Numerical Pricing of Shout Options

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Abstract

This dissertation reports on an application of the Longstaff-Schwartz Algorithm for estimating the value of 3 types of Shout Options, namely the Reset Strike Put Option, the Reset Strike Asian Put Option and the Swing Reset Strike Put Option. A method is introduced for estimating the Greeks based on an adaptation of the Pathwise Sensitivity approach and the Likelihood Ratio method. The convergence properties are analysed and the accuracy of the estimates is assessed.

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Introduction

An option is a financial contract which bestows upon the holder the right, but not the obligation to buy or sell an underlying asset for a predetermined price, called the strike during a certain period of time. Standard options tend to come in two types: a call, which entitles the owner to buy the underlying asset and a put which entitles the holder to sell the underlying asset. An added distinction is when exercise occurs. A European option allows exercise to occur only at maturity. An American option allows for exercise at any point during the life of the option, whereas a Bermudan option allows exercise on a predetermined set of dates throughout the life of the option. Obviously, the more flexible the option, the more expensive it will be.

In the last decade, there has been a growing demand for increasingly complex derivative products. One such example is a contract which allows the holder to modify certain features of the contract during the life of the contract. Such a derivative is known broadly as a Shout Option.

Prior academic research in this field has not been comprehensive. [6] and [7] analyse the Strike Reset Put Option with single and multiple shout rights using a PDE approach and consider optimal reset strategies in the presence of a dividend yield. [19] and [20] employ a finite difference approach to price and hedge more complex reset features. More complicated Shout Options have been embedded in other derivative products. [3] considers an application of a Shout Floor to index funds and describes valuation using a lattice approach and [11] considers the reset feature in Geared Equity Investments offered by Macquarie Bank. [1] analyses how resetting stock options can affect firm performance and [18] discusses Shout Options embedded in products called Segregated Funds, sold by Canadian life insurance companies. Even some energy derivative contracts called Swing Options contain a special form of shout feature as considered in [12].

In this thesis, I price a Reset Strike Put Option, a Reset Strike Asian Put Option as well as a Swing Reset Strike Put Option using the Least Squares Regression approach introduced by Longstaff and Schwartz [15]. The Swing Reset Strike Asian Put Option is an easy extension, however I do not report results for this here. I outline the extension of the Longstaff-Schwartz Algorithm to Shout Options and explain how the Longstaff-Schwartz Algorithm can be implemented in the context of Swing Options. Further, I analyse and investigate the convergence of the Longstaff-Schwartz method as applied to Shout Options by investigating the behaviour over different number of paths and time steps. This aimed to verify the result proven in [4] that the Longstaff-Schwartz value converges to the true value as the number of paths increase. I also implemented a Binomial Tree approach for valuing Shout Options to serve as a comparison. Once I was certain that the Longstaff-Schwartz method produced accurate and stable estimates, I corroborated some general properties of Shout Options as presented in [6], [7] and [19]. Lastly, I analysed and investigated the Greeks within the Longstaff-Schwartz framework by employing the Pathwise Sensitivity approach and the Likelihood Ratio method as first introduced by [17]. I must mention that in low dimensions, the Longstaff-Schwartz algorithm will not be preferable in terms of cost or efficiency. I test one-dimensional cases in order to analyse the accuracy of the method, however, in low dimensions, lattice approximations or partial differential equation approaches will be preferred. In high dimensions, these methods become considerably more costly to implement, and their tractability and efficiency is reduced. It is in this case that the Longstaff-Schwartz algorithm would be favoured.

This paper is organised as follows: In Chapter 1, I present a description of Shout Options in general as well as a detailed examination of the specific Shout contracts that this thesis analyses. Chapter 2 introduces the Longstaff-Schwartz Algorithm as applied to Bermudan options, the extension of the Longstaff-Schwartz Algorithm to Shout Options and Swing Shout Options as well as a discussion of approximation errors and convergence properties. Further, the theoretical underpinnings for pricing the Greeks is introduced and a description of the control method is reported. Chapter 3 provides specifications on the implementation of the methods as well as the numerical results which prove the robustness of this technique for modelling Shout Options. The paper ends with some conclusive remarks and a suggestion for possible extensions in the last chapter.

Chapter 1

Shout Options

1.1 Description

A Shout Option entitles the holder of the option to alter certain features of the contract during the life of the contract, according to some prespecified conditions. The time of modification, known as the reset time, may be chosen optimally by the holder or may be automatic upon fulfilment of certain preset requirements. In practice, the terms which can be reset are the strike price and the maturity of the option.

The earliest form of Shout Option presented itself as an embedded shout feature in a Call Option. This allowed the holder to lock into a profit from shouting while still enabling him to profit from any further upside moves. The terminal payoff is given by $C = \max(S_T - K, S_t - K, 0)$ where K is the strike price, S_T is the terminal asset price and S_t is the prevailing stock price at the moment of shouting. Consequently, the payoff is guaranteed to be at least $S_t - K$ and as such, it is sensible to shout only when $S_t > K$.

A more common Shout Option used in practice is the Reset Strike Put Option,which, as the name suggests is an option which enables the holder to reset the strike price to the prevailing stock price at the moment of shouting. Essentially it entitles the holder to determine when to take ownership of an at-the-money option. Denote K as the original strike price, set at initiation of the contract and let S_t and S_T denote the stock price at the shouting moment and maturity respectively. Then, as suggested in [6], the payoff of the Reset Strike Put Option will be $(K - S_T)^+$ if no shouting occurs during the life of the contract and $(S_t - S_T)^+$ if shouting occurs at time t where $0 \leq t < T$. Thus, in effect, the Reset Strike Put Option transforms into an at-the-money put option at the moment of shouting. The nature of the payoff suggests that it is suitable to shout only if $S_t > K$, which results in an increase in the terminal payoff after shouting.

As discussed in [3], another contract used often in equity markets is a Shout Floor. This option enables the holder to install a protective floor at the prevailing price. This grants investors exposure to the upside of the market while at the same time limits the possible losses they may incur. This differs from a standard Floor contract in that the holder is allowed to determine the floor level during the life of the contract as opposed to setting the level at inception of the contract. The terminal payoff in this case is $\max(S_t - S_T, 0)$ and is regarded as a generalization of the Strike Reset Put Option with initial strike set to zero.

An alternative form of Shout Option is the Reset Strike Asian Put Option. Similarly to the Reset Strike Put Option, this contract entitles the holder to reset the strike to the prevailing stock price at the moment of shouting, however in this case, the payoff is dependent on the average of the stock price over the life of the contract. We consider the average over a set of discrete dates $t_0 = 0 < t_1 < \ldots < t_M = T$. This allows the holder to determine when to take ownership of an at-the-money Asian Put Option. Denote K as the original strike price set at inception of the contract, S_{t_i} as the stock price on date t_i. The terminal payoff will then be $(K - \frac{1}{M}\sum_{i=1}^{M} S_{t_i})^+$ if no shouting occurs during the life of the contract and $(S_\tau - \frac{1}{M}\sum_{i=1}^{M} S_{t_i})^+$ if shouting occurs at time τ where $\tau \in \{t_1, t_2, \ldots, t_M\}$. An important distinction to the Reset Strike Put Option is that it may be optimal to shout at maturity of the contract. In fact, if shouting has not occurred during the life of the contract, it will always be optimal to shout at T provided $S_T > K$. Hence if shouting has not occurred prior to time T, then the terminal payoff can be adjusted to

$$\begin{aligned} \max((K - \frac{1}{M}\sum_{i=1}^{M} S_{t_i})^+, (S_T - \frac{1}{M}\sum_{i=1}^{M} S_{t_i})^+) \\ = (\max(S_T, K) - \frac{1}{M}\sum_{i=1}^{M} S_{t_i})^+ \end{aligned}$$

Asian Options are popular because they tend to have lower volatility than their vanilla counterparts, as it is believed that the average of a price series is less volatile than the price series itself. Further, it is believed that the average is more difficult to manipulate than a single price series and consequently they are thought to provide some protection against risk. There does not exist any analytical pricing formula for Arithmetic Average Options. In addition, the path dependent nature of Asian Options tends to invalidate standard pricing techniques. Consequently Monte Carlo methods are considered ideal for pricing this type of option.

I also introduce a Swing Option in a Shout Option context. Essentially, I consider a contract which has a prespecified number n of shout opportunities on n underlying put options. In this case, Each time you shout, you reset the strike price of exactly one of the underlying put options to the stock price at the moment of shouting. It is important to note that this is different from the consideration of an "n-reset" Strike Put Option which has only one underlying European Put Option and you have the right to reset the strike of that Put

Option at most n times. This paper considers the application of the Longstaff-Schwartz Algorithm to the Swing Reset Strike Put Option using a dynamic programming approach.

In this paper, I focus predominantly on the pricing of Reset Strike Put Options as well as Reset Strike Asian Put Options and Swing Reset Strike Put Options using the Longstaff-Schwartz Algorithm. [13] mentions that the valuation of a derivative with strike reset right results in an optimal stopping problem. [6] and [19] contend that Shout Options can be considered as a generalization of an American Option feature and consequently, it is essential to examine the optimal shouting policy. The Longstaff-Schwartz Method has proven to be a robust method, one that can successfully handle the optimization component inherent in an American Option and as such appears to be an appropriate technique to value Shout Options.

Chapter 2

Theory and Algorithms

Before I consider the extension of the Longstaff-Schwartz method applied to shout options, I first present the traditional Longstaff-Schwartz Algorithm as applied to Bermudan Options.

2.1 General Bermudan Option Formulation

Following a description in [9] and [14], the Bermudan option price process satisfies

$$V_t = \sup_{\tau}(\mathbb{E}^{\mathbb{Q}}{}_t(D_{t,\tau}h_\tau)) \tag{2.1}$$

where $D_{s,t} = \exp(-\int_s^t r_u du) = \exp(-r(t-s))$ is the riskless discount process and I have assumed that r is the deterministic interest rate, $h_t(S_t)$ is the payoff of the option, i.e. the option holder will receive h_t if he exercises the option at time t and τ is the set of all admissible stopping times in $\{t_1, t_2, \ldots, t_M\}$, where $\{t_1, t_2, \ldots, t_M\}$ represent the discrete set of dates on which the option can be exercised. Equation (2.1) declares that the price of a Bermudan Option should be the discounted expected value of the payoff on the date decided by the optimal exercise policy.

According to [9], a consideration of the dynamic programming approach yields the following formulation of the Bermudan Option problem:

$$V_M(S_M) = h_M(S_M) \tag{2.2}$$

$$\begin{aligned} V_i(s) &= \max(h_i(s), \mathbb{E}^{\mathbb{Q}}\left[D_{i,i+1}V_{i+1}(S_{i+1})|S_i = s\right]) \\ &= \max(h_i(s), D_{i,i+1}\mathbb{E}^{\mathbb{Q}}\left[V_{i+1}(S_{i+1})|S_i = s\right]) \\ &= \max(h_i(s), e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[V_{i+1}(S_{i+1})|S_i = s\right]) \end{aligned} \tag{2.3}$$

where S_i is the value of the underlying at time t_i, $V_{i+1}(S_{i+1})$ is the value of the option at time t_{i+1} and is defined with the assumption that it has not been exercised previously. $h_i(s)$ denotes the payoff or exercise value of the option at time t_i, $D_{i,i+1}$ is the discount factor as defined above and $\Delta t = t_{i+1} - t_i$. Equation (2.2) indicates that if the option has not been exercised before maturity, the option value is the payoff price. Equation (2.3) suggests that the value of the option is the maximum of immediate exercise and the discounted expected value of continuing.

The continuation value can be represented as

$$V_i^C(s) = e^{-r\Delta t}\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i = s\right] = e^{-r\Delta t}C_i(S_i)$$

and so we obtain the following optimal stopping rule:

$$\begin{aligned}\tau &= \min\left\{t_i : 1 \le i \le M,\ h_i(S_i) \ge V_i^C(S_i)\right\} \\ &= \min\left\{t_i : 1 \le i \le M,\ h_i(S_i) \ge e^{-r\Delta t}C_i(S_i)\right\}\end{aligned}$$

and $h_M(S_M) = V_M^C(S_M)$ by convention. This implies that the continuation value is the value of holding rather than exercising the option and by approximating the continuation value, we can obtain an approximate stopping rule. It is important to note that the algorithm values Bermudan options, but it is assumed that as the number of exercise opportunities increase, the value should approach the value of an American option.

2.2 Longstaff-Schwartz Algorithm

Longstaff and Schwartz propose a simple yet powerful approach for approximating the value of Bermudan or American Option prices by using Monte Carlo Simulation and estimating the conditional expectation of continuing using a simple least squares regression. I follow the treatment by [9],[14],[15] and [17] and represent

$$\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i = s\right] \approx \widehat{C}_i(s) = \sum_{r=0}^{R} \beta_{ir}\psi_r(s)$$

This indicates that the continuation values are estimated using a regression. I use a set of $R+1$ basis functions $\psi_0, \psi_1, \ldots, \psi_R$ to approximate the constants β_{ir} where $r = 0, 1, \ldots, R$. Typically, least squares is chosen to estimate the β_{ir} coefficients by minimising

$$\begin{aligned}&\mathbb{E}\left[(\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i\right] - \widehat{C}_i(S_i))^2\right] \\ =&\mathbb{E}\left[(\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i\right] - \sum_{r=0}^{R}\beta_{ir}\psi_r(S_i))^2\right]\end{aligned}$$

Taking the derivative with respect to β_{ir} and setting the result to zero yields

$$\mathbb{E}\left[\mathbb{E}\left(\left[V_{i+1}(S_{i+1})|S_i\right] - \widehat{C}_i(S_i)\right)\psi_r(S_i)\right] = 0$$

and hence

$$\mathbb{E}\left[\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i\right]\ \psi_r(S_i)\right] = \mathbb{E}\left[\widehat{C}_i(S_i)\psi_r(S_i)\right] = \sum_{s=0}^{R} \beta_{is}\mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right] \tag{2.4}$$

It is important to note that

$$\mathbb{E}\left[\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i\right]\psi_r(S_i)\right] = \mathbb{E}\left[\mathbb{E}\left[V_{i+1}(S_{i+1})\psi_r(S_i)|S_i\right]\right]$$

since $\psi_r(S_i)$ is measurable with respect to S_i. Further by application of the tower property of conditional expectation

$$\mathbb{E}\left[\mathbb{E}\left[V_{i+1}(S_{i+1})\psi_r(S_i)|S_i\right]\right] = \mathbb{E}\left[V_{i+1}(S_{i+1})\psi_r(S_i)\right]$$

and so equation (2.4) becomes

$$\mathbb{E}\left[V_{i+1}(S_{i+1})\psi_r(S_i)\right] = \sum_{s=0}^{R} \beta_{is}\mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right] \tag{2.5}$$

Changing to matrix formulation, define $B_{\psi\psi}(i) = B_{\psi\psi}$ and $B_{V\psi}(i) = B_{V\psi}$ for each i, then

$$(B_{\psi\psi})_{r,s} = \mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right]$$

and

$$(B_{V\psi})_r = \mathbb{E}\left[V_{i+1}(S_{i+1})\psi_r(S_i)\right]$$

Then the set of equations (2.5) can be written as

$$B_{\psi\psi}\beta_i = B_{V\psi}$$

where $\beta_i^T = (\beta_{i0}, \beta_{i1}, \ldots, \beta_{iR})$ and by inverting we obtain

$$\beta_i = B_{\psi\psi}^{-1}B_{V\psi}$$

In the numerical approximation, we perform a Monte Carlo Simulation to obtain an estimation as follows:

$$(\widehat{B}_{V\psi})_r = \frac{1}{N}\sum_{p=1}^{N} V_{i+1}(S_{i+1}^p)\psi_r(S_i^p)$$

and

$$(\widehat{B}_{\psi\psi})_{r,s} = \frac{1}{N}\sum_{p=1}^{N} \psi_r(S_i^p)\psi_s(S_i^p)$$

Hence the least squares estimate of β_i is

$$\widehat{\beta}_i = \widehat{B}_{\psi\psi}^{-1}\widehat{B}_{V\psi}$$

I am now in a position to implement the algorithm:

Algorithm 1

(1) Begin by simulating N paths of M time steps.

(2) At the terminal node set $V_M = h_M(S_M)$.

(3) Employ backward recursion, performing the following steps at each time step i:

  (i) Evaluate $\widehat{\beta}_i = \widehat{B}_{\psi\psi}^{-1}(i)\widehat{B}_{V\psi}(i)$.

