbb{R}^d_c)$, then for any constant shearing matrix A and $e_{\mathrm{res}} \in (0,1]$, there exists a unique global-in-time mild solution f to (1) satisfying*

$$\|f(t)\|_{\mathcal{M}} = \|f_0\|_{\mathcal{M}}. \tag{35}$$

Proof. Let

$$X_T := \left\{ f \in C([0,T], \mathcal{M}_+(\mathbb{R}^d_c)) \,\middle|\, \|f\|_{X_T} := \sup_{[0,T]} \|f(t)\|_{\mathcal{M}} \le 2\|f_0\|_{\mathcal{M}} \right\}$$

with $T > 0$. Consider the Picard iteration mapping $P : X_T \to X_T$ by

$$Pf(t,v) = S_f(t,0)f_0(v) + \int_0^t S_f(t,\tau)Q_e^+(f,f)(\tau,v)\,\mathrm{d}\tau.$$

We first show the boundedness of Pf for small time. By (29) and (27), for $f \in X_T$ and $t \in [0,T]$, one has

$$\begin{aligned}
\|Pf(t)\|_{\mathcal{M}} &\le \|S_f(t,0)f_0\|_{\mathcal{M}} + \int_0^t \left\|S_f(t,\tau)Q_e^+(f,f)(\tau)\right\|_{\mathcal{M}} \mathrm{d}\tau \\
&\le \|f_0\|_{\mathcal{M}} + \int_0^t \left\|Q_e^+(f,f)(\tau)\right\|_{\mathcal{M}} \mathrm{d}\tau \\
&= \|f_0\|_{\mathcal{M}} + b_0\int_0^t \|f(\tau)\|^2_{\mathcal{M}}\,\mathrm{d}\tau,
\end{aligned}$$

which, further, by the definition of X_T with norm $\|\cdot\|_{X_T}$, gives

$$\|Pf(t)\|_{\mathcal{M}} \le \|f_0\|_{\mathcal{M}} + b_0T\|f\|^2_{X_T} \le \|f_0\|_{\mathcal{M}} + 4b_0T\|f_0\|^2_{\mathcal{M}}.$$

Thus, for

$$T \le \frac{4b_0}{\|f_0\|_{\mathcal{M}} + 1}, \tag{36}$$

it holds that

$$\|Pf\|_{X_T} \le 2\|f_0\|_{\mathcal{M}}. \tag{37}$$

Furthermore, since $b \ge 0$ and $f \ge 0$, $Pf(t) \ge 0$ for all $t \in [0,T]$, which implies the positivity of Pf.

Now we prove that P is a contraction mapping on X_T for small T. Letting $f, g \in X_T$ for $0 \le t \le T$, we take the difference to get

$$\begin{aligned}
\|Pf(t) - Pg(t)\|_{\mathcal{M}} \le{}& \|S_f(t,0)f_0 - S_g(t,0)f_0\|_{\mathcal{M}} \\
&+ \int_0^t \left\|S_f(t,\tau)Q_e^+(f,f)(\tau) - S_g(t,\tau)Q_e^+(g,g)(\tau)\right\|_{\mathcal{M}} \mathrm{d}\tau.
\end{aligned}$$

It follows from (30), (27), and (28) that

$$\begin{aligned}
&\|Pf(t) - Pg(t)\|_{\mathcal{M}} \\
\le{}& b_0\|f_0\|_{\mathcal{M}} \int_0^t \|f(\tau) - g(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau \\
&+ \int_0^t \Big(\left\|Q_e^+(f,f)(\tau) - Q_e^+(g,g)(\tau)\right\|_{\mathcal{M}} + \frac{b_0}{2}\big(\left\|Q_e^+(f,f)(\tau)\right\|_{\mathcal{M}} + \left\|Q_e^+(g,g)(\tau)\right\|_{\mathcal{M}}\big) \\
&\qquad\times \int_\tau^t \|f(\tau') - g(\tau')\|_{\mathcal{M}}\,\mathrm{d}\tau'\Big)\mathrm{d}\tau \\
\le{}& b_0T\|f_0\|_{\mathcal{M}}\|f - g\|_{X_T}
\end{aligned}$$

$$+\|f-g\|_{X_T}\int_0^t \big(b_0(\|f(\tau)\|_{\mathcal{M}}+\|g(\tau)\|_{\mathcal{M}})+\frac{b_0^3}{2}(\|f(\tau)\|_{\mathcal{M}}^2+\|g(\tau)\|_{\mathcal{M}}^2)T\big)\,\mathrm{d}\tau.$$

By letting T satisfy (36), we have from (37) that

$$\|Pf(t)-Pg(t)\|_{\mathcal{M}} \leq \|f-g\|_{X_T} b_0 T\, \|f_0\|_{\mathcal{M}}\,(5+4b_0^2 T\, \|f_0\|_{\mathcal{M}}).$$

Further, we let

$$T=\min\Big\{\frac{1}{1+10b_0\,\|f_0\|_{\mathcal{M}}+8b_0^3\,\|f_0\|_{\mathcal{M}}^2},\frac{4b_0}{\|f_0\|_{\mathcal{M}}+1}\Big\},$$

to get $\|Pf-Pg\|_{X_T}\leq \frac{1}{2}\|f-g\|_{X_T}$, which implies P is a contraction on X_T. By the Banach contraction theorem, there exists a unique $f\in X_T$ such that $f=Pf$.

In order to get a global solution, we integrate in v and t on both sides of (1) to get

$$\|f(t)\|_{\mathcal{M}}=\|f_0\|_{\mathcal{M}}\,. \tag{38}$$

Hence, the mass is conserved with (35), so we can extend the local-in-time solution to be a unique global-in-time mild solution, which finishes the proof of Theorem 2.5. □

Concerning the moments of a mild solution, we have the following result.

Theorem 2.6. *Suppose f is a mild solution of (1) with initial condition $f_0(v)\in\mathcal{M}_+$ with $\int_{\mathbb{R}^d}|v|^{s_0}f_0\,\mathrm{d}v<\infty$ for some $s_0>1$. Then, for any $0<T<\infty$, it holds that*

$$\sup_{0\leq t\leq T}\|f(t)\|_{s_0}\leq C_T\|f_0\|_{s_0}<\infty, \tag{39}$$

with some constant C_T which depends on T, and the weak form (21) is satisfied on test function $\psi(t,v)\in C^1([0,T],C^1(\mathbb{R}^d))$ with $|\psi(t,v)|+|v||\nabla_v\psi(t,v)|\leq C(1+|v|^{s'})$ uniformly on $t\in[0,T]$ for some $1<s'<s_0$. Furthermore, on each $s>s_0$ and for each $0<T<\infty$, there exists f_n with $\sup_{[0,T]}\int|v|^s f_n(t)<\infty$, and f can be approximated by $f_n(t,v)\in C^0([0,T],\mathcal{M}_+)$ in the sense that $\sup_{[0,T]}\|f_n(t)-f(t)\|_{s_0}\to 0$ as $n\to\infty$.

Proof. By (33), (31), and (38), we have

$$\begin{aligned}\|f(t)\|_{s_0}&\leq \|S_f(t,0)f_0\|_{s_0}+\int_0^t\|S_f(t,\tau)Q_e^+(f,f)(\tau)\|_{s_0}\,\mathrm{d}\tau\\&\leq C_T(\|f_0\|_{s_0}+\int_0^t\|f(\tau)\|_{\mathcal{M}}\,\|f(\tau)\|_{s_0}\,\mathrm{d}\tau)\\&=C_T(\|f_0\|_{s_0}+\|f_0\|_{\mathcal{M}}\int_0^t\|f(\tau)\|_{s_0}\,\mathrm{d}\tau),\end{aligned}$$

which implies (39).

Let $\psi(t,v)\in C^1([0,T],C^1(\mathbb{R}^d))$ be a test function with $|\psi(t,v)|+|v||\nabla_v\psi(t,v)|\leq C(1+|v|^{s'})$. Then, letting $\psi(t,v)$ be approximated by the sequence $\{\psi_n(t,v)\}_{n=1}^\infty\subset$

$C^1([0,T], C^1(\mathbb{R}^d_c))$ with $|\nabla_v \psi_n| \le C(1+|v|^{s'-1} + |\nabla_v \psi|)$, one gets

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \psi_n(T,v) f(T,v)\,\mathrm{d}v - \int_{\mathbb{R}^d} \psi_n(0,v) f_0(v)\,\mathrm{d}v \\
&\qquad + \int_0^T \int_{\mathbb{R}^d} f(t,v)\partial_t \psi_n(t,v)\mathrm{d}v\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} Avf(t,v)\cdot \nabla_v \psi_n(t,v)\mathrm{d}v\,\mathrm{d}t \\
&= \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\cos\theta) f(t,v) f(t,v_*) \\
&\qquad\qquad (\psi_n(t,v') + \psi_n(t,v'_*) - \psi_n(t,v) - \psi_n(t,v_*))\,\mathrm{d}\sigma \mathrm{d}v_* \mathrm{d}v \mathrm{d}t. \qquad (40)
\end{aligned}
$$

Then, a Taylor expansion gives

$$
\begin{aligned}
&\Big|\int_{\mathbb{S}^{d-1}} b(\cos\theta)(\psi_n(t,v') + \psi_n(t,v'_*) - \psi_n(t,v) - \psi_n(t,v_*))\,\mathrm{d}\sigma\Big| \\
&\le C\Big|(v'-v)\cdot \int_0^1 (\nabla_v \psi_n(t, v+\xi(v'-v)) - \nabla_v\psi_n(t, v_* + \xi(v'_* - v_*)))\,\mathrm{d}\xi\Big|. \qquad (41)
\end{aligned}
$$

A direct calculation shows that

$$
\begin{aligned}
|v'-v| = |v'_* - v_*| &= \Big|\frac{z}{2}(v-v_*) - \frac{z}{2}|v-v_*|\sigma\Big| \\
&= \Big|\frac{z}{2}|v-v_*|(\frac{v-v_*}{|v-v_*|} - \sigma)\Big| \le z|v-v_*|,
\end{aligned}
$$

which yields

$$
\begin{aligned}
&C\Big|(v'-v)\cdot\int_0^1 (\nabla_v\psi_n(t, v+\lambda(v'-v)) - \nabla_v\psi_n(t, v_* + \lambda(v'_* - v_*)))\,\mathrm{d}\lambda\Big| \\
&\le C \sup_{|u-v|\le z|v-v_*|} |v-v_*||\nabla_v\psi_n(t,u)| + \sup_{|u-v_*|\le z|v-v_*|} |v-v_*||\nabla_v\psi_n(t,u)| \\
&\le C(|v|+|v_*|) \sup_{|u|\le (1+z)(|v|+|v_*|)} |\nabla_v \psi_n(t,u)| \\
&\le C(|v|+|v_*|)\big(1 + (|v|+|v_*|)^{s'-1} + \sup_{R\le|u|\le(1+z)(|v|+|v_*|)} |\nabla_v\psi(t,u)|\big) \\
&\le C(1+(|v|+|v_*|)^{s'}), \qquad (42)
\end{aligned}
$$

where the constant C is independent of n. The third inequality above holds since $\psi(t,v) \in C([0,T], C^1(\mathbb{R}^d))$, which gives $|\nabla_v\psi(t,v)| \le M$ on a neighborhood $B(0,R)$ of the origin for some constant $M = M(\psi) > 0$. Collecting (40), (41), and (42), then passing to the limit $n\to\infty$ on ψ_n in (40), by dominated convergence theorem one obtains (21) for $\psi(t,v) \in C^1([0,T], C^1(\mathbb{R}^d))$ with $|\psi(t,v)| + |v||\nabla_v\psi(t,v)| \le C(1+|v|^{s'})$.

Consider the higher moment approximation of f now. Let χ denote the indicator function such that $\chi_E(x) = 1$ for $x\in E$. Defining the initial data $f_{n0} = f_0\chi_{|v|\le n}$ and the corresponding mild solutions f_n, we have

$$
\|f_{n0} - f_0\|_{s_0} = \int_{\mathbb{R}^d} (1+|v|^{s_0}) f_0(v)\chi_{|v|>n}\mathrm{d}v \to 0,
$$

as $n\to\infty$. From (39), it holds that

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|f_n(t)\|_s \le C_T\|f_{n0}\|_s < \infty, \\
&\sup_{t\in[0,T]} \|f_n(t)\|_{s_0} \le C_T\|f_{n0}\|_{s_0} \le C_T\|f_0\|_{s_0} < \infty,
\end{aligned}
$$

which, together with (33), (34), and (32), yield

$$\begin{aligned}
&\|f_n(t) - f(t)\|_{s_0} \\
&= \|S_{f_n}(t,0)f_{n0} - S_f(t,0)f_0\|_{s_0} \\
&\quad + \int_0^t \|S_{f_n}(t,\tau)Q_e^+(f_n,f_n)(\tau) - S_f(t,\tau)Q_e^+(f,f)(\tau)\|_{s_0}\,\mathrm{d}\tau \\
&\le C\Big(\|f_{n0} - f_0\|_{s_0} + \|f_0\|_{s_0}\int_0^t \|f_n(\tau) - f(\tau)\|_{\mathcal{M}}\,\mathrm{d}\tau \\
&\qquad + \int_0^t (\|f_n(\tau)\|_{s_0} + \|f(\tau)\|_{s_0})\|f_n(\tau) - f(\tau)\|_{s_0}\,\mathrm{d}\tau \\
&\qquad + \int_0^t (\|f_n(\tau)\|_{s_0}^2 + \|f(\tau)\|_{s_0}^2)\int_\tau^t \|f_n(\tau') - f(\tau')\|_{\mathcal{M}}\,\mathrm{d}\tau'\,\mathrm{d}\tau\Big) \\
&\le C\|f_{n0} - f_0\|_{s_0} + C_T\|f_0\|_{s_0}(\|f_0\|_{s_0} + T)\int_0^t \|f_n(\tau) - f(\tau)\|_{s_0}\,\mathrm{d}\tau.
\end{aligned}$$

It then follows from the Gronwall inequality that $\sup_{t\in[0,T]}\|f_n(t) - f(t)\|_{s_0} \le C_T\|f_{n0} - f_0\|_{s_0} \to 0$ as $n \to \infty$. □

Theorem 2.5 and Theorem 2.6 will be very useful later in the proof of Theorem 2.8 for the existence of the stationary self-similar profile G.

2.1. **Second moment equation.** We now compute the evolution equation for the second moment tensor $\mathcal{B} = \mathcal{B}(t) := \int_{\mathbb{R}^d} v^{\otimes 2} f\,\mathrm{d}v$ on a mild solution f of (12) that has unit mass and zero momentum $\int_{\mathbb{R}^d}(1,v)f(t,v)\,\mathrm{d}v = (1,0)$. Note that, by definition, $\mathcal{B}$ is symmetric positive semidefinite.

