

A new approach to computing the asymptotics of the position of Fisher-KPP fronts

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February 12, 2018

Abstract

This paper presents a novel way of computing front positions in Fisher-KPP equations. Our method is based on an exact relation between the Laplace transform of the initial condition and some integral functional of the front position. Using singularity analysis, one can obtain the asymptotics of the front position up to the $\mathcal{O}(\log t/t)$ term. Our approach is robust and can be generalised to other front equations.

1 Introduction

The goal of this letter is to present a novel way of computing the asymptotic position of a front propagating into an unstable phase. The typical equation we consider is the Fisher-KPP equation [1, 2],

$$\partial_t h = \partial_x^2 h + h - h^2 \quad (\text{Fisher-KPP}), \quad (1)$$

but our method is general and can be adapted to a large class of other reaction-diffusion equations. An important feature of (1) is that the solution converges to a travelling wave: for an initial condition $h_0 \in [0, 1]$ such that $h_0(x) \rightarrow 1$ as $x \rightarrow -\infty$ and $h_0(x) \rightarrow 0$ exponentially fast as $x \rightarrow \infty$, then

$$h(\mu_t + z, t) \rightarrow \omega(z), \quad (2)$$

where μ_t is the position of the front (we will choose μ_t in such a way that $h(\mu_t, t) = \frac{1}{2}$ but other choices are possible) and $\omega(z)$ is the travelling wave.

In a recent work [3], we have shown how to apply our method to an equation looking like (1), but

with the non-linear term replaced by a free boundary condition. This allowed us to understand in great detail how the large t asymptotics of the position μ_t depends on the initial condition. In the present paper, we show that our method is much more general and can be applied to a large variety of non-linear equations such as (1).

The determination of the position μ_t of the Fisher-KPP equation has attracted an uninterrupted attention [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] since the equation was introduced in 1937 [1, 2]. So far all the results were obtained either by probabilistic methods, or by computing precisely how the shape $h(\mu_t + x, t)$ of the centred front converges to the travelling wave, and then to determine μ_t . Our method is different. It consists in writing a relation between the initial condition h_0 and μ_t . To do so, introduce

$$\varphi(r, t) = \int_{\mathbb{R}} dz h(\mu_t + z, t)^2 e^{rz}, \quad (3)$$

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and

$$\Psi(r) = \int_{\mathbb{R}} dx h_0(x) e^{rx}. \quad (4)$$

Then, our main relation (derived in Section 4) is

$$\Psi(r) = \int_0^\infty dt \varphi(r, t) e^{r\mu_t - (r^2+1)t}, \quad (5)$$

for any r small enough so that both sides converge. Notice from (3) that $\varphi(r, t) e^{r\mu_t}$ is independent of μ_t . Therefore, (5) holds in fact for an arbitrary choice of μ_t and, by itself, it is not sufficient to determine the position of the front. However, when μ_t is the position of the front, we then have

$$\varphi(r, t) \rightarrow \hat{\varphi}(r) \quad \text{with} \quad \hat{\varphi}(r) := \int dz \omega(z)^2 e^{rz}, \quad (6)$$

for r small enough, and we can evaluate the speed of that convergence. This eventually allows to determine the first terms of the large t asymptotics of μ_t .

2 Velocity selection

At this point, one can already understand from (5) and (6) how the asymptotic velocity of the front $v = \lim_{t \rightarrow \infty} \mu_t/t$ depends on the initial condition.

First assume that $\Psi(r)$ is singular as $r \nearrow \gamma \leq 1$, meaning (roughly speaking) that $h_0(x)$ decays as $e^{-\gamma x}$. Then, obviously, the right-hand-side of (5) must also have be singular as $r \nearrow \gamma$. This singularity must come from the large t part of the integral, when $\varphi(r, t)$ is nearly equal to $\hat{\varphi}(r)$ according to (6). When $r < \gamma$, the integral in (5) converges because $r\mu_t - (r^2+1)t$ goes to $-\infty$ linearly in t . As r crosses γ , the integral becomes singular because $r\mu_t - (r^2+1)t$ changes sign. This means that $\mu_t \sim vt$ with v such that $\gamma v - (\gamma^2+1) = 0$, which is the expected relation between the decay rate γ and the velocity v when $\gamma < 1$.

