A Mathematical Game Semantics of Concurrency and Nondeterminism

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Abstract. Concurrent games as event structures form a partial order model of concurrency where concurrent behaviour is captured by nondeterministic concurrent strategies—a class of maps of event structures. Extended with winning conditions, the model is also able to give semantics to logics of various kinds. An interesting subclass of this game model is the one considering deterministic strategies only, where the induced model of strategies can be fully characterised by closure operators. The model based on closure operators exposes many interesting mathematical properties and allows one to define connections with many other semantic models where closure operators are also used. However, such a closure operator semantics has not been investigated in the more general nondeterministic case. Here we do so, and show that some nondeterministic concurrent strategies can be characterised by a new definition of nondeterministic closure operators which agrees with the standard game model for event structures and with its extension with winning conditions.

Keywords: Concurrent games, Event structures, Closure operators.

1 Introduction

Event structures [13] are a canonical model of concurrency within which the partial order behaviour of nondeterministic concurrent systems can be represented. In event structures, the behaviour of a system is modelled via a partial order of events which are used to explicitly model the causal dependencies between the events that a computing system performs. Following this approach, in the model of event structures, the interplay between *concurrency* (independence of events) and *nondeterminism* (conflicts between events) can be naturally captured.

Event structures have a simple two-player game-theoretic interpretation [16]. Within this framework, games are represented by event structures with polarities, and a strategy on a game is a polarity-preserving map of event structures satisfying some behaviour-preserving properties. In [16], concurrent games were presented as event structures and proposed as a new, alternative basis for the semantics of concurrent systems and programming languages. The definition of strategies as presented in [16] was given using *spans of event structures*—a family of maps of event structures. This definition has been both generalised and specialised to better understand particular classes of systems/games.

For instance, in [20] the original definition of strategies was given a characterisation based on *profunctors*, and related sheaves and factorisation systems, a more abstract presentation that can provide links with other models of concurrency based on games. In the other direction, in [19], Winskel studied a subclass of concurrent systems corresponding to deterministic games. In this simpler setting, concurrent strategies were shown to correspond to *closure operators*.

In this paper, we will investigate a model of strategies that is intermediate between the representations based on closure operators (which correspond to *deterministic strategies*) and profunctors (which correspond to the general model of *nondeterministic strategies*). In particular, we provide a mathematical model, which builds on closure operators and has a simple game-theoretic interpretation, where some forms of concurrency and nondeterminism are allowed to coexist.

Semantic frameworks based on closure operators are not new. In fact, they have been used in various settings as a semantic basis, amongst other reasons, because they can provide a mathematically elegant model of concurrent behaviour—see, *e.g.*, [3, 7, 14, 17, 19], for some examples. In particular, semantics based on closure operators provide an intuitively simple *operational* reading of their behaviour. However, such a simplicity comes at a price: the interplay between concurrency and nondeterminism must be severely restricted.

The model we provide here inherits many of the desirable features of systems with closure operator semantics, but also some of its limitations. In particular, it can be used to represent concurrent systems/games represented as event structures having a property called *race-freedom*, a structural condition on event structure games which ensures that no player can interfere with the moves available to the other. Our main results are significant since most known applications of games as event structures fall within the scope of the class of race-free games (cf. Section 6). The various models of games and strategies we have described above can be organised, in terms of *expressive* power, as shown in Figure 1.

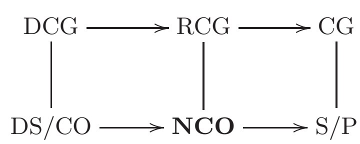

Fig. 1: The following abbreviations are used: Deterministic Concurrent Games (DCG); Race-free Concurrent Games (RCG); General Concurrent Games (CG); Deterministic Strategies (DS); Closure Operators (CO); Nondeterministic CO (NCO); Strategies as spans of event structures (S) and profunctors (P). The model of strategies in bold (**NCO**) is the one investigated in this paper.