By multiplying $v^{\otimes 2}$ on both sides of (12) and integrating over v, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} v^{\otimes 2} f\,\mathrm{d}v \\
&= \int_{\mathbb{R}^d} v^{\otimes 2}\nabla_v\cdot(A_\beta v f)\,\mathrm{d}v + \int_{\mathbb{R}^d} v^{\otimes 2} Q_e(f,f)\,\mathrm{d}v \\
&= -\int_{\mathbb{R}^d} A_\beta v f\cdot\nabla_v(v^{\otimes 2})\,\mathrm{d}v \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\sigma\cdot\hat{e})f(t,v)f(t,v_*)\big(v'^{\otimes 2} + {v'_*}^{\otimes 2} - v^{\otimes 2} - {v_*}^{\otimes 2}\big)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v,
\end{aligned}$$

where we denote $\hat{e} = \frac{v - v_*}{|v - v_*|}$. Recalling (9), (10), and (11), a direct calculation gives

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B} &= -\int_{\mathbb{R}^d}(v\otimes(A_\beta v) + (A_\beta v)\otimes v)f(t,v)\,\mathrm{d}v \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\sigma\cdot\hat{e})f(t,v)f(t,v_*)|v - v_*|^2 \\
&\qquad\qquad \times\big(z(2-z)\hat{e}^{\otimes 2} - z^2\sigma^{\otimes 2} - z(1-z)(\hat{e}\otimes\sigma + \sigma\otimes\hat{e})\big)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\
&= -\Big(\big(\int_{\mathbb{R}^d} v^{\otimes 2} f(t,v)\,\mathrm{d}v\big)A_\beta^{\mathsf{T}} + A_\beta\int_{\mathbb{R}^d} v^{\otimes 2} f(t,v)\,\mathrm{d}v\Big) \\
&\quad - \frac{1}{4}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(t,v)f(t,v_*)|v - v_*|^2\big((2\zeta + z^2 d c_{11})\hat{e}^{\otimes 2} - z^2 c_{11} I\big)\,\mathrm{d}v_*\mathrm{d}v,
\end{aligned}$$

where ζ and c_{11} are defined in (7) and (8), respectively. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B} &= -(\mathcal{B}A_\beta^\mathsf{T} + A_\beta\mathcal{B}) - \frac{1}{4}((2\zeta + z^2 dc_{11})\cdot 2\mathcal{B} - z^2 c_{11}\cdot 2\mathrm{tr}(\mathcal{B})I)\\ &= -\big(A_\beta\mathcal{B} + \mathcal{B}A_\beta^\mathsf{T} + \zeta\mathcal{B} + z^2\frac{dc_{11}}{2}(\mathcal{B} - \frac{\mathrm{tr}(\mathcal{B})}{d}I)\big),\end{aligned}$$

which yields

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B} + A_\beta\mathcal{B} + \mathcal{B}A_\beta^\mathsf{T} + \zeta\mathcal{B} + z^2\frac{dc_{11}}{2}(\mathcal{B} - \frac{\mathrm{tr}(\mathcal{B})}{d}I)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B} + A\mathcal{B} + \mathcal{B}A^\mathsf{T} + 2(\beta + \frac{\zeta}{2})\mathcal{B} + z^2\frac{dc_{11}}{2}(\mathcal{B} - \frac{\mathrm{tr}(\mathcal{B})}{d}I)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{B} + (2\beta + \zeta + \tilde{c})\mathcal{B} + L\mathcal{B},\end{aligned}\tag{43}$$

where A_β is defined in (13), $L\mathcal{B} = A\mathcal{B} + \mathcal{B}A^\mathsf{T} - \frac{\tilde{c}}{d}\mathrm{tr}(\mathcal{B})I$, and $\tilde{c} = z^2\frac{dc_{11}}{2} > 0$. Notice that our equation contains the classical elastic case where $e_{\mathrm{res}} = 1$, which leads to $z = 1$ and $\zeta = 0$, see [35, 7].

In terms of (43), we denote the steady second moment equation for symmetric B by

$$(2\tilde{\beta} + \tilde{c})B + LB = (2\beta + \zeta + \tilde{c})B + LB = 0,\tag{44}$$

where $\tilde{\beta} = \beta + \zeta/2$. Here and in what follows, for the steady case we have used the constant matrix B instead of the time-dependent matrix $\mathcal{B}$.

2.2. **Second moment for uniform shear flow.** We now consider the second moment equation (43) in the case where $A = \alpha E_{12}$, defined in (6), is the uniform shear flow matrix, look for a symmetric stationary solution, and study the eigensystem of $-L$. In such case, we can have a precise description on whether the temperature is increasing or decreasing in terms of the comparison of effects between shear heating and inelastic cooling.

Theorem 2.7. *On a given non-degenerate cutoff collision kernel satisfying (5) and inelastic parameter $z \in (1/2, 1)$, for the inelastic Boltzmann equation under uniform simple shear heating*

$$\partial_t f - \alpha v_2\partial_{v_1} f = Q_e(f, f),$$

there exists a critical shearing parameter $\alpha_0 = \alpha_0(z) > 0$ defined by (53) in the proof such that for the mild solution $f(t, v)$ with temperature $T(t) = \int_{\mathbb{R}^d}|v|^2 f(t, v)\mathrm{d}v$ on an initial probability measure with zero momentum and finite energy, the following holds:

- *If $\alpha > \alpha_0$, $T(t)$ grows exponentially with rate $2\gamma - \zeta > 0$;*
- *If $0 \le \alpha < \alpha_0$, $T(t)$ decays to zero exponentially with rate $\zeta - 2\gamma > 0$;*
- *If $\alpha = \alpha_0$, $T(t)$ converges exponentially to some finite positive value with rate $\zeta - 2\sigma \ge \tilde{c} > 0$.*

Here, ζ is defined in (7), and $\gamma = \gamma(\alpha)$ and $\sigma = \sigma(\alpha)$ are defined in (49) and (50) in the proof, respectively.

Proof. Rewrite (44) for symmetric B as

$$(2\tilde{\beta} + \tilde{c})B + \alpha E_{12}B + \alpha BE_{21} - \frac{\tilde{c}}{d}\mathrm{tr}(B)I = 0.\tag{45}$$

By enumerating the indices, one has

$$\begin{cases} (2\tilde{\beta}+\tilde{c})B_{11}+2\alpha B_{12}-\frac{\tilde{c}}{d}\mathrm{tr}(B)=0 & \text{if } i=j=1, \\ (2\tilde{\beta}+\tilde{c})B_{1j}+\alpha B_{2j}=0 & \text{if } i=1,\ j\neq 1, \\ (2\tilde{\beta}+\tilde{c})B_{ii}-\frac{\tilde{c}}{d}\mathrm{tr}(B)=0 & \text{if } i=j\neq 1, \\ (2\tilde{\beta}+\tilde{c})B_{ij}=0 & \text{if } i,j,1 \text{ are distinct to each other, } d\geq 3. \end{cases} \tag{46}$$

First, consider the degenerate case $\alpha=0$, or equivalently $A=0$. Then, (45) becomes

$$(2\tilde{\beta}+\tilde{c})B-\frac{\tilde{c}}{d}\mathrm{tr}(B)I=0.$$

It is easy to see that there are only two solutions $(\tilde{\beta},B)$ to the above equation. The first one is $\tilde{\beta}=-\tilde{c}/2$, $\mathrm{tr}(B)=0$, which corresponds to the case that the eigenspace is of codimension 1 in the space of symmetric matrices, and the second one is $\tilde{\beta}=0$, $B=cI$ for any constant c, which is the eigenspace of dimension 1.

Now assume that $\alpha>0$. In the case where $\tilde{\beta}=-\tilde{c}/2$ (corresponding to eigenvalue 0 of $-L$), we have $\mathrm{tr}(B)=0$, which implies that the only symmetric positive semidefinite solution is $B=0$. More precisely, the solution space consists exactly of symmetric matrices B with $\mathrm{tr}(B)=0$ and $B_{2j}=0$ for all j, so it has codimension $d+1$.

Before proceeding to other solutions of (45), let us consider the behavior of $(-L)^2$ for $d\geq 3$. It is straightforward to get that

$$L^2B=2\alpha^2E_{12}BE_{21}-\frac{\alpha\tilde{c}}{d}\mathrm{tr}(B)(E_{12}+E_{21})-\frac{\tilde{c}}{d}(2\alpha B_{12}-\tilde{c}\mathrm{tr}(B))I.$$

Comparing each element, it holds that the kernel of $(-L)^2$ is defined by

$$\begin{cases} 0 &= (L^2B)_{11}=-2\frac{\tilde{c}\alpha}{d}B_{12}+2\alpha^2B_{22}+\frac{\tilde{c}^2}{d}\mathrm{tr}(B), \\ 0 &= (L^2B)_{12}=-\frac{\alpha\tilde{c}}{d}\mathrm{tr}(B), \\ 0 &= (L^2B)_{ii}=-2\frac{\tilde{c}\alpha}{d}B_{12}+\frac{\tilde{c}^2}{d}\mathrm{tr}(B), \quad \text{if } i\geq 2, \end{cases}$$

or equivalently $\mathrm{tr}(B)=B_{12}=B_{22}=0$, giving a nullspace of codimension 3.

We now assume that $2\tilde{\beta}+\tilde{c}\neq 0$. It follows from (46) that

$$\begin{aligned} B_{1j}&=0 \quad \text{if } j\geq 3, \\ B_{ii}&=\frac{\tilde{c}}{d(2\tilde{\beta}+\tilde{c})}\mathrm{tr}(B) \quad \text{if } i\geq 2, \\ B_{12}&=-\frac{\alpha}{2\tilde{\beta}+\tilde{c}}B_{22}=-\frac{\alpha\tilde{c}}{d(2\tilde{\beta}+\tilde{c})^2}\mathrm{tr}(B), \\ B_{11}&=-\frac{2\alpha}{2\tilde{\beta}+\tilde{c}}B_{12}+\frac{\tilde{c}}{d(2\tilde{\beta}+\tilde{c})}\mathrm{tr}(B)=\frac{\tilde{c}}{d(2\tilde{\beta}+\tilde{c})}(1+\frac{2\alpha^2}{(2\tilde{\beta}+\tilde{c})^2})\mathrm{tr}(B). \end{aligned}$$

Hence, we have

$$\mathrm{tr}(B)=B_{11}+\sum_{i=2}^{d}B_{ii}=\frac{\tilde{c}}{d(2\tilde{\beta}+\tilde{c})}(\frac{2\alpha^2}{(2\tilde{\beta}+\tilde{c})^2}+d)\mathrm{tr}(B). \tag{47}$$

If $\mathrm{tr}(B)=0$, then $B_{ii}=0$ for all i, and $B_{12}=0$, which implies $B=0$. If $\mathrm{tr}(B)\neq 0$, then it follows from (47) that $d\tilde{\beta}(2\tilde{\beta}+\tilde{c})^2=\tilde{c}\alpha^2$, or equivalently

$$\tilde{\beta}(4\tilde{\beta}+z^2dc_{11})^2=2z^2c_{11}\alpha^2. \tag{48}$$

Direct calculation shows that the derivative of the cubic has two distinct zeros, and the cubic is negative at the smallest zero, which indicates that (48) has only one real root $\tilde{\beta} = \gamma$,

$$\gamma = \frac{\tilde{c}}{3}(-1 + \cosh Y) = \frac{2\tilde{c}}{3}\sinh^2\frac{Y}{2}$$
$$= \frac{z^2 d c_{11}}{3}\sinh^2\big(\frac{1}{6}\mathrm{arcosh}\big(1 + \frac{108}{d^3}(\frac{\alpha}{z^2 c_{11}})^2\big)\big) \in (0, \frac{\alpha}{2\sqrt{2d}}], \quad (49)$$

with $Y = \frac{1}{3}\mathrm{arcosh}(1 + \frac{27}{d}(\alpha/\tilde{c})^2) = \frac{1}{3}\mathrm{arcosh}(1 + \frac{108}{d^3}(\frac{\alpha}{z^2 c_{11}})^2) > 0$. One may refer to pages 473–477 in [34] for the formula. The solution $\alpha \mapsto \gamma$ is bijective on $\mathbb{R}^+$, strictly increasing, and has asymptotic $\gamma \sim \alpha^2$ on $\alpha \ll 1$ and $\gamma \sim \alpha^{2/3}$ on $\alpha \gg 1$.

The remaining two complex roots are

$$\gamma_\pm = \frac{\tilde{c}}{3}(-1 - \frac{1}{2}\cosh Y \pm i\frac{\sqrt{3}}{2}\sinh Y) = \sigma_R \pm i\omega, \quad (50)$$

with real part $\sigma_R = \mathbb{R}e\gamma_\pm = -\frac{\tilde{c}}{6}(2 + \cosh Y) < -\tilde{c}/2 < 0$ and imaginary part $\omega = \frac{\tilde{c}}{2\sqrt{3}}\sinh Y > 0$. Note that $2\gamma - 2\sigma_R = \frac{\tilde{c}}{3}\cosh Y > 0$ and $\gamma + 2\sigma_R = -\tilde{c} < 0$.

For a root $\tilde{\beta}$ of (48), the solution space of (45) is of dimension 1, and each solution takes the form

$$B = \mathrm{tr}(B)\Big(\frac{1}{d(1 + 2\tilde{\beta}/\tilde{c})}I - \frac{\tilde{\beta}}{\alpha}(E_{12} + E_{21}) + (1 - \frac{1}{1 + 2\tilde{\beta}/\tilde{c}})E_{11}\Big). \quad (51)$$

Particularly, for the real root γ,

$$B = \frac{\mathrm{tr}(B)}{d(1 + 2\gamma/\tilde{c})}\Big(I - \sqrt{d\frac{\gamma}{\tilde{c}}}(E_{12} + E_{21}) + 2d\frac{\gamma}{\tilde{c}}E_{11}\Big). \quad (52)$$

Thus, assuming $\mathrm{tr}(B) = d(1 + 2\gamma/\tilde{c}) > 0$, all eigenvalues of such B are, in non-increasing order,

$$\Lambda_1 = 1 + d\frac{\gamma}{\tilde{c}} + \sqrt{d\frac{\gamma}{\tilde{c}}(1 + d\frac{\gamma}{\tilde{c}})},$$
$$\Lambda_2 = \ldots = \Lambda_{d-1} = 1,$$
$$\Lambda_d = 1 + d\frac{\gamma}{\tilde{c}} - \sqrt{d\frac{\gamma}{\tilde{c}}(1 + d\frac{\gamma}{\tilde{c}})},$$

which are all positive. Thus, B is positive definite, with the property that $\Lambda_1 \to \infty$, $\Lambda_d \to 1/2$ when $\alpha \to \infty$, and $\Lambda_1^{-1} + \Lambda_d^{-1} = 2$.

