Partial sum process of orthogonal series as rough process

Danyu Yang
Keble College
University of Oxford

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Abstract

This thesis is divided into six chapters.

Chapter 1 is an introduction, an overview of rough path theory and of the contribution of this thesis.

Chapter 2 is about space of paths taking value in Banach space, where we define p-variation and set notations.

Chapter 3 is about area operator and geometric 2-rough path. (1) When continuous bounded-variation paths are equipped with 2-variation and their areas are equipped with p-variation, the area operator is closable when $p = 1$ but not closable when $p > 1$. When $p = 1$, the area operator is closable but unbounded, and the paths in the closure are not linear. (2) The area defined by Riemann-Stieltjes integral is the only possible function to enhance a vanishing 2-variation path to a geometric 2-rough path, but the integral may not exist.

Chapter 4 is about rough path theory, where we state the main theorems but without detailed proof.

In Chapter 5, we prove that the partial sum process of orthogonal series $\sum_{n\geq 0} c_n u_n$ is a geometric 2-rough process, for any orthonormal system $\{u_n\}$ in L^2 and any sequence of numbers $\{c_n\}$ satisfying $\sum_{n\geq 0} \log_2 (n+1)^2 |c_n|^2 < \infty$. Since being a geometric 2-rough process implies the existence of a limit function up to a null set, this theorem could be treated as an improvement of Menshov-Rademacher theorem. Moreover, for Fourier series and i.i.d. sequence, the condition can be strengthened to $\sum_{n\geq 0} \log_2 (n+1) |c_n|^2 < \infty$.

In Chapter 6, we work on the partial sum process of L^2 Fourier series. We prove that for $s > 0$ and $f \in L^2 \left([-\pi, \pi], \mathbb{R}^d\right)$ with Fourier series $\{c_n\}$, $\sum_{n=0}^{\infty} \log_2 (n+1)^{2s} |c_n|^2 < \infty$ is equivalent to

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(u) - f(v)|^2}{\left|\sin \frac{u-v}{2}\right|} \left(\log_2 \frac{\pi}{\left|\sin \frac{u-v}{2}\right|} \right)^{2s-1} dudv < \infty.$$

As a result, if f satisfies $\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(u)-f(v)|^2}{\left|\sin \frac{u-v}{2}\right|} dudv < \infty$, then the partial sum process of the Fourier series of f is a geometric 2-rough process. On the other hand, we construct an L^2 Fourier series whose partial sum process has infinite 2-variation a.e., by using upper semi-continuity of cumulative distribution function of p-variation.

Contents

Chapter 1

Introduction

Rough path theory develops a method to give meanings to differential equations driven by irregular paths in such a way that the solutions (to rough differential equations) are continuous with respect to the driving paths. Rough path achieves this goal by enhancing the original driving path with higher leveled functions and identifying a family of norms on these functions (called rough norms). For a continuous path with bounded variation, there exists a canonical choice for these higher leveled functions—the iterated integrals of the original path. The path and higher leveled functions are considered as a single object, called the signature (of the path).

For a continuous bounded variation path γ, its signature $S(\gamma)$ is a function on $\{(s,t)\,|0\leq s\leq t\leq T\}$, defined as the linear combination of iterated integrals ($\otimes$ is the tensor product)

$$\begin{aligned} S(\gamma)(s,t) &:= 1+\gamma^{1}(s,t)+\cdots+\gamma^{k}(s,t)+\cdots \\ \text{with } \gamma^{k}(s,t) &:= \textstyle\int\cdots\int_{s<u_1<\cdots<u_k<t} d\gamma(u_1)\otimes\cdots\otimes d\gamma(u_k). \end{aligned} \tag{1.1}$$

When γ is of bounded variation, all higher leveled iterated integrals are controlled by γ : $\left\|\gamma^{k}(s,t)\right\|\leq(k!)^{-1}\|\gamma\|_{1-var,[s,t]}^{k}$. Actually, suppose we impose the p-variation seminorm on the set of continuous paths on $[0,T]$, (D is any finite partition of $[0,T]$, i.e. $D=\{t_k\}_{k=0}^{n}$ for $0=t_0<t_1<\cdots<t_n=T$)

$$\|\gamma\|_{p-var,[0,T]}:=\left(\sup_{D\subset[0,T]}\sum_{k,t_k\in D}\|\gamma(t_{k+1})-\gamma(t_k)\|^{p}\right)^{\frac{1}{p}}. \tag{1.2}$$

Then based on Young integral [38], if γ is of finite p-variation, $p<2$, then the iterated integrals are well-defined as Riemann-Stieltjes integrals and controllable by the first level path:

$$\left\|\gamma^{k+1}(s,\cdot)\right\|_{p-var,[s,t]} \leq C_p\left\|\gamma^{k}(s,\cdot)\right\|_{p-var,[s,t]}\|\gamma\|_{p-var,[s,t]}\leq(C_p)^{k}\|\gamma\|_{p-var,[s,t]}^{k+1}.$$

However, when γ is of infinite p-variation, $p<2$, the iterated integrals may not exist as Riemann-Stieltjes integrals. Brownian motion is a typical example for such irregular paths, and the definition of its iterated integrals depends on the selection of representative points of subintervals (e.g. Itô/Stratonovich integral). Though depending on the selection of representative points, the integral exists (at least in probability), which results from the symmetry of Brownian motion and is not true for general deterministic paths (as stated below).

A continuous path γ on $[0,T]$ is said to be of vanishing 2-variation, if

$$\lim_{\delta\to 0}\sup_{D\subset[0,T],|D|\leq\delta}\sum_{k,t_k\in D}\left\|\gamma\left(t_{k+1}\right)-\gamma\left(t_k\right)\right\|^2=0. \tag{1.3}$$

Denote γ^D as the piecewisely linear approximation of γ w.r.t. finite partition D. Then based on Wiener's characterization (Thm 5.31 [9]), (1.3) is equivalent to

$$\lim_{|D|\to 0}\left\|\gamma^D-\gamma\right\|_{2-var}=0. \tag{1.4}$$

Then paths of vanishing 2-variation can be approximated by continuous bounded variation paths in 2-variation, and are of finite 2-variation.

In Chapter 3, we give an example of a continuous path $f:[0,1]\to\mathbb{R}^2$, which is of vanishing 2-variation, but for any $a\in[-\infty,\infty]$, there exists a nested sequence of finite partitions $\{D_n\}$ s.t. $\lim_{n\to\infty}\int_0^1 f^{D_n}\otimes df^{D_n}=\begin{pmatrix}0 & a\\ -a & 0\end{pmatrix}$. Thus, based on (1.4), the operator $\gamma\mapsto\int_0^T\gamma\otimes d\gamma$ is neither continuous nor bounded, when considered as defined on continuous bounded variation paths equipped with 2-variation. On the other hand, when $\gamma:[0,T]\to\mathbb{R}^2$ is continuous and of bounded variation, $t\mapsto\int_0^t\gamma_u\otimes d\gamma_u$ is the solution to the second level projection of the differential equation:

$$dS_t=S_t\otimes d\gamma_t,\ \ S_0=(1,\gamma_0,0)\in 1\oplus\mathbb{R}^2\oplus\left(\mathbb{R}^2\right)^{\otimes 2}.$$

Thus the non-continuity and unboundedness of the operator $\gamma\mapsto\int_0^T\gamma\otimes d\gamma$ seems to pose an unsurmountable obstacle to the continuity of solution map.

Rough path intelligently resolved this problem by incorporating $\int\gamma\otimes d\gamma$ as part of the driving signal, and proved the remarkable theorem that: for a continuous path γ with vanishing 2-variation, the pair $\Gamma:=\left(\gamma,\int\gamma\otimes d\gamma\right)$ contains enough information such that, (when V is regular enough) the differential equation

$$dy=V\left(y\right)d\Gamma,\ y_0=\xi,$$

has a unique solution, and the solution is continuous w.r.t. Γ (in rough path metric). However, when γ is of vanishing 2-variation, the Riemann-Stieltjes integral $\int\gamma\otimes d\gamma$ may not exist. One possible solution is to look for another function α, which behaves like $\int\gamma\otimes d\gamma$, so one could combine γ with α and define differential equation driven by (γ,α). This pair (γ,α) is called a 2-rough path. Moreover, if (γ,α) can be approximated by step-2 signatures of continuous bounded variation paths in 2-rough norm (consisting 2-variation of γ and 1-variation of α), then (γ,α) is called a geometric 2-rough path.

In Chapter 3, for a fixed γ with vanishing 2-variation, we try to find the possible α, such that (γ,α) is a geometric 2-rough path. However, as we demonstrate, the Riemann-Stieltjes integral $\int\gamma\otimes d\gamma$ is the only possible candidate to enhance γ (to a geometric

2-rough path), and the integral may not exist. While based on [22], there exists an enhancement of γ into a geometric p-rough path, for any $p > 2$.

As the regularity of the driving path further weakens, higher leveled information is needed. If γ is a continuous path with finite p-variation, then the enhancement of γ into a p-rough path involves $[p]$ functions: $\left(\gamma^1, \gamma^2, \cdots, \gamma^{[p]}\right)$ (with $\gamma^1 = \gamma$), where $(\gamma^j)_j$ satisfy an algebraic condition (Chen's identity) and an analytical condition (finite p-rough norm). The p-rough norm is induced from p-rough metric:

$$d_p\left((\gamma_1^j)_j, (\gamma_2^j)_j\right) := \max_{1\leq j\leq [p]} \left(\sup_D \sum_{k,t_k\in D} \left\|\gamma_1^j\left(t_k, t_{k+1}\right) - \gamma_2^j\left(t_k, t_{k+1}\right)\right\|^{\frac{p}{j}}\right)^{\frac{1}{p}}.$$

A p-rough path is said to be geometric if it can be approximated by step-$[p]$ signature of continuous bounded variation paths in p-rough metric.

Suppose Γ is a geometric p-rough path, then y is said to be a solution of the rough differential equation:

$$dy = V\left(y\right) d\Gamma, \quad y(0) = \xi, \tag{1.5}$$

if there exists a sequence of continuous bounded variation paths $\{\gamma_n\}$ s.t. the step-$[p]$ signature of $\{\gamma_n\}$ converge in p-rough metric to Γ, and the solution y_n of the ordinary differential equation:

$$dy_n = V\left(y_n\right) d\gamma_n, \quad y_n(0) = \xi,$$

converges in uniform norm to y. The solution to (1.5) exists when V is $Lip\left(\gamma\right)$, $\gamma > p-1$, and is unique when V is $Lip\left(\gamma\right)$, $\gamma \geq p$.

Chapter 5 is somehow detached from Chapter 2 $\sim$ 4, and lies at the junction of rough path, classical analysis and harmonic analysis. Our main object of investigation is the partial sum process of general orthogonal series. Suppose $\{u_n\}$ is an orthonormal system in $L^2\left(\Omega, \mathcal{F}, \mu; \mathcal{V}, \langle\cdot,\cdot\rangle\right)$ and $\{c_n\}$ is a sequence of numbers, then the **partial sum process** X of $\sum_{n\geq 0} c_n u_n$ is *the continuous process on the positive half-line, obtained by assigning* $X_n\left(\omega\right) := \sum_{k=0}^n c_k u_k\left(\omega\right)$, $\forall n \in \mathbb{N}$, *and interpolating linearly between adjacent integers.* Thus the partial sum process is of bounded variation on any finite interval, and the only possible oscillation occurs near infinity. We want to control the oscillation near infinity and prove that the partial sum process is a geometric 2-rough process (i.e. geometric 2-rough path almost everywhere), under some decay condition on the coefficients $\{c_n\}$. This decay condition holds for all orthonormal systems—it does not depend on the measure space, nor on the Hilbert space, and it does not depend on the orthonormal system.

Since being a geometric 2-rough process implies the existence of a limit function upto a null set, our theorem has a direct connection with almost everywhere convergence of general orthogonal series.

Menshov-Rademacher Theorem *Suppose* $\{c_n\}$ *is a sequence of numbers satisfying* $\sum_{n\geq 0} \log_2\left(n+1\right)^2 \left|c_n\right|^2 < \infty$, *then* $\sum_n c_n u_n$ *converges almost everywhere for any orthonormal system* $\{u_n\}$. *Moreover,* $\log_2\left(n+1\right)^2$ *can not be replaced by* $o(\log_2\left(n+1\right)^2)$, *and there exists an absolute constant C_1, s.t.*

$$\int_\Omega \sup_{0\leq i\leq j\leq\infty} \| \sum_{k=i}^{j} c_k u_k(\omega) \|^2 \mu(d\omega) \leq C_1 \sum_{n=0}^{\infty} \log_2(n+1)^2 |c_n|^2 . \tag{1.6}$$

We improve this theorem by replacing $\sup_{0\leq i\leq j\leq\infty} \| \sum_{k=i}^{j} c_k u_k(\omega) \|$ by 2-rough norm of the partial sum process X of $\sum_n c_n u_n$, which consists of 2-variation of X and 1-variation of the area of X. Suppose γ is a continuous bounded variation path. The area $A(\gamma)$ of γ is a function on $\{(s,t) | 0 \leq s \leq t \leq T\}$ defined as, ($[u,v] := u \otimes v - v \otimes u$)

$$A(\gamma)(s,t) := \textstyle\iint_{s<u_1<u_2<t} [d\gamma(u_1), d\gamma(u_2)] .$$

Then one can check that $A(\gamma)(s,t) + \frac{1}{2}(\gamma(t) - \gamma(s))^{\otimes 2} = \iint_{s<u_1<u_2<t} d\gamma(u_1) \otimes d\gamma(u_2)$. Thus when γ is of finite 2-variation, there is no critical difference between the 1-variation of the area and 1-variation of the second iterated integral. Note that 1-variation of area is defined differently from that of the path in (1.2). The 1-variation of area is defined as

$$\|A(\gamma)\|_{1-var} := \sup_{D\subset[0,T]} \sum_{k, t_k \in D} \|A(\gamma)(t_k, t_{k+1})\| ,$$

and generally $A(\gamma)(t_k, t_{k+1}) \neq A(\gamma)(0, t_{k+1}) - A(\gamma)(0, t_k)$.

Back to our improvement of Menshov-Rademacher theorem, we prove that

$$\int_\Omega \|X(\omega)\|_{2-var}^2 + \|A(X(\omega))\|_{1-var} \mu(d\omega) \leq C_2 \sum_{n=0}^{\infty} \log_2(n+1)^2 |c_n|^2 . \tag{1.7}$$

Thus, when $\sum_n \log_2(n+1)^2 |c_n|^2 < \infty$, the partial sum process of $\sum_n c_n u_n$ is of finite 2-rough norm a.e. and a geometric 2-rough process, because the partial sum process can be pathwisely approximated by truncations on $[0,n]$, $n \in \mathbb{N}$, which are of bounded variation.

For orthonormal system $\{u_n\}$, satisfying that there exists constant C_3 such that for any sequence of numbers $\{a_n\} \in l^2$

$$\int_\Omega \sup_{0\leq i\leq j\leq\infty} \| \sum_{k=i}^{j} a_k u_k(\omega) \|^2 \mu(d\omega) \leq C_3 \sum_{n=0}^{\infty} |a_n|^2 , \tag{1.8}$$

we can improve $\log_2(n+1)^2$ in (1.7) to $\log_2(n+1)$, and get

$$\int_\Omega \|X(\omega)\|_{2-var}^2 + \|A(X(\omega))\|_{1-var} \mu(d\omega) \leq C_4 \sum_{n=0}^{\infty} \log_2(n+1) |c_n|^2 . \tag{1.9}$$

Fourier series and i.i.d. sequences are examples of orthonormal systems satisfying (1.8), based on Carleson-Hunt inequality and Burkholder-Davis-Gundy inequality. Moreover, for Fourier series, $\log_2(n+1)$ in (1.9) can not be replaced by $o(\log_2(n+1))$ with lower bound given by Dirichlet kernel.

Suppose $f \in L^2\left([-\pi,\pi],\mathbb{R}^d\right)$ with Fourier coefficients $\{c_n\}$, we want to know what $\sum_n \log_2(n+1)|c_n|^2 < \infty$ implies about the regularity of the limit function f. In Section 6.1, we prove that: for $s \in (0,\infty)$ and any $d \geq 1$, there exist constants k_s and K_s, $0 < k_s \leq K_s < \infty$, s.t. for any $f \in L^2\left([-\pi,\pi],\mathbb{R}^d\right)$ with Fourier coefficients $\{c_n\}$,

$$k_s l \leq L \leq K_s l, \tag{1.10}$$

$$\text{where } L := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|}\left(\log_2\frac{\pi}{\left|\sin\frac{u-v}{2}\right|}\right)^{2s-1} dudv,$$

$$\text{and } l := \sum_{k=1}^{\infty}\log_2(n+1)^{2s}|c_n|^2.$$

Combining (1.9) with (1.10), we get that, if $f : [-\pi,\pi] \to \mathbb{R}^d$ satisfies

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|}dudv < \infty,$$

then $f \in L^2\left([-\pi,\pi],\mathbb{R}^d\right)$ and there exists constant C_5 (depending on d) such that, (Denote X as the partial sum process of the Fourier series of f)

$$\int_{-\pi}^{\pi}\|X(\theta)\|_{2-var}^2 + \|A(X(\theta))\|_{1-var}\,d\theta \leq C_5\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|}dudv. \tag{1.11}$$

(1.11) could be treated as a modest complement to the nontrivial result in [26], where the authors proved that, for $r > 1$, $p > \max\{2, r/(r-1)\}$,

$$\left(\int_{-\pi}^{\pi}\|X(\theta)\|_{p-var}^r\,d\theta\right)^{\frac{1}{r}} \leq C_{p,r}\left(\int_{-\pi}^{\pi}|f(\theta)|^r\,d\theta\right)^{\frac{1}{r}}.$$

Thus, the partial sum process of L^2 Fourier series is of finite p-variation a.e., $p > 2$.

One might be tempted to ask whether all L^2 Fourier series has finite 2-variation a.e., which, however, is not true. Jones and Wang [10] gave a counterexample of a bounded function on $[-\pi,\pi]$, whose Fourier series has infinite 2-variation a.e.. Their proof relies on the result by Qian [30], who gave a concrete lower bound on the growth of 2-variation of partial sum process of i.i.d. sequences. We also give a counterexample and our proof does not rely on the result by Qian. Instead, we use the upper semi-continuity of cumulative distribution function of p-variation. More specifically, suppose $\{X_n\}$ and X are stochastic processes taking value in $\mathbb{R}^d$, and X_n converge to X in distribution as n tends to infinity, in the topology induced by the metric

$$d(\omega_1,\omega_2) := \sum_{n=1}^{\infty}\frac{1}{2^n}\max_{0\leq t\leq n}\left(|\omega_1(t)-\omega_2(t)| \wedge 1\right).$$

Then we have that, for any $p \geq 1$ and constant $C > 0$,

$$\overline{\lim}_{n\to\infty}P\left(\|X_n\|_{p-var} \leq C\right) \leq P\left(\|X\|_{p-var} \leq C\right).$$

In our setting, $\{X_n\}$ are rescaled random walks, X is Brownian motion and $p = 2$. By picking out trigonometric functions $\left\{e^{in_k\theta}\right\}_k$ whose $\{n_k\}_k$ increases so fast that $\left\{e^{in_k\theta}\right\}_k$ resemble an i.i.d. sequence, we construct a L^2 function whose Fourier series has infinite 2-variation a.e..

Chapter 2

Spaces of paths

Although rough path is essentially nonlinear, we first inspect the linear part of a rough path—the first level regular path. This part is treated in detail in P. Friz and N. Victoir [9], so this chapter may not contain anything new, but mainly gives definitions and sets notations.

First, we define p-variation of a path, which dates back to Wiener [36], and widely used in rough path [19], [20], [9], [21].

Notation 2.1 *Suppose* $(\mathcal{V}, \|\cdot\|)$ *is a Banach space. Denote* $C([0,T],\mathcal{V})$ *as the space of continuous paths defined on* $[0,T]$ *taking value in* $\mathcal{V}$*.*

Definition 2.2 *A finite set of points* $D = \{t_k\}_{k=0}^n$ *is said to be a finite partition of interval* $[0,T]$*, if* $0 = t_0 < t_1 < \cdots < t_n \leq T$*. Denote the mesh of* $|D| := \max_k |t_{k+1} - t_k|$*.*

Definition 2.3 *Suppose* $\gamma \in C([0,T],\mathcal{V})$*. Then when* $1 \leq p < \infty$*, define the* p*-variation of* γ *on* $[0,T]$ *as*

$$\|\gamma\|_{p-var,[0,T]} := \left(\sup_{D\subset[0,T]} \sum_{k,t_k\in D} \|\gamma(t_{k+1}) - \gamma(t_k)\|^p \right)^{\frac{1}{p}}, \tag{2.1}$$

where we take supremum over all finite partitions.
When $p = \infty$*, define* $\|\gamma\|_{\infty-var,[0,T]} := \sup_{s,t\in[0,T]} \|\gamma(t) - \gamma(s)\|$*.*

A path defined on $[0,T]$ is of finite p-variation iff it can be reparametrised to be p^{-1}-Hölder continuous.

Notation 2.4 *For* $1 \leq p < \infty$*, denote* $C^{p-var}([0,T],\mathcal{V}) \subseteq C([0,T],\mathcal{V})$ *as the space of continuous paths with finite* p*-variation.*

Proposition 2.5 (lower semi-continuity) *Suppose* $\{\gamma_n\}_{n=1}^\infty \subset C^{1-var}([0,T],\mathcal{V})$ *is a sequence of continuous bounded variation path converging to* γ *pointwisely. Then*

$$\|\gamma\|_{p-var,[0,T]} \leq \underline{\lim}_{n\to\infty} \|\gamma_n\|_{p-var,[0,T]}.$$

Proof. By passing to a subsequence, we assume that $\|\gamma_n\|_{p-var,[0,T]}$ converge (to the lower limit) as n tends to infinity. For $\epsilon > 0$, suppose $D_\epsilon = \{t_k\}$ is a finite partition of $[0,T]$ satisfying

$$\|\gamma\|_{p-var} - \epsilon \leq \left(\sum_{k,t_k \in D_\epsilon} \|\gamma(t_{k+1}) - \gamma(t_k)\|^p \right)^{\frac{1}{p}}.$$

Then since γ is the pointwise limit of $\{\gamma_n\}_n$ and D_ϵ is finite,

$$\begin{aligned} \|\gamma\|_{p-var} - \epsilon &\leq \lim_{n\to\infty} \left(\sum_{k,t_k \in D_\epsilon} \|\gamma_n(t_{k+1}) - \gamma_n(t_k)\|^p \right)^{\frac{1}{p}} \\ &\leq \lim_{n\to\infty} \|\gamma_n\|_{p-var,[0,T]} = \underline{\lim}_{n\to\infty} \|\gamma_n\|_{p-var,[0,T]}. \end{aligned}$$

Let $\epsilon \to 0$, proof finishes. ■

Notation 2.6 *Suppose $\gamma \in C([0,T],\mathcal{V})$. For $p \in [1,\infty)$, denote*

$$\omega_p(\gamma,\delta) := \left(\sup_{D,|D|\leq\delta} \sum_{k,t_k\in D} \|\gamma(t_{k+1}) - \gamma(t_k)\|^p \right)^{\frac{1}{p}}; \tag{2.2}$$

When $p = \infty$, denote $\omega_\infty(\gamma,\delta) := \sup_{|t-s|\leq\delta} \|\gamma(t) - \gamma(s)\|$.

For fixed γ and δ, the function $p \mapsto \omega_p(\gamma,\delta)$ is non-increasing. For fixed γ and p, the function $\delta \mapsto \omega_p(\gamma,\delta)$ is non-decreasing with $\omega_p(\gamma,T) = \|\gamma\|_{p-var,[0,T]}$, so the limit $\lim_{\delta\to 0}\omega_p(\gamma,\delta)$ exists (and $< \infty$ when $\|\gamma\|_{p-var,[0,T]} < \infty$).

Definition 2.7 (paths with vanishing p-variation) *Suppose $\gamma \in C([0,T],\mathcal{V})$. For $1 \leq p < \infty$, γ is said to be of vanishing p-variation if*

$$\lim_{\delta\to 0}\omega_p(\gamma,\delta) = 0. \tag{2.3}$$

Denote $C^{0,p-var}([0,T],\mathcal{V})$ as the space of paths with vanishing p-variation.

(In [9], $C^{0,p-var}([0,T],\mathcal{V})$ denotes the closure of continuous bounded variation paths in p-variation, and these two notions coincide when $1 < p < \infty$, see Theorem 2.12 below. We define it in this way for the convenience of Chapter 2.)

Since norm satisfies triangle inequality, a path of vanishing 1-variation is a constant.

Lemma 2.8 *Suppose $1 \leq p < \infty$ and $\gamma \in C([0,T],\mathcal{V})$, we have*

$$\|\gamma\|_{p-var,[0,T]} < \infty \Leftrightarrow \lim_{\delta\to 0}\omega_p(\gamma,\delta) < \infty.$$

Proof. $\Leftarrow$ Fix $\delta > 0$ s.t. $\omega(\gamma,\delta) < \infty$. Then

$$\|\gamma\|_{\infty-var,[0,T]} := \sup_{0\leq s<t\leq T} \|\gamma(t) - \gamma(s)\| \leq \frac{T}{\delta}\omega_\infty(\gamma,\delta) \leq \frac{T}{\delta}\omega_p(\gamma,\delta).$$

For any finite partition $D = \{t_k\}$,

$$\begin{aligned}
&\sum_{k,t_k\in D} \left\|\gamma\left(t_{k+1}\right)-\gamma\left(t_k\right)\right\|^p \\
=\ & \sum_{k,\left|t_{k+1}-t_k\right|\leq\delta} + \sum_{k,\left|t_{k+1}-t_k\right|>\delta} \leq \omega_p^p\left(\gamma,\delta\right)+\frac{T}{\delta}\left\|\gamma\right\|_{\infty-var,[0,T]}^p \\
\leq\ & \left(1+\left(\frac{T}{\delta}\right)^{p+1}\right)\omega_p^p\left(\gamma,\delta\right)<\infty.
\end{aligned}$$

$\Rightarrow$ Since $\delta\mapsto\omega_p\left(\gamma,\delta\right)$ is non-decreasing,

$$\lim_{\delta\to 0}\omega_p\left(\gamma,\delta\right)\leq\omega_p\left(\gamma,T\right)=\left\|\gamma\right\|_{p-var,[0,T]}<\infty.$$

■

Thus, combine Lemma 2.8 with definition of $C^{0,p-var}\left(\left[0,T\right],\mathcal{V}\right)$ at (2.3),

$$C^{0,p-var}\left(\left[0,T\right],\mathcal{V}\right)\subseteq C^{p-var}\left(\left[0,T\right],\mathcal{V}\right).$$

Corollary 2.9 *Suppose $\gamma\in C^{p-var}\left(\left[0,T\right],\mathcal{V}\right)$ for some $p\geq 1$. Then for any $q>p$, $\lim_{\delta\to 0}\omega_q\left(\gamma,\delta\right)=0$.*

Proof. Since $q>p$,

$$\omega_q\left(\gamma,\delta\right)\leq\left(\omega_\infty\left(\gamma,\delta\right)\right)^{1-\frac{p}{q}}\left(\omega_p\left(\gamma,\delta\right)\right)^{\frac{p}{q}}.$$

γ is continuous so uniformly continuous on $[0,T]$, so $\omega_\infty\left(\gamma,\delta\right)$ tends to zero as $\delta\to 0$. $\omega_p\left(\gamma,\delta\right)$ is bounded by $\left\|\gamma\right\|_{p-var,[0,T]}$, so $\lim_{\delta\to 0}\omega_q\left(\gamma,\delta\right)=0$. ■

Thus, based on Corollary 2.9,

$$\cup_{q>p}C^{0,q-var}\left(\left[0,T\right],\mathcal{V}\right)\subseteq C^{p-var}\left(\left[0,T\right],\mathcal{V}\right).$$

Lemma 2.10

$$\lim_{\delta\to 0}\omega_p\left(\gamma,\delta\right)=0 \quad \textit{iff} \quad \lim_{\delta\to 0}\sup_{|D|\leq\delta}\sum_{k,t_k\in D}\left\|\gamma\right\|_{p-var,\left[t_k,t_{k+1}\right]}^p=0. \tag{2.4}$$

Proof. Recall the definition of ω_p :

$$\omega_p\left(\gamma,\delta\right):=\left(\sup_{|D|\leq\delta}\sum_{k,t_k\in D}\left\|\gamma\left(t_{k+1}\right)-\gamma\left(t_k\right)\right\|^p\right)^{\frac{1}{p}}.$$

Since $\left\|\gamma\left(t_{k+1}\right)-\gamma\left(t_k\right)\right\|^p\leq\left\|\gamma\right\|_{p-var,\left[t_k,t_{k+1}\right]}^p$, " $\Leftarrow$" in (2.4) is clear. For " $\Rightarrow$", firstly, fix finite partition $D=\{t_k\}_k$. Then $\sum_{k,t_k\in D}\left\|\gamma\right\|_{p-var,\left[t_k,t_{k+1}\right]}^p$ is obtained through taking supremum over all finite partitions D', $D'\supseteq D$. Since $|D'|\leq|D|$,

$$\sum_{k,t_k\in D}\left\|\gamma\right\|_{p-var,\left[t_k,t_{k+1}\right]}^p\leq\sup_{|D'|\leq|D|}\sum_{j,t_j\in D'}\left\|\gamma\left(t_{j+1}\right)-\gamma\left(t_j\right)\right\|^p.$$

Then take supremum over all D, $|D|\leq\delta$,

$$\sup_{D,|D|\leq\delta}\sum_{k,t_k\in D}\left\|\gamma\right\|_{p-var,\left[t_k,t_{k+1}\right]}^p\leq\sup_{D',|D'|\leq\delta}\sum_{j,t_j\in D'}\left\|\gamma\left(t_{j+1}\right)-\gamma\left(t_j\right)\right\|^p.$$

Let δ tends to zero. Proof finishes. ■

Notation 2.11 *Suppose $\gamma \in C([0,T],\mathcal{V})$ and $D = \{t_k\}_k$ is a finite partition of $[0,T]$. Denote γ^D as the piecewise linear approximation to γ w.r.t. D, i.e.*

$$\gamma^D(t) = \frac{t-t_k}{t_{k+1}-t_k}\gamma(t_{k+1}) + \frac{t_{k+1}-t}{t_{k+1}-t_k}\gamma(t_k),\ t \in [t_k, t_{k+1}]. \tag{2.5}$$

The following theorem is Thm 5.31 in [9], which identifies $C^{0,p-var}([0,T],\mathcal{V})$, $p>1$, as the closure of $C^{1-var}([0,T],\mathcal{V})$ in p-variation norm.

Theorem 2.12 (Wiener's characterization) *Suppose $1<p<\infty$. Then the following three statements are equivalent:*

$$\begin{gathered}(i)\ \gamma \in C^{0,p-var}([0,T],\mathcal{V});\\ (ii)\ \exists\{\gamma_n\}_n \subset C^{1-var}([0,T],\mathcal{V}),\ \lim_{n\to\infty}\|\gamma_n-\gamma\|_{p-var,[0,T]}=0;\\ (iii)\ \lim_{|D|\to 0}\left\|\gamma^D-\gamma\right\|_{p-var}=0.\end{gathered} \tag{2.6}$$

When $p=1$: $\gamma \in C^{0,1-var}([0,T],\mathcal{V})$ iff γ is a constant, (ii) and (iii) are equivalent to the absolute continuity of γ (Prop 1.32 and Corollary 1.34 [9]). When $p=\infty$, (i), (ii) and (iii) are equivalent to the continuity of γ on $[0,T]$.

Proof. $(iii)\Rightarrow(ii)$ is clear, since $\{\gamma^D\}_D \subset C^{1-var}([0,T],\mathcal{V})$.

$(ii)\Rightarrow(i)$. We want to prove $\lim_{\delta\to 0}\omega_p(\gamma,\delta)=0$. For any fixed $\epsilon>0$, there exists $N\geq 1$, s.t. $\|\gamma_N-\gamma\|_{p-var}<\epsilon$. This fixed γ_N is continuous and of bounded variation, so according to Corollary 2.9, γ_N is of vanishing p-variation (since $p>1$). Thus, for fixed $\epsilon>0$, there exists $\delta_0>0$, s.t. for any $\delta\in(0,\delta_0)$, we have $\omega_p(\gamma_N,\delta)<\epsilon$ and

$$\omega_p(\gamma,\delta) \leq \|\gamma_N-\gamma\|_{p-var} + \omega_p(\gamma_N,\delta) < 2\epsilon,\ \forall\delta\in(0,\delta_0).$$

$(i)\Rightarrow(iii)$. Fix a finite partition $D=\{t_k\}$. Denote γ^D as at (2.5), and denote $\gamma_c^D := \gamma-\gamma^D$. Since γ and γ^D coincide at $\{t_k\}$, $\gamma_c^D(t_k)=0$, $\forall k$.

Suppose $t_{k-1}<s\leq t_k\leq t_l\leq t<t_{l+1}$, then

$$\left\|\gamma_c^D(t)-\gamma_c^D(s)\right\|^p \leq 2^{p-1}\left(\left\|\gamma_c^D(t)-\gamma_c^D(t_l)\right\|^p + \left\|\gamma_c^D(t_k)-\gamma_c^D(s)\right\|^p\right),$$

where we used $\gamma_c^D(t_k)=\gamma_c^D(t_l)$. In this way, we replace $[s,t]$ by $[s,t_k]\cup[t_l,t]$, with $[s,t_k]\subset[t_{k-1},t_k]$ and $[t_l,t]\subset[t_l,t_{l+1}]$. For any finite partition of D, apply our estimates to every interval of D, sum them together, and take supremum over all finite partitions, we get:

$$\begin{aligned}\left\|\gamma^D-\gamma\right\|_{p-var,[0,T]}^p &= \left\|\gamma_c^D\right\|_{p-var,[0,T]}^p \leq 2^{p-1}\sum_{k,t_k\in D}\left\|\gamma_c^D\right\|_{p-var,[t_k,t_{k+1}]}^p\\ &\leq 2^{2p-2}\sum_{k,t_k\in D}\left(\|\gamma\|_{p-var,[t_k,t_{k+1}]}^p + \left\|\gamma^D\right\|_{p-var,[t_k,t_{k+1}]}^p\right)\\ &\leq 2^{2p-1}\sum_{k,t_k\in D}\|\gamma\|_{p-var,[t_k,t_{k+1}]}^p,\end{aligned}$$

where we used $\left\|\gamma^{D}\right\|_{p-var,[t_k,t_{k+1}]}=\|\gamma(t_{k+1})-\gamma(t_k)\|$, $\forall k$. Since γ is of vanishing p-variation, based on Lemma 2.10,

$$\lim_{\delta\to 0,|D|\le\delta}\left\|\gamma^{D}-\gamma\right\|_{p-var,[0,T]}^{p}\le 2^{2p-1}\lim_{\delta\to 0,|D|\le\delta}\sum_{k,t_k\in D}\|\gamma\|_{p-var,[t_k,t_{k+1}]}^{p}=0.$$

■

Proposition 2.13 (Separability) *As a result of Wiener's characterization (Theorem 2.12), when* $\dim(\mathcal{V})<\infty$ *and* $1<p<\infty$, $C^{0,p-var}([0,T],\mathcal{V})$ *is separable. While when* $\mathcal{V}$ *is not trivial (*$\exists e\in\mathcal{V},\ e\neq 0$*) and* $1\le p<\infty$, $\left(C^{p-var}([0,T],\mathcal{V}),\|\cdot\|_{p-var,[0,T]}\right)$ *is not separable (Example 1.26 and Example 5.26 in [9]).*

Sum it up, we have the following inclusion property:

$$\bigcup_{1\le q<p}C^{q-var}\overset{(1)}{\subset}C^{0,p-var}\overset{(2)}{\subset}C^{p-var}\overset{(3)}{\subset}\bigcap_{q>p}C^{0.q-var}.$$

All inclusions are strict (when $\dim(\mathcal{V})\ge 2$ and $1<p<\infty$) with examples:
For (1) : $f_1(x)=x^{\frac{1}{p}}\cos^2(\pi/x)/\ln x$, $x\in[0,1]$ (Exer 5.35 (ii) [9]).
For (2) : For sufficiently large c (Exer 5.35 (i) [9]),

$$f_2(x)=\sum_{n=0}^{\infty}c^{-\frac{n}{p}}\exp(2\pi i c^{n}x),x\in[0,1].$$

For (3) : For any $1<p<\infty$, the sample paths of p^{-1}-fractional brownian motion are of infinite p-variation but of vanishing q-variation a.e., for any $q>p$ (see [29]).

Chapter 3

Area operator and geometric 2-rough paths

In this chapter, we investigate the " area", which is the second level function of a rough path (in terms of Lie algebra), and is where rough paths start to be different from paths. For a path $\gamma \in C^{1-var}\left([0,T],\mathcal{V}\right)$, the area of γ, $A\left(\gamma\right)$ is defined as the Riemann-Stieltjes integral (for $v_1, v_2 \in \mathcal{V}$, $[v_1, v_2] := v_1 \otimes v_2 - v_2 \otimes v_1$ with "$\otimes$" the tensor product defined below in Section 3.1.1)

$$A\left(\gamma\right)\left(s,t\right) := \iint_{s<u_1<u_2<t} \left[d\gamma\left(u_1\right), d\gamma\left(u_2\right)\right], \ \forall\ 0 \leq s \leq t \leq T. \tag{3.1}$$

For paths in $\cup_{1\leq p<2} C^{p-var}\left([0,T],\mathcal{V}\right)$, the area is well defined based on Young integral [38]. In 2009, P. L. Lions [18] sketched a proof of the statement that: if γ_1 and γ_2 are of vanishing 2-variation, then the path $t \mapsto \int_0^t \gamma_1 \otimes d\gamma_2$ can be defined as Riemann-Stieltjes integral and is of vanishing 2-variation. His statement, however, is incorrect. We give a counterexample and demonstrate that, the integral $\int_0^T \gamma \otimes d\gamma$ may not exist when γ is of vanishing 2-variation.

As a result, when $\gamma \in C^{0,2-var}\left([0,T],\mathcal{V}\right)$ the differential equation

$$dS\left(t\right) = S(t) \otimes d\gamma(t), \quad S(0) = (1, \gamma(0), 0) \in 1 \oplus \mathcal{V} \oplus \mathcal{V}^{\otimes 2}, \tag{3.2}$$

may not have solution in the sense of ordinary differential equation. Indeed, the projection of (3.2) to $\mathcal{V}^{\otimes 2}$ is the differential equation

$$dy(t) = \gamma(t) \otimes d\gamma(t), \quad y(0) = 0 \in \mathcal{V}^{\otimes 2}. \tag{3.3}$$

To give meaning to the differential equation (3.2), one might be tempted to enhance γ to a geometric 2-rough path and solve (3.2) as a rough differential equation. However, as we demonstrate, the Riemann-Stieltjes integral at (3.1) is the only possible function to enhance γ into a geometric 2-rough path. This limitation on choices, in a sense, bridges the potential gap between paths and rough paths, and manifests the special closable property of geometric n-rough path for $n \in \mathbb{N}$, $n \geq 2$.

The area operator A is the operator which sends a continuous bounded variation path to its area. We equip continuous bounded variation paths with $\left\|\cdot\right\|_{2-var,[0,T]}$ and equip their area with $\|\cdot\|_A$. We choose different $\|\cdot\|_A$ to investigate the properties of the area operator. (The p-variation of areas is similar to but different from the p-variation of paths, see Section 3.1.3.)

As we demonstrate, even when $\|\cdot\|_A = \|\cdot\|_{\infty-var,[0,T]}$ (the uniform norm), the area operator is neither continuous nor bounded. When considering the closability, the area operator is closable when $\|\cdot\|_A = \|\cdot\|_{1-var,[0,T]}$, but not closable when $\|\cdot\|_A = \|\cdot\|_{p-var,[0,T]}, p > 1$.

When $\|\cdot\|_A = \|\cdot\|_{1-var,[0,T]}$, the area operator is closable. Denote paths in the closure as $\mathcal{G}_2(\mathcal{V})$, i.e.

$$\gamma \in \mathcal{G}_2(\mathcal{V}) \Longleftrightarrow \exists \{\gamma_n\} \subset C^{1-var}([0,T],\mathcal{V}) \text{ s.t.} \tag{3.4}$$

$$\lim_{n\to\infty} \|\gamma_n - \gamma\|_{2-var,[0,T]} = 0 \text{ and } \lim_{n_1,n_2\to\infty} \left\|A(\gamma_{n_1}) - A(\gamma_{n_2})\right\|_{1-var,[0,T]} = 0.$$

Then $\mathcal{G}_2(\mathcal{V})$ is a subset of $C^{0,2-var}([0,T],\mathcal{V})$ and is exactly the set of paths which admits an enhancement into a geometric 2-rough path.

We explore the properties of $\mathcal{G}_2(\mathcal{V})$ (e.g. non-linearity), and get an extension to Young integral when $p^{-1} + q^{-1} = 1$, by assigning a finer scale continuity.

In this chapter we mainly follow the contents of [37]. Proofs are postponed to the last section.

3.1 Area operator

In this section, we demonstrate what can go wrong for the Riemann-Stieltjes integral $\int f dg$ when f and g are of vanishing 2-variation, and investigate the properties of the area operator $A : \gamma \mapsto A(\gamma)$ when γ is restricted to be in the normed vector space:

$$\left(C^{1-var}([0,T],\mathcal{V}), \|\cdot\|_{2-var,[0,T]}\right).$$

3.1.1 Tensor product

We follow the definition of tensor product in [32].

Suppose $\mathcal{U}$ and $\mathcal{V}$ are two Banach spaces. Denote $\mathcal{B}(\mathcal{U}\times\mathcal{V})$ as the set of bilinear forms on $\mathcal{U}\times\mathcal{V}$ and denote its dual space as $\mathcal{B}(\mathcal{U}\times\mathcal{V})^*$. For $u \in \mathcal{U}$ and $v \in \mathcal{V}$, define $u \otimes v \in \mathcal{B}(\mathcal{U}\times\mathcal{V})^*$ as

$$(u\otimes v)(B) := B(u,v), \quad \forall B \in \mathcal{B}(\mathcal{U}\times\mathcal{V}). \tag{3.5}$$

Then the tensor product of $\mathcal{U}$ and $\mathcal{V}$ (denoted ad $\mathcal{U}\otimes\mathcal{V}$) is defined as the subspace of $\mathcal{B}(\mathcal{U}\times\mathcal{V})^*$ spanned by finite linear combinations of functionals in the form of (3.5):

$$\mathcal{U}\otimes\mathcal{V} := \left\{ \sum_{i=1}^{n} u_i \otimes v_i \,\middle|\, u_i \in \mathcal{U},\ v_i \in \mathcal{V},\ n \geq 1 \right\}.$$

The primary goal of defining tensor product is to linearize bilinear maps on $\mathcal{U}\times\mathcal{V}$, and it produces an isomorphism between $\mathcal{B}(\mathcal{U}\times\mathcal{V})$ and $\mathcal{L}(\mathcal{U}\otimes\mathcal{V})$ (linear forms on $\mathcal{U}\otimes\mathcal{V}$).

More specifically, suppose $B \in \mathcal{B}(\mathcal{U} \times \mathcal{V})$ is a bilinear form, then define $\widetilde{\mathcal{B}}$ on $\mathcal{U} \otimes \mathcal{V}$ by setting $\widetilde{B}(w) := w(B)$, $\forall w \in \mathcal{U} \otimes \mathcal{V}$. Then

$$\widetilde{B}(u \otimes v) = B(u, v),$$

$$\text{and } \widetilde{B}\left(\sum_{i=1}^{n} u_i \otimes v_i\right) = \left(\sum_{i=1}^{n} u_i \otimes v_i\right)(B) = \sum_{i=1}^{n} B(u_i, v_i) = \sum_{i=1}^{n} \widetilde{B}(u_i \otimes v_i).$$

On the other hand, suppose $\widetilde{B}$ is a linear map on $\mathcal{U} \otimes \mathcal{V}$. Define B on $\mathcal{U} \times \mathcal{V}$ by $B(u, v) := \widetilde{B}(u \otimes v)$, then it can be checked that B is bilinear. This bijection is an isomorphism because it preserves the linear structure.

In the special case that $\mathcal{U}$ and $\mathcal{V}$ are finite-dimensional, with bases $\{e_i\}_{i=1}^{n}$ and $\{f_j\}_{j=1}^{m}$ respectively, $\{e_i \otimes f_j\}_{1\le i\le n, 1\le j\le m}$ compose a basis of $\mathcal{U} \otimes \mathcal{V}$. In this case, any linear form $\widetilde{B}$ on $\mathcal{U} \otimes \mathcal{V}$ satisfies, for any sequence of numbers $\{c_{i,j}\}_{1\le i\le n, 1\le j\le m}$:

$$\widetilde{B}\left(\sum_{1\le i\le n, 1\le j\le m} c_{i,j}\ e_i \otimes f_j\right) = \sum_{1\le i\le n, 1\le j\le m} c_{i,j} B(e_i, f_j),$$

where B is the bilinear map on $\mathcal{U} \times \mathcal{V}$ corresponding to $\widetilde{B}$. Thus, when $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional, the tensor product $\mathcal{U} \otimes \mathcal{V}$ can be identified with the vector space of $n \times m$ matrices.

In addition to the algebraic structure, we equip $\mathcal{U} \otimes \mathcal{V}$ with a norm denoted as $\|\cdot\|_{\mathcal{U}\otimes\mathcal{V}}$, which satisfies that,

$$\|u \otimes v\|_{\mathcal{U}\otimes\mathcal{V}} \le \|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}, \quad \forall u \in \mathcal{U},\ \forall v \in \mathcal{V}. \tag{3.6}$$

(3.6) is satisfied by projective and injective tensor norms (Prop 2.1 and Prop 3.1 [32]). Moreover, we make completion of $\mathcal{U} \otimes \mathcal{V}$ w.r.t. $\|\cdot\|_{\mathcal{U}\otimes\mathcal{V}}$. The Banach space got is still denoted as $\mathcal{U} \otimes \mathcal{V}$ and is the tensor space we mainly work with. More specifically,

Notation 3.1 (normed Tensor space) *Suppose* $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$ *and* $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ *are two Banach spaces. We denote* $\left(\mathcal{U} \otimes \mathcal{V}, \|\cdot\|_{\mathcal{U}\otimes\mathcal{V}}\right)$ *as the Banach space defined as the completion of* $\{\sum_{i=1}^{n} u_i \otimes v_i,\ u_i \in \mathcal{U}, v_i \in \mathcal{V}, n \ge 1\}$ *w.r.t.* $\|\cdot\|_{\mathcal{U}\otimes\mathcal{V}}$.

Notation 3.2 *Suppose* $(\mathcal{V}, \|\cdot\|_{\mathcal{V}})$ *is a Banach spaces. Then for* $v_1, v_2 \in \mathcal{V}$*, we denote (the Lie bracket)* $[v_1, v_2] := v_1 \otimes v_2 - v_2 \otimes v_1$*, and denote* $\left([\mathcal{V}, \mathcal{V}], \|\cdot\|_{\mathcal{V}\otimes\mathcal{V}}\right)$ *as the Banach space defined as the completion of* $\{\sum_{i=1}^{n} [v_i^1, v_i^2], v_i^1, v_i^2 \in \mathcal{V}, n \ge 1\}$ *w.r.t.* $\|\cdot\|_{\mathcal{V}\otimes\mathcal{V}}$*. Similarly, we denote* $([\mathcal{V}, [\mathcal{V}, \mathcal{V}]], \|\cdot\|_{\mathcal{V}^{\otimes 3}})$ *as the Banach space defined as the completion of* $\{\sum_{i=1}^{n} [v_i^3, [v_i^1, v_i^2]], v_i^1, v_i^2, v_i^3 \in \mathcal{V}, n \ge 1\}$ *w.r.t.* $\|\cdot\|_{\mathcal{V}^{\otimes 3}}$*, so on and so forth.*

3.1.2 Riemann-Stieltjes integral

By Riemann-Stieltjes integral, we mean the existence of a strong limit of Riemann sums as the mesh of finite partitions tends to zero, which does not depend on the selection of representative points of subintervals.

Suppose γ_1 and γ_2 are continuous paths on $[0,T]$ taking value in Banach space $\mathcal{V}$, and consider the Riemann-Stieltjes integral (whenever it exists):

$$i(t)=\int_0^t \gamma_1(u)\otimes d\gamma_2(u),t\in[0,T].$$

If γ_1 is continuous and γ_2 is of bounded variation, then i is of bounded variation, and

$$\|i\|_{1-var,[0,T]}\leq\left(\|\gamma_1\|_{\infty-var,[0,T]}+\|\gamma_1(0)\|\right)\|\gamma_2\|_{1-var,[0,T]}.$$

Young [38] demonstrated that, when γ_1 is of finite p-variation, γ_2 is of finite q-variation, and $p>1$, $q>1$, $p^{-1}+q^{-1}>1$, then i is still well-defined, and

$$\|i\|_{q-var,[0,T]}\leq C_{p,q}\|\gamma_1\|_{p-var,[0,T]}\|\gamma_2\|_{q-var,[0,T]}. \tag{3.7}$$

(i is of finite q-variation, $q>1$, the same as γ_2.) However, the existence of integral is problematic when $p^{-1}+q^{-1}=1$. In the special case $\gamma_1=\gamma_2:=\gamma$, the definition of $\int\gamma\otimes d\gamma$ is problematic when γ is of (vanishing) 2-variation.

Problem 3.3 *Suppose $\mathcal{V}$ is a Banach spaces, and $\gamma\in C^{0,2-var}([0,T],\mathcal{V})$. Does the Riemann-Stieltjes integration $\int_0^T\gamma\otimes d\gamma$ exist; if it exists, what is the regularity of path $t\mapsto\int_0^t\gamma\otimes d\gamma$?*

In 2009, P. L. Lions [18] sketched a proof of the statement that: if γ_1 and γ_2 are of vanishing 2-variation, then $\int_0^\cdot\gamma_1\otimes d\gamma_2$ can be defined as Riemann-Stieltjes integral and is of vanishing 2-variation. His statement, however, is incorrect: first of all, the Riemann-Stieltjes integral may not exist (Example 3.36); secondly, (when restricted to continuous bounded variation paths equipped with 2-variation) the area operator is not bounded.

