Indifference Price and Optimal Hedging Performance for Variance Swaps

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Contents

1 Introduction

Indifference pricing and hedging of derivatives on non traded assets in a utlity based framework are now viewed as natural means to obtain values of such derivatives in an incomplete market. This valuation method has been treated and widely discussed in the literature by, among others, Delbaen et al [6], Musiela and Zariphopoulou [18, 19], Henderson [9] and Monoyios [17]. This method is due to Hodges and Neuberger [11] who introduced concepts of indifference price, which we will derive and analytically formulate for a variance swap.

Actually, as this utility based formulation of the valuation problem has a dual formulation, indifference price can be approached via both primal and dual formulation. Many papers such as [20] of Sircar and Zariphopoulou or [15, 16] due to Monoyios use principally the primal formulation, which is convenient in order to derive Hamilton-Jacobi-Bellman (HJB) equation for the value function and then deduce pricing PDEs. This kind of study even leads to nice expressions for indifference price as Musiela and Zariphopoulou [18] show that we can use a distortion method to achieve this aim when we are in a basis risk model, in which the incompleteness is due to a claim on a non-traded asset denoted Y and where we hedge using a correlated traded asset (the stock) denoted S. In this paper processes followed by the stock S and the non traded underlying asset Y are correlated geometric Brownian motions. And this work has been generalised by Grasselli and Hurd [8] via dual formulation, and by Monoyios [17] using the link between primal and dual approaches, who establish the same kind of formula for the indifference price of our derivative on the non traded asset using expectations under a particular measure, namely the minimal martingale measure.

The most popular derivatives on non traded assets which are concerned by this study are the weather derivatives as treated for instance by Davis [5] or volatility derivatives studied in [8]. We will focus our study in this context on variance swaps which are then simple volatility derivatives paying: $V_T = \int_0^T Y_s ds$ at maturity T, where we denote by Y the squared volatility or the variance. We will aslo choose a standard Heston model as defined historically in [10] for our stock S and variance Y to build our model, in order to refine more usual results with simple geometric Brownian motions. Our work will be based mainly on Grasselli and Hurd [8] derivation of analytical formulae for indifference price and optimal hedge of variance swaps. Our main contribution will be to extend their results to non discounted processes and to implement these analytical formulae to get numerical results such as the terminal hedging error when initially selling the variance swap and then hedging this position by the optimal hedging strategy. In this regard the variance swap is now actively traded on mature market where we know how to perfectly hedge it using a large number of vanilla calls and puts on S as shown by Carr, Ellis and Gupta [3]. Our study in incomplete market is indeed more adapted for emergent market where we do not have great liquidity and where we will only cover our position by trading in the stock. But we will see how to make the link between these two approaches and compare them.

We will then organize the dissertation as follows: in Section 2 we set-up the stochastic volatility model and give a review of the primal formulation to indifference pricing for a general claim $B(Y_T, S_T, V_T)$ along the lines of Sircar and Zariphopoulou [20]. Then we will specialize the problem to a claim depending only on the variance Y and the integral average V and see how we can obtain nice formulation via distortion method in Subsection 2.3. The last Subsection 2.4 will give a first formula for the indifference asking price of such a claim in terms of expectations under the minimal martingale measure and we will derive the pricing PDE depending on certainty equivalent as introduced by Grasselli and Hurd [8]. This was not done in their paper and can be considered as a new way of finding the pricing PDE in this model. Section 3 will consider the dual formulation of our utility maximization problem. In Subsection 3.1, we will introduce optimal trading strategy and derive its expression as in [8] but extended to non zero interest rate. In Subsection 3.2 we derive the pricing PDE from the dual framework and in Subsection 3.3 we define the residual risk process and see which SDE it follows in our model. In Subsection 3.4 we derive the indifference asking price p^{ask} formula and give also the formula for the certainty equivalent cer^B in abstract forms. In Section 4 we will follow calculus of [8, 13] to derive analytical formulae of: p^{ask} and cer^B using Laplace transform and the confluent hypergeometric function F_1. Our contribution is here to extend this to the optimal hedge in Subsection 4.2 by deriving the analytical formula for the hedge too. In Section 5, we conduct some new numerical investigations and verify first our analytical formulae of Section 4 via Monte-Carlo estimations for Subsection 5.1. We will then calibrate F_1 for different values of times to maturity and variance in Subsection 5.2, and once we will get correct calibration, our contribution will be to simulate the residual risk process R in Subsection 5.3 to compute the distribution of the terminal hedging error when selling the variance swap. This constitutes the implementation of an extended work of what Grasselli and Hurd do in [8] and we will go further in Subsection 5.4 by comparing this distribution with the distribution reached when adding a static position in a particular straddle to the hedge.

2 Primal Formulation

We begin by setting-up the general framework and define the main quantities which will be useful in the next parts of this dissertation. We will also derive the HJB equation for a general claim $B(Y_T, S_T, V_T)$ and then specialize our study by dropping the stock in order to obtain expectation representations of value function and indifference price via the Feyman-Kac theorem. Our main contribution will be the derivation of the pricing PDE for the asking price in the last part of the primal study.

2.1 General Model Set-up

In this section we set-up the general model in which we will derive the indifference price. We follow a similar construction as Sircar and Zariphopoulou [20] or Monoyios [15, 16] and use here an Heston model [10].
We will assume in this paper that we have a market with two assets: a riskless money market account denoted S_t^0 and a stock S_t, which follow respectively:

$$\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t \tag{1}$$

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sqrt{Y}_t S_t \mathrm{d}W_t^1 \tag{2}$$

for $0 \leq t \leq T$ .The parameters μ and the riskless interest rate r, are constant and non negative. As we choose an Heston model, the variance Y of the stock follows the correlated diffusion:

$$\mathrm{d}Y_t = (a - bY_t)\mathrm{d}t + c\sqrt{Y}_t \mathrm{d}W_t^\rho \tag{3}$$

where the Brownian motions $(W_t^1, W_t^\rho)_{0\leq t\leq T}$ have correlation ρ such that $\mathrm{d}W_t^1\mathrm{d}W_t^\rho = \rho\mathrm{d}t$ and $-1 \leq \rho \leq 1$. The dynamics (2) and (3) are thus written under the physical measure P. We can also rewrite (3) in terms of independent standard Brownian motions (W_t^1, W_t^2) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_\mathrm{t}), \mathcal{P})$, where the filtration $(\mathcal{F}_\mathrm{t})$ is the one generated by $(W_t^1, W_t^2)_{0\leq t\leq T}$. As these Brownian motions are independent, we can express W_t^ρ as :

$$W_t^\rho = \rho W_t^1 + \epsilon W_t^2 \tag{4}$$

with $\epsilon = \sqrt{1-\rho^2}$. We also denote by $(\mathcal{G}_\mathrm{t})$ the filtration generated by $(W_t^\rho)_{0\leq t\leq T}$, which drives the stock variance.

The generic derivative we worke with is a European one with payoff $B(Y_T, S_T, V_T)$, which depends then on the variance and the stock at maturity but also on $V_T = \int_0^T f(Y_s)\mathrm{d}s$, where f is a deterministic function from $\mathcal{R}^+$ to $\mathcal{R}^+$.

We now consider an agent with preferences expressed as an exponential utility function (Constant Absolute Risk Aversion) given by:

$$U(x) = -\exp(-\gamma x), \quad \gamma > 0 \tag{5}$$

where γ is the constant risk aversion parameter. The aim of our investor is to maximize the expected utility of terminal wealth at T with a random endowment of n units of claim. We assume that he can trade a dynamic portfolio in a self-financing manner, holding Δ_t units of the traded stock S_t at time t and balancing to hold the remainder in the money market account S_t^0 (We will take $S_0^0 = 1$ here such that $S_t^0 = e^{rt}$). And if we denote by π_t^Δ the total wealth invested in the stock, we have: $\pi_t^\Delta = \Delta_t S_t$ at time t such that the investor's total wealth X_t is given by:

$$\mathrm{d}X_t = (rX_t + (\mu - r)\pi_t^\Delta)\mathrm{d}t + \sqrt{Y}_t \pi_t^\Delta \mathrm{d}W_t^1 \tag{6}$$

and we will note that there is not explicit dependence on S in (6).