When $\Psi(r)$ has no singularity up to $r = 1$ (meaning that the initial condition decays “fast”), the velocity of the front cannot be larger than 2 (otherwise, there would a singularity at some $\gamma < 1$ solution to $\gamma v = \gamma^2 + 1$) so it must be equal to 2 as there are no

positive travelling waves of speed less than 2; this is also a well known fact of the Fisher-KPP equation.

3 Higher order corrections

We have just seen that the position of the singularity determines the velocity: $\mu_t \approx vt$; we are now going to see that the nature of the singularity gives the next order terms in μ_t . Let us illustrate this method by focusing on the Ebert and van Saarloos term [10].

Assume, for simplicity, that the initial condition decays fast enough for $\Psi(r)$ as given by (4) to be analytic at $r = 1$. Since Bramson’s work [8], it is known that

$$\mu_t = 2t - \frac{3}{2} \log t + a + o(1), \quad (7)$$

and we want to estimate the $o(1)$. As a first attempt, let us look at what happens as $r \nearrow 1$ in (5) when $\varphi(r, t)$ is replaced by its limit $\hat{\varphi}(r)$ and μ_t is given by $2t - \frac{3}{2} \log t + a$ for $t > t_0$, without any further corrective terms. Then, with these substitutions, $\Psi(1 - \epsilon)$ would be equal to

$$f(\epsilon) + \hat{\varphi}(1 - \epsilon) e^{(1-\epsilon)a} \int_{t_0}^\infty dt \frac{e^{-\epsilon^2 t}}{t^{\frac{3}{2}}} e^{\epsilon^{\frac{3}{2}} \log t} \quad (8)$$

where $f(\epsilon)$, which corresponds to the integral from 0 to t_0 , is obviously analytic. On the other hand, the integral above is an incomplete Gamma function, which one can expand in powers of ϵ to obtain

$$A + B\epsilon + 6\sqrt{\pi}\epsilon^2 \log \epsilon + C\epsilon^2 + \mathcal{O}(\epsilon^3),$$

where A , B and C depend on t_0 , but where the singular term in $\epsilon^2 \log \epsilon$ does not. (See also Section 6.)

Such a singular term cannot be actually present in the expansion of $\Psi(1 - \epsilon)$, because we know (from our choice of initial condition) that Ψ is analytic at 1. As in the linear case [3], the only possibility for the $\epsilon^2 \log \epsilon$ term to disappear, is that it is cancelled by another $\epsilon^2 \log \epsilon$ term coming from the $o(1)$ in (7). One finds that this $o(1)$ term must be given, to leading order, by the Ebert and van Saarloos term:

$$\mu_t = 2t - \frac{3}{2} \log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + \dots \quad (9)$$

Repeating the same procedure, one can notice that inserting $\mu_t = 2t - \frac{3}{2} \log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}}$ into (5) leads to a $\epsilon^3 \log \epsilon$ singular term in the expansion. By a careful small ϵ expansion, one finds as illustrated in Section 6 that this term is cancelled by choosing

$$\mu_t = 2t - \frac{3}{2} \log t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + \frac{9}{8}(5 - 6 \log 2) \frac{\log t}{t} + \dots, \quad (10)$$

and so on: each new term in the large t expansion of μ_t allows to remove a singularity in the small ϵ expansion of Ψ , but introduces a new, weaker, singularity.

Remark that we started this analysis by requiring that $\Psi(r)$ is analytic at $r = 1$. In fact, this hypothesis is not needed: to obtain (9), the only requirement is that there is no $\epsilon^2 \log \epsilon$ term in the expansion of $\Psi(1 - \epsilon)$:

$$\Psi(1 - \epsilon) = A + B\epsilon + o(\epsilon^2 \log \epsilon)$$

for some constants A and B . (From (4), this condition is satisfied if the initial condition decays a bit faster than $x^{-3}e^{-x}$.) Similarly, the $(\log t)/t$ term of (10) requires that there is no $\epsilon^3 \log \epsilon$ term in $\Psi(r)$, that is that the initial condition decays a bit faster than $x^{-4}e^{-x}$.