In [9](p194), the authors give an example of possible divergence of Riemann sums w.r.t. the finite partition D as $|D|\to 0$. Here we modify the example and get non-existence.

For paths of vanishing 2-variation, selecting different representative points only produces a negligible error. Actually, suppose that γ is a path defined on $[0,T]$ of vanishing 2-variation, and $D=\{t_j\}$ is a finite partition satisfying $|D|\leq\delta$. Then for any $\{\eta_j,\xi_j\}_j$ satisfying $\eta_j,\xi_j\in[t_j,t_{j+1}]$, we have (assume $\|u\otimes v\|\leq\|u\|\|v\|$):

$$\begin{aligned}&\left\|\sum_j\left(\gamma(\eta_j)-\gamma(\xi_j)\right)\otimes\left(\gamma(t_{j+1})-\gamma(t_j)\right)\right\|\\&\qquad\leq\left(\sum_j\left\|\gamma(\eta_j)-\gamma(\xi_j)\right\|^2\right)^{\frac{1}{2}}\left(\sum_j\left\|\gamma(t_{j+1})-\gamma(t_j)\right\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\{\eta_j,\xi_j\}_j$ can be treated as points in another finite partition whose mesh is less or equal to 2δ,

$$\begin{aligned}&\lim_{\delta\to0}\sup_{D,|D|\leq\delta}\left\|\sum_{j,t_j\in D}\left(\gamma(\eta_j)-\gamma(\xi_j)\right)\otimes\left(\gamma(t_{j+1})-\gamma(t_j)\right)\right\|\\&\qquad\leq\lim_{\delta\to0}\sup_{D,|D|\leq2\delta}\sum_{j,t_j\in D}\left\|\gamma(t_{j+1})-\gamma(t_j)\right\|^2=0.\end{aligned}$$

However, a problem may occur when one keeps on inserting partition points—the area generated by the added points could be infinite. In Example 3.36, we give a path $f \in C^{0,2-var}([0,1],\mathbb{C})$:

$$f(t)=\sum_{n=1}^{\infty}\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k^{\frac{1}{2}}2^k}\exp\left(2\pi i(-1)^n 2^{2k}t\right),\ \ t\in[0,1], \tag{3.8}$$
$$\text{where } c>\pi,\ \ c^n\leq\sum_{k=l_n}^{l_{n+1}}k^{-1}\leq c^n+1,\ \forall n\geq 1,$$

which satisfies that, there exists a sequence of finite partitions $\{D_n\}_n$ of $[0,1]$ satisfying ($x:=\operatorname{Re}f$, $y:=\operatorname{Im}f$)

$$D_n\subset D_{n+1}\ \forall n\in\mathbb{N},\quad \lim_{n\to\infty}|D_n|=0, \tag{3.9}$$
$$\forall a\in[-\infty,+\infty],\exists\{D_n^a\}\subset\{D_n\}\text{ s.t. }\lim_{n\to\infty}\sum_{k,t_k\in D_n^a}x(t_k)\,y(t_{k+1})-x(t_{k+1})\,y(t_k)=a.$$

As a result, since the Riemann sum of $\int_0^1 f\otimes df$ w.r.t. finite partition D is (we select the middle point because the error occurred from selecting different representative points tends to zero as $|D|\to 0$)

$$\begin{aligned}&\sum_{k,t_k\in D}\frac{1}{2}\left(f(t_k)+f(t_{k+1})\right)\otimes\left(f(t_{k+1})-f(t_k)\right)\\ =&\ \frac{1}{2}\sum_{k,t_k\in D}\left[f(t_k),f(t_{k+1})\right]+\frac{1}{2}f(T)^{\otimes 2}-\frac{1}{2}f(0)^{\otimes 2},\end{aligned}$$

which does not have a limit as $|D|\to 0$ because of (3.9), so the Riemann-Stieltjes integral $\int_0^1 f\otimes df$ does not exist.

The function f in (3.8) takes value in $\mathbb{C}$. Similar argument can be applied to a Banach space $\mathcal{V}$ when $\dim(\mathcal{V})\geq 2$. Select $e_1,\ e_2\in\mathcal{V}$, s.t. $[e_1,e_2]\neq 0$. With f at (3.8), define $\widetilde{f}=(\operatorname{Re}f)\,e_1+(\operatorname{Im}f)\,e_2$. Then following the similar reasoning, the Riemann-Stieltjes integral $\int_0^1\widetilde{f}\otimes d\widetilde{f}$ does not exist, and for any $a\in[-\infty,\infty]$, there exists a sequence of nested finite partitions $\{D_n^a\}_n$ of $[0,1]$, s.t.

$$\lim_{n\to\infty}|D_n^a|=0\text{ but }\lim_{n\to\infty}\left\|\sum_{k,t_k\in D_n^a}\left[\widetilde{f}(t_k),\widetilde{f}(t_{k+1})\right]\right\|=a. \tag{3.10}$$

When $\dim(\mathcal{V})=1$, the Riemann-Stieltjes integral $\int_0^T\gamma d\gamma$ does exist for any $\gamma\in C^{0,2-var}([0,T],\mathcal{V})$ and equals to $2^{-1}\left(\gamma^2(T)-\gamma^2(0)\right)$, because the vector field is commutative in one-dimensional case and the Lie bracket vanishes.

3.1.3 Variation norm for functions on triangle

In a similar way as defining p-variation for paths, we define p-variation of functions on the triangle $\{(s,t)\,|0\leq s\leq t\leq T\}$.

Notation 3.4 $\triangle_{[0,T]} := \{(s,t) \,|0 \leq s \leq t \leq T\}$

Notation 3.5 *Denote*

$$C\left(\triangle_{[0,T]}, \mathcal{V}\right) := \left\{\alpha | \alpha : \triangle_{[0,T]} \rightarrow \mathcal{V} \text{ is continuous, } \alpha(t,t) = 0, \forall t \in [0,T]\right\}.$$

Definition 3.6 *Suppose* $\alpha \in C\left(\triangle_{[0,T]}, \mathcal{V}\right)$*. For* $p > 0$*, define the p-variation of* α *on* $[0,T]$ *as*

$$\|\alpha\|_{p-var,[0,T]} := \left(\sup_{D\subset[0,T]} \sum_{j,t_j\in D} \|\alpha(t_j, t_{j+1})\|^p\right)^{\frac{1}{p}}, \tag{3.11}$$

where the supremum is over all finite partitions of $[0,T]$*.*
When $p = \infty$*, define* $\|\alpha\|_{\infty-var,[0,T]} := \sup_{0\leq s<t\leq T} \|\alpha(s,t)\|$*.*

Notation 3.7 *For* $p > 0$*, denote* $C^{p-var}\left(\triangle_{[0,T]}, \mathcal{V}\right) \subseteq C\left(\triangle_{[0,T]}, \mathcal{V}\right)$ *as the space of continuous functions with bounded p-variation.*

For any fixed $\alpha \in C\left(\triangle_{[0,T]}, \mathcal{V}\right)$, the function $p \mapsto \|\alpha\|_{p-var,[0,T]}$ on $p \in (0,\infty]$ is non-increasing.

Definition 3.8 *Suppose* $\alpha \in C\left(\triangle_{[0,T]}, \mathcal{V}\right)$*. Then* α *is said to be of vanishing p-variation for some* $p > 0$*, if*

$$\lim_{\delta\to 0} \omega_p(\alpha,\delta) = 0, \quad \text{where } \omega_p(\alpha,\delta) := \left(\sup_{D,|D|\leq\delta} \sum_{j,t_j\in D} \|\alpha(t_j, t_{j+1})\|^p\right)^{\frac{1}{p}}. \tag{3.12}$$

Denote $C^{0,p-var}\left(\triangle_{[0,T]}, \mathcal{V}\right) \subseteq C^{p-var}\left(\triangle_{[0,T]}, \mathcal{V}\right)$ *as the space of functions with vanishing p-variation.*

A path $\gamma \in C([0,T], \mathcal{V})$ can be treated as a function $\widetilde{\gamma} \in C\left(\triangle_{[0,T]}, \mathcal{V}\right)$, by setting $\widetilde{\gamma}(s,t) := \gamma(t) - \gamma(s)$.

3.1.4 Area operator

Suppose that γ_1 and γ_2 are two paths on $[0,T]$ taking value in Banach space $\mathcal{V}$. Consider the Riemann-Stieltjes integrals which defines $\alpha : \triangle_{[0,T]} \rightarrow \mathcal{V}^{\otimes 2}$:

$$\alpha(s,t) = \int_s^t (\gamma_1(u) - \gamma_1(s)) \otimes d\gamma_2(u), \ \forall (s,t) \in \triangle_{[0,T]}. \tag{3.13}$$

If γ_1 is continuous and γ_2 of bounded variation, then α is of bounded variation, and

$$\|\alpha\|_{1-var,[0,T]} \leq \|\gamma_1\|_{\infty-var,[0,T]} \|\gamma_2\|_{1-var,[0,T]}.$$

When γ_1 is of finite p-variation, γ_2 is of finite q-variation, $p > 1$, $q > 1$, $p^{-1} + q^{-1} > 1$, based on Young integral (and Thm 1.16 in [20]), we have

$$\left\|\int_s^t (\gamma_1(u) - \gamma_1(s)) \otimes d\gamma_2(u)\right\| \leq C'_{p,q} \|\gamma_1\|_{p-var,[s,t]} \|\gamma_2\|_{q-var,[s,t]}. \tag{3.14}$$

Then by using Hölder inequality, we get

$$\|\alpha\|_{(p^{-1}+q^{-1})^{-1}-var,[0,T]} \leq C'_{p,q} \|\gamma_1\|_{q-var,[0,T]} \|\gamma_2\|_{p-var,[0,T]} . \tag{3.15}$$

Thus, α is of finite $(p^{-1}+q^{-1})^{-1}$-variation, $(p^{-1}+q^{-1})^{-1} < 1$.

Definition 3.9 *Suppose that* $\mathcal{V}$ *is a Banach space, and* $\gamma_i \in C^{1-var}([0,T],\mathcal{V}), i = 1,2$. *Define the area produced by* γ_1 *and* γ_2 *:* $A(\gamma_1,\gamma_2) : \triangle_{[0,T]} \to [\mathcal{V},\mathcal{V}]$ *by setting*

$$A(\gamma_1,\gamma_2)(s,t) := 2^{-1} \iint_{s<u_1<u_2<t} [d\gamma_1(u_1), d\gamma_2(u_2)], \ \ \forall (s,t) \in \triangle_{[0,T]}. \tag{3.16}$$

Then it can be checked (through direct computation) that: for any $0 \leq s \leq u \leq t \leq T$,

$$\begin{aligned} A(\gamma_1,\gamma_2)(s,t) &= A(\gamma_1,\gamma_2)(s,u) + A(\gamma_1,\gamma_2)(u,t) \\ &\quad +2^{-1}[\gamma_1(u)-\gamma_1(s), \gamma_2(t)-\gamma_2(u)]. \end{aligned} \tag{3.17}$$

This property is preserved under pointwise convergence (so preserved under p-variation convergence, $1 \leq p \leq \infty$), and will be used in our proofs.

Definition 3.10 *Suppose that* $\gamma \in C^{1-var}([0,T],\mathcal{V})$*, then the area of* γ *is denoted as* $A(\gamma)$ *and defined as* $A(\gamma) := A(\gamma,\gamma)$.

Definition 3.11 *The area operator* $A : C^{1-var}([0,T],\mathcal{V}) \to C^{\frac{1}{2}-var}(\triangle_{[0,T]}, [\mathcal{V},\mathcal{V}])$ *is defined as the operator which sends* γ *to* $A(\gamma)$.

When $\gamma \in C^{1-var}([0,T],\mathcal{V})$, we have $A(\gamma) \in C^{\frac{1}{2}-var}(\triangle_{[0,T]}, [\mathcal{V},\mathcal{V}])$ because of (3.15). The area operator can be extended where the Riemann-Stieltjes integral is well-defined, e.g. to $\cup_{1\leq p<2} C^{p-var}([0,T],\mathcal{V})$ (or $\mathcal{G}_2(\mathcal{V})$ defined in the next section).

The following property is useful when we investigate the regularity of geometric 2-rough paths:

$$\overline{\{A(\gamma) \,|\gamma \in C^{1-var}([0,T],\mathcal{V})\}}^{1-var} \subseteq C^{0,1-var}(\triangle_{[0,T]}, [\mathcal{V},\mathcal{V}]). \tag{3.18}$$

Actually, when $\gamma \in C^{1-var}([0,T],\mathcal{V})$,

$$A(\gamma) \subseteq C^{\frac{1}{2}-var}(\triangle_{[0,T]}, [\mathcal{V},\mathcal{V}]) \subseteq C^{0,1-var}(\triangle_{[0,T]}, [\mathcal{V},\mathcal{V}]).$$

On the other hand, $C^{0,1-var}(\triangle_{[0,T]}, \mathcal{V})$ is closed under 1-variation, because $\omega_1(\alpha,\delta) \leq \|\alpha - \alpha_n\|_{1-var} + \omega_1(\alpha_n,\delta)$ (ω_1 defined at (3.12)), so we get (3.18).

Before starting our investigation, we clarify what do we mean by saying that a (non-linear) operator is continuous/bounded/closable. Suppose (E,d_E) and (F,d_F) are two metric spaces, and $\alpha : E \to F$ is an operator. Then α is said to be a continuous operator if it is a continuous function from (E,d_E) to (F,d_F); α is said to be bounded if it sends bounded sets in (E,d_E) to bounded sets in (F,d_F); α is said to be closable, if for any two sequences $\{x_n\}_n$ and $\{y_n\}_n$ in E which satisfy: (1) $\lim_n x_n = \lim_n y_n$ (2) both $\lim_n \alpha(x_n)$ and $\lim_n \alpha(y_n)$ exist, we have $\lim_n \alpha(x_n) = \lim_n \alpha(y_n)$. Thus, continuity implies closability and boundedness on compact subsets.

Then we investigate the properties of the area operator.

Problem 3.12 *When equipping $C^{1-var}([0,T],\mathcal{V})$ with 2-variation norm, is the area operator continuous, or bounded?*

When $\dim(\mathcal{V})=1$, the area operator is trivial. In that case it is continuous and bounded.

When $\dim(\mathcal{V})\geq 2$, as a consequence of possible non-existence and divergence of the Riemann-Stieltjes integral (3.10), the area operator is neither continuous nor bounded.

Actually, suppose that γ is a path of vanishing 2-variation on $[0,T]$, and γ^D is the piecewise linear path which coincides with γ on points in D (as defined at (2.5)). Then after direct computation, the Riemann sum of $\int\gamma\otimes d\gamma$ w.r.t. D equals to $A\left(\gamma^D\right)(0,T)$ plus a constant:

$$\begin{aligned}
&\sum_{k,t_k\in D}\frac{1}{2}\left(\gamma(t_k)+\gamma(t_{k+1})\right)\otimes\left(\gamma(t_{k+1})-\gamma(t_k)\right) \qquad (3.19)\\
=\;& \frac{1}{2}\sum_{k,t_k\in D}\left[\gamma(t_k),\gamma(t_{k+1})\right]+\frac{1}{2}\gamma^{\otimes 2}(T)-\frac{1}{2}\gamma^{\otimes 2}(0)\\
=\;& A\left(\gamma^D\right)(0,T)+\frac{1}{2}\left(\gamma(T)+\gamma(0)\right)\otimes\left(\gamma(T)-\gamma(0)\right).
\end{aligned}$$

Thus, based on (3.10), there exists a path $f:[0,1]\to\mathcal{V}$ of vanishing 2-variation, such that for any $a\in[-\infty,\infty]$, there exists a sequence of finite partitions $\{D_n^a\}$ of $[0,1]$, $\lim_{n\to\infty}|D_n^a|=0$, satisfying $\lim_{n\to\infty}\left\|A\left(f^{D_n^a}\right)(0,1)\right\|=a$. While $f^{D_n^a}$ converges to f in 2-variation when n tends to infinity (based on (2.6)). Thus, the area operator is neither continuous nor bounded, at least when area is equipped with uniform norm.

Moreover, by modifying our example, we get a sequence of continuous bounded variation paths (Example 3.39 at p38) converging to zero in 2-variation, but their area diverge at any non-trivial point: $(s,t)\in\triangle_{[0,T]}$, $s<t$. (The paths in Example 3.39 are in $C^\infty([0,1],\mathbb{C})$, and can be generalized to $C^{1-var}([0,1],\mathcal{V})$ whenever $\dim(\mathcal{V})\geq 2$.) Thus, when $C^{1-var}([0,T],\mathcal{V})$ is equipped with 2-variation, the area operator is never continuous nor bounded (when $\dim(\mathcal{V})\geq 2$).

On the other hand, 2-variation is not the clear-cut norm which fails the boundedness of the area operator. Based on our extension to Young integral (Section 3.3), there exists a sequence of norms on $C^{1-var}([0,T],\mathcal{V})$ (by adding a log term, $\log\log$ term, etc.) which fails the boundedness of the area operator. However, these norms are linear (on paths), while the area is essentially non-linear, so generally, it is not possible to describe the behavior of the area by a norm on paths (unless $\dim(\mathcal{V})=1$, in which case the area operator is trivial).

Problem 3.13 *When $C^{1-var}([0,T],\mathcal{V})$ is equipped with 2-variation norm, is the area operator closable in p-variation? In other words, if $\{\gamma_n\}_n$ and $\{\gamma_m\}_m$ are two sequences of paths in $C^{1-var}([0,T],\mathcal{V})$ converging in 2-variation to the same limit, and $\{A(\gamma_n)\}_n$ and $\{A(\gamma_m)\}_m$ converge in p-variation respectively. Then is it true that $\{A(\gamma_n)\}_n$ and $\{A(\gamma_m)\}_m$ converge to the same limit?*

When $p > 1$, it is not true. When $p = 1$, it is true. (We assume $\dim(\mathcal{V}) \geq 2$, because area vanishes for one-dimensional paths.)

For $p > 1$, an illustrative example is $r_n(t) = \left(\frac{\cos nt}{\sqrt{n}}, \frac{\sin nt}{\sqrt{n}}\right)$, $t \in [0, 2\pi]$, $n \geq 1$. r_n converges to 0 in q-variation for any $q > 2$, but their area converge to $t - s$ in p-variation for any $p > 1$:

$$\int_s^t \left(\frac{\cos nu}{\sqrt{n}} - \frac{\cos ns}{\sqrt{n}}\right) d\frac{\sin nu}{\sqrt{n}} - \left(\frac{\sin nu}{\sqrt{n}} - \frac{\sin ns}{\sqrt{n}}\right) d\frac{\cos nu}{\sqrt{n}} = t - s - \frac{\sin n(t-s)}{n}.$$

and

$$\left\|\frac{1}{\sqrt{n}} \exp(int)\right\|_{q-var} \lesssim \frac{1}{n^{\frac{1}{2}-\frac{1}{q}}}, \quad \left\|\frac{\sin n(t-s)}{n}\right\|_{p-var} \lesssim \frac{1}{n^{1-\frac{1}{p}}}.$$

Thus, $(0,0)$ and $(0, t-s)$ are two geometric q-rough paths with the same first level path for any $q \in (2,3)$. (Geometric q-rough paths $q \in [2,3)$ are elements in the closure of $\{(\gamma, A(\gamma)) \mid \gamma \in C^{1-var}([0,T], \mathcal{V})\}$ under the metric

$$d((\gamma_1, A(\gamma_1)), (\gamma_2, A(\gamma_2))) := \left(\|\gamma_1 - \gamma_2\|_{q-var}^q + \|A(\gamma_1) - A(\gamma_2)\|_{\frac{q}{2}-var}^{\frac{q}{2}}\right)^{\frac{1}{q}}.$$

However, $\{r_n\}$ are uniformly bounded in 2-variation, but do not converge in 2-variation ($\|n^{-\frac{1}{2}} \cos(nt) - (2n)^{-\frac{1}{2}} \cos(2nt)\|_{2-var} \geq \sqrt{2}$, $\forall n$). To construct our example, we add a decay factor, sum finitely many of them together to compensate the decaying effect on $t - s$, and end up with functions $\{g_n\}_n \subset C^\infty([0,1], \mathbb{C})$ (Example 3.40 at p38)

$$g_n(t) = \left(\pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k}\right)^{-\frac{1}{2}} \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k^{\frac{1}{2}} 2^k} \exp\left(2\pi i 2^{2k} t\right), \ t \in [0,1], \tag{3.20}$$

$$\text{where } \sum_{k=l_n}^{l_{n+1}-1} k^{-1} \geq 1, \ n \geq 1.$$

We prove that g_n converges in 2-variation to zero as n tends to infinity, but their areas converge to $t - s$ in p-variation, for any $p > 1$.

For Banach space $\mathcal{V}$, $\dim(\mathcal{V}) \geq 2$, select $e_1, e_2 \in \mathcal{V}$, s.t. $[e_1, e_2] \neq 0$. With g_n defined at (3.20), define $\widetilde{g_n} := (\operatorname{Re} g_n) e_1 + (\operatorname{Im} g_n) e_2$. Then $\{\widetilde{g_n}\}_n \subset C^{1-var}([0,1], \mathcal{V})$, $\lim_{n\to\infty} \|\widetilde{g_n}\|_{2-var} = 0$ and

$$\lim_{n\to\infty} \|A(\widetilde{g_n})(s,t) - (t-s)[e_1, e_2]\|_{p-var} = 0 \text{ for any } p > 1.$$

When $p = 1$, if (γ, α_1) and (γ, α_2) are two geometric 2-rough paths, then $\alpha_1 - \alpha_2 := \varphi$ is additive thus a path. Moreover, α_1 and α_2 are limits in 1-variation of areas of continuous bounded variation paths, so based on (3.18), α_1 and α_2 are of vanishing 1-variation and φ is also of vanishing 1-variation. While a path of vanishing 1-variation is constant, so $\alpha_1 = \alpha_2$.

Actually, for the same reason we have: the projection of a geometric n-rough path to its first $n - 1$ level elements is injective for any $n \in \mathbb{N}$, $n \geq 2$. While in Remark 9.13

(case ii b2) in [9], the authors commented that the projection is not an injection without providing a proof.

Thus, the area operator is closable when paths equipped with 2-variation and area equipped with 1-variation (which is the 2-rough norm). Thus, under 2-rough norm, the area operator is a closable but non-continuous and unbounded operator on continuous bounded variation paths.

3.2 Enhancible paths

As we demonstrated, when $\gamma \in C^{0,2-var}([0,T],\mathcal{V})$,

$$dS(t) = S(t) \otimes d\gamma(t), \quad S(0) = (1, \gamma(0), 0) \in 1 \oplus \mathcal{V} \oplus \mathcal{V}^{\otimes 2}, \tag{3.21}$$

may not have solution as ordinary differential equation (because its second level projection is $dy(t) = \gamma(t) \otimes d\gamma(t), y(0) = 0$). One might be tempted to enhance γ into a geometric 2-rough path, and solve (3.21) as a rough differential equation. However, as we will prove in the following, the Riemann-Stieltjes integral $\int \gamma \otimes d\gamma$ is the only possible function to enhance γ (into a geometric 2-rough path).

Definition 3.14 (weak geometric 2-rough path) *Suppose* $\gamma \in C([0,T],\mathcal{V})$ *and* $\alpha \in C\left(\triangle_{[0,T]}, [\mathcal{V},\mathcal{V}]\right)$. *Then* $\Gamma := (\gamma, \alpha) \in C\left(\triangle_{[0,T]}, \mathcal{V} \oplus [\mathcal{V},\mathcal{V}]\right)$ *is called a weak geometric 2-rough path, if for any* $0 \leq s \leq u \leq t \leq T$,

$$\alpha(s,t) = \alpha(s,u) + \alpha(u,t) + \frac{1}{2}\left[\gamma(u) - \gamma(s), \gamma(t) - \gamma(u)\right], \tag{3.22}$$
$$\text{and } \|\Gamma\|_{G^{(2)}} := \left(\|\gamma\|_{2-var}^{2} + \|\alpha\|_{1-var}\right)^{\frac{1}{2}} < \infty.$$

Property at (3.22) is called multiplicativity. $\|\cdot\|_{G^{(2)}}$ is 2-rough norm.

Definition 3.15 (geometric 2-rough path) $\Gamma := (\gamma, \alpha) \in C\left(\triangle_{[0,T]}, \mathcal{V} \oplus [\mathcal{V},\mathcal{V}]\right)$ *is a geometric 2-rough path, if there exist* $\{\gamma_n\}_n \subset C^{1-var}([0,T],\mathcal{V})$ *such that*

$$\lim_{n\to\infty} \|\Gamma - (\gamma_n, A(\gamma_n))\|_{G^{(2)}} = 0.$$

A geometric 2-rough path is automatically a weak geometric 2-rough path.

If (γ, α) is a geometric 2-rough path, then γ is of vanishing 2-variation (Theorem 2.12 at p9) and α is of vanishing 1-variation (because of (3.18) at p17).

Definition 3.16 (enhancible path) *Suppose* $\gamma \in C^{0,2-var}([0,T],\mathcal{V})$. *Then we say* γ *can be enhanced into a geometric 2-rough path (or enhancible), if there exists* $\alpha \in C^{0,1-var}\left(\triangle_{[0,T]}, [\mathcal{V},\mathcal{V}]\right)$ *such that* (γ, α) *is a geometric 2-rough path.*

Notation 3.17 *For Banach space* $\mathcal{V}$, *denote* $\mathcal{G}_2(\mathcal{V})$ *as the set of enhancible paths.*

$\mathcal{G}_2(\mathcal{V})$ is invariant under reparametrisation and contains $C^{1-var}([0,T],\mathcal{V})$.

3.2.1 Questions and answers

We investigate the properties of $\mathcal{G}_2(\mathcal{V})$ through several questions.

Based on Wiener's characterization of paths with vanishing 2-variation (Theorem 2.12),

$$\mathcal{G}_2(\mathcal{V}) \subseteq \overline{C^{1-var}([0,T],\mathcal{V})}^{2-var} = C^{0,2-var}([0,T],\mathcal{V}).$$

Problem 3.18 *Is it true that every path in $C^{0,2-var}([0,T],\mathcal{V})$ admits an enhancement into a (weak) geometric 2-rough path?*

When $\dim(\mathcal{V}) = 1$, the area vanishes for any $\gamma \in C^{1-var}([0,T],\mathcal{V})$, so

$$\mathcal{G}_2(\mathcal{V}) = C^{0,2-var}([0,T],\mathcal{V}) \text{ when } \dim(\mathcal{V}) = 1. \tag{3.23}$$

When $\dim(\mathcal{V}) \geq 2$, the proposition is not true, and an example is given in Exer 9.14 [9]. Actually, following the same reasoning as in Thm 9.12 [9], we use f defined at (3.8) on page 15 to prove that $\mathcal{G}_2(\mathcal{V}) \subsetneqq C^{0,2-var}([0,T],\mathcal{V})$ when $\dim(\mathcal{V}) \geq 2$. Select e_1, $e_2 \in \mathcal{V}$, s.t. $[e_1, e_2] \neq 0$. With f in Example 3.36, denote $\widetilde{f} := (\operatorname{Re} f)\, e_1 + (\operatorname{Im} f)\, e_2$, so $\widetilde{f} \in C^{0,2-var}([0,T],\mathcal{V})$. Assume that $\left(\widetilde{f}, \alpha\right)$ is a weak geometric 2-rough path. Then using multiplicativity of $\left(\widetilde{f}, \alpha\right)$, for any finite partitions D, we have

$$\begin{aligned}&\left\| \sum_{j, t_j \in D} \left[\widetilde{f}(t_j), \widetilde{f}(t_{j+1})\right] - \left[\widetilde{f}(0), \widetilde{f}(1)\right] \right\| = 2 \left\| \alpha(0,1) - \sum_{j, t_j \in D} \alpha(t_j, t_{j+1}) \right\| \\ &\leq 4 \|\alpha\|_{1-var} < \infty.\end{aligned}$$

Then contradiction is established, if $\sum_{j, t_j \in D} \left[\widetilde{f}(t_j), \widetilde{f}(t_{j+1})\right]$ are not uniformly bounded for all finite partitions, which is true because of (3.10). Thus, there exists a path f, which is of vanishing 2-variation, but can not be enhanced into a (weak) geometric 2-rough path.

On the other hand, in [22] the authors proved that: any continuous path with finite p-variation can be enhanced to a geometric q-rough path for any $q > p$, and can be enhanced into a weak geometric p-rough path when p is not an integer. (For the definition of (weak) geometric p-rough path, please refer to Chapter 4.) As a result, the $\widetilde{f}$ above can be enhanced to a geometric p-rough path for any $p > 2$, and that $\widetilde{f}$ can not be enhanced to a (weak) geometric 2-rough path is related to the fact that 2 is an integer.

Then a natural question arises:

Problem 3.19 *What is the condition for vanishing 2-variation paths to be enhancible (i.e. in $\mathcal{G}_2(\mathcal{V})$)?*

We prove that:

Theorem 3.20 *Suppose $\gamma \in C^{0,2-var}([0,T],\mathcal{V})$. Then $\gamma \in \mathcal{G}_2(\mathcal{V})$ if and only if $A\left(\gamma^D\right)$ converges in 1-variation as $|D| \to 0$.*

The proof is given in page 47.

In Thm 8.22 [9], the authors proved that, when $\mathcal{V}=\mathbb{R}^d$, if (γ,α) is a geometric 2-rough path, then there exists a sequence of continuous bounded variation paths $\{\gamma_n\}$, s.t. $(\gamma_n, A(\gamma_n))$ converge to (γ,α) in 2-rough norm $\|\cdot\|_{G^{(2)}}$ as n tends to ∞. However, their construction of $\{\gamma_n\}$ depends on α (i.e. Chow-Rashevskii connectivity theorem), while not $\{\gamma^D\}_D$, in general.

For any $0\le s\le t\le T$ and any finite partition D of $[s,t]$, the Riemann sums of $2^{-1}\int_s^t [\gamma(u)-\gamma(s), d\gamma(u)]$ w.r.t. $D\subset[s,t]$ is

$$\begin{aligned}&2^{-1}\sum_{k,t_k\in D}\frac{1}{2}\left[\gamma(t_k)+\gamma(t_{k+1}),\gamma(t_{k+1})-\gamma(t_k)\right]-2^{-1}\left[\gamma(s),\gamma(t)\right]\\ =\;&2^{-1}\sum_{k,t_k\in D}\left[\gamma(t_k),\gamma(t_{k+1})\right]-2^{-1}\left[\gamma(s),\gamma(t)\right].\end{aligned}$$

On the other hand, direct computation gives us

$$A\left(\gamma^D\right)(s,t)=2^{-1}\sum_{k,t_k\in D\subset[s,t]}\left[\gamma(t_k),\gamma(t_{k+1})\right]-2^{-1}\left[\gamma(s),\gamma(t)\right].$$

Thus, the Riemann-Stieltjes integral $2^{-1}\int_s^t[\gamma(u)-\gamma(s),d\gamma(u)]$ is the pointwise limit of $A\left(\gamma^D\right)$ as $|D|\to 0$. Hence, if γ is in $\mathcal{G}_2(\mathcal{V})$, then $A\left(\gamma^D\right)$ converges in 1-variation (Theorem 3.20), so converges pointwisely, to $2^{-1}\int_s^t[\gamma(u)-\gamma(s),d\gamma(u)]$.

Therefore, Riemann-Stieltjes integral is the only possible candidate to enhance γ: If the integral does not exist, or $(\gamma,\,2^{-1}\int_s^t[\gamma(u)-\gamma(s),d\gamma(u)])$ is not a geometric 2-rough path, then γ can not be enhanced into a geometric 2-rough path.

The fact that the Riemann-Stieltjes integral is the only possible candidate to enhance γ can also be derived from (3.24) below, which can be got through direct computation (also spelt out in Lemma 3.44 on p47). Suppose (γ,α) is multiplicative, then for any $s<t$ and any finite partition D of $[s,t]$, we have

$$\alpha(s,t)=\sum_{k,t_k\in D\subset[s,t]}\alpha(t_k,t_{k+1})+A(\gamma^D)(s,t). \tag{3.24}$$

If further assuming that α is of vanishing 1-variation, we have

$$\alpha(s,t)=\lim_{|D|\to 0, D\subset[s,t]}A(\gamma^D)(s,t). \tag{3.25}$$

Thus, when (γ,α) is multiplicative and α is of vanishing 1-variation, the limit on the r.h.s. of (3.25) exists and is the only possible choice for α. However, in getting (3.25), the regularity of γ does not seem to be very important. Actually, no matter what norm we place on continuous bounded variation paths, if we equip their area with 1-variation, the (path, area) graph is closable. (3.25) actually established the only possible " area " corresponding to γ. There exist paths, which are not of vanishing 2-variation, but for which the limit in (3.25) exists and is of vanishing 1-variation. (As a trivial example, one-dimensional paths.) The existence of Riemann-Stieltjes integral of the area implies the existence of limit in (3.25), but the other way is not true. They are equivalent when the error occurred from selecting different representative points is negligible, e.g. when γ is of vanishing 2-variation.

When $p>2$, the convergence of $A\left(f^{D}\right)$ as $|D|\to 0$ is not necessary to enhance a path in $C^{0,p-var}\left([0,T],\mathcal{V}\right)$. Our path f at (3.8) is in $C^{0,2-var}\left([0,T],\mathcal{V}\right)\subset C^{2-var}\left([0,T],\mathcal{V}\right)\subset\cap_{p>2}C^{0,p-var}\left([0,T],\mathcal{V}\right)$. Based on [22], a path of finite p-variation can be enhanced into a geometric q-rough path for any $q>p$, so f can be enhanced into a geometric p-rough path for any $p>2$. While $\sup_{D\subset[0,1]}A\left(f^{D}\right)(0,1)$ is not bounded, so $A\left(f^{D}\right)$ do not converge in p-variation as $|D|\to 0$, for any $p\in[1,\infty]$.

Problem 3.21 *Is $\mathcal{G}_2\left(\mathcal{V}\right)$ a linear space?*

$\mathcal{G}_2\left(\mathcal{V}\right)$ is a linear space when $\dim\left(\mathcal{V}\right)=1$; it is not a linear space when $\dim\left(\mathcal{V}\right)\geq 2$.

Based on (3.23), when $\dim\left(\mathcal{V}\right)=1$, $\mathcal{G}_2\left(\mathcal{V}\right)=C^{0,2-var}\left([0,T],\mathcal{V}\right)$ thus linear. When $\dim\left(\mathcal{V}\right)\geq 2$, select e_1, $e_2\in\mathcal{V}$, s.t. $[e_1,e_2]\neq 0$. With f defined at (3.8) on page 15, denote $\widetilde{f}:=\left(\operatorname{Re}f\right)e_1+\left(\operatorname{Im}f\right)e_2$. Then both $\left(\operatorname{Re}f\right)e_1$ and $\left(\operatorname{Im}f\right)e_2$ are one-dimensional and of vanishing 2-variation, so in $\mathcal{G}_2\left(\mathcal{V}\right)$. While based on Problem 3.18, $\widetilde{f}\notin\mathcal{G}_2\left(\mathcal{V}\right)$, so $\mathcal{G}_2\left(\mathcal{V}\right)$ is not a linear space.

The non-linearity of $\mathcal{G}_2\left(\mathcal{V}\right)$ is inherited from the non-linearity of the area.

Proposition 3.22 *When $\dim\left(\mathcal{V}\right)\geq 2$, both $\mathcal{G}_2\left(\mathcal{V}\right)$ and $C^{0,2-var}\left([0,T],\mathcal{V}\right)\backslash\mathcal{G}_2\left(\mathcal{V}\right)$ are dense in $C^{0,2-var}\left([0,T],\mathcal{V}\right)$ under 2-variation norm.*

Proof. $\mathcal{G}_2\left(\mathcal{V}\right)$ is dense in $C^{0,2-var}\left([0,T],\mathcal{V}\right)$, because

$$C^{1-var}\left([0,T],\mathcal{V}\right)\subseteq\mathcal{G}_2\left(\mathcal{V}\right)\subseteq C^{0,2-var}\left([0,T],\mathcal{V}\right)=:\overline{C^{1-var}\left([0,T],\mathcal{V}\right)}^{2-var}.$$

On the other hand, when $\dim\left(\mathcal{V}\right)\geq 2$, for any fixed $\gamma\in C^{0,2-var}\left([0,T],\mathcal{V}\right)$, we want to find a non-enhancible path $\widetilde{\gamma}$ in the 2-variation neighborhood of γ. Based on the definition of f at (3.8):

$$f\left(t\right):=\sum_{n=1}^{\infty}\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k^{\frac{1}{2}}2^{k}}\exp\left(2\pi i\left(-1\right)^{n}2^{2k}t\right),t\in[0,1].$$

Define

$$f_N\left(t\right):=\sum_{n=N}^{\infty}\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k^{\frac{1}{2}}2^{k}}\exp\left(2\pi i\left(-1\right)^{n}2^{2k}t\right),t\in[0,1].$$

Then (based on Lemma 3.35 below, which is used in the proof of the non-enhancibility of f), $\{f_N\}\subset C^{0,2-var}([0,1],\mathbb{C})$ and $\sup_N l_N^{\frac{1}{2}}\left\|f_N\right\|_{2-var}:=C<\infty$ (where l_N is the smallest index in the definition of f_N). Moreover, for each fixed N, we have $\sup_D\left|A((f_N)^{D})\left(0,1\right)\right|=\infty$ (because $\sup_D\left|A\left(f^{D}\right)\left(0,1\right)\right|=\infty$ and $f-f_N$ is smooth).

Select e_1, $e_2\in\mathcal{V}$, s.t. $[e_1,e_2]\neq 0$. For any $\epsilon>0$, choose an integer K, s.t. $2^{-2K}<T$, $\left\|\gamma\right\|_{2-var,[0,2^{-2K}]}<\epsilon$ and $\left(\left\|e_1\right\|+\left\|e_2\right\|\right)\left(Cl_K^{-\frac{1}{2}}\right)<\epsilon$. Define g as

$$g\left(t\right)=\left(\operatorname{Re}\left(f_K\left(t\right)-f_K\left(1\right)\right)\right)e_1+\left(\operatorname{Im}\left(f_K\left(t\right)-f_K\left(1\right)\right)\right)e_2,t\in[0,1].$$

Then $g \in C^{0,2-var}([0,1],\mathcal{V})$, $g(1)=0$ and

$$\begin{aligned}\|g\|_{2-var,[0,1]} &\leq (\|e_1\|+\|e_2\|)\|f_K\|_{2-var,[0,1]}\\ &\leq (\|e_1\|+\|e_2\|)(Cl_K^{-\frac{1}{2}}) < \epsilon;\\ \sup_D \left\|A(g^D)(0,1)\right\| &= \sup_D \left|A((f_K)^D)(0,1)\right| \|[e_1,e_2]\| = \infty.\end{aligned}$$

Define

$$\widetilde{\gamma}(t) = \begin{cases} g\left(2^{2(K+1)}t\right)+\gamma(\frac{1}{2^{2(K+1)}}), & t\in[0,2^{-2(K+1)}]\\ \text{linear}, & t\in[2^{-2(K+1)},2^{-2K}]\\ \gamma(t), & t\in[2^{-2K},T]\end{cases}.$$

Then $\widetilde{\gamma}$ is of vanishing 2-variation and

$$\|\gamma-\widetilde{\gamma}\|_{2-var} \leq 2\|\gamma\|_{2-var,[0,2^{-2K}]}+\|g\|_{2-var,[0,1]} < 3\epsilon.$$

On the other hand,

$$\begin{aligned}&\sup_{D\subset[0,1]}\left\|A(\widetilde{\gamma}^D)\right\|_{1-var} \geq \sup_{D\subset[0,1]}\left\|A(\widetilde{\gamma}^D)\right\|_{\infty-var}\\ \geq &\sup_{D\subset[0,\frac{1}{2^{2(K+1)}}]}\left\|A(\widetilde{\gamma}^D)(0,\frac{1}{2^{2(K+1)}})\right\| = \sup_{D\subset[0,1]}\left\|A(g^D)(0,1)\right\| = \infty.\end{aligned}$$

Thus $A(\widetilde{\gamma}^D)$ does not converge in 1-variation as $|D|\to 0$, and based on Theorem 3.20, $\widetilde{\gamma}$ is not enhancible. ■

When γ is a path of finite p-variation, $p\in[1,2)$, based on Young integral and Theorem 3.20, the enhancement of γ to geometric 2-rough path exists uniquely in the form of Riemann-Stieltjes integral. Thus $\cup_{1\leq p<2}C^{p-var}([0,T],\mathcal{V})\subseteq\mathcal{G}_2(\mathcal{V})$. Then

Problem 3.23 *Is the inclusion* $\cup_{1\leq p<2}C^{p-var}([0,T],\mathcal{V})\subseteq\mathcal{G}_2(\mathcal{V})$ *strict?*

Yes, it is. When $\dim(\mathcal{V})=1$, $\mathcal{G}_2(\mathcal{V})=C^{0,2-var}([0,1],\mathcal{V})$ (based on (3.23)). Select $e\in\mathcal{V}$, $e\neq 0$, and define $h(t)=\left(t^{\frac{1}{2}}\cos^2\left(\frac{\pi}{t}\right)/\ln t\right)e$, $t\in[0,1]$. Then

$$h\in C^{0,2-var}([0,T],\mathcal{V})\setminus\cup_{1\leq p<2}C^{p-var}([0,T],\mathcal{V}) \quad \text{(Exer5.35[9])}.$$

When $\dim(\mathcal{V})\geq 2$, the inclusion is strict because $\cup_{1\leq p<2}C^{p-var}([0,T],\mathcal{V})$ is a space, but $\mathcal{G}_2(\mathcal{V})$ is not (Problem 3.21).

Although $\mathcal{G}_2(\mathcal{V})$ is not a space, it can be shifted in any of the " Young" direction.

Proposition 3.24 $\forall\gamma_1\in\mathcal{G}_2(\mathcal{V}),\forall\gamma_2\in\cup_{1\leq p<2}C^{p-var}([0,T],\mathcal{V})$, $\gamma_1+\gamma_2\in\mathcal{G}_2(\mathcal{V})$.

Suppose $\gamma_1\in\mathcal{G}_2(\mathcal{V})$, then γ_1 is of finite 2-variation. For any γ_2 of finite p-variation, $p\in[1,2)$, according to Young integral (i.e.(3.15)), $A(\gamma_1^D,\gamma_2^D)$ (defined at (3.16)) converges in $(2^{-1}+p^{-1})^{-1}$-variation as $|D|\to 0$ ($p<2$, so converges in 1-variation). Similarly, $A(\gamma_2^D,\gamma_1^D)$ and $A(\gamma_2^D,\gamma_2^D)$ converge in 1-variation as $|D|$ tends to zero. On the other hand, $\gamma_1\in\mathcal{G}_2(\mathcal{V})$, so applying Theorem 3.20, $A(\gamma_1^D):=A\left(\gamma_1^D,\gamma_1^D\right)$ converge in 1-variation.

Therefore $A((\gamma_1+\gamma_2)^D)=\sum_{i,j=1,2}A\left(\gamma_i^D,\gamma_j^D\right)$ converge in 1-variation as $|D|\to 0$ and $\gamma_1+\gamma_2$ is enhancible (Theorem 3.20).

Moreover, we can extend the space $\cup_{1\leq p<2}C^{p-var}\left([0,T],\mathcal{V}\right)$ to a larger space in $\mathcal{G}_2\left(\mathcal{V}\right)$ (which is a special case of the extended Young integral in Section 3.3).

Theorem 3.25 *Let $\gamma:[0,1]\to\mathcal{V}$ be a continuous path. If there exists a non-decreasing function $m:[0,1]\to\overline{\mathbb{R}^{+}}$ satisfying*

$$\lim_{t\to 0}m\left(t\right)=0,\ m\left(1\right)\leq 1,\ \text{and}\ \int_0^1\frac{m^2\left(t\right)}{t}dt<\infty,$$

such that

$$\sup_{0\leq s<t\leq 1}\frac{\left\Vert\gamma\left(t\right)-\gamma\left(s\right)\right\Vert}{\left|t-s\right|^{\frac{1}{2}}m\left(t-s\right)}<\infty. \tag{3.26}$$

Then $\gamma\in\mathcal{G}_2(\mathcal{V})$.

The proof of Theorem 3.25 is given in p48.

Remark 3.26 *In Theorem 3.25, by adding a* log *term and* log-log *term so on and so forth to function " m ", one can get a sequence of nested linear spaces in $\mathcal{G}_2\left(\mathcal{V}\right)$. Because they are nested, their union is still a linear space in $\mathcal{G}_2\left(\mathcal{V}\right)$.*

Since being enhancible is invariant under reparametrisation, as a variational form of Theorem 3.25, we have:

Corollary 3.27 *Let $\gamma:[0,T]\to\mathcal{V}$ be a continuous path. If there exists continuous function $\varphi:[0,\infty)\to[0,\infty)$ which is strictly increasing and onto, satisfies $\varphi\left(0\right)=0$,*

$$u\mapsto\frac{u^2}{\varphi(u)}\text{is non-decreasing , and}\ \int_0^1\frac{u}{\varphi\left(u\right)}du<\infty\ , \tag{3.27}$$

such that

$$\sup_D\sum_{k,t_k\in D}\varphi\left(\left\Vert\gamma\left(t_{k+1}\right)-\gamma\left(t_k\right)\right\Vert\right)<\infty. \tag{3.28}$$

Then $\gamma\in\mathcal{G}_2\left(\mathcal{V}\right)$.

Corollary 3.27 is proved on p49.

Moreover, the condition $\int_0^1\frac{m^2(t)}{t}dt<\infty$ is necessary:

Example 3.28 *For any Banach space $\mathcal{V}$ satisfying $\dim\left(\mathcal{V}\right)\geq 2$, and any non-decreasing function $m:[0,1]\to\overline{\mathbb{R}^{+}}$ satisfying*

$$\lim_{t\to 0}m\left(t\right)=0,m\left(1\right)\leq 1\ \text{and}\ \int_0^1\frac{m^2\left(t\right)}{t}dt=\infty,$$

there exists $\gamma\in C\left([0,1],\mathcal{V}\right)$ satisfies (3.26) but not in $\mathcal{G}_2\left(\mathcal{V}\right)$.

Example 3.28 is a special case ($p=q=2$) of Example 3.33 in Section 3.3.

3.3 Extension to Young integral

In the way of exploring paths in $\mathcal{G}_2$, we get an extension to Young integral [38] by assigning a finer scale continuity. We prove convergence in q-variation, while similar pointwise convergence was obtained in [4]. More resembling of our theorem is Corollary 3.90 in [7].

Before stating the theorem, we define the iterated integral of two paths, which is used in our estimation.

Definition 3.29 *For $i = 1, 2$, suppose $\mathcal{V}_i$ are Banach spaces, and $\gamma_i \in C([0,T], \mathcal{V}_i)$. Define the iterated integral of γ_1 and γ_2, $I(\gamma_1,\gamma_2) : \triangle_{[0,T]} \to \mathcal{V}_1 \otimes \mathcal{V}_2$ by setting (whenever the Riemann-Stieltjes integral exists)*

$$I(\gamma_1,\gamma_2)(s,t) := \iint_{s<u_1<u_2<t} d\gamma_1(u_1) \otimes d\gamma_2(u_2), \ \forall (s,t) \in \triangle_{[0,T]}. \tag{3.29}$$

Then we have, for any $(s,t) \in \triangle_{[0,T]}$,

$$I(\gamma_1,\gamma_2)(s,t) = \int_s^t \gamma_1(u) \otimes d\gamma_2(u) - \gamma_1(s) \otimes (\gamma_2(t) - \gamma_2(s)).$$

Theorem 3.30 *Let $\mathcal{V}_i$, $i = 1, 2$, be two Banach spaces and $\gamma_i : [0,1] \to \mathcal{V}_i$ be two continuous paths. If there exist $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, and two non-decreasing functions $m_i : [0,1] \to \overline{\mathbb{R}^+}$, $i = 1, 2$, satisfying*

$$\lim_{t\to 0} m_i(t) = 0,\ m_i(1) \le 1, \text{ and } \int_0^1 \frac{m_1(t)\, m_2(t)}{t} dt < \infty,$$

such that

$$\sup_{0\le s<t\le 1} \frac{\|\gamma_1(t) - \gamma_1(s)\|}{|t-s|^{\frac{1}{p}} m_1(t-s)} := C_1 < \infty, \ \sup_{0\le s<t\le 1} \frac{\|\gamma_2(t) - \gamma_2(s)\|}{|t-s|^{\frac{1}{q}} m_2(t-s)} := C_2 < \infty. \tag{3.30}$$

Then the Riemann-Stieltjes integral $\int_0^t \gamma_1(t) \otimes d\gamma_2(t)$, $t \in [0,1]$, exists, and

$$\left\| \int_0^{\cdot} \gamma_1(t) \otimes d\gamma_2(t) \right\|_{q-var} \le 8C_1C_2 \left(2 + \int_0^1 \frac{m_1(t)\, m_2(t)}{t} dt \right).$$

The proof for Theorem 3.30 starts on page 42.

Remark 3.31 *When $m_1(x) = x^a$, $m_2(x) = x^b$, $a > 0$, $b > 0$, we get Young integral.*

Remark 3.32 *In the proof of Theorem 3.30, we get an estimate of the iterated integral of γ_1 and γ_2:*

$$\|I(\gamma_1,\gamma_2)\|_{1-var} \le C_1C_2 \left(15 + 8 \int_0^1 \frac{m_1(t)\, m_2(t)}{t} dt \right).$$

On the other hand, the condition $\int_0^1 \frac{m_1(t)m_2(t)}{t} dt < \infty$ in Theorem 3.30 is necessary:

Example 3.33 *Suppose $m_i : [0,1] \to \overline{\mathbb{R}^+}$ are two non-decreasing functions, satisfying $\lim_{t\to 0} m_i(t) = 0$, $m_i(1) \le 1$, $i = 1,2$, and $\int_0^1 \frac{m_1(t)m_2(t)}{t}dt = \infty$. Then for any $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, there exist two continuous real-valued paths $\gamma_i : [0,1] \to \mathbb{R}$, $i = 1,2$, s.t.*

$$\sup_{0\le s<t\le 1} \frac{|\gamma_1(t) - \gamma_1(s)|}{|t-s|^{\frac{1}{p}} m_1(t-s)} < \infty, \quad \sup_{0\le s<t\le 1} \frac{|\gamma_2(t) - \gamma_2(s)|}{|t-s|^{\frac{1}{q}} m_2(t-s)} < \infty,$$

but the Riemann-Stieltjes integral $\int_0^1 \gamma_1(t)\, d\gamma_2(t)$ does not exist.