Thus, on a given self-similar parameter β, the operator $B \mapsto (2\beta + \zeta + \tilde{c})B + LB$ has eigenvalues $2\beta+\zeta-2\gamma, 2\beta+\zeta-2\gamma_+, 2\beta+\zeta-2\gamma_-,$ *and* $2\beta+\zeta+\tilde{c}$, where all except the last are simple eigenvalues, and $2\beta + \zeta + \tilde{c}$ has $D - d - 1$ linearly independent eigenvectors and $d - 2$ generalized eigenvectors of order 2 with $D = \frac{d(d+1)}{2}$ being the dimension of the space of symmetric $d \times d$ complex matrices. Combined, the solution $\mathcal{B}(t)$ of (43) on the self-similar parameter β takes the form

$$\mathcal{B}(t) = c_0 e^{-(2\beta+\zeta-2\gamma)t}B_0 + c_+ e^{-(2\beta+\zeta-2\sigma_R-2i\omega)t}B_+ + c_- e^{-(2\beta+\zeta-2\sigma_R+2i\omega)t}B_-$$
$$+ e^{-(2\beta+\zeta+\tilde{c})t}(c_1 B_1 + \ldots + c_{D-d-1}B_{D-d-1})$$
$$+ te^{-(2\beta+\zeta+\tilde{c})t}(c_1' B_1' + \ldots + c_{d-2}' B_{d-2}')$$

for some constants $c_0, c_+, c_-, c_1, \ldots, c_{D-d-1}, c_1', \ldots, c_{d-2}' \in \mathbb{C}$, nonzero matrices B_0, B_+, B_- of the form of (52) and (51), nonzero symmetric matrices $B_1, \ldots, B_{D-d-1}$

which have trace and $(2,j)$-entries zero for all j, and nonzero symmetric matrices $B_1',\dots,B_{d-2}'$ which have trace, $(1,2)$-entry, and $(2,2)$-entry zero. Assuming B_0, B_+, B_- are normalized to have unit trace and noting that all except these three terms have zero trace, taking the trace on $\mathcal{B}$ yields the temperature $T \coloneqq \mathrm{tr}(\mathcal{B})$ of the form

$$T(t) = c_0 e^{-(2\beta+\zeta-2\gamma)t} + e^{-(2\beta+\zeta-2\sigma_R)t}(c_+ e^{2i\omega t} + c_- e^{-2i\omega t}).$$

As $T(t)$ should be a real number for all t, we have that $c_0 \in \mathbb{R}$ and $c_+ = \overline{c_-}$. On $c_+ = r + im$ with $r, m \in \mathbb{R}$, we have

$$e^{2\beta t} T(t) = c_0 e^{(2\gamma-\zeta)t} + e^{(2\sigma_R-\zeta)t}(2r\cos(2\omega t) - 2m\sin(2\omega t))$$

for some constants $c_0, r, m \in \mathbb{R}$. Similarly, by considering the equation for $\mathcal{B}_{12}$ and $\mathcal{B}_{22}$, the three coefficients satisfy the relation

$$\begin{aligned}
T(0) &= c_0 + 2r,\\
\mathcal{B}_{12}(0) &= -\frac{\gamma}{\alpha}c_0 - \frac{2\sigma_R}{\alpha}r + \frac{2\omega}{\alpha}m,\\
\mathcal{B}_{22}(0) &= \frac{\tilde{c}}{d(2\gamma+\tilde{c})}c_0 + \frac{2\tilde{c}(2\sigma_R+\tilde{c})}{d((2\sigma_R+\tilde{c})^2+4\omega^2)}r + \frac{4\tilde{c}\omega}{d((2\sigma_R+\tilde{c})^2+4\omega^2)}m.
\end{aligned}$$

Solving the system gives

$$\begin{pmatrix} c_0 \\ r \\ m \end{pmatrix} = \frac{d}{\tilde{c}^2\alpha^2}\left(27 + 2d\frac{\tilde{c}^2}{\alpha^2}\right)^{-\frac{1}{2}}$$
$$\begin{pmatrix} 4\tilde{c}\omega(\sigma_R^2+\omega^2) & 4\tilde{c}\alpha\sigma_R\omega & 2\tilde{c}\alpha^2\omega \\ -\frac{2}{3}\tilde{c}\omega(\tilde{c}\gamma - 8\omega^2) & -2\tilde{c}\alpha\sigma_R\omega & -\tilde{c}\alpha^2\omega \\ \frac{1}{6}\tilde{c}^2(\tilde{c}\gamma-8\omega^2) & -\frac{1}{2}\tilde{c}\alpha(\tilde{c}\gamma-8\omega^2) & \frac{1}{2}\tilde{c}\alpha^2(\tilde{c}+3\gamma) \end{pmatrix}\begin{pmatrix} T(0) \\ \mathcal{B}_{12}(0) \\ \mathcal{B}_{22}(0) \end{pmatrix}.$$

Particularly, we can see that c_0, r, m, and the precise behavior of $T(t)$ are determined by the initial second moment matrix $\mathcal{B}_0 = \mathcal{B}(0)$ only via $\mathrm{tr}(\mathcal{B}_0) = \int |v|^2 f_0$, $(\mathcal{B}_0)_{12} = \int v_1 v_2 f_0$, and $(\mathcal{B}_0)_{22} = \int v_2^2 f_0$, and the oscillating terms in $T(t)$ vanish if these quantities satisfy (52).

Furthermore, we can see that in the long-time asymptotic, the energy

- grows exponentially if $2\gamma > \zeta + 2\beta$ (shearing dominant),
- decays exponentially if $2\gamma < \zeta + 2\beta$ (cooling dominant), and
- converges to some constant $c_0 > 0$ with rate $\zeta - 2\sigma_R > 0$ if $2\gamma = \zeta + 2\beta$,

where γ depending on z, α, c_{11}, and d is defined in (49), and ζ depending only on z is given in (7). Without self-similar scaling, the last case happens if $\beta = 0$ solves (48), or equivalently

$$\alpha = \alpha_0 \coloneqq (\zeta + z^2\frac{dc_{11}}{2})\sqrt{\frac{\zeta}{z^2 c_{11}}} = ((1-z)(b_0-b_1) + z\frac{dc_{11}}{2})\sqrt{z(1-z)\frac{b_0-b_1}{c_{11}}} > 0, \tag{53}$$

for $1/2 < z < 1$. It is also direct to see $\alpha_0 \to 0$ as $z \to 1-$. This completes the proof of Theorem 2.7. $\square$

Note that α_0 as in (53) has a finite upper bound

$$\alpha_0 < \frac{1}{4}(b_0 - b_1 + dc_{11}/2)\sqrt{(b_0-b_1)/c_{11}} < \infty.$$

2.3. **Stationary self-similar Radon profile.** In [35], the existence of a stationary self-similar profile for elastic collision can be shown under the assumption that the shear strength $\|A\|$ is sufficiently small. However, due to the appearance of the cooling effect for inelastic collision, we need to further impose the smallness of ζ as in (7), or equivalently the condition that the inelastic parameter z is close to 1, to establish the existence. Here, the shearing matrix A is not necessarily associated with a uniform shear.

Theorem 2.8. *There exists $\epsilon > 0$ such that if $\|A\| < \epsilon$ and $1 - z < \epsilon$, then there is some $\beta \in \mathbb{R}$ and a symmetric positive definite matrix B with $\operatorname{tr}(B) = 1$ such that for all $\Theta > 0$, a self-similar profile $G \in \mathcal{M}_+$ exists where (14) is satisfied in the sense of measure with $\int_{\mathbb{R}^d}(1, v, v^{\otimes 2})G(v)\,\mathrm{d}v = (1, 0, \Theta B)$. In particular, $\int_{\mathbb{R}^d}(1, v, |v|^2)G(v)\,\mathrm{d}v = (1, 0, \Theta)$.*

Proof. First, we need to show the existence of B for the stationary second moment equation (44) with the scaling parameter β. By similar perturbation arguments as in [35, Lemma 4.16], for sufficiently small $\|A\| \ll 1$, there exist a $\beta \in \mathbb{R}$ and a nontrivial symmetric positive definite stationary solution B to (44). Moreover, β is chosen to be the complex number with the largest real part for which (44) holds, and B is normalized in this theorem with $\operatorname{tr}(B) = 1$.

Take $C_* > 0$ large enough and $s \in (2, 4]$. Let $K \subseteq \mathcal{M}_+$ be the set of measures G that satisfies $\int_{\mathbb{R}^d}(1, v, v^{\otimes 2})G(v)\mathrm{d}v = (1, 0, \Theta B)$ and $\int |v|^s G \le C_*$. Note that K depends on C_*, s, and Θ. It is easy to see that K is nonempty, bounded in $\mathcal{M}_+$, and convex. Furthermore, as K is weakly closed, then by the Banach–Alaoglu theorem it is weak-* compact.

Let $S : [0, \infty) \times \mathcal{M}_+(\mathbb{R}^d_c) \to \mathcal{M}_+(\mathbb{R}^d_c)$ be the mild solution operator such that $G(t) = S(t)G_0$ is the mild solution from Theorem 2.5 with initial condition G_0. From Theorem 2.6, we can see that $t \mapsto S(t)$ is a semigroup on $\mathcal{M}_+(\mathbb{R}^d_c)$.

First, we show that S is uniformly weakly continuous on $[0, T] \times K$ for each $T < \infty$. Let $0 \le t_1 \le t_2 \le T < \infty$, and $G_0(v) \in \mathcal{M}_+$ with $\int_{\mathbb{R}^d} |v|^s G_0(v)\mathrm{d}v < \infty$. By Theorem 2.6, $G(t) = S(t)G_0$ also has finite s-moment on $[0, T]$. Let $\varphi \in C^0(\mathbb{R}^d_c)$. Then, for a cutoff function $\varphi_n \in C^2(\mathbb{R}^d_c)$ with $|v||\nabla_v \varphi_n| + |v|^2|\nabla_v^2 \varphi_n| \le C(1 + |v|^s)$ for some constant C uniformly in n, we have

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}^d} \varphi_n(v)G(t_1, v)\mathrm{d}v - \int_{\mathbb{R}^d} \varphi_n(v)G(t_2, v)\mathrm{d}v \Big| \\
&\le \int_{t_1}^{t_2} \int_{\mathbb{R}^d} |Av \cdot \nabla_v \varphi_n| G(t, v)\,\mathrm{d}v\mathrm{d}t \\
&\quad + \frac{1}{2}\int_{t_1}^{t_2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\cos\theta)G(t, v)G(t, v_*) \\
&\qquad\qquad \times |\varphi_n(v') + \varphi_n(v'_*) - \varphi_n(v) - \varphi_n(v_*)|\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}t,
\end{aligned}
$$

which, together with (39), yields

$$
\begin{aligned}
&\Big| \int_{\mathbb{R}^d} \varphi_n(v)G(t_1, v)\,\mathrm{d}v - \int_{\mathbb{R}^d} \varphi_n(v)G(t_2, v)\,\mathrm{d}v \Big| \\
&\le C\int_{t_1}^{t_2} \int_{\mathbb{R}^d} |v||\nabla_v \varphi_n| G(t, v)\,\mathrm{d}v\mathrm{d}t \\
&\quad + C\int_{t_1}^{t_2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(t, v)G(t, v_*)|v - v_*|^2 |\nabla_v^2 \varphi_n|\,\mathrm{d}v_*\mathrm{d}v\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_{t_1}^{t_2}(1+|v|^s)G(t,v)\,\mathrm{d}t + C\int_{t_1}^{t_2}(\int_{\mathbb{R}^d}(1+|v|^s)G(t,v)\,\mathrm{d}v)^2\mathrm{d}t\\
&\le C_T|t_2-t_1|(1+\|G_0\|_s)^2,
\end{aligned}
$$

for some constant C_T depending on T, $|\nabla_v\varphi_n|$, and $|\nabla_v^2\varphi_n|$. By the density of our test functions in $C^0(\mathbb{R}^d_c)$, the continuity holds for all test functions $\varphi \in C^0(\mathbb{R}^d_c)$.

We now show the weak continuity of $S(t)$ on K with fixed $t \le T < \infty$. Motivated by [35], this will be done via the 1-Wasserstein distance on $\mathcal{M}$:

$$
\mathscr{W}_1(f,g) = \sup_{\mathrm{Lip}(\varphi)\le 1}\int_{\mathbb{R}^d}\varphi(v)(f(v)-g(v))\mathrm{d}v,
$$

where $\mathrm{Lip}(\varphi) = \sup_{x\neq y}\frac{|\varphi(x)-\varphi(y)|}{|x-y|}$ is the Lipschitz constant of φ. Let $f(t) = S(t)f_0$ *and* $g(t) = S(t)g_0$ be mild solutions with $f_0, g_0 \in \mathcal{M}_+$, $\int_{\mathbb{R}^d}(f_0(v), g_0(v))\,\mathrm{d}v = (1,1)$, finite s-moments, and $\varphi \in C(\mathbb{R}^d_c)$ with finite Lipschitz constant that satisfies $\mathrm{Lip}(\varphi)\le 1$. Then, for fixed $0\le t\le T<\infty$ and $\psi(t',v) = \varphi(e^{(t-t')A}v)$ on $t'\in[0,t]$ which satisfies $\partial_{t'}\psi + Av\cdot\nabla_v\psi = 0$, we first write

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\varphi(v)(f(t,v)-g(t,v))\,\mathrm{d}v - \int_{\mathbb{R}^d}\varphi(e^{tA}v)(f_0(v)-g_0(v))\,\mathrm{d}v\\
&\qquad\qquad = \int_{\mathbb{R}^d}\psi(t,v)(f(t,v)-g(t,v))\,\mathrm{d}v - \int_{\mathbb{R}^d}\psi(0,v)(f_0(v)-g_0(v))\,\mathrm{d}v,
\end{aligned}
$$

and it follows from the weak formulation (21) that

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\varphi(f(t,v)-g(t,v))\mathrm{d}v - \int_{\mathbb{R}^d}\varphi(e^{tA}v)(f_0(v)-g_0(v))\mathrm{d}v\\
&= \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)(f(\tau,v_*)-g(\tau,v_*))\\
&\qquad\qquad \times(\psi(\tau,v')+\psi(\tau,v_*')-\psi(\tau,v)-\psi(\tau,v_*))\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}\tau\\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)(f(\tau,v)-g(\tau,v))g(\tau,v_*)\\
&\qquad\qquad \times(\psi(\tau,v')+\psi(\tau,v_*')-\psi(\tau,v)-\psi(\tau,v_*))\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}\tau\\
&\quad - \int\psi(0,v)(f_0(v)-g_0(v))\,\mathrm{d}v.
\end{aligned}\tag{54}
$$

Since $\mathrm{Lip}(\varphi)\le 1$, on $\tau\in[0,T]$ we have

$$
\mathrm{Lip}(\psi(\tau)) = \mathrm{Lip}(\varphi(e^{(t-\tau)A}v)) \le \mathrm{Lip}(\varphi)\|e^{(t-\tau)A}\| \le \sup_{\tau\in[0,T]}\|e^{\tau A}\|
$$

which is a constant that depends only on T. Hence, one has

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\varphi(e^{tA}v)(f_0(v)-g_0(v))\,\mathrm{d}v\\
&\qquad\qquad = \sup_{\tau\in[0,T]}\|e^{\tau A}\|\int_{\mathbb{R}^d}\frac{\varphi(e^{tA}v)}{\sup_{[0,T]}\|e^{\tau' A}\|}(f_0(v)-g_0(v))\,\mathrm{d}v \le C\mathscr{W}_1(f_0,g_0).
\end{aligned}
$$

Notice that

$$
\mathrm{Lip}_{v_*}\Big(\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)\psi(\tau,v_*)\,\mathrm{d}\sigma\mathrm{d}v\Big) \le C_T\mathrm{Lip}_{v_*}(\psi(\tau,v_*)) \le C_T,\tag{55}
$$

$$\mathrm{Lip}_{v_*}\Big(-\int_{\mathbb{R}^d}\int_{\mathbb{S}^{-\nVdash}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)\psi(\tau,v)\,\mathrm{d}\sigma\mathrm{d}v\Big)=0. \tag{56}$$