Structure of the Paper. The rest of the paper is organised as follows. Section 2 presents some background material on concurrent games as event structures and Section 3 introduces nondeterministic closure operators. Section 4 describes when and how concurrent strategies can be characterised as nondeterministic closure operators and Section 5 extends such a characterisation to games with winning conditions. Section 6 concludes, describes some relevant related work, and puts forward a number of potential interesting application domains.

2 Concurrent Games as Event Structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy axioms:

$$\begin{array}{l} \{e' \mid e' \leq e\} \text{ is finite for all } e \in E, \\ \{e\} \in \mathrm{Con} \text{ for all } e \in E, \\ Y \subseteq X \in \mathrm{Con} \implies Y \in \mathrm{Con}, \quad \text{and} \\ X \in \mathrm{Con}\ \&\ e \leq e' \in X \implies X \cup \{e\} \in \mathrm{Con}. \end{array}$$

The *configurations* of E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\implies X \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \implies e' \in x$.

We write $\mathcal{C}(E)$ for the set of configurations of E. We say that an event structure is well-founded if all its configurations are finite. We only consider well-founded event structures. Two events e_1, e_2 which are both consistent and incomparable with respect to causal dependency in an event structure are regarded as *concurrent*, written $e_1\ co\ e_2$. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$ and no event in between plays an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note that if $X \in \mathrm{Con}$ then $[X] \in \mathrm{Con}$. We use $x {-\!\!\subset} y$ to mean y covers x in $\mathcal{C}(E)$, *i.e.*, $x \subset y$ with nothing in between, and $x \stackrel{e}{-\!\!\!-\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}(E)$ and event $e \notin x$. We use $x \stackrel{e}{-\!\!\!-\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \stackrel{e}{-\!\!\!-\!\!\subset} y$ for some configuration y.

Let E and E' be event structures. A *map* of event structures is a partial function on events $f : E \to E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and if $e_1, e_2 \in x$ and $f(e_1) = f(e_2)$ (with both defined) then $e_1 = e_2$. The map expresses how the occurrence of an event e in E induces the coincident occurrence of the event $f(e)$ in E' whenever it is defined. Maps of event structures compose as partial functions, with identity maps given by identity functions. Thus, we say that the map is *total* if the function f is total.

The category of event structures is rich in useful constructions on processes. In particular, *pullbacks* are used to define the composition of *strategies*, while *restriction* (a form of equalizer) and the *defined part* of maps will be used in defining strategies. Any map of event structures $f : E \to E'$, which may be a partially defined on events, has a *defined part* the total map $f_0 : E_0 \to E'$, in which the event structure E_0 has events those of E at which f is defined, with causal dependency and consistency inherited from E, and where f_0 is simply f restricted to its domain of definition. Given an event structure E and a subset $R \subseteq E$ of its events, the *restriction* $E \upharpoonright R$ is the event structure comprising events $\{e \in E \mid [e] \subseteq R\}$ with causal dependency and consistency inherited from E; we sometimes write $E \setminus S$ for $E \upharpoonright (E \setminus S)$, where $S \subseteq E$.

Event Structures with Polarity Both a game and a strategy in a game are represented with event structures with polarity, comprising an event structure E together with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ (Player) or $-$ (Opponent) to its events; the events correspond to moves. Maps of event structures with polarity, are maps of event structures which preserve polarities. An event structure with polarity E is *deterministic* iff

$$\forall X \subseteq_{\text{fin}} E.\ Neg[X] \in \text{Con}_E \implies X \in \text{Con}_E\,,$$

where $Neg[X] =_{\text{def}} \{e' \in E \mid pol(e') = - \ \&\ \exists e \in X.\ e' \leq e\}$. We write $Pos[X]$ if $pol(e') = +$. The *dual*, $E^{\perp}$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities.

Given two sets of events x and y, we write $x \subset^{+} y$ to express that $x \subset y$ and $pol(y \setminus x) = \{+\}$; similarly, we write $x \subset^{-} y$ iff $x \subset y$ and $pol(y \setminus x) = \{-\}$.