Proof of Example 3.33 is give in page 45.

Suppose $\gamma_i = (x_i, y_i)$, $i = 1,2$, are two 2-dimensional paths. Then the existence of Riemann-Stieltjes integral $\int \gamma_i \otimes d\gamma_i$ for $i = 1,2$, only implies certain compensation between the regularities of x_i and y_i, $i = 1,2$, while it does not provide information about the relationship between x_1 and y_2 or x_2 and y_1. Therefore, there is no reason to expect the Riemann-Stieltjes integral $\int (x_1 + x_2)\, d(y_1 + y_2)$ to exist. Actually, when adding two paths together, the rougher one counts for the regularity, so it is possible that the integral $\int (x_1 + x_2)\, d(y_1 + y_2)$ does not exist. This compensating property also manifests itself in rough paths, and is one of the potential reasons for the set of rough paths (and $\mathcal{G}_2(\mathcal{V})$) to be non-linear.

3.4 Proofs

Recall $\triangle_{[0,1]} = \{(s,t)\,|0 \le s \le t \le 1\}$.

Lemma 3.34 *For any $p > 1$ and any $a > 0$, there exists a constant $C_{a,p} > 0$, such that for any integer $m \ge 1$,*

$$\sum_{k=1}^{m} \frac{2^{2\left(1-\frac{1}{p}\right)k}}{k^a} \le C_{a,p} \frac{2^{2\left(1-\frac{1}{p}\right)m}}{m^a}.$$

Proof. L'Hospital's rule. ■

The following lemma is in the form of Exercise 9.14 in [9], only that we give uniform estimates.

Lemma 3.35 *Suppose $\mathcal{V}$ is a Banach space, $\{\varphi_n\}_{n\ge 1} \subset C\left(\triangle_{[0,1]}, \mathcal{V}\right)$, and there exists constant $M > 0$ s.t.*

$$\|\varphi_n(s,t)\| \le M(1 \wedge |t-s|), \ \forall (s,t) \in \triangle_{[0,T]}, \ \forall n \ge 1.$$

For $p \in (1,\infty)$, $a \in (0,\infty)$ and integers $1 \le N_1 \le N_2 \le \infty$, define

$$g^{a,p}_{N_1,N_2}(s,t) = \sum_{k=N_1}^{N_2} \frac{1}{k^a 2^{\frac{2k}{p}}} \varphi_k\left(2^{2k}s, 2^{2k}t\right), \ t \in [0,1].$$

Then

$$(i) \quad \sup_{1\le N_1\le N_2\le\infty} \sup_{0\le s<t\le 1} \frac{\left\|g^{a,p}_{N_1,N_2}(s,t)\right\|}{|t-s|^{\frac{1}{p}} \left(\ln \frac{2}{t-s}\right)^{-a}} \le C_{a,p.M} < \infty; \qquad (3.31)$$

for any $\delta \in (0,1)$ *(recall* $\omega_p(\gamma,\delta)$ *defined at* (3.12)*), we have*

$$(ii) \sup_{1\le N_1\le N_2\le\infty} \omega_p\left(g^{a,p}_{N_1,N_2},\delta\right) \le C_{a,p,M}\left(\ln\frac{2}{\delta}\right)^{-a}; \tag{3.32}$$

and for any fixed $N_1 \ge 1$,

$$(iii) \sup_{N_1\le N_2\le\infty} \left\|g^{a,p}_{N_1,N_2}\right\|_{p-var,[0,1]} \le \frac{\widetilde{C_{a,p,M}}}{N_1^a}, \tag{3.33}$$

where $C_{a,p,M} = (\ln 4)^a\, 2^{-\frac{1}{p}} M\left(8C_{a,p} + \left(2^{\frac{2}{p}}-1\right)^{-1}\right)$ *with* $C_{a,p}$ *from Lemma 3.34, and* $\widetilde{C_{a,p,M}} = \left((\ln 4)^{-ap} C^p_{a,p.M} + 2M^p\left(1-2^{-\frac{2}{p}}\right)^{-p}\right)^{\frac{1}{p}}$.

Proof. We prove (3.31) first. Fix $0 \le s < t \le 1$. Denote $n := \left[\log_4 \frac{8}{t-s}\right]$, then using $\left\|\varphi_k(s,t)\right\| \le M(1\wedge|t-s|)$, we get

$$\begin{aligned}\left\|g^{a,p}_{N_1,N_2}(s,t)\right\| &\le \sum_{k=1}^{n}\frac{1}{k^a 2^{\frac{2k}{p}}}\left\|\varphi_k\left(2^{2k}s,2^{2k}t\right)\right\| + \sum_{k=n+1}^{\infty}\frac{1}{k^a 2^{\frac{2k}{p}}}\left\|\varphi_k\left(2^{2k}s,2^{2k}t\right)\right\| \\ &\le M\sum_{k=1}^{n}\frac{2^{2\left(1-\frac{1}{p}\right)k}}{k^a}|t-s| + \sum_{k=n+1}^{\infty}\frac{M}{k^a 2^{\frac{2k}{p}}}.\end{aligned}$$

Based on Lemma 3.34, there exists $C_{a,p}$, s.t. for any $m \ge 1$, $\sum_{k=1}^{m} k^{-a}2^{2\left(1-\frac{1}{p}\right)k} \le C_{a,p}m^{-a}2^{2\left(1-\frac{1}{p}\right)m}$. Thus ($n > \log_4\frac{2}{t-s}$ and $\frac{2}{t-s} < 2^{2n} \le \frac{8}{t-s}$),

$$\begin{aligned}\left\|g^{a,p}_{N_1,N_2}(s,t)\right\| &\le MC_{a,p}\frac{2^{2\left(1-\frac{1}{p}\right)n}}{n^a}|t-s| + \frac{M}{2^{\frac{2}{p}}-1}\frac{1}{n^a 2^{\frac{2n}{p}}} \\ &\le M\left(8C_{a,p} + \frac{1}{2^{\frac{2}{p}}-1}\right)\frac{1}{n^a 2^{\frac{2n}{p}}} \\ &\le \frac{(\ln 4)^a M}{2^{\frac{1}{p}}}\left(8C_{a,p} + \frac{1}{2^{\frac{2}{p}}-1}\right)|t-s|^{\frac{1}{p}}\left(\ln\frac{2}{t-s}\right)^{-a}.\end{aligned}$$

Since our estimates holds for any $0 \le s < t \le 1$ and any integers $1 \le N_1 \le N_2 \le \infty$, (3.31) is done.

Based on (3.31), for any $\delta \in (0,1)$, and any finite partition $D = \{t_j\}$, $|D| \le \delta$, we have

$$\sum_{j,t_j\in D}\left\|g^{a,p}_{N_1,N_2}(t_j,t_{j+1})\right\|^p \le C^p_{a,p.M}\left(\ln\frac{2}{\delta}\right)^{-ap}\sum_{j,t_j\in D}|t_{j+1}-t_j| = C^p_{a,p,M}\left(\ln\frac{2}{\delta}\right)^{-ap}.$$

It holds for any D, $|D| \le \delta$, and any integers $1 \le N_1 \le N_2 \le \infty$, so (3.32) holds.

Then we prove (3.33). Fix N_1. Finite partitions whose mesh are less than 2^{-2N_1} are done in (3.32):

$$\sup_{N_1\le N_2\le\infty}\ \sup_{|D|\le 2^{-2N_1}}\sum_{j,t_j\in D}\left\|g^{a,p}_{N_1,N_2}(t_j,t_{j+1})\right\|^p \le \frac{C^p_{a,p.M}}{(\ln 4)^{ap}}\frac{1}{N_1^{ap}}. \tag{3.34}$$

For finite partitions $D=\{t_j\}$, $|D|>2^{-2N_1}$, denote $J_{N_1+}:=\left\{j\,|\,|t_{j+1}-t_j|>2^{-2N_1}\right\}$. Since there can not be more than 2×2^{2N_1} many subintervals in J_{N_1+} (and using $\|\varphi_n(s,t)\|\le M$)

$$\sum_{t_j\in D,j\in J_{N_1+}}\left\|g^{a,p}_{N_1,N_2}(t_j,t_{j+1})\right\|^p\le 2^{2N_1+1}\left(\sum_{k=N_1}^{\infty}\frac{M}{k^a2^{\frac{2k}{p}}}\right)^p\le 2\left(\frac{2^{\frac{2}{p}}M}{2^{\frac{2}{p}}-1}\right)^p\frac{1}{N_1^{ap}}.$$

The intervals in D which are not in J_{N_1+} can be treated as subintervals in another finite partition D', $|D'|\le 2^{-2N_1}$, so using (3.34) to bound them, we get

$$\sum_{t_j\in D}\left\|g^{a,p}_{N_1,N_2}(t_j,t_{j+1})\right\|^p\le\sum_{j\notin J_{N_1+}}+\sum_{j\in J_{N_1+}}\le\left(\frac{C^p_{a,p,M}}{(\ln 4)^{ap}}+2\left(\frac{2^{\frac{2}{p}}M}{2^{\frac{2}{p}}-1}\right)^p\right)\frac{1}{N_1^{ap}}.$$

Our estimates hold for any finite partition D, and for any integer $N_2\ge N_1$, so (3.33) holds. ■

Example 3.36 *Suppose $c>\pi$ is a constant, and $\{l_n\}$ is a sequence of increasing integers, satisfying*

$$c^n\le\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k}\le c^n+1,\ \forall n\ge 1. \tag{3.35}$$

If define $f:[0,1]\to\mathbb{C}$ as

$$f(t)=\sum_{n=1}^{\infty}\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k^{\frac{1}{2}}2^k}\exp\left(2\pi i(-1)^n2^{2k}t\right),\ \ t\in[0,1].$$

Then f is of vanishing 2-variation, and for any $a\in[-\infty,\infty]$, there exists a sequence of finite partition $\{D^a_n\}$ of $[0,1]$ satisfying (with $x:=\operatorname{Re}f$, $y:=\operatorname{Im}f$)

$$\lim_{n\to\infty}|D^a_n|=0\text{ and }\lim_{n\to\infty}\sum_{l,t_l\in D^a_n}\left(x(t_l)\,y(t_{l+1})-y(t_l)\,x(t_{l+1})\right)=a. \tag{3.36}$$

The $(-1)^n$ ensures that the limit oscillates. Without $(-1)^n$ we only get divergence, while not non-existence.

Proof. f of vanishing 2-variation follows from (3.32) in Lemma 3.35 (with $a=\frac{1}{2}$, $p=2$, $M=1$, $N_1=1$, $N_2=\infty$). Suppose $N\ge 1$ is an integer, denote

$$D_N:=\left\{l2^{-2N}\right\}_{l=0}^{2^{2N}},\ t^N_l:=l2^{-2N},\ l=0,1,\dots,2^{2N}, \tag{3.37}$$

$$\text{and }\langle f,D_N\rangle:=\sum_{l=0}^{2^{2N}-1}\left(x\left(t^N_l\right)y\left(t^N_{l+1}\right)-y\left(t^N_l\right)x\left(t^N_{l+1}\right)\right). \tag{3.38}$$

We want to prove that for each $a\in[-\infty,\infty]$, there exists a sequence of finite partitions $\{D^a_n\}_n\subset\{D_N\}_N$, satisfying $\lim_{n\to\infty}\langle f,D^a_n\rangle=a$.

Denote

$$\epsilon_k:=(-1)^n,\ \text{for }k=l_n,\dots,l_{n+1}-1;$$

$$c_k^N := 2\pi 2^{2k-2N}\epsilon_k, \text{ for } k = l_1, \dots, N-1.$$

Then $2\pi\epsilon_k 2^{2k} t_l^N = l c_k^N$, and

$$\begin{aligned}
& x\left(t_l^N\right) y\left(t_{l+1}^N\right) - y\left(t_l^N\right) x\left(t_{l+1}^N\right) \\
= & \left(\sum_{j=l_1}^{N-1} \frac{1}{j^{\frac{1}{2}} 2^j} \cos\left(2\pi\epsilon_j 2^{2j} t_l^N\right)\right)\left(\sum_{k=l_1}^{N-1} \frac{1}{k^{\frac{1}{2}} 2^k} \sin\left(2\pi\epsilon_k 2^{2k} t_{l+1}^N\right)\right) \\
& - \left(\sum_{j=l_1}^{N-1} \frac{1}{j^{\frac{1}{2}} 2^j} \sin\left(2\pi\epsilon_j 2^{2j} t_l^N\right)\right)\left(\sum_{k=l_1}^{N-1} \frac{1}{k^{\frac{1}{2}} 2^k} \cos\left(2\pi\epsilon_k 2^{2k} t_{l+1}^N\right)\right) \\
= & \sum_{k,j=l_1}^{N-1} \frac{1}{k^{\frac{1}{2}} j^{\frac{1}{2}} 2^{k+j}} \sin\left(\left(l+1\right) c_k^N - l c_j^N\right) \\
= & \sum_{k=l_1}^{N-1} \frac{1}{k 2^{2k}} \sin\left(2\pi\epsilon_k 2^{2k-2N}\right) \\
& + \sum_{l_1 \le k < j \le N-1} \frac{1}{k^{\frac{1}{2}} j^{\frac{1}{2}} 2^{k+j}} \left(\sin\left(l\left(c_k^N - c_j^N\right) + c_k^N\right) + \sin\left(l\left(c_j^N - c_k^N\right) + c_j^N\right)\right)
\end{aligned}$$

Sum l from 0 to $2^{2N}-1$,

$$\begin{aligned}
\langle f, D_N \rangle &= \sum_{l=0}^{2^{2N}-1} x\left(t_l^N\right) y\left(t_{l+1}^N\right) - y\left(t_l^N\right) x\left(t_{l+1}^N\right) \\
&= \sum_{k=l_1}^{N-1} \frac{1}{k 2^{2k-2N}} \sin\left(2\pi\epsilon_k 2^{2k-2N}\right) \\
&\quad + \sum_{l_1 \le k < j \le N-1} \frac{1}{k^{\frac{1}{2}} j^{\frac{1}{2}} 2^{k+j}} \sum_{l=0}^{2^{2N}-1} \left(\sin\left(l\left(c_k^N - c_j^N\right) + c_k^N\right) + \sin\left(l\left(c_j^N - c_k^N\right) + c_j^N\right)\right)
\end{aligned}$$

Since

$$\sum_{l=0}^{2^{2N}-1} \sin\left(l\left(c_k^N - c_j^N\right) + c_k^N\right) = \sum_{l=0}^{2^{2N}-1} \sin\left(l\left(c_j^N - c_k^N\right) + c_j^N\right) = 0,$$

we have

$$\begin{aligned}
\langle f, D_N \rangle &= \sum_{k=l_1}^{N-1} \frac{1}{k 2^{2k-2N}} \sin\left(2\pi\epsilon_k 2^{2k-2N}\right) \\
&=: \sum_{j=1}^{J-1} \left(-1\right)^j s_j^N + \left(-1\right)^J \sum_{k=l_J}^{N-1} \frac{1}{k 2^{2k-2N}} \sin\left(2\pi 2^{2k-2N}\right).
\end{aligned}$$

where $l_J + 1 \le N \le l_{J+1}$, and

$$s_j^N := \sum_{k=l_j}^{l_{j+1}-1} \frac{1}{k 2^{2k-2N}} \sin\left(2\pi 2^{2k-2N}\right), \; 1 \le j \le J-1.$$

Using $\frac{2}{\pi}\theta \leq \sin\theta \leq \theta$ when $\theta \in \left[0, \frac{\pi}{2}\right]$ and condition (3.35), we have, for any $j \geq 1$, and any $N \geq l_{j+1}$,

$$4 \times c^j \leq s_j^N \leq 2\pi \times \left(c^j + 1\right).$$

Thus using $s_j^N - s_{j-1}^N \geq (4c - 2\pi)\, c^{j-1} - 2\pi$, we estimate $\sum_{j=1}^{m-1} (-1)^j s_j^N$. When m is even and $m \geq 4$, for any $N \geq l_m$,

$$\begin{aligned} \sum_{j=1}^{m-1} (-1)^j s_j^N &= -\left(s_{m-1}^N - s_{m-2}^N\right) - \cdots - s_1^N \\ &\leq -\frac{4c - 2\pi}{c^2 - 1}\left(c^m - c^2\right) + \pi\,(m-2) - 4c. \end{aligned} \tag{3.39}$$

Similarly, when m is odd and $m \geq 5$, for any $N \geq l_m$,

$$\begin{aligned} \sum_{j=1}^{m-1} (-1)^j s_j^N &= \left(s_{m-1}^N - s_{m-2}^N\right) + \cdots + \left(s_2^N - s_1^N\right) \\ &\geq \frac{4c - 2\pi}{c^2 - 1}\left(c^m - c\right) - \pi\,(m-1)\,; \end{aligned} \tag{3.40}$$

and when m is odd and $m \geq 5$, for any $N \geq l_m$, the upper bound:

$$\begin{aligned} \sum_{j=1}^{m-1} (-1)^j s_j^N &= s_{m-1}^N - \left(s_{m-2}^N - s_{m-3}^N\right) - \cdots - s_1^N \\ &\leq 2\pi \times \left(c^{m-1} + 1\right) - \frac{4c - 2\pi}{c^2 - 1}\left(c^{m-1} - c^2\right) + \pi\,(m-3) - 4c \\ &= \left(\frac{2\pi}{c} - \frac{4c - 2\pi}{c\,(c^2 - 1)}\right) c^m + \pi\,(m-1) + \frac{4c - 2\pi}{c^2 - 1} c^2 - 4c. \end{aligned} \tag{3.41}$$

Since we assumed $c > \pi$, in (3.39) and (3.40) we have $\frac{4c-2\pi}{c^2-1} > 0$. On the other hand, since $\langle f, D_{l_m} \rangle = \sum_{j=1}^{m-1} (-1)^j s_j^{l_m}$, based on (3.39) and (3.40), we have

$$\lim_{n\to\infty} \langle f, D_{l_{2n}} \rangle = -\infty \text{ and } \lim_{n\to\infty} \left\langle f, D_{l_{2n+1}} \right\rangle = +\infty.$$

Thus, when $a = +\infty$ let $D_n^a := D_{l_{2n+1}}$, when $a = -\infty$ let $D_n^a := D_{l_{2n}}$.

Fix $a \in (-\infty, \infty)$.

Firstly, we assumed $c > \pi$, so

$$0 < \frac{2\pi}{c} - \frac{4c - 2\pi}{c\,(c^2 - 1)} < \frac{2\pi}{c} < 2.$$

For our fixed $c > \pi$, choose integer $M_c \geq 1$, s.t. for any $m \geq M_c$,

$$\left(\frac{2\pi}{c} - \frac{4c - 2\pi}{c\,(c^2 - 1)}\right) c^m + \pi\,(m-1) + \frac{4c - 2\pi}{c^2 - 1} c^2 - 4c \leq 2c^m.$$

Thus, combined with (3.41), when m is odd and $m \geq 5 \vee M_c$, for any $N \geq l_m$, we have

$$\sum_{j=1}^{m-1} (-1)^j s_j^N \leq 2c^m. \tag{3.42}$$

Then for our fixed $a \in (-\infty, \infty)$, choose odd integer $M(a) \geq 5 \vee M_c$ such that, for any odd integer $m \geq M(a)$, and any $N \geq l_m$, we have

$$\sum_{j=1}^{m-1} (-1)^j s_j^N > |a| + 10\pi, \tag{3.43}$$

which is possible because of (3.40).

We prove that for any odd integer $m \geq M(a)$, there exists $N_m(a)$, $l_m < N_m(a) < l_{m+1}$, s.t.

$$\left|\left\langle f, D_{N_m(a)} \right\rangle - a\right| \leq \frac{\pi}{l_m}.$$

Fix odd integer $m \geq M(a)$. For any $N \geq l_m$ (use $c^m \leq \sum_{k=l_m}^{l_{m+1}-1} k^{-1}$, i.e.(3.35)),

$$|a| + 10\pi < \sum_{j=1}^{m-1} (-1)^j s_j^N \leq 2c^m \leq 2 \sum_{k=l_m}^{l_{m+1}-1} k^{-1}. \tag{3.44}$$

Thus, when $N = l_{m+1}$ in (3.44), we have

$$\begin{aligned} \left\langle f, D_{l_{m+1}} \right\rangle &= \sum_{j=1}^{m-1} (-1)^j s_j^{l_{m+1}} - \sum_{k=l_m}^{l_{m+1}-1} \frac{\sin\left(2\pi 2^{2k-2l_{m+1}}\right)}{k 2^{2k-2l_{m+1}}} \\ &\leq \sum_{j=1}^{m-1} (-1)^j s_j^{l_{m+1}} - 4 \sum_{k=l_m}^{l_{m+1}-1} k^{-1} \\ &\leq -2 \sum_{k=l_m}^{l_{m+1}-1} k^{-1} < -|a| - 10\pi\ . \end{aligned} \tag{3.45}$$

While in (3.44) let $N = l_m$, we have

$$\langle f, D_{l_m} \rangle = \sum_{j=1}^{m-1} (-1)^j s_j^m > |a| + 10\pi. \tag{3.46}$$

Combine (3.45) with (3.46), if $|\langle f, D_N\rangle - \langle f, D_{N+1}\rangle|$ is uniformly small when $l_m \leq N \leq l_{m+1} - 1$, then $\exists N_m(a)$, $l_m \leq N_1(a) \leq l_{m+1}$, s.t. $\left\langle f, D_{N_m(a)}\right\rangle$ is in the neighborhood of a.

Actually, for any $N \geq l_1 + 1$,

$$\begin{aligned} &|\langle f, D_{N+1}\rangle - \langle f, D_N\rangle| \\ \leq\ &\left|\sum_{k=l_1}^{N-1} \frac{1}{k 2^{2k-2(N+1)}} \left(\frac{\sin\left(2\pi 2^{2k-2N}\right)}{4} - \sin\left(2\pi 2^{2k-2(N+1)}\right)\right)\right| + \frac{4}{N}. \end{aligned}$$

For any $\theta \in \left[0, \frac{\pi}{2}\right]$, using $\sin(2\theta) = 2\sin\theta\cos\theta$, we have

$$\begin{aligned} \frac{1}{\theta}\left|\frac{\sin(4\theta)}{4} - \sin\theta\right| &= \frac{\sin\theta}{\theta} |\cos\theta\cos 2\theta - 1| \\ \leq \left|\left(1 - 2\sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\theta\right) - 1\right| &\leq 14\sin^2\frac{\theta}{2} \leq \frac{7}{2}\theta^2. \end{aligned}$$

Thus let $\theta = 2\pi 2^{2k-2(N+1)}$, we have

$$\frac{1}{2^{2k-2(N+1)}}\left|\frac{\sin\left(2\pi 2^{2k-2N}\right)}{4} - \sin\left(2\pi 2^{2k-2(N+1)}\right)\right| \leq 28\pi^3\left(\frac{1}{2^{2(N+1)-2k}}\right)^2.$$

Thus, when $l_m \leq N \leq l_{m+1} - 1$,

$$\left|\langle f, D_{N+1}\rangle - \langle f, D_N\rangle\right| \leq 28\pi^3 \sum_{k=l_1}^{N-1} \frac{1}{k}\left(\frac{1}{2^{2(N+1)-2k}}\right)^2 + \frac{4}{N}. \tag{3.47}$$

While one can prove that for any $m \geq 2$, $\sum_{k=1}^{m-1} \frac{2^{4k}}{k} \leq \frac{2^{4m}}{m}$ by using mathematical induction, so for any $N \geq l_1 + 1$,

$$\sum_{k=l_1}^{N-1} \frac{1}{k}\left(\frac{1}{2^{2(N+1)-2k}}\right)^2 \leq \frac{1}{2^{4N+4}} \sum_{k=1}^{N-1} \frac{2^{4k}}{k} \leq \frac{1}{16N}. \tag{3.48}$$

Then, combined (3.47) with (3.48), we get when $l_m \leq N \leq l_{m+1} - 1$,

$$\left|\langle f, D_{N+1}\rangle - \langle f, D_N\rangle\right| \leq \left(\frac{7}{4}\pi^3 + 4\right)\frac{1}{N} < \frac{20\pi}{l_m}.$$

Thus, combined with (3.45) and (3.46), there exists integer $N_m(a)$, $l_m \leq N_m(a) \leq l_{m+1}$, s.t.

$$\left|\left\langle f, D_{N_m(a)}\right\rangle - a\right| < \frac{10\pi}{l_m}.$$

Moreover, since $\langle f, D_{l_m}\rangle > |a| + 10\pi \geq |a| + \frac{10\pi}{l_m}$, $\left\langle f, D_{l_{m+1}}\right\rangle < -|a| - 10\pi \leq -|a| - \frac{10\pi}{l_m}$, we have $l_m < N_m(a) < l_{m+1}$.

Therefore, if let $D_m^a := D_{N_m(a)}$, $m \geq 1$, then $\{D_m^a\}_m$ is a sequence of finite partitions, whose mesh tends to zero, but the limit of the corresponding Riemann sum is a. ■

Next, we demonstrate that when the space of smooth paths is equipped with 2-variation, the area operator is unbounded, and non-closable when the area is equipped with p-variation, $p > 1$.

Lemma 3.37 *Suppose $\{l_n\}_n$ is a sequence of strictly increasing integers. Then*

$$\lim_{n\to\infty}\left\|\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin\left(2\pi 2^{2k}(t-s)\right)\right\|_{p-var,[0,1]} = 0 \text{ for any } p > 1.$$

Proof. We do estimation for fixed $p > 1$ and fixed sufficiently large n.

For integer $m \geq l_n$, denote $I_m := (2^{-2p(m+1)}, 2^{-2pm}]$, and denote $I_{l_n+} := (2^{-2pl_n}, 1]$. Suppose $D = \{t_j\}$ is a finite partition satisfying that $\{|t_{j+1} - t_j|\}_j \subset \cup_{i=1}^s I_{m_i} \cup I_{l_n+}$ with $\min_{1\leq i\leq s} m_i \geq l_n$. Denote $J_{m_i} := \{j | t_{j+1} - t_j \in I_{m_i}\}$ and $J_{l_n+} := \{j | t_{j+1} - t_j \in I_{l_n+}\}$. We assume that J_{m_i} is not empty for each i. For J_{l_n+}, since we can not have more than $2^{2pl_n+1}\sum_{j,j\in J_{l_n+}}(t_{j+1} - t_j)$ intervals in J_{l_n+},

$$\begin{aligned}
&\sum_{j,j\in J_{l_n+}}\left(\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin\left(2\pi 2^{2k}(t_{j+1} - t_j)\right)\right)^p \\
\leq\ & 2^{2pl_n+1}\left(\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}}\right)^p \sum_{j,j\in J_{l_n+}}(t_{j+1} - t_j) \leq \frac{2^{2p+1}}{3^p l_n^p} \sum_{j,j\in J_{l_n+}}(t_{j+1} - t_j).
\end{aligned} \tag{3.49}$$

Then we do estimation for fixed i, $i = 1, 2, \ldots, s$. Suppose $t_{j+1} - t_j \in I_{m_i}$, then

$$\begin{aligned}
&\left(\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin\left(2\pi 2^{2k}\left(t_{j+1}-t_j\right)\right)\right)^p \\
\leq\ & 2^{p-1}\left(\left(2\pi \sum_{k=l_n}^{m_i} \frac{1}{k}\right)^p \left|t_{j+1}-t_j\right|^p + \left(\sum_{k=m_i+1}^{\infty} \frac{1}{k2^{2k}}\right)^p\right) \\
\leq\ & 2^{p-1}\left(\left(2\pi\left(1+\ln m_i\right)\right)^p \left(\frac{1}{2^{2p^2 m_i}}\right) + \frac{1}{3^p m_i^p 2^{2pm_i}}\right).
\end{aligned}$$

Since there can not be more than $2 \times 2^{2p(m_i+1)} \sum_{j,j\in J_{m_i}} \left(t_{j+1}-t_j\right)$ many intervals whose length fail into the category I_{m_i},

$$\begin{aligned}
&\sum_{j,j\in J_{m_i}} \left(\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin\left(2\pi 2^{2k}\left(t_{j+1}-t_j\right)\right)\right)^p \qquad (3.50) \\
\leq\ & 2^{p-1}\left(\left(2\pi\right)^p \frac{\left(1+\ln m_i\right)^p}{2^{2p^2 m_i}} + \frac{1}{3^p m_i^p 2^{2pm_i}}\right) \times 2^{2p(m_i+1)+1} \sum_{j,j\in J_{m_i}} \left(t_{j+1}-t_j\right) \\
\leq\ & 2^{3p}\left(\left(2\pi\right)^p \frac{\left(1+\ln m_i\right)^p}{2^{2p(p-1)m_i}} + \frac{1}{3^p m_i^p}\right) \sum_{j,j\in J_{m_i}} \left(t_{j+1}-t_j\right).
\end{aligned}$$

Since $\{l_n\}$ are strictly increasing integers, $\lim_{n\to\infty} l_n = +\infty$. Thus, for our fixed $p > 1$, there exists $N(p) \geq 1$, s.t. for any $n \geq N(p)$ and any $m_i \geq l_n$, we have

$$\frac{\left(1+\ln m_i\right)^p}{2^{2p(p-1)m_i}} \leq \frac{1}{m_i^p}.$$

Therefore, for any fixed finite partition $D = \{t_j\}$ of $[0,1]$, when $n \geq N(p)$, we have (using (3.49), (3.50) and $\sum_{i=1}^{s} \sum_{j\in J_{m_i}} \left(t_{j+1}-t_j\right) + \sum_{j\in J_{l_n+}} \left(t_{j+1}-t_j\right) = 1$, $\min_{1\leq i\leq s} m_i \geq l_n$)

$$\begin{aligned}
&\sum_{j,t_j\in D} \left(\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin\left(2\pi 2^{2k}\left(t_{j+1}-t_j\right)\right)\right)^p \\
\leq\ & 2^{3p}\left(\left(2\pi\right)^p + \frac{1}{3^p}\right)\left(\sum_{i=1}^{s} \frac{1}{m_i^p} \sum_{j,j\in J_{m_i}} \left(t_{j+1}-t_j\right)\right) + \frac{2^{2p+1}}{3^p l_n^p} \sum_{j,j\in J_{l_n+}} \left(t_{j+1}-t_j\right) \\
\leq\ & 2^{3p}\left(\left(2\pi\right)^p + \frac{1}{3^p}\right)\frac{1}{l_n^p}.
\end{aligned}$$

Hence, for any fixed $p > 1$, there exists integer $N(p)$, s.t. for any $n \geq N(p)$,

$$\left\|\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}} \sin\left(2\pi 2^{2k}\left(t-s\right)\right)\right\|_{p-var,[0,1]}^p \leq 2^{3p}\left(\left(2\pi\right)^p + \frac{1}{3^p}\right)\frac{1}{l_n^p}.$$

■

Lemma 3.38 *Suppose $\{l_n\}_n$ is a sequence of strictly increasing integers. Define*

$$g_n(t) = \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k^{\frac{1}{2}} 2^k} \exp\left(2\pi i 2^{2k} t\right), \ t \in [0,1].$$

Then $\lim_{n\to\infty} \|g_n\|_{2-var} = 0$, and for any $p > 1$,

$$\lim_{n\to\infty} \left\| A(g_n)(s,t) - \left(\pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k}\right)(t-s) \right\|_{p-var,[0,1]} = 0.$$

Proof. Since trigonometric functions are Lipschitz and bounded, according to (3.33) in Lemma 3.35 with $p = 2$, $\lim_{n\to\infty} \|g_n\|_{2-var,[0,1]} = 0$.

According to the definition of area, if we denote $x_n := \operatorname{Re} g_n$, $y_n := \operatorname{Im} g_n$, and

$$\begin{aligned} p_n(s,t) &: = \int_s^t x_n(u)\, dy_n(u) - y_n(u)\, dx_n(u), \\ q_n(s,t) &: = y_n(s)\, x_n(t) - x_n(s)\, y_n(t), \end{aligned}$$

we have

$$A(g_n)(s,t) = \frac{1}{2}\left(p_n(s,t) + q_n(s,t)\right).$$

Firstly, for $p_n(s,t)$,

$$\begin{aligned} & p_n(s,t) \\ = \ & 2\pi \int_s^t \left(\sum_{i=l_n}^{l_{n+1}-1} \frac{1}{i^{\frac{1}{2}} 2^i} \cos\left(2\pi 2^{2i} u\right)\right)\left(\sum_{j=l_n}^{l_{n+1}-1} \frac{2^j}{j^{\frac{1}{2}}} \cos\left(2\pi 2^{2j} u\right)\right) \\ & \qquad + \left(\sum_{j=l_n}^{l_{n+1}-1} \frac{1}{j^{\frac{1}{2}} 2^j} \sin\left(2\pi 2^{2j} u\right)\right)\left(\sum_{j=l_n}^{l_{n+1}-1} \frac{2^i}{i^{\frac{1}{2}}} \sin\left(2\pi 2^{2i} u\right)\right) du \\ = \ & 2\pi \sum_{i,j=l_n}^{l_{n+1}-1} \int_s^t \frac{2^{j-i}}{i^{\frac{1}{2}} j^{\frac{1}{2}}} \cos\left(2\pi 2^{2i} u\right) \cos\left(2\pi 2^{2j} u\right) + \frac{2^{i-j}}{i^{\frac{1}{2}} j^{\frac{1}{2}}} \sin\left(2\pi 2^{2j} u\right) \sin\left(2\pi 2^{2i} u\right) du \\ = \ & \left(2\pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k}\right)(t-s) + 2\pi \sum_{l_n \le i < j \le l_{n+1}-1} \left(\frac{2^{j-i}}{i^{\frac{1}{2}} j^{\frac{1}{2}}} + \frac{2^{i-j}}{i^{\frac{1}{2}} j^{\frac{1}{2}}}\right) \int_s^t \cos\left(2\pi\left(2^{2j} - 2^{2i}\right) u\right) du \\ = \ & \left(2\pi \sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k}\right)(t-s) + \sum_{l_n \le i < j \le l_{n+1}-1} \frac{1}{i^{\frac{1}{2}} j^{\frac{1}{2}} 2^{i+j}} p_{i,j}(s,t), \end{aligned}$$

where

$$p_{i,j}(s,t) := \left(\frac{2^{2j} + 2^{2i}}{2^{2j} - 2^{2i}}\right)\left(\sin\left(2\pi\left(2^{2j} - 2^{2i}\right) t\right) - \sin\left(2\pi\left(2^{2j} - 2^{2i}\right) s\right)\right).$$

While, for $q_n(s,t)$,

$$\begin{aligned}
& q_n(s,t) = y_n(s)\,x_n(t) - x_n(s)\,y_n(t) \\
= & \left(\sum_{i=l_n}^{l_{n+1}-1} \frac{1}{i^{\frac{1}{2}}2^i}\sin\left(2\pi 2^{2i}s\right)\right)\left(\sum_{j=l_n}^{l_{n+1}-1} \frac{1}{j^{\frac{1}{2}}2^j}\cos\left(2\pi 2^{2j}t\right)\right) \\
& -\left(\sum_{i=l_n}^{l_{n+1}-1} \frac{1}{i^{\frac{1}{2}}2^i}\cos\left(2\pi 2^{2i}s\right)\right)\left(\sum_{j=l_n}^{l_{n+1}-1} \frac{1}{j^{\frac{1}{2}}2^j}\sin\left(2\pi 2^{2j}t\right)\right) \\
= & \sum_{i,j=l_n}^{l_{n+1}-1} \frac{1}{i^{\frac{1}{2}}j^{\frac{1}{2}}2^{i+j}}\sin\left(2\pi\left(2^{2i}s-2^{2j}t\right)\right) \\
= & -\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}}\sin\left(2\pi 2^{2k}(t-s)\right) + \sum_{l_n\le i<j\le l_{n+1}-1} \frac{1}{i^{\frac{1}{2}}j^{\frac{1}{2}}2^{i+j}} q_{i,j}(s,t),
\end{aligned}$$

where

$$q_{i,j}(s,t) = \sin\left(2\pi\left(2^{2i}s-2^{2j}t\right)\right) + \sin\left(2\pi\left(2^{2j}s-2^{2i}t\right)\right).$$

Thus

$$\begin{aligned}
& A(g_n)(s,t) - \left(\pi\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k}\right)(t-s) \\
= & \frac{1}{2}\left(-\sum_{k=l_n}^{l_{n+1}-1} \frac{1}{k2^{2k}}\sin\left(2\pi 2^{2k}(t-s)\right) + \sum_{l_n\le i<j\le l_{n+1}-1} \frac{1}{i^{\frac{1}{2}}j^{\frac{1}{2}}2^{i+j}}\left(p_{i,j}(s,t)+q_{i,j}(s,t)\right)\right).
\end{aligned}$$

Based on Lemma 3.37, $\sum_{k=l_n}^{l_{n+1}-1} k^{-1}2^{-2k}\sin\left(2\pi 2^{2k}(t-s)\right)$ converge to 0 as n tends to infinity in p-variation for any $p>1$, so we are left with

$$\sum_{l_n\le i<j\le l_{n+1}-1} \frac{1}{i^{\frac{1}{2}}j^{\frac{1}{2}}2^{i+j}}\left(p_{i,j}(s,t)+q_{i,j}(s,t)\right).$$

While

$$\begin{aligned}
& p_{i,j}(s,t)+q_{i,j}(s,t) \\
= & \left(\frac{2^{2j}+2^{2i}}{2^{2j}-2^{2i}}\right)\left(\sin\left(2\pi\left(2^{2j}-2^{2i}\right)t\right)-\sin\left(2\pi\left(2^{2j}-2^{2i}\right)s\right)\right) \\
& +\sin\left(2\pi\left(2^{2i}s-2^{2j}t\right)\right)+\sin\left(2\pi\left(2^{2j}s-2^{2i}t\right)\right) \\
= & \left(\frac{2\times 2^{2i}}{2^{2j}-2^{2i}}\right)\left(\sin\left(2\pi\left(2^{2j}-2^{2i}\right)t\right)-\sin\left(2\pi\left(2^{2j}-2^{2i}\right)s\right)\right) \\
& +\left(\sin\left(2\pi\left(2^{2j}-2^{2i}\right)t\right)+\sin\left(2\pi\left(2^{2i}s-2^{2j}t\right)\right)\right) \\
& +\left(\sin\left(2\pi\left(2^{2j}s-2^{2i}t\right)\right)-\sin\left(2\pi\left(2^{2j}-2^{2i}\right)s\right)\right) \\
= & \left(\frac{2\times 2^{2i}}{2^{2j}-2^{2i}}\right)\left(\sin\left(2\pi\left(2^{2j}-2^{2i}\right)t\right)-\sin\left(2\pi\left(2^{2j}-2^{2i}\right)s\right)\right) \\
& -2\cos\left(2\pi\left(2^{2j}t-2^{2i}\frac{t+s}{2}\right)\right)\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)
\end{aligned}$$

$$\begin{aligned}
& -2\cos\left(2\pi\left(2^{2j}s-2^{2i}\frac{t+s}{2}\right)\right)\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right) \\
= & \left(\frac{4\times 2^{2i}}{2^{2j}-2^{2i}}\right)\cos\left(2\pi\left(2^{2j}-2^{2i}\right)\frac{t+s}{2}\right)\sin\left(2\pi\left(2^{2j}-2^{2i}\right)\frac{t-s}{2}\right) \\
& -4\cos\left(2\pi\left(2^{2j}-2^{2i}\right)\frac{t+s}{2}\right)\cos\left(2\pi 2^{2j}\frac{t-s}{2}\right)\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right) \\
= & \ 4\cos\left(2\pi\left(2^{2j}-2^{2i}\right)\frac{t+s}{2}\right)\left(\left(\frac{2^{2i}}{2^{2j}-2^{2i}}\right)\sin\left(2\pi\left(2^{2j}-2^{2i}\right)\frac{t-s}{2}\right)\right. \\
& \left. -\cos\left(2\pi 2^{2j}\frac{t-s}{2}\right)\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \left|p_{i,j}(s,t)+q_{i,j}(s,t)\right| \leq 4\left(\frac{2^{2i+2j}}{2^{2j}-2^{2i}}\right) \\
& \times\left|\frac{\sin\left(2\pi 2^{2j}\frac{t-s}{2}\right)}{2^{2j}}\cos\left(2\pi 2^{2i}\frac{t-s}{2}\right)-\frac{\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)}{2^{2i}}\cos\left(2\pi 2^{2j}\frac{t-s}{2}\right)\right|.
\end{aligned} \tag{3.51}$$

While, for any θ, and any integer $n\geq 1$,

$$\sin\theta\prod_{k=0}^{n-1}\cos\left(2^k\theta\right)=\frac{\sin\left(2^n\theta\right)}{2^n},$$

so, when $j>i$,

$$\frac{\sin\left(2^{2j}\theta\right)}{2^{2j}}=\frac{\sin\left(2^{2i}\theta\right)}{2^{2i}}\prod_{k=2i}^{2j-1}\cos\left(2^k\theta\right).$$

Thus when $\theta=\pi(t-s)$, continue with (3.51), we have

$$\begin{aligned}
& \left|p_{i,j}(s,t)+q_{i,j}(s,t)\right| \\
= & \ 4\left(\frac{2^{2i+2j}}{2^{2j}-2^{2i}}\right)\left|\frac{\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)}{2^{2i}}\right| \\
& \times\left|\cos\left(2\pi 2^{2i}\frac{t-s}{2}\right)\prod_{k=2i}^{2j-1}\cos\left(2\pi 2^{k}\frac{t-s}{2}\right)-\cos\left(2\pi 2^{2j}\frac{t-s}{2}\right)\right| \\
\leq & \ 4\left(\frac{2^{2i+2j}}{2^{2j}-2^{2i}}\right)\frac{\left|\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)\right|}{2^{2i}}\times 2 \\
\leq & \ \frac{32}{3}\left|\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)\right|. \quad \left(\frac{2^{2j}}{2^{2j}-2^{2i}}\leq\frac{4}{3}\text{ when } j>i\right)
\end{aligned}$$

Therefore, for any $p\in(1,2)$,

$$\begin{aligned}
& \sum_{l_n\leq i<j\leq l_{n+1}-1}\frac{1}{i^{\frac{1}{2}}j^{\frac{1}{2}}2^{i+j}}\left|p_{i,j}(s,t)+q_{i,j}(s,t)\right| \\
\leq & \ \frac{32}{3}\sum_{j=l_n+1}^{l_{n+1}-1}\frac{1}{j^{\frac{1}{2}}2^j}\left(\sum_{i=l_n}^{j-1}\frac{1}{i^{\frac{1}{2}}2^i}\left|\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)\right|\right) \\
\leq & \ \frac{32}{3}\sum_{j=l_n+1}^{l_{n+1}-1}\frac{1}{j^{\frac{1}{2}}2^{\left(2-\frac{2}{p}\right)j}}\left(\sum_{i=l_n}^{j-1}\frac{1}{i^{\frac{1}{2}}2^{\frac{2}{p}i}}\left|\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)\right|\right).
\end{aligned} \tag{3.52}$$

While since $\left|\sin\left(t-s\right)\right| \leq 1 \wedge |t-s|$, based on (3.33) in Lemma 3.35, for any $p>1$, there exists a constant $\widetilde{C_{\frac{1}{2},p,1}}$, s.t. for any l_n and any $j>l_n$, we have,

$$\left\|\sum_{i=l_n}^{j-1}\frac{1}{i^{\frac{1}{2}}2^{\frac{2}{p}i}}\left|\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)\right|\right\|_{p-var} \leq \frac{\widetilde{C_{\frac{1}{2},p,1}}}{l_n^{\frac{1}{2}}}.$$

Therefore, for any $p\in(1,2)$, since $\left\|\cdot\right\|_{p-var}$ is a norm, combined with (3.52),

$$\begin{aligned}
&\left\|\sum_{l_n\leq i<j\leq l_{n+1}-1}\frac{1}{i^{\frac{1}{2}}j^{\frac{1}{2}}2^{i+j}}\left(p_{i,j}\left(s,t\right)+q_{i,j}\left(s,t\right)\right)\right\|_{p-var}\\
\leq\ & \frac{32}{3}\sum_{j=l_n+1}^{l_{n+1}-1}\frac{1}{j^{\frac{1}{2}}2^{2\left(1-\frac{1}{p}\right)j}}\left\|\sum_{i=l_n}^{j-1}\frac{1}{i^{\frac{1}{2}}2^{\frac{2}{p}i}}\left|\sin\left(2\pi 2^{2i}\frac{t-s}{2}\right)\right|\right\|_{p-var}\\
\leq\ & \frac{32\widetilde{C_{\frac{1}{2},p,1}}}{3l_n^{\frac{1}{2}}}\sum_{j=l_n+1}^{l_{n+1}-1}\frac{1}{j^{\frac{1}{2}}2^{2\left(1-\frac{1}{p}\right)j}} \leq \frac{32\widetilde{C_{\frac{1}{2},p,1}}}{3\left(2^{2\left(1-\frac{1}{p}\right)}-1\right)}\frac{1}{l_n2^{2\left(1-\frac{1}{p}\right)l_n}}\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Thus, for any $p>1$ (since p-variation is non-increasing, so if converge in p-variation, $p\in(1,2)$, then converge in p-variation, $p>1$)

$$\lim_{n\to\infty}\left\|A\left(g_n\right)\left(s,t\right)-\left(\pi\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k}\right)\left(t-s\right)\right\|_{p-var}=0.$$

■

Example 3.39 *Suppose $\{l_n\}$ is a sequence of increasing integers, satisfying that for any $n\geq 1$, $\sum_{k=l_n}^{l_{n+1}-1}k^{-1}\geq n$. Define*

$$f_n\left(t\right)=\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k^{\frac{1}{2}}2^k}\exp\left(2\pi i 2^{2k}t\right),\ t\in\left[0,1\right]. \tag{3.53}$$

Then $\lim_{n\to\infty}\left\|f_n\right\|_{2-var,[0,1]}=0$, but for any $0\leq s<t\leq 1$, $\lim_{n\to\infty}A\left(f_n\right)\left(s,t\right)=+\infty$.

Proof. Follows from Lemma 3.38:

$$\lim_{n\to\infty}\left\|A\left(f_n\right)\left(s,t\right)-\left(\pi\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k}\right)\left(t-s\right)\right\|_{p-var}=0, \text{ for any } p>1.$$

■

As a clear consequence of this example, when the space of smooth paths is equipped with 2-variation, the area operator is not continuous, nor bounded.

Example 3.40 *Suppose $\{l_n\}$ is a sequence of increasing integers, satisfying that for any $n\geq 1$, $\sum_{k=l_n}^{l_{n+1}-1}k^{-1}\geq 1$. Define*

$$g_n\left(t\right)=\left(\pi\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k}\right)^{-\frac{1}{2}}\sum_{k=l_n}^{l_{n+1}-1}\frac{1}{k^{\frac{1}{2}}2^k}\exp\left(2\pi i 2^{2k}t\right),\ t\in\left[0,1\right].$$

Then $\lim_{n\to\infty}\|g_n\|_{2-var,[0,1]}=0$*, and for any* $p>1$*,*

$$\lim_{n\to\infty}\|A(g_n)(s,t)-(t-s)\|_{p-var,[0,1]}=0.$$

Proof. Follows from Lemma 3.38. ■

The convergence of $A(g_n)$ to $t-s$ can not hold in 1-variation, because h_n is a sequence of smooth paths, so the limit of $A(g_n)$ in 1-variation is of vanishing 1-variation, while $t-s$ is not. Actually, g_n converges to zero in 2-variation, so if $A(g_n)$ converges in 1-variation then should converge to 0 (closable when area equipped with 1-variation).

Example 3.40 demonstrates that when the space of smooth paths is equipped with 2-variation and their area with p-variation, $p>1$, the area operator is not closable.

Next, we extend Young integral [38] to the case $p^{-1}+q^{-1}=1$ by assigning a finer scale continuity (e.g. logarithmic). Before that, we prove a lemma. Suppose $\mathcal{V}_1$ and $\mathcal{V}_2$ are two Banach spaces. For simplicity, we denote $\|\cdot\|_{\mathcal{V}_1}$, $\|\cdot\|_{\mathcal{V}_2}$ and $\|\cdot\|_{\mathcal{V}_1\otimes\mathcal{V}_2}$ as $\|\cdot\|$. Recall definition of $\omega_p(\gamma,\delta)$ at (3.12).