  (ii) Set the continuation value at time t_i for each path p, $p = 1, \ldots, N$ according to $V_i^C(S_i^p) = e^{-r\Delta t}\widehat{C}_i(S_i^p) = e^{-r\Delta t}\sum_{r=0}^{R}\widehat{\beta}_{ir}\psi_r(S_i^p)$

  (iii) Estimate the function value:

$$\widehat{V}_i^p = \begin{cases} h_i(S_i^p) & h_i(S_i^p) \geq V_i^C(S_i^p) \\ e^{-r\Delta t}V_{i+1}^p & h_i(S_i^p) < V_i^C(S_i^p) \end{cases}$$

(4) Average over the N paths in order to find an estimate for V_0 according to

$$V_0 = \frac{1}{N}\sum_{p=1}^{N}\widehat{V}_0^p$$

This process yields an estimate for β_i as well as a possible value for the price. Theoretically, the estimate for the price will tend to be biased low as a result of the fact that we are approximating an optimal stopping rule and any particular realisation will be suboptimal. However, if one uses identical paths for the regression and evaluation, this may result in the estimate being biased high. Consequently, to ensure the results match our intuition, one should use two sets of paths: one to estimate the continuation value (i.e in the regression step) and the other set for valuation. If we follow this approach then it is possible to work forwards in time.

The forwards approach entails the following specification:

Algorithm 2

(1) Determine $\widehat{\beta}_i$ for $i = 1, \ldots, M$ as discussed in Algorithm 1.

(2) Simulate another N paths of M time steps. It is essential that these paths are independent of the paths used to calculate the $\beta's$ in the regression step.

(3) Calculate for each path p, $V_i^C(S_i^p) = e^{-r\Delta t}\widehat{C}_i(S_i^p) = e^{-r\Delta t}\sum_{r=0}^{R}\widehat{\beta}_{ir}\psi_r(S_i^p)$

(4) Determine the optimal exercise date along each path p, $p = 1, \ldots, N$ according to $\widehat{\tau}^p = \min\left\{t_i : 1 \leq i \leq M,\ h_i(S_i^p) \geq V_i^C(S_i^p)\right\}$.

(5) Calculate V_0 by averaging over the N values according to
$V_0 = \frac{1}{N}\sum_{p=1}^{N} e^{-r\hat{\tau}^p} h_{\hat{\tau}^p}(S^p_{\hat{\tau}^p})$
where $h_{\hat{\tau}^p}$ represents the exercise value at time $\hat{\tau}^p$ and $S^p_{\hat{\tau}^p}$ represents the asset price in path p at time $\hat{\tau}^p$.

Remark 2.2.1. It is important to note that in the forward approach, step (3) and (4) from Algorithm (2) can be combined and once an exercise point is determined according to (4), it is possible to stop at that point instead of performing calculations on all further possible exercise dates.

As mentioned in [9] the Longstaff-Schwartz Algorithm is relatively fast and applicable to an extensive range of problems. The flexibility to choose the basis functions is a distinct advantage as it provides the opportunity to make use of specific knowledge or intuition about the problem one is considering. It is important to note however, that restrictions are placed on the basis functions as discussed in [4] and [10]. For our purposes, I follow the approach of [15] in which the set of weighted Laguerre polynomials is used. They are defined as

$$\psi_r(s) = e^{-\frac{s}{2}} e^s \frac{d^r}{r!ds^r}(s^r e^{-s}) \tag{2.6}$$

Another concept worth noting is that in [15], the regression is performed only when the option is in-the-money since the exercise decision is only relevant when the option is in-the-money. By in-the-money, I mean that at time t_i the regression is performed including paths such that $h_i(S^p_i) > 0$. It is important to note that paths that are excluded at time t_i may be included at another time t_j if at that point $h_j(S^p_j) > 0$. I follow this notion here, however, with regards to Reset Strike Put Options, the Option can be regarded as in-the-money when the current asset price is higher than the initial strike price K, that is $S^p_i > K$. It will then be optimal to shout at time t_i as shouting will result in an increase in the terminal payoff.

2.3 Longstaff-Schwartz: Extension to Shout Options

In the formulation of the Bermudan problem, the price of a Bermudan Option could be represented as

$$V_i(s) = \max(h_i(s), e^{-r\Delta t}\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i = s\right])$$

where $h_i(s)$ is the payoff of the option at time t_i and $e^{-r\Delta t}\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i = s\right]$ is the discounted expected value of continuing. In the case of a general Shout Option, one can write the following equation:

$$V_i(s) = \max(h_i(s), e^{-r\Delta t}\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i = s\right])$$

where

$$h_i(s) = \begin{cases} e^{-r(t_M - t_i)}\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i = s\right] & i < M \\ \max\left\{F(S_M, S_{M-1}, \ldots, S_0, K), G_M\right\} & i = M \end{cases}$$

where G_i is the payoff of the underlying option that is received at time $T = t_M$ from shouting at time t_i and F is the payoff if no shouting occurs during the life of the contract with some prespecified initial strike K. In particular, for the case of the Reset Strike Put Option

$$F(S_M, K) = (K - S_M)^+$$

Then the price of a Shout Option at time t_i can be represented as

$$V_i(s) = \max(e^{-r(t_M - t_i)}\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i = s\right], e^{-r\Delta t}\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i = s\right])$$

where G_i is the payoff of the underlying option that is received at time $T = t_M$ from shouting at time t_i. Specifically, if we consider the Reset Strike Put Option described in Chapter 1 where shouting occurs at time t_i when the share price is $S_i = s$ then

$$G_i(S_M, s) = (s - S_M)^+ \tag{2.7}$$

As before, the continuation value can be represented as

$$V_i^C(s) = e^{-r\Delta t}C_i(s) = e^{-r\Delta t}\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i = s\right]$$

and I define the shouting value as

$$V_i^{SH}(s) = e^{-r(t_M - t_i)}\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i = s\right]$$

and I obtain the following optimal shouting rule:

$$\tau = \begin{cases} \min\{t_i : 1 \le i \le M,\ V_i^{SH}(S_i) \ge V_i^C(S_i)\} & \text{if } \{i, V_i^{SH}(S_i) \ge V_i^C(S_i)\} \ne \emptyset \\ t_M & \text{otherwise} \end{cases}$$

In this case, the continuation value is the value of not shouting rather than shouting and changing the contract specified strike K to the price of the asset at the moment of shouting. Since both the shouting value and the continuation value involve conditional expectations, in this case it is necessary to approximate both the shouting value as well as the continuation value in order to obtain an approximate stopping rule. In an analogous fashion to the treatment of the Bermudan Option, I employ a least squares regression technique in order to estimate the shouting values. Specifically,

$$\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i = s\right] \approx \sum_{r=0}^{R} \alpha_{ir}\psi_r(s)$$

where $\psi_0, \psi_1, \ldots, \psi_R$ are $R+1$ basis functions and α_{ir} are the regression coefficients where $r = 0, 1, \ldots, R$. Again, using least squares, I estimate the α_{ir} coefficients by minimising

$$\mathbb{E}\left[\left(\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i\right] - \sum_{r=0}^{R} \alpha_{ir}\psi_r(S_i)\right)^2\right]$$

Taking the derivative with respect to α_{ir} and setting to zero yields

$$\mathbb{E}\left[\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i\right]\psi_r(S_i)\right] = \sum_{s=0}^{R} \alpha_{is}\mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right]$$

Since $\psi_r(S_i)$ is measurable with respect to S_i and using the tower rule of conditional expectation allows us to write

$$\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, , S_i)\psi_r(S_i)\right] = \sum_{s=0}^{R} \alpha_{is}\mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right]$$

Changing to matrix formulation, define $B_{\psi\psi}(i) = B_{\psi\psi}$ and $B_{G\psi}(i) = B_{G\psi}$ for each i, then it is possible to write

$$B_{\psi\psi}\alpha_i = B_{G\psi}$$

where

$$\alpha_i^T = \{\alpha_{i0}, \alpha_{i1}, \ldots, \alpha_{iR}\}$$

$$(B_{\psi\psi})_{r,s} = \mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right]$$

$$(B_{G\psi})_r = \mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)\psi_r(S_i)\right]$$

and by inverting we obtain

$$\alpha_i = B_{\psi\psi}^{-1}B_{G\psi}$$

In the numerical approximation, I obtain the following estimations

$$(\widehat{B}_{G\psi})_r = \frac{1}{N}\sum_{p=1}^{N} G_i(S_M^p, S_{M-1}^p, \ldots, S_i^p)\psi_r(S_i^p)$$

$$(\widehat{B}_{\psi\psi})_{r,s} = \frac{1}{N}\sum_{p=1}^{N} \psi_r(S_i^p)\psi_s(S_i^p)$$

and hence the least squares estimate of α_i is

$$\widehat{\alpha}_i = \widehat{B}_{\psi\psi}^{-1}\widehat{B}_{G\psi}$$

I am now in a position to implement the Algorithm:

Algorithm 1B

(1) Simulate N paths of M time steps.

(2) At the terminal node set $V_M = h_M(S_M)$.

(3) Employ backward recursion performing the following steps at et each time step i:

   (i) Evaluate $\widehat{\alpha}_i = \widehat{B}^{-1}_{\psi\psi}(i)\widehat{B}_{G\psi}(i)$

   (ii) Evaluate $\widehat{\beta}_i = \widehat{B}^{-1}_{\psi\psi}(i)\widehat{B}_{V\psi}(i)$

   (iii) Set the shouting value at time t_i for each path p, $p = 1, \ldots, N$ according to: $V_i^{SH}(S_i^p) = e^{-r(t_M - t_i)} \sum_{r=0}^{R} \widehat{\alpha}_{ir}\psi_r(S_i^p)$ and the continuation value at time t_i for each path p, $p = 1, \ldots, N$ according to: $V_i^C(S_i^p) = e^{-r\Delta t}\widehat{C}_i(S_i^p) = e^{-r\Delta t} \sum_{r=0}^{R} \widehat{\beta}_{ir}\psi_r(S_i^p)$

   (iv) Estimate the function value:

$$\widehat{V}_i^p = \begin{cases} e^{-r(t_M - t_i)} G_i(S_M^p, S_{M-1}^p, \ldots, S_i^p) & V_i^{SH}(S_i^p) \geq V_i^C(S_i^p) \\ e^{-r\Delta t} V_{i+1}^p & V_i^{SH}(S_i^p) < V_i^C(S_i^p) \end{cases}$$

(4) Average over the N paths in order to find an estimate for V_0 according to

$$V_0 = \frac{1}{N} \sum_{p=1}^{N} \widehat{V}_0^p$$

Following the same reasoning as discussed for the traditional Longstaff-Schwartz algorithm applied to Bermudan Options, if one uses the same paths for regression and evaluation, it may lead to the estimate being biased high. As such, it is necessary to generate another independent set of N paths of M time steps and perform the regression and evaluation steps separately by following the forwards procedure described in Algorithm 2B.

Algorithm 2B

(1) Determine $\widehat{\alpha}_i$ and $\widehat{\beta}_i$ for $i = 1, \ldots, M$ as discussed in Algorithm 1B.

(2) Simulate another N paths of M time steps. It is essential that these paths are independent of the paths used in the regression step above.

(3) Calculate $V_i^{SH}(S_i^p) = e^{-r(t_M - t_i)} \sum_{r=0}^{R} \widehat{\alpha}_{ir}\psi_r(S_i^p)$ and $V_i^C(S_i^p) = e^{-r\Delta t}\widehat{C}_i(S_i^p) = e^{-r\Delta t} \sum_{r=0}^{R} \widehat{\beta}_{ir}\psi_r(S_i^p)$ the shout and continuation values respectively at time t_i for each path p, $p = 1, \ldots, N$.

(4) Determine optimal shouting dates along each path p, $p = 1, \ldots, N$ according to

$$\widehat{\tau}^p = \begin{cases} \min\{t_i : 1 \leq i \leq M,\ V_i^{SH}(S_i^p) \geq V_i^C(S_i^p)\} & \text{if } \{i, V_i^{SH}(S_i^p) \geq V_i^C(S_i^p)\} \neq \emptyset \\ t_M & \text{otherwise} \end{cases}$$

(5) Calculate V_0 by averaging over the N values according to
$V_0 = \frac{1}{N}\sum_{p=1}^{N} e^{-rt_M} G_{\widehat{\tau}^p}(S_M^p, S_{M-1}^p, \ldots, S_{\widehat{\tau}^p}^p)$
where $S_{\widehat{\tau}^p}^p$ represents the asset price at time $\widehat{\tau}^p$ and $G_{\widehat{\tau}^p}$ represents the payoff of the underlying option at time $T = t_M$ when shouting occurs at time $\widehat{\tau}^p$.

Remark 2.3.1. The above analysis was conducted on a very general Shout Option with payoff of the underlying option received at maturity when shouting occurs at time t_i given by $G_i(S_M, S_{M-1}, \ldots, S_i)$. The dependence of the payoff on $S_M, S_{M-1}, \ldots, S_i$ does not impact the analysis in any way and as such the arguments can be carried out exactly as for the Bermudan Option case. Further, the identical arguments will apply to a Reset Strike Asian Option if you change the state space to $Y_i = (S_i, A_i)$ where $A_i = \frac{1}{i}\sum_{j=1}^{i} S_j$ and we note that Y_i is then a Markov Process. By replacing S_i by Y_i all the arguments follow and as such it is possible to apply the Longstaff-Schwartz Algorithm to the path dependent Reset Strike Asian Put Option.

Remark 2.3.2. In the standard Longstaff-Schwartz method, regression is performed only on the in-the-money paths. I incorporate this element in that I consider only the paths where shouting will be relevant. In particular, for the Reset Strike Put Option, I consider only paths where $S_i > K$.

2.4 Approximation Errors and Convergence Properties

[4] distinguishes two types of approximations that the Longstaff-Schwartz algorithm entails. The first type of approximation involves estimating the conditional expectations by projections on a finite set of functions drawn from a suitable basis. The second type of approximation uses Monte Carlo simulations and Least Squares Regression to estimate the value functions and is known as the Monte Carlo Procedure. [4] proves the convergence of the value function of the first type of approximation to the value function of the initial stopping problem as the number of basis functions tends to infinity. Unfortunately, in practice, it is not feasible to consider an infinite number of basis functions and this results in a low biased estimate. In addition, [4] proves that for a finite number of functions, the Monte Carlo Procedure converges almost surely to the value function of the first approximation.

In the Longstaff-Schwartz Method as applied to the American Option problem, there is a further source of error as a result of a discrete number of time steps. This is essentially the problem of approximating an American Option with a Bermudan Option. However, if you include a sufficiently large number of exercise opportunities, the Bermudan Option value will approximate the American Option value, however this too implies a suboptimal estimate. [10] considers situations in which the number of basis functions and number of paths increase together. They consider Brownian Motion or Geometric Brownian Motion underlying processes and deduce that the number of paths must grow exponentially with the number of basis functions in the first case and even faster in the second case. Thus, when implementing the algorithm, it is essential to use a sufficient number of paths in the regression step in order to obtain accurate and stable $\beta's$.

With regard to the Shout Option case, an additional source of error is introduced as in this case, I estimate the shouting value by projections on a finite set of functions taken from an appropriate basis. Then I use Monte Carlo simulations and both Least Squares Regression approximations to estimate the value functions. There has been no formal publication justifying the convergence of this approximation, and I do not provide a rigorous proof here, however the numerical results obtained in Chapter 3 do provide some justification of the convergence. This may, however be subject to further research and investigation.

As mentioned in [10] it must be noted that in low dimensions, solving the optimal stopping problem and evaluating the American Option price is relatively straightforward and many methods such as lattice approximations and partial differential equation formulations have been suggested which will be preferable to Longstaff-Schwartz algorithm in terms of cost and efficiency. However, in high dimensions, these methods become considerably more costly to implement, less tractable and efficient and it is in this case that the Longstaff-Schwartz Method demonstrates its value.

2.5 Longstaff-Schwartz: Extension to Swing Shout Options

As discussed in [8] and [16], it is possible to directly apply the Longstaff-Schwartz Algorithm to the valuation of a Swing Reset Strike Put Option. The swing part of the contract grants the owner the opportunity to shout up to a maximum of n times and set the strike to the asset price at the moment of shouting. The predominant difference to the consideration in which one shout opportunity is allowed (corresponding to the 1 Swing Reset Strike Shout Option) is that, when multiple shouts are allowed, the benefit from shouting is not only the discounted expected value from shouting but also the value of the option with one shout opportunity less than the current one. The existence of more than one shout opportunity introduces an additional dimension to the problem as one needs to consider the number of paths, the number of time steps and the number of shout rights left.