Games and Strategies Let A be an event structure with polarity—a game; its events stand for the possible moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game.

A *strategy (for Player)* in A is a total map $\sigma : S \to A$ from an event structure with polarity S, which is both *receptive* and *innocent*. Receptivity ensures an openness to all possible moves of Opponent. Innocence, on the other hand, restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game.
Receptivity: A map σ is *receptive* iff
$\sigma x \overset{a}{-\!\!\!\subset}\ \&\ pol_A(a) = - \implies \exists! s \in S.\ x \overset{s}{-\!\!\!\subset}\ \&\ \sigma(s) = a\,.$
Innocence: A map σ is *innocent* iff
$s \rightarrowtriangle s'\ \&\ (pol(s) = + \text{ or } pol(s') = -) \implies \sigma(s) \rightarrowtriangle \sigma(s')$.
Say a strategy $\sigma : S \to A$ is *deterministic* if S is deterministic.

Composing Strategies Suppose that $\sigma : S \to A$ is a strategy in a game A. A counter-strategy is a strategy of Opponent, so a strategy $\tau : T \to A^{\perp}$ in the dual game. The effect of playing-off a strategy σ against a counter-strategy τ is described via a pullback. Ignoring polarities, we have total maps of event structures $\sigma : S \to A$ and $\tau : T \to A$. Form their pullback,

$$\begin{array}{ccc} P & \xrightarrow{\Pi_2} & T \\ {\scriptstyle \Pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow[\sigma]{} & A\,. \end{array}$$

The event structure P describes the play resulting from playing-off σ against τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}(P)$. A maximal z images to a configuration $\sigma\Pi_1 z = \tau\Pi_2 z$ in $\mathcal{C}(A)$. Define the set of *results* of playing-off σ against τ to be

$$\langle \sigma, \tau \rangle =_{\text{def}} \{\sigma\Pi_1 z \mid z \text{ is maximal in } \mathcal{C}(P)\}\,.$$

Winning Conditions A *game with winning conditions* comprises $G = (A, W)$ where A is an event structure with polarity and the set $W \subseteq \mathcal{C}(A)$ consists of the *winning configurations (for Player)*. Define the *losing conditions (for Player)* to be $L = \mathcal{C}(A) \setminus W$. The dual $G^{\perp}$ of a game with winning conditions $G = (A, W)$ is defined to be $G^{\perp} = (A^{\perp}, L)$, a game where the roles of Player and Opponent are reversed, as are correspondingly the roles of winning and losing conditions.

A strategy in G is a strategy in A. A strategy in G is regarded as *winning* if it always prescribes moves for Player to end up in a winning configuration, no matter what the activity or inactivity of Opponent. Formally, a strategy $\sigma : S \to A$ in G is *winning (for Player)* if $\sigma x \in W$ for all $\oplus$-maximal configurations $x \in \mathcal{C}(S)$—a configuration x is $\oplus$-maximal if whenever $x \stackrel{s}{\longrightarrow\!\!\!\subset}$ then the event s has $-$ve polarity. Equivalently, a strategy σ for Player is winning if when played against any counter-strategy τ of Opponent, the final result is a win for Player; precisely, it can be shown [5] that a strategy σ is a winning for Player iff all the results $\langle \sigma, \tau \rangle$ lie within W, for any counter-strategy τ of Opponent.