Lemma 3.41 *Suppose* $\gamma_1\in C^{p-var}([0,1],\mathcal{V}_1)$*,* $\gamma_2\in C^{q-var}([0,1],\mathcal{V}_2)$*,* $p^{-1}+q^{-1}=1$*.* $D_1=\{t_k\}_k$ *and* $D_2=\{s_j\}_j$ *are two finite partitions of* $[0,1]$*, and* D_2 *is a refinement of* D_1*, i.e. for any* k*, there exist integers* $n_k<n_{k+1}$*, s.t.* $t_k=s_{n_k}<s_{n_k+1}<\cdots<s_{n_{k+1}}=t_{k+1}$*. If denote* $I^D:=I(\gamma_1^D,\gamma_2^D)$ *(see definition at* (3.29)*) and suppose* $|D_1|\le\delta$*, then we have*

$$\begin{aligned}\left\|I^{D_1}-I^{D_2}\right\|_{1-var,[0,1]} &\le \sum_k\left\|I^{D_1}\right\|_{1-var,[t_k,t_{k+1}]}+\sum_k\left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]}\\ &\quad+2\omega_p(\gamma_1,\delta)\|\gamma_2\|_{q-var,[0,1]}+2\omega_q(\gamma_2,\delta)\|\gamma_1\|_{p-var,[0,1]}.\end{aligned}$$

Proof. Denote $\triangle\gamma_i:=\gamma_i^{D_1}-\gamma_i^{D_2}$, $i=1,2$, and denote $\triangle I:=I^{D_1}-I^{D_2}$. For any $(s,t)\in\triangle_{[0,T]}$,

$$\begin{aligned}\triangle I(s,t) &= \int_s^t\left(\gamma_1^{D_1}(u)-\gamma_1^{D_1}(s)\right)\otimes d\gamma_2^{D_1}(u)-\int_s^t\left(\gamma_1^{D_2}(u)-\gamma_1^{D_2}(s)\right)\otimes d\gamma_2^{D_2}(u)\\ &= \int_s^t(\triangle\gamma_1(u)-\triangle\gamma_1(s))\otimes d\gamma_2^{D_1}(u)+\int_s^t\left(\gamma_1^{D_2}(u)-\gamma_1^{D_2}(s)\right)\otimes d\triangle\gamma_2(u)\\ &=: I\left(\triangle\gamma_1,\gamma_2^{D_1}\right)(s,t)+I\left(\gamma_1^{D_2},\triangle\gamma_2\right)(s,t).\end{aligned}$$

Suppose $t_{k_1-1}<s\le t_{k_1}\le t_{k_2}\le t<t_{k_2+1}$, then

$$\begin{aligned}I\left(\triangle\gamma_1,\gamma_2^{D_1}\right)(s,t) &= I\left(\triangle\gamma_1,\gamma_2^{D_1}\right)(s,t_{k_1})+I\left(\triangle\gamma_1,\gamma_2^{D_1}\right)(t_{k_1},t_{k_2})+I\left(\triangle\gamma_1,\gamma_2^{D_1}\right)(t_{k_2},t)\\ &\quad+(\triangle\gamma_1(t_{k_1})-\triangle\gamma_1(s))\otimes\left(\gamma_2^{D_1}(t)-\gamma_2^{D_1}(t_{k_1})\right)\\ &\quad+(\triangle\gamma_1(t_{k_2})-\triangle\gamma_1(t_{k_1}))\otimes\left(\gamma_2^{D_1}(t)-\gamma_2^{D_1}(t_{k_2})\right).\end{aligned}$$

where the last term vanishes, because $\triangle\gamma_1(t_{k_1})=\triangle\gamma_1(t_{k_2})$. Similar result holds for $I(\gamma_2,\triangle\gamma)(s,t)$:

$$\begin{aligned}I\left(\gamma_1^{D_2},\triangle\gamma_2\right)(s,t) &= I\left(\gamma_1^{D_2},\triangle\gamma_2\right)(s,t_{k_1})+I\left(\gamma_1^{D_2},\triangle\gamma_2\right)(t_{k_1},t_{k_2})+I\left(\gamma_1^{D_2},\triangle\gamma_2\right)(t_{k_2},t)\\ &\quad+\left(\gamma_1^{D_2}(t_{k_2})-\gamma_1^{D_2}(s)\right)\otimes(\triangle\gamma_2(t)-\triangle\gamma_2(t_{k_2})).\end{aligned}$$

Thus (since $\triangle I = I(\triangle\gamma, \gamma_1) + I(\gamma_2, \triangle\gamma)$, $\|v_1 \otimes v_2\|_{\mathcal{V}_1\otimes\mathcal{V}_2} \leq \|v_1\|_{\mathcal{V}_1}\|v_2\|_{\mathcal{V}_2}$)

$$\begin{aligned} \|\triangle I(s,t)\| &\leq \|\triangle I(s,t_{k_1})\| + \|\triangle I(t_{k_1},t_{k_2})\| + \|\triangle I(t_{k_2},t)\| \\ &\quad + \|\triangle\gamma_1(t_{k_1}) - \triangle\gamma_1(s)\| \left\|\gamma_2^{D_1}(t) - \gamma_2^{D_1}(t_{k_1})\right\| \\ &\quad + \left\|\gamma_1^{D_2}(t_{k_2}) - \gamma_1^{D_2}(s)\right\| \|\triangle\gamma_2(t) - \triangle\gamma_2(t_{k_2})\|. \end{aligned} \tag{3.54}$$

For $\triangle I(t_{k_1}, t_{k_2})$, by using multiplicativity and $\triangle\gamma_i(t_k) = 0, \forall k$, $i = 1,2$, we get

$$\triangle I(t_{k_1}, t_{k_2}) = \sum_{j=k_1}^{k_2-1} \triangle I(t_j, t_{j+1}). \tag{3.55}$$

Thus, combining (3.54) with (3.55), we decompose $[s,t]$ into the union of three kinds of subintervals: $[s,t_{k_1}]$, $[t_j,t_{j+1}]$ and $[t_{k_2},t]$, and each of them is a subinterval of some $[t_k,t_{k+1}]$. Thus, for any finite partition, applying our estimates to each subinterval, summing them together, and taking supremum over all finite partitions. By using Hölder inequality, we get

$$\begin{aligned} \|\triangle I\|_{1-var,[0,1]} &\leq \sum_k \|\triangle I\|_{1-var,[t_k,t_{k+1}]} \\ &\quad + \left(\sum_k \|\triangle\gamma_1\|_{p-var,[t_k,t_{k+1}]}^p\right)^{\frac{1}{p}} \left\|\gamma_2^{D_1}\right\|_{q-var,[0,1]} \\ &\quad + \left(\sum_k \|\triangle\gamma_2\|_{q-var,[t_k,t_{k+1}]}^q\right)^{\frac{1}{q}} \left\|\gamma_1^{D_2}\right\|_{p-var,[0,1]} \end{aligned} \tag{3.56}$$

On the other hand, when $i = 1,2$,

$$\sup_D \left\|\gamma_i^D\right\|_{p-var,[0,1]} \leq \|\gamma_i\|_{p-var,[0,1]}, \tag{3.57}$$

and since $\triangle\gamma_i := \gamma_i^{D_1} - \gamma_i^{D_2}$,

$$\begin{aligned} \|\triangle\gamma_i\|_{p-var,[t_k,t_{k+1}]} &\leq \left\|\gamma_i^{D_1}\right\|_{p-var,[t_k,t_{k+1}]} + \left\|\gamma_i^{D_2}\right\|_{p-var,[t_k,t_{k+1}]} \\ &\leq 2\|\gamma_i\|_{p-var,[t_k,t_{k+1}]}. \end{aligned} \tag{3.58}$$

Therefore, combine (3.56), (3.57) with (3.58),

$$\begin{aligned} \|\triangle I\|_{1-var,[0,1]} &\leq \sum_k \|\triangle I\|_{1-var,[t_k,t_{k+1}]} + 2\left(\sum_k \|\gamma_1\|_{p-var,[t_k,t_{k+1}]}^p\right)^{\frac{1}{p}} \|\gamma_2\|_{q-var,[0,1]} \\ &\quad +2\left(\sum_k \|\gamma_2\|_{q-var,[t_k,t_{k+1}]}^q\right)^{\frac{1}{q}} \|\gamma_1\|_{p-var,[0,1]}. \end{aligned}$$

Since $\|\triangle I\|_{1-var,[t_k,t_{k+1}]} \leq \left\|I^{D_1}\right\|_{1-var,[t_k,t_{k+1}]} + \left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]}$ and $|D_1| \leq \delta$, use $\omega_p(\gamma,\delta)$ (defined at (3.12)). ■

The following lemma will be used in the proof of Theorem 3.30.

Lemma 3.42 *Suppose* $\gamma_i : [0,T] \to \mathcal{V}_i$*,* $i = 1,2$*, are two continuous piecewise linear paths obtained by interpolating on the same finite partition of* $[0,T]$*. Then for any* $p > 1$*,* $q > 1$*,* $p^{-1} + q^{-1} = 1$*, there exists a finite partition* $D = \{t_k\}$ *of* $[0,T]$ *s.t.* $|D| \leq 2^{-1}T$ *and*

$$\|I(\gamma_1,\gamma_2)\|_{1-var,[0,T]} \leq \sum_{k,t_k\in D} \|I(\gamma_1,\gamma_2)\|_{1-var,[t_k,t_{k+1}]} + 2\|\gamma_1\|_{p-var,[0,T]}\|\gamma_2\|_{q-var,[0,T]}.$$

If γ_i *are linear on* $[0,T]$ *then*

$$\|I(\gamma_1,\gamma_2)\|_{1-var,[0,T]} \leq \|\gamma_1\|_{p-var,[0,T]}\|\gamma_2\|_{q-var,[0,T]}.$$

Proof. Denote $I := I(\gamma_1,\gamma_2)$ and denote $D' = \{t_j\}_{j=0}^n$ as the finite partition on which γ_i, $i = 1,2$, are interpolated.

When $n = 1$, $\{t_j\}_{j=0}^n = \{0,T\}$, then γ_i are linear on $[0,T]$, $i = 1,2$. After computation, one gets

$$\begin{aligned}\|I\|_{1-var,[0,T]} &= \|(\gamma_1(T)-\gamma_1(0))\otimes(\gamma_2(T)-\gamma_2(0))\| \\ &\leq \|\gamma_1(T)-\gamma_1(0)\|\|\gamma_2(T)-\gamma_2(0)\| \\ &\leq \|\gamma_1\|_{p-var,[0,T]}\|\gamma_2\|_{q-var,[0,T]}.\end{aligned}\tag{3.59}$$

When $n \geq 2$, denote $t_{j_1} := \min_j \{t_j | t_j \leq 2^{-1}T\}$.

If $t_{j_1} = 0$, then $j_1 = 0$, and $t_{j_1+1} = t_1 > 2^{-1}T$. Thus

$$\begin{aligned}\|I\|_{1-var,[0,T]} \leq\ & \|I\|_{1-var,[0,t_1]} + \|I\|_{1-var,[t_1,T]} \\ & + \|\gamma_1\|_{p-var,[0,t_1]}\|\gamma_2\|_{q-var,[t_1,T]}.\end{aligned}$$

Use (3.59) for $\|I\|_{1-var,[0,t_1]}$,

$$\begin{aligned}\|I\|_{1-var,[0,T]} &\leq \|I\|_{1-var,[t_1,T]} + \|\gamma_1\|_{p-var,[0,t_1]}\|\gamma_2\|_{q-var,[0,t_1]} \\ &\quad + \|\gamma_1\|_{p-var,[0,t_1]}\|\gamma_2\|_{q-var,[t_1,T]}. \\ &\leq \|I\|_{1-var,[t_1,T]} + 2^{\frac{1}{p}}\|\gamma_1\|_{p-var,[0,T]}\|\gamma_2\|_{q-var,[0,T]}.\end{aligned}$$

Since $T - t_1 < 2^{-1}T$, lemma holds.

If $t_{j_1} > 0$, then

$$\begin{aligned}\|I\|_{1-var,[0,T]} \leq\ & \|I\|_{1-var,\left[0,t_{j_1}\right]} + \|I\|_{1-var,\left[t_{j_1},T\right]} \\ & + \|\gamma_1\|_{p-var,\left[0,t_{j_1}\right]}\|\gamma_2\|_{q-var,\left[t_{j_1},T\right]}.\end{aligned}\tag{3.60}$$

Then if $t_{j_1+1} = T$, γ_i are linear on $[t_{j_1},T]$, $i = 1,2$, so similar as above,

$$\|I\|_{1-var,[0,T]} \leq \|I\|_{1-var,\left[0,t_{j_1}\right]} + 2^{\frac{1}{q}}\|\gamma_1\|_{p-var,[0,T]}\|\gamma_2\|_{q-var,[0,T]}.$$

Since $t_{j_1} \leq 2^{-1}T$, lemma holds.

If $t_{j_1+1} < T$, then $0 < t_{j_1} \leq 2^{-1}T < t_{j_1+1} < T$, continue with (3.60),

$$\begin{aligned} \|I\|_{1-var,\left[t_{j_1},T\right]} &\leq \|I\|_{1-var,\left[t_{j_1},t_{j_1+1}\right]} + \|I\|_{1-var,\left[t_{j_1+1},T\right]} \\ &\quad + \|\gamma_1\|_{p-var,\left[t_{j_1},t_{j_1+1}\right]} \|\gamma_2\|_{q-var,\left[t_{j_1+1},T\right]}. \end{aligned} \tag{3.61}$$

While γ_i are linear on $[t_{j_1}, t_{j_1+1}]$, so

$$\|I\|_{1-var,\left[t_{j_1},t_{j_1+1}\right]} \leq \|\gamma_1\|_{p-var,\left[t_{j_1},t_{j_1+1}\right]} \|\gamma_2\|_{q-var,\left[t_{j_1},t_{j_1+1}\right]} . \tag{3.62}$$

Thus, combining (3.60), (3.61) with (3.62), and using Hölder inequality, we get

$$\begin{aligned} \|I\|_{1-var,[0,T]} &\leq \|I\|_{1-var,\left[0,t_{j_1}\right]} + \|I\|_{1-var,\left[t_{j_1+1},T\right]} \\ &\quad +2 \|\gamma_1\|_{p-var,[0,T]} \|\gamma_2\|_{q-var,[0,T]}. \end{aligned}$$

Since $t_{j_1} \leq 2^{-1}T$ and $t_{j_1+1} > 2^{-1}T$, lemma holds. ■

Theorem 3.30 *Let $\gamma_i : [0,1] \to \mathcal{V}_i$, $i = 1,2$, be two continuous paths. If there exist $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, and two non-decreasing functions $m_i : [0,1] \to \overline{\mathbb{R}^+}$, $i = 1,2$, satisfying*

$$\lim_{t\to 0} m_i(t) = 0,\ m_i(1) \leq 1,\ i = 1,2, \text{ and } \int_0^1 \frac{m_1(t)\, m_2(t)}{t} dt < \infty.$$

such that

$$C_1 := \sup_{0\leq s<t\leq 1} \frac{\|\gamma_1(t) - \gamma_1(s)\|}{|t-s|^{\frac{1}{p}}\, m_1(t-s)} < \infty,\ C_2 := \sup_{0\leq s<t\leq 1} \frac{\|\gamma_2(t) - \gamma_2(s)\|}{|t-s|^{\frac{1}{q}}\, m_2(t-s)} < \infty.$$

Then the Riemann-Stieltjes integral $\int_0^t \gamma_1(t) \otimes d\gamma_2(t)$, $t \in [0,1]$, exists, and

$$\left\| \int_0^{\cdot} \gamma_1(t) \otimes d\gamma_2(t) \right\|_{q-var} \leq 8C_1C_2 \left(2 + \int_0^1 \frac{m_1(t)\, m_2(t)}{t} dt \right).$$

Proof. Recall the definition of $I\left(\gamma_1^D, \gamma_2^D\right)$ at (3.29):

$$I\left(\gamma_1^D, \gamma_2^D\right)(s,t) = \int_s^t \left(\gamma_1^D(u) - \gamma_1^D(s)\right) \otimes d\gamma_2^D(u),\ \ 0 \leq s < t \leq 1.$$

Denote $I^{D_i} := I\left(\gamma_1^{D_i}, \gamma_2^{D_i}\right)$, $i = 1,2$. Firstly, we prove that I^D converges in 1-variation as $|D| \to 0$.

Since m_i are non-decreasing, (ω_p defined at (3.12))

$$\begin{gathered} \omega_p(\gamma_1, \delta) \leq C_1 m_1(\delta),\ \omega_q(\gamma_2, \delta) \leq C_2 m_2(\delta); \\ \text{since } |m_i| \leq 1 \text{ so } \|\gamma_1\|_{p-var,[0,T]} \leq C_1,\ \|\gamma_2\|_{q-var,[0,T]} \leq C_2. \end{gathered} \tag{3.63}$$

Based on Lemma 3.41, for any finite partition $D_1 \subset D_2 \subset [0,1]$, if $|D_1| \leq \delta$ then

$$\begin{aligned} & \left\|I^{D_1} - I^{D_2}\right\|_{1-var} \\ \leq\ & \sum_k \left\|I^{D_1}\right\|_{1-var,[t_k,t_{k+1}]} + \sum_k \left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]} \\ & +2\omega_p(\gamma_1,\delta)\|\gamma_2\|_{q-var,[0,1]} + 2\omega_q(\gamma_2,\delta)\|\gamma_1\|_{p-var,[0,1]}. \end{aligned} \tag{3.64}$$

Combined with (3.63), we get

$$\begin{aligned} & 2\omega_p(\gamma_1,\delta)\|\gamma_2\|_{q-var,[0,1]} + 2\omega_q(\gamma_2,\delta)\|\gamma_1\|_{p-var,[0,1]} \\ \leq\ & 2C_1C_2(m_1(\delta)+m_2(\delta)) \end{aligned} \tag{3.65}$$

For $\sum_k \left\|I^{D_1}\right\|_{1-var,[t_k,t_{k+1}]}$ in (3.64), since D_1 is linear on $[t_k, t_{k+1}]$,

$$\begin{aligned} \left\|I^{D_1}\right\|_{1-var,[t_k,t_{k+1}]} & \leq \|\gamma_1(t_{k+1}) - \gamma_1(t_k)\| \|\gamma_2(t_{k+1}) - \gamma_2(t_k)\| \\ & \leq \|\gamma_1\|_{p-var,[t_k,t_{k+1}]} \|\gamma_2\|_{q-var,[t_k,t_{k+1}]}. \end{aligned}$$

Therefore, using Hölder inequality,

$$\begin{aligned} & \sum_k \left\|I^{D_1}\right\|_{1-var,[t_k,t_{k+1}]} \\ \leq\ & \left(\sum_k \|\gamma_1\|^p_{p-var,[t_k,t_{k+1}]}\right)^{\frac{1}{p}} \left(\sum_k \|\gamma_2\|^q_{q-var,[t_k,t_{k+1}]}\right)^{\frac{1}{q}} \\ \leq\ & m_p(\gamma_1,\delta)\, m_q(\gamma_2,\delta) \leq C_1C_2 m_1(\delta)\, m_2(\delta). \end{aligned} \tag{3.66}$$

For $\sum_k \left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]}$, applying Lemma 3.42 to $\left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]}$, $\forall k$, then there exists a partition $D^{(1)} = \left\{u_j^1\right\}_j$, $\left|D^{(1)}\right| \leq 2^{-1}\delta$, s.t.

$$\begin{aligned} & \sum_k \left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]} \\ \leq\ & \sum_{j,u_j^1\in D^{(1)}} \left\|I^{D_2}\right\|_{1-var,\left[u_j^1,u_{j+1}^1\right]} + 2\sum_k \|\gamma_1\|_{p-var,[t_k,t_{k+1}]} \|\gamma_2\|_{q-var,[t_k,t_{k+1}]} \\ \leq\ & \sum_{j,u_j^1\in D^{(1)}} \left\|I^{D_2}\right\|_{1-var,\left[u_j^1,u_{j+1}^1\right]} + 2\left(\sum_k \|\gamma_1\|^p_{p-var,[t_k,t_{k+1}]}\right)^{\frac{1}{p}} \left(\sum_k \|\gamma_2\|^q_{q-var,[t_k,t_{k+1}]}\right)^{\frac{1}{q}} \\ \leq\ & \sum_{j,u_j^1\in D^{(1)}} \left\|I^{D_2}\right\|_{1-var,\left[u_j^1,u_{j+1}^1\right]} + 2C_1C_2 m_1(\delta)\, m_2(\delta). \end{aligned}$$

Continue the process: applying Lemma 3.42 to $\left\|I^{D_2}\right\|_{1-var,\left[u_j^1,u_{j+1}^1\right]}$, $\forall j$, then there exists a finite partition $D^{(2)} = \left\{u_j^2\right\}$, $\left|D^{(2)}\right| \leq 2^{-2}\delta$, s.t.

$$\sum_{j,u_j^1\in D^{(1)}} \left\|I^{D_2}\right\|_{1-var,\left[u_j^1,u_{j+1}^1\right]} \leq \sum_{j,u_j^2\in D^{(2)}} \left\|I^{D_2}\right\|_{1-var,\left[u_j^2,u_{j+1}^2\right]} + 2C_1C_2 m_1\left(\frac{\delta}{2}\right) m_2\left(\frac{\delta}{2}\right).$$

So on and so forth, and we get (for fixed D_2, I^{D_2} is of vanishing 1-variation)

$$\sum_k \left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]} \leq 2C_1C_2\sum_{n=0}^{\infty} m_1\left(\frac{\delta}{2^n}\right) m_2\left(\frac{\delta}{2^n}\right). \tag{3.67}$$

Since m_1 and m_2 are non-decreasing, when $n \geq 1$,

$$m_1\left(\frac{\delta}{2^n}\right) m_2\left(\frac{\delta}{2^n}\right) \leq \left(\frac{\delta}{2^n}\right)^{-1}\int_{\frac{\delta}{2^n}}^{\frac{\delta}{2^{n-1}}} m_1(t)\, m_2(t)\, dt \leq 2\int_{\frac{\delta}{2^n}}^{\frac{\delta}{2^{n-1}}} \frac{m_1(t)\, m_2(t)}{t} dt.$$

$$\text{Thus } \sum_{n=0}^{\infty} m_1\left(\frac{\delta}{2^n}\right) m_2\left(\frac{\delta}{2^n}\right) \leq m_1(\delta)\, m_2(\delta) + 2\int_0^{\delta}\frac{m_1(t)\, m_2(t)}{t} dt.$$

Combined with (3.67),

$$\sum_k \left\|I^{D_2}\right\|_{1-var,[t_k,t_{k+1}]} \leq 2C_1C_2\left(m_1(\delta)\, m_2(\delta) + 2\int_0^{\delta}\frac{m_1(t)\, m_2(t)}{t} dt\right). \tag{3.68}$$

Therefore, combining (3.64), (3.65), (3.66) with (3.68), we get

$$\begin{aligned} & \left\|I^{D_1} - I^{D_2}\right\|_{1-var} \\ \leq\ & C_1C_2\left(2\left(m_1(\delta) + m_2(\delta)\right) + 3m_1(\delta)\, m_2(\delta) + 4\int_0^{\delta}\frac{m_1(t)\, m_2(t)}{t} dt\right). \end{aligned}$$

In the above we assume $D_2 \subset D_1$. For two general finite partitions D and D', $|D| \vee |D'| \leq \delta$, denote $D'' := D \cup D'$, apply our estimates to D, D'' and D', D'', we get

$$\begin{aligned} & \left\|I^{D} - I^{D'}\right\|_{1-var} \\ \leq\ & 2C_1C_2\left(2\left(m_1(\delta) + m_2(\delta)\right) + 3m_1(\delta)\, m_2(\delta) + 4\int_0^{\delta}\frac{m_1(t)\, m_2(t)}{t} dt\right). \end{aligned}$$

Because we assumed that $\lim_{t\to 0} m_i(t) = 0$ and $\int_0^1 \frac{m_1(t)m_2(t)}{t} dt < \infty$, so the Riemann-Stieltjes integral $I(\gamma_1, \gamma_2)$ exists, $I\left(\gamma_1^D, \gamma_2^D\right)$ converge in 1-variation to $I(\gamma_1, \gamma_2)$ as $|D| \to 0$, and ($|m_i| \leq 1$, $i = 1, 2$)

$$\sup_D \left\|I(\gamma_1, \gamma_2) - I\left(\gamma_1^D, \gamma_2^D\right)\right\|_{1-var} \leq 2C_1C_2\left(7 + 4\int_0^1\frac{m_1(t)\, m_2(t)}{t} dt\right).$$

Moreover, if denote finite partition $D_0 := \{0, 1\}$ then

$$\left\|I^{D_0}\right\|_{1-var} \leq \left\|\left(\gamma_1(1) - \gamma_1(0)\right) \otimes \left(\gamma_2(1) - \gamma_2(0)\right)\right\| \leq C_1C_2.$$

$$\text{Thus, } \left\|I(\gamma_1, \gamma_2)\right\|_{1-var} \leq C_1C_2\left(15 + 8\int_0^1\frac{m_1(t)\, m_2(t)}{t} dt\right) \tag{3.69}$$

Then we work out $\left\|\int_0^{\cdot} \gamma_1(u) \otimes d\gamma_2(u)\right\|_{q-var}$ from $\left\|I(\gamma_1, \gamma_2)\right\|_{1-var}$. Since

$$\begin{aligned} I(\gamma_1, \gamma_2)(s, t) \ :\ &= \int_s^t \left(\gamma_1(u) - \gamma_1(s)\right) \otimes d\gamma_2(u) \\ &= \int_s^t \gamma_1(u) \otimes d\gamma_2(u) - \gamma_1(s) \otimes \left(\gamma_2(t) - \gamma_2(s)\right) \end{aligned}$$

Therefore, if define function $\beta : \triangle_{[0,1]} \to \mathcal{V}_1 \otimes \mathcal{V}_2$ by setting

$$\beta(s,t) := \gamma_1(s) \otimes (\gamma_2(t) - \gamma_2(s)), \forall (s,t) \in \triangle_{[0,1]}.$$

Then

$$\|\beta\|_{q-var} \le \|\gamma_1\|_{\infty-var} \|\gamma_2\|_{q-var} \le C_1 C_2.$$

Thus, combined with (3.69), we get

$$\begin{aligned} \left\| \int_0^{\cdot} \gamma_1(u) \otimes d\gamma_2(u) \right\|_{q-var} &\le \|I(\gamma_1, \gamma_2)\|_{1-var} + \|\beta\|_{q-var} \\ &\le 8C_1C_2 \left(2 + \int_0^1 \frac{m_1(t) m_2(t)}{t} dt \right). \end{aligned}$$

■

When $m_1(t) = t^a$, $m_2(t) = t^b$, $a > 0$, $b > 0$, we get Young integral.

The condition $\int_0^1 \frac{m_1(t)m_2(t)}{t} dt < \infty$ is necessary in the sense of following example.

Example 3.33 *Suppose* $m_i : [0,1] \to \overline{\mathbb{R}^+}$ *are two non-decreasing functions, satisfying* $\lim_{t\to 0} m_i(t) = 0$, $|m_i| \le 1$, $i = 1,2$, *and* $\int_0^1 \frac{m_1(t)m_2(t)}{t} dt = \infty$. *Then for any* $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$, *there exist two continuous real-valued paths* $\gamma_i : [0,1] \to \mathbb{R}$, $i = 1,2$, *s.t.*

$$C_1 := \sup_{0\le s<t\le 1} \frac{|\gamma_1(t) - \gamma_1(s)|}{|t-s|^{\frac{1}{p}} m_1(t-s)} < \infty, C_2 := \sup_{0\le s<t\le 1} \frac{|\gamma_2(t) - \gamma_2(s)|}{|t-s|^{\frac{1}{q}} m_2(t-s)} < \infty, \tag{3.70}$$

but the Riemann-Stieltjes integral $\int_0^1 \gamma_1(t) d\gamma_2(t)$ *does not exist.*

Proof. Let $\epsilon_k = 1$ or -1, $\forall k$, and define

$$\begin{aligned} \gamma_1(t) &= \sum_{k=1}^{\infty} \frac{m_1(2^{-2k})}{2^{\frac{2k}{p}}} \cos(2\pi 2^{2k} t), t \in [0,1], \\ \gamma_2(t) &= \sum_{k=1}^{\infty} \epsilon_k \frac{m_2(2^{-2k})}{2^{\frac{2k}{q}}} \sin(2\pi 2^{2k} t), t \in [0,1]. \end{aligned}$$

Then γ_i satisfy (3.70). Take γ_1 as an example. For $0 \le s < t \le 1$, let $n = \left[\log_4 \frac{4}{|t-s|}\right]$, we have

$$\begin{aligned} &|\gamma_1(t) - \gamma_1(s)| \\ \le\ & 2\pi \left(\sum_{k=1}^{n} m_1(2^{-2k}) 2^{2(1-\frac{1}{p})k} \right) |t-s| + 2 \sum_{k=n+1}^{\infty} \frac{m_1(2^{-2k})}{2^{\frac{2k}{p}}}. \end{aligned} \tag{3.71}$$

Since $\lim_{t\to 0} \frac{m_1(4t)}{m_1(t)} = 1$ ($\int_0^1 \frac{m_i(t)}{t} dt \ge \int_0^1 \frac{m_1(t)m_2(t)}{t} dt = \infty$ so $\lim_{t\to 0} \frac{m_i(t)}{\left(\ln \frac{1}{t}\right)^{-2}} = \infty$), using L'Hospital's rule, there exists constant C_1, s.t. for any $n \ge 1$,

$$\sum_{k=1}^{n} m_1(2^{-2k}) 2^{2(1-\frac{1}{p})k} \le C_1 m_1(2^{-2n}) 2^{2(1-\frac{1}{p})n},$$

Continue with (3.71), since m_1 is non-decreasing ($n = \left[\log_4 \frac{4}{t-s}\right]$, $2^{-2n} < |t-s| \leq 2^{-2(n-1)}$)

$$\begin{aligned}
|\gamma_1(t) - \gamma_1(s)| &\leq 2\pi C_1 m_1\left(2^{-2n}\right) 2^{2\left(1-\frac{1}{p}\right)n} |t-s| + \frac{2}{2^{\frac{2}{p}} - 1} m_1\left(2^{-2n}\right) 2^{-\frac{2}{p}n} \\
&\leq \left(8\pi C_1 + \frac{2}{2^{\frac{2}{p}} - 1}\right) |t-s|^{\frac{1}{p}} m_1(t-s).
\end{aligned}$$

Then we prove the Riemann-Stieltjes integral $\int_0^1 \gamma_1(t)\, d\gamma_2(t)$ does not exist. First, the limit of Riemann sum as $|D| \to 0$ does not depend on the selection of representative points, because $\gamma_1 \in C^{0,p-var}$, $\gamma_2 \in C^{0,q-var}$. Actually, since γ_i satisfy (3.70) and m_i are non-decreasing, we have $\omega_p(\gamma_1, \delta) \leq C_1 m_1(\delta)$ and $\omega_q(\gamma_2, \delta) \leq C_2 m_2(\delta)$. On the other hand, since

$$\begin{aligned}
&\sum_{k,} \frac{1}{2}\left(\gamma_1(t_{k+1}) + \gamma_1(t_k)\right)\left(\gamma_2(t_{k+1}) - \gamma_2(t_k)\right) \\
= &\ \frac{1}{2}\sum_k \left(\gamma_1(t_k)\gamma_2(t_{k+1}) - \gamma_2(t_k)\gamma_1(t_{k+1})\right) + \frac{1}{2}\gamma_1(1)\gamma_2(1) - \frac{1}{2}\gamma_1(0)\gamma_2(0),
\end{aligned}$$

so the existence of Riemann-Stieltjes integral $\int_0^1 \gamma_1(t)\, d\gamma_2(t)$ is equivalent to the existence of

$$\lim_{|D|\to 0} \sum_{k, t_k \in D} \left(\gamma_1(t_k)\gamma_2(t_{k+1}) - \gamma_2(t_k)\gamma_1(t_{k+1})\right).$$

Similar as the estimates in Example 3.36, if denote finite partition $D_{2N} := \left\{t_l^N\right\}_{l=0}^{2^{2N}}$ with $t_l^N := l2^{-2N}$, we get

$$\begin{aligned}
\left\langle \int \gamma_1 d\gamma_2, D_{2N} \right\rangle &:= \sum_{l=0}^{2^{2N}-1} \left(\gamma_1\left(t_l^N\right)\gamma_2\left(t_{l+1}^N\right) - \gamma_2\left(t_l^N\right)\gamma_1\left(t_{l+1}^N\right)\right) \qquad (3.72) \\
&= \sum_{k=1}^{N-1} \epsilon_k \frac{m_1\left(2^{-2k}\right) m_2\left(2^{-2k}\right)}{2^{2k-2N}} \sin\left(2\pi 2^{2k-2N}\right).
\end{aligned}$$

While since m_i are non-decreasing, for any $k \geq 1$,

$$m_1\left(2^{-2k}\right) m_2\left(2^{-2k}\right) \geq \frac{1}{3}\int_{2^{-2(k+1)}}^{2^{-2k}} \frac{m_1(t)\, m_2(t)}{t} dt,$$

so based on our assumption $\int_0^1 \frac{m_1(t) m_2(t)}{t} dt = \infty$, we have

$$\sum_{k=1}^{\infty} m_1\left(2^{-2k}\right) m_2\left(2^{-2k}\right) = \infty.$$

Thus, since m_i are non-decreasing and $\lim_{t\to 0} m_i(t) = 0$, using exactly the same estimates as in Example 3.36, for any sequence of strictly increasing integers $\{l_n\}$ satisfying for some $c > \pi$

$$c^n \leq \sum_{k=l_n}^{l_{n+1}-1} m_1\left(2^{-2k}\right) m_2\left(2^{-2k}\right) \leq c^n + 1, \ \forall n \geq 1,$$

we let $\epsilon_k = (-1)^n$, when $l_n \leq k \leq l_{n+1} - 1$.

Then, for any $a \in [-\infty, \infty]$, there exists a sequence of finite partitions $\{D_n^a\}_n \subset \{D_{2N}\}_N$, $\lim_{n\to\infty} |D_n^a| = 0$, but $\lim_{n\to\infty} \left\langle \int \gamma_1 d\gamma_2, D_n^a \right\rangle = a$. ■

Next, we want to prove that a vanishing 2-variation path γ can be enhanced into a geometric 2-rough path, if and only if $A\left(\gamma^D\right)$ (the areas of piecewisely linear approximation) converge in 1-variation as $|D| \to 0$.

Lemma 3.43 *Suppose $\gamma \in C^{0,2-var}([0,T], \mathcal{V})$. $D_1 = \{t_k\}_k$ and $D_2 = \{s_j\}_j$ are two finite partitions of $[0,T]$, and D_2 is a refinement of D_1, i.e. for any k, there exist integers $n_k < n_{k+1}$, s.t. $t_k = s_{n_k} < s_{n_k+1} < \cdots < s_{n_{k+1}} = t_{k+1}$. Then if $|D_1| \le \delta$, we have*

$$\left\|A\left(\gamma^{D_1}\right) - A\left(\gamma^{D_2}\right)\right\|_{1-var} \le \sum_k \left\|A\left(\gamma^{D_2}\right)\right\|_{1-var,[t_k,t_{k+1}]} + 4\left\|\gamma\right\|_{2-var,[0,T]} \omega_2(\gamma, \delta).$$

Proof. Almost the same as that of Lemma 3.41 when $p = q = 2$, by using $\|[u,v]\| \le 2\|u\|\|v\|$. $\sum_k \left\|A^{D_1}\right\|_{1-var,[t_k,t_{k+1}]} = 0$ because γ^{D_1} is linear on $[t_k, t_{k+1}]$, $\forall k$. ■

Lemma 3.44 *Suppose (γ, α) is a weak geometric 2-rough path, and $D = \{t_k\}_{k=0}^n$ is a finite partition of $[0,T]$. Then*

$$A\left(\gamma^D\right)(0,T) = \frac{1}{2}\sum_{k=0}^{n-1}\left[\gamma(t_k), \gamma(t_{k+1})\right] - \frac{1}{2}\left[\gamma(0), \gamma(T)\right],$$

$$\alpha(0,T) = \sum_{k=0}^{n-1} \alpha(t_k, t_{k+1}) + A\left(\gamma^D\right)(0,T).$$

Proof. The first is obtained from directly computation, the second is got by using multiplicativity of (γ, α) (i.e.(3.22)). ■

Theorem 3.20 *Suppose $\gamma \in C^{0,2-var}([0,T], \mathcal{V})$. Then $\gamma \in \mathcal{G}_2(\mathcal{V})$ if and only if $A\left(\gamma^D\right)$ converges in 1-variation as $|D| \to 0$.*

Proof. $\Leftarrow$ is clear; we prove $\Rightarrow$. Suppose (γ, α) is a geometric 2-rough path. Then γ is of vanishing 2-variation, and α is of vanishing 1-variation. Thus, for any $\epsilon > 0$, there exists $\delta > 0$, s.t. for any finite partition D of $[0,T]$ satisfying $|D| \le \delta$, we have $\sum_{k,t_k\in D} \|\gamma\|_{2-var,[t_k,t_{k+1}]}^2 < \epsilon$ and $\sum_{k,t_k\in D} \|\alpha(t_k, t_{k+1})\| < \epsilon$.

Suppose $D_1 = \{t_k\}_k$ and $D_2 = \{s_j\}_j$ are two finite partitions of $[0,T]$ satisfying $|D_1| \le \delta$, $|D_2| \le \delta$, D_2 is a refinement of D_1. Based on Lemma 3.43,

$$\left\|A\left(\gamma^{D_1}\right) - A\left(\gamma^{D_2}\right)\right\|_{1-var} \le \sum_k \left\|A\left(\gamma^{D_2}\right)\right\|_{1-var,[t_k,t_{k+1}]} + 4\left\|\gamma\right\|_{2-var,[0,T]} \epsilon^{\frac{1}{2}}.$$

Then we estimate $\sum_k \left\|A\left(\gamma^{D_2}\right)\right\|_{1-var,[t_k,t_{k+1}]}$. Since γ^{D_2} is a piecewisely linear path on each $[t_k, t_{k+1}]$, we only consider finite partitions, whose points are all “ corner ” points. Suppose D_3 is a finite partition satisfying $D_1 \subset D_3 = \{u_i\} \subset D_2$. Suppose $u_i = s_{m_i} < s_{m_i+1} < \cdots < s_{m_{i+1}} = u_{i+1}$, then based on Lemma 3.44, for each i,

$$\left\|A\left(\gamma^{D_2}\right)(u_i, u_{i+1})\right\| \le \left\|\alpha(u_i, u_{i+1})\right\| + \sum_{j=m_i}^{m_{i+1}-1} \left\|\alpha(s_j, s_{j+1})\right\|$$

Sum over i, then

$$\sum_{i,u_i\in D_3}\left\|A\left(\gamma^{D_2}\right)(u_i,u_{i+1})\right\|\leq\sum_{i,u_i\in D_3}\|\alpha(u_i,u_{i+1})\|+\sum_i\sum_{j=m_i}^{m_{i+1}-1}\|\alpha(s_j,s_{j+1})\|.$$

Since $|D_2|\leq|D_3|\leq|D_1|\leq\delta$, based on the selection of δ, $\sum_{i,u_i\in D_3}\|\alpha(u_i,u_{i+1})\|<\epsilon$, $\sum_i\sum_{j=m_i}^{m_{i+1}-1}\|\alpha(s_j,s_{j+1})\|=\sum_{j,s_j\in D_2}\|\alpha(s_j,s_{j+1})\|<\epsilon$. Thus

$$\sum_{i,u_i\in D_3}\left\|A\left(\gamma^{D_2}\right)(u_i,u_{i+1})\right\|<2\epsilon.$$

Therefore, taking supremum over all possible D_3, we get

$$\sum_k\left\|A\left(\gamma^{D_2}\right)\right\|_{1-var,[t_k,t_{k+1}]}\leq 2\epsilon.$$

Thus

$$\left\|A\left(\gamma^{D_1}\right)-A\left(\gamma^{D_2}\right)\right\|_{1-var}\leq 2\epsilon+4\|\gamma\|_{2-var,[0,T]}\epsilon^{\frac{1}{2}}.$$

For any finite partition D and D', denote $D''=D\cup D'$, and use the above estimates for D, D'' and D', D''. ■

Therefore, if a vanishing 2-variation path γ can be enhanced into a geometric weak geometric 2-rough path, then $A\left(\gamma^D\right)$ converges in 1-variation as $|D|\to 0$, so converges pointwisely to the Riemann-Stieltjes integral $2^{-1}\int_s^t[\gamma(u)-\gamma(s),d\gamma(u)]$.

Lemma 3.45 *Suppose $\gamma:[0,T]\to\mathcal{V}$, is a continuous finitely piecewise linear path. Then for any $p>1$, $q>1$, $p^{-1}+q^{-1}=1$, there exists finite partition $D=\{t_k\}$ s.t. $|D|\leq 2^{-1}T$ and*

$$\|A(\gamma)\|_{1-var,[0,T]}\leq\sum_{k,t_k\in D}\|A(\gamma)\|_{1-var,[t_k,t_{k+1}]}+2\|\gamma\|_{2-var,[0,T]}^2.$$

Proof. Almost the same as that of Lemma 3.42 when $p=q=2$, by using $\|[u,v]\|\leq 2\|u\|\|v\|$. ■

Theorem 3.25 *Let $\gamma:[0,1]\to\mathcal{V}$ be a continuous paths. Then if there exists an non-decreasing function $m:[0,1]\to\overline{\mathbb{R}^+}$ satisfying*

$$\lim_{t\to 0}m(t)=0,\ m(1)\leq 1,\text{ and }\int_0^1\frac{m^2(t)}{t}dt<\infty,$$

such that

$$\sup_{0\leq s<t\leq 1}\frac{\|\gamma(t)-\gamma(s)\|}{|t-s|^{\frac{1}{2}}m(t-s)}<\infty.$$

Then $\gamma\in\mathcal{G}_2(\mathcal{V})$.

Proof. Denote

$$C:=\sup_{0\leq s<t\leq 1}\frac{\|\gamma(t)-\gamma(s)\|}{|t-s|^{\frac{1}{2}}m(t-s)}<\infty.$$

Then $\|\gamma\|_{2-var,[0,T]} \leq C$, $\omega_2(\gamma,\delta) \leq Cm(\delta)$. Using Lemma 3.43,

$$\begin{aligned}\left\|A\left(\gamma^{D_1}\right) - A\left(\gamma^{D_2}\right)\right\|_{1-var} &\leq \sum_k \left\|A\left(\gamma^{D_2}\right)\right\|_{1-var,[t_k,t_{k+1}]} + 4\|\gamma\|_{2-var,[0,T]}\,\omega_2(\gamma,\delta). \\ &\leq \sum_k \left\|A\left(\gamma^{D_2}\right)\right\|_{1-var,[t_k,t_{k+1}]} + 4C^2 m(\delta).\end{aligned}$$

While, applying Lemma 3.45 to bisect intervals, and using similar reasoning as that lead to (3.67) in proof of Theorem 3.30 (starting from page 42), we get

$$\sum_k \left\|A^{D_2}\right\|_{1-var,[t_k,t_{k+1}]} \leq 2C^2 \sum_{n=0}^{\infty} m^2\left(\frac{\delta}{2^n}\right) \leq 2C^2\left(m^2(\delta) + 2\int_0^\delta \frac{m^2(t)}{t}dt\right).$$

$$\text{Thus, } \left\|A\left(\gamma^{D_1}\right) - A\left(\gamma^{D_2}\right)\right\|_{1-var} \leq 2C^2\left(2m(\delta) + m^2(\delta) + 2\int_0^\delta \frac{m^2(t)}{t}dt\right).$$

Since $\lim_{\delta\to 0} m(\delta) = 0$ and $\int_0^1 \frac{m^2(t)}{t}dt < \infty$, $A\left(\gamma^D\right)$ converge in 1-variation as $|D| \to 0$. Based on Theorem 3.20, γ is in $\mathcal{G}_2(\mathcal{V})$. ■

Corollary 3.27 *Let $\gamma : [0,T] \to \mathcal{V}$ be a continuous path. If there exists a continuous function $\varphi : [0,\infty) \to [0,\infty)$ which is strictly increasing and onto, satisfies $\varphi(0) = 0$,*

$$u \mapsto \frac{u^2}{\varphi(u)} \text{ is non-decreasing, and } \int_0^1 \frac{u}{\varphi(u)}du < \infty, \tag{3.73}$$

such that

$$\sup_D \sum_{k,t_k\in D} \varphi\left(\|\gamma(t_{k+1}) - \gamma(t_k)\|\right) < \infty. \tag{3.74}$$

Then $\gamma \in \mathcal{G}_2(\mathcal{V})$.

Proof. Based on Prop 5.39 in [9], there exists control ω on $[0,T]$, such that

$$\|\gamma(t) - \gamma(s)\| \leq \varphi^{-1}(\omega(s,t)). \tag{3.75}$$

Then because ω is a control and φ^{-1} is increasing, we have

$$\|\gamma(t) - \gamma(s)\| \leq \varphi^{-1}(\omega(0,t) - \omega(0,s)).$$

Thus, if define $\widetilde{\gamma}(\omega(0,t)) := \gamma(t)$, then $\widetilde{\gamma}$ is well-defined, because if $\omega(0,s) = \omega(0,t)$ then $\omega(s,t) \leq \omega(0,t) - \omega(0,s) = 0$ and (3.75) implies that γ is constant on $[s,t]$. Therefore,

$$\|\widetilde{\gamma}(t) - \widetilde{\gamma}(s)\| \leq \varphi^{-1}(t-s),\ \forall\, 0 \leq s \leq t \leq \omega(0,T).$$

Then we only need to check that $t^{-\frac{1}{2}}\varphi^{-1}(t)$ satisfies the condition of Theorem 3.25. (Since the integral in (3.73) is finite, so $\lim_{u\to 0}\frac{u^2}{\varphi(u)} = 0$)

$$\begin{aligned}\int_0^1 \frac{u}{\varphi(u)}du &= \int_0^1 \frac{1}{\varphi(u)}d\frac{1}{2}u^2 = \frac{u^2}{2\varphi(u)}|_0^1 + \frac{1}{2}\int_0^1 \frac{u^2}{\varphi^2(u)}d\varphi(u) \\ &= \frac{1}{2\varphi(1)} + \frac{1}{2}\int_0^{\varphi(1)} \frac{\left(\varphi^{-1}(t)\right)^2}{t^2}dt\end{aligned}$$

which is finite because $\int_0^1 \frac{u}{\varphi(u)}du < \infty$. ■

Chapter 4

Rough Path Theory

Rough path is a path taking value in topological group and controllable in rough norm. Normally a rough path is composed of several levels of functions, with the first level function a path taking value in vector space, and higher leveled function defined as the iterated integrals (or something equivalent) of the first level path.

When the first level path is regular enough (e.g. with finite p-variation, $1 \leq p < 2$), all higher leveled functions (the canonical choices) are controlled by the first level path. In that case, the increment of the driving path already contains sufficient information to determine the solution to a differential equation (under some assumptions on the vector field), which is the story of classical ordinary differential equation.

However, as the regularity of the first level path weakens, it could fail to control the higher leveled functions. Suppose $\mathcal{V}$ is a Banach space, and consider the space of continuous bounded variation paths $C^{1-var}\left([0,T],\mathcal{V}\right)$ equipped with p-variation norm. The area operator A, as we defined before, is the operator which sends $\gamma \in C^{1-var}\left([0,T],\mathcal{V}\right)$ to $A\left(\gamma\right) : \triangle_{[0,T]} \rightarrow [\mathcal{V},\mathcal{V}]$ defined by

$$A\left(\gamma\right)\left(s,t\right) = \iint_{s<u_1<u_2<t} \left[d\gamma\left(u_1\right), d\gamma\left(u_2\right)\right], \ \forall \left(s,t\right) \in \triangle_{[0,T]}.$$

According to Young integral, when $p \in [1,2)$, there exists a constant C_p, s.t.

$$\left\|A\left(\gamma\right)\right\|_{\frac{p}{2}-var,[0,T]} \leq C_p \left\|\gamma\right\|_{p-var,[0,T]}, \ \forall \gamma \in C^{1-var}\left([0,T],\mathcal{V}\right).$$

While based on Section 3.1.4, when $p = 2$ (and $\dim \mathcal{V} \geq 2$), there does *not* exist C,

$$\left\|A\left(\gamma\right)\right\|_{\infty-var,[0,T]} \leq C \left\|\gamma\right\|_{2-var,[0,T]}, \ \forall \gamma \in C^{1-var}\left([0,T],\mathcal{V}\right).$$

Thus, when the norm on $C^{1-var}\left([0,T],\mathcal{V}\right)$ is weakened to 2-variation, the area is no longer controllable by the path. Since $C^{0,2-var}\left([0,T],\mathcal{V}\right) = \overline{C^{1-var}\left([0,T],\mathcal{V}\right)}^{2-var}$, when $\gamma \in C^{0,2-var}\left([0,T],\mathcal{V}\right)$, if we want to give meaning to differential equation as simple as

$$dy\left(t\right) = \gamma(t) \otimes d\gamma(t), \ y\left(0\right) = 0 \in \mathcal{V}^{\otimes 2},$$

which is the second level projection of

$$dS\left(t\right) = S\left(t\right) \otimes d\gamma\left(t\right), \ S\left(0\right) = \left(1, \gamma(0), 0\right) \in 1 \oplus \mathcal{V} \oplus \mathcal{V}^{\otimes 2},$$

then we have to control the area of γ (or something equivalent).

More generally, as the norm on $C^{1-var}([0,T],\mathcal{V})$ weakened to p-variation, $p\geq 1$, we need to control the regularity of kth iterated integral of γ, $k=1,2,\dots,[p]$, which are defined as

$$\gamma^{k}(s,t):=\int\dots\int_{s<u_1<u_2<\dots<u_k<t}d\gamma(u_1)\otimes\dots\otimes d\gamma(u_k),\forall(s,t)\in\triangle_{[0,T]}.$$

Although relying on and is closely related to rough path theory, this manuscript is not intended to give a full-coverage of rough path theory. For more general and systematical treatments of rough path, please refer to [20], [21] and [9].

In this Chapter, we mainly use the structure and follow the line of reasoning in [9] and [20].

4.1 Definition of rough paths

For Banach space $\mathcal{V}$, recall the definition of normed tensor space $\mathcal{V}^{\otimes k}$ and $[\mathcal{V},[\mathcal{V},\dots[\mathcal{V},\mathcal{V}]]]$ in Section 3.1.1.

Notation 4.1 *Suppose $\mathcal{V}$ is a Banach space and $N\geq 1$ is an integer. Denote*

$$T^{N}(\mathcal{V}):=\left(0,\mathcal{V},\mathcal{V}^{\otimes 2},\dots,\mathcal{V}^{\otimes N}\right).$$

Denote $\mathcal{V}^{1}:=\mathcal{V}$, $\mathcal{V}^{k+1}:=\left[\mathcal{V},\mathcal{V}^{k}\right]$, and

$$L^{N}(\mathcal{V}):=\left(0,\mathcal{V},\mathcal{V}^{2},\dots,\mathcal{V}^{N}\right). \tag{4.1}$$

$\mathcal{V}^{\otimes k}$ is called the space of homogeneous polynomial of degree k. $\mathcal{V}^{k}$ is called the space of homogeneous Lie polynomial of degree k. $\mathcal{V}^{k}$ is a subspace of $\mathcal{V}^{\otimes k}$. $L^{N}(\mathcal{V})$ is a subspace of $T^{N}(\mathcal{V})$.

Denote

$$1+T^{N}(\mathcal{V}):=\left(1,\mathcal{V},\dots,\mathcal{V}^{\otimes N}\right).$$

Then $1+T^{N}(\mathcal{V})$ is closed and associative under $\otimes$, and is invertible:

$$(1+a)^{-1}:=1+\sum_{k=1}^{N}(-1)^{k}a^{\otimes k},\ \ \forall a\in T^{N}(\mathcal{V}).$$

Thus, $1+T^{N}(\mathcal{V})$ is a group.