At the beginning of the current section, we have replaced $\varphi(r, t)$ in (5) by its limit $\hat{\varphi}(r)$ to obtain (8). It is now time to justify this simplification. The term we neglected until now is

$$\Delta(r) = \int_0^\infty dt \left[\varphi(r, t) - \hat{\varphi}(r) \right] e^{r\mu_t - (r^2+1)t}. \quad (11)$$

We claim that

$$\Delta(1 - \epsilon) = \tilde{A} + \tilde{B}\epsilon + \tilde{C}\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (12)$$

which means that the first singularity in the small $\epsilon > 0$ expansion of $\Delta(1 - \epsilon)$ is smaller than ϵ^3 . Then, the result (10) still holds as it was obtained by suppressing a singularity $\epsilon^3 \log \epsilon$, bigger than ϵ^3 .

To justify (12), we argue in Section 5 that, when μ_t is defined as the position where the front is $1/2$, one has

$$\varphi(r, t) = \hat{\varphi}(r) + \mathcal{O}\left(\frac{1}{t}\right). \quad (13)$$

Then, inserting (13) and Bramson's estimate (7) for the position μ_t of the front into (11), one obtains

$$\Delta(1 - \epsilon) = \int_1^\infty dt \frac{e^{-\epsilon^2 t + \frac{3}{2}\epsilon \log t}}{t^{3/2}} \times \mathcal{O}\left(\frac{1}{t}\right).$$

One checks directly that the integral on the right hand side satisfies (12).

4 Derivation of (5)

From its definition (3), it is obvious that $\varphi(r, t)e^{r\mu_t}$ is independent of the choice of μ_t . Thus, it is sufficient to establish (5) for $\mu_t = 0$. Define, for r small enough,

$$g(r, t) = \int_{\mathbb{R}} dx h(x, t) e^{rx}. \quad (14)$$

(Of course $\Psi(r) = g(r, 0)$ from (4).) Then, from (1) and (3) with $\mu_t = 0$ one has

$$\partial_t g(r, t) = (1 + r^2)g(r, t) - \varphi(r, t) \quad (15)$$

where we integrated by parts $\int dx \partial_x^2 h e^{rx}$. One can solve (15) to get

$$g(r, t) = e^{(1+r^2)t} \left[\Psi(r) - \int_0^t ds \varphi(r, s) e^{-(1+r^2)s} \right].$$

It only remains to show that

$$g(r, t) e^{-(1+r^2)t} \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad (16)$$

to conclude. The solution $h(x, t)$ to (1) is smaller than $L(x, t)$, the solution to the linearised equation $\partial_t L(x, t) = \partial_x^2 L(x, t) + L(x, t)$ with $L(x, 0) = h_0(x)$. For any β such that $\Psi(\beta) < \infty$,

$$\begin{aligned} L(x, t) &= \int_{\mathbb{R}} dy h_0(y) e^t \frac{e^{-\frac{(x-y)^2}{4t}}}{\sqrt{4\pi t}} \\ &= \int_{\mathbb{R}} dy h_0(y) e^{(1+\beta^2)t - \beta(x-y)} \frac{e^{-\frac{(x-y-2\beta t)^2}{4t}}}{\sqrt{4\pi t}} \\ &\leq \frac{e^{(1+\beta^2)t}}{\sqrt{4\pi t}} e^{-\beta x} \Psi(\beta). \end{aligned} \quad (17)$$

Then, we write that $h(x, t) \leq \min[1, L(x, t)]$. By using the bound (17), one has

$$h(x, t) \leq \begin{cases} 1 & \text{if } x < d_{\beta, t}, \\ \frac{e^{(1+\beta^2)t}}{\sqrt{4\pi t}} e^{-\beta x} \Psi(\beta) & \text{if } x > d_{\beta, t}, \end{cases} \quad (18)$$

where $d_{\beta, t}$ is the position where the second bound is also equal to 1. Then, for $r < \beta$, one gets

$$g(r, t) \leq \left(\frac{1}{r} + \frac{1}{\beta - r} \right) e^{rd_{\beta, t}}.$$

Using $e^{rd_{\beta, t}} = \left(\frac{e^{(1+\beta^2)t}}{\sqrt{4\pi t}} \Psi(\beta) \right)^{r/\beta}$, this leads for $t > 1$ to

$$g(r, t) \leq C e^{r(\beta+\beta^{-1})t} \quad (19)$$

for some constant C . Choose furthermore $\beta \leq 1$. With $r < \beta$, one checks that $r(\beta + \beta^{-1}) < 1 + r^2$, and one concludes that (16) and (5) hold for all $r < 1$ such that $r < \sup[\beta; \Psi(\beta) < \infty]$.