3 Nondeterministic Closure Operators

It is often useful to think "operationally" of a strategy $\sigma : S \to A$ as an function that associates to a configuration of A another configuration of A that, potentially, can be played next. Since, in general, a concurrent strategy can be nondeterministic then such a function may not be between configurations of A, but rather a function from $\mathcal{C}(A)$ to the powerset of $\mathcal{C}(A)$, denoted by $\wp(\mathcal{C}(A))$. In particular, for *race-free* concurrent games—those games which satisfy a structural condition called race-freedom, to be defined in the following section—given a strategy $\sigma : S \to A$, we define $\sigma_\mu : \mathcal{C}(A) \to \wp(\mathcal{C}(A))$ with respect to σ as follows:

$$y' \in \sigma_\mu(y) \text{ iff } \exists x, x' \in \mathcal{C}(S).\ \sigma x = y \ \&\ x' \in f_\mu^{\rightarrow}(x) \ \&\ \sigma x' = y'$$

for some operator $f_\mu^{\rightarrow} : \mathcal{C}(S) \to \wp(\mathcal{C}(S))$, also defined with respect to $\sigma : S \to A$, as a *nondeterministic closure operator* $f^{\rightarrow} : \mathcal{C}(S) \to \wp(\mathcal{C}(S))$, that is, as an operator from $\mathcal{C}(S)$ to $\wp(\mathcal{C}(S))$ that satisfies the following properties:

1. $\forall x' \in f^{\rightarrow}(x).\ x \subseteq^{+} x'$,
2. $\forall x' \in f^{\rightarrow}(x).\ \{x'\} = f^{\rightarrow}(x')$,
3. $x_1 \subseteq^{-} x_2 \Longrightarrow f^{\rightarrow}(x_1) \subseteq f^{\rightarrow}(x_2)$

In fact (for 3):

$$\begin{array}{l} \forall x_1' \in f^{\rightarrow}(x_1).\ \exists\, x_2' \in f^{\rightarrow}(x_2).\ x_1' \subseteq x_2' \text{ and} \\ \forall x_2' \in f^{\rightarrow}(x_2).\ \exists! x_1' \in f^{\rightarrow}(x_1).\ x_1' \subseteq x_2'. \end{array}$$

That is, such that for some x, x' in $\mathcal{C}(S)$ and $f_\mu^{\rightarrow}$, the diagram below commutes:

$$\begin{array}{ccc} x & \xrightarrow{f_\mu^{\rightarrow}} & x' \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle\sigma} \\ y & \xrightarrow{\sigma_\mu} & y' \end{array}$$

Remark *If $f_\mu^{\rightarrow}$ is* deterministic *in the sense that the image of $f_\mu^{\rightarrow}(x)$ is a singleton set, for every $x \in \mathcal{C}(S)$, then $f_\mu^{\rightarrow}$ can be regarded as a usual closure operator on the configurations of S, with the order given by set inclusion. To see this, simply let x', $\{x'\}$, and $f_\mu^{\rightarrow}(x)$ be $\mathrm{cl}^{\rightarrow}(x)$, where $\mathrm{cl}^{\rightarrow}(x) = \bigcup f_\mu^{\rightarrow}(x)$, and eliminate quantifiers as they are no longer needed. Moreover, the condition that $Pos[x_1] = Pos[x_2]$ (given by $x_1 \subseteq^- x_2$ in 3) can be eliminated too as no positive event of $\bigcup f_\mu^{\rightarrow}(x_1)$ is inconsistent with a positive event of x_2. And since $f_\mu^{\rightarrow}(x)$ is the set of maximal configurations in $\{x' \in \mathcal{C}(S) \mid x \subseteq^+ x'\}$ we know that $f_\mu^{\rightarrow}$ preserves negative events; then we can also omit all references to polarities so as to yield the following presentation: 1. $x \subseteq \mathrm{cl}^{\rightarrow}(x)$; 2. $\mathrm{cl}^{\rightarrow}(x) = \mathrm{cl}^{\rightarrow}(\mathrm{cl}^{\rightarrow}(x))$; 3. $x_1 \subseteq x_2 \Longrightarrow \mathrm{cl}^{\rightarrow}(x_1) \subseteq \mathrm{cl}^{\rightarrow}(x_2)$. These facts are formally presented below.*

Proposition 1 (Deterministic games). *Let A be a game and $\sigma : S \to A$ a concurrent strategy. If S is deterministic, then $f_\mu^{\rightarrow}$ is a closure operator.*

4 Strategies as Nondeterministic Closure Operators

In [5] it was shown that in order to build a bicategory of concurrent games, where the objects are event structures and the morphisms are concurrent strategies (that is, innocent and receptive maps of event structures), a structural property called *race-freedom* had to be satisfied by the 'copy-cat' strategy in order to behave as an identity in such a bicategory. Race-freedom proved again to be a fundamental structural property of games as event structures when studying games with winning conditions: it was, in [5], shown to be a necessary and sufficient condition for the existence of winning strategies in well-founded games.