Notation 4.2 *Denote $\pi_k:1+T^{N}(\mathcal{V})\to\mathcal{V}^{\otimes k}$ as the projection to homogeneous polynomial of degree k.*

Definition 4.3 *For $N\geq 1$, we define norm $\|\cdot\|$ on $1+T^{N}(\mathcal{V})$ as*

$$\|t\|:=\sum_{k=1}^{N}\|\pi_k(t)\|_{\mathcal{V}^{\otimes k}}^{\frac{1}{k}},\ \ \forall t\in 1+T^{N}(\mathcal{V}). \tag{4.2}$$

Then, it can be checked that $\left(1+T^{N}\left(\mathcal{V}\right),\left\Vert \cdot\right\Vert \right)$ is a topological group. Next, we define a metric on continuous paths $\mathfrak{g}:[0,T]\rightarrow\left(1+T^{N}\left(\mathcal{V}\right),\left\Vert \cdot\right\Vert \right)$.

Definition 4.4 (metric on $C\left([0,T],\left(1+T^{N}\left(\mathcal{V}\right)\right),\left\Vert \cdot\right\Vert \right)$) *Suppose* $\mathfrak{g}$ *is a continuous path defined on* $[0,T]$ *taking value in* $\left(1+T^{N}\left(\mathcal{V}\right),\left\Vert \cdot\right\Vert \right)$, *(denote* $\mathfrak{g}_{s,t}:=\mathfrak{g}_{s}^{-1}\otimes\mathfrak{g}_{t}$*)*

$$\text{when } 1\leq p<\infty,\ \text{define}\ \ \left\Vert \mathfrak{g}\right\Vert _{p-var,[0,T]}:=\left(\sup_{D\subset[0,T]}\sum_{k,t_{k}\in D}\left\Vert \mathfrak{g}_{t_{k},t_{k+1}}\right\Vert ^{p}\right)^{\frac{1}{p}},\tag{4.3}$$

$$\text{when } p=\infty,\ \text{define}\ \ \left\Vert \mathfrak{g}\right\Vert _{\infty-var,[0,T]}:=\sup_{0\leq s\leq t\leq T}\left\Vert \mathfrak{g}_{\mathfrak{s},\mathfrak{t}}\right\Vert .$$

For $\mathfrak{g}^{1},\mathfrak{g}^{2}\in C\left([0,T],\left(1+T^{N}\left(\mathcal{V}\right),\left\Vert \cdot\right\Vert \right)\right)$, *define*

$$d_{p-var,[0,T]}\left(\mathfrak{g}^{1},\mathfrak{g}^{2}\right):=\left(\sup_{D\subset[0,T]}\sum_{k,t_{k}\in D}\left\Vert \mathfrak{g}_{t_{k},t_{k+1}}^{1}-\mathfrak{g}_{t_{k},t_{k+1}}^{2}\right\Vert ^{p}\right)^{\frac{1}{p}}.\tag{4.4}$$

Definition 4.5 (Rough path) *Suppose* $p\geq1$ *and* $\mathcal{V}$ *is a Banach space. Then* $\mathfrak{g}\in C\left([0,T],\left(1+T^{[p]}\left(\mathcal{V}\right),\left\Vert \cdot\right\Vert \right)\right)$ *is called a p-rough path, if* $\left\Vert \mathfrak{g}\right\Vert _{p-var,[0,T]}<\infty$. *Denote the set of p-rough paths as* $\Omega_{p}\left(1+T^{[p]}\left(\mathcal{V}\right),\left\Vert \cdot\right\Vert \right)$.

Thus, rough paths are continuous paths with bounded p-variation taking value in topological group $\left(1+T^{[p]}\left(\mathcal{V}\right),\left\Vert \cdot\right\Vert \right)$.

Definition 4.6 (Signature) *Suppose* $\gamma\in C^{1-var}\left([0,T],\mathcal{V}\right)$. *Then for any integer* $N\geq1$, *define the step-N signature of* γ, $S_{N}\left(\gamma\right):\triangle_{[0,T]}\rightarrow1+T^{N}\left(\mathcal{V}\right)$ *by setting*

$$S_{N}\left(\gamma\right)\left(s,t\right)=\left(1,\gamma^{1}\left(s,t\right),\ldots,\gamma^{N}\left(s,t\right)\right),\ \forall\left(s,t\right)\in\triangle_{[0,T]},$$

where $\gamma^{k}:\triangle_{[0,T]}\rightarrow\mathcal{V}^{\otimes k}$ *are defined as*

$$\gamma^{k}\left(s,t\right):=\int\cdots\int_{s<u_{1}<u_{2}<\cdots<u_{n}<t}d\gamma\left(u_{1}\right)\otimes\cdots\otimes d\gamma\left(u_{n}\right),\ \ \forall\left(s,t\right)\in\triangle_{[0,T]}.\tag{4.5}$$

Properties of $\left\{ \gamma^{k}\right\}$:

- $d\gamma^{k+1}\left(s,t\right)=\gamma^{k}\left(s,t\right)\otimes d\gamma\left(t\right)$.
- $\left\Vert \gamma^{k}\left(s,t\right)\right\Vert _{\mathcal{V}^{\otimes k}}\leq\left(k!\right)^{-1}\left\Vert \gamma\right\Vert _{1-var,[s,t]}^{k}$.
- For any $0\leq s\leq u\leq t\leq T$,

$$\gamma^{k}\left(s,t\right)=\gamma^{k}\left(s,u\right)+\gamma^{k}\left(u,t\right)+\sum_{j=1}^{k-1}\gamma^{j}\left(s,u\right)\otimes\gamma^{k-j}\left(u,t\right)\tag{4.6}$$

Properties of $\left\{ S_{N}\left(\gamma\right)\right\}$:

(i) $dS_{N+1}\left(\gamma\right)\left(s,t\right)=S_{N}\left(\gamma\right)\left(s,t\right)\otimes d\gamma\left(t\right)$.

(ii) For any $0 \leq s \leq u \leq t \leq T$,

$$S_N(\gamma)(s,t) = S_N(\gamma)(s,u) \otimes S_N(\gamma)(u,t). \tag{4.7}$$

(iii) Denote $\overleftarrow{\gamma}$ as γ run backwards, then

$$S_N(\overleftarrow{\gamma})(0,T) = (S_N(\gamma)(0,T))^{-1}.$$

(4.6) is truncation of (4.7) at degree k.

Based on *(i)*, signature could be treated as non-commutative exponential of a path; based on *(ii)* and *(iii)*, signature takes value in a group (step-N free nilpotent Lie group as defined below).

(4.7) is called Chen's identity, which states: for $\gamma_i \in C^{1-var}([0,T],\mathcal{V})$, $i = 1,2$, if define the concatenation of γ_1 and γ_2 as

$$(\gamma_1 \sqcup \gamma_2)(t) = \left\{ \begin{array}{cc} \gamma_1(2t) & t \in [0, 2^{-1}T] \\ \gamma_2(2t - T) - \gamma_2(0) + \gamma_1(T) & t \in [2^{-1}T, T] \end{array} \right. ,$$

then $S_N(\gamma_1 \sqcup \gamma_2)(0,T) = S_N(\gamma_1)(0,T) \otimes S_N(\gamma_2)(0,T)$ for any $N \geq 1$.

Define $\exp : T^N(\mathcal{V}) \to 1 + T^N(\mathcal{V})$ and $\log : 1 + T^N(\mathcal{V}) \to T^N(\mathcal{V})$ by

$$\exp(a) := 1 + \sum_{k=1}^{N} \frac{a^{\otimes k}}{k!}, \ \forall a \in T^N(\mathcal{V}),$$
$$\log(1+a) := \sum_{k=1}^{N} (-1)^{k+1} \frac{a^{\otimes k}}{k}, \ \forall a \in T^N(\mathcal{V}).$$

Then we have

$$\exp(\log(1+a)) = 1 + a, \quad \log(\exp(a)) = a, \quad \forall a \in T^N(\mathcal{V}).$$

Definition 4.7 (Free nilpotent lie group) *We call $L^N(\mathcal{V})$ (defined at* (4.1)*) the free nilpotent Lie algebra of step N, and call*

$$G^N(\mathcal{V}) := \exp\left(L^N(\mathcal{V})\right)$$

the free nilpotent Lie group of step N.

The topology of $G^N(\mathcal{V})$ is inherited from $1 + T^N(\mathcal{V})$. That $G^N(\mathcal{V})$ is a group follows from Campbell-Baker-Hausdorff formula, which states that

$$\log(\exp(a)\exp(b)) = a + b + \frac{1}{2}[a,b] + \frac{1}{12}[a,[a,b]] \cdots \in L^N(\mathcal{V}),$$
$$\forall a \in L^N(\mathcal{V}), b \in L^N(\mathcal{V}).$$

Based on the definition of signature, one has

Theorem 4.8 *$\{S_N(\gamma)(0,T) \,|\, \gamma \in C^{1-var}([0,T],\mathcal{V})\} \subseteq G^N(\mathcal{V})$, and they are equal when $\mathcal{V} = \mathbb{R}^d$.*

Thus, for $\gamma \in C^{1-var}([0,T],\mathcal{V})$, $S_N(\gamma)$ is a path on $[0,T]$ taking value in $G^N(\mathcal{V})$. When $\mathcal{V} = \mathbb{R}^d$, based on Chow-Rashevskii connectivity theorem, for any $g \in G_N(\mathbb{R}^d)$ there exist $\gamma_g \in C^{1-var}([0,T],\mathbb{R}^d)$ s.t. $S(\gamma_g)(0,T) = g$. Thus,

$$\left\{S_N(\gamma)(0,T) \,|\, \gamma \in C^{1-var}\left([0,T],\mathbb{R}^d\right)\right\} = G^N\left(\mathbb{R}^d\right)$$

However, whether or not Chow-Rashevskii can be extended to infinite dimensional spaces is not clear.

Proposition 4.9 *Suppose $\gamma \in C^{1-var}([0,T],\mathcal{V})$. Then for any integer $N \geq 1$, $S_N(\gamma) \in C^{1-var}\left([0,T],\left(G^N(\mathcal{V}),\|\cdot\|\right)\right)$, and*

$$\|S_N(\gamma)\|_{1-var,[0,T]} \leq N\,\|\gamma\|_{1-var,[0,T]}\,.$$

Proof. For $k = 1,2,\ldots,N$ and any $(s,t) \in \triangle_{[0,T]}$,

$$\begin{aligned} \left\|\gamma^k(s,t)\right\|_{\mathcal{V}^{\otimes k}} &\leq \frac{1}{k!}\,\|\gamma\|_{1-var,[s,t]}^k\,. \\ \text{Thus, } \|S_N(\gamma)\|_{1-var,[0,T]} &\leq \sum_{k=1}^N \left\|\gamma^k\right\|_{\frac{1}{k}-var,[0,T]}^{\frac{1}{k}} \leq N\,\|\gamma\|_{1-var,[0,T]}\,. \end{aligned}$$

■

Therefore, when signatures are considered as paths taking value in $\left(G^N(\mathcal{V}),\|\cdot\|\right)$, they are of bounded variation:

$$\left\{S_N(\gamma) \,|\, \gamma \in C^{1-var}([0,T],\mathcal{V})\right\} \subseteq C^{1-var}\left([0,T],\left(G^N(\mathcal{V}),\|\cdot\|\right)\right).$$

Denote $[p]$ as the biggest integer which is less or equal to p. Then we define geometric p-rough paths as $\overline{\left\{S_{[p]}(\gamma) \,|\, \gamma \in C^{1-var}([0,T],\mathcal{V})\right\}}^{d_{p-var,[0,T]}}$, with d_{p-var} defined at (4.4).

Definition 4.10 (Geometric rough paths) *Suppose $p \geq 1$ and $\mathcal{V}$ is a Banach space. Then $\mathfrak{g} \in C\left([0,T],\left(G^{[p]}(\mathcal{V}),\|\cdot\|\right)\right)$ is said to be a geometric p-rough path, if there exist $\{\gamma_n\}_{n=1}^\infty \subset C^{1-var}([0,T],\mathcal{V})$ s.t.*

$$\lim_{n\to\infty} d_{p-var}\left(\mathfrak{g}, S_{[p]}(\gamma_n)\right) = 0.$$

Denote the set of geometric p-rough paths as $G\Omega_p\left(G^{[p]}(\mathcal{V}),\|\cdot\|\right)$.

When $p = 1$, $G\Omega_1$ is the set of absolutely continuous paths (Prop 1.32 [9]); when $p \in (1,2)$, $G\Omega_p = C^{0,p-var}([0,T],\mathcal{V})$ (based on Theorem 2.12 on page 9). When $p \geq 2$,

$$G\Omega_p\left(G^{[p]}(\mathcal{V}),\|\cdot\|\right) \subseteq C^{0,p-var}\left([0,T],\left(G^{[p]}(\mathcal{V}),\|\cdot\|\right)\right).$$

Its proof is based on Proposition 4.9 (signatures of continuous bounded variation paths are of bounded variation so of vanishing p-variation, $p > 1$) and the inequality (ω_p as defined at (2.2)):

$$\omega_p(\mathfrak{g}, \delta) \leq d_{p-var}(\mathfrak{g}, \mathfrak{g}_n) + \omega_p(\mathfrak{g}_n, \delta).$$

In the special case $\mathcal{V} = \mathbb{R}^d$, based on Theorem 8.22 [9] and that homogeneous norms are equivalent on finite dimensional spaces, we have, for any $p > 1$,

$$G\Omega_p\left(G^{[p]}\left(\mathbb{R}^d\right), \|\cdot\|\right) = C^{0,p-var}\left([0,T], \left(G^{[p]}\left(\mathbb{R}^d\right), \|\cdot\|\right)\right).$$

Definition 4.11 (Weak geometric rough paths) *Suppose $p \geq 1$ and $\mathcal{V}$ is a Banach space. Then $\mathfrak{g} \in C\left([0,T], \left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)\right)$ is said to be a weak geometric p-rough path, if $\|\mathfrak{g}\|_{p-var,[0,T]} < \infty$. Denote the set of weak geometric p-rough paths as $WG\Omega_p\left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)$.*

Thus, weak geometric p-rough paths are continuous paths taking value in step-$[p]$ free nilpotent Lie group and of finite p-variation.

Then, we have the inclusion relation:

$$G\Omega_p\left(G^{[p]}(\mathcal{V}), \|\cdot\|\right) \subseteq WG\Omega_p\left(G^{[p]}(\mathcal{V}), \|\cdot\|\right) \subseteq \Omega_p\left(1 + T^{[p]}(\mathcal{V}), \|\cdot\|\right),$$

and both inclusions are strict (when $p \geq 2$, $\dim(\mathcal{V}) \geq 2$).

The relationship between $G\Omega_p$ and $WG\Omega_p$ is comparable to the relationship between $C^{0,p-var}([0,T], \mathcal{V})$ and $C^{p-var}([0,T], \mathcal{V})$ (e.g. when $p = 1$, the relationship between absolutely continuous path and continuous paths of bounded variation). While the difference between $WG\Omega_p$ and Ω_p is more substantial, which is comparable to the difference between anti-symmetric polynomials and general non-commutative polynomials.

$G\Omega_p$ is separable when $\dim(\mathcal{V}) < \infty$ (based on its definition). While $WG\Omega_p$ and Ω_p are not separable. For Ω_p, we have

$$\forall \mathfrak{g} \in \Omega_p\left(1 + T^{[p]}(\mathcal{V}), \|\cdot\|\right), \ \forall \gamma \in C^{\frac{p}{[p]}-var}\left([0,T], \mathcal{V}^{\otimes [p]}\right), \ \mathfrak{g} + \gamma \in \Omega_p\left(1 + T^{[p]}(\mathcal{V}), \|\cdot\|\right).$$

Thus, because $C^{\frac{p}{[p]}-var}\left([0,T], \mathcal{V}^{\otimes [p]}\right)$ is not separable, $\Omega_p\left(T^{[p]}(\mathcal{V}), \|\cdot\|\right)$ is not separable. Similarly, when $\dim(\mathcal{V}) \geq 2$, using the non-separability of $C^{\frac{p}{[p]}-var}\left([0,T], \mathcal{V}^{[p]}\right)$, $WG\Omega_p\left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)$ is not separable. When $\dim(\mathcal{V}) = 1$, select $e \in \mathcal{V}, e \neq 0$, then (one can prove that)

$$\begin{aligned} \mathfrak{g} &\in WG\Omega_p\left(G^{[p]}(\mathcal{V}), \|\cdot\|\right) \\ &\Leftrightarrow \exists f \in C^{p-var}([0,T], \mathbb{R}) \text{ s.t. } \mathfrak{g}(t) = \exp(f(t)e), t \in [0,T]. \end{aligned}$$

Since $C^{p-var}([0,T], \mathbb{R})$ is not separable (Proposition 2.13 at p10), $WG\Omega_p\left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)$ is not separable.

When $p \geq 2$ and $\dim(\mathcal{V}) \geq 2$, $G\Omega_p$, $WG\Omega_p$ and Ω_p are non-linear, but they are cones w.r.t. dilation (π_k is the projection to $\mathcal{V}^{\otimes k}$):

$$\delta_\lambda g := \left(1, \lambda \pi_1(g), \lambda^2 \pi_2(g), \ldots, \lambda^N \pi_N(g)\right), \ \forall g \in 1 + T^N(\mathcal{V}), \forall \lambda \in \mathbb{R}. \tag{4.8}$$

Theorem 4.12 (Lyons Lift) *Suppose* $p \geq 1$ *and* $\mathfrak{g} \in \Omega_p\left(1+T^{[p]}(\mathcal{V}), \|\cdot\|\right)$. *Then for any integer* $N \geq [p]+1$, *there exists a unique lift of* $\mathfrak{g}$ *into a path with finite p-variation taking value in* $1+T^N(\mathcal{V})$.

This theorem follows from Thm 3.7 in [20]. The neo-classical inequality used in Thm 3.7 is proved in [12].

Theorem 4.12 explains the reason why, in Definition 4.10 and 4.11, paths are members of $C\left([0,T], \left(G^{[p]}(\mathcal{V}), \|\cdot\|\right)\right)$ while not $C\left([0,T], \left(G^{N}(\mathcal{V}), \|\cdot\|\right)\right)$ for $N \geq [p]+1$—because the extension exists uniquely. The following two theorems explain why we did not use $C\left([0,T], \left(G^{[p]-1}(\mathcal{V}), \|\cdot\|\right)\right)$.

Theorem 4.13 *Suppose* $p \geq 2$ *and* $\mathcal{V}$ *is a Banach space satisfying* $\dim(\mathcal{V}) \geq 2$. *For integer* $N \geq 1$ *denote*

$$\overline{S_N(\mathcal{V})}^p := \overline{\left\{S_N(\gamma) \,|\, \gamma \in C^{1-var}([0,T], \mathcal{V})\right\}}^{d_{p-var,[0,T]}}.$$

Define $\Pi_{[p]-1} : \overline{S_{[p]}(\mathcal{V})}^p \to \overline{S_{[p]-1}(\mathcal{V})}^p$ *by*

$$\Pi_{[p]-1}(\mathfrak{g}) := 1 + \sum_{k=1}^{[p]-1} \pi_k(\mathfrak{g}), \ \ \forall \mathfrak{g} \in \overline{S_{[p]}(\mathcal{V})}^p.$$

Then when p is not an integer, $\Pi_{[p]-1}$ *is not injective but surjective; when p is an integer,* $\Pi_{[p]-1}$ *is injective but not surjective.*

This theorem is (a modified) part of Thm 9.12 in [9].

In Remark 9.13 [9], the authors point out that $\Pi_{[p]-1}$ is not injective when p is an integer, which is incorrect. It is because that when p is an integer, different choices of level $[p]$ function can only differ by a path of vanishing 1-variation, so a constant.

Theorem 4.14 *Suppose* $p \geq 2$ *and* $\mathcal{V}$ *is a Banach space satisfying* $\dim(\mathcal{V}) \geq 2$. *Define* $\Pi_{[p]-1} : C^{p-var}\left([0,T], G^{[p]}(\mathcal{V})\right) \to C^{p-var}\left([0,T], G^{[p]-1}(\mathcal{V})\right)$ *by*

$$\Pi_{[p]-1}(\mathfrak{g}) := 1 + \sum_{k=1}^{[p]-1} \pi_k(\mathfrak{g}), \ \ \forall \mathfrak{g} \in C^{p-var}\left([0,T], G^{[p]}(\mathcal{V})\right).$$

Then $\Pi_{[p]-1}$ *is not injective; is surjective when p is not an integer; is not surjective when p is an integer.*

This theorem is also part of Theorem 9.12 in [9].

Thus, when $p \geq 2$ and $\dim(\mathcal{V}) \geq 2$, if a path in $C^{p-var}\left([0,T], G^{[p]-1}(\mathcal{V})\right)$ can be lifted to a path in $C^{p-var}\left([0,T], G^{[p]}(\mathcal{V})\right)$ (i.e. a weak geometric p-rough path), then the lift is *not* unique—(logarithm of) the lifted path could differ by a path $\gamma \in C^{\frac{p}{[p]}-var}\left([0,T], \mathcal{V}^{[p]}\right)$.

4.2 Carnot-Caratheodory norm

The Carnot-Caratheodory norm is the right norm on $G^N(\mathcal{V})$ when $\dim\mathcal{V} < \infty$. When $\dim\mathcal{V} < \infty$, based on Chow-Rashevskii theorem, $G^N(\mathcal{V})$ is a geodesic space when equipped with Carnot-Caratheodory norm. As a consequence, we have explicit characterization of weak geometric rough paths and better characterization of geometric rough paths, in term of signatures of continuous bounded variation paths. When $\dim\mathcal{V} = \infty$, it is not clear whether $G_N(\mathcal{V})$ is a geodesic space or not, because the extension of Chow-Rashevskii theorem to infinite dimensional spaces is not easy. For this reason, in this section, we work with $G_N\left(\mathbb{R}^d\right)$.

4.2.1 Geodesic space

Definition 4.15 (geodesic space) *Suppose (E,d) is a metric space. Then (E,d) is said to be a geodesic space, if for any $g,h \in E$, there exists a path $\gamma : [0,1] \to (E,d)$ s.t. $\gamma(0) = g$, $\gamma(1) = h$ and*

$$d(\gamma(s), \gamma(t)) = |t-s|\, d(g,h), \ \forall (s,t) \in \triangle_{[0,1]}.$$

The path γ is called a geodesic connecting g and h.

Definition 4.16 (geodesic approximation) *Suppose (E,d) is a geodesic space, and $\mathfrak{g} \in C([0,T], (E,d))$. For finite partition $D = \{t_k\}$ of $[0,T]$, define geodesic approximation to $\mathfrak{g}$ w.r.t. D as*

$$\mathfrak{g}^D(t) := \gamma_k\left(\frac{t-t_k}{t_{k+1}-t_k}\right), \ \forall t \in [t_k, t_{k+1}],$$

with $\gamma_k : [0,1] \to (E,d)$ a geodesic connecting $\mathfrak{g}(t_k)$ and $\mathfrak{g}(t_{k+1})$.

Lemma 4.17 *(E,d) is a geodesic space and $\mathfrak{g} \in C([0,T], (E,d))$. Then*

$$\lim_{|D|\to 0} \sup_{0\le t\le T} d\left(\mathfrak{g}_t^D,\ \mathfrak{g}_t\right) = 0. \tag{4.9}$$

Moreover, if $\mathfrak{g} \in C^{p-var}([0,T], (E,d))$, then

$$\sup_{D\subset[0,T]} \left\|\mathfrak{g}^D\right\|_{p-var,[0,T]} \le 3^{1-\frac{1}{p}} \|\mathfrak{g}\|_{p-var,[0,T]}, \tag{4.10}$$

and for any finite partition D and any $\delta > |D|$,

$$\sup_{|D'|\le\delta} \sum_{i,t_i\in D'} \left\|\mathfrak{g}^D\right\|^p_{p-var,[t_i,t_{i+1}]} \le 3^{p-1} \sup_{|D'|\le\delta} \sum_{i,t_i\in D'} \|\mathfrak{g}\|^p_{p-var,[t_i,t_{i+1}]}. \tag{4.11}$$

Which is Lemma 5.19, Prop 5.20 and Remark 5.22 in [9].

4.2.2 Carnot-Caratheodory norm

Definition 4.18 (Carnot-Caratheodory) *Define the Carnot-Caratheodory norm (C-C norm) on* $G^N\left(\mathbb{R}^d\right)$ *as, for any* $g \in G^N\left(\mathbb{R}^d\right)$*,*

$$|||g||| := \inf\left\{\|\gamma\|_{1-var,[0,T]} \,|\gamma \in C^{1-var}\left([0,T],\mathbb{R}^d\right), S_N(\gamma)(0,T) = g\right\},$$

and the infimum can be obtained at some $\gamma_* \in C^{1-var}\left([0,T],\mathbb{R}^d\right)$*.*

Based on Theorem 4.8, for any $g \in G^N\left(\mathbb{R}^d\right)$, the set

$$\left\{\gamma \in C^{1-var}\left([0,T],\mathbb{R}^d\right) |S_N(\gamma)(0,T) = g\right\}$$

is not empty, and the existence of γ_* follows from Theorem 7.32 in [9].

Carnot-Caratheodory norm is a norm on the equivalent classes of paths sharing the same step-N signature on a fixed time interval. In Hambly and Lyons [11], they demonstrate that: if $\gamma_1, \gamma_2 \in C^{1-var}\left([0,T],\mathbb{R}^d\right)$ satisfy $S_N(\gamma_1)(0,T) = S_N(\gamma_2)(0,T)$ for all $N \geq 1$, then γ_1 and γ_2 differ by a tree-like path (i.e. a path which cancels itself out, in lattice setting a tree-like path is comparable to different formal representations of the trivial word, e.g. $aa^{-1}b^{-1}b$).

Properties of Carnot-Caratheodory norm (Prop 7.40 [9])

Suppose $g, h \in G^N\left(\mathbb{R}^d\right)$. Then

- $|||g||| = 0$ iff $g = (1, 0, \ldots, 0)$.
- $|||\delta_\lambda g||| = |\lambda|\, |||g|||$ for all $\lambda \in \mathbb{R}$.
- $|||g||| = |||g^{-1}|||$.
- $|||g \otimes h||| \leq |||g||| + |||h|||$.

We do not have $|||g \otimes h||| = |||h \otimes g|||$ (or equivalently $|||h \otimes g \otimes h^{-1}||| = |||g|||$), because for $\gamma_i \in C^{1-var}\left([0,T],\mathbb{R}^d\right)$, $i = 1, 2$, the concatenated path $\overleftarrow{\gamma_2} \sqcup \gamma_1 \sqcup \gamma_2$, generally, does not have the same (tree-reduced) length as γ_1.

$G^N\left(\mathbb{R}^d\right)$ is a group, so is not linear (when $N \geq 2$), but is a geodesic space when equipped with C-C norm.

Proposition 4.19 $\left(G^N\left(\mathbb{R}^d\right), |||\cdot|||\right)$ *is a geodesic space.*

Which is Prop 7.42 in [9]. The geodesic connecting g and h can be chosen as $g \otimes S_N(\gamma_*)(0,t)$, where γ_* is a continuous bounded variation path satisfying $S_N(\gamma_*)(0,T) = g^{-1} \otimes h$, $\|\gamma_*\|_{1-var} = |||g^{-1} \otimes h|||$ and is reparametrised to be of unit speed.

Since $G_N\left(\mathbb{R}^d\right)$ is finite dimensional, all norms on $G_N\left(\mathbb{R}^d\right)$ are equivalent. Especially, there exists a constant $C(d,N)$ s.t. (with $\|\cdot\|$ defined at (4.2))

$$C(d,N)^{-1}\|g\| \leq |||g||| \leq C(d,N)\|g\|,\ \forall g \in G^N\left(\mathbb{R}^d\right).$$

However, in an infinite dimensional space, not all norms are equivalent, and the set of rough paths depends on the choice of norm (see [34] for an infinite dimensional path, which can be enhanced into a geometric rough path in one norm, but can not be enhanced into a rough path in another norm).

We omit the norm used on $G^N\left(\mathbb{R}^d\right)$, and denote the set of weak geometric p-rough paths/geometric p-rough paths on $\mathbb{R}^d$ as $WG\Omega_p\left(\mathbb{R}^d\right)$ and $G\Omega_p\left(\mathbb{R}^d\right)$.

Corollary 4.20 *Suppose* $\mathfrak{g} \in WG\Omega_p\left(\mathbb{R}^d\right)$. *Then there exist a sequence of continuous bounded variation paths* $\{\gamma^n\}$ *on* $[0,T]$ *taking value in* $\mathbb{R}^d$, *such that*

$$\lim_{n\to\infty} d_{\infty-var}\left(S_{[p]}\left(\gamma^n\right),\mathfrak{g}\right)=0 \text{ and } \sup_n \left\|S_{[p]}\left(\gamma^n\right)\right\|_{p-var,[0,T]} \le C_{d,p}\left\|\mathfrak{g}\right\|_{p-var,[0,T]}.$$

Proof. For finite partition $D=\{t_k\}$, denote $\gamma^{t_k,t_{k+1}}$ as the geodesic connecting $\mathfrak{g}_{t_k}$ and $\mathfrak{g}_{t_{k+1}}$. Using Chen's identity at (4.7), geodesic approximation $\mathfrak{g}^D$ (Definition 4.16) is the signature of the concatenated bounded variation path $\gamma^{0,t_1}\sqcup\gamma^{t_1,t_2}\sqcup\cdots\sqcup\gamma^{t_{n-1},T}$. Since $\left(G^{[p]}\left(\mathbb{R}^d\right),|||\cdot|||\right)$ is a geodesic space and all homogeneous norms are equivalent, combining Lemma 4.17 with Prop 8.15 in [9]. ■

Using interpolation, the convergence in Corollary 4.20 holds in d_{q-var} , $q>p$.

Under C-C norm, signatures have the same length as paths (Prop 7.59 [9]):

Proposition 4.21 *Suppose* $\gamma \in C^{1-var}([0,T],\mathbb{R}^d)$. *Then for any integer* $N\ge 1$, $S_N(\gamma)\in C^{1-var}\left([0,T],G^N\left(\mathbb{R}^d\right)\right)$, *and*

$$\left\|S_N(\gamma)\right\|_{1-var,[0,T]}=\left\|\gamma\right\|_{1-var,[0,T]}.$$

Proof. For any $0\le s<t\le T$, we have

$$\left\|\gamma_t-\gamma_s\right\|\le\left|\left|\left|S_N\left(\gamma\right)_{s,t}\right|\right|\right|\le\left\|\gamma\right\|_{1-var,[s,t]}.$$

The first inequality is because that the linear path is the shortest path which shares the same step 1 signature with γ, so $\left\|\gamma_t-\gamma_s\right\|=|||S_1\left(\gamma\right)_{s,t}|||\le|||S_N\left(\gamma\right)_{s,t}|||$. The second inequality is based on the definition of C-C norm. ■

Recall that the notation $\mathfrak{g}\in C^{0,p-var}\left([0,T],G^N\left(\mathbb{R}^d\right)\right)$ means

$$\lim_{\delta\to 0}\omega_p\left(\mathfrak{g},\delta\right):=\lim_{\delta\to 0}\left(\sup_{D,|D|\le\delta}\sum_{k,t_k\in D}\left\|\mathfrak{g}_{t_k,t_{k+1}}\right\|^p\right)^{\frac{1}{p}}=0.$$

Theorem 4.22 (Wiener's characterization) *Suppose* $p>1$. *Then the following three statements are equivalent when* $\mathfrak{g}\in C^{p-var}\left([0,T],G^{[p]}\left(\mathbb{R}^d\right)\right)$:

$$(i)\ \mathfrak{g}\in C^{0,p-var}\left([0,T],G^{[p]}\left(\mathbb{R}^d\right)\right),$$
$$(ii)\ \exists\left\{\gamma_n\right\}\subset C^{1-var}\left([0,T],\mathbb{R}^d\right)\ s.t.\ \lim_{n\to\infty}d_{p-var,[0,T]}\left(S_{[p]}\left(\gamma_n\right),\mathfrak{g}\right)=0,$$
$$(iii)\ \lim_{|D|\to 0}d_{p-var,[0,T]}\left(\mathfrak{g}^D,\mathfrak{g}\right)=0.$$

This theorem is Theorem 8.22 in [9].

Proof. $(iii) \Rightarrow (ii)$ Similar reasoning as that of Corollary 4.20.

$(ii) \Rightarrow (i)$ Based on Proposition 4.21, $S_{[p]}(\gamma_n) \in C^{1-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$ when $\gamma_n \in C^{1-var}\left([0,T], \mathbb{R}^d\right)$. Suppose $\{\mathfrak{g}_n\} \subset C^{1-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$ converge to $\mathfrak{g}$ in $d_{p-var,[0,T]}$ as n tends to infinity. Then for $\epsilon > 0$, choose n, s.t. $d_{p-var,[0,T]}(\mathfrak{g}, \mathfrak{g}_n) < \epsilon$. This selected g_n is continuous and of bounded variation, so of vanishing p-variation for any $p > 1$. Thus, there exists $\delta_0 > 0$ s.t. for any $\delta \in (0, \delta_0)$, $\omega_p(\mathfrak{g}_n, \delta) < \epsilon$. As a consequence,

$$\omega_p(\mathfrak{g}, \delta) \leq \omega_p(\mathfrak{g}_n, \delta) + d_{p-var,[0,T]}(\mathfrak{g}, \mathfrak{g}_n) < 2\epsilon, \quad \forall \delta \in (0, \delta_0).$$

$(i) \Rightarrow (iii)$ Firstly, it can be verified that condition (i) is equivalent to

$$\lim_{\delta \to 0} \sup_{|D| \leq \delta} \sum_{k, t_k \in D} \|\mathfrak{g}\|^p_{p-var,[t_k,t_{k+1}]} = 0. \tag{4.12}$$

For any finite partition D, $\mathfrak{g}$ and $\mathfrak{g}^D$ are in $C^{p-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$. Thus, for any $\delta > |D|$ and any finite partition $D' = \{t_i\}_i$

$$\begin{aligned}
&\sum_{i, t_i \in D'} d\left(\mathfrak{g}_{t_i,t_{i+1}}, \mathfrak{g}^D_{t_i,t_{i+1}}\right)^p \\
\leq\ & \sum_{i, t_i \in D', |t_{i+1}-t_i| \leq \delta} d\left(\mathfrak{g}_{t_i,t_{i+1}}, \mathfrak{g}^D_{t_i,t_{i+1}}\right)^p + \sum_{i, t_i \in D', |t_{i+1}-t_i| > \delta} d\left(\mathfrak{g}_{t_i,t_{i+1}}, \mathfrak{g}^D_{t_i,t_{i+1}}\right)^p \\
\leq\ & 2^{p-1} \sum_{i, t_i \in D', |t_{i+1}-t_i| \leq \delta} \left(\|\mathfrak{g}\|^p_{p-var,[t_i,t_{i+1}]} + \left\|\mathfrak{g}^D\right\|^p_{p-var,[t_i,t_{i+1}]}\right) + \frac{T}{\delta}\left(d_{\infty-var,[0,T]}\left(\mathfrak{g}, \mathfrak{g}^D\right)\right)^p.
\end{aligned}$$

Fixing $\delta > |D|$, and taking supremum over all D', we get

$$\begin{aligned}
&\left(d_{p-var,[0,T]}\left(\mathfrak{g}, \mathfrak{g}^D\right)\right)^p \\
\leq\ & 2^{p-1}\left(\sup_{|D'| \leq \delta} \sum_{i, t_i \in D'} \left\|\mathfrak{g}^D\right\|^p_{p-var,[t_i,t_{i+1}]} + \sup_{|D'| \leq \delta} \sum_{i, t_i \in D'} \|\mathfrak{g}\|^p_{p-var,[t_i,t_{i+1}]}\right) \\
& + \frac{T}{\delta}\left(d_{\infty-var,[0,T]}\left(\mathfrak{g}, \mathfrak{g}^D\right)\right)^p.
\end{aligned}$$

On the other hand, based on (4.11) and that all homogeneous norms are equivalent on $G^N\left(\mathbb{R}^d\right)$, for any $\delta > |D|$,

$$\sup_{|D'| \leq \delta} \sum_{i, t_i \in D'} \left\|\mathfrak{g}^D\right\|^p_{p-var,[t_i,t_{i+1}]} \leq C_{p,d,1} \sup_{|D'| \leq \delta} \sum_{i, t_i \in D'} \|\mathfrak{g}\|^p_{p-var,[t_i,t_{i+1}]}.$$

Thus, for any finite partition D and any $\delta > |D|$,

$$\left(d_{p-var,[0,T]}\left(\mathfrak{g}, \mathfrak{g}^D\right)\right)^p \leq C_{p,d,2} \sup_{|D'| \leq \delta} \sum_{i, t_i \in D'} \|\mathfrak{g}\|^p_{p-var,[t_i,t_{i+1}]} + \frac{T}{\delta}\left(d_{\infty-var,[0,T]}\left(\mathfrak{g}, \mathfrak{g}^D\right)\right)^p.$$

Since $\mathfrak{g}$ is of vanishing p-variation, (based on (4.12)) for any $\epsilon > 0$ choose $\delta_0 > 0$,

$$C_{p,d,2} \sup_{|D'| \leq \delta_0} \sum_{i, t_i \in D'} \|\mathfrak{g}\|^p_{p-var,[t_i,t_{i+1}]} < \epsilon.$$

Based on (4.9) and Prop 8.15 in [9], $\lim_{|D|\to 0} d_{\infty-var,[0,T]}\left(\mathfrak{g},\mathfrak{g}^D\right) = 0$. Thus, for this fixed δ_0, choose $\delta_1 < \delta_0$, s.t. for any D, $|D| \leq \delta_1$, we have

$$\frac{T}{\delta_0}\left(d_{\infty-var,[0,T]}\left(\mathfrak{g},\mathfrak{g}^D\right)\right)^p < \epsilon.$$

Therefore, for any $\epsilon > 0$, there exists $\delta_1 > 0$, s.t.

$$\sup_{D,|D|\leq\delta_1} d_{p-var,[0,T]}\left(\mathfrak{g},\mathfrak{g}^D\right) < (2\epsilon)^{\frac{1}{p}}.$$

■

Based on the proof, Wiener's characterization holds for paths taking value in any geodesic space. Since all homogeneous norms on $G^N\left(\mathbb{R}^d\right)$ are equivalent, geometric p-rough paths on $\mathbb{R}^d$, $p > 1$, are exactly the set of paths with vanishing p-variation taking value in $G^{[p]}\left(\mathbb{R}^d\right)$.

4.3 Rough Differential Equations

4.3.1 Operator norm

Suppose $\mathcal{U}$ and $\mathcal{V}$ are two Banach spaces. Denote $L(\mathcal{U},\mathcal{V})$ as the set of linear functions from $\mathcal{U}$ to $\mathcal{V}$. Let x be a path taking value in $\mathcal{V}$, $\xi \in \mathcal{U}$ and $f : \mathcal{U} \to L(\mathcal{V},\mathcal{U})$. Consider the differential equation:

$$dy_t = f(y_t)\,dx_t, \quad y_0 = \xi.$$

That f taking value in $L(\mathcal{V},\mathcal{U})$ explains the notation $f(y_t)\,dx_t$. In the special case $\mathcal{V} = \mathbb{R}^d$,

$$f : y \in \mathcal{U} \mapsto \left\{(a_1, a_2, \ldots, a_d) \in \mathbb{R}^d \mapsto \sum_{i=1}^d f_i(y)\,a_i \in \mathcal{U}\right\},$$

$$f(y_t)\,dx_t := \sum_{i=1}^d f_i(y_t)\,dx_t^i.$$

Definition 4.23 (operator norm) *Suppose* $\mathcal{U}$ *and* $\mathcal{V}$ *are two Banach spaces, and* $f : \mathcal{U} \to L(\mathcal{V},\mathcal{U})$. *Denote*

$$\|f\|_\infty := \sup_{y\in\mathcal{U}}\sup_{x\in\mathcal{V}}\frac{\|f(y)\,x\|_{\mathcal{U}}}{\|x\|_{\mathcal{V}}}.$$

For $\alpha \in (0,1]$, *denote*

$$\|f\|_{\alpha-H\ddot{o}l} := \sup_{y,z\in\mathcal{U},y\neq z}\sup_{x\in\mathcal{V}}\frac{\|(f(y)-f(z))\,x\|_{\mathcal{U}}}{\|y-z\|_{\mathcal{U}}^{\alpha}\|x\|_{\mathcal{V}}}.$$

Suppose $f : \mathcal{U} \to L(\mathcal{V},\mathcal{U})$. Suppose f is Fréchet differentiable with derivative f', i.e. for all $y \in \mathcal{U}$, $f'(y)$ is a bounded linear operator from $\mathcal{U}$ to $L(\mathcal{V},\mathcal{U})$ s.t.

$$\lim_{\|h\|_{\mathcal{U}}\to 0}\frac{\|f(y+h)-f(y)-f'(y)h\|_\infty}{\|h\|_{\mathcal{U}}} = 0,$$

with $\|\cdot\|_{\infty}$ is the operator norm we just defined. Then f' is an operator defined on $\mathcal{U}$ taking value in $L(\mathcal{U}, L(\mathcal{V},\mathcal{U})) \sim L(\mathcal{U}\otimes\mathcal{V},\mathcal{U})$. Similarly, the kth derivative $f^{(k)}$ is an operator defined on $\mathcal{U}$ and takes value in $L(\mathcal{U}^{\otimes(k-1)}\otimes\mathcal{V},\mathcal{U})$.

Definition 4.24 ($Lip(\gamma)$ vector field) *Suppose $\mathcal{U}$ and $\mathcal{V}$ are two Banach spaces, $f : \mathcal{U} \to L(\mathcal{V},\mathcal{U})$ and $\gamma > 0$. Denote $\lfloor\gamma\rfloor$ is the largest integer which is strictly less than γ. Then f is said to be $Lip(\gamma)$ if f is $\lfloor\gamma\rfloor$-times differentiable, and*

$$\|f\|_{Lip(\gamma)} := \left(\max_{k=0,1,\ldots,\lfloor\gamma\rfloor}\left\|f^{(k)}\right\|_{\infty}\right)\vee\left\|f^{(\lfloor\gamma\rfloor)}\right\|_{(\gamma-\lfloor\gamma\rfloor)-H\ddot{o}l} < \infty.$$

Thus, when $\gamma = n$ for integer $n \geq 2$, f is $Lip(n)$ implies that f is $(n-1)$-times differentiable and $f^{(n-1)}$ is Lipschitz (with a global constant bound).

4.3.2 Solution of Rough Differential Equations

Based on Corollary 4.20, for any $\mathbf{x}\in C^{p-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$, there exists a sequence of continuous bounded variation paths $\{x^n\}_n$ satisfying

$$\lim_{n\to\infty} d_{\infty-var}\left(S_{[p]}(x^n), \mathbf{x}\right) = 0 \text{ and } \sup_n\left\|S_{[p]}(x^n)\right\|_{p-var,[0,T]} < \infty. \tag{4.13}$$

Firstly, we define what we mean by a (full) solution to a rough differential equation (RDE): the solution is a path taking value in $\mathbb{R}^e$, the full solution is a path taking value in free nilpotent Lie group.

Definition 4.25 *Suppose $\mathbf{x}\in C^{p-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$, $f:\mathbb{R}^e\to L\left(\mathbb{R}^d,\mathbb{R}^e\right)$ is continuous, $\eta\in\mathbb{R}^e$, and $\boldsymbol{\xi} = \left(1,\xi^1,\ldots,\xi^{[p]}\right)\in G^{[p]}(\mathbb{R}^e)$.*

- *Then $y\in C([0,T],\mathbb{R}^e)$ is said to be a solution to the RDE*

  $$dy_t = f(y_t)\,d\mathbf{x}_t,\ \ y_0 = \eta,$$

  driven by $\mathbf{x}$ along the vector field f and starts from $\eta\in\mathbb{R}^e$, if there exists $\{x^n\}_n\subset C^{1-var}\left([0,T],\mathbb{R}^d\right)$ s.t.

  (i) (4.13) holds;

  (ii) $\lim_{n\to\infty} d_{\infty-var}(y,y^n) = 0$, with y^n a solution to the ODE

  $$dy^n_t = f(y^n_t)\,dx^n_t,\ \ y^n_0 = \eta.$$

- *Then $\mathbf{y}\in C\left([0,T], G^{[p]}(\mathbb{R}^e)\right)$ is said to be a Full solution to the RDE*

  $$d\mathbf{y}_t = f(y_t)\,d\mathbf{x}_t,\ \ \mathbf{y}_0 = \boldsymbol{\xi} = \left(1,\xi^1,\ldots,\xi^{[p]}\right),$$

  driven by $\mathbf{x}$ along the vector field f and starts from $\boldsymbol{\xi}\in G^{[p]}(\mathbb{R}^e)$, if there exists $\{x^n\}_n\subset C^{1-var}\left([0,T],\mathbb{R}^d\right)$ s.t.

  (i) (4.13) holds;

  (ii) $\lim_{n\to\infty} d_{\infty-var}\left(\mathbf{y},\boldsymbol{\xi}\otimes S_{[p]}(y^n)\right) = 0$, with y^n a solution to the ODE

  $$dy^n_t = f(y^n_t)\,dx^n_t,\ \ y^n_0 = \xi^1.$$

Based on Definition 4.25, the existence of a full solution implies the existence of a solution to the same RDE. On the other hand, (as demonstrated in the following theorem, which is Thm 10.35 in [9]) the full solution could be viewed as a solution driven by the same rough path along a modified vector field.

Theorem 4.26 *Suppose* $\mathbf{x} \in C^{p-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$, $f : \mathbb{R}^e \to L\left(\mathbb{R}^d, \mathbb{R}^e\right)$ *is* $Lip(\gamma)$ *for* $\gamma + 1 > p \geq 1$, *initial condition* $\boldsymbol{\xi} = \left(1, \xi^1, \xi^2, \dots, \xi^{[p]}\right) \in G^{[p]}(\mathbb{R}^e)$, *and* $\mathbf{y}$ *is a full solution to the RDE*

$$d\mathbf{y}_t = f(y_t)\, d\mathbf{x}_t, \ \ \mathbf{y}_0 = \boldsymbol{\xi},$$

Then, $\mathbf{y}$ *is a path in* $C\left([0,T], 1 + T^{[p]}(\mathbb{R}^e)\right)$ *and is a solution to the RDE*

$$dz_t = W(z)\, d\mathbf{x}_t, \ \ z_0 = \boldsymbol{\xi},$$

with $W(z) = z \otimes f(\pi_1(z)) : 1 + T^{[p]}(\mathbb{R}^e) \to L\left(\mathbb{R}^d, 1 + T^{[p]}(\mathbb{R}^e)\right)$ *a local* $Lip(\gamma)$ *vector field.*

Proof. Based on the characterization of weak geometric rough paths in Corollary 4.20, we assume $\mathbf{x} = S_{[p]}(x)$ with $x \in C^{1-var}\left([0,T], \mathbb{R}^d\right)$. Then, if y is a solution to the ODE

$$dy_t = f(y_t)\, dx_t, \ \ y_0 = \xi^1 \in \mathcal{U}, \tag{4.14}$$

then (based on the continuity of f), y is in $C^{1-var}([0,T], \mathbb{R}^e)$. Denote $\mathbf{y}_t := \boldsymbol{\xi} \otimes S_{[p]}(y)_{0,t}$. We have

$$d\mathbf{y}_t = \boldsymbol{\xi} \otimes S_{[p]}(y)_{0,t} \otimes dy_{0,t} = \boldsymbol{\xi} \otimes S_{[p]}(y)_{0,t} \otimes f(y_t)\, dx_t. = \mathbf{y}_t \otimes f(y_t)\, dx_t.$$

Thus, if y is a solution of the ODE (4.14), then $\boldsymbol{\xi} \otimes S_{[p]}(y)$ is a solution to the ODE

$$dz_t = z_t \otimes f(\pi_1(z_t))\, dx_t, \ \ z_0 = \boldsymbol{\xi} \in 1 + T^{[p]}(\mathbb{R}^e).$$

Based on the definitions of full solution and solution, theorem holds. ■

Theorem 4.26 is a key remark in [9] in getting full solution from solution.

Since we are working with finite dimensional spaces, all homogeneous norms are equivalent. Thus, on one hand, one can use Carnot-Caratheodory norm to get good approximation to the driving rough path (Corollary 4.20); on the other hand, one can use the norm (which is inherited from $1 + T^N(\mathbb{R}^e)$)

$$\|g\| := \sum_{k=1}^{N} \|\pi_k(g)\|_{(\mathbb{R}^e)^{\otimes k}}^{\frac{1}{k}}, \ \ g \in G^N(\mathbb{R}^e)$$

to get explicit estimation of the solution.

The theorems in the following are for (the existence and uniqueness of) full solutions; the parallel results for solutions follows trivially.

Theorem 4.27 (existence $\gamma > p-1$) *Suppose* $\mathbf{x} \in C^{p-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$*, initial condition* $\boldsymbol{\xi} = \left(1, \xi^1, \xi^2, \dots, \xi^{[p]}\right) \in G^{[p]}\left(\mathbb{R}^e\right)$*, and* $f : \mathbb{R}^e \to L\left(\mathbb{R}^d, \mathbb{R}^e\right)$ *is* $Lip(\gamma)$ *for* $\gamma + 1 > p \geq 1$*. Then the RDE*

$$d\mathbf{y}_t = f(y_t)\, d\mathbf{x}_t, \quad \mathbf{y}_0 = \boldsymbol{\xi},$$

admits a full solution in $C^{p-var}\left([0,T], G^{[p]}\left(\mathbb{R}^e\right)\right)$*, and each solution satisfies*

$$\|\mathbf{y}\|_{p-var,[0,T]} \leq C(p,\gamma)\left(\|f\|_{Lip(\gamma)}\|\mathbf{x}\|_{p-var,[0,T]} \vee \|f\|_{Lip(\gamma)}^p \|\mathbf{x}\|_{p-var,[0,T]}^p\right).$$

Which is Theorem 10.36 in [9].