The price of the Swing Shout Option with j shout opportunities remaining can now be represented as:

$$V_{i,0}(S_i) = 0 \text{ for } i = 1, \ldots, M \text{ by convention,}$$
$$V_{M,j}(S_M) = h_{M,j}(S_M) \text{ for } j = 1, \ldots, n \text{ by convention}$$
$$V_{i,j}(s) = \max\left\{h_{i,j}(s) + e^{-r\Delta t}\mathbb{E}\left[V_{i+1,j-1}(S_{i+1})|S_i = s\right], e^{-r\Delta t}\mathbb{E}\left[V_{i+1,j}(S_{i+1})|S_i = s\right]\right\}$$

where $V_{i,j}(s)$ denotes the value of the Swing Option at time i, stock price S_i with j shout opportunities remaining and $h_{i,j}$ is the discounted expected value of shouting at time i with j shout opportunities left and is given as in Section 2.3. Hence, following from our usual treatment of Shout Options as outlined in Section 2.3, I can write

$$\begin{aligned} V_{i,j}(s) =& \max\Big\{e^{-r(t_M-t_i)}\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i = s\right] + e^{-r\Delta t}\mathbb{E}\left[V_{i+1,j-1}(S_{i+1})|S_i = s\right], \\ & e^{-r\Delta t}\mathbb{E}\left[V_{i+1,j}(S_{i+1})|S_i = s\right]\Big\} \\ =& \max\left\{V_i^{SH}(s) + V_{i,j-1}^C(s), V_{i,j}^C(s)\right\} \end{aligned}$$

where

$$V_{i,j}^C(s) = e^{-r\Delta t}\left[V_{i+1,j}(S_{i+1})|S_i = s\right] \approx e^{-r\Delta t}\sum_{r=0}^{R}\beta_{ijr}\psi_r(s)$$

Following the same approach as for the original description of the Longstaff-Schwartz method in Section 2.3, I obtain a similar representation to Equation (2.5)

$$\mathbb{E}\left[V_{i+1,j}(S_{i+1})\psi_r(S_i)\right] = \sum_{s=0}^{R}\beta_{ijs}\mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right]$$

and so in matrix formulation, by defining $B_{\psi\psi}(i,j) = B_{\psi\psi}$ and $B_{V\psi}(i,j) = B_{V\psi}$ for each i and for each j, then

$$(B_{\psi\psi})_{r,s} = \mathbb{E}\left[\psi_r(S_i)\psi_s(S_i)\right]$$

and

$$(B_{V\psi})_r = \mathbb{E}\left[V_{i+1,j}(S_{i+1})\psi_r(S_i)\right]$$

and so

$$B_{\psi\psi}\beta_{ij} = B_{V\psi}$$

where $\beta_{ij}^T = (\beta_{ij0}, \beta_{ij1}, \ldots, \beta_{ijR})$ and so

$$\beta_{ij} = B_{\psi\psi}^{-1}B_{V\psi}$$

Denote τ_j as the time when the $j-$th remaining shout opportunity is used, then it is possible to represent the optimal shouting policy for each path as:

$$\tau_n = \begin{cases} \min\{t_i : 1 \leq i \leq M-1,\ V_i^{SH} + V_{i,n-1}^C \geq V_{i,n}^C\} & \text{if the set is not empty} \\ t_M & \text{otherwise} \end{cases}$$

and

$$\tau_j = \begin{cases} \min\{t_i : t_i > \tau_{j+1}, 1 \leq i \leq M-1,\ V_i^{SH} + V_{i,j-1}^C \geq V_{i,j}^C\} & \text{if the set is not empty,} \\ t_M & \text{if empty and } \tau_{j+1} < t_M \\ t_{M+1} & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, n-1$, where $V_i^{SH} = V_i^{SH}(S_i)$, and $V_{i,k}^C = V_{i,k}^C(S_i)$ for $k = 1, \ldots, n$ and for technical reasons I define $h_{M+1}(S_{M+1}) = 0$. Thus it is possible to implement the Swing Shout Option within the Longstaff-Schwartz framework as follows:

Algorithm 1C

(1) Simulate N paths of M time steps.

(2) At the terminal node set $V_{M,j} = h_{M,j}(S_M)$ for $j = 1, \ldots, n$

(3) Employ backward recursion performing the following steps at et each time step i, $i = M, \ldots, 1$ for $j = 1, \ldots, n$:

  (i) Evaluate $\widehat{\alpha}_i = \widehat{B}_{\psi\psi}^{-1}(i)\widehat{B}_{G\psi}(i)$

  (ii) Evaluate $\widehat{\beta}_{ij} = \widehat{B}_{\psi\psi}^{-1}(i,j)\widehat{B}_{V\psi}(i,j)$

  (iii) Set the shouting value at time t_i for each path p, $p = 1, \ldots, N$ according to: $V_i^{SH}(S_i^p) = e^{-r(t_M - t_i)} \sum_{r=0}^R \widehat{\alpha}_{ir}\psi_r(S_i^p)$
  and the continuation value at time t_i for each path p, $p = 1, \ldots, N$ according to: $V_{i,j}^C(S_i^p) = e^{-r\Delta t}\widehat{C}_{i,j}(S_i^p) = e^{-r\Delta t}\sum_{r=0}^R \widehat{\beta}_{ijr}\psi_r(S_i^p)$

  (iv) Estimate the function value:

$$\widehat{V}_{i,j}^p = \begin{cases} e^{-r(t_M - t_i)}G_i(S_M^p, S_{M-1}^p, \ldots, S_i^p) + e^{-r\Delta t}V_{i+1,j-1}^p \\ \qquad \text{if } V_i^{SH}(S_i^p) + V_{i,j-1}^C(S_i^p) \geq V_{i,j}^C(S_i^p) \\ e^{-r\Delta t}V_{i+1,j}^p \qquad \text{if } V_i^{SH}(S_i^p) + V_{i+1,j-1}^C(S_i^p) < V_{i,j}^C(S_i^p) \end{cases}$$

(4) Average over the N paths in order to find an estimate for $V_{0,n}$ according to

$$V_{0,n} = \frac{1}{N}\sum_{p=1}^N \widehat{V}_{0,n}^p$$

It is then possible to work forwards in time according to the following Algorithm:

Algorithm 2C

(1) Determine $\widehat{\alpha}_i$ and $\widehat{\beta}_{ij}$ for $i = 1, \dots, M$ and $j = 1, \dots, n$ as discussed in Algorithm 1C.

(2) Simulate another N paths of M time steps. It is essential that these paths are independent of the paths used in the regression step above.

(3) Calculate $V_i^{Sh}(S_i^p) = e^{-r(t_M - t_i)} \sum_{r=0}^{R} \widehat{\alpha}_{ir} \psi_r(S_i^p)$ and
$V_{i,j}^{C}(S_i^p) = e^{-r\Delta t} \widehat{C}_{i,j}(S_i^p) = e^{-r\Delta t} \sum_{r=0}^{R} \widehat{\beta}_{ijr} \psi_r(S_i^p)$
the shout and continuation values respectively at time t_i for p, $p = 1, \dots, N$.

(4) Determine optimal shouting dates along each path p, $p = 1, \dots, N$ according to

$$\widehat{\tau}_n^p = \begin{cases} \min\{t_i : 1 \le i \le M-1,\ V_i^{SH} + V_{i,n-1}^C \ge V_{i,n}^C\} & \text{if the set is not empty} \\ t_M & \text{otherwise} \end{cases}$$

$$\widehat{\tau}_j^p = \begin{cases} \min\{t_i : t_i > \widehat{\tau}_{j+1}^p, 1 \le i \le M-1,\ V_i^{SH} + V_{i,j-1}^C \ge V_{i,j}^C\} & \text{if the set is not empty,} \\ t_M & \text{if empty and } \tau_{j+1} < t_M \\ t_{M+1} & \text{otherwise} \end{cases}$$

for $j = 1, \dots, n-1$, where $V_i^{SH} = V_i^{SH}(S_i^p)$ and $V_{i,k}^C = V_{i,k}^C(S_i^p)$ for $k = 1, \dots, n$.

(5) Calculate V_0 by averaging over the N values according to
$V_{0,n} = \frac{1}{N} \sum_{p=1}^{N} \widehat{V}_{0,n}^p$
where $\widehat{V}_{0,n}^p$ denotes the value of the option in path p at time zero when there are n shout opportunities remaining. It is given by
$\widehat{V}_{0,1}^p = e^{-rt_M} G_{\widehat{\tau}_1^p}(S_M^p, S_{M-1}^p, \dots, S_{\widehat{\tau}_1^p}^p)$ and
$\widehat{V}_{0,j}^p = \left\{ e^{-rt_M} G_{\widehat{\tau}_j^p}(S_M^p, S_{M-1}^p, \dots, S_{\widehat{\tau}_j^p}^p) + V_{0,j-1}^p \right\}$ recursively for $j = 2, \dots, n$ where
$S_{\widehat{\tau}^p}^p$ denotes the asset price in path p at time $\widehat{\tau}^p$ and $G_{\widehat{\tau}^p}$ denotes the payoff of the underlying option at time T when shouting occurs at time $\widehat{\tau}^p$ and $G_{M+1} = 0$ by convention.

2.6 Estimating the Greeks within the Longstaff-Schwartz Framework

In this section, I outline how the standard techniques from estimating Greeks in a Monte Carlo Simulation can be applied within the Longstaff-Schwartz framework. I consider both the Pathwise Sensitivity approach and the Likelihood Ratio method. The ideas for the section follow [17].

2.6.1 Pathwise Sensitivity Approach

A general Shout option can be described by

$$V_i(S_i) = \max(h_i(S_i), e^{-r\Delta t}\mathbb{E}\left[V_{i+1}(S_{i+1})|S_i\right]) \tag{2.8}$$

where

$$h_i(s) = \begin{cases} e^{-r(t_M - t_i)}\mathbb{E}\left[G_i(S_M, S_{M-1}, \ldots, S_i)|S_i = s\right] & i < M \\ \max\left\{F(S_M, S_{M-1}, \ldots, S_0, K), G_M\right\} & i = M \end{cases}$$

where $G_i(S_M, S_{M-1}, \ldots, S_i)$ is the payoff of the underlying option received at time $T = t_M$ when shouting occurs at time t_i and $F(S_M, S_{M-1}, \ldots, S_0, K)$ is the payoff at maturity if no shouting occurs during the life of the contract. $h_i(S_i)$ is regarded as the discounted expected value of shouting at time t_i or the original payoff of the option if no shouting occurs during the life of the option and S_i denotes the asset price at time t_i. Equation (2.8) implies that the option value depends only on the stock price at time t_i. $(S_i)_{i\geq 0}$ is a Markov process and as such, both the conditional expectations depend only on S_i and not $S_0, S_1, \ldots, S_{i-1}$. Consequently, backward recursion implies that $V_i(S_i)$ depends only on S_i.

Following the approach in [9] and [17], let θ be a parameter. We consider that $S_i(\theta)$ is the share price as a function of parameter θ. $S_i(\theta)$ is a stochastic process indexed by θ. Then we can write the option price as

$$\begin{aligned} V_0(S_0(\theta), \theta) = \mathbb{E}\big[& e^{-rt_1}h_1(S_1(\theta))\mathbf{1}_{\{h_1(S_1(\theta))\geq V_1(S_1(\theta),\theta)\}} \\ &+ e^{-rt_2}h_2(S_2(\theta))\mathbf{1}_{\{h_1(S_1(\theta))<V_1(S_1(\theta),\theta)\}}\mathbf{1}_{\{h_2(S_2(\theta))\geq V_2(S_2(\theta),\theta)\}} \\ &+ \ldots + e^{-rt_M}h_M(S_M(\theta))\mathbf{1}_{\{h_1(S_1(\theta))<V_1(S_1(\theta),\theta)\}}\cdots \\ &\mathbf{1}_{\{h_{M-1}(S_{M-1}(\theta))<V_{M-1}(S_{M-1}(\theta),\theta)\}}\mathbf{1}_{\{h_M(S_M(\theta))>0\}}\big] \end{aligned}$$

It is important to remark that if we can consider that $(S_i(\theta))_{i\geq 0}$ is almost surely continuous with respect to θ for each i, then on each path $\exists\ \epsilon(\omega)$ such that

$$\mathbf{1}_{\{h_i(S_i(\theta))<V_i(S_i(\theta),\theta)\}} = \mathbf{1}_{\{h_i(S_i(\hat{\theta}))<V_i(S_i(\hat{\theta}),\hat{\theta})\}}$$

so long as h_i is continuous and $|\theta - \hat{\theta}| < \epsilon$ where ϵ is small. Then it will be possible to estimate unbiased sensitivities provided one can interchange differentiation and expectation, the justification of which is discussed extensively in [9]. I do not provide a theoretical proof for this interchangeability in the case of the particular examples considered in the next chapter, however, my numerical results do suggest that this assumption is valid. Hence, one can write for each i

$$\frac{\partial}{\partial\theta}\mathbb{E}\left[e^{-rt_i}h_i(S_i(\theta))\mathbf{1}_{F_i}\right] = e^{-rt_i}\mathbb{E}\left[\mathbf{1}_{F_i}\frac{\partial}{\partial\theta}h_i(S_i(\theta))\right]$$

where

$F_i = \{h_j(S_j(\theta)) < V_j(S_j(\theta),\theta), j = 1,\dots,i-1 \text{ and } h_i(S_i(\theta)) \geq V_i(S_i(\theta),\theta)\}$. Then

$$\frac{\partial V_0(S_0(\theta),\theta)}{\partial\theta} = \mathbb{E}\left[\sum_{i=1}^{M} e^{-rt_i}\frac{\partial h_i(S_i(\theta))}{\partial\theta}\mathbf{1}_{F_i}\right]$$

Now

$$\begin{aligned}\frac{\partial}{\partial\theta}h_i(S_i(\theta)) &= \frac{\partial}{\partial\theta}\left(e^{-r(t_M-t_i)}\mathbb{E}\left[G_i(S_M(\theta),S_{M-1}(\theta),\dots,S_i(\theta))|S_i=s\right]\right)\\ &= e^{-r(t_M-t_i)}\mathbb{E}\left[\frac{\partial}{\partial\theta}G_i(S_M(\theta),S_{M-1}(\theta),\dots,S_i(\theta))|S_i=s\right]\\ &= e^{-r(t_M-t_i)}\mathbb{E}\left[\frac{\partial G_i}{\partial S_M}\frac{\partial S_M}{\partial\theta}+\dots+\frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial\theta}|S_i=s\right]\end{aligned}$$

where it is assumed that $G_i = G_i(S_M(\theta), S_{M-1}(\theta),\dots S_i(\theta))$ and so

$$\begin{aligned}\frac{\partial V_0(S_0(\theta),\theta)}{\partial\theta} &= \mathbb{E}\left[\sum_{i=1}^{M} e^{-rt_i}\frac{\partial h_i(S_i(\theta))}{\partial\theta}\mathbf{1}_{F_i}\right]\\ &= \mathbb{E}\left[\sum_{i=1}^{M} e^{-rt_i}e^{-r(t_M-t_i)}\mathbb{E}\left[\frac{\partial G_i}{\partial S_M}\frac{\partial S_M}{\partial\theta}+\dots+\frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial\theta}|S_i=s\right]\mathbf{1}_{F_i}\right]\\ &= e^{-rt_M}\sum_{i=1}^{M}\mathbb{E}\left[\mathbb{E}\left[\frac{\partial G_i}{\partial S_M}\frac{\partial S_M}{\partial\theta}+\dots+\frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial\theta}|S_i=s\right]\mathbf{1}_{F_i}\right]\\ &= e^{-rt_M}\sum_{i=1}^{M}\mathbb{E}\left[\left(\frac{\partial G_i}{\partial S_M}\frac{\partial S_M}{\partial\theta}+\dots+\frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial\theta}\right)\mathbf{1}_{F_i}\right]\end{aligned}$$

where the last equality is due to the Markov property of $(S_i)_{i\geq 0}$, the fact that F_i is measurable with respect to S_i and the tower property of conditional expectation.