Race-freedom, formally defined below, is satisfied by all concurrent games we are aware of. Informally, race-freedom is a condition that prevents one player from interfering with the moves available to the other player. Formally, a game A is *race-free* if and only if for all configurations $y \in \mathcal{C}(A)$ the following holds:

$$y \xrightarrow{a}\subset \;\&\; y \xrightarrow{a'}\subset \;\&\; pol(a) \neq pol(a') \implies y \cup \{a, a'\} \in \mathcal{C}(A)\,.$$

Race-freedom proves to be useful again. It is shown to be a necessary and sufficient condition characterising strategies as nondeterministic closure operators. To see that race-freedom is necessary, consider the following simple example.

Example 2 (Race-freedom). Let A be the game depicted below. The wiggly line means conflict, that is, that the set of events $\{\ominus, \oplus\}$ is not a configuration of A.

$$\ominus \sim\!\sim\!\sim \oplus$$

This game is not race-free. Moreover, there is a strategy for Player that cannot be represented as a nondeterministic closure operator, namely, the strategy $\sigma : S \to A$ that plays $\oplus$. To see that this is the case, consider condition 3 of nondeterministic closure operators (the other two conditions are satisfied). Let $f^{\rightarrow} : \mathcal{C}(S) \to \wp(\mathcal{C}(S))$ be a candidate nondeterministic closure operator to represent σ. Observe that even though $\emptyset \subseteq \{\ominus\}$, it is not the case that $f^{\rightarrow}(\emptyset) \subseteq f^{\rightarrow}(\{\ominus\})$; indeed, $f^{\rightarrow}(\emptyset) = \{\{\oplus\}\}$ and $f^{\rightarrow}(\{\ominus\}) = \{\{\ominus\}\}$. □

Proposition 3. *Let A be a concurrent game that is not race-free. Then, there is a nondeterministic strategy $\sigma : S \rightarrow A$ for Player that do not determine a nondeterministic closure operator on $\mathcal{C}(S)$—and similarly for Opponent.*

Then, if one wants to build a model where *every* strategy has a nondeterministic closure operator representation for *every* game, race-freedom will be a *necessary* condition. This is not a surprising result since, as mentioned before, copy-cat strategies, which can be represented as conventional closure operators, require this condition. What is, therefore, much more interesting is that race-freedom is in fact a *sufficient* condition too, as shown by the result below.

Theorem 4 (Closure operator characterisation). *Let $\sigma : S \rightarrow A$ be a nondeterministic concurrent strategy in a race-free concurrent game A. Then, the strategy σ determines a nondeterministic closure operator on $\mathcal{C}(S)$.*

Proof (Sketch). Since A is race-free then S is race-free (because S cannot introduce inconsistencies between events of opposite polarity). Then, $f_{\mu}^{\rightarrow}(x)$ is the set of $\oplus$-maximal configurations that cover x, namely $f_{\mu}^{\rightarrow}$ is the nondeterministic closure operator determined by σ, as shown next.