Notation 4.28 *Suppose* $p \geq 1$ *and* $\mathfrak{g}, \mathfrak{h} \in C\left([0,T], 1 + T^N\left(\mathbb{R}^e\right)\right)$*. Denote the inhomogeneous norm:*

$$\omega_{p-var,[0,T]}(\mathfrak{g},\mathfrak{h}) := \max_{k=1,2,\dots,N}\left(\sup_D \sum_{j,t_j\in D}\left\|\pi_k\left(\mathfrak{g}_{t_j,t_{j+1}} - \mathfrak{h}_{t_j,t_{j+1}}\right)\right\|_{(\mathbb{R}^e)^{\otimes k}}^{\frac{p}{k}}\right)^{\frac{k}{p}}.$$

Theorem 4.29 (uniqueness of full solution $\gamma > p$) *Suppose* $\boldsymbol{\xi}, \boldsymbol{\eta} \in G^{[p]}\left(\mathbb{R}^e\right)$ *are initial conditions, for* $i = 1, 2$*,* $\mathbf{x}^i \in C^{p-var}\left([0,T], G^{[p]}\left(\mathbb{R}^d\right)\right)$*,* $f^i : \mathbb{R}^e \to L\left(\mathbb{R}^d, \mathbb{R}^e\right)$ *are* $Lip(\gamma)$ *for* $\gamma > p \geq 1$*, and* $\mathbf{y}_t^i$ *are full solutions to the RDEs*

$$\begin{aligned} d\mathbf{y}_t^1 &= f^1\left(y_t^1\right) d\mathbf{x}_t^1, \ \ \mathbf{y}_0^1 = \boldsymbol{\xi}. \\ d\mathbf{y}_t^2 &= f^2\left(y_t^2\right) d\mathbf{x}_t^2, \ \ \mathbf{y}_0^2 = \boldsymbol{\eta}. \end{aligned}$$

Denote $v := \|f^1\|_{Lip(\gamma)} \vee \|f^2\|_{Lip(\gamma)}$ *and* $l_p := \|\mathbf{x}^1\|_{p-var,[0,T]} \vee \|\mathbf{x}^2\|_{p-var,[0,T]}$*. Then* ($\delta$ *is the dilation operator defined at* (4.8)*)*

$$\begin{aligned} &\omega_{p-var,[0,T]}(\mathbf{y}_1, \mathbf{y}_2) \\ \leq\ & Cvl_p\left(\left|\xi^1 - \eta^1\right| + \frac{1}{v}\left\|f^1 - f^2\right\|_{Lip(\gamma-1)} + \omega_{p-var,[0,T]}\left(\delta_{\frac{1}{l_p}}\mathbf{x}^1, \delta_{\frac{1}{l_p}}\mathbf{x}^2\right)\right)\exp\left(Cv^p l_p^p\right), \end{aligned}$$

where C *depending on* p *and* γ*, and* $\xi^1 = \pi_1(\boldsymbol{\xi})$*,* $\eta^1 = \pi_1(\boldsymbol{\eta})$*.*

Which is corollary 10.39 in [9].

Theorem 4.30 (uniqueness $\gamma = p$) *Same as the condition of the above Theorem. Then when* $\gamma = p \geq 1$*, the full RDE solution is still unique, and for any* $\epsilon > 0$*, there exists* $\delta > 0$ *depending on* ϵ*,* p*,* v *and* l_p*, such that*

$$\left|\xi^1 - \eta^1\right| + \left\|f^1 - f^2\right\|_{Lip(p-1)} + d_{\infty-var}\left(\mathbf{x}^1, \mathbf{x}^2\right) < \delta \Rightarrow d_{\infty-var,[0,T]}\left(\mathbf{y}^1, \mathbf{y}^2\right) < \epsilon.$$

Which is Theorem 10.41 in [9]. Theorem 10.41 is actually formulated to incorporate driving noise with finite $\psi_{p,1}$-variation, where $\psi_{p,1}(t) = t^p / (\ln^* \ln^* 1/t)$ with $\ln^* := \max(1, \ln)$. The $\psi_{p,1}$-variation of $\gamma \in C([0,T], (E,d))$ is defined as:

$$\|\gamma\|_{\psi_{p,1}-var} := \inf\left\{M \,\middle|\, \sup_{D\subset[0,T]} \sum_{k,t_k\in D}\psi_{p,1}\left(M^{-1}d\left(\gamma_{t_k}, \gamma_{t_{k+1}}\right)\right) < \infty\right\}.$$

Since enhanced Brownian motion (denoted as $\mathbf{B}$) has finite $\psi_{2,1}$-variation, the RDE $d\mathbf{y} = f(y)\, d\mathbf{B}$ has a unique solution when f is $Lip(2)$.

On the other hand, the condition $\gamma \geq p$ (for uniqueness) and $\gamma > p-1$ (for existence) are necessary. In [6], examples are given that the solution may not unique when $1 < \gamma < p < 2$ and $2 < \gamma < p < 3$, and the solution may not exist when $\gamma = p - 1$, $1 < p < 2$.

Chapter 5

Partial sum process of orthogonal series as rough process

In this chapter, we prove that the partial sum process of orthogonal expansion $\sum_{n\geq 0} c_n u_n$ is a geometric 2-rough process, for any orthonormal system $\{u_n\}_{n\geq 0}$ in L^2 and any sequence of numbers $\{c_n\}$ satisfying $\sum_{n\geq 0} (\log_2(n+1))^2 |c_n|^2 < \infty$. Since being a geometric 2-rough process implies the existence of a limit function up to a null set, this theorem could be treated as an improvement of Menshov-Rademacher theorem. For Fourier series, the condition can be strengthened to $\sum_{n\geq 0} \log_2(n+1) |c_n|^2 < \infty$, which is equivalent to $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|} dudv < \infty$ (with f the limit function).

This chapter mainly follows the content of [23].

5.1 Introduction

Definition 5.1 *$\{u_n\}_{n=0}^{\infty}$ is said to be an orthonormal system in L^2 and denoted as $\{u_n\} \in L^2$, if there exist measure space $(\Omega, \mathcal{F}, \mu)$ and Hilbert space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$, such that $u_n : (\Omega, \mathcal{F}, \mu) \to (\mathcal{V}, \langle\cdot,\cdot\rangle)$, $\forall n \in \mathbb{N}$, and $\int_\Omega \langle u_n(\omega), u_m(\omega)\rangle \mu(d\omega) = \delta_{mn}$, $\forall n, m \in \mathbb{N}$.*

Generally, an orthonormal system is denoted as $\{u_n\} \in L^2(\Omega, \mathcal{F}, \mu; \mathcal{V}, \langle\cdot,\cdot\rangle)$. In the above definition, we use L^2 and omit explicit dependence on spaces, because we want to identify a condition which does not depends on spaces.

Definition 5.2 *Suppose $\{u_n\}_{n=0}^{\infty}$ is an orthonormal system in L^2, and $\{c_n\}_{n=0}^{\infty}$ is a sequence of numbers. Then the partial sum process X of $\sum_{k=0}^{\infty} c_n u_n$ is a continuous process indexed by $[0,\infty)$, got by defining for each $\omega \in \Omega$,*

$$X_n(\omega) := \sum_{k=0}^{n} c_k u_k(\omega), \ \forall n \in \mathbb{N}, \tag{5.1}$$

and interpolating linearly between adjacent integers.

We will identify a condition on $\{c_n\}$, under which X is a rough path with finite 2-variation on the half line, almost everywhere on Ω and for every choice of orthogonal sequence. Since almost everywhere finiteness of 2-variation of partial sum process implies the existence of a limit function upto a null set, our topic has a direct connection with a.e. convergence of general orthonormal series, which dates back to Weyl[35].

Definition 5.3 (Weyl multiplier for property p) *Suppose* $\{w(n)\}_{n=0}^{\infty}$ *is a sequence of positive non-decreasing numbers.* $\{w(n)\}$ *is said to be a Weyl multiplier for property* p*, if* p *holds for all orthogonal series* $\sum_{n=0}^{\infty} c_n u_n$*, for any orthonormal system* $\{u_n\}$ *in* L^2 *and any sequence of numbers* $\{c_n\}$ *satisfying* $\sum_{n=0}^{\infty} w(n)|c_n|^2 < \infty$*.*

Not every orthogonal series with coefficients in l^2 is convergent. There exists an L^2 Fourier series which diverges a.e. after some rearrangement, [39]. In fact, for any complete orthonormal system in $L^2((0,1),\mathbb{R})$, there exists a continuous function, whose expansion diverges unboundedly almost everywhere after some rearrangement, [27]. Moreover, Banach[2] proved that, if we equip $L^2((0,1),\mathbb{R})$ with the metric

$$d(\{u_n\},\{v_n\}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{\|u_n - v_n\|_{L^2}}{1 + \|u_n - v_n\|_{L^2}},\ \|u\|_{L^2} = \left(\int_0^1 u^2(x)\,dx\right)^{\frac{1}{2}}, \tag{5.2}$$

then the set of orthonormal systems, whose expansions of all bounded variation functions diverge unboundedly almost everywhere, is a G_δ and everywhere second category subset of $L^2((0,1),\mathbb{R})$.

The exact Weyl multiplier for almost everywhere convergence of general orthogonal series is found by Menshov[25] and Rademacher[31].

Theorem 5.4 (Menshov-Rademacher) *The orthogonal series* $\sum_{n=0}^{\infty} c_n u_n$ *converges almost everywhere, for any* $\{u_n\}_{n=0}^{\infty} \in L^2$ *and any sequence of numbers* $\{c_n\}_{n=0}^{\infty}$ *satisfying*

$$\sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2 < \infty. \tag{5.3}$$

Furthermore, $(\log_2(n+1))^2$ *in* (5.3) *can not be replaced by* $o\left((\log_2(n+1))^2\right)$*, and there exists an absolute constant* C *such that*

$$\int_\Omega \max_{0\le i\le j<\infty} \|\sum_{n=i}^{j} c_n u_n(\omega)\|^2 \mu(d\omega) \le C \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2. \tag{5.4}$$

Although its estimation is rough using Cauchy–Schwarz inequality (p251[15]), the Weyl multiplier $\{(\log_2(n+1))^2\}$ is exact: For any Weyl multiplier $\{w(n)\}$ satisfying $w(n) = o((\log_2(n+1))^2)$, there exists an a.e. divergent orthogonal series $\sum_n c_n u_n$, whose coefficients satisfy $\sum_n w(n)|c_n|^2 < \infty$ (p254[15]). (The main idea is to glue independent pieces of finite orthogonal sequences together, where each piece provides a constant increment on a sufficiently large set, then almost everywhere divergence follows from Borel-Cantelli lemma.)

Moreover, as a remarkable improvement of the counter-example, Tandori[33] showed that: if the absolute value of c_n is monotone decreasing and $\sum_n (\log_2(n+1))^2 |c_n|^2 = \infty$, then there exists $\{u_n\} \in L^2$ such that $\sum_n c_n u_n$ diverges a.e.. Thus, if the absolute value of $\{c_n\}$ is monotone decreasing, then $\sum_n c_n u_n$ converge almost everywhere for all $\{u_n\} \in L^2$ if and only if $\sum_n (\log_2 (n+1))^2 |c_n|^2 < \infty$.

A recent improvement of Menshov-Rademacher Theorem by A. Lewko and M. Lewko [17] strengthened a.e. finite ∞-variation to a.e. finite 2-variation. They decompose the partial sum process into the sum of two, one of which encodes long range displacement, while the other keeps returning to origin. The power of this decomposition already manifested itself in the proof of Menshov-Rademacher theorem. We will use this decomposition, and show that the partial sum process is a geometric rough process.

For a specific orthonormal system, Weyl multiplier for a.e. convergence can be strengthened, even $w(n) = 1$ for all n. In that case, the orthonormal system is called a convergent system. Among those convergent systems, almost everywhere convergence of L^2 Fourier series came as a deep theorem by Carleson[5]. Hunt[14] extended Carleson's result to L_r, $1 < r < \infty$, and proved:

$$\left(\int_{-\pi}^{\pi} \|X(\theta)\|_{\infty-var}^r \, d\theta\right)^{\frac{1}{r}} \leq C_r \left(\int_{-\pi}^{\pi} |f(\theta)|^r \, d\theta\right)^{\frac{1}{r}}, \tag{5.5}$$

where $X(\theta)$ is the partial sum process of Fourier series of f at θ. Moreover, in a recent paper by Oberlin, Seeger, Tao, Chiele and Wright[26], they proved a p-variation version of Carleson's theorem, which is a deep result and mainly the inequality: when $r > 1$ and $p > \max\{2, r/(r-1)\}$,

$$\left(\int_{-\pi}^{\pi} \|X(\theta)\|_{p-var}^r \, d\theta\right)^{\frac{1}{r}} \leq C_{p,r} \left(\int_{-\pi}^{\pi} |f(\theta)|^r \, d\theta\right)^{\frac{1}{r}}.$$

Thus, the partial sum process of L^2 Fourier series has finite p-variation a.e., for any $p > 2$. As a complement to [26], in [17], the authors proved that $\{\log_2 (n+1)\}$ is a Weyl multiplier for a.e. finite 2-variation of partial sum process of Fourier series.

We strengthen Menshov-Rademacher theorem by identifying $\{(\log_2 (n+1))^2\}$ as the exact Weyl multiplier for the partial sum process to be a geometric 2-rough process, and for Fourier series, the Weyl multiplier can be improved to $\{(\log_2 (n+1))\}$.

5.2 Geometric 2-rough path on $[0, \infty)$

Before proceeding to our proofs, we recall (clarify) the definition of geometric 2-rough path on $[0, \infty)$, following [20] with small modifications. (Rough paths on $[0, \infty)$ is just a reparametrisation of rough paths on $[0, 1]$. See also Chapter 4.)

Notation 5.5 *Denote* $\triangle_{[0,\infty)} := \{(s,t) \,| 0 \leq s \leq t < \infty\}$

In this chapter, intervals instead of points are used in definition of finite partition to simply notations in our proof.

Definition 5.6 $D = \{[t_j, t_{j+1}]\}_{j=0}^{n-1}$ *is said to be a finite partition of* $[0, T]$*, if* $0 = t_0 < t_1 \cdots < t_n = T$*. Denote the set of finite partitions of* $[0, T]$ *as* $D_{[0,T]}$.

Definition 5.7 (p-variation) *Suppose $(\mathcal{V}, \|\cdot\|)$ is a Banach space, $\alpha : \triangle_{[0,\infty)} \to \mathcal{V}$ is jointly continuous satisfying $\alpha(t,t) = 0$, $\forall t \in [0,\infty)$. Denote $\alpha([s,t]) := \alpha(s,t)$. Then for $p \in [1,\infty)$, define the p-variation of α on $[0,\infty)$ as*

$$\|\alpha\|_{p-var,[0,\infty)} := \lim_{n\to\infty} \|\alpha\|_{p-var,[0,n]} := \lim_{n\to\infty} \left(\sup_{\{I_k\}\in D_{[0,n]}} \sum_k \|\alpha(I_k)\|^p \right)^{\frac{1}{p}}.$$

Continuous path γ on $[0,\infty)$ can be treated as a function $\widetilde{\gamma}$ on $\triangle_{[0,\infty)}$ by setting $\widetilde{\gamma}(s,t) := \gamma(t) - \gamma(s)$, $\forall (s,t) \in \triangle_{[0,\infty)}$.

Definition 5.8 *path γ on $[0,\infty)$ is said to be of locally bounded variation if it is of bounded variation on any finite interval.*

Definition 5.9 *Suppose $\gamma_i : [0,\infty) \to (\mathcal{V}, \|\cdot\|)$, $i = 1,2$, are continuous and of locally bounded variation. Then define the area produced by γ_1 and γ_2, $A(\gamma_1,\gamma_2) : \triangle_{[0,\infty)} \to [\mathcal{V},\mathcal{V}]$ by setting*

$$A(\gamma_1,\gamma_2)(s,t) := \frac{1}{2} \iint_{s<u_1<u_2<t} [d\gamma_1(u_1), d\gamma_2(u_2)], \ \forall (s,t) \in \triangle_{[0,\infty)}.$$

One can check that, if there exist $c \in \mathcal{V}$ and $i = 1$ or 2, such that $\gamma_i(t_k) = c$ on a finite sequence of times $0 \le t_0 \le t_1 \le \cdots \le t_n < \infty$, then

$$A(\gamma_1,\gamma_2)(t_0,t_n) = \sum_{k=0}^{n-1} A(\gamma_1,\gamma_2)(t_k,t_{k+1}). \tag{5.6}$$

Definition 5.10 (area process) *Suppose γ defined on $[0,\infty)$ is continuous and of locally bounded variation, then we call $A(\gamma) := A(\gamma,\gamma)$ the area of γ.*

Suppose X is a process indexed by $[0,\infty)$ such that $X(\omega)$ is of locally bounded variation for every ω. Define the area process of X as $(A(X))(\omega) := A(X(\omega))$.

One can verify that, for any $0 \le t_1 \le t_2 \le t_3 < \infty$,

$$A(\gamma)(t_1,t_3) = A(\gamma)(t_1,t_2) + A(\gamma)(t_2,t_3) + \frac{1}{2}[\gamma(t_2)-\gamma(t_1), \gamma(t_3)-\gamma(t_2)], \tag{5.7}$$

which is called multiplicativity of $(\gamma, A(\gamma))$.

(5.6) and (5.7) will be used in our proofs.

Definition 5.11 (geometric 2-rough path) *Suppose $(\mathcal{V}, \|\cdot\|)$ is a Banach space. $\Gamma : \triangle_{[0,\infty)} \to \mathcal{V} \oplus [\mathcal{V},\mathcal{V}] = (\gamma,\alpha)$ is called a geometric 2-rough path, if there exists a sequence of continuous bounded variation paths $\gamma_n : [0,\infty) \to \mathcal{V}$, $n \ge 1$, s.t.*

$$\begin{aligned} &\lim_{n\to\infty} \|\Gamma - (\gamma_n, A(\gamma_n))\|_{G^{(2)}} \\ : &= \lim_{n\to\infty} \left(\|\gamma - \gamma_n\|^2_{2-var,[0,\infty)} + \|\alpha - A(\gamma_n)\|_{1-var,[0,\infty)} \right)^{\frac{1}{2}} = 0. \end{aligned}$$

We call $\|\cdot\|_{G^{(2)}}$ the 2-rough norm. Here we define geometric 2-rough path in term of Lie algebra, which is equivalent to the definition given in Chapter 4.

Definition 5.12 (geometric 2-rough process) *(X, A) is called a geometric 2-rough process if $(X(\omega), A(\omega))$ is a geometric 2-rough path for almost every ω.*

5.3 Main Result

Suppose $\{u_n\}$ is an orthonormal system in L^2 and $\{c_n\}$ a sequence of numbers. Using techniques in rough analysis (e.g. [21], [11], [24]), we proved:

Theorem 5.13 *The partial sum process of* $\sum_n c_n u_n$, *when enhanced by its area process, is a geometric* 2*-rough process (denoted as* $\mathbf{X}$*) for any* $\{u_n\} \in L^2$ *and any* $\{c_n\}$ *satisfying* $\sum_{n=0}^{\infty} (\log_2 (n+1))^2 |c_n|^2 < \infty$. *Moreover,* $\log_2 (n+1)^2$ *can not be replaced by* $o(\log_2 (n+1)^2)$, *and*

$$\int_\Omega \|\mathbf{X}(\omega)\|_{G^{(2)}}^2 \mu(d\omega) \leq 768 \sum_{n=0}^{\infty} (\log_2 (n+1))^2 |c_n|^2 . \tag{5.8}$$

That $(\log_2(n+1))^2$ can not be replaced by $o((\log_2(n+1))^2)$ follows from Menshov-Rademacher theorem. It is an improvement of Menshov-Rademacher Theorem since $\|X(\omega)\|_{\infty-var} \leq \|\mathbf{X}(\omega)\|_{G^{(2)}}$, $\forall \omega$.

Definition 5.14 $\{u_n\} \in L^2$ *is said to have the Hardy property with constant* C, *if for any sequence of numbers* $\{a_n\}_{n=0}^{\infty}$ *satisfying* $\sum_{n=0}^{\infty} |a_n|^2 < \infty$,

$$\int_\Omega \sup_{0 \leq i \leq j < \infty} \left\| \sum_{k=i}^{j} a_k u_k(\omega) \right\|^2 \mu(d\omega) \leq C \left(\sum_{n=0}^{\infty} |a_n|^2 \right).$$

Theorem 5.15 *Suppose* $\{u_n\} \in L^2$ *has the Hardy property with constant* C. *Then, for* $\{c_n\}$ *satisfying* $\sum_n \log_2 (n+1) |c_n|^2 < \infty$, *the partial sum process of* $\sum_n c_n u_n$, *when enhanced by its area process, is a geometric* 2*-rough process (denoted as* $\mathbf{X}$*). Moreover,*

$$\int_\Omega \|\mathbf{X}(\omega)\|_{G^{(2)}}^2 \mu(d\omega) \leq (3580 + 40C) \sum_{n=0}^{\infty} \log_2 (n+1) |c_n|^2 . \tag{5.9}$$

Almost everywhere finiteness of 2-variation of the partial sum process in Theorem 5.13 and Theorem 5.15 are proved in [17]. Thus, since area vanishes if the orthonormal system is one-dimensional, our result is an improvement only in multi-dimensional case.

Corollary 5.16 *Suppose* $(\Omega, \mathcal{F}, P)$ *is a probability space, and* $\xi_n : (\Omega, \mathcal{F}, P) \to (\mathcal{V}, \langle \cdot, \cdot \rangle)$ *is an i.i.d. sequence satisfying* $\mathbb{E}(\xi_n) = 0$ *and* $\mathbb{E} \|\xi_n\|^2 = 1$. *Then Theorem 5.15 applies to* $\{\xi_n\}$.

This corollary follows from Burkholder-Davis-Gundy inequality. Actually, based on [28] and Thm 14.12 in [9], if $\{\xi_n\}$ is an i.i.d. sequence taking value in $\mathbb{R}^d$ with zero mean and unit variance, then for any $p \geq 1$, $p \neq 2$, there exists constant $C(d,p)$ s.t. for any $\{a_n\} \in l^{(p \vee 2)}$, (denote X as the partial sum process of $\sum_{n=0}^{\infty} a_n \xi_n$)

$$\mathbb{E}\left(\|X\|_{p-var,[0,\infty)}^2 \right) \leq C(d,p) \left(\sum_{n=0}^{\infty} |a_n|^2 \right) \vee \left(\sum_{n=0}^{\infty} |a_n|^p \right)^{\frac{2}{p}} . \tag{5.10}$$

However, based on Donsker's theorem, (5.10) can *not* be extended to $p = 2$. Indeed, Qian [30] further proved that, if in addition $\{\xi_n\}$ are of finite $2+\delta$ moment for $\delta > 0$ and denote X_n as the partial sum process of $n^{-\frac{1}{2}} \sum_{k=1}^{n} \xi_k$, then there exists $c > 0$ s.t. $\lim_{n\to\infty} P\left(\|X_n\|_{2-var}^2 \geq c \ln\ln n\right) = 1$. It is not surprising if the Weyl multiplier on the r.h.s. of (5.9) can be improved, with the best possible $\{\ln\ln n\}$ based on Qian's result.

Corollary 5.17 *Theorem 5.15 holds for Fourier system, where* $\log_2(n+1)$ *in* (5.9) *can not be replaced by* $o(\log_2(n+1))$.

This corollary follows from Carleson–Hunt's inequality (5.5) (see also [8]) and Theorem 5.15. The lower bound, as indicated in [26] or [17], can be obtained in the case of de la Vallée-Poussin kernel, or say, Dirichlet kernel.

5.4 Proof of Theorem 5.13 and Theorem 5.15

Definition 5.18 *Denote the set of integers* $\mathbb{N} := \{0, 1, 2, \ldots\}$. *Interval* I *is said to be an integer interval, if* $I = [m, n]$ *for* $m \in \mathbb{N}$, $n \in \mathbb{N}$, $m < n$.

If two intervals only intersect on their boundary, then we abuse the notion and label them as " *disjoint* ".

Since the process we are considering is piecewise linear obtained by interpolating on integers, we assume that all intervals in finite partitions are integer intervals, without decreasing the 2-variation of the path, or 1-variation of the area. In the rest of this section, unless otherwise specified, " I is an interval" means " I is an integer interval".

Definition 5.19 *Interval* I *is called a dyadic interval of level* $n \in \mathbb{N}$, *if* $I = [k2^n, (k+1)2^n]$ *for some* $k \in \mathbb{N}$. *Integer* m *is called a dyadic point of level* $n \in \mathbb{N}$, *if* $m = k2^n$ *for some* $k \in \mathbb{N}$.

Notation 5.20 *For interval* J, *denote the level of biggest dyadic interval in* J *as* $n(J)$, *i.e.* $n(J) = \max\{\text{level of dyadic interval } I | I \subseteq J\}$. *Similarly, for* $P \in \mathbb{N}$, *denote* $N(P) := \max\{n | P = k2^n \text{ for } n \in \mathbb{N},\ k \in \mathbb{N}\}$.

Thus, $2^{n(J)} \leq |J|$, so $n(J) \leq \log_2 |J|$; $N(0) = \infty$; $N(m) \geq 0$, $\forall m \in \mathbb{N}$.

Notation 5.21 *Suppose* J *is a finite interval. Denote* B_J *as the set of dyadic intervals in* J, *i.e.* $B_J := \{I | \text{ interval } I \text{ is dyadic, and } I \subseteq J\}$, *and* $B_J^j := \{I | I \in B_J,\ n(I) = j\}$.

Then two properties of $B_J (B_J^j)$.

(*i*) Suppose $\{I_k\} \in D_J$ (i.e.$\{I_k\}$ is a finite partition of interval J), then $B_{I_{k_1}} \cap B_{I_{k_2}} = \varnothing$ when $k_1 \neq k_2$, and

$$\sqcup_k B_{I_k} \subseteq B_J. \tag{5.11}$$

Similar result holds for B_J^j for any level j:

$$\sqcup_k B_{I_k}^j \subseteq B_J^j. \tag{5.12}$$

Proof. Only prove (5.11); (5.12) is similar. $I_k \subseteq J$ so $B_{I_k} \subseteq B_J$. I_{k_1} and I_{k_2} are disjoint when $k_1 \neq k_2$, so $B_{I_{k_1}} \cap B_{I_{k_2}} = \varnothing$. ■

(ii) Let X be the partial sum process of $\sum_{n=0}^{\infty} c_n u_n$. Then for any interval J,

$$\sum_{I \in B_J} \int_\Omega \|X_\omega(I)\|^2 \mu(d\omega) \leq 2\log_2(|J|+1) \sum_{k,[k-1,k]\subseteq J} |c_k|^2 . \tag{5.13}$$

Proof. Since each $[k-1,k] \subseteq J$ can only be included in one dyadic interval of level j, $0 \leq j \leq n(J)$, so in $\cup_I \{I | I \in B_J\}$ (the union of all dyadic intervals in J), $[k-1,k]$ is counted at most $n(J)+1 \leq \log_2|J|+1 \leq 2\log_2(|J|+1)$ times. While for any interval I,

$$\int_\Omega \|X_\omega(I)\|^2 \mu(d\omega) = \int_\Omega \| \sum_{k,[k-1,k]\subseteq I} c_k u_k(\omega) \|^2 \mu(d\omega) = \sum_{k,[k-1,k]\subseteq I} |c_k|^2 ,$$

so sum over all dyadic intervals I in B_J,

$$\begin{aligned} \sum_{I\in B_J} \int_\Omega \|X_\omega(I)\|^2 \mu(d\omega) &= \sum_{I\in B_J} \sum_{k,[k-1,k]\subseteq I} |c_k|^2 \\ &= \sum_{k,[k-1,k]\subseteq J} \#\{I | [k-1,k] \subseteq I, I \in B_J\} |c_k|^2 \\ &\leq 2\log_2(|J|+1) \sum_{k,[k-1,k]\subseteq J} |c_k|^2 . \end{aligned}$$

■

The following two Lemmas give a method of decomposing an interval as union of dyadic intervals: each time, we cut out biggest dyadic interval available, and the number of dyadic sub-intervals is bounded above by logarithm of the length of the interval. (The decomposition is in the same spirit in Prop 4.1.1. in [21].)

Lemma 5.22 *Suppose J is an interval with one boundary point a level n dyadic point $k2^n$, for some $k \geq 0$, $n \geq 1$, and $|J| < 2^n$. Then, J can be decomposed as union of disjoint dyadic intervals, in such a way that the level of dyadic intervals is strictly monotone with respect to their position in J (strictly increasing when $k2^n$ is the right boundary point of J; strictly decreasing when $k2^n$ is the left boundary point of J).*

Proof. Suppose $k2^n$ is the right boundary point of J (result for left boundary point can be obtained by symmetry). Translate J by $-(k-1)2^n$ (translating J, $|J| < 2^n$, by $j2^n$, $j \in \mathbb{N}$, will not affect our conclusion) and assume $J = [m, 2^n]$ for some $m \geq 1$. We use mathematical induction on n. As a clear consequence of $|J| < 2^n$, any dyadic interval in J is of level strictly less than n.

When $n = 1$, since $|J| < 2^n$, $|J| = 1$. Thus J itself is a 0 level interval.

Suppose the result is true for $n = l$, i.e. for any $1 \leq m < 2^l$, $[m, 2^l]$ can be decomposed as union of dyadic intervals such that their level is strictly increasing with respect to their position in $[m, 2^l]$. Then we will prove that the same statement holds when $n = l+1$.

Suppose $J = \left[m, 2^{l+1}\right]$ for some $1 \leq m < 2^{l+1}$. If $m < 2^l$, then according to inductive hypothesis, there exists a dyadic partition of $\left[m, 2^l\right] = \cup_k I_k$, such that the level of I_k is strictly increasing. In that case, decompose $\left[m, 2^{l+1}\right] = \cup_k I_k \cup \left[2^l, 2^{l+1}\right]$. Since each I_k has level strictly less than l, $\cup_k I_k \cup \left[2^l, 2^{l+1}\right]$ is still strictly increasing in their level. If $m = 2^l$, then J is itself a dyadic interval $\left[2^l, 2^{l+1}\right]$. If $m > 2^l$, then by translation, use inductive hypothesis on $\left[m - 2^l, 2^l\right]$, proof finishes. ■

Lemma 5.23 *Suppose J is an interval, then there exists a decomposition of J as union of disjoint dyadic intervals, in a way that there exists a point P in the dyadic partition, such that $N(P) \geq n(J) + 1$, and to the left and right side of P, the level of dyadic intervals is strictly decreasing. As a result, no more than two dyadic intervals of any given level are included, and the number of dyadic intervals is bounded by $4 \log_2 (|J| + 1)$.*

Proof. Denote $n_0 := n(J)$ (the level of biggest dyadic interval in J). Then there exists at least one dyadic interval of level n_0 in J, and there can be two adjacent ones, but there can not be more than two of them. If there are two n_0 intervals, then they are adjacent, because the gap between n_0 intervals is a union of n_0 intervals, so if there were gap between them, there would be at least three consecutive n_0 intervals in J (because J is an interval, all intermediate intervals are also in J), thus two of them will constitute a level $n_0 + 1$ interval, contradicting with maximal assumption for n_0. Same reasoning applies that one can verify that there are no more than two n_0 intervals in J. In this way, we decompose J as union of three intervals: the interval to the left of n_0 interval(s), denoted as I_l; n_0 interval(s); the interval to the right of n_0 interval(s), denoted as I_r.

We take I_l as example. Suppose I_l is not empty. According to the construction of I_l, it is an interval with right boundary point a n_0 level dyadic point, and $|I_l| < 2^{n_0}$ (otherwise I_l contains an n_0 interval—since it shares a n_0 level dyadic point with the n_0 interval(s)—contradicting with the construction of I_l). Thus according to Lemma 5.22, I_l can be decomposed as union of dyadic intervals, whose level are less than n_0, and strictly increasing with respect to their position in I_l. Same reasoning applies to I_r, i.e. I_r can be decomposed as union of dyadic intervals of strictly decreasing level, with highest level less than n_0. Then we select P as one of the boundary points of n_0 interval(s). If there are two n_0 intervals, then select P as the point which parts them. The reason that these two n_0 intervals do not constitute an $n_0 + 1$ interval is that the point parting them is a dyadic point of level $n_0 + 1$, so $N(P) \geq n(J) + 1$. Otherwise there is only one n_0 interval, denoted as $[k 2^{n_0}, (k+1) 2^{n_0}]$ (there are no more than two n_0 intervals, as we verified). If k is even, let $P = k 2^{n_0}$; if k is odd, let $P = (k+1) 2^{n_0}$. Then $2^{n_0+1} | P$, $N(P) \geq n(J) + 1$. In this way, J is decomposed as disjoint union of dyadic intervals, s.t. there exists point $P \in J$ satisfying $N(P) \geq n(J) + 1$, and the level of dyadic intervals is strictly decreasing from P to left and right. As a result, no more than two dyadic intervals of any given level are included. Since $2^{n_0} \leq |J|$, the number of dyadic intervals is bounded by $2n_0 + 2 \leq 2 \log_2 |J| + 2 \leq 4 \log_2 (|J| + 1)$. ■

Remark 5.24 *Since the level of dyadic intervals to the left/right of P is strictly decreasing and $\sum_{k=0}^{n-1} 2^k = 2^n - 1$ (thus one can not get new dyadic interval through union), one* *can check that, the decomposition will be the same, if one repeatedly cut out the biggest dyadic interval available in J.*

Definition 5.25 *Suppose $\gamma : [0,\infty) \to (\mathcal{V}, \|\cdot\|)$ is continuous, and $\{t_n\}_{n=0}^{\infty}$ is a sequence of strictly increasing positive real numbers satisfying $\lim_{n\to\infty} t_n = +\infty$. Then we say γ^1 is the piecewisely linear path which coincides with γ on $\{t_n\}$ and denote $\gamma^1 = L(\gamma, \{t_n\})$, if $\gamma^1(t) = \gamma(t_0)$ on $t \in [0, t_0]$, and for any $n \in \mathbb{N}$,*

$$\gamma^1(t) = \frac{t_{n+1}-t}{t_{n+1}-t_n}\gamma(t_n) + \frac{t-t_n}{t_{n+1}-t_n}\gamma(t_{n+1}), t \in [t_n, t_{n+1}].$$

Suppose X is a continuous process indexed by $[0,\infty)$. Then we say X^1 is the piecewisely linear process which coincides with X on $\{t_n\}$ and denoted as $X^1 = L(X, \{t_n\})$, if for any $\omega \in \Omega$, $X^1(\omega) = L(X(\omega), \{t_n\})$.

Lemma 5.26 *Suppose $\gamma : [0,\infty) \to (\mathcal{V}, \|\cdot\|)$ is a continuous path, and $\{t_n\}_{n=0}^{\infty}$ is a sequence of strictly increasing positive numbers satisfying $\lim_{n\to\infty} t_n = +\infty$. Let $\gamma^1 = L(\gamma, \{t_n\})$, then*

$$\|\gamma\|_{2-var}^2 \le 3\left(\|\gamma\|_{2-var,[0,t_0]}^2 + \sum_{n=0}^{\infty} \|\gamma\|_{2-var,[t_n,t_{n+1}]}^2 + \left\|\gamma^1\right\|_{2-var}^2\right),$$

Proof. For any finite interval $[s,t] \subset [0,\infty)$, if there exists $k \le l$, s.t. $s < t_k \le t_l < t$. Then

$$\|\gamma(s,t)\|^2 \le 3\left(\|\gamma(s,t_k)\|^2 + \|\gamma(t_k,t_l)\|^2 + \|\gamma(t_l,t)\|^2\right). \tag{5.14}$$

In this way, we divide $[s,t]$ into $[s,t_k] \cup [t_k,t_l] \cup [t_l,t]$. Therefore, for any $N \ge 1$ and any fixed finite partition $\{[s_k, s_{k+1}]\}_k \in D_{[0,t_N]}$, apply (5.14) to each $[s_k, s_{k+1}]$, sum over k,

$$\sum_k \|\gamma(s_k, s_{k+1})\|^2 \le 3\left(\|\gamma\|_{2-var,[0,t_0]}^2 + \sum_{n=0}^{N-1} \|\gamma\|_{2-var,[t_n,t_{n+1}]}^2 + \left\|\gamma^1\right\|_{2-var,[0,t_N]}^2\right).$$

Take supremum over all possible finite partitions of $[0,t_N]$, and let N tends to infinity. ■

Lemma 5.27 *Suppose $\gamma, \gamma^2 : [0,\infty) \to (\mathcal{V}, \|\cdot\|)$ are of locally bounded variation, and there exist $c \in \mathcal{V}$ and a sequence of positive strictly increasing numbers $\{t_n\}_{n=0}^{\infty}$ satisfying $\lim_{n\to\infty} t_n = +\infty$, such that $\gamma^2(t_n) = c$, $\forall n \in \mathbb{N}$. Then if denote $\gamma^1 := \gamma - \gamma^2$, and $A := A(\gamma)$, $A^{11} := A(\gamma^1)$, we have*

$$\begin{aligned}\|A\|_{1-var} &\le \|A\|_{1-var,[0,t_0]} + 2\|\gamma\|_{2-var}^2 \\ &\quad + 2\sum_{n=0}^{\infty} \|A\|_{1-var,[t_n,t_{n+1}]} + 2\sup_{\{n_k\}} \sum_{n_k < n_{k+1}} \left\|A^{11}\left(t_{n_k}, t_{n_{k+1}}\right)\right\|.\end{aligned}$$

Proof. We estimate $\|A\|_{1-var,[0,t_N]}$, $N \geq 1$, by inserting partition points $\{t_n\}_{n=0}^{N-1}$ into any finite partition of $[0,t_N]$. Fix $\{[s_k,s_{k+1}]\} \in D_{[0,t_N]}$. If there exist integers n_1, n_2, such that $s_k < t_{n_1} \leq t_{n_2} < s_{k+1}$. Then using multiplicativity,

$$\begin{aligned}
& \|A(s_k,s_{k+1})\| \\
\leq\ & \|A(s_k,t_{n_1})\| + \|A(t_{n_1},t_{n_2})\| + \|A(t_{n_2},s_{k+1})\| \\
& \quad +2\|\gamma(s_k,t_{n_1})\|^2 + 2\|\gamma(t_{n_1},t_{n_2})\|^2 + 2\|\gamma(t_{n_2},s_{k+1})\|^2 \\
\leq\ & \|A(s_k,t_{n_1})\| + \|A(t_{n_1},t_{n_2})\| + \|A(t_{n_2},s_{k+1})\| + 2\|\gamma\|^2_{2-var,[s_k,s_{k+1}]}.
\end{aligned}$$

In this way, we cut out those big intervals $[t_{n_1},t_{n_2}]$, $n_1 < n_2$, which have no partition points, and divide $[s_k,s_{k+1}]$ which contains one t_n into $[s_k,t_n]\cup[t_n,s_{k+1}]$. Therefore, apart from intervals in the form $[t_{n_1},t_{n_2}]$, we have three kinds of intervals: $[s_{k_1},t_{n_1}]$, $[t_{n_2},s_{k_1+1}]$ and $[s_{k_2},s_{k_2+1}]$, each of which is included in some $[t_n,t_{n+1}]$. Thus,

$$\begin{aligned}
\|A\|_{1-var} \leq\ & \|A\|_{1-var,[0,t_0]} + \sum_{n=0}^{\infty}\|A\|_{1-var,[t_n,t_{n+1}]} + 2\|\gamma\|^2_{2-var} \\
& + \sup_{\{n_k\}}\sum_{n_k<n_{k+1}}\left\|A\left(t_{n_k},t_{n_{k+1}}\right)\right\|.
\end{aligned} \tag{5.15}$$

Denote $A^{11,c} := A - A^{11}$. Since $\gamma^2(t_n) = c$, $\forall n \in \mathbb{N}$, using identity (5.6),

$$\left\|A^{11,c}\left(t_{n_k},t_{n_{k+1}}\right)\right\| \leq \sum_{k=n_k}^{n_{k+1}-1}\left\|A^{11,c}(t_k,t_{k+1})\right\|.$$

$$\begin{aligned}
\text{Hence, } \sup_{\{n_k\}}\sum_{n_k<n_{k+1}}\left\|A^{11,c}\left(t_{n_k},t_{n_{k+1}}\right)\right\| \leq\ & \sum_{n=0}^{\infty}\left\|A^{11,c}(t_n,t_{n+1})\right\| \\
\leq\ & \sum_{n=0}^{\infty}\|A(t_n,t_{n+1})\| + \sum_{n=0}^{\infty}\left\|A^{11}(t_n,t_{n+1})\right\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \sup_{\{n_k\}}\sum_{n_k<n_{k+1}}\left\|A\left(t_{n_k},t_{n_{k+1}}\right)\right\| \\
\leq\ & \sup_{\{n_k\}}\sum_{n_k<n_{k+1}}\left\|A^{11,c}\left(t_{n_k},t_{n_{k+1}}\right)\right\| + \sup_{\{n_k\}}\sum_{n_k<n_{k+1}}\left\|A^{11}\left(t_{n_k},t_{n_{k+1}}\right)\right\| \\
\leq\ & \sum_{n=0}^{\infty}\|A(t_n,t_{n+1})\| + \sum_{n=0}^{\infty}\left\|A^{11}(t_n,t_{n+1})\right\| + \sup_{\{n_k\}}\sum_{n_k<n_{k+1}}\left\|A^{11}\left(t_{n_k},t_{n_{k+1}}\right)\right\| \\
\leq\ & \sum_{n=0}^{\infty}\|A\|_{1-var,[t_n,t_{n+1}]} + 2\sup_{\{n_k\}}\sum_{n_k<n_{k+1}}\left\|A^{11}\left(t_{n_k},t_{n_{k+1}}\right)\right\|.
\end{aligned}$$

Combined with (5.15), proof finishes. ■

Corollary 5.28 *Suppose X is a process defined on $(\Omega,\mathcal{F},\mu)$ taking value in $(\mathcal{V},\|\cdot\|)$, and indexed by $[0,\infty)$. Further assume that X_ω is of locally bounded variation for every ω, with* *area process $A := A(X)$, and $X^1 = L(X, \{t_n\})$ (Definition 5.25) with $A^{11} := A(X^1)$. Then if (X^1, A^{11}) is a geometric 2-rough process, we have (X, A) is a geometric 2-rough process provided*

$$\sum_{n=0}^{\infty} \left(\|X_\omega\|^2_{2-var,[t_n,t_{n+1}]} + \|A_\omega\|_{1-var,[t_n,t_{n+1}]} \right) < \infty, \text{ a.e..}$$

Moreover,

$$\begin{aligned} \int_\Omega \|X_\omega\|^2_{2-var} \mu(d\omega) &\leq 3\int_\Omega \|X_\omega\|^2_{2-var,[0,t_0]} \mu(d\omega) \\ &\quad +3\sum_{n=0}^{\infty} \int_\Omega \|X_\omega\|^2_{2-var,[t_n,t_{n+1}]} \mu(d\omega) + 3\int_\Omega \left\|X^1_\omega\right\|^2_{2-var} \mu(d\omega), \end{aligned} \tag{5.16}$$

$$\begin{aligned} \int_\Omega \|A_\omega\|_{1-var} \mu(d\omega) &\leq \int_\Omega \|A_\omega\|_{1-var,[0,t_0]} \mu(d\omega) + 2\int_\Omega \|X_\omega\|^2_{2-var} \mu(d\omega) \\ &\quad +2\sum_{n=0}^{\infty} \int_\Omega \|A_\omega\|_{1-var,[t_n,t_{n+1}]} \mu(d\omega) + 2\int_\Omega \left\|A^{11}_\omega\right\|_{1-var} \mu(d\omega). \end{aligned} \tag{5.17}$$

Proof. Since X_ω is of locally bounded variation for every ω, thus if (X_ω, A_ω) has finite 2-rough norm a.e., (X, A) is a geometric 2-rough process. $X^1 = L(X, \{t_n\})$, so for each $\omega \in \Omega$, $(X_\omega - X^1_\omega)(t_n) = 0$, $\forall n \in \mathbb{N}$, and the rest of this corollary follows from Lemma 5.26 and Lemma 5.27. ■

The following Lemma works in the same spirit as the Lemma used in the proof of Menshov-Rademacher theorem, but replace ∞-variation by 2-variation.

Lemma 5.29 *Suppose X is the partial sum process of $\sum_{k=0}^{n} c_n u_n$, then,*

$$\int_\Omega \|X_\omega\|^2_{2-var,[0,n]} \mu(d\omega) \leq 8(\log_2(n+1))^2 \sum_{k=1}^{n} |c_k|^2.$$

Proof. Suppose interval $J \subseteq [0, n]$. By Lemma 5.23, decompose J as union of disjoint dyadic intervals, denote them as I_k, $1 \leq k \leq l$, with $l \leq 4\log_2(|J|+1)$. B_J is the set of dyadic intervals included in J (Notation 5.21). I_k are disjoint as k varies since $\{I_k\}$ is a finite partition of J, and each I_k is a member of B_J, so $\sum_{k=1}^{l} \|X_\omega(I_k)\|^2 \leq \sum_{I\in B_J} \|X_\omega(I)\|^2$ for each $\omega \in \Omega$. Then using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \|X_\omega(J)\|^2 &= \|\sum_{k=1}^{l} X_\omega(I_k)\|^2 \leq l\sum_{k=1}^{l} \|X_\omega(I_k)\|^2 \\ &\leq 4\log_2(|J|+1)\sum_{k=1}^{l} \|X_\omega(I_k)\|^2 \leq 4\log_2(n+1)\sum_{I\in B_J} \|X_\omega(I)\|^2. \end{aligned} \tag{5.18}$$

Suppose $\{I_j\} \in D_{[0,n]}$ (the set of finite partitions of $[0, n]$). Use (5.18) for each I_j, and $\sqcup_j B_{I_j} \subseteq B_{[0,n]}$ (according to (5.11)),

$$\|X_\omega\|^2_{2-var,[0,n]} = \sup_{\{I_j\}\in D_{[0,n]}} \sum_j \|X_\omega(I_j)\|^2 \leq 4\log_2(n+1) \sum_{I\in B_{[0,n]}} \|X_\omega(I)\|^2.$$

Integrate both sides, and use property at (5.13), i.e. $\sum_{I\in B_{[0,n]}}\int_{\Omega}\|X_{\omega}(I)\|^{2}\mu(d\omega)\leq 2\log_{2}(n+1)\sum_{k=1}^{n}|c_{k}|^{2}$, we get

$$\int_{\Omega}\|X_{\omega}\|_{2-var,[0,n]}^{2}\mu(d\omega)\leq 8(\log_{2}(n+1))^{2}\sum_{k=1}^{n}|c_{k}|^{2}.$$

■

This inequality is interesting when taking into account that: (p255[15]) there exists $c_0>0$ such that, for any $n\geq 1$ there exists an orthonormal sequence $\{\varphi_k\}_{k=1}^{n}$ on $(0,1)$, s.t. the partial sum process X^n of $\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\varphi_k$ satisfies

$$P\left(\|X^{n}\|_{\infty-var}\geq c_{0}\log_{2}n\right)\geq\frac{1}{4}.$$

The following result is proved in [17], we put it here for completeness.

Lemma 5.30 *The partial sum process of $\sum_n c_n u_n$ (denoted as X) is of finite 2-variation a.e. for any orthonormal system $\{u_n\}$ in L^2 and any sequence of numbers $\{c_n\}$ satisfying $\sum_n(\log_2(n+1))^2|c_n|^2<\infty$. Moreover, $(\log_2(n+1))^2$ can not be replaced by $o\left((\log_2(n+1))^2\right)$ and*

$$\int_{\Omega}\|X_{\omega}\|_{2-var}^{2}\mu(d\omega)\leq 36\sum_{n=0}^{\infty}(\log_{2}(n+1))^{2}|c_{n}|^{2}. \tag{5.19}$$

Proof. Since $\|X_\omega\|_{\infty-var}\leq\|X_\omega\|_{2-var}$, $\forall\omega\in\Omega$, based on Menshov-Rademacher Theorem, we only have to prove (5.19). Define process $X^1:=L(X,\{2^n\})$, then according to (5.16) in Corollary 5.28,

$$\begin{aligned}\int_{\Omega}\|X_{\omega}\|_{2-var}^{2}\mu(d\omega) &\leq 3\int_{\Omega}\|X_{\omega}\|_{2-var,[0,1]}^{2}\mu(d\omega) \\ &\quad+3\sum_{n=0}^{\infty}\int_{\Omega}\|X_{\omega}\|_{2-var,[2^{n},2^{n+1}]}^{2}\mu(d\omega)+3\int_{\Omega}\left\|X_{\omega}^{1}\right\|_{2-var}^{2}\mu(d\omega).\end{aligned} \tag{5.20}$$

While if denote f as the limit function (according to Menshov-Rademacher theorem, $f(\omega)=\lim_{n\to\infty}X_\omega(n)$ exists a.e., set $f(\omega)=0$ elsewhere), we have

$$\begin{aligned}\int_{\Omega}\left\|X_{\omega}^{1}\right\|_{2-var}^{2}\mu(d\omega) &= \int_{\Omega}\sup_{\{m_k\}}\sum_{k}\left\|X_{\omega}(2^{m_{k+1}})-X_{\omega}(2^{m_k})\right\|^{2}\mu(d\omega) \\ &\leq 2\int_{\Omega}\sum_{n=0}^{\infty}\|X_{\omega}(2^{n})-f(\omega)\|^{2}\mu(d\omega)=2\sum_{n=0}^{\infty}\sum_{k\geq 2^{n}+1}|c_{k}|^{2} \\ &\leq 4\sum_{n=2}^{\infty}(\log_{2}(n+1))^{2}|c_{n}|^{2}.\end{aligned}$$

Combined with Lemma 5.29 for estimation of $\|X_\omega\|_{2-var,[2^n,2^{n+1}]}^2$, $n\geq 0$, and (5.20). ■

We will use Lemma 5.30 in the proof of Theorem 5.13.