5 Justification of (13)

With μ_t the position where the front is $1/2$, define

$$\delta(x, t) = h(\mu_t + x, t) - \omega(x)$$

one obtains from (1) that

$$\begin{aligned} \partial_t \delta &= \partial_x^2 \delta + 2\partial_x \delta + (1 - 2\omega)\delta - (2 - \dot{\mu}_t)(\partial_x \delta + \omega') - \delta^2 \\ &\approx \partial_x^2 \delta + 2\partial_x \delta + (1 - 2\omega)\delta - (2 - \dot{\mu}_t)\omega' \end{aligned}$$

where one neglected two second order terms (recall that $\delta \rightarrow 0$ and $2 - \dot{\mu}_t \rightarrow 0$). With $\mu_t \approx 2t - \frac{3}{2} \log t$, one expects $(2 - \dot{\mu}_t) \sim 3/(2t)$ for large times. This means that

$$\delta(x, t) \sim \frac{3}{2t} \eta(x) \quad \text{as } t \rightarrow \infty,$$

with $\eta(x)$ the unique solution to

$$\eta'' + 2\eta' + (1 - 2\omega)\eta = \omega' \quad \eta(0) = 0 \quad \eta(-\infty) = 0.$$

(The $\partial_t \delta = \mathcal{O}(t^{-2})$ term is also negligible compared to δ , so that δ satisfies a nonhomogeneous second order linear equation. We eliminate other solutions by using $\delta(0, t) = 0$, and $\delta(\pm\infty, t) = 0$.)

One checks that $\eta(x) \sim -Ax^3 e^{-x}$ for large x , so that the difference

$$\begin{aligned} \varphi(r, t) - \hat{\varphi}(r) &= \int dx e^{rx} \left[h(\mu_t + x, t)^2 - \omega(x)^2 \right] \\ &= \int dx e^{rx} \delta(x, t) \left[h(\mu_t + x, t) + \omega(x) \right] \end{aligned}$$

converges nicely for r around 1 (and even up to $r = 2 - \epsilon$), so that one obtains (13).

6 A small ϵ expansion

To illustrate the methods used in the present paper to obtain the asymptotic expansion of μ_t , we give here (without going into the details of the computation) the small ϵ expansion of

$$I = \int_0^\infty dt e^{-\epsilon^2 t + (1-\epsilon)(\mu_t - 2t)},$$

where μ_t is an arbitrary function such that, as $t \rightarrow \infty$,

$$\mu_t = 2t - \frac{3}{2} \log t + a + \frac{b}{\sqrt{t}} + \frac{c \log t + d}{t} + o(t^{-1}),$$

for arbitrary constants a, b, c, d . One finds

$$\begin{aligned} I &= A_0 + A_1 \epsilon + 2e^a (b + 3\sqrt{\pi}) \epsilon^2 \log \epsilon + A_2 \epsilon^2 \\ &\quad - 3e^a (b + 3\sqrt{\pi}) \epsilon^3 \log^2 \epsilon \\ &\quad + e^a \left[\left(15 - \frac{8}{3}c - 18 \log 2 \right) \sqrt{\pi} \right. \\ &\quad \left. - (3\gamma_E + 2a - 1)(b + 3\sqrt{\pi}) \right] \epsilon^3 \log \epsilon \\ &\quad + A_3 \epsilon^3 + o(\epsilon^3), \end{aligned}$$

with γ_E the Euler constant. Notice that the singular terms only depend on the asymptotic behaviour of μ_t , while the regular terms A_0, A_1, \dots depend on the whole function μ_t . For instance, $A_0 = \int_0^\infty dt e^{\mu_t - 2t}$ and $A_1 = -e^a 2\sqrt{\pi} + \int_0^\infty dt e^{\mu_t - 2t} (2t - \mu_t)$. The value of A_0 is obvious, the value of A_1 is maybe less obvious, and A_2 and A_3 have complicated expressions.

To remove the singularities in the expansion of I , the only possible choice is $b = -3\sqrt{\pi}$ and $c = \frac{9}{8}(5 - 6 \log 2)$.

7 Conclusion

In this letter, we have presented a new method to study the Fisher-KPP equation. It relies on a single relation (5) between the initial condition h_0 (through Ψ) and the position μ_t of the front. A careful analysis of the singularities in (5) leads to the large time asymptotics of the position of the front.