Suppose $\sigma x = y$ & $\sigma x' = y'$ & $y' \in \sigma_{\mu}(y)$. Then (for 1) $x \subseteq x'$ and $Neg[x] = Neg[x']$, for every $x' \in f_{\mu}^{\rightarrow}(x)$. And, (for 2) as every x' is $\oplus$-maximal, then it cannot be extended positively by any configuration; hence, $f_{\mu}^{\rightarrow}(x') = \{x'\}$. Now, (for 3) suppose $\sigma x_1 = y_1$ & $\sigma x_2 = y_2$ & $y_2 \in \sigma_{\mu}(y_1)$, with $x_1 \subseteq^{-} x_2$ (and therefore $y_1 \subseteq^{-} y_2$). Thus, since

- $f_{\mu}^{\rightarrow}(x_1)$ is the set of $\oplus$-maximal configurations that cover x_1, and
- $Pos[x_1] = Pos[x_2]$, and
- S is race-free,

then $f_{\mu}^{\rightarrow}(x_1) \subseteq f_{\mu}^{\rightarrow}(x_2)$, because x_2 enables at least as many $\oplus$-events as x_1; recall that $x_1 \subseteq^{-} x_2$ means that $Neg[x_1] \subseteq Neg[x_2]$ and $Pos[x_1] = Pos[x_2]$. □

Informally, what Theorem 4 shows is that whereas in the deterministic case, a strategy $\sigma : S \rightarrow A$ can be seen as a partial function between the configurations of A which satisfies the axioms of a closure operator, in the nondeterministic race-free setting, a strategy can be seen as a partial function from $\mathcal{C}(A)$ to $\wp(\mathcal{C}(A))$ which satisfies the axioms of a nondeterministic closure operator. This, we believe, gives a more operational view of strategies than the one given by strategies as maps of event structures [16] or as certain fibrations and profunctors [20].

Race-free/Probabilistic Games. Because our nondeterministic closure operator characterisation of strategies only applies to race-free games, a natural question is whether race-freedom is either a mild or a severe modelling restriction. (We already know that race-freedom is not a real restriction with respect to sequential systems, but it is a restriction with respect to concurrent ones.) Even though we do not address such a question in this paper, we would like to note that a possible way to relax the race-freedom structural condition is by moving to a quantitative setting where races were allowed but only in a probabilistic manner, that is, to a setting where players' choices are associated with a probability distribution.

5 Characterising Winning Strategies

Theorem 4 provides a key closure operator (game semantic) characterisation of the model of nondeterministic concurrent strategies in games as event structures. It relies, in particular, in the fact that the games are race-free. Under the same conditions, other general theorems for games with winning conditions can also be given with respect to the new closure operator game semantics. In particular, we extend the characterisation of strategies as nondeterministic closure operators to games with winning conditions. We start by providing the following result.

Theorem 5. *Let A be a race-free game. A strategy $\sigma : S \to A$ in (A, W) is winning iff $\sigma_\mu(y) \subseteq W$ for all $y \in \mathcal{C}(A)$ under σ.*

Based on Theorem 5, which relates the standard definition of strategies as maps of event structures with strategies as nondeterministic closure operators, known techniques to characterise winning strategies can be used so that such concurrent strategies can be characterised, instead, with respect to the existence of nondeterministic closure operators. First, let us define the set of results of a concurrent game via nondeterministic closure operators.

Given A and two nondeterministic closure operators σ_μ and τ_μ for Player and Opponent, their *one-step composition* at $y \in \mathcal{C}(A)$, denoted by $(\sigma_\mu \bowtie \tau_\mu)(y)$, induces the following set of configurations: $\{\sigma_\mu(y') \subseteq \mathcal{C}(A) \mid y' \in \tau_\mu(y)\}$. Now, let the set R be the *partial results* of playing-off σ_μ against τ_μ, which is inductively defined as follows: $(\sigma_\mu \bowtie \tau_\mu)(\emptyset) \subseteq R$ and if $y \in R$ then $(\sigma_\mu \bowtie \tau_\mu)(y) \subseteq R$. Finally, similar to the case where the results of a concurrent game are computed using a pullback construction, we define the set of *results* of the game as the maximal elements of R, which we simply denote by $\sigma_\mu \bowtie \tau_\mu$. Using these definitions one can show that $\bowtie$ is a commutative operator, that is, that the following holds.