To estimate $\mathrm{Lip}_{v_*}(\iint bf(\tau,v)\psi(\tau,v')\mathrm{d}\sigma\mathrm{d}v)$, let $\hat{e}\in\mathbb{S}^{d-1}$ be a fixed unit vector, and $R=R(v,v_*)$ be a rotation such that $R^{\mathsf{T}}\frac{v-v_*}{|v-v_*|}=\hat{e}$. Then, we use a change of variables to get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)\psi(\tau,v)\,\mathrm{d}\sigma\mathrm{d}v\\
=&\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot R\sigma)f(\tau,v)\\
&\qquad\times\psi\Big(\tau,\frac{v+v_*}{2}+\frac{1-z}{2}(v-v_*)+\frac{z}{2}|v-v_*|R\sigma\Big)\,\mathrm{d}\sigma\mathrm{d}v\\
=&\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\hat{e}\cdot\sigma)f(\tau,v)\psi(\tau,\tilde{v}')\,\mathrm{d}\sigma\mathrm{d}v,
\end{aligned}\tag{57}$$

with $\tilde{v}'(v,v_*,\sigma)=\frac{v+v_*}{2}+\frac{1-z}{2}(v-v_*)+\frac{z}{2}|v-v_*|R(v,v_*)\sigma$. Since

$$\|\nabla_{v_*}(|v-v_*|R(v,v_*)\sigma)\|\le\Big\|(R(v,v_*)\sigma)\otimes\frac{v-v_*}{|v-v_*|}\Big\|+|v-v_*|\Big\|\partial_{v_*}\frac{v-v_*}{|v-v_*|}\Big\|\le C,$$

uniformly on v_*, we have

$$\begin{aligned}
\mathrm{Lip}_{v_*}(\tilde{v}')&\le\frac{1}{2}+\frac{1-z}{2}+\frac{z}{2}\mathrm{Lip}_{v_*}(|v-v_*|R(v,v_*)\sigma)\\
&\le C+C\sup_{v_*}\|\nabla_{v_*}(|v-v_*|R(v,v_*)\sigma)\|\le C,
\end{aligned}$$

which, together with (57), implies

$$\begin{aligned}
&\mathrm{Lip}_{v_*}(\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)\psi(\tau,v')\,\mathrm{d}\sigma\mathrm{d}v)\\
\le&C_T\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\hat{e}\cdot\sigma)f(\tau,v)\mathrm{Lip}_{\tilde{v}'}(\psi(\tau,\tilde{v}'))\mathrm{Lip}_{v_*}(\tilde{v}')\,\mathrm{d}\sigma\mathrm{d}v\le C_T.
\end{aligned}\tag{58}$$

Similar arguments give that

$$\mathrm{Lip}_{v_*}(\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)\psi(\tau,v_*')\,\mathrm{d}\sigma\mathrm{d}v)\le C_T. \tag{59}$$

Combining (55), (56), (58), and (59), we have

$$\mathrm{Lip}_{v_*}(\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)(\psi'(\tau)+\psi_*'(\tau)-\psi(\tau)-\psi'(\tau))\,\mathrm{d}\sigma\mathrm{d}v)\le C_T,$$

which gives

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)f(\tau,v)(f(\tau,v_*)-g(\tau,v_*))\\
&\quad\times(\psi(\tau,v')+\psi(\tau,v_*')-\psi(\tau,v)-\psi(\tau,v_*))\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\le C_T\mathscr{W}_1(f(\tau),g(\tau)),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} b(\frac{v-v_*}{|v-v_*|}\cdot\sigma)(f(\tau,v)-g(\tau,v))g(\tau,v_*)\\
&\quad\times(\psi(\tau,v')+\psi(\tau,v_*')-\psi(\tau,v)-\psi(\tau,v_*))\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\le C_T\mathscr{W}_1(f(\tau),g(\tau)).
\end{aligned}$$

It holds from the above two inequalities and (54) that

$$\mathscr{W}_1(f(t),g(t)) \le C_T(\mathscr{W}_1(f_0,g_0) + \int_0^t \mathscr{W}_1(f(\tau),g(\tau))\,\mathrm{d}\tau),$$

which further yields by the Gronwall inequality that $\mathscr{W}_1(f(t),g(t)) \le C_T\mathscr{W}_1(f_0,g_0)$ with the constant C_T depending only on T. Hence, for each $t>0$, $S(t): K \to \mathcal{M}_+$ is weakly continuous.

Recalling that $K \subseteq \mathcal{M}_+$ is the set of measures G with $\int(1,v,v^{\otimes 2})G = (1,0,\Theta B)$ and $\int |v|^s G \le C_*$, we now show that $S(t)K \subseteq K$ if C_* is sufficiently large. By the properties of mild solutions and the selection of β, it suffices to show that $\int_{\mathbb{R}^d} |v|^s G(v)\mathrm{d}v \le C_*$ on $G(t) = S(t)G_0$ for $G_0 \in K$. Let $G_{n0} = G_0\chi_{|v|\le n}$ and $G_n(t) = S(t)G_{n0}$. As in Theorem 2.6, we have $\|G_n(t) - G(t)\|_2 \to 0$ when $n\to\infty$. By the selection of B and the fact that $\int_{\mathbb{R}^d}(1+|v|^2)G(t,v)\mathrm{d}v = 1+\Theta > 0$ is constant, then for sufficiently large n we have

$$\int_{\mathbb{R}^d}(1+|v|^2)G_n(t,v)\mathrm{d}v \le C\int_{\mathbb{R}^d}(1+|v|^2)G(t,v)\mathrm{d}v \le C_T$$

at the given time $t \le T < \infty$ with constant independent of n.

For $s>2$, [31, Lem. 3.3] gives the following Povzner inequality:

$$\int_{\mathbb{S}^{d-1}} b(\cos\theta)(|v'|^s + |v'_*|^s - |v|^s - |v_*|^s)\,\mathrm{d}\sigma = P - N, \tag{60}$$

with

$$P \le C(|v|^2|v_*|^{s-2} + |v_*|^2|v|^{s-2}), \tag{61}$$

$$N \ge c(|v|^2 + |v_*|^2)^{s/2}, \tag{62}$$

where the constants C and c depend only on s and b_0. It follows from (21), (60), (61), and (62) that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}|v|^s G_n(t,v)\,\mathrm{d}v\\
&= -s\int_{\mathbb{R}^d} v^{\mathsf{T}}A_\beta v|v|^{s-2}G_n(t,v)\,\mathrm{d}v\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_n(t,v)G_n(t,v_*)\int_{\mathbb{S}^{d-1}} b(\cos\theta)(|v'|^s + |v'_*|^s - |v|^s - |v_*|^s)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\\
&\le s(\|A\| - \beta)\int_{\mathbb{R}^d}|v|^s G_n(t,v)\,\mathrm{d}v\\
&\quad + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_n(t,v)G_n(t,v_*)(C(|v|^2|v_*|^{s-2} + |v_*|^2|v|^{s-2}) - c(|v|^s + |v_*|^s))\mathrm{d}v_*\mathrm{d}v\\
&= C(\int_{\mathbb{R}^d}|v|^2 G_n(t,v)\,\mathrm{d}v)(\int_{\mathbb{R}^d}|v|^{s-2}G_n(t,v)\,\mathrm{d}v) + (s\,\|A\| - s\beta - c)\int_{\mathbb{R}^d}|v|^s G_n(t,v)\,\mathrm{d}v.
\end{aligned} \tag{63}$$

The first term on the right-hand side above is controlled by

$$\begin{aligned}
&(\int_{\mathbb{R}^d}|v|^2 G_n(v)\,\mathrm{d}v)(\int_{\mathbb{R}^d}|v|^{s-2}G_n(v)\,\mathrm{d}v)\\
&\le (\int_{\mathbb{R}^d}(1+|v|^2)G_n(v)\,\mathrm{d}v)^2 \le C_T(\int_{\mathbb{R}^d}(1+|v|^2)G_{n0}(v)\,\mathrm{d}v)^2\\
&\le C_T(\int_{\mathbb{R}^d}(1+|v|^2)G_0(v)\,\mathrm{d}v)^2.
\end{aligned} \tag{64}$$

Recalling $|\tilde{\beta}| = |\beta + \zeta/2| \le C\|A\|$ and ζ defined in (7), we choose z, A such that $1-z$ and $\|A\|$ are sufficiently small with

$$\|A\| - \beta \le \|A\| + |\tilde{\beta}| + \zeta/2 \le C(\|A\| + (1-z)) < c/s. \tag{65}$$

Collecting (63), (64), and (65), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |v|^s G_n(t,v)\,\mathrm{d}v \le C_T\Big(\int_{\mathbb{R}^d}(1+|v|^2)G_0\,\mathrm{d}v\Big)^2 - \delta\int_{\mathbb{R}^d}|v|^s G_n(t,v)\,\mathrm{d}v$$

for some $C_T, \delta > 0$, and thus, by the Gronwall inequality, $\int_{\mathbb{R}^d}|v|^s G_n(t,v) \le C_*$ if $\int_{\mathbb{R}^d}|v|^s G_0(t) \le C_*$ with $C_* > 0$ sufficiently large. Passing to the limit, we have $\int_{\mathbb{R}^d}|v|^s G(t,v) \le C_*$ on each finite $t \in [0,T]$. This implies that for each $t > 0$, $S(t)K \subseteq K$.

By the Schauder fixed-point theorem, for each $0 < t < T < \infty$, there exists a fixed-point $G_t \in K$ such that $S(t)G_t = G_t$. By the semigroup property of $S(t)$, $S(nt)G_t = G_t$ for all $n \in \mathbb{Z}^+$. Let $t_n > 0$ be a sequence such that $t_n \to 0$ when $n \to \infty$. Since K is closed and compact, up to a subsequence, we have that $G_{t_n} \to G$ when $n \to \infty$ for some $G \in K$.

Let $t > 0$. Then, there exists $n_k \in \mathbb{Z}^+$ such that $n_k t_k \to t$ when $k \to \infty$. Therefore, by the convergence of G_{t_k} and continuity of S in time, it holds that

$$G_{t_k} = S(n_k t_k)G_{t_k} = (S(n_k t_k) - S(t))G_{t_k} + S(t)G_{t_k} \to 0 + S(t)G, \quad k \to \infty,$$

which yields $S(t)G = G$ for all $t > 0$. Hence, the proof of Theorem 2.8 is finished. ☐

The above theorem gives us the existence of the stationary Radon profile noted in Theorem 1.1. Also, note that by construction, β is chosen such that the temperature of the stationary profile is a finite positive number. For the uniform shear flow matrix, the discussion in Theorem 2.7 implies the sign relation in Theorem 1.2.

From the proof, we can also see that the profile has finite moment of order $s \in (2,4]$, given that A and $1-z$ are sufficiently small.

We remark that the smallness assumption posed on $\|A\|$ and $1-z$ are needed in only two places:

- the existence of stationary solution to the second moment equation (43),
- the control $\|A\| - \beta < c/s$ for the Gronwall inequality in (65).

2.3.1. *Sufficient condition to self-similar solutions for uniform shear flow.* In [35], a variant of Povzner inequality is proposed to tackle the existence problem of a self-similar profile for elastic USF. The same variant also works for inelastic collision and a more generic class of shearing matrix, assuming that additional conditions on the kernel and the parameters are satisfied.

Lemma 2.9. *Let $\beta \in \mathbb{R}$ be chosen as in Theorem 2.8. Suppose the kernel b is continuous in the sense that*

$$(v,w) \mapsto \int_{\mathbb{S}^{d-1}} b\big(\sigma\cdot\frac{v-w}{|v-w|}\big)g(v,w,\sigma)\,\mathrm{d}\sigma$$

is continuous for all continuous functions $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1} \to \mathbb{R}$, (44) holds for A^T, and the self-similar parameter β has a symmetric positive definite stationary solution $\tilde{B} \in \mathbb{R}^{d\times d}$. If

$$\max_{|v|=1} \mathcal{W}(v;W) + \mathcal{H}(v) < 0, \tag{66}$$

for some matrix $W \in \mathbb{R}^{d\times d}$, where

$$\mathcal{W}(v;W) := \int_{\mathbb{S}^{d-1}} b\big(\sigma \cdot \frac{v}{|v|}\big)(W^+ + W^- - W)\,\mathrm{d}\sigma - A_\beta v \cdot \nabla_v W,$$

and

$$\mathcal{H}(v) := \int_{\mathbb{S}^{d-1}} b\big(\sigma \cdot \frac{v}{|v|}\big)(\tilde{B}^+ \ln \tilde{B}^+ + \tilde{B}^- \ln \tilde{B}^- - \tilde{B}\ln \tilde{B})\,\mathrm{d}\sigma - (1+\ln \tilde{B})A_\beta v \cdot \nabla_v \tilde{B},$$

with matrices acting as the induced quadratic form $W(v) = v^\mathsf{T} W v$ and

$$v^- = \frac{z}{2}(v - |v|\sigma), \quad v^+ = v - v^-,$$

then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, $\Phi(v) = (\tilde{B}(v) + \epsilon W(v))^{1+\epsilon}$ is homogeneous of degree $s = 2(1+\epsilon) > 2$ and satisfies

$$\begin{aligned} U\Phi(v,v_*) &:= \int_{\mathbb{S}^{d-1}} b\big(\sigma \cdot \frac{v - v_*}{|v - v_*|}\big)(\Phi(v') + \Phi(v'_*) - \Phi(v) - \Phi(v_*))\,\mathrm{d}\sigma - A_\beta v \cdot \nabla_v \Phi \\ &\le C|v|^{s/2}|v_*|^{s/2} - \kappa\Phi, \end{aligned}$$

for some $C, \kappa > 0$, $|v_*| \le |v|$, and $\Phi(v) \le C|v|^s$.

Note that we can always choose ϵ sufficiently small such that $s \in (2,3)$.