Consider the Reset Strike Put Option in which G_i, the payoff of the option at time T from shouting at time t_i corresponds to equation (2.7) and is given by $G_i = (S_i - S_M)^+$. Then

$$\begin{aligned}\frac{\partial G_i}{\partial S_0} &= \frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial S_0}+\frac{\partial G_i}{\partial S_M}\frac{\partial S_M}{\partial S_0}\\ &= \mathbf{1}_{\{S_i>S_M\}}\frac{\partial S_i}{\partial S_0}-\mathbf{1}_{\{S_i>S_M\}}\frac{\partial S_M}{\partial S_0}\end{aligned}$$

Now $S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$ since it is assumed the share price follows Geometric Brownian Motion and so

$$\frac{\partial S_t}{\partial S_0} = e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t} = \frac{S_t}{S_0} \tag{2.9}$$

and hence

$$\frac{\partial V_0(S_0)}{\partial S_0} = \Delta(S_0) = e^{-rt_M} \sum_{i=1}^{M} \mathbb{E}\left[\left(\mathbf{1}_{\{S_i>S_M\}}\frac{S_i}{S_0} - \mathbf{1}_{\{S_i>S_M\}}\frac{S_M}{S_0}\right)\mathbf{1}_{F_i}\right]$$

For the Reset Strike Asian Option, the payoff received at maturity when shouting occurs at time t_i is given by $G_i = (S_i - \bar{S})^+$ where $\bar{S} = \frac{1}{M}\sum_{i=1}^{M} S_i$. Remark 2.3.1 remains true and as such, all the above arguments for the general shout option can be applied to calculate the Greeks of the Reset Strike Asian Option.

$$\begin{aligned}\frac{\partial G_i}{\partial S_0} &= \frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial S_0} + \frac{\partial G_i}{\partial \bar{S}}\frac{\partial \bar{S}}{\partial S_0} \\ &= \mathbf{1}_{\{S_i>\bar{S}\}}\frac{\partial S_i}{\partial S_0} - \mathbf{1}_{\{S_i>\bar{S}\}}\frac{\partial \bar{S}}{\partial S_0}\end{aligned}$$

From equation (2.9), $\frac{\partial S_i}{\partial S_0} = \frac{S_i}{S_0}$ and similarly

$$\frac{\partial \bar{S}}{\partial S_0} = \frac{1}{M}\sum_{i=1}^{M}\frac{\partial S_i}{\partial S_0} = \frac{1}{M}\sum_{i=1}^{M}\frac{S_i}{S_0} = \frac{\bar{S}}{S_0}$$

Hence

$$\begin{aligned}\Delta_{\text{Asian}}(S_0) &= e^{-rt_M}\sum_{i=1}^{M}\mathbb{E}\left[\frac{\partial G_i}{\partial S_0}\mathbf{1}_{F_i}\right] \\ &= e^{-rt_M}\sum_{i=1}^{M}\mathbb{E}\left[\left(\mathbf{1}_{\{S_i>\bar{S}\}}\frac{S_i}{S_0} - \mathbf{1}_{\{S_i>\bar{S}\}}\frac{\bar{S}}{S_0}\right)\mathbf{1}_{F_i}\right]\end{aligned}$$

For calculating Vega of the Reset Strike Put Option:

$$\begin{aligned}\frac{\partial G_i}{\partial \sigma} &= \frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial \sigma} + \frac{\partial G_i}{\partial S_M}\frac{\partial S_M}{\partial \sigma} \\ &= \mathbf{1}_{\{S_i>S_M\}}\frac{\partial S_i}{\partial \sigma} - \mathbf{1}_{\{S_i>S_M\}}\frac{\partial S_M}{\partial \sigma}\end{aligned}$$

where $\frac{\partial S_t}{\partial \sigma} = (W_t - \sigma t)S_t$. Hence,

$$\begin{aligned}\text{Vega}(S_0) &= e^{-rt_M}\sum_{i=1}^{M}\mathbb{E}\left[\frac{\partial G_i}{\partial \sigma}\mathbf{1}_{F_i}\right] \\ &= e^{-rt_M}\sum_{i=1}^{M}\mathbb{E}\left[\left(\mathbf{1}_{\{S_i>S_M\}}(W_{t_i} - \sigma t_i)S_i - \mathbf{1}_{\{S_i>S_M\}}(W_{t_M} - \sigma t_M)S_M\right)\mathbf{1}_{F_i}\right]\end{aligned}$$

For calculating Vega of the Reset Strike Asian Put Option:

$$\begin{aligned}\frac{\partial G_i}{\partial \sigma} &= \frac{\partial G_i}{\partial S_i}\frac{\partial S_i}{\partial \sigma} + \frac{\partial G_i}{\partial \bar{S}}\frac{\partial \bar{S}}{\partial \sigma} \\ &= \mathbf{1}_{\{S_i>\bar{S}\}}\frac{\partial S_i}{\partial \sigma} - \mathbf{1}_{\{S_i>\bar{S}\}}\frac{\partial \bar{S}}{\partial \sigma}\end{aligned}$$

where $\frac{\partial \bar{S}}{\partial \sigma} = \frac{1}{M}\sum_{i=1}^{M} \frac{\partial S_i}{\partial \sigma}$ and so

$$\begin{aligned} \text{Vega}_{\text{Asian}}(S_0) &= e^{-rt_M} \sum_{i=1}^{M} \mathbb{E}\left[\frac{\partial G_i}{\partial \sigma} \mathbf{1}_{F_i}\right] \\ &= e^{-rt_M} \sum_{i=1}^{M} \mathbb{E}\left[\left(\mathbf{1}_{\{S_i > \bar{S}\}} \frac{\partial S_i}{\partial \sigma} - \mathbf{1}_{\{S_i > \bar{S}\}} \frac{\partial \bar{S}}{\partial \sigma}\right) \mathbf{1}_{F_i}\right] \end{aligned}$$

The above arguments hold for the calculation of Delta and Vega. The consideration of Rho of the Reset Strike Put Option is marginally more complex:

$$\begin{aligned} \rho(S_0) = \frac{\partial V_0(S_0(r), r)}{\partial r} &= \frac{\partial}{\partial r}\mathbb{E}\left[\sum_{i=1}^{M} e^{-rt_i} h_i(S_i(r)) \mathbf{1}_{F_i}\right] \\ &= \frac{\partial}{\partial r}\mathbb{E}\left[\sum_{i=1}^{M} e^{-rt_i} e^{-r(t_M - t_i)} G_i(S_M(r), S_i(r)) \mathbf{1}_{F_i}\right] \\ &= \sum_{i=1}^{M} \mathbb{E}\left[\frac{\partial}{\partial r}\left(e^{-rt_i} e^{-r(t_M - t_i)} G_i(S_M(r), S_i(r))\right) \mathbf{1}_{F_i}\right] \\ &= \sum_{i=1}^{M} \mathbb{E}\left[\frac{\partial}{\partial r}\left(e^{-rt_M}(S_i - S_M)^+\right) \mathbf{1}_{F_i}\right] \\ &= \sum_{i=1}^{M} \mathbb{E}\left[\left(-t_M e^{-rt_M}(S_i - S_M)^+ \right.\right. \\ &\left.\left. + e^{-rt_M}\left(\mathbf{1}_{\{S_i > S_M\}} \frac{\partial S_i}{\partial r} - \mathbf{1}_{\{S_i > S_M\}} \frac{\partial S_M}{\partial r}\right)\right) \mathbf{1}_{F_i}\right] \\ &= \sum_{i=1}^{M} \mathbb{E}\left[\left(-t_M e^{-rt_M}(S_i - S_M)^+ + \right.\right. \\ &\left.\left. e^{-rt_M}\left(\mathbf{1}_{\{S_i > S_M\}} t_i S_i - \mathbf{1}_{\{S_i > S_M\}} t_M S_M\right)\right) \mathbf{1}_{F_i}\right] \end{aligned}$$

since $\frac{\partial S_t}{\partial r} = tS_t$.

For the Rho of the Reset Strike Asian Put Option, we have a similar representation:

$$\begin{aligned} \rho_{\text{Asian}}(S_0) &= \sum_{i=1}^{M} \mathbb{E}\left[\left(-t_M e^{-rt_M}(S_i - \bar{S})^+ + e^{-rt_M}\left(\mathbf{1}_{\{S_i > \bar{S}\}} \frac{\partial S_i}{\partial r} - \mathbf{1}_{\{S_i > \bar{S}\}} \frac{\partial \bar{S}}{\partial r}\right)\right) \mathbf{1}_{F_i}\right] \\ &= \sum_{i=1}^{M} \mathbb{E}\left[\left(-t_M e^{-rt_M}(S_i - \bar{S})^+ + e^{-rt_M}\left(\mathbf{1}_{\{S_i > \bar{S}\}} t_i S_i - \mathbf{1}_{\{S_i > \bar{S}\}} \frac{1}{M}\sum_{i=1}^{M} t_i S_i\right)\right) \mathbf{1}_{F_i}\right] \end{aligned}$$

since $\frac{\partial \bar{S}}{\partial r} = \frac{1}{M}\sum_{i=1}^{M} t_i S_i$.

The primary drawback of the Pathwise Sensitivity Approach is that the payoffs are required to be differentiable. Furthermore, when considering the Gamma, one requires the payoff to be twice differentiable. Obviously, the Reset Strike Put Option and the Reset Strike Asian Put Option do not satisfy this constraint, and as such, one needs to consider an alternative approach in order to determine Gamma.

2.6.2 Likelihood Ratio Method

The Likelihood Ratio Method provides an alternative approach to Pathwise Sensitivity. It does not require continuity or differentiability of the payoffs as it involves differentiating probabilities as opposed to payoffs. Using the arguments presented in the section 2.6.1, I can write

$$\begin{aligned} V_0(S_0) &= \mathbb{E}\left[e^{-r\Delta t}V_1(S_1)|S_0\right] \\ &= \int e^{-r\Delta t}V_1(S_1)p(S_0,S_1)dS_1 \end{aligned}$$

where $p(S_0, S_1)$ is the transition density of (S_0, S_1). Setting $H(S_1) = e^{-r\Delta t}V_1(S_1)$ I write

$$V_0(S_0) = \int H(S_1)p(S_0,S_1)dS_1$$

This method relies on the interchange of integration and differentiation. As discussed in [9], probability density functions are typically smooth functions of the parameters and hence this rarely hinders the use of the Likelihood Ratio Method. Thus I can write

$$\begin{aligned} \frac{\partial V_0(S_0)}{\partial\theta} &= \frac{\partial}{\partial\theta}\int H(S_1)p(S_0,S_1)dS_1 \\ &= \int H(S_1)\frac{\partial p(S_0,S_1)}{\partial\theta}dS_1 \\ &= \int H(S_1)\frac{\partial \log p(S_0,S_1)}{\partial\theta}p(S_0,S_1)dS_1 \\ &= \mathbb{E}\left[H(S_1)\frac{\partial \log p(S_0,S_1)}{\partial\theta}\right] \end{aligned}$$

Hence

$$\Delta(S_0) = \frac{\partial V_0}{\partial S_0} = \mathbb{E}\left[H(S_1)\frac{\partial \log p(S_0,S_1)}{\partial S_0}\right] \tag{2.10}$$

Denoting $p(S_0, S_1)$ as p, to calculate second derivatives I use

$$\frac{\partial^2 \log p}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{1}{p}\frac{\partial p}{\partial\theta}\right) = \frac{1}{p}\frac{\partial^2 p}{\partial\theta^2} - \frac{1}{p}\left(\frac{\partial p}{\partial\theta}\right)^2$$

and this implies $\frac{1}{p}\left(\frac{\partial p}{\partial\theta}\right)^2 = \frac{\partial^2 \log p}{\partial\theta^2} + \left(\frac{\partial \log p}{\partial\theta}\right)^2$. Hence

$$\begin{aligned} \Gamma(S_0) &= \int H(S_1)\left[\frac{\partial^2 \log p(S_0,S_1)}{\partial S_0^2} + \left(\frac{\partial \log p}{\partial S_0}\right)^2\right]p(S_0,S_1)dS_1 \\ &= \mathbb{E}\left[H(S_1)\left(\frac{\partial^2 \log p(S_0,S_1)}{\partial S_0^2} + \left(\frac{\partial \log p}{\partial S_0}\right)^2\right)\right] \end{aligned} \tag{2.11}$$

For the usual Geometric Brownian Motion, I can write the lognormal probability distribution as

$$p(S_0, S_1) = \frac{1}{\sqrt{2\pi\Delta t\sigma^2}}\frac{1}{S_1}\exp\left(-\frac{(\log\frac{S_1}{S_0} - (r - \frac{1}{2}\sigma^2)\Delta t)^2}{2\sigma^2\Delta t}\right)$$

Consequently

$$\log p(S_0, S_1) = -\log S_1 - \log\sigma - \frac{1}{2}\log(2\pi\Delta t) - \frac{1}{2}\frac{\left(\log(\frac{S_1}{S_0}) - (r - \frac{1}{2}\sigma^2)\Delta t\right)^2}{\sigma^2\Delta t}$$

and so

$$\frac{\partial \log p}{\partial S_0} = \frac{\log(S_1) - \log(S_0) - (r - \frac{1}{2}\sigma^2)\Delta t}{S_0\sigma^2\Delta t} \tag{2.12}$$

and

$$\frac{\partial^2 \log p}{\partial S_0^2} = -\frac{1 + \log(S_1) - \log(S_0) - (r - \frac{1}{2}\sigma^2)\Delta t}{S_0^2\sigma^2\Delta t} \tag{2.13}$$

Thus, one can easily consider the Delta and Gamma for the Reset Strike Put Option, the Reset Strike Asian Put Option and the Swing Reset Strike Put Option by inserting equations (2.12) and (2.13) into (2.10) and (2.11) where appropriate.

2.7 Control Method: Binomial Tree pricing of Shout Options

In order to provide some measure of comparison, I present the Binomial Tree Method for the pricing of Reset Strike Put Options. The Binomial Tree Method of pricing options has become a standard technique in the literature and was first introduced by [5]. I will outline the basic method as applied to Shout Option pricing. The time T is divided into M_1 steps, each of size $\Delta t = \frac{T}{M_1}$ and a tree of asset prices is generated by assuming that at each step t_m the asset price S_m can move up to $S_m u$ or down to $S_m d$ with probabilities p and $(1-p)$ respectively. The parameters u and p are chosen so as to match the first and second moments of a Geometric Brownian Motion distribution with drift r and volatility σ. One choice, common in the literature is $u = e^{\sigma \Delta t}$, $d = \frac{1}{u}$ and $p = \frac{e^{r\Delta t} - d}{u-d}$. It is possible to shout on M dates $\{\tau_1, \tau_2, \ldots, \tau_M\} \subseteq \{t_1, t_2, \ldots, t_{M_1}\}$. The option value is then determined by working backwards according to the following equation

$$V_M(S_{M,i}) = (K - S_{M,i})^+$$

$$V_{m-1}(S_{m-1,i}) = \begin{cases} \max(e^{-r\Delta t}(pV_m(uS_{m-1,i}) + (1-p)V_m(dS_{m-1,i})), & \\ e^{-r(t_M - t_{m-1})}G_{m-1}(S_{M,i}, S_{m-1,i})) & m \in \{\tau_1, \tau_2, \ldots, \tau_M\} \\ e^{-r\Delta t}(pV_m(uS_{m-1,i}) + (1-p)V_m(dS_{m-1,i})) & m \notin \{\tau_1, \tau_2, \ldots, \tau_M\} \end{cases}$$

where G_{m-1} represents the payoff of the option received at T from shouting at time t_{m-1}. For the case of the Strike Reset Put Option, the shout value given by $e^{-r(t_M - t_{m-1})}G_{m-1}(S_{M,i}, S_{m-1,i})$ is the value of an at-the-money put and can be valued using the at-the-money Black-Scholes value for a put as given by the following formula

$$e^{-r(t_M - t_{m-1})}G_{m-1}(S_M, S_{m-1}) = e^{-r(t_M - t_{m-1})}S_{m-1}N(-d_-) - S_{m-1}N(-d_+)$$

where $d\pm = (r \pm \frac{1}{2}\sigma^2)\frac{\sqrt{t_M - t_{m-1}}}{\sigma}$ and $N(\cdot)$ represents the cumulative normal density function. As another, related control case, I can use the Black-Scholes at-the-money value as described above instead of generating the $\alpha's$ as described in Section 2.3 and then implement the Longstaff-Schwartz algorithm to obtain the $\beta's$ and values as discussed in Section 2.3. This control case gives me the opportunity to consider the impact or the error introduced from estimating the shout value using the least squares regression technique. It is important to mention that, although in this case we have a closed form solution for the shout value at each step, which in this case is linear in S_{m-1}, in general this will not be the case and as such it is interesting to analyse the error that the additional regression estimation produces.

Remark 2.7.1. Asian Option pricing cannot easily be performed using the Binomial Tree Approach as the path dependency of the payoff causes difficulties within the Binomial Tree framework. As such, I do not apply any control case to the examination of the Strike Reset Asian Put Option.