In [3, 12], we already used a similar method to study, respectively, a linear front equation with a free boundary or on the lattice. The $(\log t)/t$ term was first identified, for the lattice case in [17]. The main progress of the present work is to show that this method is not limited to linear fronts, but works also in the non-linear case. Our main relation in [3] was simpler than (5) because the term $\varphi(r, t)$ was absent. However, we argue in Section 5 that $\varphi(r, t)$ converges fast enough as $t \rightarrow \infty$ for the large time analysis in [3] to apply equally in the present setting, for the Fisher-KPP equation.

The method presented here is robust, and can be adapted to a wide variety of front equations. If one writes an equation such as

$$\partial_t h = \partial_x^2 h + h - F(h),$$

with the h^2 term replaced by an arbitrary non-linearity, very little is needed to make sure that (5) still holds (after changing the definition of φ). In fact, $F(h)$ could even be a functional of h rather than a function, and (5) holds for instance for the non-local Fisher-KPP [18]

$$\partial_t h = \partial_x^2 h + h - h\rho * h,$$

where $\rho > 0$ is some well-behaved kernel with $\int \rho = 1$. One could also work with equations discrete in space and/or time [9, 12].

J.B. was partially supported by ANR grants ANR-14-CE25-0014 (ANR GRAAL) and ANR-14-CE25-0013 (ANR NONLOCAL)

References

- [1] R.A. Fisher, The wave of advance of advantageous genes, *Annals of Eugenics* **7**, 355 (1937)
- [2] A. Kolmogorov, I. Petrovsky, N. Piscounov, étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bull. Univ. État Moscou, A* **1**, 1 (1937)
- [3] J. Berestycki, E. Brunet, B. Derrida, Exact solution and precise asymptotics of a Fisher-KPP type front, *Journal of Physics A: Mathematical and Theoretical* **51**, 035204 (2017)
- [4] D.G. Aronson, H.F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, in *Partial Differential Equations and Related Topics*, Lecture Notes in Mathematics volume 446 (Springer-Verlag, 1975), pp. 5–49
- [5] H.P. McKean, Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov, *Communications on Pure and Applied Mathematics* **28**, 323 (1975)
- [6] M.D. Bramson, Maximal displacement of branching Brownian motion, *Communications on Pure and Applied Mathematics* **31**, 531 (1978)
- [7] K. Uchiyama, The behavior of solutions of some non-linear diffusion equations for large time, *Journal of Mathematics of Kyoto University* **18**, 453 (1978)
- [8] M.D. Bramson, Convergence of solutions of the Kolmogorov equation to travelling waves, *Memiors of the American Mathematical Society* **44** (1983)
- [9] E. Brunet, B. Derrida, Shift in the velocity of a front due to a cutoff, *Physical Review E* **56**, 2597 (1997)

- [10] U. Ebert, W. van Saarloos, Front propagation into unstable states: universal algebraic convergence towards uniformly translating pulled fronts, *Physica D* **146**, 1 (2000)
- [11] A.H. Mueller, S. Munier, Phenomenological picture of fluctuations in branching random walks, *Physical Review E* **90**, 042143 (2014)
- [12] E. Brunet, B. Derrida, An exactly solvable travelling wave equation in the Fisher-KPP class, *Journal of Statistical Physics* **161**, 801 (2015)
- [13] C. Henderson, Population stabilization in branching brownian motion with absorption and drift, *Communications in Mathematical Sciences* **14**, 973 (2016)
- [14] J. Berestycki, E. Brunet, S.C. Harris, M. Roberts, Vanishing corrections for the position in a linear model of FKPP fronts, *Communications in Mathematical Physics* **349**, 857 (2016)
- [15] J. Nolen, J.M. Roquejoffre, L. Ryzhik. Refined long time asymptotics for Fisher-KPP fronts (2017). URL <http://arxiv.org/abs/1607.08802>
- [16] C. Graham. Precise asymptotics for Fisher-KPP fronts (2017). URL <https://arxiv.org/abs/1712.02472>
- [17] J. Berestycki, E. Brunet. A note on the convergence of the Fisher-KPP front centred around its α -level (2016). URL <http://arxiv.org/abs/1603.06005>
- [18] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, The non-local Fisher-KPP equation: travelling waves and steady states, *Nonlinearity* **22**, 2813 (2009)