Assuming that we started from $X_t = x, S_t = s, Y_t = y$ and $V_t = v$ at $t \in [0, T]$, we can thus formulate this optimization problem for our investor in terms of the primal value function $F^n(t, x, y, s, v)$ (using the same notation as Monoyios in [15]) as finding the optimal strategy $\pi^{*,n}$ which achieves the following supremum:

$$F^n(t, x, y, s, v) = \sup_{\pi \in \mathcal{A}} E_{t,x,y,s,v}[U(X_T + nB(Y_T, S_T, V_T))] \tag{7}$$

where $E_{t,x,y,s,v}$ is the conditional expectation given $X_t = x, S_t = s, Y_t = y, V_t = v$ and $\mathcal{A}$ is the set of all relevant admissible strategies. We use here the set $\mathcal{A}$ as defined by Grasselli and Hurd [8]: $\mathcal{A} = \{\pi \in \mathrm{L(S)} : \int_0^{\mathrm{t}} \pi_{\mathrm{s}} \mathrm{dS_s}$ is a Q-martingale$\}$ with $L(S)$ the set of predictable Sintegrable processes and Q is an absolutely continuous measure with finite relative entropy with respect to P. And we will remark that, to be correctly defined the optimization problem need to assume that the random endowment $nB(Y_T, S_T, V_T)$ is bounded below. It is further discussed in [15, 16, 8, 20] and we will assume here that we work with suitable claims.

From this set-up, we now define the utility based indifference price for our claim.

Definition 1 *(Indifference price). It is specified a priori as the price $p^n(t,x,y,s,v)$ which makes the investor indifferent between making the deal or not. In other words, if n claims are at stake, the indifference price is obtained via:*

$$F^n(t, x - np^n(t,x,y,s,v), y, s, v) = F^0(t,x,y) \tag{8}$$

The second term in the equality comes from no clear dependence of F^0 on s,v as $n = 0$. For this particular case where there is no claim at stake, the problem reduces to the classical Merton's problem studied in [14]. The particular configuration of selling the claim and then hedging this position will be studied in the next parts of this dissertation. We use then a special notation for this asking price: $p^{ask}(t,x,y,s,v)$ which is defined as above for $n = -1$ via:

$$F^{-1}(t, x + p^{ask}(t,x,y,s,v), y, s, v) = F^0(t,x,y) \tag{9}$$

Now that we defined the indifference price in this utility maximization framework, we will write the Hamilton-Jacobi-Bellman equation for the primal value function F^n.

2.2 Hamilton Jacobi Bellman equation for $B(Y_T, S_T, V_T)$

In order to clarify the presentation we are using a similar notation as Sircar and Zariphopoulou in [20] and introducing the following operators:

$$J^{(s,y,v)}F^n = (a - by)F^n_y + \frac{1}{2}c^2 y F^n_{yy} + \mu s F^n_s + \frac{1}{2} y s^2 F^n_{ss} + f(y)F^n_v + \rho c y s F^n_{sy}$$

$$H^{(s,y)}(F^n_x, F^n_{xx}, F^n_{xs}, F^n_{xy}) = (rx + (\mu - r)\pi)F^n_x + \frac{1}{2}\pi^2 y F^n_{xx} + \rho c y \pi F^n_{xy} + y s \pi F^n_{xs}$$

and we can note that $J^{(s,y,v)}$ is actually the infinitesimal generator of the Markov process (S, Y, V) and $H^{(s,y)}$ is the appropriate Hamiltonian. It does not depend on V_t as this follows the dynamics: $\mathrm{d}V_t = f(Y_t)\mathrm{d}t$ and then has no volatility term. Then:

Proposition 2 *(HJB equation for $B(S_T, Y_T, V_T)$). The Hamilton-Jacobi-Bellman equation for the value function $F^n(t,x,y,s,v)$ is :*

$$F^n_t + J^{(s,y,v)}F^n + \max_{\pi} H^{(s,y)}(F^n_x, F^n_{xx}, F^n_{xs}, F^n_{xy}) = 0 \tag{10}$$

with terminal condition

$$F^n(T,x,y,s,v) = -e^{-\gamma(x+nB(y,s,v))} \tag{11}$$

This can be justified by the uniqueness of viscosity solution and we can refer to [20] for a clear proof, but as we will focus on claim paying $B(Y_T, V_T)$ a classical Bellman optimality principle argument with the fact that if one can find a solution to which Ito's lemma is applicable, leads us to a proof of optimality via the standard verification theorem.

More precisely, we can obtain the optimal strategy $\pi^{*,n}$ for n claims by deriving the Hamiltonian. Formally this is solution to:

$$\frac{\partial H^{(s,y)}(F_x^n, F_{xx}^n, F_{xs}^n, F_{xy}^n)}{\partial \pi} = (\mu - r)F_x^n + \pi^{*,n} y F_{xx}^n + \rho c y F_{xy}^n + s y F_{xs}^n = 0$$

which gives

$$\pi^{*,n} = -\frac{(\mu - r)F_x^n + \rho c y F_{xy}^n + s y F_{xs}^n}{y F_{xx}^n} \tag{12}$$

Substituting into (10), the HJB equation can be rewritten as:

$$F_t^n + J^{(s,y,v)}F^n + rxF_x^n - \frac{(\lambda_t F_x^n + \rho c\sqrt{y}F_{xy}^n + s\sqrt{y}F_{xs}^n)^2}{2F_{xx}^n} = 0 \tag{13}$$

where: $\lambda_t = \frac{\mu - r}{\sqrt{y}}$ is the market price of risk, and terminal boundary condition given by (11).

We rediscover in (13) the equation derived by Sircar and Zariphopoulou [20] for a claim $B(Y_T, S_T)$ with an additional term , namely $f(Y_t)F_v^n$, due to the dependence on V_t of our studied claim.

2.3 Distortion Solution for the particular case of $B(Y_T, V_T)$

From this point we are reducing our study to the case of a claim paying out $B(Y_T, V_T)$ at time T i.e it does not depend on S_T anymore. This restriction avoids a far more complicated study in terms of HJB equation solutions which involve viscosity solutions when there is an S-dependency as mentioned earlier. And as we will later focus on variance swaps, it suffices to look at claim $B(Y_T, V_T)$. For more discussion on the case of $B(Y_T, S_T)$, one can refer to [20].

We begin by rewriting the simplified HJB equation for $B(Y_T, V_T)$ where all the s-terms disappear:

$$F_t^n + J^{(y,v)}F^n + \max_{\pi} H^{(y)}(F_x^n, F_{xx}^n, F_{xs}^n, F_{xy}^n) = 0 \tag{14}$$

with the same notations as (10). And the optimal $\pi^{*,n}$ is in this case:

$$\pi^{*,n} = -\frac{(\mu - r)F_x^n + \rho c y F_{xy}^n}{y F_{xx}^n} \tag{15}$$

such that (14) can be rewritten as:

$$F_t^n + J^{(y,v)}F^n + rxF_x^n - \frac{(\lambda_t F_x^n + \rho c\sqrt{y}F_{xy}^n)^2}{2F_{xx}^n} = 0 \tag{16}$$

with terminal boundary condition:

$$F^n(T, x, y, v) = -e^{-\gamma(x+nB(y,v))} \tag{17}$$

As we use an exponential utility, we can factor out the initial endowment x. It comes from the Constant Absolute Risk Aversion property of $U(x) = -e^{-\gamma(x)}$. To quickly justify this, we recall (6):

$$\mathrm{d}X_t = (rX_t + (\mu - r)\pi_t^{\Delta})\mathrm{d}t + \sqrt{Y}_t \pi_t^{\Delta}\mathrm{d}W_t^1$$

If we sart from $X_t = x$, it leads to: $X_T = b(t,T)x + G$ where $b(t,T) = e^{r(T-t)}$ and the gain process: $G = (\mu - r)e^{rT}\int_t^T e^{-ru}\pi_u^{\Delta}\mathrm{d}u + \sqrt{y}e^{rT}\int_t^T e^{-ru}\pi_u^{\Delta}\mathrm{d}W_u^1$. We have then:

$$U(X_T + nB(Y_T, V_T)) = e^{-\gamma b(t,T)x}(-e^{-\gamma(G+nB(Y_T,V_T))})$$

Therfore we can work with a reduced value function $C^n(t,y,v) = F^n(t,0,y,v)$. And from (16) we derive the PDE for this reduced value function C^n:

Corollary 3 *(PDE for the reduced value function). C^n follows:*

$$C_t^n + (a - by - c\sqrt{(}y)\rho\lambda)C_y^n + \frac{1}{2}c^2 y C_{yy}^n + f(y)C_v^n - \frac{1}{2}\lambda^2 C^n - \frac{1}{2}\rho^2 c^2 y\frac{(C_y^n)^2}{C^n} = 0$$

With terminal boundary condition: $C^n(t,y,v) = -e^{-\gamma nB(y,v)}$. It is a semi-linear PDE where all x-terms have disappeared due to the previous factorisation.