Chapter 3

Implementation of Longstaff-Schwartz Algorithm and Numerical Results

3.1 Implementation and Results of Longstaff-Schwartz applied to Reset Strike Put Option

I executed the Longstaff-Schwartz Algorithm applied to a Reset Strike Put Option according to the technique outlined in Section 2.3 in MATLAB. I considered a contract with the following specifications:

- Initial Asset price $S_0 = 36$
- Initial Strike price $K = 40$
- Interest rate $r = 0.06$
- Maturity $T = 1$
- Volatility $\sigma = 0.2$

I performed the analysis with various underlying test functions and concluded that the weighted Laguerre Polynomials as described in equation (2.6) were the most suitable. Further, I examined the weighted Laguerre Polynomials for $R = 2, 3, 4, 5, 6$ in order to determine the most appropriate for the problem at hand. I concluded that using 4 basis functions is sufficient in this model in that the estimates are close to the control case and have small errors. It is possible to add additional explanatory basis functions, however this does not change the prices in a meaningful manner so as to justify the inclusion of these basis functions. In addition, I considered applying different basis functions to estimate the $\alpha's$ and $\beta's$. This however had very little impact and as such, to maintain tractability, I consider here the same basis functions for the regressions of the $\alpha's$ and the $\beta's$. I did consider incorporating the payoff as one of the basis functions, however, contrary to intuition, this yielded inferior results and tended to vastly overestimate the value of the option. As is consistent with [15] and will be shown in the convergence results below, 100000 paths was found to produce satisfactory convergence in that it generated a relatively small Monte Carlo Error(MCE) which is defined in the literature as

$$MCE = \sqrt{\frac{\sigma^2}{N}} \tag{3.1}$$

where

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(V_n(0) - \frac{1}{N}\sum_{i=0}^{N} V_i(0))^2$$

Table 3.1 below considers convergence results for the value of a Reset Strike Put Option evaluated by Longstaff-Schwartz. I consider $M = 10$ and $N = 10^2, 10^3, 10^4, 10^5, 10^6$ for $R = 2, 4, 6$ in order to examine convergence. The same set of paths were used for $R = 2, 4, 6$. I have also reported the Monte Carlo errors as defined in equation (3.1) in parentheses next to the values. Convergence is apparent as the number of paths increase. The Monte Carlo errors seem reasonable and tend to decrease significantly as the number of paths increase. Further, the results appear to be consistent in that the values reported for fewer paths tend to be within ±3MCE bands of the more accurate value based on the next increased number of paths. The Monte Carlo errors appear to be very similar for different R's but the same N. This corroborates the results obtained in [17]. Further,it appears to support [10] which states that when more basis functions are used, it is necessary to increase the number of paths in order to obtain a given level of accuracy. This is apparent from Table 3.1 as the estimate for $R = 6$ with $N = 1000$ is close to the values obtained for $R = 2$ and $R = 4$ with $N = 100$. Similarly, the estimate when $R = 6, N = 10000$ is closer to the values obtained for $R = 2, 4$ and $N = 1000$ than $R = 2, 4$ and $N = 10000$. This trend proceeds until a large number of paths, $N = 1000000$ is employed at which point the estimates do appear to converge to similar values. Further, one notices that increasing the number of basis functions beyond $R = 4$, when N is large does not contribute significantly to the results. In fact, when the number of basis functions increases beyond $R = 5$, it is possible for the matrices to become singular when using $N = 100000$. This may destabilize the β values. Hence, one would be forced to increase the number of paths in this case in order to maintain stability of the results. This increases computational cost. Figure 3.1 shows the convergence result with the Monte Carlo error bounds for $R = 2$,$R = 4$ and $R = 6$.

Remark 3.1.1. The variance, represented by σ^2 is conditional to a particular set of $\alpha' s$ and $\beta' s$. An even more interesting test that one might consider is to repeat this many times in order to obtain an estimate of the total variance. However, this will be considerably more costly and I do not report any values for this here as I lacked the computational capacity to compute this. However, I do propose this as a possible suggestion for further research.

Table 3.1: Convergence Results for value of Reset Strike Put Option by Longstaff-Schwartz, $M = 10$, Binomial Tree Estimate = 4.2914

N	R=2	R=4	R=6
100	4.3762(0.4397)	4.4378(0.4362)	4.5406(0.4796)
1000	4.3588(0.1389)	4.3358(0.1371)	4.4080(0.1382)
10000	4.2825(0.0421)	4.2773(0.0421)	4.3748(0.0426)
100000	4.2800(0.0133)	4.2831(0.0133)	4.2855(0.0133)
1000000	4.2874(0.0042)	4.2894(0.0042)	4.2899(0.0042)

Table 3.2 considers the impact of increasing the number of time steps. In this case, I consider $N = 100000$ and examine the impact of $M = 10, 50, 100$ for $R = 2, 4, 6$. It is apparent from the table that increasing the number of time steps used in the simulation increases the value. Since the Longstaff-Schwartz Algorithm produces suboptimal estimates, this increase in the price is advantageous in this case. However, it is important to weigh up the benefit of obtaining marginally increased accuracy with additional cost and memory required by increasing the number of paths.

Table 3.2: Impact of Number of time steps, $N = 100000$

M	R=2	R=4	R=6	Binomial Estimate
10	4.2800(0.0133)	4.2831(0.0133)	4.2855(0.0133)	4.2914
50	4.3092(0.0133)	4.3187(0.0133)	4.3202(0.0133)	4.3246
100	4.3204(0.0133)	4.3280(0.0133)	4.3286(0.0133)	4.3287

Table 3.3 shows the value of a Reset Strike Put Option obtained using Longstaff-Schwartz Method and compares it to the value obtained by the two control cases for five different initial seeds of the random number generator as represented by the five rows in the table. The first column is the value obtained using pure Longstaff-Schwartz Algorithm. The second column is the value obtained using a Longstaff-Schwartz Method using the at-the-money Black-Scholes Value for the shout value at each time step. The Longstaff-Schwartz Algorithms use $N = 100000$, $M = 100$ and $R = 4$. The Binomial approximation has 1000 time steps, with the same $M = 100$ possible shout dates as in the Longstaff-Schwartz Algorithm and produces an estimate of 4.3287. The parentheses hold Monte Carlo Error Values where appropriate. The values obtained appear to be relatively close to the control case. Further, the estimates obtained from the Binomial control case do fall within ± 3 Monte Carlo error bounds of the Longstaff-Schwartz Algorithm values. The third column, labelled Error1 reports the difference between the Binomial Tree estimate and the value obtained using the

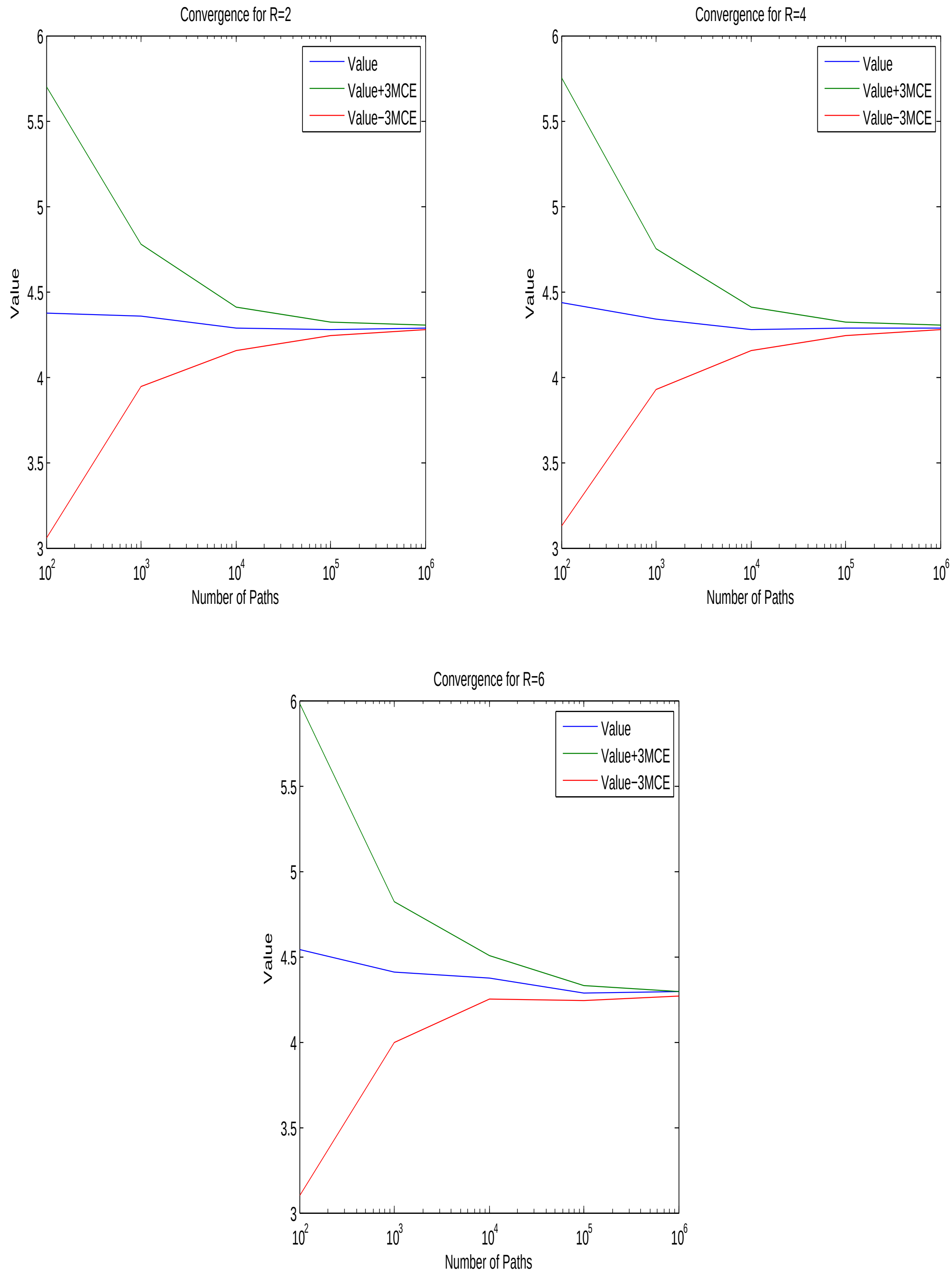

Figure 3.1: Convergence of Longstaff-Schwartz Algorithm for $R = 2, 4, 6$

pure Longstaff-Schwartz algorithm which is reported in column 1 and the fourth column, labelled Error2 reports the difference between the Binomial Tree estimate and the value obtained using the Longstaff-Schwartz algorithm with Black-Scholes input for the shouting value at each node which is reported in column 2. The last column, labelled Error3 is the difference between the first and second columns and indicates a measure of the error introduced by performing the additional regression on the $\alpha's$ as opposed to using the analytic value obtained from the Black-Scholes value for an at-the-money put. As already mentioned, the Longstaff-Schwartz Algorithm tends to lead to a suboptimal estimate and the table appears to support this. Specifically, the values reported in column 1 are consistently lower than the values reported in column 2 and the Monte Carlo Errors reported for column 1 are larger than those reported for column 2 indicating the additional error that is introduced from performing the regression on the $\alpha's$. Further, for all of the observations, the value in column 1 and column 2 are lower than the Binomial Tree estimate of 4.3287. The differences between the Longstaff-Schwartz estimates and the Binomial Tree estimates are the same order of magnitude as those reported in [15] and are typically less than or equal to 1 cent. Lastly, the values reported in column 5 are small and are the same order of magnitude as the values reported in Error2 indicating that the Longstaff-Schwartz Algorithm is able to successfully approximate the at-the-money put option value. This provides a justification for the α regression.

It is important to note that in the case of a Reset Strike Put Option, there are analytic formulas available to value the shout value. As such, it is a convenient framework in which to analyse how well the Longstaff-Schwartz Method estimates the shout value. The method appears to produce satisfactory results and consequently, I feel comfortable applying this approach to other payoffs in which no analytic formulas are available.

Table 3.3: Comparison of Longstaff-Schwartz Method to Control Methods, M=100, N=100000, R=4, Binomial Tree Estimate=4.3287

Longstaff-Schwartz	Longstaff-Schwartz with BS input	Error1	Error2	Error3
4.3184(0.0133)	4.3199(0.0123)	0.0103	0.0088	0.0015
4.3226(0.0133)	4.3275(0.0123)	0.0061	0.0012	0.0049
4.3175(0.0133)	4.3240(0.0123)	0.0112	0.0047	0.0065
4.3223(0.0133)	4.3225(0.0123)	0.0064	0.0062	0.0002
4.3240(0.0133)	4.3280(0.0123)	0.0047	0.0007	0.0040

The apparent convergence indicates the suitability of this model for pricing Shout Options and I now show that it is possible to produce some general features of Reset Strike Put

Options within the Longstaff-Schwartz framework. Figure 3.2 plots the Reset Strike Put Option with initial strike $K = 40$ against the Reset Strike Put Option with initial strike $K = 0$ and against the European Put Option price. The simulations were run with the same random noise in each case. One notices that both these options form a lower bound for the Reset Strike Put Option with $K = 40$. This makes sense intuitively since, as mentioned in [19], the holder can always choose not to shout in which case the Reset Strike Put Option is a European Put Option. Obviously, as the asset price increases, it becomes profitable for the holder of the Reset Strike Put Option to shout and reset the strike price to the prevailing asset price at the moment of shouting as it is unlikely that the original strike setting will result in an in-the-money contract at maturity. Further, the Reset Strike Put Option with initial strike setting of $K = 0$ is equivalent to an at-the-money Put Option, the contract that is received upon shouting. The fact that these two contracts become tangential indicates that it will be optimal to shout at some point during the life of the contract.

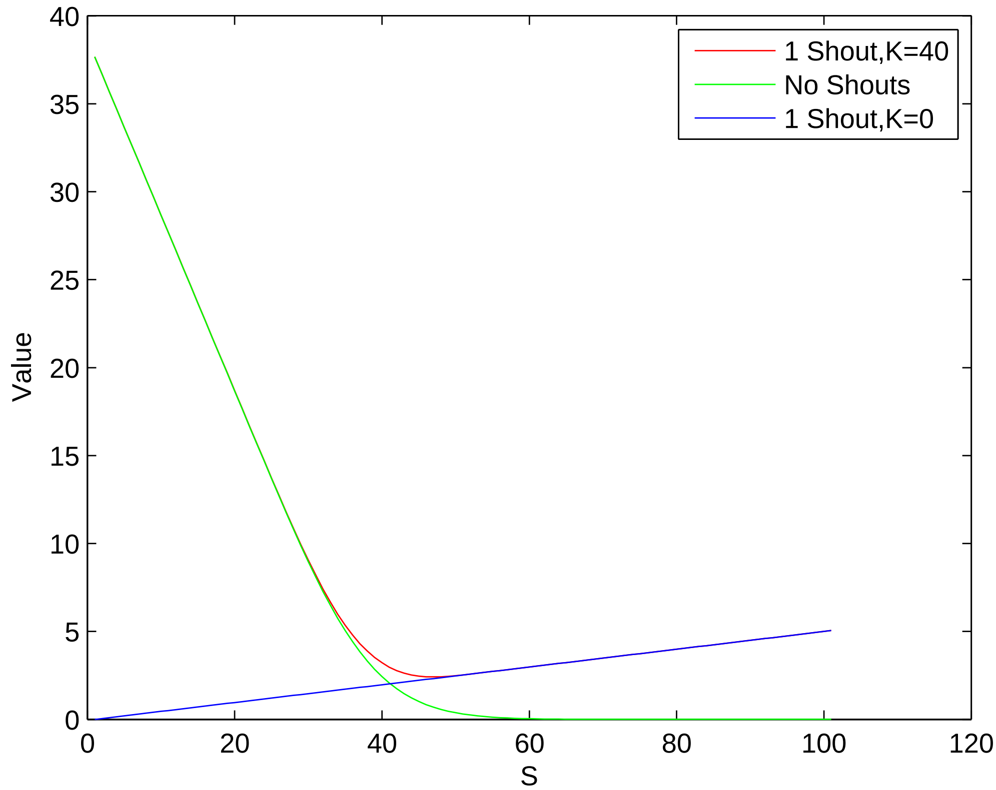

Figure 3.2: 1 Shout VS No Shouts

Figure 3.3 presents the value of the Reset Strike Put Option for various initial strike settings. One notices that the higher the initial strike, the higher the value of the contract, however, as discussed in [19], as $S \to \infty$ they all tend to the same value since, as $S \to \infty$, the holders will be more likely to shout and will then each receive the identical contract upon shouting, independent of the initial strike setting. In our case, for $S \approx 80$, it is no more valuable to have an initial strike setting of $K = 60$ than it is to have an initial strike setting of $K = 20$.

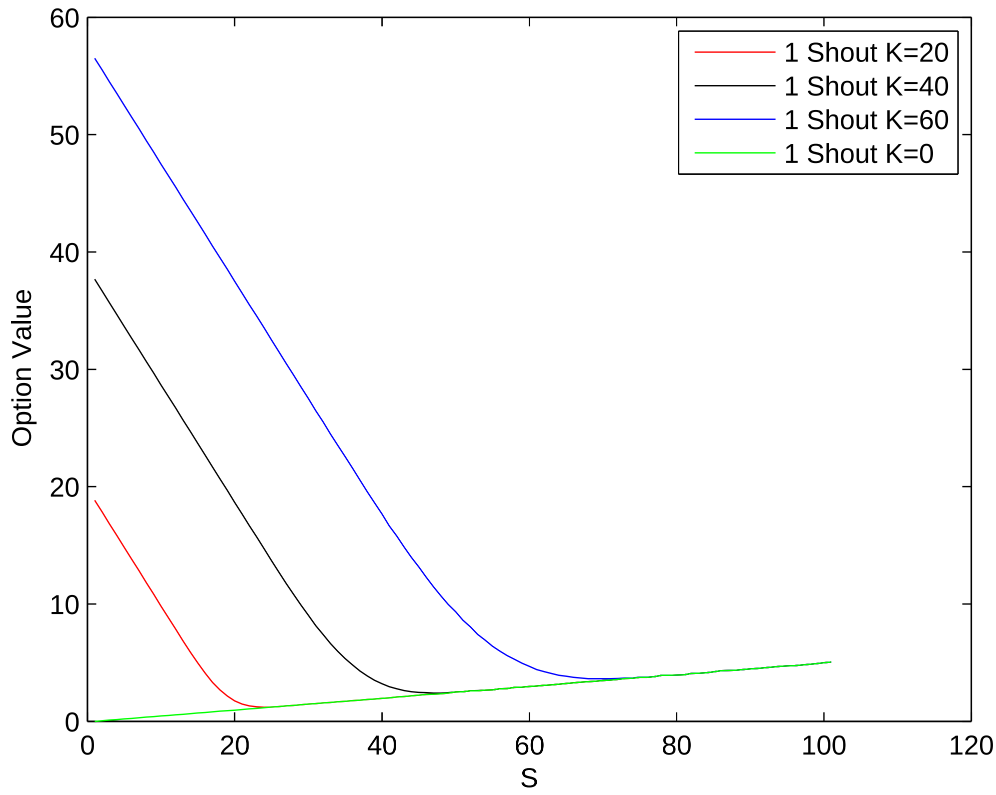

Figure 3.3: Various Initial Strike Settings

Lastly, Figure 3.4 presents the value of the Reset Strike Put Option with various times to maturity remaining. As discussed in [6], the value of the Reset Strike Put Option does not exhibit monotonicity in τ where τ is the time to maturity. Thus the Reset Strike Put Option displays different behaviour to the American Option which is monotonically increasing in τ. This can be explained by the fact that, upon shouting, the holder receives an at-the-money European Put Option which is not monotonic in τ. Figure 3.4 also suggests that with a long time left till maturity, the value profile is roughly U-shaped and becomes more V-shaped as τ decreases. This corroborates results seen in [20] and may imply that it may be more difficult to hedge these options close to maturity.