Proof of Theorem 5.13. Denote the partial sum process of $\sum_{n=0}^{\infty} c_n u_n$ as X, and $A := A(X)$ as the area process of X. Since $\|X_\omega\|_{\infty-var} \leq \|X_\omega\|_{2-var}$, $\forall \omega \in \Omega$, so based on Menshov-Rademacher Theorem(in p67), we only need to prove $\int_\Omega \|X_\omega\|_{2-var}^2 + \|A_\omega\|_{1-var} \mu(d\omega) \leq 768 \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2$. While $\int_\Omega \|X_\omega\|_{2-var}^2 \mu(d\omega)$ is done in Lemma 5.30, so we concentrate on 1-variation of area. Denote $X^1 := L(X, \{2^n\})$ (Definition 5.25) and $A^{11} := A(X^1)$, then use (5.17) in Corollary 5.28:

$$\begin{aligned} \int_\Omega \|A_\omega\|_{1-var} \mu(d\omega) &\leq 2\int_\Omega \|X_\omega\|_{2-var}^2 \mu(d\omega) + \int_\Omega \|A_\omega\|_{1-var,[0,1]} \mu(d\omega) \\ &\quad + 2\sum_{l=0}^{\infty} \int_\Omega \|A_\omega\|_{1-var,[2^l,2^{l+1}]} \mu(d\omega) + 2\int_\Omega \|A_\omega^{11}\|_{1-var} \mu(d\omega). \end{aligned} \tag{5.21}$$

Since X_ω is linear on $[0,1]$ and $[1,2]$, $\|A_\omega\|_{1-var,[0,1]} = \|A_\omega\|_{1-var,[1,2]} = 0$, $\forall \omega \in \Omega$. Thus, we are done if we can prove

$$\int_\Omega \|A_\omega\|_{1-var,[2^l,2^{l+1}]} \mu(d\omega) \leq 10 \sum_{n=2^l+1}^{2^{l+1}} (\log_2(n+1))^2 |c_n|^2, \forall l \geq 1 \tag{5.22}$$

and

$$\int_\Omega \|A_\omega^{11}\|_{1-var} \mu(d\omega) \leq 32\pi^2 \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2. \tag{5.23}$$

($\pi^2 \leq 10$, $2 \times 36 + 2 \times 10 + 2 \times 32\pi^2 \leq 732$, $732 + 36 = 768$.)

In the following, we do analysis for fixed $\omega \in \Omega$.

Using multiplicativity of (X_ω, A_ω) (identity (5.7) at p69), for any finite interval I and any disjoint decomposition $\{I_1, I_2\} \in D_I$, we have

$$\|A_\omega(I)\| \leq \|A_\omega(I_1)\| + \|A_\omega(I_2)\| + \|X_\omega(I_1)\| \|X_\omega(I_2)\|.$$

Therefore, for A_ω on dyadic interval $I = [m2^n, (m+1)2^{n+1}]$, repeatedly bisecting I down to 0 level dyadic intervals on which X_ω is linear and the area vanishes, we get (B_I is the set of dyadic intervals included in I, Notation 5.21),

$$\begin{aligned} \|A_\omega(I)\| &= \left\|A_\omega\left(m2^n, (m+1)2^{n+1}\right)\right\| \\ &\leq \sum_{j=0}^{n-1} \sum_{k=0}^{2^{n-j}-1} \left\|X_\omega\left(\left[m2^n + k2^j, m2^n + (k+1)2^j\right]\right)\right\|^2 \\ &\leq \sum_{I' \in B_I \setminus \{I\}} \|X_\omega(I')\|^2. \end{aligned} \tag{5.24}$$

This estimation of A_ω on dyadic intervals will be used repeatedly.

For interval J which is not dyadic, decomposing it as union of dyadic intervals $\{I_k\}_{k=1}^{l}$ by Lemma 5.23, then $l \leq 4\log_2(|J|+1)$. We estimate $A_\omega(J)$ by successively removing dyadic partition points from J. Suppose $\{I_k\}$ are numbered that $k \mapsto I_k$ is increasing from left to right of J, then the accumulated error incurred to $\|A_\omega(J)\|$ from removing point between I_k and $\cup_{j=k+1}^{l} I_j$, $1 \leq k \leq l-1$, is bounded by

$$\begin{aligned}\sum_{k=1}^{l-1} \left\| X_\omega \left(I_k \right) \right\| \left\| X_\omega \left(\cup_{j=k+1}^{l} I_j \right) \right\| &\leq \sum_{k=1}^{l-1} \left(l-k \right) \left\| X_\omega \left(I_k \right) \right\|^2 \qquad (5.25) \\ + \sum_{k=1}^{l-1} \sum_{j=k+1}^{l} \left\| X_\omega \left(I_j \right) \right\|^2 \leq l \sum_{k=1}^{l} \left\| X_\omega \left(I_k \right) \right\|^2 &\leq 4 \log_2 \left(\left| J \right| + 1 \right) \sum_{k=1}^{l} \left\| X_\omega \left(I_k \right) \right\|^2 .\end{aligned}$$

After removing all dyadic partition points from J, we are left with area on I_k, $1 \leq k \leq l$, so

$$\left\| A_\omega \left(J \right) \right\| \leq \sum_{k=1}^{l} \left\| A_\omega \left(I_k \right) \right\| + 4 \log_2 \left(\left| J \right| + 1 \right) \sum_{k=1}^{l} \left\| X_\omega \left(I_k \right) \right\|^2 .$$

While apply (5.24) to each I_k, and use $\sqcup_{k=1}^{l} \left\{ I_k \right\} \subseteq \sqcup_{k=1}^{l} B_{I_k} \subseteq B_J$ (since I_k are dyadic and $\left\{ I_k \right\}_{k=1}^{l}$ is a finite partition of J, use (5.11)),

$$\sum_{k=1}^{l} \left\| A_\omega \left(I_k \right) \right\| \leq \sum_{k=1}^{l} \sum_{I \in B_{I_k}} \left\| X_\omega \left(I \right) \right\|^2 \leq \sum_{I \in B_J} \left\| X_\omega \left(I \right) \right\|^2 .$$

$$\begin{aligned}\text{Thus, } \left\| A_\omega \left(J \right) \right\| &\leq \sum_{I \in B_J} \left\| X_\omega \left(I \right) \right\|^2 + 4 \log_2 \left(\left| J \right| + 1 \right) \sum_{I \in B_J} \left\| X_\omega \left(I \right) \right\|^2 \qquad (5.26) \\ &\leq 5 \log_2 \left(\left| J \right| + 1 \right) \sum_{I \in B_J} \left\| X_\omega \left(I \right) \right\|^2 .\end{aligned}$$

Therefore, suppose $\left\{ I_j \right\} \in D_{\left[2^l, 2^{l+1} \right]}$, $l \geq 1$, use (5.26) for each I_j, and $\sqcup_j B_{I_j} \subseteq B_{\left[2^l, 2^{l+1} \right]}$,

$$\sum_j \left\| A_\omega \left(I_j \right) \right\| \leq \sum_j 5 \log_2 \left(\left| I_j \right| + 1 \right) \sum_{I \in B_{I_j}} \left\| X_\omega \left(I \right) \right\|^2 \leq 5 \log_2 \left(2^l + 1 \right) \sum_{I \in B_{\left[2^l, 2^{l+1} \right]}} \left\| X_\omega \left(I \right) \right\|^2 .$$

Taking supremum over all finite partitions,

$$\left\| A_\omega \right\|_{1-var, \left[2^l, 2^{l+1} \right]} = \sup_{\left\{ I_j \right\} \in D_{\left[2^l, 2^{l+1} \right]}} \sum_j \left\| A_\omega \left(I_j \right) \right\| \leq 5 \log_2 \left(2^l + 1 \right) \sum_{I \in B_{\left[2^l, 2^{l+1} \right]}} \left\| X_\omega \left(I \right) \right\|^2 .$$

Integrate both sides, use (5.13), i.e.

$$\sum_{I \in B_{\left[2^l, 2^{l+1} \right]}} \int_\Omega \left\| X_\omega \left(I \right) \right\|^2 \mu \left(d\omega \right) \leq 2 \log_2 \left(2^l + 1 \right) \sum_{k=2^l+1}^{2^{l+1}} \left| c_k \right|^2 ,$$

and $\log_2 \left(2^l + 1 \right) \leq \log_2 \left(k+1 \right)$ when $k \in \left[2^l, 2^{l+1} \right]$, we get

$$\int_\Omega \left\| A_\omega \right\|_{1-var, \left[2^l, 2^{l+1} \right]} \mu \left(d\omega \right) \leq 10 \sum_{k=2^l+1}^{2^{l+1}} \left(\log_2 \left(k+1 \right) \right)^2 \left| c_k \right|^2 . \qquad (5.27)$$

Then, what left is the estimation of the long-time behavior, i.e.(5.23) about $A^{11}_{\omega} := A\left(X^1_{\omega}\right)$. Since $X^1 := L\left(X, \{2^n\}\right)$, if denote

$$v_n\left(\omega\right) = \sum_{k=2^n+1}^{2^{n+1}} \frac{c_k u_k\left(\omega\right)}{\sqrt{\sum_{k=2^n+1}^{2^{n+1}} \left|c_k\right|^2}} \text{ and } b_n = \sqrt{\sum_{k=2^n+1}^{2^{n+1}} \left|c_k\right|^2}, \tag{5.28}$$

then $\{v_n\}$ is an orthonormal system in L^2, and X^1 is the reparametrised partial sum process of $\sum_n b_n v_n$.

Notation: Denote the partial sum process of $\sum_n b_n v_n$ as Z, $A^Z := A\left(Z\right)$.

Since 1-variation is invariant under reparametrisation and

$$\sum_{n=1}^{\infty} n^2 \left|b_n\right|^2 \leq \sum_{n=0}^{\infty} \left(\log_2\left(n+1\right)\right)^2 \left|c_n\right|^2,$$

so our aim is to prove $\int_{\Omega} \left\|A^Z_{\omega}\right\|_{1-var} \mu\left(d\omega\right) \leq 32\pi^2 \sum_{n=1}^{\infty} n^2 \left|b_n\right|^2$, which is actually a modest version of our theorem.

Suppose J is an interval, we decompose J as union of dyadic intervals by Lemma 5.23 with small modifications: if $[0, 2^n]$, $n \geq 1$, appear according to Lemma 5.23, decompose it as $[0,1] \cup \cup_{k\geq 1}^{n} \left[2^{k-1}, 2^k\right]$, which does not affect the statement that there are no more than two dyadic intervals of the same level in our decomposition for any level $n \geq 1$, but it does affect level 0 with one more possible copy. The reason is that, if the interval $[0, 2^n]$ does appear in our decomposition, then it must be at the left most of some interval in the finite partition. According to Lemma 5.23, in our decomposition, the level of dyadic intervals is strictly decreasing from biggest dyadic interval(s) to left/right. Therefore, since $[0, 2^n]$ is at the left most, replacing $[0, 2^n]$ by $\cup_{k\geq 1}^{n} \left[2^{k-1}, 2^k\right] = [1, 2^n]$ does not affect the monotonicity of dyadic intervals (the level of $\left[2^{k-1}, 2^k\right]$, $1 \leq k \leq n$, is strictly increasing and less than n), thus does not affect the fact that there are no more than two dyadic intervals of any given level. However, we are left with $[0,1]$, so one more possible level 0 interval.

Having fixed the dyadic partition, we estimate $\left\|A^Z_{\omega}\left(J\right)\right\|$ for interval J by systematically removing dyadic partition points from J. We will first remove the point which define the biggest dyadic subinterval, and kill subintervals level by level, from high to low. We will not touch end points of dyadic intervals of certain level until all the dyadic intervals of higher levels disappear. In this way, the bigger dyadic subintervals merge into one, and absorb the smaller intervals as the process goes.

Denote the dyadic partition of J as $\{I_k\}_{k=1}^{l}$ (as we specified, $[0, 2^n]$, $n \geq 1$, is not a member of $\{I_k\}$).

For fixed k, $1 \leq k \leq l$. If we remove one point which parts our merged big interval J_k and dyadic interval I_k, then the change of area is bounded by

$$\left\|Z_{\omega}\left(J_k\right)\right\| \left\|Z_{\omega}\left(I_k\right)\right\| \leq \frac{1}{2}\left(\left\|Z_{\omega}\left(J_k\right)\right\|^2 + \left\|Z_{\omega}\left(I_k\right)\right\|^2\right). \tag{5.29}$$

Denote the level of I_k as $n(I_k)$, then according to the way we remove points, $J_k \subseteq \cup_i \{I_i | I_i \in \{I_k\}, n(I_i) \geq n(I_k)\}$. Use Hölder inequality,

$$\|Z_\omega(J_k)\|^2 \leq 3 \left(\sum_{j \geq n(I_k)} \frac{1}{(j+1)^2} \right) \left(\sum_{j \geq n(I_k)} (j+1)^2 \sum_{n(I_i)=j} \|Z_\omega(I_i)\|^2 \right),$$

When we removed all dyadic partition points from J, $I_i \in \{I_k\}$ is counted in $\cup_{k=1}^{l} J_k$ at most $\#\{j | I_j \in \{I_k\}, n(I_j) \leq n(I_i)\} \leq 2n(I_i) + 3$ times, where we used the fact that there are no more than two intervals of level n for any $n \geq 1$, and no more than three intervals of level 0. Therefore

$$\begin{aligned} & \sum_{k=1}^{l} \|Z_\omega(J_k)\|^2 && (5.30) \\ \leq\ & 3 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \left(\sum_{j \geq 0} (j+1)^2 \sum_{n(I_i)=j} \#\{k | n(k) \leq n(I_i)\} \|Z_\omega(I_i)\|^2 \right) \\ \leq\ & \frac{\pi^2}{2} \sum_{j \geq 0} (j+1)^2 (2j+3) \sum_{n(I_i)=j} \|Z_\omega(I_i)\|^2 . \end{aligned}$$

After removing all the partition points, combining the estimation of the A_ω^Z on dyadic interval I_k (i.e.(5.24)), the error produced by removing points from dyadic partition (i.e.(5.29) and (5.30)), we get

$$\begin{aligned} \left\|A_\omega^Z(J)\right\| \leq\ & \sum_{k=1}^{l} \left\|A_\omega^Z(I_k)\right\| + \frac{1}{2} \sum_{k=1}^{l} \|Z_\omega(I_k)\|^2 + \frac{1}{2} \sum_{k=1}^{l} \|Z_\omega(J_k)\|^2 && (5.31) \\ \leq\ & \sum_{k=1}^{l} \sum_{I \in B_{I_k} \setminus \{I_k\}} \|Z_\omega(I)\|^2 + \sum_{k=1}^{l} \|Z_\omega(I_k)\|^2 \\ & + \frac{\pi^2}{4} \sum_{j=0}^{\infty} (j+1)^2 (2j+3) \sum_{n(I_i)=j} \|Z_\omega(I_i)\|^2 \\ =\ & \sum_{k=1}^{l} \sum_{I \in B_{I_k}} \|Z_\omega(I)\|^2 + \frac{\pi^2}{2} \sum_{j=0}^{\infty} (j+1)^2 \left(j + \frac{3}{2} \right) \sum_{n(I_k)=j} \|Z_\omega(I_k)\|^2 , \end{aligned}$$

where $\{I_k\}$ is are dyadic, so $\sqcup_k (B_{I_k} \setminus \{I_k\}) \sqcup_k \{I_k\} = \sqcup_k B_{I_k}$.

Recall B_J^j is the set of level j dyadic intervals included in J as defined in Notation 5.21 in p71, with property (5.12): If $\{I_k\}$ is a finite partition of J, then for any level $j \geq 0$, $\sqcup_k B_{I_k}^j \subseteq B_J^j$. Moreover, if I_k are all dyadic, then

$$\{I_k | n(I_k) = j\} \subseteq \sqcup_k B_{I_k}^j \subseteq B_J^j. \tag{5.32}$$

As we modified the dyadic partition of J, $\{I_k\}$ does not include $[0, 2^j]$, $j \geq 1$. Thus, $\sqcup_k B_{I_k}$ does not include $[0, 2^j]$, $j \geq 1$ (otherwise if $[0, 2^j]$, $j \geq 1$, is included in B_{I_k}, so $[0,2^j] \subseteq I_k = [0,2^{j'}]$ for some $j' \geq 1$ (since I_k is dyadic), contradictory with our modification). Then in addition to (5.32), we have

$$\{I_k | n(I_k) = j\} \subseteq \sqcup_k B^j_{I_k} \subseteq B^j_J \backslash \{[0,2^j]\} \text{ when } j \geq 1.$$

$$\text{Hence, } \sum_{k=1}^{l} \sum_{I \in B_{I_k}} \|Z_\omega(I)\|^2 \leq \sum_{I \in B^0_J} \|Z_\omega(I)\|^2 + \sum_{j=1}^{\infty} \sum_{I \in B^j_J \backslash \{[0,2^j]\}} \|Z_\omega(I)\|^2,$$

$$\text{and when } j \geq 1, \sum_{n(I_k)=j} \|Z_\omega(I_k)\|^2 \leq \sum_{I \in B^j_J \backslash \{[0,2^j]\}} \|Z_\omega(I)\|^2.$$

Continue with (5.31),

$$\begin{aligned} \left\|A^Z_\omega(J)\right\| &\leq \left(\frac{3}{4}\pi^2 + 1\right) \sum_{I \in B^0_J} \|Z_\omega(I)\|^2 \qquad (5.33) \\ &\quad + \sum_{j=1}^{\infty} \left(\frac{\pi^2}{2}(j+1)^2\left(j + \frac{3}{2}\right) + 1\right) \sum_{I \in B^j_J \backslash \{[0,2^j]\}} \|Z_\omega(I)\|^2 \\ &\leq \pi^2 \sum_{I \in B^0_J} \|Z_\omega(I)\|^2 + \pi^2 \sum_{j=1}^{\infty} (j+1)^3 \sum_{I \in B^j_J \backslash \{[0,2^j]\}} \|Z_\omega(I)\|^2. \end{aligned}$$

where in the last step, when $j \geq 1$, $\frac{\pi^2}{4}(j+1)^2 + 1 \leq \frac{\pi^2}{2}(j+1)^2 \leq \frac{\pi^2}{2}(j+1)^3$.

Suppose $\{I_m\}_{m=1}^M \in D_{[0,2^N]}$, apply (5.33) to each I_m and use (property (5.12))

$$\sqcup_m B^j_{I_m} \subseteq B^j_{[0,2^N]}, 0 \leq j \leq N$$

(any dyadic interval in $[0,2^N]$ is of level less or equal to N),

$$\begin{aligned} \sum_{m=1}^{M} \left\|A^Z_\omega(I_m)\right\| &\leq \pi^2 \sum_{m=1}^{M} \sum_{I \in B^0_{I_m}} \|Z_\omega(I)\|^2 + \pi^2 \sum_{j=1}^{N} (j+1)^3 \sum_{m=1}^{M} \sum_{I \in B^j_{I_m} \backslash \{[0,2^j]\}} \|Z_\omega(I)\|^2 \\ &\leq \pi^2 \sum_{I \in B^0_{[0,2^N]}} \|Z_\omega(I)\|^2 + \pi^2 \sum_{j=1}^{N} (j+1)^3 \sum_{I \in B^j_{[0,2^N]} \backslash \{[0,2^j]\}} \|Z_\omega(I)\|^2 \\ &= \pi^2 \|Z_\omega([0,1])\|^2 + \pi^2 \sum_{j=0}^{N-1} (j+1)^3 \sum_{k=1}^{2^{N-j}-1} \left\|Z_\omega\left(\left[k2^j, (k+1)2^j\right]\right)\right\|^2. \end{aligned}$$

where $B^0_{[0,2^N]} = \{[0,1]\} \cup \left\{B^0_{[0,2^N]} \backslash \{[0,1]\}\right\}$. Take supremum over all finite partitions and integrate

$$\int_\Omega \left\|A^Z_\omega\right\|_{1-var,[0,2^N]} \mu(d\omega) \leq \pi^2 |b_1|^2 + \pi^2 \sum_{j=0}^{N-1} (j+1)^3 \sum_{n=2^j+1}^{2^N} |c_n|^2.$$

Take limit $N \to \infty$ and use Fatou's Lemma,

$$\begin{aligned}\int_\Omega \left\|A^{11}_\omega\right\|_{1-var} \mu(d\omega) &= \int_\Omega \left\|A^{Z}_\omega\right\|_{1-var} \mu(d\omega) \le \pi^2 |b_1|^2 + \pi^2 \sum_{j=0}^{\infty} (j+1)^3 \sum_{n\ge 2^j+1} |b_n|^2 \\ &= \pi^2 |b_1|^2 + \pi^2 \sum_{n=2}^{\infty} \left(\sum_{j=0}^{[\log_2(n-1)]} (j+1)^3 \right) |b_n|^2 .\end{aligned}$$

When $n \ge 2$, $\log_2(n-1)+1 \le 2\log_2 n$, and $(\log_2 n)^4 \le 2n^2$, so $\sum_{j=0}^{[\log_2(n-1)]} (j+1)^3 \le (\log_2(n-1)+1)^4 \le 16(\log_2 n)^4 \le 32n^2$. Hence, using $|b_n|^2 = \sum_{k=2^n+1}^{2^{n+1}} |c_k|^2$,

$$\int_\Omega \left\|A^{11}_\omega\right\|_{1-var} \mu(d\omega) \le 32\pi^2 \sum_{n=1}^{\infty} n^2 |b_n|^2 \le 32\pi^2 \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2 . \tag{5.34}$$

Therefore, combine (5.21), (5.27), (5.34) and Lemma 5.30, use Fatou's lemma, we get,

$$\begin{aligned}&\int_\Omega \|A_\omega\|_{1-var} \mu(d\omega) \qquad (5.35)\\ \le\ & 2\sum_{n=1}^{\infty} \int_\Omega \|A_\omega\|_{1-var,[2^n,2^{n+1}]} \mu(d\omega) + 2\int_\Omega \left\|A^{11}_\omega\right\|_{1-var} + \|X_\omega\|^2_{2-var} \mu(d\omega) \\ \le\ & 2\left(10+32\pi^2+36\right) \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2 \le 732 \sum_{n=0}^{\infty} (\log_2(n+1))^2 |c_n|^2 .\end{aligned}$$

As a result, we proved

$$\begin{aligned}\int_\Omega \|\mathbf{X}_\omega\|_{G^{(2)}} \mu(d\omega) &= \int_\Omega \|X_\omega\|^2_{2-var} \mu(d\omega) + \int_\Omega \|A_\omega\|_{1-var} \mu(d\omega) \\ &\le 768 \sum_n (\log_2(n+1))^2 |c_n|^2 .\end{aligned}$$

■

Remark 5.31 *Suppose X_ω and A_ω are as defined in the proof above. One can check that for any $(s,t) \in \triangle_{[0,\infty)}$ and any $\omega \in \Omega$,*

$$\begin{aligned}&A_\omega(s,t) + \frac{1}{2}(X_\omega(t) - X_\omega(s))^{\otimes 2} \\ =\ & \frac{1}{2}\int_{s<u_1<u_2<t} dX_\omega(u_1) \otimes dX_\omega(u_2) - dX_\omega(u_2) \otimes dX_\omega(u_1) \\ & + \frac{1}{2}\int_{s<u_1<u_2<t} dX_\omega(u_1) \otimes dX_\omega(u_2) + dX_\omega(u_2) \otimes dX_\omega(u_1) \\ =\ & \int_{s<u_1<u_2<t} dX_\omega(u_1) \otimes dX_\omega(u_2) := \left(\int dX_\omega \otimes dX_\omega\right)(s,t),\end{aligned}$$

which is a function on $\triangle_{[0,\infty)}$, we call the " second iterated integral" of X_ω. Thus

$$\left| \|A_\omega\|_{1-var} - \left\| \int dX_\omega \otimes dX_\omega \right\|_{1-var} \right| \le \frac{1}{2} \|X_\omega\|^2_{2-var} .$$

Therefore, use (5.35) *in Theorem 5.13 and Lemma 5.30, we get*

$$\int_{\Omega}\left\|\int dX_{\omega}\otimes dX_{\omega}\right\|_{1-var}\mu\left(d\omega\right)\leq 750\sum_{n=0}^{\infty}\left(\log_{2}\left(n+1\right)\right)^{2}\left|c_{n}\right|^{2}.$$

Then $\left\{\left(\log_{2}\left(n+1\right)\right)^{2}\right\}$ *is also a Weyl multiplier for the second iterated integral of partial sum process to have finite variation a.e..*

The following decomposition is used in Theorem 16 [17] to prove the first part of our Theorem 5.15 (finiteness of 2-variation of partial sum process of Fourier series).

Lemma 5.32 *Every non-dyadic interval* J *can be decomposed as disjoint union of two intervals* $J=J^{1}\cup J^{2}$*, such that there exist two disjoint dyadic intervals* I_{1} *and* I_{2}*, satisfying* $J^{i}\subseteq I^{i}$ *and* $\left|J^{i}\right|>\frac{1}{2}\left|I^{i}\right|$, $i=1,2$.

Proof. First suppose that K is such an interval, that it can be decomposed as disjoint union of dyadic intervals which are strictly monotone in their level, with biggest dyadic interval I_{n} of level n. We want to find a dyadic interval I, s.t. $K\subseteq I$ and $\left|K\right|>\frac{1}{2}\left|I\right|$. If $K=I_{n}$, then I_{n} is the dyadic interval we want. If $K\neq I_{n}$, then I_{n} is a strict subset of K. Since the dyadic intervals in K are strictly monotone in their level, with I_{n} of level n the biggest dyadic interval at one end, so $K\backslash I_{n}$ is an interval satisfying $\left|K\backslash I_{n}\right|\leq\sum_{k=0}^{n-1}2^{k}=2^{n}-1$. Thus, since $K\backslash I_{n}$ shares a level n dyadic boundary point with I_{n}, $K\backslash I_{n}$ is contained in another level n dyadic interval (denoted as $I_{n}^{\prime}$). Therefore, if $I_{n}\cup I_{n}^{\prime}$ constitute a level $n+1$ interval (denoted as I_{n+1}), then K is included I_{n+1}, and $\left|K\right|>\left|I_{n}\right|=\frac{1}{2}\left|I_{n+1}\right|$. While the condition for $I_{n}\cup I_{n}^{\prime}$ to constitute a level $n+1$ dyadic interval is that the boundary point of I_{n} which it shares with K is a $n+1$ level dyadic point. Thus, if K satisfies $C1$: dyadic, or $C2$: (1) the level of dyadic intervals changes monotonically, and (2) the boundary point which it shares with its biggest dyadic subinterval (of level n) is not only a level n dyadic point, but also a level $n+1$ dyadic point. Then there exists a dyadic interval I, s.t. $K\subseteq I$ and $\left|K\right|>\frac{1}{2}\left|I\right|$.

Suppose J is decomposed by Lemma 5.23, then we select a point in the dyadic partition of J which part J^{1} and J^{2}, such that J^{1} and J^{2} satisfy $C1$ or $C2$. According to Lemma 5.23, there exists a point P in the dyadic partition of J, s.t. $N\left(P\right)\geq n\left(J\right)+1$, and the level of dyadic interval to the left and right side of P is strictly decreasing. Thus, if P divides J into two intervals with positive length, then we select P as the point parting J^{1} and J^{2} and both J^{1} and J^{2} satisfy $C2(1)$ and $C2(2)$. While if P is a boundary point of J, then the level of dyadic intervals in J is already monotone. Denote the biggest dyadic interval in J as I_{n} (of level n), and we select the other boundary point of I_{n} which divides J into two intervals of positive length: I_{n} and $J\backslash I_{n}$. I_{n} is dyadic so satisfies $C1$. $J\backslash I_{n}$ is union of monotone dyadic intervals, so satisfies $C2(1)$; the biggest dyadic interval in $J\backslash I_{n}$ is of level less than n, but shares a level n boundary point with I_{n}, so $J\backslash I_{n}$ satisfies $C2(2)$. ■

Remark 5.33 *As we selected, the point in* J *dividing* J^{1} *and* J^{2} *is one of the boundary points of biggest dyadic sub-interval(s) of* J*, and the level of dyadic intervals is strictly decreasing to left and right side of this point.*

Lemma 5.34 *Suppose J is a finite non-dyadic interval. If we bisect $J = J^1 \cup J^2$ according to Lemma 5.32, and continue to bisect non-dyadic J^1 and/or J^2, so on and so forth, until all intervals left are dyadic. Then the dyadic intervals left constitute the dyadic partition of J by Lemma 5.23.*

Proof. Suppose the dyadic partition of J by Lemma 5.23 is $\{I_k\}_{k=1}^n$, where $\{I_k\}$ are numbered that $k \mapsto I_k$ is increasing from left to right of J. Denote P as the point bisecting $J = J^1 \cup J^2$ by Lemma 5.32. Based on Remark 5.33, P is one of the boundary points of some I_k, $1 \leq k \leq n$, and the level of dyadic intervals is strictly decreasing to the left and right side of P. Since $\{I_k\}_{k=1}^n$ is a finite partition of J, J^1 and J^2 are union of I_ks: there exists m, $1 \leq m \leq n-1$, such that $J^1 = \cup_{k=1}^m I_k$ and $J^2 = \cup_{k=m+1}^n I_k$. We continue to bisect non-dyadic J^1 and/or J^2. Take $J^1 = \cup_{k=1}^m I_k$ for example. Since I_k, $k = 1, 2, \ldots, m$ are strictly increasing in their level, according to Lemma 5.32, bisecting J^1 is to cut I_m out (the biggest dyadic subinterval). While $J^1 \backslash I_m = \cup_{k=1}^{m-1} I_k$ is still composed of strictly increasing dyadic subintervals, so bisecting $J^1 \backslash I_m$ is to cut I_{m-1} out, so on and so forth. In this way, bisecting J^1 down to dyadic intervals, one gets back $\{I_k\}_{k=1}^m$. Similar is true for J^2. Thus, $\{I_k\}_{k=1}^m \cup \{I_k\}_{k=m+1}^n = \{I_k\}_{k=1}^n$ is the dyadic partition of J by Lemma 5.23. ■

Before proceeding to the proof of Theorem 5.15, we define $\widetilde{B}_J$ for finite interval J as the set of dyadic intervals which contain " part" of J.

Notation 5.35 *Suppose J is a finite interval, denote*

$$\widetilde{B}_J := \left\{ I | I \text{ is dyadic, } |I \cap J| > \frac{1}{2} |I| \right\}. \tag{5.36}$$

Four properties of $\widetilde{B}_J$:

(*i*) $B_J \subseteq \widetilde{B}_J$.

Proof. Recall B_J is the set of dyadic intervals included in J. Suppose $I \in B_J$, then I is dyadic and $I \subseteq J$, so $|I \cap J| = |I| > \frac{1}{2} |I|$, $I \in \widetilde{B}_J$. ■

(*ii*) when J is dyadic, $\widetilde{B}_J = B_J$.

Proof. For two dyadic intervals, either one is wholly included in another, or they are disjoint, bar boundary points. Thus, suppose J and I are dyadic intervals and $|I \cap J| > 0$, then either $I \subseteq J$, or $J \subset I$. If $I \subseteq J$, then $I \in B_J \subseteq \widetilde{B}_J$. If $J \subset I$, and $I \in \widetilde{B}_J$, then $|J| < |I| < 2|I \cap J| = 2|J|$, which is not possible since I and J are dyadic. Therefore, when J is dyadic, $\widetilde{B}_J$ is the set of dyadic intervals included in J, thus coincides with B_J. ■

(*iii*) If $J' \subseteq J$, then $\widetilde{B}_{J'} \subseteq \widetilde{B}_J$.

Proof. Suppose $I \in \widetilde{B}_{J'}$, then $|I \cap J| \geq |I \cap J'| > \frac{1}{2} |I|$, so $I \in \widetilde{B}_J$. ■

(iv) Suppose $\{I_k\}$ is a finite partition of J, then $\sqcup_k \widetilde{B}_{I_k} \subseteq \widetilde{B}_J$.

Proof. $\widetilde{B}_{I_k} \subseteq \widetilde{B}_J$ is from *(iii)*. If $I \in \widetilde{B}_{I_{k_1}} \cap \widetilde{B}_{I_{k_2}}$, $k_1 \neq k_2$, then

$$\begin{aligned} |I_{k_1} \cap I_{k_2}| &\geq |(I \cap I_{k_1}) \cap (I \cap I_{k_2})| \\ &= |I \cap I_{k_1}| + |I \cap I_{k_2}| - |(I \cap I_{k_1}) \cup (I \cap I_{k_2})| \\ &> \frac{1}{2}|I| + \frac{1}{2}|I| - |I| = 0, \end{aligned}$$

contradictory with that I_k are disjoint since $\{I_k\}$ is a finite partition of J. ■

Proof of Theorem 5.15. Denote the partial sum process of $\sum_n c_n u_n$ as X, and $A := A(X)$. Denote $X^1 := L(X, \{2^n\})$ and $A^{11} := A(X^1)$. If let

$$v_n(\omega) = \sum_{k=2^n+1}^{2^{n+1}} \frac{c_k u_k(\omega)}{\sqrt{\sum_{k=2^n+1}^{2^{n+1}} |c_k|^2}}, \; b_n = \sqrt{\sum_{k=2^n+1}^{2^{n+1}} |c_k|^2},$$

then X^1 is reparametrised partial sum process of $\sum_{n=0}^{\infty} b_n v_n$. Since being geometric rough process is invariant under reparametrisation, according to Theorem 5.13, X^1 is a geometric 2-rough process when $\sum_{n\geq 0} (\log_2(n+1))^2 |b_n|^2 < \infty$. On the other hand, (use $(\log_2(n+1))^2 \leq 2n$, $\forall n \in \mathbb{N}$)

$$\sum_{n\geq 0} (\log_2(n+1))^2 |b_n|^2 \leq 2\sum_{n\geq 1} n |b_n|^2 \leq 2\sum_{n\geq 0} \log_2(n+1) |c_n|^2 .$$

Thus when $\sum_n \log_2(n+1)|c_n|^2 < \infty$, X^1 is a geometric 2-rough process, and (according to Lemma 5.30 and (5.35))

$$\begin{aligned} \int_\Omega \left\|X^1_\omega\right\|^2_{2-var} \mu(d\omega) &\leq 72\sum_{n=0}^{\infty} \log_2(n+1)|c_n|^2, \qquad (5.37) \\ \int_\Omega \left\|A^{11}_\omega\right\|_{1-var} \mu(d\omega) &\leq 1464\sum_{n=0}^{\infty} \log_2(n+1)|c_n|^2. \end{aligned}$$

Therefore, if we can prove that for any $n \geq 1$,

$$\begin{aligned} \int_\Omega \|X_\omega\|^2_{2-var,[2^n,2^{n+1}]} \mu(d\omega) &\leq 4C \sum_{k=2^n+1}^{2^{n+1}} \log_2(k+1)|c_k|^2, \qquad (5.38) \\ \int_\Omega \|A_\omega\|_{1-var,[2^n,2^{n+1}]} \mu(d\omega) &\leq 2(C+1) \sum_{k=2^n+1}^{2^{n+1}} \log_2(k+1)|c_k|^2. \end{aligned}$$

Then according to Corollary 5.28 (in p75),

$$\begin{aligned} &\int_\Omega \|X_\omega\|^2_{2-var} + \|A_\omega\|_{1-var} \mu(d\omega) \\ \leq\; & 9|c_1|^2 + 9|c_2|^2 + \int_\Omega 9\left\|X^1_\omega\right\|^2_{2-var} + 2\left\|A^{11}_\omega\right\|_{1-var} \mu(d\omega) \\ & + \sum_{n=1}^{\infty} \int_\Omega 9\|X_\omega\|^2_{2-var,[2^n,2^{n+1}]} + 2\|A_\omega\|_{1-var,[2^n,2^{n+1}]} \mu(d\omega). \end{aligned}$$

Substitute in (5.37) and (5.38), we get

$$\int_{\Omega} \|X_{\omega}\|_{2-var}^{2} + \|A_{\omega}\|_{1-var} \mu(d\omega) \leq (3580 + 40C) \sum_{n=0}^{\infty} \log_2 (n+1) |c_n|^2,$$

where $3580 + 40C = 9 \times 72 + 2 \times 1464 + 36C + 4(1 + C)$. Thus, if the two inequalities in (5.38) are true, then (X, A) is a geometric 2-rough process under the condition $\sum_{n=0}^{\infty} \log_2 (n+1) |c_n|^2 < \infty$. Therefore, in the following, we concentrate on two inequalities in (5.38).

Suppose we are working on $[2^n, 2^{n+1}]$ for some fixed integer $n \geq 1$.

For any fixed finite partition $D = \{[m_k, m_{k+1}]\}_k$ of $[2^n, 2^{n+1}]$, denote the dyadic intervals in D as $\{I_j\}$ (i.e.$[m_k, m_{k+1}]$ which are dyadic), denote the non-dyadic intervals in D as $\{J_k\}$. Use Lemma 5.32 to bisect non-dyadic intervals: every J_k can be decomposed as disjoint union of J_k^1 and J_k^2, such that J_k^1 and J_k^2 are intervals of positive length, and there exists two disjoint dyadic intervals I_k^1, I_k^2, satisfying $J_k^i \subseteq I_k^i$ and $|J_k^i| > \frac{1}{2} |I_k^i|$, $i = 1, 2$. As a result, when bisecting a set of *disjoint* non-dyadic intervals $\{J_k\}$, in the set of related dyadic intervals $\{I_k^1, I_k^2\}$, each dyadic interval is counted at most once. (Otherwise, there are two disjoint J_k^i share the same dyadic interval I, so there must be one J_k^i satisfies $|J_k^i| \leq \frac{1}{2} |I|$, contradicting with the selection of I.) Denote $\|X\|_{\infty,I} := \sup_{I' \subseteq I} \|X(I')\|$. Then,

$$\begin{aligned}
&\sum_{[m_k, m_{k+1}] \in D} \|X_{\omega}([m_k, m_{k+1}])\|^2 = \sum_k \|X_{\omega}(J_k)\|^2 + \sum_j \|X_{\omega}(I_j)\|^2 \qquad (5.39)\\
&\leq 2 \sum_k \left(\left\|X_{\omega}\left(J_k^1\right)\right\|^2 + \left\|X_{\omega}\left(J_k^2\right)\right\|^2 \right) + \sum_j \|X_{\omega}(I_j)\|^2\\
&\leq 2 \sum_k \left(\|X_{\omega}\|_{\infty, I_k^1}^2 + \|X_{\omega}\|_{\infty, I_k^2}^2 \right) + \sum_j \|X_{\omega}\|_{\infty, I_j}^2 \leq 2 \sum_{I \in B_{[2^n, 2^{n+1}]}} \|X_{\omega}\|_{\infty, I}^2,
\end{aligned}$$

where we used that I_k^1, I_k^2 and I_j are dyadic, and $\{I_k^1\} \sqcup \{I_k^2\} \sqcup \{I_j\} \subseteq B_{[2^n, 2^{n+1}]}$. That I_k^i are different as k and i vary, as we stated, is because J_k^i are disjoint, thus there can not be two J_k^i share the same I; while I_k^i differs from I_j is because if $I_k^i = I_j$ for some i, j, k, then $J_k^i \subseteq I_k^i = I_j$, so $0 < |J_k^i| = |J_k^i \cap I_j| \leq |J_k \cap I_j|$, contradicting with that J_k and I_j are disjoint since they are elements of finite partition D. Thus, use (5.39) and take supremum over all finite partitions of $[2^n, 2^{n+1}]$, we get,

$$\|X_{\omega}\|_{2-var, [2^n, 2^{n+1}]}^2 \leq 2 \sum_{I \in B_{[2^n, 2^{n+1}]}} \|X_{\omega}\|_{\infty, I}^2 .$$

Using the assumption (Hardy property) that for any interval I,

$$\int_{\Omega} \|X_{\omega}\|_{\infty, I}^2 \mu(d\omega) \leq C \int_{\Omega} \|X_{\omega}(I)\|^2 \mu(d\omega)$$

and (5.13), i.e.

$$\sum_{I \in B_{[2^n, 2^{n+1}]}} \int_{\Omega} \|X_{\omega}(I)\|^2 \mu(d\omega) \leq 2 \log_2 (2^n + 1) \sum_{k=2^n+1}^{2^{n+1}} |c_k|^2,$$

we get, for any integer n,

$$\int_{\Omega}\|X_{\omega}\|_{2-var,[2^n,2^{n+1}]}^{2}\mu(d\omega)\leq 2\int_{\Omega}\sum_{I\in B_{[2^n,2^{n+1}]}}\|X_{\omega}\|_{\infty,I}^{2}\mu(d\omega) \tag{5.40}$$

$$\leq 2C\sum_{I\in B_{[2^n,2^{n+1}]}}\int_{\Omega}\|X_{\omega}(I)\|^{2}\mu(d\omega)\leq 4C\log_2(2^n+1)\sum_{k=2^n+1}^{2^{n+1}}|c_k|^2$$

$$\leq 4C\sum_{k=2^n+1}^{2^{n+1}}\log_2(k+1)|c_k|^2 .$$

Then we estimate 1-variation of A_{ω} on $[2^n,2^{n+1}]$. On dyadic interval $I\subseteq[2^n,2^{n+1}]$, use (5.24):

$$\|A_{\omega}(I)\|\leq\sum_{I'\in B_I\backslash\{I\}}\|X_{\omega}(I')\|^{2}. \tag{5.41}$$

Suppose $J\subseteq[2^n,2^{n+1}]$ is a non-dyadic interval. Use Lemma 5.32 to bisect $J=J^1\cup J^2$, with associated dyadic intervals I^i, then $|I^i\cap J|=|J^i|>\frac{1}{2}|I^i|$. Thus $I^i\in\widetilde{B}_J$ ($\widetilde{B}_J$ is defined at (5.36)), and

$$\begin{aligned}\|A_{\omega}(J)\| &\leq \left\|A_{\omega}(J^1)\right\|+\left\|A_{\omega}(J^2)\right\|+\left\|X_{\omega}(J^1)\right\|\left\|X_{\omega}(J^2)\right\|\\ &\leq \left\|A_{\omega}(J^1)\right\|+\left\|A_{\omega}(J^2)\right\|+\|X_{\omega}\|_{\infty,I^1}^{2}+\|X_{\omega}\|_{\infty,I^2}^{2}.\end{aligned}$$

The bisecting process terminates if both J^1 and J^2 are dyadic, otherwise, continue to bisect non-dyadic J^1 and/or J^2, so on and so forth, until all the intervals left are dyadic. According to Lemma 5.34, all the dyadic intervals left constitute the dyadic partition of J in Lemma 5.23.

The dyadic intervals, which are by-products of our sequence of bisections (e.g. I^1 and I^2 from bisecting J), are elements of $\widetilde{B}_J$, because if dyadic interval I is obtained from bisecting interval $J'\subseteq J$, then $I\in\widetilde{B}_{J'}\subseteq\widetilde{B}_J$ ($I\in\widetilde{B}_{J'}$ is the same reason as I^1, $I^2\in\widetilde{B}_J$; $\widetilde{B}_{J'}\subseteq\widetilde{B}_J$ is *(iii)* in p85). Moreover, these by-product dyadic intervals differ from one another. Otherwise, suppose $J^{(1)}$ and $J^{(2)}$ are two different intervals generated in the bisecting process, sharing the same dyadic interval I, i.e. $J^{(i)}\subseteq I$, and $\left|J^{(i)}\right|>\frac{1}{2}|I|$, then $\left|J^{(1)}\cap J^{(2)}\right|>0$, and I is the smallest dyadic interval which includes $J^{(1)}(J^{(2)})$. Since $J^{(1)}$ and $J^{(2)}$ are sub-intervals generated in the process of decomposing J, so if $\left|J^{(1)}\cap J^{(2)}\right|>0$, then one is wholly included in another. Thus, without loss of generality, suppose $J^{(2)}\subset J^{(1)}$, then $J^{(2)}$ is obtained from further bisecting $J^{(1)}$. When bisecting $J^{(1)}$, according to Lemma 5.32, there exist two disjoint dyadic intervals I' and I'', s.t. $\left|J^{(1)}\cap I'\right|>0$, $\left|J^{(1)}\cap I''\right|>0$. Since $J^{(2)}$ is obtained from further bisecting $J^{(1)}$, so without loss of generality, assume $J^{(2)}\subseteq I'$. As we denoted, I is the smallest dyadic interval containing $J^{(2)}$, so $I\subseteq I'$, while I is also the smallest dyadic interval containing $J^{(1)}$, so $J^{(1)}\subseteq I$, thus $J^{(1)}\subseteq I'$, contradictory with that I' and I'' are disjoint and $\left|J^{(1)}\cap I''\right|>0$.

Therefore, if denote the dyadic partition of J in Lemma 5.23 as $\cup_k I_k$, use the estimation for A_ω on dyadic intervals (i.e.(5.24)), we get

$$\sum_k \left\| A_\omega \left(I_k \right) \right\| \le \sum_k \sum_{I \in B_{I_k} \setminus \{I_k\}} \left\| X_\omega \left(I \right) \right\|^2 \le \sum_{I \in B_J} \left\| X_\omega \left(I \right) \right\|^2 .$$

Thus (all by-products dyadic intervals are elements of $\widetilde{B}_J$, and they are different from one another),

$$\left\| A_\omega \left(J \right) \right\| \le \sum_k \left\| A_\omega \left(I_k \right) \right\| + \sum_{I \in \widetilde{B}_J .} \left\| X_\omega \right\|_{\infty, I}^2 \le \sum_{I \in B_J} \left\| X_\omega \left(I \right) \right\|^2 + \sum_{I \in \widetilde{B}_J} \left\| X_\omega \right\|_{\infty, I}^2 . \tag{5.42}$$

Therefore, suppose $\{I_j\}$ is a finite partition of $[2^n, 2^{n+1}]$, combine estimation on dyadic intervals($I_j \in B_{[2^n,2^{n+1}]}$) in (5.41) and on non-dyadic intervals($I_j \notin B_{[2^n,2^{n+1}]}$) in (5.42),

$$\begin{aligned}
& \sum_j \left\| A_\omega \left(I_j \right) \right\| \\
= \; & \sum_{j, I_j \in B_{[2^n,2^{n+1}]}} \left\| A_\omega \left(I_j \right) \right\| + \sum_{j, I_j \notin B_{[2^n,2^{n+1}]}} \left\| A_\omega \left(I_j \right) \right\| \\
\le \; & \sum_{j, I_j \in B_{[2^n,2^{n+1}]}} \sum_{I \in B_{I_j} \setminus \{I_j\}} \left\| X_\omega \left(I \right) \right\|^2 + \sum_{j, I_j \notin B_{[2^n,2^{n+1}]}} \Big(\sum_{I \in B_{I_j}} \left\| X_\omega \left(I \right) \right\|^2 + \sum_{I \in \widetilde{B}_{I_j}} \left\| X_\omega \right\|_{\infty, I}^2 \Big) \\
\le \; & \sum_j \sum_{I \in B_{I_j}} \left\| X_\omega \left(I \right) \right\|^2 + \sum_j \sum_{I \in \widetilde{B}_{I_j}} \left\| X_\omega \right\|_{\infty, I}^2 .
\end{aligned}$$

Use $\sqcup_j B_{I_j} \subseteq B_{[2^n,2^{n+1}]}$ (according to (5.11)), $\sqcup_j \widetilde{B}_{I_j} \subseteq \widetilde{B}_{[2^n,2^{n+1}]}$ (according to (*iv*) in p86), and $B_{[2^n,2^{n+1}]} = \widetilde{B}_{[2^n,2^{n+1}]}$ for dyadic interval $[2^n, 2^{n+1}]$ (according to (*ii*) in p85), we get

$$\left\| A_\omega \right\|_{1-var,[2^n,2^{n+1}]} = \sup_{\{I_j\} \in D_{[2^n,2^{n+1}]}} \sum_j \left\| A_\omega \left(I_j \right) \right\| \le \sum_{I \in B_{[2^n,2^{n+1}]}} \left(\left\| X_\omega \left(I \right) \right\|^2 + \left\| X_\omega \right\|_{\infty, I}^2 \right) .$$

Integrate both sides, use $\int_\Omega \left\| X \right\|_{\infty, I}^2 \mu \left(d\omega \right) \le C \int_\Omega \left\| X \left(I \right) \right\|^2 \mu \left(d\omega \right)$, and (5.13), i.e.

$$\sum_{I \in B_{[2^n,2^{n+1}]}} \int_\Omega \left\| X_\omega \left(I \right) \right\|^2 \mu \left(d\omega \right) \le 2 \log_2 \left(2^n + 1 \right) \sum_{k=2^n+1}^{2^{n+1}} \left| c_k \right|^2 ,$$

we get, for any $n \ge 1$,

$$\begin{aligned}
\int_\Omega \left\| A_\omega \right\|_{1-var,[2^n,2^{n+1}]} \mu \left(d\omega \right) & \le \left(1 + C \right) \sum_{I \in B_{[2^n,2^{n+1}]}} \int_\Omega \left\| X_\omega \left(I \right) \right\|^2 \mu \left(d\omega \right) \\
& \le 2 \left(1 + C \right) \sum_{k=2^n+1}^{2^{n+1}} \log_2 \left(k + 1 \right) \left| c_k \right|^2 .
\end{aligned} \tag{5.43}$$

Combined with reasoning at the beginning of the proof and (5.40). ■

Chapter 6

Partial sum process of L^2 Fourier series

In this chapter, we mainly follow the content of [23].

In Chapter 5, $\{\log_2(n+1)\}$ is identified as a Weyl multiplier for the partial sum process of L^2 Fourier series to be a geometric rough process. More specifically, as stated in Corollary 5.17, for any $f \in L^2([-\pi,\pi],\mathbb{R}^d)$ with Fourier coefficients $\{c_n\}$, (denote $\mathbf{X}$ as the partial sum process of the Fourier series of f),

$$\int_{-\pi}^{\pi} \|\mathbf{X}(\theta)\|_{G^{(2)}}^2 \, d\theta \leq (3580+40C_0) \sum_{n=0}^{\infty} \log_2(n+1) |c_n|^2, \tag{6.1}$$

with C_0 the constant in Carleson-Hunt inequality for L^2 Fourier series. Moreover, in (6.1), $\log_2(n+1)$ can not be replaced by $o(\log_2(n+1))$, with lower bound given by Dirichlet kernel.

It is interesting to know what does the decay condition $\sum_n \log_2(n+1)|c_n|^2 < \infty$ imply about the regularity of the limit function f. For this reason, we define sobolev space H_{Log}^s for $s>0$, as the space of functions in $L^2([-\pi,\pi],\mathbb{R}^d)$, whose Fourier coefficients satisfy

$$\sum_{n=0}^{\infty} (\log_2(n+1))^{2s} |c_n|^2 < \infty.$$

We establish the following identification of functions in H_{Log}^s (when $s=\frac{1}{2}$, the equivalency is proved in Thm 4 [3]).