To solve this we use a technique called distortion by Zariphopoulou [18], which is frequently used in the literature as one can see in [15, 16, 17]. This involves a power transformation to write the reduced value function as: $C^n(t,y,v) = (k^n(t,y,v))^\delta$ where δ is a constant parameter. It leads us to another PDE for the function k^n derived from (18):

$$\begin{aligned} k_t^n + (a - by - c\sqrt(y)\rho\lambda)k_y^n + \frac{1}{2}c^2 y k_{yy}^n + f(y)k_v^n \\ - \frac{\lambda^2}{2\delta}k^n + \frac{1}{2}c^2 y[(\delta - 1) - \rho^2\delta]\frac{(k_y^n)^2}{k^n} = 0 \end{aligned} \tag{19}$$

The terminal boundary condition is: $k^n(T,y,v) = -e^{-\frac{\gamma n B(y,v)}{\delta}}$

For a general δ, we still have a non-linear PDE but with an appropriate choice, i.e $\delta = \frac{1}{1-\rho^2}$, we can reduce the equation (19) to a linear one (as it makes $(\delta - 1) - \rho^2\delta$ equals to 0). It is stated in the next proposition as:

Proposition 4 *(PDE for k^n via distortion method). For this particular choice of the power transformation, k^n satisfies:*

$$k_t^n + Op^{(y,v)}k^n - \omega k^n = 0 \tag{20}$$

The terminal condition is: $k^n(T,y,v) = -e^{-\gamma(1-\rho^2)nB(y,v)}$.
We have defined here ω as a function of $Y_t = y$ such that:

$$\omega = \frac{1}{2}\lambda^2(1-\rho^2) = \frac{1}{2}\frac{(\mu - r)^2}{y}(1-\rho^2) \tag{21}$$

and $Op^{(y,v)}$ is a differential operator which is given via:

$$Op^{(y,v)}k^n = (a - by - c\sqrt{y}\rho\lambda)k_y^n + \frac{1}{2}c^2 y k_{yy}^n + f(y)k_v^n \tag{22}$$

The idea is then to recognize in $Op^{(y,v)}$ the infinitesimal generator of (Y_t, V_t) under a particular measure $\tilde{P}$ where dynamics can be written as:

$$\mathrm{d}Y_t = (a - by - c\sqrt{y}\rho\lambda)\mathrm{d}t + c\sqrt{Y}_t \mathrm{d}\tilde{W}_t^\rho \tag{23}$$

$$\mathrm{d}V_t = f(Y_t)\mathrm{d}t \tag{24}$$

where $\tilde{W}^\rho$ is a Brownian motion with respect to the just mentioned $\tilde{P}$. The process (24) for V_t is exactly the same as the one we defined at the beginning. But for Y_t, we need to adjust the drift to rediscover the process given earlier in (3). In other words (23) and (3) are the same under a carefully chosen σ-algebra. We follow here the discussion [15] of Monoyios for the choice of an adapted σ-algebra. Considering σ-algebras large enough to contain all necesseray information about Y_t up to time T (V_t being not involved we focus only here on Y_t), we can isolate two different choices for this σ-algebra. The first one is $\mathcal{F}_{\mathrm{t}}$ defined in Subsection 2.1. By Girsanov theorem, we can consider the following Radom-Nikodym derivative:
$Z_T = \frac{\partial \tilde{P}}{\partial P}$ with $Z_t = \exp(-\int_0^t m_u \mathrm{d}W_u^1 - \int_0^t g_u \mathrm{d}W_u^2 - \frac{1}{2}\int_0^t m_u^2 \mathrm{d}u - \frac{1}{2}\int_0^t g_u^2 \mathrm{d}u)$ where m and g are $\mathcal{F}_{\mathrm{t}}$-adapted processes.
Then we know that:

$$\tilde{W}_t^1 = W_t^1 + \int_0^t m_u \mathrm{d}u$$

and

$$\tilde{W}_t^2 = W_t^2 + \int_0^t g_u \mathrm{d}u$$

are Brownian motions under $\tilde{P}$. Now, rewriting the dynamics of Y_t in (3) under the measure $\tilde{P}$:

$$\begin{aligned}\mathrm{d}Y_t &= (a - bY_t)\mathrm{d}t + c\sqrt{Y}_t(\rho \mathrm{d}\tilde{W}_t^1 + \epsilon \mathrm{d}\tilde{W}_t^2 - \rho m_t \mathrm{d}t - \epsilon g_t \mathrm{d}t) \\ &= [a - bY_t - c\sqrt{Y}_t(\rho m_t + \epsilon g_t)]\mathrm{d}t + c\sqrt{Y}_t \mathrm{d}\tilde{W}_t^\rho \end{aligned} \quad (25)$$

Looking at this process, we see that we can identify it to (23) by setting: $\rho\lambda_t = \rho m_t + \epsilon g_t$

The second possible choice is $\mathcal{G}_{\mathrm{t}}$ defined in Subsection 2.1. Considering here:
$\check{Z}_T = \frac{\partial \check{P}}{\partial P}$ with $\check{Z}_t = exp(-\int_0^t \theta_u \mathrm{d}W_u^\rho - \frac{1}{2}\int_0^t \theta_u^2 \mathrm{d}u)$, where θ is a $\mathcal{G}_{\mathrm{t}}$-adapted process.
The Girsanov theorem yields to:

$$\tilde{W}_t^\rho = W_t^\rho + \int_0^t \theta_u \mathrm{d}u$$

which is a Brownian motion with respect to $\check{P}$. The dynamics of Y_t in (3) is then under $\check{P}$:

$$\mathrm{d}Y_t = [a - bY_t - c\sqrt{Y}_t\theta_t]\mathrm{d}t + c\sqrt{Y}_t\mathrm{d}\tilde{W}_t^\rho \tag{26}$$

The identification with (23) leads us to setting up: $\rho\lambda_t = \theta_t$. Monoyios [15] defines then the set: $N = \{\tilde{P} \sim P$ on $\mathcal{F}_\mathrm{t}/\rho\mathrm{m}_t + \epsilon\mathrm{g}_t = \rho\lambda_\mathrm{t}, \mathrm{t} \leq T\}$, and since $\mathcal{G}_\mathrm{T} \subset \mathcal{F}_\mathrm{T}$, this set N also includes $\check{P} \sim P$ on $\mathcal{G}_\mathrm{T}$. Monoyios shows then that $(m_t, g_t) = (\rho^2\lambda_t, \rho\epsilon\lambda_t)$ for $0 \leq t \leq T$ and that measures $\tilde{P}$ from N can be used to express k^n as an expectation due to the Feynman-Kac theorem. To simplify this study, we are going to choose a particular $\tilde{P}$ from N to express our expectations in the next part of this dissertation. We then use Q^m, the minimal martingale measure corresponding to $(m_t, g_t) = (\lambda_t, 0)$ as done by Henderson [9]. For more details on the choice of the measure, one can refer to [15, 17].