Proposition 6. *Let σ and τ be two strategies for Player and Opponent. Then*

$$\sigma_\mu \bowtie \tau_\mu = \tau_\mu \bowtie \sigma_\mu$$

The equivalence relation given by Proposition 6 ensures that the two strategies can be played in *parallel* while preserving the same set of results—a property of the composition of strategies in the model of games as event structures.

Based on the above results, one can also show that winning strategies, when represented as nondeterministic closure operators, can be characterised with respect to the sets of results obtained when composing them with every deterministic strategy, represented as closure operators, for the other player. Finally, the following result fully captures the notion of winning in race-free games.

Theorem 7 (Winning strategies). *Let A be a race-free concurrent game. The nondeterministic closure operator σ_μ is winning for Player if and only if $(\sigma_\mu \bowtie \tau_\mu) \subseteq W$, for all closure operators τ_μ for Opponent.*

Theorem 7 follows from results about winning strategies [5], and the fact that not only every strategy $\sigma : S \to A$ determines a unique (partial) nondeterministic closure operator $\sigma_\mu : \mathcal{C}(A) \to \wp(\mathcal{C}(A))$, but also every operator σ_μ is determined by some (total) nondeterministic closure operator $f^{\to} : \mathcal{C}(S) \to \wp(\mathcal{C}(S))$.

6 Conclusions, Application Domains, and Related Work

In this paper, we studied a mathematical model, which builds on closure operators and has a game-theoretic interpretation, where some forms of concurrency and nondeterminism are allowed to coexist. In particular, the model extends those based on deterministic games—and hence on closure operators too.

Indeed, deterministic games/strategies are already important in the model of games as event structures. Strategies in this kind of games can represent stable spans and stable functions [18], Berry's dI-domains [4], closure operator models of CCP [17], models of fragments of Linear Logic [1,3], and innocent strategies in simple games [8], which underlie Hyland–Ong [9] and AJM [2] games. Strategies in deterministic games are also equivalent to those in Melliès and Mimram [11] model of asynchronous games with receptive ingenious strategies.

However, none of the models above mentioned allow a free interplay of nondeterminism and concurrency: either nondeterminism is allowed in a sequential setting, or concurrency is studied in a deterministic setting. Still, nondeterminism is needed in certain scenarios, or may be a desirable property. We would like to mention three prominent cases: concurrent game models of logical systems [5], formal languages with nondeterministic behaviour [12, 15], and concurrent systems with partial order behaviour—also called 'true-concurrency' systems [13].

Logical systems. In order to give a concurrent game semantics of logical systems such as classical or modal logics, the power to express nondeterministic choices is needed, in particular, in order to be able to interpret disjunctions in a concurrent way—a "parallel or" operator. Deterministic strategies—and hence conventional closure operators—are unable to do this in a full and faithful way.

Formal languages. Another example where nondeterminism is allowed is within formal languages such as ntcc [12], a nondeterministic extension of CCP, and in simple programming languages with nondeterminism as the one initially studied by Plotkin using powerdomains [15]. Whereas in the former case no game theoretic model has been studied, in the latter case no closure operator semantics has been investigated. Indeed, to the best of our knowledge, no game theoretic characterisation of powerdomains has been defined so far. An interesting potential application would be a (nondeterministic closure operator) game characterisation of Kahn–Plotkin concrete domains [10] given the simpler structure of nondeterministic choices allowed in such a denotational model.

True-concurrency. In concurrent systems with partial order behaviour, such as Petri nets or asynchronous transition systems, both concurrency and nondeterminism are allowed at the same time, which prevents the use of conventional closure operators as the basis for the definition of a fully abstract model. In all of these cases, the model of concurrent games as event structures could be used as an underlying semantic framework, and in particular our nondeterministic closure operator characterisation/semantics when restricted to race-free systems. A good starting point would be to consider free-choice nets [6], since in this case race-freedom can be easily imposed by associating it with conflicts in the net.

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