Theorem 6.1 *For $s \in (0,\infty)$ and $d \geq 1$, there exist constants $0 < k_s \leq K_s < \infty$, such that for any $f \in L^2([-\pi,\pi],\mathbb{R}^d)$, with Fourier coefficients $\{c_n\}$,*

$$\text{if denote } L := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|} \left(\log_2 \frac{\pi}{\left|\sin\frac{u-v}{2}\right|}\right)^{2s-1} du dv$$

$$\text{and } l := \sum_{n=0}^{\infty} (\log_2(n+1))^{2s} |c_n|^2, \text{ then } k_s l \leq L \leq K_s l.$$

Corollary 6.2 *Suppose* $f:[-\pi,\pi]\to\mathbb{R}^d$ *satisfying*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|f(u)-f(v)\right|^2}{\left|\sin\frac{u-v}{2}\right|}dudv<\infty.$$

Then f *is in* L^2*, and the partial sum process of the Fourier series of* f*, when enhanced by its area process, is a geometric* 2*-rough process (denoted as* $\mathbf{X}$*). Moreover,*

$$\int_{-\pi}^{\pi}\left\|\mathbf{X}(\theta)\right\|_{G^{(2)}}^2 d\theta\leq\left(3580+40C_0\right)k_{\frac{1}{2}}^{-1}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\left|f(u)-f(v)\right|^2}{\left|\sin\frac{u-v}{2}\right|}dudv,$$

where C_0 *is the constant in Carleson-Hunt inequality for* L^2 *Fourier series and* $k_{\frac{1}{2}}$ *is defined in Theorem 6.1.*

This corollary follows trivially from Corollary 5.17 and Theorem 6.1.

Corollary 6.2 could be treated as a modest complement to [26], where the authors proved the non-trivial result that, for any $r>1$ and $p>\max\{2,r/(r-1)\}$,

$$\int_{-\pi}^{\pi}\left\|X(\theta)\right\|_{p-var}^2 d\theta\leq C_{r,p}\int_{-\pi}^{\pi}\left|f(\theta)\right|^2 d\theta.$$

However, although $\log_2(n+1)$ in (6.1) can not be replaced by $o(\log_2(n+1))$, the condition $\sum_n\log_2(n+1)\left|c_n\right|^2<\infty$ is not necessary for the partial sum process of Fourier series to be a geometric 2-rough process. In fact, we give a little stronger statement.

Example 6.3 *Suppose* $\{w(n)\}$ *is a Weyl multiplier that* $n\mapsto\frac{w(n)}{(\log_2\log_2 n)^2}$ *is strictly increasing from some point on, and* $\lim_{n\to\infty}\frac{w(n)}{(\log_2\log_2 n)^2}=\infty$*. Then there exists a* 2*-dimensional Fourier series* $\sum_{n=1}^{\infty}c_ne^{in\theta}$*, such that its partial sum process is a geometric* 2*-rough process, but* $\sum_{n=1}^{\infty}w(n)\left|c_n\right|^2=\infty$*.*

The above example is 2-dimensional, so area is non-trivial.

One might be tempted to ask whether all L^2 Fourier series have finite 2-variation a.e., which, however, is not true. It is proved in [10] that there exists a bounded function, whose Fourier series has infinite 2-variation a.e.. Their proof relies on nontrivial estimates on 2-variation of partial sum process of i.i.d. sequences in [30]. In this paper, we provide a self-contained proof, where we use the upper semi-continuity of cumulative distribution function of p-variation. This example is constructed without knowledge of [10], nor the result in [30].

Example 6.4 *There exists an* L^2 *Fourier series whose partial sum process has infinite* 2*-variation almost everywhere.*

6.1 Sobolev spaces H^s_{Log}

In this section, we identify an equivalent norm on the space of functions whose Fourier coefficients satisfy $\sum_n (\log_2(n+1))^{2s}|c_n|^2 < \infty$ for some $s > 0$. We also construct an example to show that $\sum_n w(n)|c_n|^2 < \infty$ is not necessary for the partial sum process of Fourier series to be a geometric 2-rough process, for any Weyl multiplier $\{w(n)\}$ increasing strictly faster than $\{(\log_2\log_2 n)^2\}$.

Let H^δ be the sobolev space $W^{\delta,2}$. The fact that $f : [-\pi,\pi] \to \mathbb{R}^d$ belongs to H^δ for some $0 < \delta < 1$, can be stated equivalently in the following two ways (Theorem 8.6 in [16]):

$$\sum_{n=0}^{\infty} n^{2\delta}|c_n|^2 < \infty, \tag{6.2}$$

and

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|^{2\delta+1}}dudv < \infty. \tag{6.3}$$

with $\{c_n\}$ the Fourier coefficients of f (if $f = (f_1, f_2, \ldots, f_d)$, then $c_n = \left(c_n^1, c_n^2, \ldots, c_n^d\right) \in \mathbb{R}^{2d}$, with $c_n^k = \int_{-\pi}^{\pi} f_k(\theta)e^{in\theta}d\theta$). When $\delta = 0$, the space defined by (6.3) is strictly included in L^2, which, as we will prove (also proved in Thm4 [3]), is equivalent to

$$\sum_{n=0}^{\infty}\log_2(n+1)|c_n|^2 < \infty.$$

To fit the framework of our theorems,

Definition 6.5 *Define sobolev spaces H^s_{Log}, $-\infty < s < \infty$, as the linear space of $\mathbb{R}^d$ valued functions on $[-\pi,\pi]$ with finite the following norm:*

$$\|f\|_{Log,s} := \left(\sum_{n=1}^{\infty}(\log_2(n+1))^{2s}|c_n|^2\right)^{\frac{1}{2}}, \tag{6.4}$$

where $\{c_n\}$ are Fourier coefficients of f.

Similar to H^s, H^s_{Log} is a separable Hilbert space for any $-\infty < s < \infty$, with trigonometric polynomials as a dense subset; H^{-s}_{Log} is the dual space of H^s_{Log}; and H^q_{Log} can be compactly embedded into H^p_{Log} for any $q > p$. Moreover, for the interpolation space $\left(H^p_{Log}, H^q_{Log}\right)_{\theta,2} = H^r_{Log}$, where $r = (1-\theta)p + \theta q$, Hölder inequality holds:

$$\|f\|_{Log,r} \leq \|f\|^{1-\theta}_{Log,p}\|f\|^{\theta}_{Log,q}.$$

All these properties can be proved as counterparts as those of H^δ (e.g. p108-p117, [16]).

As an example, the function

$$f_{s,\epsilon}(x) = \frac{1}{x^{\frac{1}{2}}\left|\log_2\frac{x}{2}\right|^{s+\frac{1}{2}}\left|\log_2\left(2\left|\log_2\frac{x}{2}\right|\right)\right|^{\frac{1}{2}+\epsilon}}, \quad x \in (0,1),$$

(according to Theorem 2.24 at p190 in Vol I [40]) belongs to H^s_{Log} when $\epsilon > 0$, not belongs to H^s_{Log} when $\epsilon \leq 0$.

Next, we prove that there exists an equivalent norm on H^s_{Log} as the one for H^s in (6.3), which is inspired by Theorem 8.6 in [16].

Theorem 6.1 *For $0 < s < \infty$ and $d \geq 1$, there exist constants $0 < k_s \leq K_s < \infty$, such that for any $f \in L^2\left(\left[-\pi, \pi\right], \mathbb{R}^d\right)$ with Fourier coefficients $\{c_n\}$,*

$$\text{if denote } L(f) := \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{|f(u) - f(v)|^2}{\left|\sin\frac{u-v}{2}\right|}(\log_2 \frac{\pi}{\left|\sin\frac{u-v}{2}\right|})^{2s-1} du dv \tag{6.5}$$

$$\text{and } l(f) := \sum_{n=0}^{\infty}\left(\log_2(n+1)\right)^{2s}|c_n|^2, \text{ then } k_s l(f) \leq L(f) \leq K_s l(f).$$

Proof. Fix s. Without loss of generality, we assume f is one-dimensional. Since trigonometric polynomials are dense in H^s_{Log}, we only prove the theorem for trigonometric polynomials. One can verify that e^{inx}, $n \in \mathbb{Z}$ are orthogonal w.r.t. this inner product:

$$\langle f_1, f_2\rangle = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\mathrm{Re}\left(\left(f_1(u) - f_1(v)\right)\overline{\left(f_2(u) - f_2(v)\right)}\right)}{\left|\sin\frac{u-v}{2}\right|}(\log_2 \frac{\pi}{\left|\sin\frac{u-v}{2}\right|})^{2s-1} du dv.$$

Thus, the problem boils down to: for any $s \in (0, \infty)$, there exists integer N_s and constants $0 < b_s \leq B_s < \infty$, s.t. for any $n \geq N_s$,

$$b_s\left(\log_2(\pi n)\right)^{2s} \leq R^s_n := \int_0^1 \frac{\sin^2\left(\frac{1}{2}\pi n t\right)}{t}\left|\log_2\frac{t}{2}\right|^{2s-1} dt \leq B_s\left(\log_2(\pi n)\right)^{2s}.$$

Denote

$$R^s_n = \int_0^{\pi n}\frac{\sin^2\frac{1}{2}t}{t}\left|\log_2\frac{t}{2} - \log_2(\pi n)\right|^{2s-1} dt = \int_0^2 + \int_2^{\pi n} := R^s_1(n) + R^s_2(n).$$

For $R^s_1(n)$,

$$\frac{R^s_1(n)}{\left(\log_2(\pi n)\right)^{2s-1}} = \int_0^2 \frac{\sin^2\frac{1}{2}t}{t}\left|\frac{1}{\log_2(\pi n)}\log_2\frac{t}{2} - 1\right|^{2s-1} dt.$$

When $n \geq 1$,

$$1 \leq \left|\frac{1}{\log_2(\pi n)}\log_2\frac{t}{2} - 1\right| \leq 1 + \left|\log_2\frac{t}{2}\right| \quad \text{when } t \in (0,2).$$

Thus when $s \geq \frac{1}{2}$,

$$0 < \int_0^2\frac{\sin^2\frac{1}{2}t}{t}dt \leq \frac{R^s_1(n)}{\left(\log_2(\pi n)\right)^{2s-1}} \leq \int_0^2\frac{\sin^2\frac{1}{2}t}{t}\left(1 + \left|\log_2\frac{t}{2}\right|\right)^{2s-1} dt < \infty. \tag{6.6}$$

When $0 < s < \frac{1}{2}$, the upper bound and lower bound in (6.6) exchange. Thus, $R^s_1(n) \sim \left(\log_2(\pi n)\right)^{2s-1}$, and for any $\epsilon > 0$, there exists $N_\epsilon \geq 1$, s.t.

$$\left|R^s_1(n)\right| \leq \epsilon\left(\log_2(\pi n)\right)^{2s}, \forall n \geq N_\epsilon. \tag{6.7}$$

For $R_2^s(n)$,

$$\frac{R_2^s(n)}{(\log_2(\pi n))^{2s}} = \frac{1}{\log_2(\pi n)} \int_2^{\pi n} \frac{\sin^2 \frac{1}{2}t}{t} \left| \frac{1}{\log_2(\pi n)} \log_2 \frac{t}{2} - 1 \right|^{2s-1} dt. \tag{6.8}$$

For the lower bound: When $2 \le t \le \sqrt{n}\pi$,

$$0 \le \frac{1}{\log_2(\pi n)} \log_2 \frac{t}{2} \le \frac{1}{\log_2(\pi n)} \left(\frac{1}{2} \log_2 n + \log_s \frac{\pi}{2} \right) \le \frac{1}{2},$$

so

$$\frac{1}{2} \le \left| \frac{1}{\log_2(\pi n)} \log_2 \frac{t}{2} - 1 \right| \le 1 \text{ when } 2 \le t \le \sqrt{n}\pi.$$

Denote $[\sqrt{n}\,]$ as the integer part of $\sqrt{n}$, then when $s \ge \frac{1}{2}$, $n \ge 1$,

$$\frac{R_2^s(n)}{(\log_2(\pi n))^{2s}} \ge \frac{1}{2^{2s-1}\log_2(\pi n)} \sum_{k=1}^{[\sqrt{n}\,]-1} \int_{k\pi}^{(k+1)\pi} \frac{\sin^2 \frac{1}{2}t}{t} dt \ge \frac{1}{2^{2s}\log_2(\pi n)} \sum_{k=1}^{[\sqrt{n}\,]-1} \frac{1}{k+1}.$$

While $\displaystyle\sum_{k=1}^{[\sqrt{n}\,]-1} \frac{1}{k+1} = \sum_{k=1}^{[\sqrt{n}\,]} \frac{1}{k} - 1 \ge \int_1^{[\sqrt{n}\,]+1} \frac{1}{x} dx - 1 = \ln\left([\sqrt{n}\,]+1\right) - 1 \ge \frac{1}{2}\ln(n) - 1,$

so for $s \ge \frac{1}{2}$, when $n \ge [e^4\pi]+1$, we have $\frac{\ln n - 2}{\ln n + \ln \pi} \ge \frac{1}{2}$, and

$$\frac{R_2^s(n)}{(\log_2(\pi n))^{2s}} \ge \frac{\ln 2 (\ln n - 2)}{2^{2s+1}(\ln n + \ln \pi)} \ge \frac{\ln 2}{2^{2s+2}}.$$

Similarly, for $0 < s < \frac{1}{2}$, when $n \ge [e^4\pi]+1$,

$$\frac{R_2^s(n)}{(\log_2(\pi n))^{2s}} \ge \frac{\ln 2}{8}.$$

For the upper bound of $\frac{R_2^s(n)}{(\log_2(\pi n))^{2s}}$, in (6.8) let $y = \frac{\log_2 \frac{t}{2}}{\log_2(\pi n)}$, then

$$\frac{R_2^s(n)}{(\log_2(\pi n))^{2s}} \le \ln 2 \int_0^1 \sin^2\left((\pi n)^y\right)(1-y)^{2s-1} dy \le \ln 2 \int_0^1 (1-y)^{2s-1} dy_1 = \frac{\ln 2}{2s}.$$

Thus, when $n \ge [e^4\pi]+1$,

$$\frac{\ln 2}{2^{2(s \vee \frac{1}{2})+2}} = \min\{\frac{\ln 2}{2^{2s+2}}, \frac{\ln 2}{8}\} \le \frac{R_2^s(n)}{(\log_2(\pi n))^{2s}} \le \frac{\ln 2}{2s}.$$

Therefore, if for $s > 0$ let $\epsilon(s) = \frac{\ln 2}{2^{2(s \vee \frac{1}{2})+3}}$, then according to (6.7), there exists integer $N_{\epsilon(s)} \ge 1$, s.t. for any $n \ge N_{\epsilon(s)}$, $|R_1^s(n)| \le \epsilon(s)(\log_2(\pi n))^{2s}$. Thus, we get: for any $n \ge N_s := \max\left\{N_{\epsilon(s)}, [e^4\pi]+1\right\}$,

$$\frac{\ln 2}{2^{2(s \vee \frac{1}{2})+3}} \le \frac{R_n^s}{(\log_2(\pi n))^{2s}} \le \frac{\ln 2}{2s} + \frac{\ln 2}{2^{2(s \vee \frac{1}{2})+3}},$$

where we used $R_2^s(n) - |R_1^s(n)| \le R_n^s \le R_2^s(n) + |R_1^s(n)|$. Combined with reasoning at the beginning of the proof. ■

Remark 6.6 *Similarly, one can prove that there exist constants* $0 < k < K < \infty$*, s.t. for any* $f \in L^2\left([-\pi,\pi],\mathbb{R}^d\right)$,

$$k\int_{-\pi}^{\pi}|f(x)|^2\,dx \leq \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(u)-f(v)|^2\,dudv \leq K\int_{-\pi}^{\pi}|f(x)|^2\,dx.$$

Then $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|f(u)-f(v)|^2\,dudv < \infty$ *iff* f *is in* L^2*. Since* $\left|\sin\frac{u-v}{2}\right| \leq 1$*, so from this perspective, one can also get that*

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|}dudv < \infty \Longrightarrow f \textit{ is an } L^2 \textit{ function.}$$

Combine Theorem 6.1 with Corollary 5.17(in p71), we get that if

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|}dudv < \infty,$$

then f is in L^2(also Remark 6.6), and the partial sum process of Fourier series of f is a geometric 2-rough process (denoted as $\mathbf{X}$). Moreover, there exists absolute constant C,

$$\int_{-\pi}^{\pi}\|\mathbf{X}(\theta)\|_{G^{(2)}}^2\,d\theta \leq C\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{|f(u)-f(v)|^2}{\left|\sin\frac{u-v}{2}\right|}dudv \sim \sum_{n=0}^{\infty}\log_2(n+1)|c_n|^2 \tag{6.9}$$

However, although in (6.9) $\log_2(n+1)$ can not be replaced by $o(\log_2(n+1))$, as we show below, the Weyl multiplier $\{\log_2(n+1)\}$ is not necessary for the partial sum process of Fourier series to be a geometric 2-rough process.

Example 6.7 *There exists a* 2*-dimensional* L^2 *Fourier series* $\sum_{n=1}^{\infty}c_ne^{in\theta}$*, s.t. its partial sum process is a geometric* 2*-rough process, but* $\sum_n\log_2(n+1)|c_n|^2 = \infty$.

The same example can be modified to any Weyl multiplier growing strictly faster than $\left\{(\log_2\log_2 n)^2\right\}$, as in Example 6.3 proved after this example.

Proof. Define

$$f(\theta) = \sum_{n=1}^{\infty}\frac{1}{n2^{\frac{n}{2}}}\sum_{k=2^n+1}^{2^{n+1}}e^{ik\theta},\ \ \theta \in [0,2\pi). \tag{6.10}$$

Then $|c_n|^2 \sim n^{-1}(\log_2 n)^{-2}$, so f is in L^2 and $\sum_n\log_2(n+1)|c_n|^2 = \infty$. Denote X as the partial sum process of f, then

$$X_\theta(k) = \frac{e^{i(2^n+1)\theta}-e^{i(k+1)\theta}}{n2^{\frac{n}{2}}\left(1-e^{i\theta}\right)} + X_\theta(2^n),\ k = 2^n+1,\ldots,2^{n+1}. \tag{6.11}$$

Suppose $X^1 = L(X,\{2^n\})$. Then X^1 can be enhanced into a geometric 2-rough process (if denote $v_n = 2^{-\frac{n}{2}}\sum_{k=2^n+1}^{2^{n+1}}e^{ik\theta}$, then X^1 is the reparametrised partial sum process of $\sum_n n^{-1}v_n$, and based on Theorem 5.13). Denote $A := A(X)$ as the area process of X. Based on Corollary 5.28(in p75), we are done if we can prove,

$$\sum_{n=1}^{\infty}\left(\|X_\theta\|_{2-var,[2^n,2^{n+1}]}^2 + \|A_\theta\|_{1-var,[2^n,2^{n+1}]}\right) < \infty \text{ a.e..} \tag{6.12}$$

When $\theta = 0$, $\|X_\theta\|_{1-var} = \infty$, so (X_0, A_0) is not a geometric 2-rough path. In the following, we prove that (6.12) holds for any $\theta \in (0, 2\pi)$. More specifically, we prove that, for any $\theta \in (0, 2\pi)$, there exists constant C_θ, s.t.

$$\|X_\theta\|^2_{2-var,[2^n,2^{n+1}]} + \|A_\theta\|_{1-var,[2^n,2^{n+1}]} \le \frac{C_\theta}{n^2}, \forall n \ge \max\left\{\log_2(\frac{2\pi}{\theta}), 1\right\}.$$

We do analysis for fixed $\theta \in (0, 2\pi)$ and on fixed interval $[2^n, 2^{n+1}]$ for integer $n \ge \max\{\log_2(\frac{2\pi}{\theta}), 1\}$.

Denote $N := \left[\frac{2^n\theta}{2\pi}\right]$ (the integer part of $\frac{2^n\theta}{2\pi}$), so $N \ge 1$ since $n \ge \log_2(\frac{2\pi}{\theta})$. Denote $t_j := \frac{2j\pi}{\theta}$ for $j = 0, 1, \dots, N$.

In (6.11) for fixed θ and fixed n: $X_\theta(2^n)$, n, $2^{\frac{n}{2}}$, $1 - e^{i\theta}$ and $e^{i(2^n+1)\theta}$ are constants, so denote

$$\widetilde{Y}(t) = \begin{cases} e^{it\theta} & t \in [0, 2^n] \\ e^{i2^n\theta} & t \in (2^n, \infty) \end{cases} \quad \text{and } Y(t) = \begin{cases} \widetilde{Y}(t) & t = 0, 1, \dots, 2^n \\ \text{linear} & \text{elsewhere on } [0, 2^n] \\ \widetilde{Y}(2^n) & t \in (2^n, \infty) \end{cases}.$$

First, we estimate 2-variation of Y. since $\|Y\|^2_{2-var} \le \|\widetilde{Y}\|^2_{2-var}$, suppose $s < t_{j_1} \le t_{j_2} < t$, then

$$\|\widetilde{Y}(s,t)\|^2 \le 2\left(\|\widetilde{Y}(s, t_{j_1})\|^2 + \|\widetilde{Y}(t_{j_2}, t)\|^2\right),$$

where $\widetilde{Y}(t_{j_1}) = \widetilde{Y}(t_{j_2})$. Thus, when $N \ge 1$, using $\|\widetilde{Y}\|^2_{2-var,[t_j,t_{j+1}]} \le \|\widetilde{Y}\|^2_{1-var,[t_j,t_{j+1}]} = 4\pi^2$, $j = 0, 1, \dots, N$,

$$\|\widetilde{Y}\|^2_{2-var} \le 2\sum_{j=0}^{N} \|\widetilde{Y}\|^2_{2-var,[t_j,t_{j+1}]} \le 8\pi^2(N+1) \le 16\pi^2 N. \tag{6.13}$$

For area of Y, denote

$$Y_1(t) := \begin{cases} \widetilde{Y}(t) & t = 0, 1, \dots, 2^n \text{ or } t_0, t_1, \dots, t_N \\ \text{linear} & \text{elsewhere on } [0, 2^n] \\ \widetilde{Y}(2^n) & t \in (2^n, \infty) \end{cases},$$

$$Y_2 := Y - Y_1.$$

Denote $A_Y := A(Y)$ and $A_{Y,2} := A(Y_2)$. According to Lemma 5.27(at p74), since $Y_1(t_j) = 1$, $j = 0, 1, \dots, N$, we have (here $t_0 = 0$)

$$\|A_Y\|_{1-var} \le 2\|\widetilde{Y}\|^2_{2-var} + 2\sum_{j=0}^{N} \|A_Y\|_{1-var,[t_j,t_{j+1}]} + 2\sup_{\{m_k\}} \sum_{m_k < m_{k+1}} \left\|A_{Y,2}\left(t_{m_k}, t_{m_{k+1}}\right)\right\| \tag{6.14}$$

where we used $\|Y\|^2_{2-var} \le \|\widetilde{Y}\|^2_{2-var}$.

First, we prove that, $\|A_Y\|_{1-var,[t_j,t_{j+1}]} \le \pi^2 + 2\|Y\|^2_{2-var,[t_j,t_{j+1}]}$, $j = 0, 1, \dots, N$. Denote

$$n_j := [t_j] \text{ (the integer part of } t_j). \tag{6.15}$$

Since $t_{j+1}-t_j = \frac{2\pi}{\theta} > 1$, so $n_{j_1} \neq n_{j_2}$ when $j_1 \neq j_2$, and $n_j+1 \leq n_{j+1}$, $j = 0, 1, \ldots, N-1$. Using multiplicativity of (Y, A_Y) at (5.7) and taking supremum over all integer finite partitions, we get

$$\|A_Y\|_{1-var,[t_j,t_{j+1}]} \leq \|A_Y\|_{1-var,[n_j+1,n_{j+1}]} + 2\,\|Y\|^2_{2-var,[t_j,t_{j+1}]}\,. \tag{6.16}$$

On $[n_j+1, n_{j+1}] \subseteq [t_j, t_{j+1}]$: If $n_{j+1} = n_j+1$ or n_j+2, then Y on $[n_j+1, n_{j+1}]$ is a point or describes a cord of unit circle, so $\|A_Y\|_{1-var,[n_j+1,n_{j+1}]} = 0$; If $n_{j+1} \geq n_j+3$, then Y on $[n_j+1, n_{j+1}]$ — after connecting $Y(n_{j+1})$ with $Y(n_j+1)$ — draws a simple convex polygon, with unit circle its circumcircle, so $\|A_Y\|_{1-var,[n_j+1,n_{j+1}]} \leq \pi^2$. Combined with (6.16),

$$\|A_Y\|_{1-var,[t_j,t_{j+1}]} \leq \pi^2 + 2\,\|Y\|^2_{2-var,[t_j,t_{j+1}]}\,.$$

Thus, we have, when $N \geq 1$, (use $\|Y\|^2_{2-var} \leq \|\widetilde{Y}\|^2_{2-var}$ and (6.13))

$$\sum_{j=0}^{N} \|A_Y\|_{1-var,[t_j,t_{j+1}]} \leq \pi^2(N+1) + 2\|\widetilde{Y}\|^2_{2-var} \leq 34\pi^2 N. \tag{6.17}$$

Therefore, what left in (6.14) is $\sup_{\{m_k\}} \sum_{n_k<n_{k+1}} \left\|A_{Y,2}\left(t_{m_k}, t_{m_{k+1}}\right)\right\|$.

For $\|A_{Y,2}(t_{m_k}, t_{m_{k+1}})\|$, using multiplicativity and $Y_2(n_{m_k}+1) = Y_2\left(n_{m_{k+1}}\right)$,

$$\begin{aligned}\|A_{Y,2}(t_{m_k}, t_{m_{k+1}})\| \ \leq\ & \left\|A_{Y,2}\left(n_{m_k}+1, n_{m_{k+1}}\right)\right\| \\ & +2^{-1}\|Y_2(n_{m_k}+1) - Y_2(t_{m_k})\|^2 + 2^{-1}\|Y_2(t_{m_{k+1}}) - Y_2(n_{m_{k+1}})\|^2.\end{aligned} \tag{6.18}$$

For $A_{Y,2}\left(n_{m_k}+1, n_{m_{k+1}}\right)$, since $Y_2(k) = 0$, $k = 0, 1, \ldots, 2^n$, use property (5.6) and that Y_2 can only be non-trivial on $[n_j, n_j+1]$,

$$\left\|A_{Y,2}\left(n_{m_k}+1, n_{m_{k+1}}\right)\right\| \leq \sum_{j,[n_j,n_j+1]\subseteq[n_{m_k}+1,n_{m_{k+1}}]} \|A_{Y,2}(n_j, n_j+1)\|\,.$$

Since Y_2 is linear on $[n_j, t_j]$ and $[t_j, n_j+1]$ ($n_{j_1} \neq n_{j_2}$ when $j_1 \neq j_2$),

$$\|A_{Y,2}(n_j, n_j+1)\| \leq 2^{-1}\|Y_2(t_j) - Y_2(n_j)\|^2 + 2^{-1}\|Y_2(n_j+1) - Y_2(t_j)\|^2\,.$$

Thus

$$\begin{aligned}&\left\|A_{Y,2}\left(n_{m_k}+1, n_{m_{k+1}}\right)\right\| \\ \leq\ & \sum_{j=m_k+1}^{m_{k+1}-1} 2^{-1}\left(\|Y_2(t_j) - Y_2(n_j)\|^2 + \|Y_2(n_j+1) - Y_2(t_j)\|^2\right).\end{aligned} \tag{6.19}$$

Combine (6.18) with (6.19),

$$\begin{aligned}&\left\|A_{Y,2}(t_{m_k}, t_{m_{k+1}})\right\| \\ \leq\ & 2^{-1}\sum_{j=m_k+1}^{m_{k+1}-1}(\|Y_2(t_j) - Y_2(n_j)\|^2 + \|Y_2(n_j+1) - Y_2(t_j)\|^2) \\ & +2^{-1}\|Y_2(n_{m_k}+1) - Y_2(t_{m_k})\|^2 + 2^{-1}\left\|Y_2(t_{m_{k+1}}) - Y_2(n_{m_{k+1}})\right\|^2.\end{aligned} \tag{6.20}$$

Thus, because $Y_2(t_j) = Y(t_j) - Y_1(t_j) = Y(t_j) - 1$, $j = 0, \dots, N$, and $Y_2(k) = 0$, $k = 0, 1, \dots, 2^n$, so use (6.20), we get

$$\begin{aligned} & \sup_{\{m_k\}} \sum_k \left\| A_{Y,2}(t_{m_k}, t_{m_{k+1}}) \right\| \\ \leq\ & 2^{-1} \sum_{j=0}^{N} \left(\|Y_2(t_j) - Y_2(n_j)\|^2 + \|Y_2(n_j+1) - Y_2(t_j)\|^2 \right) = \sum_{j=0}^{N} \|Y(t_j) - 1\|^2 . \end{aligned} \tag{6.21}$$

Since Y is linear on $[n_j, n_j+1]$, so there exists $\rho_j \in [0,1]$, s.t. $Y(t_j) = \rho_j Y(n_j) + \left(1 - \rho_j\right) Y(n_j+1)$. Hence

$$\begin{aligned} \|Y(t_j) - 1\|^2 &= \left\| \rho_j \left(Y(n_j) - 1\right) + \left(1 - \rho_j\right)\left(Y(n_j+1) - 1\right) \right\|^2 \\ &\leq \|Y(n_j) - 1\|^2 + \|Y(n_j+1) - 1\|^2 \\ &= \|\widetilde{Y}(t_j) - \widetilde{Y}(n_j)\|^2 + \|\widetilde{Y}(n_j+1) - \widetilde{Y}(t_j)\|^2, \end{aligned} \tag{6.22}$$

where we used that Y coincides with $\widetilde{Y}$ at integers, and $\widetilde{Y}(t_j) = 1$. Therefore, combine (6.21) and (6.22), we get

$$\begin{aligned} & \sup_{\{m_k\}} \sum_k \|A_{Y,2}(t_{m_k}, t_{m_{k+1}})\| \\ \leq\ & \sum_{j=0}^{N} \left(\|\widetilde{Y}(t_j) - \widetilde{Y}(n_j)\|^2 + \|\widetilde{Y}(n_j+1) - \widetilde{Y}(t_j)\|^2 \right) \leq \|\widetilde{Y}\|_{2-var}^2 . \end{aligned} \tag{6.23}$$

Therefore, when $N \geq 1$, combine (6.14) and (6.23), we get

$$\begin{aligned} \|A_Y\|_{1-var} &\leq 2\|\widetilde{Y}\|_{2-var}^2 + 2\sum_{j=0}^{N} \|A_Y\|_{1-var,[t_j,t_{j+1}]} + 2 \sup_{\{m_k\}} \sum_k \left\| A_{Y,2}(t_{m_k}, t_{m_{k+1}}) \right\| \\ &\leq 2\sum_{j=0}^{N} \|A_Y\|_{1-var,[t_j,t_{j+1}]} + 4\|\widetilde{Y}\|_{2-var}^2 . \end{aligned}$$

Use $\|\widetilde{Y}\|_{2-var}^2 \leq 16\pi^2 N$ at (6.13) and $\sum_{j=0}^{N} \|A_Y\|_{1-var,[t_j,t_{j+1}]} \leq 34\pi^2 N$ at (6.17), we get

$$\|A_Y\|_{1-var} \leq 132\pi^2 N. \tag{6.24}$$

Finally, since

$$\|X_\theta\|_{2-var,[2^n,2^{n+1}]}^2 = \frac{1}{4n^2 2^n \sin^2 \frac{\theta}{2}} \|Y\|_{2-var}^2 \leq \frac{1}{4n^2 2^n \sin^2 \frac{\theta}{2}} \|\widetilde{Y}\|_{2-var}^2 ,$$

combined with $\|\widetilde{Y}\|_{2-var}^2 \leq 16\pi^2 N$ at (6.13) and $N \leq \frac{2^n \theta}{2\pi}$,

$$\|X_\theta\|_{2-var,[2^n,2^{n+1}]}^2 \leq \frac{4\pi^2 N}{n^2 2^n \sin^2 \frac{\theta}{2}} \leq \frac{2\pi\theta}{\sin^2 \frac{\theta}{2}} \frac{1}{n^2} , \quad \forall n \geq \max\{\log_2(\frac{2\pi}{\theta}), 1\}.$$

Similarly, for area of X_θ on $[2^n, 2^{n+1}]$, use (6.24),

$$\|A_\theta\|_{1-var,[2^n,2^{n+1}]} = \frac{1}{4n^2 2^n \sin^2 \frac{\theta}{2}} \|A_Y\|_{1-var} \leq \frac{33\pi\theta}{2\sin^2 \frac{\theta}{2}} \frac{1}{n^2}, \forall n \geq \max\{\log_2(\frac{2\pi}{\theta}), 1\}.$$

Therefore,

If for $\theta \in (0, 2\pi)$, define $C_\theta = \dfrac{37\pi\theta}{2\sin^2 \frac{\theta}{2}}$, then

$$\|X_\theta\|^2_{2-var,[2^n,2^{n+1}]} + \|A_\theta\|_{1-var,[2^n,2^{n+1}]} \leq \frac{C_\theta}{n^2}, \forall n \geq \max\{\log_2(\frac{2\pi}{\theta}), 1\}. \quad (6.25)$$

Combined with the remark at the beginning of this example. ■

Although in the example above, (X_θ, A_θ) is of finite 2-rough norm when $\theta \neq 0$, the integration $\int_\Omega \|\mathbf{X}_\theta\|_{G^{(2)}} d\theta$ is not finite, and the problem occurs at 0 or 2π, as one may see. After some modifications, we can push the result a little bit further. The convergent factor n^{-2} only appeared in (6.25), so one could modify the example to

$$\sum_{n=1}^{\infty} \frac{1}{a^{\frac{1}{2}}(n)\, 2^{\frac{n}{2}}} \sum_{k=2^n+1}^{2^{n+1}} e^{ik\theta}, \quad (6.26)$$

for any positive $\{a(n)\}$ satisfying $\sum \frac{1}{a(n)} < \infty$. However, the long time behavior will then cause a problem. Denote X^1 as the piecewisely linear process which coincides with X at $\{2^n\}$. According to Theorem 5.13, we know that if $\sum_n (\log_2 n)^2 / a(n) < \infty$, then X^1 is a geometric 2-rough process, so it will not be a problem under that condition, while the local regularity is controlled by (6.25). In that case, based on Corollary 5.28(in p75), the partial sum process of (6.26) is a geometric 2-rough process. Therefore, we can generalize Example 6.7:

Example 6.3 *Suppose $\{w(n)\}$ is a Weyl multiplier that $n \mapsto \frac{w(n)}{(\log_2 \log_2 n)^2}$ is strictly increasing from some point on and $\lim_{n\to\infty} \frac{w(n)}{(\log_2 \log_2 n)^2} = \infty$. Then there exists a 2-dimensional Fourier series $\sum_{n=1}^{\infty} c_n e^{in\theta}$, s.t. its partial sum process is a geometric 2-rough process, but $\sum_n w(n)|c_n|^2 = \infty$.*

Proof. In light of Example 6.7, we only have to prove the statement for $\{w(n)\}$ growing slower than $\{\log_2(n+1)\}$. Thus, assume $\lim_{n\to\infty} \frac{w(2^{n+1})}{w(2^n)} = 1$. According to the condition of this example, assume $N \geq 2$ is such an integer, that $n \mapsto \frac{w(2^n)}{(\log_2 n)^2}$ is strictly increasing for all $n \geq N$. Let $r : [N-1, \infty) \to \mathbb{R}^+$ be a differentiable path satisfying $r'(t) \geq 0$ for all $t \geq N-1$, and

$$r(n) = \frac{w(2^n)}{(\log_2 n)^2}, \ n \geq N, \text{ with } r(N-1) = \frac{1}{2} r(N). \quad (6.27)$$

Moreover, we assume,

$$r'(n) = \frac{r(n+1) - r(n-1)}{2}, \ n \geq N, \text{ with } r'_+(N-1) = \frac{1}{2} r'(N). \quad (6.28)$$

Such kind of function r exists: The problem boils down to, for fixed real numbers $k > 0$, $k_1 > 0$, $k_2 > 0$, constructing a one dimensional non-decreasing differentiable function f, defined on $[0, 1]$, satisfying $f(0) = 0$, $f(1) = k$, $f'_+(0) = k_0$, $f'_-(1) = k_1$, and we further require that $\int_0^t f'(s)\,ds = f(t)$, $\forall t \in [0,1]$. Then f exists, if there exists a continuous function ρ, defined on $[0,1]$, satisfying $\rho(t) \geq 0$, $\rho(0) = k_0$, $\rho(1) = k_1$, $\int_0^1 \rho(t)\,dt = k$. Such ρ clearly exists, so $f(t) = \int_0^t \rho(s)\,ds$ satisfies the condition of f. Thus, we can construct r by first setting its value at integers by (6.27) and (6.28), then on $[n, n+1]$ for integer $n \geq N-1$ use the construction of f as above. In this way, r is absolutely continuous on any finite interval $[a,b] \subseteq [N-1, \infty)$ (its derivative is continuous, so r is Lipschitz on any finite interval), thus we have $\int_a^b r'(t)\,dt = r(b) - r(a)$. As an application, use (6.28),

$$r'(n) = \frac{1}{2}\int_{n-1}^{n+1} r'(t)\,dt. \tag{6.29}$$

$$\text{Let } \frac{1}{a(n)} = \frac{r'(n)}{r(n)\sqrt{(\log_2 n)^2 w(2^n)}}\text{; define } f(\theta) := \sum_{n=N}^{\infty} \frac{1}{a^{\frac{1}{2}}(n)\,2^{\frac{n}{2}}} \sum_{k=2^n+1}^{2^{n+1}} e^{ik\theta}.$$

Then since $\lim_{n\to\infty} \frac{w(2^{n+1})}{w(2^n)} = 1$, so $\lim_{n\to\infty} \frac{r(n+1)}{r(n)} = 1$, and using (6.29),

$$\begin{aligned}
\sum_{n\geq N} \frac{(\log_2 n)^2}{a(n)} &= \sum_{n\geq N} \frac{(\log_2 n)^2\, r'(n)}{r(n)\sqrt{(\log_2 n)^2 w(2^n)}} = \sum_{n\geq N} \frac{r'(n)}{(r(n))^{\frac{3}{2}}} \sim \sum_{n\geq N} \frac{r'(n)}{(r(n+1))^{\frac{3}{2}}} \\
&\leq \lim_{M\to\infty} \sum_{n=N}^{M} \frac{1}{2}\int_{n-1}^{n+1} \frac{r'}{r^{\frac{3}{2}}}dt = \lim_{M\to\infty} \int_{N-1}^{M+1} \frac{dr}{r^{\frac{3}{2}}} = \frac{2}{\sqrt{r(N-1)}} < \infty.
\end{aligned}$$

Thus, by following exactly the same reasoning of Example 6.7, the partial sum process of f is a geometric 2-rough process. On the other hand, since $\{w(n)\}$ is non-decreasing, so

$$\begin{aligned}
\sum_{n\geq 2^N+1} w(n)\,|c_n|^2 &\geq \sum_{n\geq N} \left(\sum_{k=2^n+1}^{2^{n+1}} |c_k|^2\right) w(2^n) = \sum_{n\geq N} \frac{w(2^n)}{a(n)} \\
= \sum_{n\geq N} \frac{r'(n)}{\sqrt{r(n)}} &\overset{(6.29)}{\geq} \lim_{M\to\infty} \sum_{n=N}^{M} \frac{1}{2}\int_{n}^{n+1} \frac{r'}{\sqrt{r}}dt = \lim_{M\to\infty} \frac{1}{2}\int_{N}^{M+1} \frac{dr}{\sqrt{r}} = \infty.
\end{aligned}$$

■

6.2 L^2 Fourier series with infinite 2-variation almost everywhere

Before construction, we prove the upper semi-continuity of the cumulative distribution function of p-variation.

Lemma 6.8 *Suppose $\{X_n\}_{n=1}^{\infty}$ and X are continuous processes, defined on probability space $(\Omega, \mathcal{F}, P)$, taking value in $\mathbb{R}^d$, and X_n converge to X in distribution as n tends to infinity. Then for any $p \geq 1$, $C \geq 0$,*

$$\overline{\lim}_{n\to\infty} P\left(\|X_n\|_{p-var} \leq C\right) \leq P\left(\|X\|_{p-var} \leq C\right).$$

Proof. $C[0,\infty)$, the space of continuous $\mathbb{R}^d$-valued functions on $[0,\infty)$, is a complete, separable metric space when equipped with the metric:

$$\rho(\omega_1,\omega_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0\leq t\leq n} \left(|\omega_1(t) - \omega_2(t)| \wedge 1\right).$$

X_n and X are random variables taking values in $(C[0,\infty), \mathcal{B}(C[0,\infty)))$. According to Skorohod's theorem, there exists $\widetilde{X_n}$ and $\widetilde{X}$ on an auxiliary space, s.t. $X_n \overset{D}{=} \widetilde{X_n}$, $X \overset{D}{=} \widetilde{X}$, and $\widetilde{X_n}$ converges to $\widetilde{X}$ a.e.. Use Fatou's lemma and lower semi-continuity of p-variation,

$$\begin{aligned} &\underline{\lim}_{n\to\infty} P\left(\|X_n\|_{p-var} > C\right) = \underline{\lim}_{n\to\infty} P\left(\left\|\widetilde{X_n}\right\|_{p-var} > C\right) \\ \geq\ & P\left(\underline{\lim}_{n\to\infty}\left\{\left\|\widetilde{X_n}\right\|_{p-var} > C\right\}\right) = P\left(\underline{\lim}_{n\to\infty}\left\|\widetilde{X_n}\right\|_{p-var} > C\right) \\ \geq\ & P\left(\left\|\widetilde{X}\right\|_{p-var} > C\right) = P\left(\|X\|_{p-var} > C\right). \end{aligned}$$

■

As a trivial Corollary, for any $\alpha > 0$, $p \geq 1$,

$$\underline{\lim}_{n\to\infty} E\left(\|X_n\|_{p-var}^{\alpha}\right) \geq E\left(\|X\|_{p-var}^{\alpha}\right) \tag{6.30}$$

Corollary 6.9 *Suppose S_k is the sum of first k terms of a sequence of i.i.d. random variables with mean 0 and variance 1. Define ξ_n as the continuous process on $[0,1]$ obtained by interpolating $S_k/n^{\frac{1}{2}}$ at k/n, $k = 0,1,\ldots,n$. Then for any $C \geq 0$,*

$$\lim_{n\to\infty} P\left(\|\xi_n\|_{2-var} > C\right) = 1.$$

Proof. ξ_n converge in distribution to the Wiener process W, use Lemma 6.8 and that Wiener process is of infinite 2-variation a.e., we get

$$\underline{\lim}_{n\to\infty} P\left(\|\xi_n\|_{2-var,[0,1]} > C\right) \geq P\left(\|W\|_{2-var,[0,1]} > C\right) = 1.$$

■

It is proved in [30] that there exists constant $c > 0$ s.t., if the i.i.d. random variables have finite $2+\delta$ moment for some $\delta > 0$, then $\lim_{n\to\infty} P\left(\|\xi_n\|_{2-var}^2 \geq c\ln\ln n\right) = 1$.

If we were working with Rademacher functions ($r_k(t) = sgn \sin\left(2^k\pi t\right)$, $t \in [0,1]$, $k \geq 1$), the construction would be clearer, because r_k are independent. Glue pieces of rescaled random walks together, where each piece provides sufficiently large 2-variation, then a.e. infinite 2-variation follows from Borel-Cantelli lemma. It is similar for Fourier series, only that we pick out those trigonometric functions which resemble an i.i.d. sequence. (For any m and n, $e^{2\pi in\theta}$ and $e^{2\pi im\theta}$ are never independent: suppose θ is uniformly distributed on $[0,1]$, with a binary expansion $\sum_{k=1}^{\infty}\theta_k 2^{-k}$, then both $\{n\theta\}$ and $\{m\theta\}$ – the fractional part of $n\theta$ and $m\theta$ – depend on $\sigma(\{\theta_k\}_{k\geq K})$ for some $K \geq 1$, comparing to Rademacher system, which is independent because $r_k = -2\theta_k + 1$.) However, there are far more trigonometric sequences, which do not exhibit random behavior, but with a heavy L^2 tail and infinite 2-variation almost everywhere.

Suppose we have a sequence of integers $\overbrace{n_1, n_1, \ldots, n_1}^{m_1}, \ldots, \overbrace{n_k, n_k, \ldots, n_k}^{m_k}, \ldots$ where n_k, m_k, $k \geq 1$ are integers. Denote the partial sum of this sequence as $s_0 = 0$, $s_k = \sum_{j=1}^{k} m_j n_j$. Suppose θ is uniformly distributed on $[0,1]$, and θ_k is the kth digit of the binary expansion of θ, i.e. $\theta = \sum_{k=1}^{\infty} \theta_k 2^{-k}$. One can check that $\{\theta_k\}_{k\geq 1}$ are i.i.d. random variables satisfying $P(\theta_k = 1) = P(\theta_k = 0) = \frac{1}{2}$.

Definition 6.10 *Define a sequence of random variables*

$$\varsigma_i^{(n_k)} = \cos\left(2\pi \sum_{j=1}^{n_k} \frac{\theta_{s_{k-1}+(i-1)n_k+j}}{2^j}\right), \; i = 1, 2, \ldots, m_k, \; k \geq 1, \tag{6.31}$$

where m_k, n_k, s_k, and θ_k are defined above.

$\{\varsigma_i^{(n_k)}, 1 \leq i \leq m_k, k \geq 1\}$ are independent with mean 0 variance $\frac{1}{2}$, and for each fixed k, $\{\varsigma_i^{(n_k)}, 1 \leq i \leq m_k\}$ are identically distributed. Moreover,

$$\left|\varsigma_i^{(n_k)} - \cos\left(2\pi 2^{s_{k-1}+(i-1)n_k}\theta\right)\right| \leq \frac{\pi}{2^{n_k-1}}. \tag{6.32}$$

Suppose X and Y are respectively the partial sum process of

$$f(\theta) = \sum_{k=1}^{\infty} \frac{1}{k\sqrt{m_k}} \sum_{j=1}^{m_k} \cos\left(2\pi 2^{s_{k-1}+(j-1)n_k}\theta\right) \text{ and } \varsigma = \sum_{k=1}^{\infty} \frac{1}{k\sqrt{m_k}} \sum_{j=1}^{m_k} \varsigma_j^{(n_k)}.$$

Then by showing that Y is of infinite 2-variation a.e., and choosing n_k and m_k to control the cumulated error produced by (6.32), we can prove that X of infinite 2-variation a.e.. However, the estimation in Example 6.4 (re-stated below) forces us to choose m_k before n_k. Therefore, we need a result of uniform growth of 2-variation of random walks produced by $\varsigma_i^{(n_k)}$ for different ks.

Definition 6.11 *Define Y_m^n as the continuous process on* $[0,1]$ *obtained by interpolating* $\sum_{i=1}^{k} \varsigma_i^{(n)}/m^{\frac{1}{2}}$ *at k/m, $k = 1, 2, \ldots, m$, where $\varsigma_i^{(n)}$, $i = 1, 2, \ldots, m$, are as defined in* (6.31).

Lemma 6.12 *For any constant $C \geq 0$,*

$$\underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} P\left(\|Y_m^n\|_{2-var} > C\right) = 1.$$

Proof. Suppose $\{\theta_i\}_{i=1}^{m}$ are independent random variables uniformly distributed on $[0,1]$, and Y_m the continuous process got by interpolating $(\sum_{i=1}^{k} \cos\theta_i)/m^{\frac{1}{2}}$ at k/m. Since $P(\varsigma_i^{(n)} = \cos(2\pi k 2^{-n})) = 2^{-n}$, $k = 0, \ldots, 2^n - 1$, so $\varsigma_i^{(n)}$ converge to $\cos\theta_i$ in distribution as $n \to \infty$. Noting that m is fixed, and $\varsigma_i^{(n)}$, $i = 1, 2, \ldots m$, are independent, so Y_m^n converge to Y_m in distribution as $n \to \infty$. Use Lemma 6.8 and Corollary 6.9,

$$\underline{\lim}_{m\to\infty} \underline{\lim}_{n\to\infty} P\left(\|Y_m^n\|_{2-var,[0,1]} > C\right) \geq \underline{\lim}_{m\to\infty} P\left(\|Y_m\|_{2-var,[0,1]} > C\right) = 1.$$

■

Now, we are prepared to construct our series.

Example 6.4 *There exists an L^2 Fourier series whose partial sum process has infinite 2-variation almost everywhere.*

Proof. According to Lemma 6.12, there exists a sequence of integers, $\{M_s\}_{s\geq 2}$, s.t. $\forall m \geq M_s$, $\exists N(s,m)$, s.t. $\forall n \geq N(s,m)$,

$$P\left(\|Y_m^n\|_{2-var}^2 > s^2\right) \geq \frac{1}{s}.$$

Set $m_k := \max_{1\leq s\leq k} M_s$. Choose $\{n_k\}_{k=1}^{\infty}$, s.t. $n_k \geq N(k, m_k)$, $2^{n_k} > k\sqrt{m_k}$, and $n_{k+1} > n_k$. Hence,

$$P\left(\left\|Y_{m_k}^{n_k}\right\|_{2-var}^2 > k^2\right) \geq \frac{1}{k}, \text{ and } \sum_{k=2}^{\infty} \frac{\sqrt{m_k}}{k 2^{n_k}} < \infty.$$

Denote Y as the continuous process constructed on $[0,\infty)$ by patching up $Y_{m_k}^{n_k}/k$, $k \geq 2$. Then based on the elementary inequality: $a^2 \geq b^2/2 - (a-b)^2$, we have: (X is the partial sum process of corresponding Fourier series)

$$\|X\|_{2-var}^2 \geq \frac{1}{2}\|Y\|_{2-var}^2 - \left(2\pi \sum_{k=2}^{\infty} \frac{\sqrt{m_k}}{k 2^{n_k}}\right)^2 \geq \frac{1}{2}\|Y\|_{2-var}^2 - C.$$

Noting that $Y_{m_k}^{n_k}$, $k \geq 1$, are independent, use Borel-Cantelli lemma,

$$\begin{aligned} & P\left(\|X\|_{2-var}^2 = \infty\right) \\ \geq\ & P\left(\|Y\|_{2-var}^2 = \infty\right) \geq P\left(\overline{\lim}_{k\to\infty}\left\{\left\|\frac{Y_{m_k}^{n_k}}{k}\right\|_{2-var}^2 > 1\right\}\right) = 1. \end{aligned}$$

■

In fact, the method above can be applied to all orthogonal systems in the form $\{\varphi(nx)\}_{n\geq 1}$, $x \in [0,1]$, where φ is an α-Hölder continuous function, $0 < \alpha \leq 1$.

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