Recalling that k^n follows the linear PDE (19), under Q^m the Feynman-Kac theorem allows us to write:

$$k^n(t, y, v) = E_{t,y,v}^{Q^m}[-e^{-\gamma(1-\rho^2)nB(Y_T,V_T)}e^{-\int_t^T \omega \mathrm{d}s}]$$

$$= E_{t,y,v}^{Q^m}[-e^{-\gamma(1-\rho^2)nB(Y_T,V_T)}e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}] \tag{27}$$

And then, by definition, we derive formulae for the reduced value function C^n and the actual value function F^n:

$$C^n(t, y, v) = (E_{t,y,v}^{Q^m}[-e^{-\gamma(1-\rho^2)nB(Y_T,V_T)}e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}])^{\frac{1}{1-\rho^2}} \tag{28}$$

and

$$F^n(t, y, v) = e^{-\gamma b(t,T)x}(E_{t,y,v}^{Q^m}[-e^{-\gamma(1-\rho^2)nB(Y_T,V_T)}e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}])^{\frac{1}{1-\rho^2}} \tag{29}$$

This last expression for the value function gives a formula for the asking price $p^{ask}(t, y, v)$ derived in the next section.

2.4 Indifference Price and Pricing PDE

We begin here by formulating an expression for the asking price, which will be the one from which we will derive an analytical and efficient formula in Section 4 when the claim is a variance swap.

Recalling the definition of $p^{ask}(t,y,v)$ in (9), it leads to:

$$e^{-\gamma b(t,T)x}(E^{Q^m}_{t,y}[-e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}])^{\frac{1}{1-\rho^2}}$$
$$= e^{-\gamma b(t,T)x}e^{-\gamma b(t,T)p^{ask}(t,y,v)}(E^{Q^m}_{t,y,v}[-e^{\gamma(1-\rho^2)B(Y_T,V_T)}e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}])^{\frac{1}{1-\rho^2}}$$

And finally, we can state this formula as the following theorem:

Theorem 5 *(Asking Price Formula). The asking price $p^{ask}(t,y,v)$ for the sale of one claim $B(Y_T,V_T)$ can be expressed as:*

$$p^{ask}(t,y,v) = \frac{1}{\gamma(1-\rho^2)b(t,T)}log\Big(\frac{E^{Q^m}_{t,y,v}[e^{\gamma(1-\rho^2)B(Y_T,V_T)}e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}]}{E^{Q^m}_{t,y}[e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}]}\Big) \tag{30}$$

We note once again that the indifference asking price is independent of the initial endowment $X_t = x$ due to the exponential utility function. And it corresponds to a generalization of prices respectively derived by Monoyios [17] for a claim paying $B(Y_T)$ and by Grasselli and Hurd [8] with $B(Y_T,V_T)$ and discounted processes, both using a dual approach. Regarding this last price we will follow a similar study as Grasselli and Hurd in their paper, and use the dual framework in Section 3 to rediscover this formula for p^{ask}. We also define at once one of the main quantity used in [8] and coming out naturally in the pricing PDE for p^{ask}. It is the certainty equivalent, denoted $(cer^B_t)_{0\leq t\leq T}$ for the sale of one claim or $(cer^0_t)_{0\leq t\leq T}$ if there is no claim at stake. These quantity could be awkward as we do not yet know how to get an expression for them but we will show in Section 3 that we can derive a similar formula as (30) in terms of Q^m-expectations. We generalize then the definition of Grasselli and Hurd for a non-zero interest rate:

Definition 6 *(Certainty Equivalents). We define certainty equivalents taking into account $b(t,T)$ as:*

$$e^{-\gamma b(t,T)(x-cer^B(t,y,v))} = F^{-1}(t,x,y,v) \tag{31}$$

and

$$e^{-\gamma b(t,T)(x-cer^0(t,y))} = F^0(t,x,y) \tag{32}$$

The definition of the asking price (9) is then equivalent to:

$$e^{-\gamma b(t,T)(x+p^{ask}(t,y,v)-cer^B(t,y,v))} = e^{-\gamma b(t,T)(x-cer^0(t,y))}$$

From which we can express p^{ask} in terms of certainty equivalent processes such that:

$$p^{ask}(t,y,v) = cer^B(t,y,v) - cer^0(t,y) \qquad 0 \leq t \leq T \tag{33}$$

We can now go further and derive the PDE followed by p^{ask} from its representation in (30). To achieve this aim, we recall the notation:

$$k^{-1}(t,y,v) = E^Q_{t,y,v}[-e^{\gamma(1-\rho^2)B(Y_T,V_T)}e^{-\int_t^T \omega \mathrm{d}s}]$$
$$k^0(t,y) = E^Q_{t,y}[-e^{-\int_t^T \omega \mathrm{d}s}]$$

and we use the shorthand:

$$\Phi(t,y,v) = e^{\gamma(1-\rho^2)b(t,T)p^{ask}(t,y,v)}$$

Rewriting (9) as: $k^{-1}(t,y,v) = \Phi(t,y,v)k^0(t,y)$. Arguing that k^{-1} and k^0 satisfy (19) respectively with terminal boundary conditions:

$$\begin{aligned} &k^{-1}(T,y,v) = -e^{\gamma(1-\rho^2)B(y,v)} \\ &k^0(T,y) = -1 \end{aligned} \tag{34}$$

we show that p^{ask} satisfies the following pricing pde:

Proposition 7 *(Pricing PDE for the Asking Price). The PDE followed by p^{ask} is given by:*

$$p^{ask}_t + Op^{y,v}p^{ask} - rp^{ask} + \frac{1}{2}\gamma(1-\rho^2)b(t,T)c^2y((p^{ask}_y)^2 + 2p^{ask}_y cer^0_y) = 0 \tag{35}$$

and terminal condition: $p^{ask}(T,y,v) = B(y,v)$

Proof. As $k^{-1}(t,y,v) = \phi(t,y,v)k^0(t,y)$ satisfies (19), it gives:

$$\Phi(k^0_t + Op^{y,v}k^0 - \omega k^0) + k^0(\Phi_t + Op^{y,v}\Phi) + c^2y\Phi_y k^0_y = 0$$

But as we have just said k^0 also follows (19) so it reduces to: $k^0(\Phi_t + Op^{y,v}\Phi) + c^2y\Phi_y k^0_y = 0$ i.e

$$\begin{aligned} &k^0(\gamma(1-\rho^2)b(t,T)(p^{ask}_t - rp^{ask})\Phi + \gamma(1-\rho^2)b(t,T)\Phi Op^{y,v}p^{ask}) \\ &\quad + c^2y\gamma(1-\rho^2)b(t,T)p^{ask}_y\Phi k^0_y = 0 \end{aligned} \tag{36}$$

It remains to compute k_y^0, which can be done in terms of the certainty equivalent cer^0. From the definition (32) and using (29):

$$e^{-\gamma b(t,T)(x-cer^0(t,y))} = e^{-\gamma b(t,T)x}(E_{t,y}^Q[-e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}])^{\frac{1}{1-\rho^2}}$$

or

$$\begin{aligned} e^{\gamma(1-\rho^2)b(t,T)cer^0(t,y)} &= E_{t,y}^Q[-e^{-\int_t^T \frac{1}{2}(1-\rho^2)\frac{(\mu-r)^2}{Y_s}\mathrm{d}s}] \\ &= k^0(t,y) \end{aligned}$$

We must have then: $k_y^0 = \gamma(1-\rho^2)b(t,T)cer_y^0 k^0$. Rewriting (35) as:

$$\begin{aligned} &k^0(\gamma(1-\rho^2)b(t,T)(p_t^{ask} - rp^{ask})\Phi + \gamma(1-\rho^2)b(t,T)\Phi O p^{y,v} p^{ask}) \\ &+ c^2 y \gamma(1-\rho^2)b(t,T)p_y^{ask}\Phi\gamma(1-\rho^2)b(t,T)cer_y^0 k^0 = 0 \end{aligned}$$

Factorising by $\gamma(1-\rho^2)b(t,T)k^0\Phi$ concludes the demonstration. ■

In the next section we are going to rediscover this pricing pde using dual arguments and we will also derive at the same time the optimal hedge in an elegant way due to Grasselli and Hurd [8]. It will confort us in the primal derivation of the pricing PDE, which has never been done for a claim $B(Y_T, V_T)$ and which will be very useful when computing the basis risk process at the end of the next section.