Proof. Since $\tilde{B}$ is positive definite, on sufficiently small $\epsilon > 0$ we have

$$\min_{|v|=1} \tilde{B}(v) + \epsilon W(v) > 0,$$

and so $\min_{|v|=1} \Phi(v) > 0$. Using a Taylor expansion on $\Phi(v) = (\tilde{B}(v) + \epsilon W(v))^{1+\epsilon}$ gives $\Phi = \tilde{B} + \epsilon(\tilde{B}\ln\tilde{B} + W) + O(\epsilon^2)$ and $\nabla_v \Phi = \nabla_v \tilde{B} + \epsilon((1+\ln\tilde{B})\nabla_v\tilde{B} + \nabla_v W) + O(\epsilon^2)$. Then, with $|v| = 1$, it holds that

$$\begin{aligned} &V\Phi := U\Phi(v,0) \\ &= \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \frac{v}{|v|})(\Phi^+ + \Phi^- - \Phi)\,\mathrm{d}\sigma - A_\beta v \cdot \nabla_v \Phi \\ &= \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \frac{v}{|v|})(\tilde{B}^+ + \tilde{B}^- - \tilde{B})\,\mathrm{d}\sigma - A_\beta v \cdot \nabla_v \tilde{B} \\ &\quad + \epsilon\bigg(\int_{\mathbb{S}^{d-1}} b(\sigma \cdot \frac{v}{|v|})(\tilde{B}^+ \ln \tilde{B}^+ + \tilde{B}^- \ln \tilde{B}^- - \tilde{B}\ln\tilde{B})\,\mathrm{d}\sigma - (1+\ln\tilde{B})A_\beta v \cdot \nabla_v \tilde{B} \\ &\qquad + \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \frac{v}{|v|})(W^+ + W^- - W)\,\mathrm{d}\sigma - A_\beta v \cdot \nabla_v W\bigg) + O(\epsilon^2) \\ &= \mathcal{W}(v;\tilde{B}) + \epsilon(\mathcal{H}(v) + \mathcal{W}(v;W)) + O(\epsilon^2). \end{aligned}$$

Since $\tilde{B}$ is a stationary solution to (18) on shear A^T,

$$\begin{aligned} &\mathcal{W}(v;\tilde{B}) \\ &= \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \frac{v}{|v|})(\tilde{B}^+ + \tilde{B}^- - \tilde{B})\,\mathrm{d}\sigma - A_\beta v \cdot \nabla_v \tilde{B} \\ &= \tilde{B} : \int_{\mathbb{S}^{d-1}} b(\sigma \cdot \frac{v}{|v|})((v^+)^{\otimes 2} + (v^-)^{\otimes 2} - v^{\otimes 2})\,\mathrm{d}\sigma - A_\beta v \cdot \nabla_v \tilde{B} \\ &= \tilde{B} : \frac{1}{2}\int_{\mathbb{S}^{d-1}} b(\sigma \cdot \frac{v}{|v|})(z(z-2)v^{\otimes 2} + z^2|v|^2\sigma^{\otimes 2} + z(1-z)|v|(v\otimes\sigma + \sigma\otimes v))\,\mathrm{d}\sigma \\ &\quad - A_\beta v \cdot (\tilde{B} + \tilde{B}^\mathsf{T})v \end{aligned}$$

$$= \tilde{B} : \frac{1}{2}(z^2 dc_{11}(\frac{|v|^2}{d} I - v^{\otimes 2}) + 2\zeta v^{\otimes 2}) - (A_\beta^\mathsf{T} \tilde{B} + \tilde{B} A_\beta) : v^{\otimes 2}$$
$$= -v^{\otimes 2} : (A^\mathsf{T} \tilde{B} + \tilde{B} A + (2\beta + \zeta)\tilde{B} + z^2 \frac{dc_{11}}{2}(\tilde{B} - \frac{\mathrm{tr}(\tilde{B})}{d} I)) = 0$$

with $X : Y = \sum_{ij} X_{ij} Y_{ij}$. Furthermore, by assumption (66), taking $0 < \epsilon \ll 1$ sufficiently small, we have that $\max_{|v|=1} V\Phi < 0$.

By the continuity of the kernel and Φ, $U\Phi$ is continuous in v_*, so on $|v_*| \le \delta \ll 1$, we have $\max_{|v|=1} U\Phi < 0$ as well. Since $\max_{|v|=1} \Phi > 0$, we also have $U\Phi \le -\kappa\Phi$ for some $\kappa > 0$ sufficiently small.

On $\delta \le |v_*| \le |v| = 1$, as $|v'|, |v'_*| \le |v| + |v_*| \le 2$, $|\Phi(v') + \Phi(v'_*) - \Phi(v) - \Phi(v_*)| \le C$. Also, $|\nabla_v \Phi| \le (\tilde{B} + \epsilon W)^\epsilon (|\nabla_v \tilde{B}| + \epsilon |\nabla_v W|) \le C$ on $|v| = 1$. These imply that

$$U\Phi \le \int_{\mathbb{S}^{d-1}} b\big(\sigma \cdot \frac{v - v_*}{|v - v_*|}\big) |\Phi(v') + \Phi(v'_*) - \Phi(v) - \Phi(v_*)| \,\mathrm{d}\sigma + \|A_\beta\| |v| |\nabla_v \Phi| \le C\delta^{s/2}.$$

Combining the two cases and using the homogeneity of Φ, we have that $U\Phi \le C|v|^{s/2}|v_*|^{s/2} - \kappa\Phi$ for some $C, \kappa > 0$ on $|v_*| \le |v|$. Hence, $\Phi \le C|v|^s$ follows directly from the homogeneity of Φ. □

There are few results on the existence of matrices satisfying (66). However, we note that, assuming Lemma 2.9 holds, such $\tilde{B}$ exists, and then

- $\mathcal{W}(v; W)$ is linear in W
- as $|v^\pm| \le C|v|$, $\tilde{B}, \tilde{B}^\pm \le C$. Since $x \mapsto x \ln x$ is locally bounded, the integrand in $\mathcal{H}$ is bounded. Furthermore, $\tilde{B}$ is positive definite, so $\min_{|v|=1} \ln \tilde{B}(v) > 0$. This implies that $\max_{|v|=1} \mathcal{H} < \infty$.

Hence, it would suffice to find W such that $\max_{|v|=1} \mathcal{W}(v; W) < 0$, with $\lambda > 0$ being sufficiently large,

$$\max_{|v|=1} \mathcal{H}(v) + \mathcal{W}(v; \lambda W) \le \max_{|v|=1} \mathcal{H}(v) + \lambda \max_{|v|=1} \mathcal{W}(v; W) < 0,$$

and thus (66) holds. As seen in the proof of Theorem 2.9, one has

$$\mathcal{W}(v; W) = -v^{\otimes 2} : \big(A^\mathsf{T} W + WA + (2\beta + \zeta)W + z^2 \frac{dc_{11}}{2}(W - \frac{\mathrm{tr}(W)}{d} I)\big).$$

Then, $\max_{|v|=1} \mathcal{W}(v; W) < 0$ is equivalent to

$$A^\mathsf{T} W + WA + (2\beta + \zeta)W + z^2 \frac{dc_{11}}{2}(W - \frac{\mathrm{tr}(W)}{d} I) \succ 0, \tag{67}$$

with $\succ$ defined in the sense that $A \succ 0$ if and only if $v^\mathsf{T} A v > 0$ for all $v \in \mathbb{R}^d$.

So, to use Theorem 2.9, it suffices to show that such W exists. For consistency with notations in (48), on $\tilde{\beta} = \beta + \zeta/2$ and $\tilde{c} = z^2 \frac{dc_{11}}{2}$, we rewrite (67) as

$$R(W) = A_{\tilde{\beta}}^\mathsf{T} W + W A_{\tilde{\beta}} + \tilde{c}(W - \frac{\mathrm{tr}(W)}{d} I) \succ 0. \tag{68}$$

In what follows, we would point out that, while the approach above may yield desired results on some shearing matrices, it fails in case of 2D USF.

Theorem 2.10. *If $d = 2$ and (48) holds, then no $W \in \mathbb{R}^{2\times 2}$ satisfies (68) for uniform shear flow where $A = \alpha E_{12}$.*

Proof. On $x = (\cos\theta, \sin\theta)^\mathsf{T} \in S^1$ where S^1 denotes the one-dimensional sphere,

$$x^\mathsf{T} R x = C + C_1 \cos(2\theta) + C_2 \sin(2\theta)$$

has minimum value $\min_{x\in S^1} x^\mathsf{T} Rx = C - \sqrt{C_1^2 + C_2^2}$ with

$$C = (\tilde\beta, \frac{\alpha}{2}, \frac{\alpha}{2}, \tilde\beta) \cdot \vec W,$$
$$C_1 = (\tilde\beta + \frac{\tilde c}{2}, -\frac{\alpha}{2}, -\frac{\alpha}{2}, \tilde\beta + \frac{\tilde c}{2}) \cdot \vec W,$$
$$C_2 = (\alpha, \tilde\beta + \frac{\tilde c}{2}, \tilde\beta + \frac{\tilde c}{2}, 0) \cdot \vec W,$$

and $\vec W = (W_{11}, W_{12}, W_{21}, W_{22})^\mathsf{T}$. To have $\min_{x\in S^1} x^\mathsf{T} Rx > 0$, it is equivalent to require

$$C > 0,$$
$$C^2 - C_1^2 - C_2^2 = \vec W^\mathsf{T} S \vec W > 0,$$

to hold for some $\vec W$ with

$$S = \begin{pmatrix} -\alpha^2 - \frac14 \tilde c(4\tilde\beta + \tilde c) & -\frac14\alpha\tilde c & -\frac14\alpha\tilde c & \frac14\tilde c(4\tilde\beta+\tilde c) \\ -\frac14\alpha\tilde c & -\frac14(2\tilde\beta+\tilde c)^2 & -\frac14(2\tilde\beta+\tilde c)^2 & -\frac14\alpha\tilde c \\ -\frac14\alpha\tilde c & -\frac14(2\tilde\beta+\tilde c)^2 & -\frac14(2\tilde\beta+\tilde c)^2 & -\frac14\alpha\tilde c \\ 2\tilde\beta^2 + \frac14\tilde c(4\tilde\beta+\tilde c) & -\frac14\alpha\tilde c & -\frac14\alpha\tilde c & -\frac14\tilde c(4\tilde\beta+\tilde c) \end{pmatrix}$$

and such $\vec W$ exists if and only if $-S$ is not positive semidefinite. However, we can enumerate all 15 principal minors of $-S$ and note that

$$M_1 = \alpha^2 + \frac14\tilde c(4\tilde\beta+\tilde c), \quad M_2 = M_3 = \frac14(2\tilde\beta+\tilde c)^2, \quad M_4 = \frac14\tilde c(4\tilde\beta+\tilde c),$$
$$M_{12} = M_{13} = \frac{1}{16\tilde c}(4\tilde\beta+\tilde c)^2(2\tilde\beta+\tilde c)^3, \quad M_{14} = \frac12\tilde\beta(2\tilde\beta+\tilde c)^3,$$
$$M_{23} = 0, \quad M_{24} = M_{34} = \frac{1}{16}\tilde c(2\tilde\beta+\tilde c)^3,$$
$$M_{123} = M_{124} = M_{134} = M_{234} = 0$$

are all nonnegative with $\det(-S) = 0$. Then, $-S$ is positive semidefinite, and no such $\vec W$ exists. $\square$

This implies that in order to have (66) on 2D USF, it is necessary to study the behavior of the more complicated part $\mathcal{H}$ of the condition. If $\mathcal{H}(v) \geq 0$ on $|v| = 1$, (66) fails to hold.

In the current section, following the same strategy as in [35], we have constructed the global-in-time solutions as nonnegative measures for the Cauchy problem (1) in Theorem 2.5, and further obtained the propagation of moment bounds in Theorem 2.6. Moreover, we have provided a characterization of the long-time behavior of temperature in the case of simple uniform shear flow in Theorem 2.7, and established the existence of stationary self-similar Radon profile in Theorem 2.8. In the next section, we will follow the approach in [7] for further obtaining the long-time asymptotics toward the self-similar profile for global-in-time solutions of (1).

3. **Fourier solution.** In this part we consider the solutions in the Fourier space as characteristic functions. The natural solution space is the space $\mathcal{K}$ of characteristic functions, the Fourier transform of probability measures on $\mathbb{R}^d$. For two characteristic functions $\varphi, \psi \in \mathcal{K}$, their p-Toscani distance for $p > 0$ is

$$\|\varphi - \psi\|_p = \sup_{k\neq 0} \frac{|\varphi(k) - \psi(k)|}{|k|^p}. \tag{69}$$

The Cannone–Karch space $\mathcal{K}^p \subseteq \mathcal{K}$ with $p \in (0,2]$, the space of characteristic functions with finite p-Toscani distance with 1, that is $\|\varphi - 1\|_p < \infty$, contains all characteristic functions of probability measures with finite absolute p-moment and (if $p > 1$) zero mean [16].

Similar to the elastic case in [7], we have the following lemma for properties of the operator $\widehat{Q_e^+}$.

Lemma 3.1. *Let $\widehat{Q_e^+}$ be defined in (16). Then, $b_0^{-1}\widehat{Q_e^+}(\mathcal{K},\mathcal{K}) \subseteq \mathcal{K}$, and the $\mathcal{L}_e$-Lipschitz property of $\widehat{Q_e^+}$ holds:*

$$|\widehat{Q_e^+}(\varphi,\varphi)(k) - \widehat{Q_e^+}(\psi,\psi)(k)| \leq \mathcal{L}_e|\varphi(k) - \psi(k)| \leq 2b_0\|\varphi - \psi\|_{L^\infty}, \tag{70}$$

where $\mathcal{L}_e$ is defined in (17).

Proof. The proof is very similar to [7, Lem. 3.1], see also [10]. For brevity, we only prove the $\mathcal{L}_e$-Lipschitz property as follows:

$$\begin{aligned}
&|\widehat{Q_e^+}(\varphi,\varphi) - \widehat{Q_e^+}(\psi,\psi)| \\
&\leq \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)|\varphi(k^+)\varphi(k^-) - \psi(k^+)\psi(k^-)|\,\mathrm{d}\sigma \\
&\leq \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)|\varphi(k^+)||\varphi(k^-) - \psi(k^-)|\,\mathrm{d}\sigma \\
&\quad + \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)|\varphi(k^+) - \psi(k^+)||\varphi(k^-)|\,\mathrm{d}\sigma \\
&\leq \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)|\varphi(k^-) - \psi(k^-)|\,\mathrm{d}\sigma + \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)|\varphi(k^+) - \psi(k^+)|\,\mathrm{d}\sigma \\
&= \mathcal{L}_e|\varphi - \psi| \leq 2b_0\|\varphi - \psi\|_{L^\infty},
\end{aligned}$$

for $\varphi, \psi \in \mathcal{K}$. In the third inequality above, we use the fact that $|\phi(k)| \leq 1$ for any k. □

Then, we integrate equation (15) to get the mild form of the solution defined as follows.

Definition 3.2. A function $\varphi(t,k) \in C([0,\infty),\mathcal{K})$ is a mild solution of (15) on the initial condition $\varphi_0(k) = \varphi(0,k) \in \mathcal{K}$ if

$$\varphi(t,k) = e^{-b_0 t}\varphi_0(e^{-tA^{\mathsf{T}}}k) + \int_0^t e^{-b_0(t-\tau)}\widehat{Q_e^+}(\varphi,\varphi)(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau. \tag{71}$$

We can prove the existence of the Cauchy problem solution of (15) in terms of the mild solution of (15) in the frequency space.