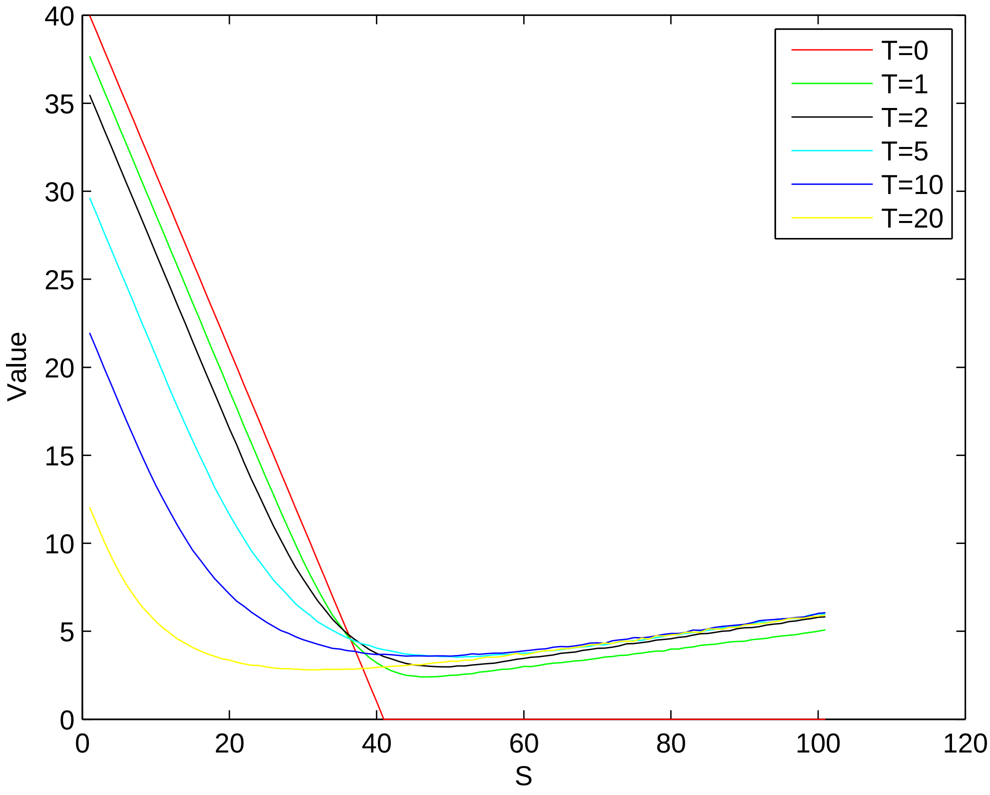

Figure 3.4: Impact of Time to Maturity

When implementing the Greeks, I used the Pathwise Sensitivity Approach as described in Section 2.6.1 to estimate Delta, Vega and Rho. I also used the Likelihood Ratio Method to estimate Delta and Gamma. For both techniques, I applied forward evaluation along N paths of M time steps and implemented the stopping rule as determined by the $\alpha's$ and $\beta's$ in the regression step.

Table 3.4 and 3.5 present the convergence results for Delta of the the Reset Strike Put Option obtained using the Pathwise Sensitivity and Likelihood Ratio Method's respectively. The convergence results are plotted in Figure 3.5. A noteworthy result is that although convergence to similar values is obtained using both methods, the Monte Carlo Error bounds are significantly larger in the Likelihood Ratio Method compared to the Pathwise Sensitivity approach. In particular, the Monte Carlo Errors reported for the Likelihood Ratio Method appear to be approximately 6 times larger than those reported for the Pathwise Sensitivity approach. This matches our intuition. The Monte Carlo Errors decrease in both methods as the number of paths increase. Another interesting observation is that the Pathwise Sensitivity method approaches the Delta value from above with increasing number of paths whereas the Likelihood Ratio method approaches the Delta value from below with an increasing number of paths. Further, the estimates appear to be consistent in that the Delta's reported for fewer paths tend to be within ±3 MCE bounds of the next, more accurate Delta, based on more paths. Additionally, the MCE bounds seem to be very close for the different R's but the same number of paths. This is consistent with the results obtained for the value of the Reset Strike Put Option.

Table 3.4: Convergence Results for Delta of Reset Strike Put Option by Longstaff-Schwartz using Pathwise Sensitivity Approach for M=10, Binomial Tree Estimate = -0.4580

N	R=2	R=4	R=6
100	-0.4435(0.0457)	-0.4312(0.0458)	-0.4214(0.0456)
1000	-0.4467(0.0144)	-0.4479(0.0144)	-0.4598(0.0143)
10000	-0.4608(0.0046)	-0.4588(0.0046)	-0.4571(0.0046)
100000	-0.4612(0.0014)	-0.4587(0.0014)	-0.4572(0.0014)
1000000	-0.4627(0.0004561)	-0.4603(0.0004565)	-0.4590(0.0004566)

Table 3.5: Convergence Results for Delta of Reset Strike Put Option by Longstaff-Schwartz using Likelihood Ratio Method for M=10, Binomial Tree Estimate = -0.4580

N	R=2	R=4	R=6
100	-0.7196(0.2861)	-0.7644(0.2768)	-0.6757(0.2953)
1000	-0.5473(0.0938)	-0.5431(0.0939)	-0.5426(0.0939)
10000	-0.4659(0.0276)	-0.4595(0.0277)	-0.4628(0.0276)
100000	-0.4550(0.0088)	-0.4548(0.0088)	-0.4548(0.0088)
1000000	-0.4564(0.0028)	-0.4560(0.0028)	-0.4560(0.0028)

Table 3.6 reports the convergence results for the Gamma of the Reset Strike Put Option using the Likelihood Ratio Method and Figure 3.6 presents a graphical representation of these results. The results appear to be consistent and exhibit similar properties as discussed for the value and Delta of the Reset Strike Put Option.

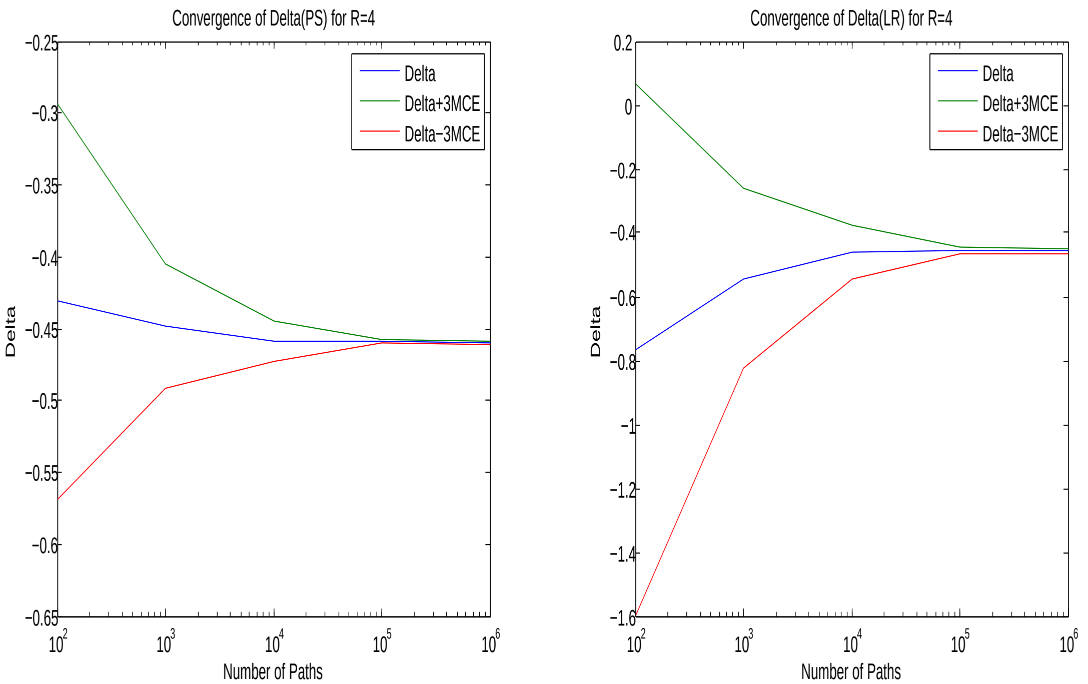

Figure 3.5: Delta Convergence of Longstaff-Schwartz Algorithm using Pathwise Sensitivity and Likelihood Ratio Methods

Table 3.6: Convergence Results for Gamma of Reset Strike Put Option by Longstaff-Schwartz using Likelihood Ratio Method for M=10, Binomial Tree Estimate = 0.0542

N	R=2	R=4	R=6
100	0.3676(0.1817)	0.3449(0.1677)	0.3904(0.1894)
1000	0.1921(0.0677)	0.1928(0.0675)	0.1910(0.0675)
10000	0.0432(0.0186)	0.0461(0.0187)	0.0433(0.0185)
100000	0.0647(0.0059)	0.0645(0.0059)	0.0645(0.0059)
1000000	0.0660(0.0019)	0.0659(0.0019)	0.0660(0.0019)

Table 3.7 and 3.8 present the convergence results for Vega and Rho. The results display the apparent convergence and, as seen in Table 3.9, the values do appear to be realistic as they are consistent with the control values. Further, the results appear to be consistent in that the values reported for fewer paths tend to be within ±3MCE bands of the more accurate value based on the next increased number of paths.

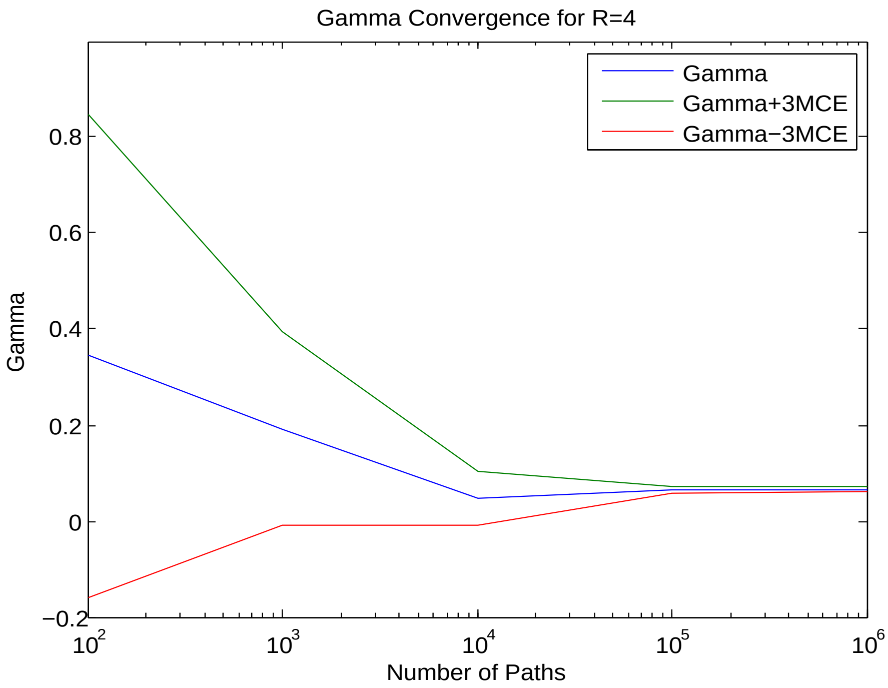

Figure 3.6: Convergence of Likelihood Ratio Gamma for R=4

Table 3.7: Convergence Results for Vega of Reset Strike Put Option by Longstaff-Schwartz using Pathwise Sensitivity Method for M=10, Binomial Tree Estimate = 18.14

N	R=2	R=4	R=6
100	16.1906(1.7742)	15.539(1.7561)	16.0858(1.7299)
1000	16.9772(0.5231)	16.862(0.5205)	17.3459(0.5272)
10000	17.7431(0.1702)	17.8047(0.1707)	17.8725(0.1711)
100000	17.7634(0.0540)	17.824(0.0541)	17.8392(0.0541)
1000000	17.7738(0.017)	17.8204(0.0171)	17.8296(0.0171)

Table 3.8: Convergence Results for Rho of Reset Strike Put Option by Longstaff-Schwartz using Pathwise Sensitivity Method for M=10, Binomial Tree Estimate =-23.06

N	R=2	R=4	R=6
100	-24.0091(1.6516)	-24.6800(1.6430)	-24.1740(1.6305)
1000	-22.7370(0.5447)	-22.9822(0.5427)	-22.7028(0.5462)
10000	-23.0469(0.1731)	-23.0773(0.1724)	-23.0549(0.1721)
100000	-22.7524(0.0549)	-22.7623(0.0548)	-22.7275(0.0547)
1000000	-22.7700(0.0174)	-22.7678(0.0173)	-22.7457(0.0173)

Table 3.9 presents the Greeks of the Reset Strike Put Option estimated using Longstaff-Schwartz method and compares it to the values obtained from the control case. The Longstaff-Schwartz method utilizes 100000 paths and 100 time steps. The values in parentheses represent Monte Carlo Errors where appropriate. The first line states the estimates of the Greeks obtained from using the Binomial Method and the second line is the pure Longstaff-Schwartz reported Greeks. The Greeks appear close to each other, an indication of the accuracy of the Longstaff-Schwartz method. Another important observation is that the estimates of the Greeks were calculated using $M = 100$ time steps as opposed to $M = 10$ time steps that was considered in Tables $3.4, 3.5, 3, 6, 3.7$ and 3.8 above. An unexpected result is that the Monte Carlo Estimates for the Greeks appear to increase as the number of time steps increases holding the number of paths constant. This is a surprising result and I believe an area that may require further investigation.

Table 3.9: Greeks of Reset Strike Put Option by Longstaff-Schwartz and its Control Method,$N =$ 100000, $M = 100$

Method	Delta	Gamma	Vega	Rho
Binomial	-0.4561	0.0535	18.44	-23.11
Longstaff-Schwartz	-0.4617(0.0014)	0.0618(0.0530)	18.2874(0.0541)	-23.5850(0.0535)

Figure 3.7 and 3.8 provides the plot of Delta and Gamma of the Reset Strike Put Option against Delta and Gamma of a European Put Option. The Delta appears to follow a similar trend to that of a vanilla put option. One noticeable distinction is that, as the underlying asset value increases, the option values both become linear, but the Reset Strike Put Option achieves a positive slope. This is consistent with the literature. In particular, [20] indicates that the Delta and Gamma of a Reset Strike Put Option will tend to display a similar shape to the Delta and Gamma of a European put, however Delta will tend to be steeper and Gamma will tend to have a higher peak than that of a vanilla put. This is corroborated by Figures 3.7 and 3.8. Consequently, this implies that Shout Options should not be significantly more difficult to hedge than standard European Options, however, since the Delta is not strictly negative (as for the European Put Option), this implies that the hedging strategy may switch between taking long and short positions at different times.