3 Dual Formulation

After studying the primal formulation for the indifference price, it seems natural to look at the dual formulation and see not only if it matches results of the Section 2 but also try to derive other results and particularly for the optimal hedge. Links between dual framework and distortion technique used in Subsection 2.3 are fully explained by Monoyios [17]. Another study of the dual problem for a claim $B(Y_T, V_T)$ that we already introduced is given by Grasselli and Hurd [8]. We will focus on it and using their approach, we are going to generalize their results for non-discounted processes in setting up their calculus in our general model described in Subsection 2.1.

3.1 Optimal Trading Strategy

Since our aim is not to construct fully dual problem, we will admit some results and focus on what can be drawn from them for our study. One can refer to [17, 8] for demonstrations. Recalling that the optimal trading problem is to find the optimal trading strategy $\pi_t^{*,-1}$ when selling one claim B at 0 and then hedging the position using it until expiry T. We will remark that we already derived a formula for it in (15) thanks to the maximization of the Hamiltonian. Now the dual formulation implies that $\pi_t^{*,-1}$ satisfies:

$$U'(X_T^* - B(Y_T, V_T)) = \xi \frac{dQ^*}{dP} \tag{37}$$

or

$$\gamma e^{-\gamma(x+\int_0^T dX_s^* ds - B(Y_T,V_T))} = \xi \frac{dQ^*}{dP} \tag{38}$$

where X_T^* is the optimal wealth at maturity reached using the optimal trading strategy $\pi_t^{*,-1}$, ξ is equal to the derivative of the value function. And Q^* is the unique maximizer in the definition of the dual problem:

$$\sup_{Q\in M^f} E^Q[B - \log\left(\frac{dQ}{dP}\right)] \tag{39}$$

where M^f denotes the set of absolutely continuous and equivalent local martingale measures for S, with finite relative entropy with respect to P.

Considering then the density process which is defined as:

Definition 8 *(Density process). Grasselli and Hurd define the following density process:*

$$\Lambda_t^* = E[\frac{dQ^*}{dP} \mid \mathcal{F}_{\mathrm{t}}], \qquad \mathrm{t} \leq \mathrm{T}$$

Grasselli and Hurd argue that as it is a stricytly positive martingale, it satisfies the dynamics given via:

$$\frac{d\Lambda_t^*}{\Lambda_t^*} = -(\lambda_t^* dW_t^1 + \nu_t^* ddW_t^2) \tag{40}$$

with adapted processes λ_t^* and ν_t^*. As Q^* belongs to M^f, S_t is a local martingale under Q^*, which implies: $\lambda_t^* = \lambda_t$ already defined in Subsection 2.2. This approach allows us to prove the next proposition:

Proposition 9 *(Optimal Trading Strategy). From this density process and under some assumptions discussed in [8] consistent with our model, the optimal trading strategy $\pi_t^{*,-1} = \pi^{*,-1}(t,y,v)$ is given by:*

$$\pi^{*,-1}(t,y,v) = c\rho(p_y^{ask}(t,y,v) + cer_y^0(t,y)) + \frac{\mu - r}{\gamma y b(t,T)} \tag{41}$$

which involves the certainty equivalent cer^0 defined in (32).

It is the formula due to Grasselli and Hurd with the additional discount factor $\frac{1}{b(t,T)}$.

Proof. We follow the same demonstration as in [8] taking into account $b(t,T) = e^{r(T-t)}$. We first rewrite Λ_t^* as:

$$\begin{aligned}\Lambda_t^* &= \frac{1}{\xi} E[U'(X_T^* - B(Y_T, V_T)) \mid \mathcal{F}_\mathrm{t}], \qquad \mathrm{t} \le \mathrm{T} \\ &= \frac{e^{-\gamma(X_t^* - cer_t^B)}}{e^{-\gamma(X_0^* - cer_0^B)}}\end{aligned}$$

by recognising in ξ the derivative $U'(X_0^* - cer_0^B)$ and definition of the certainty equivalent in (31). Now if we apply Ito's Lemma to this expression of Λ_t^*, we get:

$$\begin{aligned}d\Lambda_t^* &= \frac{\partial \Lambda_t^*}{\partial t} dt + \frac{\partial \Lambda_t^*}{\partial X} dX + \frac{\partial \Lambda_t^*}{\partial Y} dY + \frac{\partial \Lambda_t^*}{\partial V} dV \\ &+ \frac{1}{2} \frac{\partial^2 \Lambda_t^*}{\partial X^2} d < X > + \frac{1}{2} \frac{\partial^2 \Lambda_t^*}{\partial Y^2} d < Y > + \frac{\partial^2 \Lambda_t^*}{\partial X \partial Y} d < X, Y >\end{aligned}$$

And we retain only the dW^1-terms such that:

$$\frac{d\Lambda_t^*}{\Lambda_t^*} = (...)dt + (...)dW_t^2 + \gamma b(t,T)(cer_y^B c\sqrt{y}\rho - \sqrt{y}\pi^{*,-1})dW_t^1 \tag{42}$$

Identifying the dW^1-term to λ in (40) and using $p^{ask} = cer^B - cer^0$, we conclude our proof. ■

We can see that this formula is the sum of the optimal trading strategy when hedging the sale of one claim and the corresponding one when the claim is absent. It leads us to define the optimal hedge for the sale of one claim.

Definition 10 *As defined by Monoyios [16], it is optimal to hold in the stock at t:*

$$\pi^\Delta(t,y,v) = \pi^{*,-1}(t,y,v) - \pi^{*,0}(t,y) \tag{43}$$

In other words our optimal hedge is given via:

Proposition 11

$$\pi^\Delta(t,y,v) = c\rho p_y^{ask}(t,y,v) = \Delta_t S_t, \quad 0 \le t \le T.$$

3.2 Dual derivation of the Pricing PDE

To obtain the pricing PDE in this dual framework we can derive a first PDE for the certainty equivalent cer^B and then use its relation with the asking price to achieve this end. A simple martingale argument, well-known to derive pricing PDE, can be first applied to the density process Λ_t^*. So as it is a martingale, the drift term in equation (42) must be equal to zero.
This condition yields to the PDE for $cer^B(t, y, v)$ such that:

Proposition 12 *(PDE for the Certainty Equivalent). cer^B satisfies:*

$$cer_t^B + Op^{y,v} cer^B - rcer^B - \frac{(\mu - r)^2}{2\gamma b(t,T)y} + \frac{1}{2}\gamma(1-\rho^2)b(t,T)c^2 y(cer_y^B)^2 = 0 \tag{44}$$

with terminal boundary condition: $cer^B(T, y, v) = B(y, v)$

Once again it is a generalization of the result obtained by Grasselli and Hurd with non-zero r.

Proof. To prove this result we equate the dt term in (42) to zero, and we use the expression for $\pi^{*,-1}(t, y, v)$ just derived in (41). Full calculus are not mentioned here as it leads to quite long expressions and presents no difficulties as it is simply the Ito's formula. We will just mention that the term in $-rcer^B$ comes from the derivation of $b(t, T)$ with respect to t and then it is not surprising to recognize in it, the only additional term comparing to the Grasselli PDE. ■

A simple comment is now to note that cer^0 follows the same PDE but for a different terminal condition: $cer^0(T, y) = 0$. This remark allows us to derive the pricing PDE for the asking price from the relation between p^{ask} and certainty equivalent with and without the claim as follows:

$$p_t^{ask} + cer_t^0 - r(p^{ask} + cer^0) + Op^{y,v}(p^{ask} + cer^0) - \frac{(\mu - r)^2}{2\gamma b(t,T)y}$$
$$+ \frac{1}{2}\gamma(1-\rho^2)b(t,T)c^2 y(p^{ask} + cer^0)^2 = 0$$

with: $(p^{ask} + cer^0)^2 = (p^{ask})^2 + (cer^0)^2 + 2p^{ask}cer^0$ and as:

$$cer_t^0 + Op^{y,v} cer^0 - rcer^0 - \frac{(\mu - r)^2}{2\gamma b(t,T)y} + \frac{1}{2}\gamma(1-\rho^2)b(t,T)c^2 y(cer_y^0)^2 = 0$$

we finally conclude:

$$p_t^{ask} + Op^{y,v}p^{ask} - rp^{ask} + \frac{1}{2}\gamma(1-\rho^2)b(t,T)c^2y((p_y^{ask})^2 + 2p_y^{ask}cer_y^0) = 0 \quad (45)$$

As expected we rediscover our result from the primal formulation in proposition 7 with the non-linear term involving the certainty equivalent cer^0.