Theorem 3.3. *If $\varphi_0(k) \in \mathcal{K}$, then there exists a unique mild solution $\varphi(t,k) \in C([0,\infty),\mathcal{K})$ globally in time and satisfying (71) with initial condition $\varphi(t,k) = \varphi_0(k)$.*

Proof. Consider the Picard map $P : \mathcal{X}_T \to \mathcal{X}_T$ on $\mathcal{X}_T := C([0,T],\mathcal{K})$ equipped with metric $d(\varphi,\psi) = \sup_{[0,T]}\|\varphi(t) - \psi(t)\|_{L^\infty}$ defined by

$$P\varphi(t,k) = e^{-b_0 t}\varphi_0(e^{-tA^{\mathsf{T}}}k) + \int_0^t e^{-b_0(t-\tau)}\widehat{Q_e^+}(\varphi,\varphi)(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau.$$

We first show that P maps $\mathcal{X}_T$ into itself. For $\varphi_0, \varphi(\tau) \in \mathcal{K}$, it is straight-forward to see that $\varphi_0(e^{-tA^\intercal}k)$ and $\widehat{Q_e^+}(\varphi, \varphi)(\tau, e^{-(t-\tau)A^\intercal}k)$ are characteristic functions. $P\varphi(t)$, which is a convex combination of characteristic functions, is also a characteristic function for each $t \in [0, T]$. Furthermore, since $\mathcal{K}$ is a subset of all continuous functions, and $t \mapsto e^{-b_0 t}\varphi_0(e^{-tA^\intercal}k)$ and $(t, \tau) \mapsto e^{-b_0(t-\tau)}\widehat{Q_e^+}\varphi(\tau, e^{-(t-\tau)A^\intercal}k)$ are continuous, then $t \mapsto P\varphi(t)$ is continuous. These imply that P is well-defined.

For $\varphi, \psi \in \mathcal{X}_T$ and $t \in [0, T]$, it holds that

$$\begin{aligned}
|P\varphi(t,k) - P\psi(t,k)| &\leq \int_0^t e^{-b_0(t-\tau)}|\widehat{Q_e^+}(\varphi, \varphi)(\tau) - \widehat{Q_e^+}(\psi, \psi)(\tau)|(e^{-(t-\tau)A^\intercal}k)\,\mathrm{d}\tau \\
&\leq \int_0^t 2b_0\|\varphi(\tau) - \psi(\tau)\|_{L^\infty}\,\mathrm{d}\tau \\
&\leq 2b_0 T\, d(\varphi, \psi).
\end{aligned}$$

We choose $T = \frac{1}{4b_0}$, which is independent of φ_0. Hence, P is a Banach contraction, and there exists $\varphi \in \mathcal{X}_T$ such that $P\varphi = \varphi$. By a standard extension argument, there exists $\varphi \in C([0, \infty), \mathcal{K})$ such that $P\varphi = \varphi$. $\square$

We show that the distance of two mild solutions φ, ψ for the elastic equation on initial conditions $\varphi_0, \psi_0 \in \mathcal{K}$ is controlled by a corresponding solution on the linearized equation (15). The proof is similar to that in [7, Thm. 4.8], noting that the key points are using the bound $|\widehat{Q_e^+}(\varphi, \varphi) - \widehat{Q_e^+}(\psi, \psi)| \leq \mathcal{L}_e|\varphi - \psi|$ and the comparison principle.

Theorem 3.4. *Let $\varphi(t,k)$ and $\psi(t,k)$ be two mild solutions of (15) with initial conditions $\varphi_0(k), \psi_0(k) \in \mathcal{K}$. Then, if $y(t,k)$ satisfies*

$$y(t,k) = e^{-b_0 t}y_0(e^{-tA^\intercal}k) + \int_0^t e^{-b_0(t-\tau)}\mathcal{L}_e y(\tau, e^{-(t-\tau)A^\intercal}k)\,\mathrm{d}\tau, \tag{72}$$

which is the mild solution of the linearized equation

$$\partial_t y + A^\intercal k \cdot \nabla_k y = \mathcal{L}_e y - b_0 y,$$

with initial condition $y_0(k) = |\varphi_0(k) - \psi_0(k)|$, then it holds that

$$|\varphi(t,k) - \psi(t,k)| \leq y(t,k). \tag{73}$$

Proof. We have from (71) and (70) that for $z(t,k) = |\varphi(t,k) - \psi(t,k)|$,

$$\begin{aligned}
z(t,k) &= |\varphi(t,k) - \psi(t,k)| \\
&\leq e^{-b_0 t}|\varphi_0(e^{-tA^\intercal}k) - \psi_0(e^{-tA^\intercal}k)| \\
&\quad + \int_0^t e^{-b_0(t-\tau)}|\widehat{Q_e^+}(\varphi, \varphi) - \widehat{Q_e^+}(\psi, \psi)(\tau, e^{-(t-\tau)A^\intercal}k)|\,\mathrm{d}\tau \\
&\leq e^{-b_0 t}y_0(e^{-tA^\intercal}k) + \int_0^t e^{-b_0(t-\tau)}\mathcal{L}_e|\varphi - \psi|(\tau, e^{-(t-\tau)A^\intercal}k)\,\mathrm{d}\tau \\
&= e^{-b_0 t}y_0(e^{-tA^\intercal}k) + \int_0^t e^{-b_0(t-\tau)}\mathcal{L}_e z(\tau, e^{-(t-\tau)A^\intercal}k)\,\mathrm{d}\tau.
\end{aligned}$$

Therefore, (73) holds by comparison principle. $\square$

In order to control the growth of the p-Toscani distance in time, we need to define the kernel constant

$$\begin{aligned}\lambda_p &= \int_{\mathbb{S}^{d-1}} b\big(\sigma \cdot \frac{k}{|k|}\big)\big(1 - \frac{|k^+|^p + |k^-|^p}{|k|^p}\big)\,\mathrm{d}\sigma \\ &= \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)\Big(1 - (z^2 \sin^2\frac{\vartheta}{2})^{p/2} - (1 - z(2-z)\sin^2\frac{\vartheta}{2})^{p/2}\Big)\,\mathrm{d}\sigma, \end{aligned} \tag{74}$$

for $p \geq 0$ as in [12, 7]. Note that $\lambda_0 = -b_0 < 0$, $\lambda_2 = \zeta \geq 0$, and $\lambda_p \leq b_0$ for all p. Furthermore, a direct calculation shows that

$$\begin{aligned}\frac{\partial \lambda_p}{\partial p} = \int_{\mathbb{S}^{d-1}} & b(\cos\vartheta)(-(z^2\sin^2\frac{\vartheta}{2})^{p/2}\ln(z^2\sin^2\frac{\vartheta}{2}) \\ & - (1 - z(2-z)\sin^2\frac{\vartheta}{2})^{p/2}\ln(1 - z(2-z)\sin^2\frac{\vartheta}{2}))\,\mathrm{d}\sigma > 0 \end{aligned} \tag{75}$$

by the facts that $z^2\sin^2\frac{\vartheta}{2} \leq 1$ and $1 - z(2-z)\sin^2\frac{\vartheta}{2} \leq 1$, which implies that λ_p is strictly increasing in p. This gives that the unique root $p_0 > 0$ that $\lambda_{p_0} = 0$ is strictly smaller than 2 in the case of inelastic collision $z < 1$, and $\lambda_p > 0$ with $p > 2$ for all $z \in (1/2, 1]$.

Now we can justify the existence and show some properties of such $y(t)$ for appropriate an initial condition y_0 with the framework developed in [7] in the elastic case.

Theorem 3.5. *Let $\mathcal{C}_p = \big\{f \in C(\mathbb{R}^d, \mathbb{R}) |\ \|f\|_p < \infty\big\}$ equipped with p-Toscani norm for some $p > 0$. Then, it holds that:*

- *For every $y_0(k) \in \mathcal{C}_p$, there exists a unique $y(t,k)$ to (72) denoted by*
$$y(t,k) = \exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))y_0 \in C([0,\infty), \mathcal{C}_p).$$
- *If $y_0(k) \in \mathcal{C}_p$ and $u(t,k) \in C([0,\infty), \mathcal{C}_p)$ are both nonnegative, and*
$$u(t,k) \geq e^{-b_0 t} y_0(e^{-tA^{\mathsf{T}}}k) + \int_0^t e^{-b_0(t-\tau)}\mathcal{L}_e u(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau, \tag{76}$$
then one has
$$u(t,k) \geq y(t,k) = \exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))y_0(k). \tag{77}$$
- *If $u_0 = u_0(k), v_0 = v_0(k) \in \mathcal{C}_p$ and $0 \leq u_0 \leq v_0$, then it holds that*
$$0 \leq \exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))u_0 \leq \exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))v_0. \tag{78}$$

Proof. First, consider the Picard iteration

$$\begin{aligned} & z_0(t,k) = 0, \\ & z_{n+1}(t,k) = e^{-b_0 t} y_0(e^{-tA^{\mathsf{T}}}k) + \int_0^t e^{-b_0(t-\tau)}\mathcal{L}_e z_n(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau, \quad n = 0, 1, \cdots, \end{aligned}$$

with $t \in [0,T]$ for some $T > 0$ to be determined later. Then, one gets

$$\begin{aligned}\frac{|z_{n+1}(t,k)|}{|k|^p} &\leq e^{-b_0 t}\|e^{-tA^{\mathsf{T}}}\|^p\|y_0\|_p + \int_0^t e^{-b_0(t-\tau)}\|e^{-(t-\tau)A^{\mathsf{T}}}\|^p \frac{|\mathcal{L}_e z_n(\tau,k)|}{|k|^p}\,\mathrm{d}\tau \\ &\leq \|y_0\|_p \sup_{[0,T]}\|e^{-\tau A^{\mathsf{T}}}\|^p \\ &\quad + \sup_{[0,T]}\|e^{-\tau A^{\mathsf{T}}}\|^p \int_0^t e^{-b_0(t-\tau)}\|z_n(\tau)\|_p \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)\frac{|k^+|^p + |k^-|^p}{|k|^p}\,\mathrm{d}\sigma\mathrm{d}\tau \end{aligned}$$

$$\leq \|y_0\|_p \sup_{[0,T]} \|e^{-\tau A^\intercal}\|^p + T \sup_{[0,T]} \|e^{-\tau A^\intercal}\|^p (b_0 - \lambda_p) \sup_{[0,T]} \|z_n(\tau)\|_p,$$

which yields

$$\sup_{[0,T]} \|z_{n+1}(\tau)\|_p \leq \|y_0\|_p \sup_{[0,T]} \|e^{-\tau A^\intercal}\|^p + (b_0 - \lambda_p) T \sup_{[0,T]} \|e^{-\tau A^\intercal}\|^p \sup_{[0,T]} \|z_n(\tau)\|_p.$$

By induction, $z_n \in C([0,T], \mathcal{C}_p)$ for each $n > 0$ and $T > 0$.

Furthermore, direct calculations show that

$$\sup_{[0,T]} \|z_1(t) - z_0(t)\|_p = \sup_{[0,T]} \|z_1(t)\|_p < \infty,$$

and

$$\begin{aligned}
&\|z_{n+1}(t) - z_n(t)\|_p \\
&\leq \int_0^t e^{-b_0(t-\tau)} \|(\mathcal{L}_e z_n - \mathcal{L}_e z_{n-1})(\tau, e^{-(t-\tau)A^\intercal} k)\|_p \,\mathrm{d}\tau \\
&\leq \int_0^t e^{-b_0(t-\tau)} \|e^{-(t-\tau)A^\intercal}\|^p \cdot \|\mathcal{L}_e(z_n(\tau) - z_{n-1}(\tau))\|_p \,\mathrm{d}\tau \\
&\leq \sup_{[0,T]} \|e^{-\tau A^\intercal}\|^p \int_0^t e^{-b_0(t-\tau)} \int_{\mathbb{S}^{d-1}} b(\cos\vartheta) \|z_n(\tau) - z_{n-1}(\tau)\|_p \frac{|k^+|^p + |k^-|^p}{|k|^p} \,\mathrm{d}\sigma \mathrm{d}\tau \\
&\leq (b_0 - \lambda_p) T e^{pT\|A\|} \sup_{[0,T]} \|z_n(\tau) - z_{n-1}(\tau)\|_p.
\end{aligned}$$

We choose $T > 0$ small enough such that $(b_0 - \lambda_p) T e^{pT\|A\|} < 1$. This implies that $z_n \to z$ as $\to \infty$ in $\sup_{[0,T]} \|\cdot\|_p$-norm for some $z \in C([0,T], \mathcal{C}_p)$ satisfying (72). As the selection of T is independent from y_0, by a standard extension argument, we obtain that there exists some $y \in C([0,\infty), \mathcal{C}_p)$ that solves (72).

To prove the uniqueness of the solution obtained in Theorem 3.5, for two solutions $\varphi_1(t,k), \varphi_2(t,k)$ of (72) with the same initial condition, we have

$$\begin{aligned}
\frac{|\varphi_1(t,k) - \varphi_2(t,k)|}{|k|^p} &\leq \int_0^t e^{-b_0(t-\tau)} |k|^{-p} |\mathcal{L}_e(\varphi_1 - \varphi_2)(\tau, e^{-(t-\tau)A^\intercal} k)| \,\mathrm{d}\tau \\
&\leq (b_0 - \lambda_p) \int_0^t e^{-b_0(t-\tau)} \|e^{-(t-\tau)A^\intercal}\|^p \|\varphi_1(\tau) - \varphi_2(\tau)\|_p \,\mathrm{d}\tau \\
&\leq (b_0 - \lambda_p) \int_0^t e^{-b_0(t-\tau)} e^{(t-\tau)p\|A\|} \|\varphi_1(\tau) - \varphi_2(\tau)\|_p \,\mathrm{d}\tau,
\end{aligned}$$

which gives

$$e^{b_0 t} e^{-tp\|A\|} \|\varphi_1(t) - \varphi_2(t)\|_p \leq (b_0 - \lambda_p) \int_0^t e^{b_0 \tau} e^{-\tau p\|A\|} \|\varphi_1(\tau) - \varphi_2(\tau)\|_p \,\mathrm{d}\tau.$$

Then, it follows from the Gronwall inequality that $\|\varphi_1(t) - \varphi_2(t)\|_p = 0$ for all t.

To show the second part of the proposition, we note that $z_0(t,k) = 0 \leq u(t,k)$ and

$$\begin{aligned}
z_{n+1}(t,k) &= e^{-b_0 t} y_0(e^{-tA^\intercal} k) + \int_0^t e^{-b_0(t-\tau)} \mathcal{L}_e z_n(\tau, e^{-(t-\tau)A^\intercal} k) \,\mathrm{d}\tau \\
&\leq u(t,k) - \int_0^t e^{-b_0(t-\tau)} \mathcal{L}_e(u - z_n)(\tau, e^{-(t-\tau)A^\intercal} k) \,\mathrm{d}\tau
\end{aligned}$$

by (76) and the fact that $\mathcal{L}_e$ is a positive operator in the sense that $\mathcal{L}_e f \geq 0$ if $f \geq 0$. Hence, by an induction argument, one gets $z_n(t,k) \leq u(t,k)$ for all $n > 0$. Taking the limit $n \to \infty$, we have (77).

We turn to the third part. Notice that with $v(t,k) = \exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))v_0$, (72) and $0 \leq u_0 \leq v_0$ imply that

$$\begin{aligned} v(t,k) &= e^{-b_0 t} v_0(e^{-tA^{\mathsf{T}}}k) + \int_0^t e^{-b_0(t-\tau)} \mathcal{L}_e v(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau \\ &\geq e^{-b_0 t} u_0(e^{-tA^{\mathsf{T}}}k) + \int_0^t e^{-b_0(t-\tau)} \mathcal{L}_e v(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau. \end{aligned}$$

It follows from the above inequality and applying the conclusion of the second part that $v(t,k) \geq \exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))u_0(k)$, which yields (78). Nonnegativity comes directly from the fact that $z_n = 0$ for all $n > 0$ in the case when $y_0 = 0$. Hence, the proof of Theorem 3.5 is complete. □

We have the following bound of p-Toscani distance, which will be used in the next section when we consider the stationary profile.