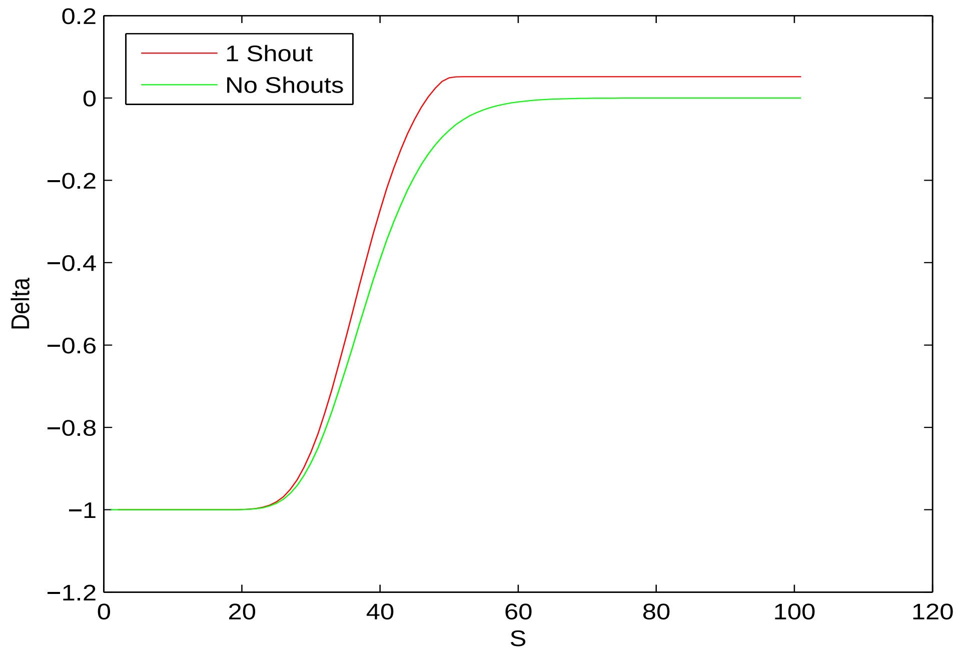

Figure 3.7: Delta of Reset Strike Put Option Vs Delta of European Put Option

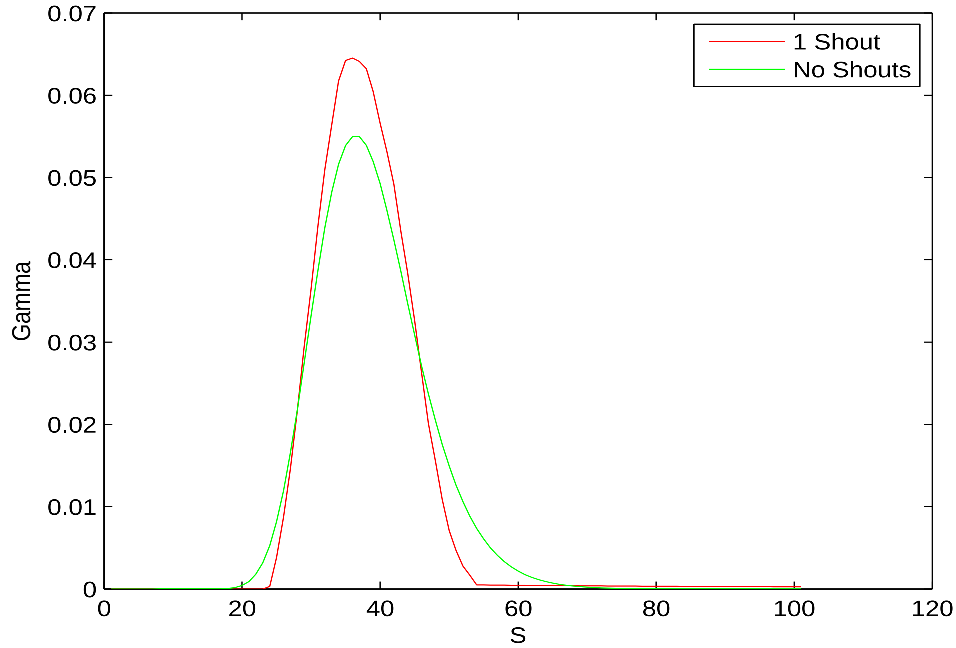

Figure 3.8: Gamma of Reset Strike Put Option Vs Gamma of European Put Option

3.2 Implementation and Results of Longstaff-Schwartz applied to Reset Strike Asian Put Option

I applied the Longstaff-Schwartz Algorithm to a Reset Strike Asian Put Option according to the technique outlined in Section 2.3 in MATLAB. I considered a contract with identical specifications as for the Reset Strike Put Option considered in Section 3.1 above. I performed the analysis with 8 basis functions as suggested in [15] and [2] including: a constant, the first two weighted Laguerre Polynomials evaluated at the share price, the first two weighted Laguerre Polynomials evaluated at the average share price to date and the cross products of the weighted Laguerre Polynomials up to third order. Similar to the case of the Reset Strike Put Option, I employed the same basis functions for both the α and β regressions. I used the forward approach in the valuation step and 100000 paths was found to produce satisfactory convergence.

Table 3.10 considers convergence results for the Reset Strike Asian Put Option with $M = 10$ and $N = 10^2, 10^3, 10^4, 10^5, 10^6$ and Figure 3.9 provides a graphical representation of this. The same results hold as for the case of the Reset Strike Put Option and as such I will not repeat the details here. An interesting result however is that the Reset Strike Asian Put Option appears to be more expensive than the Reset Strike Put Option which can be considered as its "vanilla" counterpart. This appears to contradict a generally accepted principle that Asian options should be cheaper than their vanilla counterparts because the average of a price series is less volatile than the price series itself at a point in time. As discussed in [21], this supposition overlooks an essential condition in that the option price will only be an increasing function of volatility if all other determinants remain unchanged. [21] argues that the stock price at maturity and the average of a stock price are two different random variables and are based on different probability distributions. [21] agrees that the average will be less volatile but states that even in a risk-neutral world, the means may be different. In particular, the average of a non-dividend paying stock is expected to grow more slowly than the stock price itself and thus for average put options on non-dividend paying stocks (the case we are considering), there are two alternative forces at play. The lower volatility drives the average price down whereas the lower expected growth drives the put price up. This can result in higher prices for average options when the latter effect dominates. Consequently, there exists a theoretical justification for the higher price I observe in the Reset Strike Asian Put Option. The Monte Carlo Errors of the Reset Strike Asian Put Option are approximately 1.5 times smaller than those of the Reset Strike Put Option. This can be explained by the fact that in the Reset Strike Asian Option, we use 8 basis functions as opposed to the 4 we used for the Reset Strike Put Option.

Table 3.10: Convergence Results for value of Reset Strike Asian Put Option by Longstaff-Schwartz, $M = 10$

N	Value	MCE
100	4.2774	0.2839
1000	4.6101	0.0829
10000	4.9393	0.0290
100000	4.9364	0.0092
1000000	4.9341	0.0029

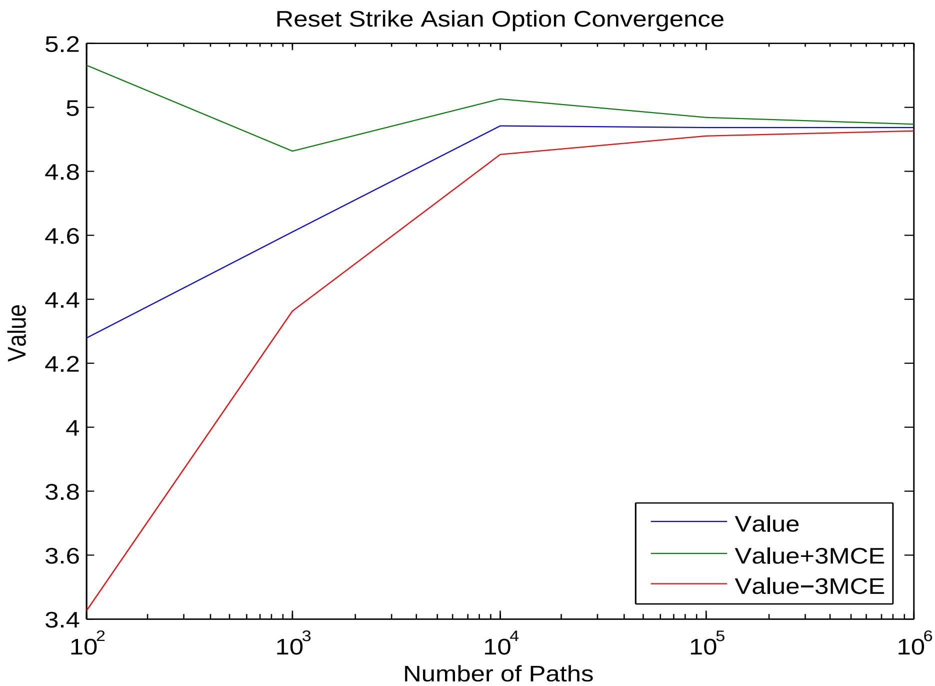

Figure 3.9: Convergence of Longstaff-Schwartz Algorithm for Reset Strike Asian Option

Table 3.11 considers the impact of increasing the number of time steps while holding the number of paths fixed at $N = 100000$. I observed than in the case of the Reset Strike Put Option, an increase in the number of time steps led to an increase in the price. For the Reset Strike Asian Put Option, there is also a change in value as the number of time steps increases however the change is so minimal that it does not justify the increase in cost and memory that will accompany it.

Table 3.12 presents the convergence results for the Delta of the Reset Strike Asian Put Option obtained using the Pathwise Sensitivity Approach and Likelihood Ratio Method and Figure 3.10 provides an illustration of this. Again, in this case, the same results apply

Table 3.11: Impact of Number of time steps, $N = 100000$

M	Value	MCE
10	4.9364	0.0092
50	4.9361	0.0094
100	4.9422	0.0095

as for the Reset Strike Put Option. A noticeable distinction however, is that the Delta of the Reset Strike Asian Put Option appears to be smaller (more negative) than the Delta of its ”vanilla” counterpart.

Table 3.12: Convergence Results for Delta of Reset Strike Asian Put Option by Longstaff-Schwartz using Pathwise Sensitivity and Likelihood Ratio Approach for M=10

N	Delta(PS)	MCE	Delta(LR)	MCE
100	-0.5297	0.0470	-0.7083	0.2792
1000	-0.5270	0.0157	-0.5012	0.0785
10000	-0.5009	0.0050	-0.4904	0.0269
100000	-0.4969	0.0016	-0.4920	0.0084
1000000	-0.4964	5.05E-04	-0.4951	0.0028

Table 3.13 presents the convergence results for the Gamma of the Reset Strike Asian Put Option using the Likelihood Ratio Method and Figure 3.11 plots the Gamma with its Monte Carlo Error confidence bands. The result appears to be consistent and exhibit similar properties to the Gamma of the Reset Strike Put Option. In particular, the magnitudes do appear to be similar.

Table 3.13: Convergence Results for Gamma, Vega and Rho of Reset Strike Asian Put Option by Longstaff-Schwartz for M=10

N	Gamma	MCE	Vega	MCE	Rho	MCE
100	0.1565	0.2014	15.5302	1.2178	-12.3356	1.5429
1000	0.0976	0.0511	15.8756	0.6096	-9.5669	0.5617
10000	0.0728	0.0186	15.0831	0.1588	-8.2561	0.1853
100000	0.0658	0.0056	15.0911	0.0493	-8.0633	0.0591
1000000	0.0627	0.0019	15.0605	0.0156	-8.0290	0.0187

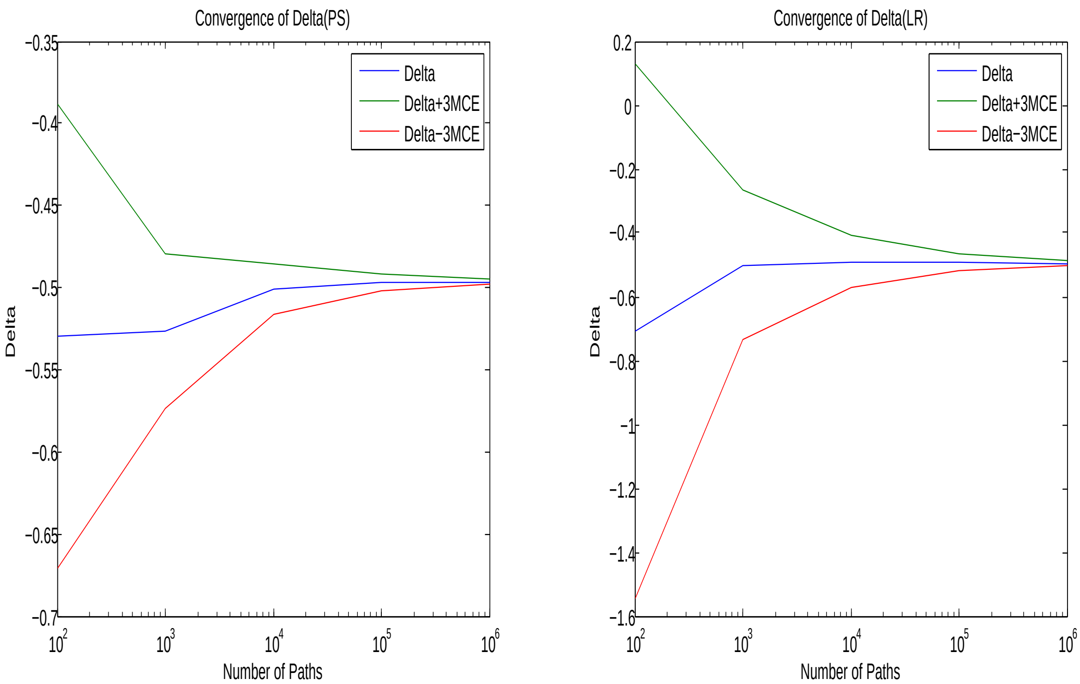

Figure 3.10: Delta Convergence of Longstaff-Schwartz Algorithm using Pathwise Sensitivity and Likelihood Ratio Methods

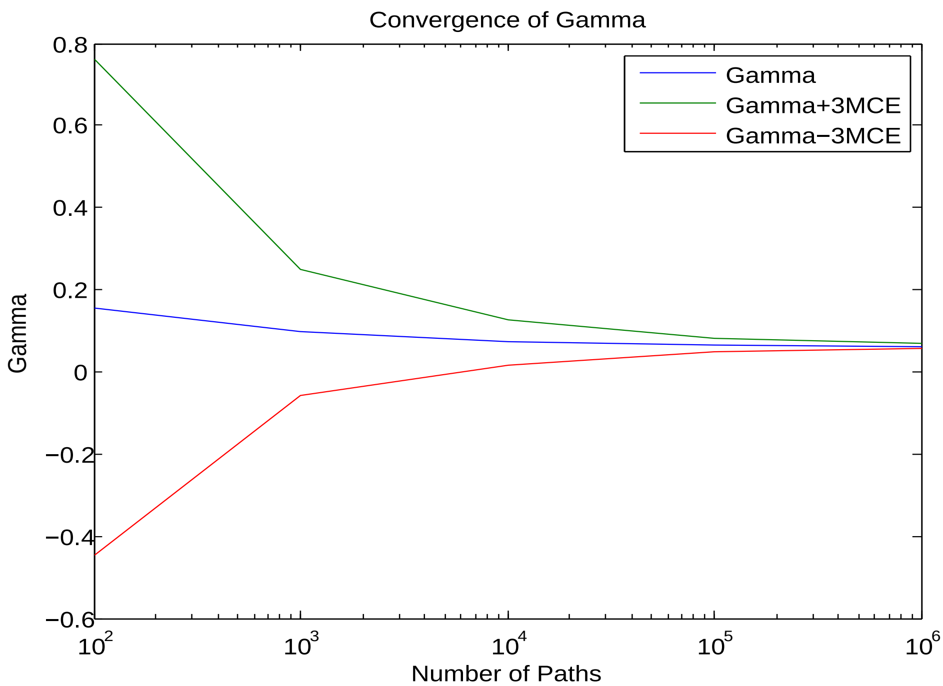

Figure 3.11: Convergence of Likelihood Ratio Gamma

The convergence results for Vega and Rho are also reported in Table 3.13. The values appear plausible, however the value for Vega appears smaller than its "vanilla" counterpart whereas the value for Rho appears larger (less negative) than its "vanilla" counterpart.

Lastly, Table 3.14 provides the Greek values with $N = 100000$ and $M = 10, 50, 100$ so that I can deduce the impact of increasing the number of time steps on the Greek values. The magnitude of the Greeks does not appear to change significantly (with a possible exception of Gamma). A noticeable result is that the Monte Carlo errors tend to increase with the number of time steps. This effect was observed in the calculation of the Greeks for the Reset Strike Put Option and appears to characterize Greek calculations for Shout Options using the Longstaff-Schwartz Algorithm, however, it was absent from the value calculations of both options.

Table 3.14: Impact of Number of time steps on Greeks, $N = 100000$

M	Delta	Gamma	Vega	Rho
10	-0.4964(5.05E-04)	0.0658(0.0056)	15.0605(0.0156)	-8.0290(0.0187)
50	-0.4953(0.0016)	0.0839(0.0261)	15.0471(0.0516)	-6.7064(0.0592)
100	-0.4956(0.0016)	0.0827(0.0507)	15.1315(0.0520)	-6.5580(0.0592)

3.3 Implementation and Results of Longstaff-Schwartz applied to Swing Reset Strike Put Option

I applied the Longstaff-Schwartz Algorithm to a Swing Reset Strike Put Option according to the technique outlined in Section 2.5 in MATLAB. I considered a contract with identical specifications as for the Reset Strike Put Option described in Section 3.1 above. I performed the analysis using 4 basis functions as was the case for the Reset Strike Put Option. I employed the same basis functions for both the α and β regressions. I consider only the case of the Reset Strike Put Option but it would be an easy extension to consider the Reset Strike Asian Put Option.