3.3 The Residual Risk Process

We define here the residual risk process similarly as Monoyios [16] such that:

Definition 13 *(Residual Risk Process). We suppose the investor sells one claim at time* 0 *for price* $p^{ask}(0, Y_0, V_0)$ *and then hedges his position using a number* $(\Delta_t)_{0\leq t\leq T}$ *of stock defined in (43) up to time* T*. His overall position is then described via the residual risk process* $(R_t)_{0\leq t\leq T}$ *such that:*

$$\begin{aligned} R_t &= X_t^* - p^{ask}(t, Y_t, V_t) \\ R_0 &= 0 \end{aligned} \quad (46)$$

Where the process for optimal wealth X_t^* is given by:

$$\begin{aligned} dX_t^* &= \Delta_t dS_t + r(X_t^* - \Delta_t S_t)dt \\ X_0^* &= p^{ask}(0, Y_0, V_0) \end{aligned} \quad (47)$$

In order to simulate later this residual risk and compute the terminal hedging error we need to derive the dynamics of the R process. This is given by the following SDE, which concludes some calculus in [8] and is one of our main contribution:

Proposition 14 *(SDE for the Residual Risk).* R_t *satisfies the following dynamics:*

$$dR_t = rR_t dt - p_y^{ask}c\sqrt{Y_t}\epsilon dW_t^2 + \frac{1}{2}\gamma\epsilon^2 b(t,T)c^2 Y_t((p_y^{ask})^2 + 2p_y^{ask}cer_y^0)dt \quad (48)$$

$$R_0 = 0$$

As expected we cannot perfectly hedge the risk associated with W_t^2 by hedging our position only by trading in the stock driven by the othogonal Brownian motion W_t^1. As obtained by Monoyios [16], the risk associated with the stock variance Y cannot be fully replicated due to the fact that Y

involved $W_t^\rho = \rho W_t^1 + \epsilon W_t^2$. This was expected as the market is incomplete.

Proof. Differentiating (46), we have: $dR_t = dX_t^* - dp^{ask} = \Delta_t dS_t + r(X_t^* - \Delta_t S_t)dt - dp^{ask}$. Then using (47) for the dynamic of X_t^*, we get:

$$dR_t = \rho c p_y^{ask}(\mu dt + \sqrt{Y_t}dW_t^1) + rX_t^* dt - r\rho c p_y^{ask} dt - dp^{ask}$$

Using the pricing PDE for p^{ask} given by proposition 7, we recognize in dp^{ask} some terms of this PDE, which leads us to simplify:

$$\begin{aligned} -dp^{ask} &= -rp^{ask}dt + \frac{1}{2}\gamma\epsilon^2 b(t,T)c^2 Y_t((p_y^{ask})^2 + 2p_y^{ask}cer_y^0)dt \\ &\quad - p_y^{ask}c\sqrt{Y_t}\epsilon dW_t^\rho - \rho c(\mu - r)p_y^{ask}dt \end{aligned}$$

Simplification of the term in $\rho c(\mu - r)p_y^{ask}dt$ yields to the above result. ■

We remark that the certainty equivalent cer^0 is still present in the residual risk process which forces us to also derive an analytical formula for this certainty equivalent too. Actually we can show that the representation (30) for the asking price using expectations under the minimal martingale measure Q^m can be rediscovered in another way due to Grasselli and Hurd [8] which gives at the same time a similar representation for cer^0, i.e in using an expectation under Q^m. It is the object of the next section.

3.4 Generalized Formulae for Asking Price and Certainty Equivalents

We are going here to rediscover the formula (30) for the asking price in terms of expectations under the minimal martingale measure Q^m, using a dual approach as done by Grasselli and Hurd [8]. In this paper they use a method of derivation that allows us to express certainty equivalents in a similar way. In other words we are going to generalize their approach to non-discounted processes.

The initial and fundamental result they prove is stated in the following proposition, which is generalized taking into account the discount factor:

Proposition 15 *(Grasselli and Hurd Martingale). If the claim is $B(Y_T, V_T)$ and the certainty equivalent cer^B satisfies the PDE of Proposition 12, then the process defined via:*

$$\Upsilon_t = e^{\gamma\epsilon^2 b(t,T)cer^B - \frac{1}{2}\int_0^t 2\lambda_s ds} \tag{49}$$

is a martingale with respect to Q^m.

We will give below only a short proof that Υ_t is a local Q^m-martingale. To prove that it is a true Q^m-martingale, one can refer to [8].
Proof. First, we apply Ito's lemma to Υ_t, which provides:

$$\frac{d\Upsilon_t}{\Upsilon_t} = \gamma\epsilon^2 b(t,T) dcer^B - \gamma\epsilon^2 b(t,T) rcer^B dt + \frac{1}{2}\gamma^2\epsilon^4 b(t,T)^2 (dcer^B)^2 - \frac{1}{2}\epsilon^2\lambda_t dt \tag{50}$$

we then compute $dcer^B$ by applying Ito's lemma once again to the certainty equivalent, which gives:

$$dcer^B = (cer^B_t + (a-by)cer^B_y + \frac{1}{2}c^2 y cer^B_{yy} + f(y)cer^B_v)dt + c\sqrt{y} cer^B_y dW^\rho_t = 0 \tag{51}$$

Recalling that $dW^\rho_t = \rho dW^1_t + \epsilon dW^2_t$ we get for (50):

$$\begin{aligned}\frac{d\Upsilon_t}{\Upsilon_t} &= \gamma\epsilon^2 b(t,T)[(cer^B_t + (a-by)cer^B_y + \frac{1}{2}c^2 y cer^B_{yy} + f(y)cer^B_v - rcer^B \\ &\quad - \frac{\lambda_t^2}{2\gamma b(t,T)} + \frac{1}{2}\gamma\epsilon^2 b(t,T) c^2 y (cer^B_y)^2)dt + c\sqrt{y} cer^B_y(\rho dW^1_t + \epsilon dW^2_t)]\end{aligned} \tag{52}$$

we now introduce the Brownian motion: $\tilde{W}^1_t = W^1_t + \lambda_t t$ which is actually a Brownian motion under Q^m by Girsanov theorem, we can rewrite (52) as:

$$\begin{aligned}\frac{d\Upsilon_t}{\Upsilon_t} &= \gamma\epsilon^2 b(t,T)\Big(\big(cer^B_t + (a-by-\rho c(\mu-r))cer^B_y + \frac{1}{2}c^2 y cer^B_{yy} + f(y)cer^B_v \\ &\quad - rcer^B - \frac{\lambda_t^2}{2\gamma b(t,T)} + \frac{1}{2}\gamma\epsilon^2 b(t,T) c^2 y (cer^B_y)^2\big)dt + c\sqrt{y} cer^B_y(\rho d\tilde{W}^1_t + \epsilon dW^2_t)\Big)\end{aligned} \tag{53}$$

And we recognize in the drift term the PDE followed by cer^B, which makes the drift term equals to zero and as the density process Λ_t defined in Definition 8 is a martingale under the physical measure P, we can conclude that Υ_t is a local martingale under Q^m. ■

Using now the previous Proposition 15, the Q^m-martingale property of Υ_t is written as:

$$e^{\gamma\epsilon^2 b(t,T) cer^B(t,y,v)} = E^{Q^m}_{t,y,v}[e^{\gamma(\epsilon^2)B(Y_T,V_T)} e^{-\int_t^T \frac{1}{2}(\epsilon^2)\lambda_s^2 \mathrm{d}s}] \tag{54}$$

When $B = 0$, the martingale property leads to:

$$e^{\gamma\epsilon^2 b(t,T) cer^0(t,y)} = E^{Q^m}_{t,y}[e^{-\int_t^T \frac{1}{2}(\epsilon^2)\lambda_s^2 \mathrm{d}s}] \tag{55}$$

This now provides the following theorem, which gives formulae for certainty equivalents and rediscover the already known formula (30) for the asking price:

Theorem 16 *(Certainty equivalent and Asking Price formulae). Considering our claim $B(Y_T, V_T)$, certainty equivalents are given by:*

$$cer^B(t,y,v) = \frac{1}{\gamma\epsilon^2 b(t,T)} log E^{Q^m}_{t,y,v}[e^{\gamma\epsilon^2 B(Y_T,V_T)} e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}] \tag{56}$$

$$cer^0(t,y) = \frac{1}{\gamma\epsilon^2 b(t,T)} log E^{Q^m}_{t,y}[e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}] \tag{57}$$

and obviously as: $p^{ask}(t,y,v) = cer^B(t,y,v) - cer^0(t,y)$ *we find again:*

$$p^{ask}(t,y,v) = \frac{1}{\gamma\epsilon^2 b(t,T)} log\Big(\frac{E^{Q^m}_{t,y,v}[e^{\gamma\epsilon^2 B(Y_T,V_T)} e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}]}{E^{Q^m}_{t,y}[e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}]}\Big) \tag{58}$$

We clearly observe that if we can find a way to compute analytically both expectations:

$$E^{Q^m}_{t,y,v}[e^{\gamma\epsilon^2 B(Y_T,V_T)} e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}] \tag{59}$$

$$and \quad E^{Q^m}_{t,y}[e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}] \tag{60}$$

we will derive analytical expressions for certainty equivalents and asking price, from which explicit derivatives will be computed and then we will get an efficient and analytical formula for the hedge. It will be done in the next section using calculus due to Hurd and Kuznetsov [13].

4 Analytical Price and Hedge for the Variance Swap

In this section, we begin by formulate our whole study for the specific case of a variance swap paying $B(V_T) = \int_0^T Y_s ds$ at maturity T, i.e it pays the integral of the variance over the interval $[0, T]$ of investisment with no strike (we will take then $f(Y_s) = Y_s$). Many papers as [12, 2, 7, 3, 4] study the pricing and hedging of this now actively traded derivative. And we will discuss further their approach in Subsection 5.4 when we will compare the hedging performance of our optimal strategy with other methods when we are allowed to trade in plain vanilla options on the stock. But now we give an analytical expression for expectations under Q^m, considering them as Laplace trasforms. This is due to the work of Hurd and Kuznetsov [13]. This gives then our analytical formulae for asking price and certainty equivalent and later our contribution will be to give the analytical formula for the optimal hedge by differentiation of the just quoted formulae.

4.1 Analytical formulae for Variance Swap Price and Certainty Equivalents

Hurd and Kuznetsov show the following result in [13] which we will admit here:

Proposition 17 *Let Y_t be a stochastic process satisfying the Cox-Ingersoll-Ross dynamic:*

$$dY_t = (a - bY_t)dt + c\sqrt{Y_t}dW_t, \qquad t \leq T \tag{61}$$

under a given measure P. And assuming that this process has an unattainable boundary at 0, which is the case if $\alpha = \frac{2a}{c^2} - 1 \geq 0$, if we use the same notations as in [13, 8], we have:

$$\begin{aligned} G(T-t, y, d1, d2) &= E^P_{t,y}[e^{-d1\int_t^T Y_s ds - d2\int_t^T \frac{1}{Y_s}ds}] \\ &= exp(-y(v1 + \frac{\Omega(-v1)}{\Omega - v1}e^{-(\frac{\beta}{2}+v1)c^2(T-t)})) \\ &\quad \times e^{-(av1+bv2+c^2v1v2)(T-t)}y^{v2}(\Omega - v1)^{-\alpha-v2-1}\Omega^{\alpha+2v2+1} \\ &\quad \times \frac{\Gamma(\alpha+v2+1)}{\Gamma(\alpha+2v2+1)}F_1(v2, \alpha+2v2+1; -\frac{\Omega^2 y e^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega - v1}) \end{aligned} \tag{62}$$

where we have defined:

$$\beta = \frac{2b}{c^2} \tag{63}$$

$$v1 = \frac{1}{2}(-\beta + \sqrt{\beta^2 + \frac{8d1}{c^2}}) \tag{64}$$

$$v2 = \frac{1}{2}(-\alpha + \sqrt{\alpha^2 + \frac{8d2}{c^2}}) \tag{65}$$

$$\Omega = (\beta + 2v1)(1 - e^{-(\frac{\beta}{2}+v1)c^2(T-t)})^{-1} \tag{66}$$

F_1 is the confluent hypergeometric function and Γ is the well known Γ-function. We have also assumed:

$$d1 \geq -\frac{b^2}{2c^2}, \qquad \textit{and} \qquad d2 \geq -\frac{\alpha^2 c^2}{8} \tag{67}$$

For full proof of this important expression, we can see [13] and their proof for a CIR process. And we will discuss in Subsection 5.2 which form of the confluent hypergeometric function should be employed depending on input values.

We now look at our expectations (59) and (60) under Q^m and verify if they satisfy conditions of the above proposition. First recalling the dynamics (23) followed by the variance Y_t under Q^m:

$$dY_t = [a - bY_t - \rho c(\mu - r)]dt + c\sqrt{Y_t}dW_t^\rho,$$

Y_t satisifes then to the boundary non-attainment condition if: $2(a - \rho c(\mu - r)) \geq c^2$. Regarding $d1$ and $d2$, which are respectively equals here to:

$$d1 = \gamma\epsilon^2, \quad \textit{and} \quad d2 = \frac{1}{2}\epsilon^2(\mu - r)^2 \tag{68}$$

must satisfy conditions:

$$\gamma\epsilon^2 \geq -\frac{b^2}{2c^2} \tag{69}$$

$$\frac{1}{2}\epsilon^2(\mu - r)^2 \geq -\frac{\alpha^2 c^2}{8} \tag{70}$$

We can see that condition (70) on $d2$ and (69) on $d1$ are automatically satisfied for our model. But for the non-attainable condition we will need to choose carefully our parameters. This choice will be discussed before the implementation in Subsection 5.1.

Assuming for the moment that the Proposition 17 is effectively applicable, we use it to derive the analytical formulae to compute Q^m-expectations (59) and (60) in the following corollary:

Corollary 18 *(Analytical Formulae for our Expectations). Expectations used to compute asking price and certainty equivalents are given by:*

$$E_{t,y}^{Q^m}[e^{\gamma\epsilon^2 B(V_T)} e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}] = G(T - t, y, d1, d2) \tag{71}$$

$$\textit{and} \quad E_{t,y}^{Q^m}[e^{-\int_t^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s} \mathrm{d}s}] = G(T - t, y, 0, d2) \tag{72}$$

Then we recall expressions of theorem 16 for certainty equivalent respectively with and without the sale of the variance swap derived at the end of Subsection 3.4. We note that with the variance swap, B does not depend on Y_T but is simply a function $B(V_T)$. From this, we can infer that as the integral of the variance at t is known when we have the evolution of Y up to t, expectation (59) does not depend on v anymore. That is, the choice of the variance swap allows us to make price and certainty equivalents independent on $V_t = v$. And we can state immediately from the previous corollary, the theorem:

Theorem 19 *(Analytical Formulae for Certainty Equivalents). Certainty equivalents cer^B and cer^0 in our model for the variance swap are obtained via:*

$$cer^B(t,y) = \frac{1}{\gamma\epsilon^2 b(t,T)} logG(T-t, y, d1, d2) \tag{73}$$

for the sale of one variance swap. And in the absence of it:

$$cer^0(t,y) = \frac{1}{\gamma\epsilon^2 b(t,T)} logG(T-t, y, 0, d2) \tag{74}$$

where $d1 = \gamma\epsilon^2$, and $d2 = \frac{1}{2}\epsilon^2(\mu - r)^2$

and obviously as the asking price is given by: $p^{ask}(t,y) = cer^B_{t,y} - cer^0_{t,y}$, its analytical formula is stated in the following theorem, which constitutes the crucial point of our further implementation:

Theorem 20 *(Analytical Formula for the Asking Price). The indifference price for selling a varaince swap is given analytically via:*

$$p^{ask}(t,y) = \frac{1}{\gamma\epsilon^2 b(t,T)} log\Big(\frac{G(T-t, y, d1, d2)}{G(T-t, y, 0, d2)}\Big) \tag{75}$$

These formulae will be verified in Subsection 5.2 by Monte-Carlo simulation of Q^m-expectations.