Lemma 3.6. *For $p \geq 0$, define*

$$u_p(t,k) := |k|^p \exp(-t(\lambda_p - p\|A\|)). \tag{79}$$

Then, the solution $\exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))|k|^p$ of (72) with initial data $|k|^p$ satisfies

$$\exp(t(-b_0 + \mathcal{L}_e - A^{\mathsf{T}}k \cdot \nabla_k))|k|^p \leq u_p(t,k). \tag{80}$$

In particular, for A_β defined in (13), it holds that

$$\exp(t(-b_0 + \mathcal{L}_e - A_\beta^{\mathsf{T}}k \cdot \nabla_k))|k|^p \leq e^{-p\beta t} u_p(t,k). \tag{81}$$

Proof. By direct computation, one gets

$$\begin{aligned} \mathcal{L}_e |k|^p &= \int_{\mathbb{S}^{d-1}} b(\cos\vartheta)(|k^+|^p + |k^-|^p)\,\mathrm{d}\sigma \\ &= |k|^p \int_{\mathbb{S}^{d-1}} b(\cos\vartheta) \frac{|k^+|^p + |k^-|^p}{|k|^p}\,\mathrm{d}\sigma \\ &= |k|^p(b_0 - \lambda_p), \end{aligned}$$

which yields $\mathcal{L}_e u_p = (b_0 - \lambda_p)u_p$. Then, we have

$$\begin{aligned} & e^{-b_0 t}|e^{-tA^{\mathsf{T}}}k|^p + \int_0^t e^{-b_0(t-\tau)} \mathcal{L}_e u_p(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau \\ &\leq e^{-b_0 t}\|e^{-tA^{\mathsf{T}}}\|^p|k|^p + (b_0 - \lambda_p)\int_0^t e^{-b_0(t-\tau)} u_p(\tau, e^{-(t-\tau)A^{\mathsf{T}}}k)\,\mathrm{d}\tau \\ &\leq e^{-b_0 t}\|e^{-tA^{\mathsf{T}}}\|^p|k|^p + (b_0 - \lambda_p)\int_0^t e^{-b_0(t-\tau)} e^{-\tau(\lambda_p - p\|A\|)} \|e^{-(t-\tau)A^{\mathsf{T}}}\|^p |k|^p\,\mathrm{d}\tau \\ &\leq |k|^p(e^{-b_0 t}e^{pt\|A\|} + (b_0 - \lambda_p)\int_0^t e^{-b_0(t-\tau)} e^{-\tau(\lambda_p - p\|A\|)} e^{p(t-\tau)\|A\|}\,\mathrm{d}\tau) \\ &= |k|^p e^{-t(\lambda_p - p\|A\|)} = u_p, \end{aligned}$$

which, together with (77) in Theorem 3.5 by substituting $y_0(k) = |k|^p$ and $u = u_p$, gives (80). □

With the help of the results above, we have the stability result on the mild solution of (15).

Theorem 3.7. *Let $\varphi(t,k)$ and $\psi(t,k)$ be mild solutions of (15) satisfying (71) with initial conditions $\varphi_0(k), \psi_0(k) \in \mathcal{K}$. Suppose $|\varphi_0(k)-\psi_0(k)| \leq C|k|^p$ for some $p>0$ and $C>0$. Then,*

$$|\varphi(t,k)-\psi(t,k)| \leq Cu_p(t,k), \tag{82}$$

where u_p is defined in (79). In particular,

$$\|\varphi(t)-\psi(t)\|_p \leq Ce^{-t(\lambda_p - p\|A\|)} \to 0 \quad \text{as } t\to\infty \tag{83}$$

if $\lambda_p/p > \|A\|$.

Proof. By our assumption, $\varphi_0-\psi_0, C|k|^p \in \mathcal{C}_p$, it holds by Theorem 3.4 and Theorem 3.5 that

$$\begin{aligned}|\varphi(t,k)-\psi(t,k)| &\leq \exp(t(-b_0+\mathcal{L}_e - A^{\mathsf{T}}k\cdot\nabla_k))|\varphi_0(k)-\psi_0(k)|\\ &\leq C\exp(t(-b_0+\mathcal{L}_e - A^{\mathsf{T}}k\cdot\nabla_k))|k|^p\\ &\leq Cu_p(t,k),\end{aligned}$$

which gives (82). Furthermore, (83) is an immediate consequence of (82) provided $\|A\| < \lambda_p/p$. □

Remark 3.8. In particular, since the Dirac mass at origin $\varphi=1$ has unit mass, zero momentum, and finite energy, and is a stationary solution to (71) for $\|A\| < \lambda_2/2 = \zeta/2$ where ζ is defined in (7), every initial condition also with unit mass, zero momentum, and finite energy weakly converges to the Dirac mass $\varphi=1$, and hence, in such case, there is no nontrivial stationary solution of (15) with finite energy for $z<1$.

We summarize some differences between the inelastic and elastic cases. Let f, g be solutions to (1) with the initial data f_0, g_0. On elastic collision, $\lambda_2=0$. Then, due to the monotonicity of λ_p in (75), if we still want (83) to hold for $(\varphi,\psi)=(\mathcal{F}f,\mathcal{F}g)$, it is necessary to consider $p>2$ as in [7], which requires the initial conditions $f_0(v)$ and $g_0(v)$ to have the same second moments. On the other hand, for inelastic collision, one can consider the case $p=2$ since we have $\lambda_2=\zeta>0$, which implies that assuming $\|A\|<\zeta/2$, on two initial probability measures f_0, g_0 with unit mass, zero momentum, and finite energy, (83) still holds. Notice that now we no longer require the second moments of f_0 and g_0 to be the same.

3.1. **Stationary self-similar Fourier profile.** The existence of a self-similar profile with finite second moments under small parameters can also be done in the Fourier framework. The stationary solution $\Phi=\Phi(k)$ for (18) with self-similar parameter $\beta\in\mathbb{R}$ solves the equation

$$\Phi(k) = \int_0^\infty e^{-b_0 t}\widehat{Q_e^+}(\Phi,\Phi)(e^{-\beta t}e^{-tA^{\mathsf{T}}}k)\,\mathrm{d}t. \tag{84}$$

Define

$$B : k^{\otimes 2} = \sum_{i,j=1}^d B_{ij}k_ik_j,$$

and

$$\mathcal{S}_p := \left\{\varphi\in\mathcal{K} \mid \|\varphi - (1-\frac{1}{2}B:k^{\otimes 2})\|_p < \infty\right\}. \tag{85}$$

Theorem 3.9. *Let $p \in (2,4]$. Then, there exists $\epsilon_p > 0$ such that for $\|A\| < \epsilon_p$ and $1 - z < \epsilon_p$, there exist $\beta \in \mathbb{R}$, $B \in \mathbb{R}^{d\times d}$ which is positive definite, and a unique $\Phi \in \mathcal{S}_p$ that solves (84).*

Proof. We still choose $\beta \in \mathbb{R}$ and $B \in \mathbb{R}^{d\times d}$ as in Theorem 2.8 such that (44) is satisfied with β having the largest real part. Notice that we no longer require $\mathrm{tr}(B) = 1$ now.

Since $\Psi_0(k) := \exp(-\frac{1}{2}B : k^{\otimes 2})$ is the characteristic function of a normal distribution, we then have $\Psi_0 \in \mathcal{K}$, and thus $\Psi_0 \in \mathcal{S}_p$ on $2 < p \le 4$ with $\mathcal{S}_p$ nonempty.

Consider the Picard mapping $P : \mathcal{S}_p \to \mathcal{K}$ by

$$P\Phi(t,k) = \int_0^\infty e^{-b_0 t}\widehat{Q_e^+}(\Phi,\Phi)(e^{-\beta t}e^{-tA^\intercal}k)\,\mathrm{d}t.$$

The mapping is well-defined since $P\Phi$ is a convex combination of characteristic functions

$$k \mapsto b_0^{-1}\widehat{Q_e^+}(\Phi,\Phi)(e^{-\beta t}e^{-tA^\intercal}k).$$

We first show that $P\Psi_0 \in \mathcal{S}_p$. Since $\Psi_0, P\Psi_0 \in \mathcal{K}$, we have $|P\Psi_0 - \Psi_0| \le C$. We expand Ψ_0 in k such that $\Psi_0(k) = 1 - \frac{1}{2}B : k^{\otimes 2} + O(|k|^4)$, and using $|k^\pm| \le |k|$, it holds that

$$\begin{aligned}
&\widehat{Q_e^+}(\Psi_0,\Psi_0)(k)\\
&= \int_{\mathbb{S}^{d-1}} b(\cos\theta)\big(1 - \frac{1}{2}B : (k^+)^{\otimes 2} + O(|k^+|^4)\big)\big(1 - \frac{1}{2}B : (k^-)^{\otimes 2} + O(|k^-|^4)\big)\,\mathrm{d}\sigma\\
&= \int_{\mathbb{S}^{d-1}} b(\cos\theta)\big(1 - \frac{1}{2}B : ((k^+)^{\otimes 2} + (k^-)^{\otimes 2}) + O(|k|^4)\big)\,\mathrm{d}\sigma\\
&= b_0 - \frac{1}{2}B : \big((b_0 - \zeta)k^{\otimes 2} + z^2\frac{dc_{11}}{2}(\frac{|k|^2}{d}I - k^{\otimes 2})\big) + O(|k|^4),
\end{aligned}$$

which gives

$$\begin{aligned}
&\widehat{Q_e^+}(\Psi_0,\Psi_0)(k) - b_0\Psi_0(k)\\
&= -\frac{1}{2}B : (-\zeta k^{\otimes 2} + z^2\frac{dc_{11}}{2}(\frac{|k|^2}{d}I - k^{\otimes 2})) + O(|k|^4)\\
&= \frac{1}{2}k^{\otimes 2} : \big(\zeta B + \tilde{c}(B - \frac{\mathrm{tr}(B)}{d}I)\big) + O(|k|^4),
\end{aligned} \tag{86}$$

where $\tilde{c} = z^2\frac{dc_{11}}{2} > 0$. Further, we have

$$\begin{aligned}
\Psi_0(e^{-\beta t}e^{-tA^\intercal}k) &= 1 - \frac{1}{2}e^{-2\beta t}e^{-tA}Be^{-tA^\intercal} : k^{\otimes 2} + O(|e^{-\beta t}e^{-tA^\intercal}k|^4)\\
&= \Psi_0(k) - \frac{1}{2}(e^{-2\beta t}e^{-tA}Be^{-tA^\intercal} - B) : k^{\otimes 2}\\
&\quad + O(|e^{-\beta t}e^{-tA^\intercal}k|^4) + O(|k|^4).
\end{aligned} \tag{87}$$

To bound the second term on the right-hand side above, by the selection of B such that (44) holds, we have

$$\zeta B + \tilde{c}(B - \frac{\mathrm{tr}(B)}{d}I) = -2\beta B - AB - BA^\intercal,$$

and a direct calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-ct}e^{-tA}Be^{-tA^\intercal}) = -e^{-ct}e^{-tA}(cB + AB + BA^\intercal)e^{-tA^\intercal}, \tag{88}$$

for all $c \in \mathbb{R}$. Thus, by choosing $\|A\|$ and $1-z$ to be sufficiently small such that $2\|A\| + \zeta - 2\tilde{\beta} \le 2\|A\| + 2|\tilde{\beta}| + \zeta < b_0$, and then choosing $c = 2\beta + b_0$ in the above equation, one gets from integrating in t on both sides of (88) that

$$\int_0^\infty e^{-(b_0+2\beta)t} e^{-tA}(\zeta B + \tilde{c}(B - \frac{\operatorname{tr}(B)}{d} I) - b_0 B) e^{-tA^\intercal} \, \mathrm{d}t = -B. \tag{89}$$

Notice that to get the above equality, we use the fact that $\|e^{-(2\beta+b_0)t} e^{-tA} e^{-tA^\intercal}\| \le e^{-(b_0+2\tilde{\beta}-\zeta-2\|A\|)t} \to 0$. For the last two terms on the right-hand side of (87), one has $e^{-b_0 t}|e^{-\beta t} e^{-tA^\intercal} k|^4 \le e^{-b_0 t} e^{-4(\tilde{\beta}-\zeta/2-\|A\|)t}|k|^4$. Then, for sufficiently small $\|A\|$ and $1-z$, it holds that $4(\tilde{\beta} - \zeta/2 - \|A\|) < b_0$, which yields

$$\int_0^\infty e^{-b_0 t}(O(|k|^4) + O(|e^{-\beta t} e^{-tA^\intercal} k|^4)) \, \mathrm{d}t = O(|k|^4). \tag{90}$$

It follows from (87), (89), and (90) that

$$\begin{aligned}
&P\Psi_0(k) - \Psi_0(k) \\
&= \int_0^\infty e^{-b_0 t}\big((\widehat{Q_e^+}(\Psi_0, \Psi_0) - b_0 \Psi_0)(e^{-\beta t} e^{-tA^\intercal} k) + b_0(\Psi_0(e^{-\beta t} e^{-tA^\intercal} k) - \Psi_0(k))\big) \, \mathrm{d}t \\
&= \frac{1}{2}\int_0^\infty e^{-b_0 t}\big(e^{-2\beta t} e^{-tA}(\zeta B + \tilde{c}(B - \frac{\operatorname{tr}(B)}{d} I) - b_0 B) e^{-tA^\intercal} : k^{\otimes 2} \\
&\qquad\qquad + b_0 B : k^{\otimes 2} + O(|k|^4) + O(|e^{-\beta t} e^{-tA^\intercal} k|^4)\big) \, \mathrm{d}t \\
&= O(|k|^4).
\end{aligned}$$

Hence, $|P\Psi_0(k) - \Psi_0(k)| \le C \min(1, |k|^4)$, and $\|P\Psi_0 - \Psi_0\|_p \le C \sup_{k \neq 0} \frac{\min(1,|k|^4)}{|k|^p} < \infty$ for $2 < p \le 4$, which implies $P\Psi_0 \in \mathcal{S}_p$.

Now we can prove that $P : \mathcal{S}_p \to \mathcal{S}_p$ is a contraction mapping. Let $\varphi(k), \psi(k) \in \mathcal{S}_p$. Then, $\|\varphi - \psi\|_p \le \|\varphi - (1 - \frac{1}{2} B : k^{\otimes 2})\|_p + \|\psi - (1 - \frac{1}{2} B : k^{\otimes 2})\|_p < \infty$, which, combining with (70), further yields

$$\begin{aligned}
|P\varphi(k) - P\psi(k)| &\le \int_0^\infty e^{-b_0 t}|\widehat{Q_e^+}(\varphi, \varphi) - \widehat{Q_e^+}(\psi, \psi)(e^{-\beta t} e^{-tA^\intercal} k)| \, \mathrm{d}t \\
&\le \int_0^\infty e^{-b_0 t} \mathcal{L}_e |\varphi - \psi|(e^{-\beta t} e^{-tA^\intercal} k) \, \mathrm{d}t \\
&\le \|\varphi - \psi\|_p \int_0^\infty e^{-b_0 t} \mathcal{L}_e |k|^p (e^{-\beta t} e^{-tA^\intercal} k) \, \mathrm{d}t \\
&\le (b_0 - \lambda_p)\|\varphi - \psi\|_p \int_0^\infty e^{-b_0 t} e^{-p\beta t} \|e^{-tA^\intercal}\|^p |k|^p \, \mathrm{d}t \\
&\le (b_0 - \lambda_p)\|\varphi - \psi\|_p |k|^p \int_0^\infty e^{-pt(b_0/p + \tilde{\beta} - \zeta/2 - \|A\|)} \, \mathrm{d}t.
\end{aligned} \tag{91}$$

Recalling the definitions of λ_p and ζ in (74) and (7), for any $p \in (2, 4]$, the mapping $z \mapsto \lambda_p/p - \zeta/2$ is continuous, and at $z = 1$ it holds that $(\lambda_p/p - \zeta/2)|_{z=1} > 0$. Therefore, for $1 - z > 0$ sufficiently small, one has $\lambda_p/p - \zeta/2 > 0$.