Table 3.15 considers convergence results for the Swing Reset Strike Put Option with 1, 2 and 3 shout opportunities for $M = 10$ and $N = 10^2, 10^3, 10^4, 10^5$ and Figure 3.12 displays the convergence results graphically for 2 and 3 shout opportunities. As expected, as the number of shout opportunities increases, the value of the contract increases. This corresponds with our intuition. Another important observation is that the Monte Carlo errors for larger number of shout opportunities tends to be larger. This can be explained by the fact that, as described in Section 2.5, the value of the option with j shout opportunities remaining depends on the value of the option with $j-1$ shout opportunities remaining. There was already calculation error introduced for pricing the value with $j-1$ shout opportunities remaining because of the Monte Carlo Simulation and the least squares regression approximation. This error tends to propagate through and results in a higher Monte Carlo error for the contract with j shout opportunities remaining. Hence, the contract with 3 shout opportunities has higher Monte Carlo Error than the contract with 2 shout opportunities which has higher Monte Carlo error than the contract with 1 shout opportunity.

Table 3.15: Convergence Results for Value of Reset Strike Put Option with 1,2 and 3 shout opportunities by Longstaff-Schwartz for M=10

N	1 Shout	MCE	2 Shouts	MCE	3 Shouts	MCE
100	4.7542	0.4807	5.5059	0.5532	6.3016	0.6966
1000	4.4154	0.1344	5.4197	0.1563	6.1104	0.189
10000	4.3051	0.042	5.3334	0.0498	6.0006	0.0603
100000	4.2935	0.0133	5.2974	0.0157	5.9295	0.0188

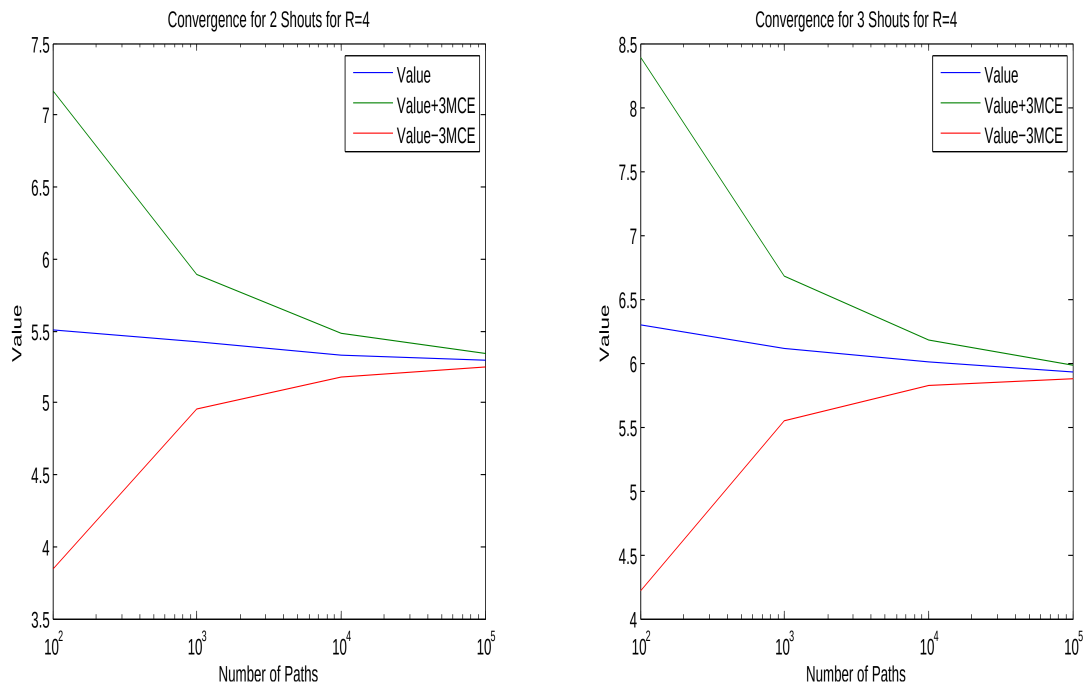

Figure 3.12: Convergence of Reset Strike Put Option with 2 and 3 Shout opportunities from Longstaff-Schwartz Algorithm

Figure 3.13 is similar to Figure 3.2 but also includes the contract in which 3 shout opportunities are available. It is apparent from the figure that $V_1 < V_3$ where V_j denotes the value of the Reset Strike Put Option with j shout opportunities remaining. This monotonic property is consistent with our intuition that a contract with more shout opportunities should be more valuable.

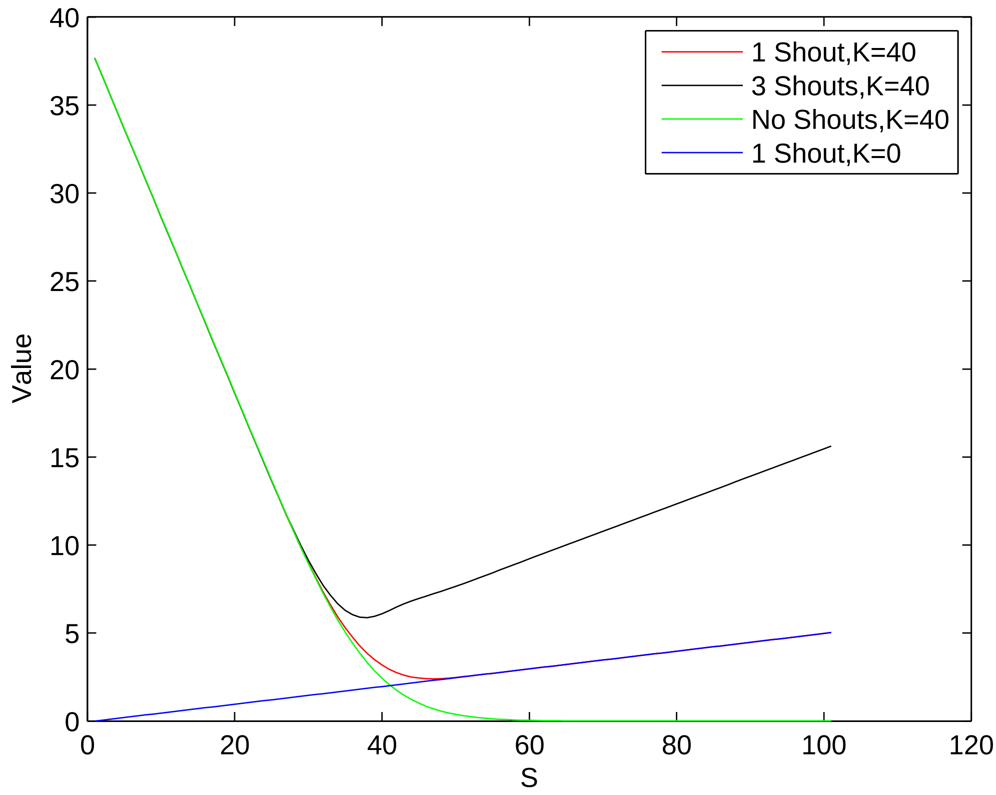

Figure 3.13: Reset Strike Put Option with different number of shout opportunities

Table 3.16 presents convergence results for Delta of the Reset Strike Put Option with 1, 2 and 3 shout opportunities using the Likelihood Ratio Method and Figure 3.14 provides an illustration of this. A noticeable observation is that the contracts with more shout opportunities tend to have a larger delta. Further, the Monte Carlo errors also appear to increase as the number of shout opportunities increase. This is consistent with what I observed for the value of the Swing Reset Strike Put Option.

Table 3.16: Convergence Results for Delta of Reset Strike Put Option with 1,2 and 3 shout opportunities by Longstaff-Schwartz for M=10

N	1 Swing	MCE	2 Swings	MCE	3 Swings	MCE
100	-0.9745	0.3708	-0.834	0.4065	-0.7068	0.4624
1000	-0.4468	0.0892	-0.197	0.1107	-0.0808	0.1302
10000	-0.4818	0.0283	-0.2599	0.035	-0.1171	0.0417
100000	-0.4733	0.0088	-0.2566	0.0109	-0.0989	0.0128

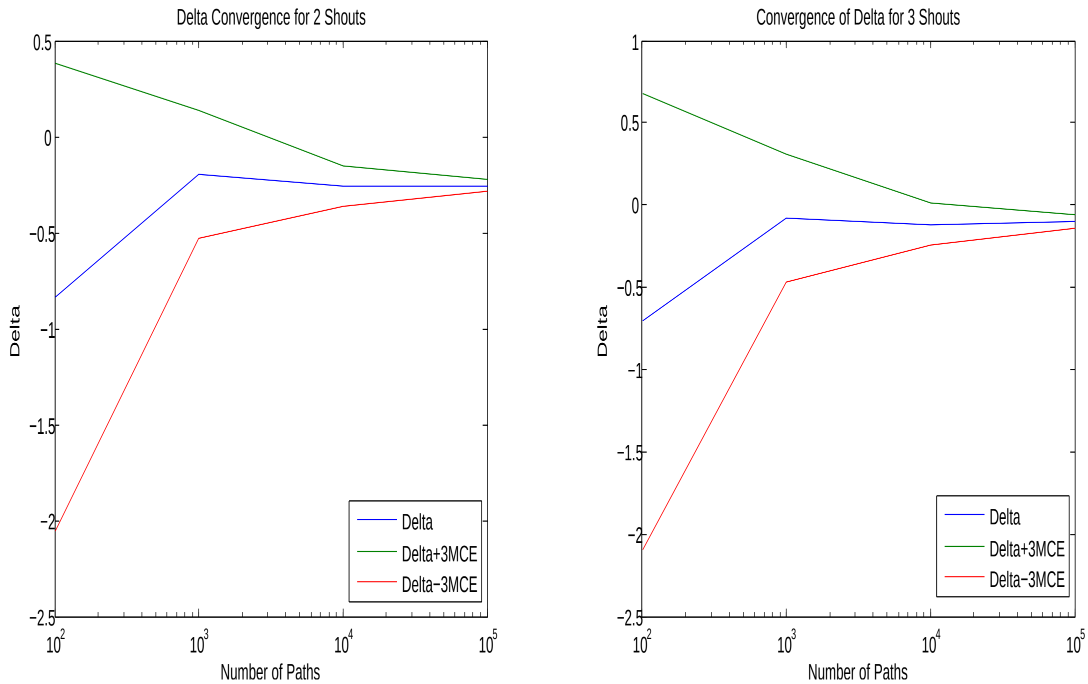

Figure 3.14: Delta Convergence of Reset Strike Put Option with 2 and 3 Shout opportunities from Longstaff-Schwartz Algorithm

Table 3.17 presents convergence results for Gamma of the Reset Strike Put Option with 1, 2 and 3 shout opportunities using the Likelihood Ratio Method and Figure 3.15 provides a graphical illustration of this. A noticeable observation is that the contracts with more shout opportunities tend to have a larger Gamma. Further, the Monte Carlo errors also appear to increase as the number of shout opportunities increase. This is consistent with what I observed for the value and Delta of the Swing Reset Strike Put Option. Table 3.18 presents a comparison of the value and Greeks of the Reset Strike Put Option with 2 shout opportunities reported by the Binomial Tree method and the Longstaff-Schwartz Algorithm. The values appear to be close and the control values do fall within 3 MCE bands of the Longstaff-Schwartz estimates. These results imply that the Longstaff-Schwartz Algorithm can produce accurate and stable results.

Table 3.17: Convergence Results for Gamma of Reset Strike Put Option with 1,2 and 3 shout opportunities by Longstaff-Schwartz for M=10

N	1 Swing	MCE	2 Swings	MCE	3 Swings	MCE
100	0.46	0.2777	0.4366	0.2886	0.4275	0.3067
1000	0.0605	0.0578	0.0928	0.0705	0.112	0.0791
10000	0.0964	0.0199	0.1248	0.0235	0.1598	0.0276
100000	0.0715	0.0061	0.0957	0.0073	0.1203	0.0086

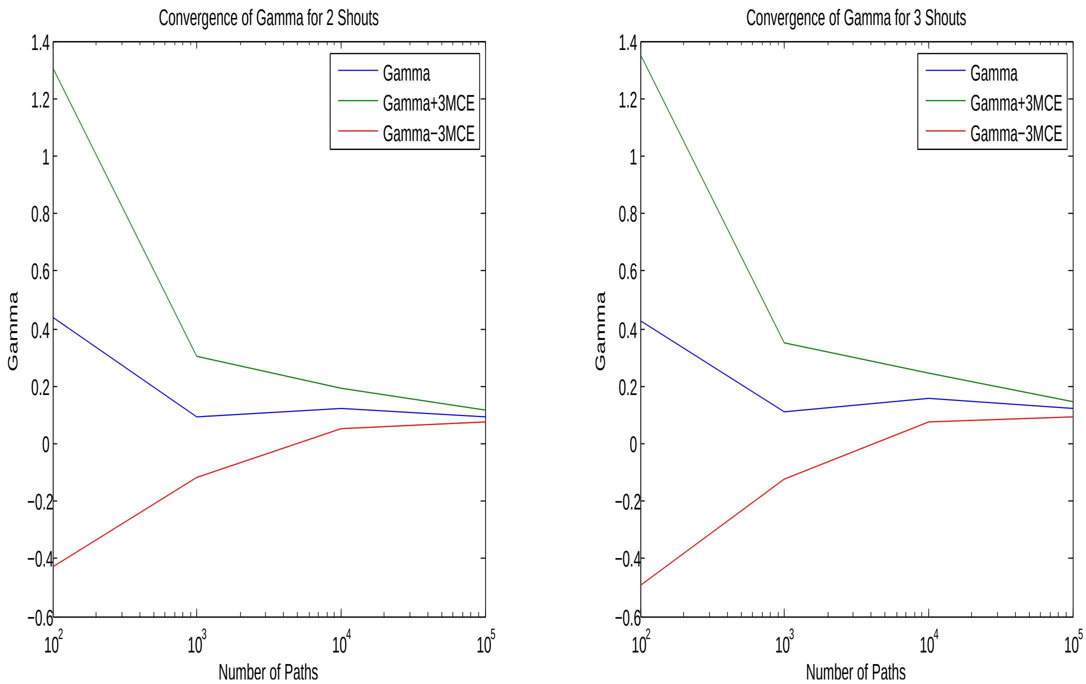

Figure 3.15: Gamma Convergence of Reset Strike Put Option with 2 and 3 Shout opportunities from Longstaff-Schwartz Algorithm

Table 3.18: Value and Greeks of Reset Strike Put Option with 2 Shouts by Longstaff-Schwartz and its Control Method, $N = 100000$, $M = 10$

Method	Value	Delta	Gamma
Binomial	5.3139	-0.2516	0.0664
Longstaff-Schwartz	5.2831(0.0157)	-0.2406(0.0108)	0.0699(0.0070)

Chapter 4

Conclusion, Final Remarks and Possible Extensions

In this thesis, I have developed and implemented the Longstaff-Schwartz approach to pricing Shout Options and their Greeks. In particular, I considered the Reset Strike Put Option, a shout contract common in the literature, the Reset Strike Asian Put Option, a more complex path dependent Shout Option and the Swing Reset Strike Put Option which allows for multiple shout opportunities. The convergence properties were investigated by comparing the Longstaff-Schwartz estimates to the results obtained from the Binomial Tree control case and Chapter 3 verifies the result in [4] that as the number of paths increase, the Longstaff-Schwartz value converges to the true value. I confirmed some general properties of Shout Options could be obtained using the Longstaff-Schwartz approach. I presented an approach for obtaining the Greeks within the Longstaff-Schwartz framework using the Pathwise Sensitivity approach and the Likelihood Ratio Method. Again, in this case, Chapter 3 verifies the convergence of these estimates. This paper serves to confirm that the Longstaff-Schwartz Algorithm can be regarded as a robust method, one that can effectively handle the optimisation component inherent in a Shout Option.

As mentioned in Chapter 1, the Swing Reset Strike Put Option is distinct form the ”n-reset” Strike Put Option. The case of the ”n-reset” Strike Put Option is interesting but requires comprehensive analysis, similar to that which I have performed on the single Strike Reset Put Option in order to confirm the convergence of the Longstaff-Schwartz estimate to the true value. The Longstaff-Schwartz Algorithm may be expensive if there are many paths included and not accurate enough if there are too few paths included. Consequently, finding a suitable number of paths to ensure convergence to an accurate value is an interesting task, one that is beyond the scope of this thesis. However, I suggest it as an area for possible extension of the work presented in this thesis.

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