Moreover, we choose $\|A\|$ to be small enough such that

$$\|A\| - \tilde{\beta} \le \|A\| + |\tilde{\beta}| < \lambda_p/p - \zeta/2 \le b_0/p - \zeta/2,$$

and the last integral in (91) is finite with

$$(b_0 - \lambda_p)\int_0^\infty e^{-t(b_0 + p\tilde{\beta} - p\frac{\zeta}{2} - p\|A\|)} \, \mathrm{d}t = \frac{b_0 - \lambda_p}{b_0 + p(\tilde{\beta} - \zeta/2 - \|A\|)} < 1,$$

which yields

$$\|P\varphi - P\psi\|_p \le \frac{b_0 - \lambda_p}{b_0 + p(\tilde{\beta} - \zeta/2 - \|A\|)} \|\varphi - \psi\|_p. \tag{92}$$

For $\varphi \in \mathcal{S}_p$, since $P\Psi_0 \in \mathcal{S}_p$, we have from (92) that

$$\|P\varphi - \Psi_0\|_p \le \|P\varphi - P\Psi_0\|_p + \|P\Psi_0 - \Psi_0\|_p \le C\|\varphi - \Psi_0\|_p + \|P\Psi_0 - \Psi_0\|_p < \infty.$$

Then, $P\varphi \in \mathcal{S}_p$, which, together with (92), implies that $P : \mathcal{S}_p \to \mathcal{S}_p$ is a Banach contraction mapping, and has a unique fixed-point $\Phi \in \mathcal{S}_p$. The proof of Theorem 3.9 is complete. □

Remark 3.10. Recalling (85), note that $\mathcal{S}_p \supseteq \mathcal{S}_q$ on $p \le q \le 4$. This implies that while the smallness conditions on $\|A\|$ and $1 - z$ depend on p, once $\|A\|$ and $1 - z$ are fixed, the stationary solution Φ_p obtained is the same for all $p \in (2, 4]$, for which the required smallness conditions are satisfied.

The convergence rate to a stationary profile can be improved from $\lambda_p + 2\beta - p\|A\|$, indicated by (83) with the following theorem.

Theorem 3.11. *Assume the same condition in Theorem 3.9, and let $\varphi(t)$ be the mild solution of (18) with self-similar scaling parameter β and initial condition $\varphi_0 \in \mathcal{K}^2$ satisfying $\|\varphi_0 - (1 - \frac{1}{2}C_0 : k^{\otimes 2})\|_p < \infty$ for some positive semidefinite matrix C_0. Then, there exist $\lambda > 0$ depending on C_0, some constant $\nu > 0$, and*

$$\eta = \frac{1}{2}\min\{\lambda_p + p(\beta - \|A\|), \nu + \zeta + 2(\beta - \|A\|)\} > 0, \tag{93}$$

such that

$$|\varphi(t,k) - \Phi(\lambda k)| \le Ce^{-\eta t}(|k|^p + |k|^2). \tag{94}$$

Proof. By assumption, C_0 is the second moment matrix of the probability measure corresponding to φ_0. Due to Theorem 2.6, $\varphi(t,k)$ also has a corresponding finite second moment matrix $C(t)$, which is positive semi-definite on $t \ge 0$, and satisfies (43) with initial condition $C(0) = C_0$. By the selection of β, B, there exists $\lambda, \nu, C \ge 0$ such that $\|C(t) - \lambda^2 B\| \le Ce^{-\nu t}$.

Let $\phi(t,k) = \exp(-\frac{1}{2}C(t) : k^{\otimes 2})$. By a similar calculation as in (86), one has

$$\begin{aligned}\widehat{Q_e}(\phi,\phi)(t,k) &= \widehat{Q_e^+}(\phi,\phi)(t,k) - b_0\phi(t,k)\\ &= \frac{1}{2}k^{\otimes 2} : (\zeta C(t) + \tilde{c}(C(t) - \frac{\mathrm{tr}(C(t))}{d}I)) + O(|k|^4),\end{aligned}$$

where the $O(|k|^4)$ term is uniformly bounded in time since $C(t)$ converges to $\lambda^2 B$. Thus, it further holds that

$$\begin{aligned}&\partial_t\phi + A_\beta^\intercal k \cdot \nabla_k\phi - \widehat{Q_e}(\phi,\phi)\\ &= -\frac{1}{2}k^{\otimes 2} : \big(\frac{\mathrm{d}}{\mathrm{d}t}C(t) + A_\beta C(t) + C(t)A_\beta^\intercal + \zeta C(t) + \tilde{c}(C(t) - \frac{\mathrm{tr}(C(t))}{d}I)\big) + O(|k|^4),\end{aligned}$$

which implies

$$(\partial_t + A_\beta^\intercal \cdot \nabla_k + b_0)(\varphi - \phi) = (\widehat{Q_e^+}(\varphi,\varphi) - \widehat{Q_e^+}(\phi,\phi)) + \delta(t,k)$$

for some $\delta(t,k)$ satisfying $|\delta(t,k)| \le C|k|^4$ for some constant $C > 0$ uniformly in time. With the $\mathcal{L}_e$-Lipschitz property (70), one gets

$$|\varphi(t,k) - \phi(t,k)|$$

$$
\begin{aligned}
&\leq e^{-t(b_0+A_\beta^\intercal k\cdot\nabla_k)}|\varphi_0(k)-\phi_0(k)| \\
&\quad + \int_0^t e^{-(t-\tau)(b_0+A_\beta^\intercal k\cdot\nabla_k)}|\widehat{Q_e^+}(\varphi,\varphi)(\tau,k)-\widehat{Q_e^+}(\phi,\phi)(\tau,k)|\,\mathrm{d}\tau \\
&\quad + \int_0^t e^{-(t-\tau)(b_0+A_\beta^\intercal k\cdot\nabla_k)}|\delta(\tau,k)|\,\mathrm{d}\tau \\
&\leq e^{-t(b_0+A_\beta^\intercal k\cdot\nabla_k)}|\varphi_0(k)-\phi_0(k)| \\
&\quad + \int_0^t e^{-(t-\tau)(b_0+A_\beta^\intercal k\cdot\nabla_k)}\mathcal{L}_e|\varphi(\tau,k)-\phi(\tau,k)|\,\mathrm{d}\tau \\
&\qquad + \int_0^t e^{-(t-\tau)(b_0+A_\beta^\intercal k\cdot\nabla_k)}|\delta(\tau,k)|\,\mathrm{d}\tau.
\end{aligned}
$$

Letting

$$
\begin{aligned}
Y(t,k) = e^{-t(b_0+A_\beta^\intercal k\cdot\nabla_k)}|\varphi_0(k)-\phi_0(k)| &+ \int_0^t e^{-(t-\tau)(b_0+A_\beta^\intercal k\cdot\nabla_k)}\mathcal{L}_e Y(\tau,k)\,\mathrm{d}\tau \\
&+ \int_0^t e^{-(t-\tau)(b_0+A_\beta^\intercal k\cdot\nabla_k)}|\delta(\tau,k)|\,\mathrm{d}\tau,
\end{aligned}
$$

it can be solved as

$$
Y(t,k) = e^{-t(A_\beta^\intercal k\cdot\nabla_k+b_0-\mathcal{L}_e)}|\varphi_0(k)-\phi_0(k)| + \int_0^t e^{-(t-\tau)(A_\beta^\intercal k\cdot\nabla_k+b_0-\mathcal{L}_e)}|\delta(\tau,k)|\,\mathrm{d}\tau.
$$

By the comparison principle, with the assumption that $|\varphi_0(k)-\phi_0(k)| = O(|k|^p)$, we have by a similar computation as in Theorem 3.6 that

$$
\begin{aligned}
&|\varphi(t,k)-\phi(t,k)| \\
&\leq Y(t,k) = e^{-t(A_\beta^\intercal k\cdot\nabla_k+b_0-\mathcal{L}_e)}|\varphi_0(k)-\phi_0(k)| \\
&\quad + \int_0^t e^{-(t-\tau)(A_\beta^\intercal k\cdot\nabla_k+b_0-\mathcal{L}_e)}|\delta(\tau,k)|\,\mathrm{d}\tau \\
&\leq Ce^{-t(A_\beta^\intercal k\cdot\nabla_k+b_0-\mathcal{L}_e)}|k|^p + C\int_0^t e^{-(t-\tau)(A_\beta^\intercal k\cdot\nabla_k+b_0-\mathcal{L}_e)}|k|^4\,\mathrm{d}\tau \\
&\leq Ce^{-p\beta t}u_p(t,k) + C\int_0^t e^{-4\beta(t-\tau)}u_4(t-\tau,k)\,\mathrm{d}\tau \\
&= Ce^{-pt(\lambda_p/p+\beta-\|A\|)}|k|^p + C\int_0^t e^{-4\tau(\lambda_4/4+\beta-\|A\|)}|k|^4\,\mathrm{d}\tau,
\end{aligned}
$$

with constants independent of t, where u_p is defined in (79). By similar arguments as in the proof of Theorem 3.9, for $\|A\|$ *and* $1-z$ sufficiently small, both exponents are negative, and so $|\varphi(t,k)-\phi(t,k)| \leq C|k|^p + C|k|^4$ for some constant C independent of t.

If $|k| \geq 1$, then $|\varphi(t,k)-\phi(t,k)| \leq C \leq C|k|^p$. If $|k|<1$, then $|\varphi(t,k)-\phi(t,k)| \leq C|k|^p + C|k|^4 \leq C|k|^p$. Then, $|\varphi(t,k)-\phi(t,k)| \leq C|k|^p$. Therefore, for $T>0$,

$$
\begin{aligned}
|\varphi(T/2,k)-\Phi(\lambda k)| &\leq |\varphi(T/2,k)-\phi(T/2,k)| + |\phi(T/2,k)-\Phi(\lambda k)| \\
&\leq C|k|^p + C\|C(T/2)-\lambda^2 B\||k|^2 \\
&\leq C|k|^p + Ce^{-\nu T/2}|k|^2,
\end{aligned}
$$

which, together with (73) for initial time $t_0 = T/2$ and (81), yields

$$\begin{aligned}
|\varphi(T,k) - \Phi(\lambda k)| &\le \exp(-\frac{T}{2}(b_0 + A_\beta^\intercal k \cdot \nabla_k - \mathcal{L}_e))|\varphi(T/2,k) - \Phi(\lambda k)| \\
&\le C\exp(-\frac{T}{2}(b_0 + A_\beta^\intercal k \cdot \nabla_k - \mathcal{L}_e))(|k|^p + e^{-\nu T/2}|k|^2) \\
&\le Ce^{-p\beta T/2}u_p(T/2,k) + e^{-\nu T/2}e^{-\beta T}u_2(T/2,k) \\
&= Ce^{-T\frac{p}{2}(\frac{\lambda_p}{p}+\beta-\|A\|)}|k|^p + e^{-T(\frac{\nu}{2}+\frac{\zeta}{2}+\beta-\|A\|)}|k|^2 \\
&\le Ce^{-\eta T}(|k|^p + |k|^2),
\end{aligned}$$

with η defined in (93), and $\|A\|$ *and* $1-z$ chosen to be sufficiently small. Hence, we get (94), and the proof of Theorem 3.11 is complete. ☐

Remark 3.12. Note that by splitting $[0,T]$ into $[0,rT] \cup [rT,T]$ with $r \in (0,1)$ instead of $[0,T/2] \cup [T/2,T]$ in the proof above, we can obtain

$$\begin{aligned}
|\varphi(T,k) - \Phi(\lambda k)| &\le Ce^{-(1-r)T(\lambda_p + p(\beta-\|A\|))}|k|^p + Ce^{-\nu rT}e^{-(1-r)T(\zeta+2(\beta-\|A\|))}|k|^2 \\
&\le e^{-\eta(r)T}(|k|^p + |k|^2)
\end{aligned}$$

with

$$\eta(r) = \min((1-r)(\lambda_p + p(\beta - \|A\|)), r\nu + (1-r)(\zeta + 2(\beta - \|A\|))),$$

which is positive if $\|A\|$ is sufficiently small. If $\|A\| - \beta < \frac{\lambda_p - \zeta}{p-2}$, then it holds that

$$\max \eta(r) = \frac{\nu(\lambda_p + p(\beta - \|A\|))}{\nu + \lambda_p - \zeta + (p-2)(\beta - \|A\|)},$$

otherwise, by selecting r sufficiently small, we have $\eta(r) = \lambda_p + p(\beta - \|A\|) - \epsilon$ for arbitrary ϵ, provided that it is small enough.

Both Theorem 3.9 and Theorem 3.11 are given in the sense of the Fourier transform. We also have the corresponding results in terms of distribution functions.

Theorem 3.13. *Let $p \in (2,4]$. Then, there exists $\epsilon_p > 0$ such that for $\|A\| < \epsilon_p$ and $1 - z < \epsilon_p$, there exists a self-similar solution f to (1) in the form*

$$f(t,v) = e^{-d\beta t}G(e^{-\beta t}v),$$

with G being a Radon probability measure and

$$\int_{\mathbb{R}^d} G(v)|v|^p \mathrm{d}v < \infty.$$

The proof of Theorem 3.13 is a direct consequence of Theorem 3.9 and thus omitted here. The following theorem gives the convergence for solutions of (1) to the self-similar profile.

Theorem 3.14. *Suppose $f \in C([0,\infty), \mathcal{M}_+)$ is a weak solution of (1) on initial condition $f_0(v) \in \mathcal{M}_+$ satisfying $\int_{\mathbb{R}^d} f_0(v)|v|^p \mathrm{d}v < \infty$ for any $p > 2$. Then, for β and λ chosen in Theorem 3.11 and G defined in Theorem 3.13, it holds that*

$$e^{d\beta t} f(t, e^{\beta t}v) \to \lambda^{-d} G(\lambda^{-1}v), \tag{95}$$

in the weak topology of $\mathcal{M}_+$ as $t \to 0$.

Proof. The weak convergence (95) follows from the combination of Theorem 3.11, the properties of the Fourier transform that relates the group of translations with the multiplication by a phase, the scaling properties of the Fourier transform, and the uniform convergence of characteristic functions in compact sets. See also [7, 28]. □

Combining Theorem 3.13 and Theorem 3.14, we obtain the result in Theorem 1.3.

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