4.2 Analytical formula for Optimal Hedge

We then need to derive from Theorems 19 and 20, another analytical formula but this time for the optimal hedge π^{Δ}_t. This can be done easily by differentiating p^{ask} and cer^0 with respect to y which implies differentiating the expresion G of proposition 17. Our contribution is here to complete the paper [8] of Grasselli and Hurd and give the corollar analytical formula for the optimal hedge and therefore a way to efficiently compute the residual risk process.

First, we recall that the optimal hedging strategy (43) is given via: $\pi^{\Delta} = c\rho p_y^{ask}$. It raises the necessity to compute the derivative of the asking price with respect to y. Analytically, from the formula in Theorem 20, we have immediately: $\mathrm{p}_y^{ask} = \frac{1}{\gamma\epsilon^2 b(t,T)}\left(\frac{G_y(T-t,y,d1,d2)}{G(T-t,y,d1,d2)} - \frac{G_y(T-t,y,0,d2)}{G(T-t,y,0,d2)}\right)$ where the derivative of G can be computed analytically as:

$$G_y(T-t,y,d1,d2) = -(v1 + \frac{\Omega(-v1)}{\Omega - v1}e^{-(\frac{\beta}{2}+v1)c^2(T-t)})G(T-t,y,d1,d2)$$
$$+\frac{v2}{y}G(T-t,y,d1,d2) + \frac{F_{1_y}(v2,\alpha+2v2+1; -\frac{\Omega^2 y e^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega - v1})}{F_1(v2,\alpha+2v2+1; -\frac{\Omega^2 y e^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega - v1})}G(T-t,y,d1,d2) \tag{76}$$

We are going to see that it let us write the optimal hedging strategy as:

Theorem 21 *(Our Analytical Formula for π^{Δ}). We prove that the optimal hedging strategy when selling a variance swap is to invest the following amount in the stock:*

$$\pi^{\Delta} = \frac{\rho c}{\gamma\epsilon^2 b(t,T)}\Big([-(v1 + \frac{\Omega(-v1)}{\Omega - v1}e^{-(\frac{\beta}{2}+v1)c^2(T-t)}) + \frac{v2}{y}$$
$$+ \frac{v2 F_1(v2+1,\alpha+2v2+2; -\frac{\Omega^2 y e^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega - v1})}{(\alpha+2v2+1)F_1(v2,\alpha+2v2+1; -\frac{\Omega^2 y e^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega - v1})}]$$
$$-[-(v1* + \frac{\Omega(-v1*)}{\Omega * - v1*}e^{-(\frac{\beta}{2}+v1*)c^2(T-t)}) + \frac{v2}{y}$$
$$+ \frac{v2 F_1(v2+1,\alpha+2v2+2; -\frac{\Omega*^2 y e^{-(\frac{\beta}{2}+v1*)c^2(T-t)}}{\Omega * - v1})}{(\alpha+2v2+1)F_1(v2,\alpha+2v2+1; -\frac{\Omega*^2 y e^{-(\frac{\beta}{2}+v1*)c^2(T-t)}}{\Omega * - v1*})}]\Big) \tag{77}$$

where we have denoted $v1*$ and $\Omega*$, the quantities $v1$ and Ω defined in Proposition 17 but for $d1 = 0$.

We give below important steps of derivation of our theorem.

Proof. We begin with using the derivative property of the confluent hypergeometric function F_1 which states that:

$$\frac{dF_1}{dy}(a,b,y) = \frac{a}{b}F_1(a+1,b+1,y) \tag{78}$$

it leads to:

$$\frac{dF_{1_y}}{F1}(a,b,y) = \frac{aF_1(a+1,b+1,y)}{bF_1(a,b,y)} \tag{79}$$

Then we have the simplified analytical formula for the derivative of $logG$ such that:

$$\frac{G_y(T-t,y,d1,d2)}{G(T-t,y,d1,d2)} = -(v1+\frac{\Omega(-v1)}{\Omega-v1}e^{-(\frac{\beta}{2}+v1)c^2(T-t)}) + \frac{v2}{y}$$
$$+\frac{v2F_1(v2+1,\alpha+2v2+2;-\frac{\Omega^2ye^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega-v1})}{(\alpha+2v2+1)F_1(v2,\alpha+2v2+1;-\frac{\Omega^2ye^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega-v1})} \tag{80}$$

To conclude these technical calculus, we get the analytical formula for p_y^{ask}:

$$p_y^{ask} = \frac{1}{\gamma\epsilon^2 b(t,T)}\Big([-(v1+\frac{\Omega(-v1)}{\Omega-v1}e^{-(\frac{\beta}{2}+v1)c^2(T-t)}) + \frac{v2}{y}$$
$$+\frac{v2F_1(v2+1,\alpha+2v2+2;-\frac{\Omega^2ye^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega-v1})}{(\alpha+2v2+1)F_1(v2,\alpha+2v2+1;-\frac{\Omega^2ye^{-(\frac{\beta}{2}+v1)c^2(T-t)}}{\Omega-v1})}]$$
$$-[-(v1*+\frac{\Omega(-v1*)}{\Omega*-v1*}e^{-(\frac{\beta}{2}+v1*)c^2(T-t)}) + \frac{v2}{y}$$
$$+\frac{v2F_1(v2+1,\alpha+2v2+2;-\frac{\Omega*^2ye^{-(\frac{\beta}{2}+v1*)c^2(T-t)}}{\Omega*-v1})}{(\alpha+2v2+1)F_1(v2,\alpha+2v2+1;-\frac{\Omega*^2ye^{-(\frac{\beta}{2}+v1*)c^2(T-t)}}{\Omega*-v1*})}]\Big) \tag{81}$$

And we conclude the proof by multiplying by ρc. ■

As an obvious corollary, we give the analytical formula for the certainty equivalent cer^0 when there is no claim at stake. This will be useful when implementing the residual risk process as the SDE for R given in Proposition 14 involved it.

Corollary 22 *(Our Analytical Forrmula for cer_y^0). The y-derivative of the certainty equivalent without claim is given via:*

$$cer_y^0 = \frac{1}{\gamma\epsilon^2 b(t,T)}\Big(-(v1*+\frac{\Omega(-v1*)}{\Omega*-v1*}e^{-(\frac{\beta}{2}+v1*)c^2(T-t)}) + \frac{v2}{y}$$
$$+\frac{v2F_1(v2+1,\alpha+2v2+2;-\frac{\Omega*^2ye^{-(\frac{\beta}{2}+v1*)c^2(T-t)}}{\Omega*-v1})}{(\alpha+2v2+1)F_1(v2,\alpha+2v2+1;-\frac{\Omega*^2ye^{-(\frac{\beta}{2}+v1*)c^2(T-t)}}{\Omega*-v1*})}\Big) \tag{82}$$

More precisely, these formulae will be implemented in Subsection 5.3, where we will simulate the residual risk process: $dR_t = rR_t dt - p_y^{ask} c\sqrt{Y_t}\epsilon dW_t^2 + \frac{1}{2}\gamma\epsilon^2 b(t,T)c^2 Y_t((p_y^{ask})^2 + 2p_y^{ask} cer_y^0)dt$. We have now all materials to proceed to numerical implementation with final aim of computing the terminal hedging error with the simulation of R. But this calculus is not completely direct, since as we will see in next section, we need to calibrate our confluent hypergeometric function F_1 depending on the value y and time to maturity $T - t$.

5 Numerical Implementation and Investigations

In this section we will conclude our study in first verifying that analytical formulae obtained in Subsection 4.2 are actually correct by conducting a Monte-Carlo simulation of corresponding Q^m-expectations. Then we will calibrate the confluent hypergeometric function which can be approximated by two different forms depending on input values. To be accurate we need to know when switching from a form to another. Finally we will simulate the terminal hedging error using the optimal strategy and compare it with another strategy when we complete the market with vanilla options.

5.1 Numerical Verification of Analytical Formulae

It seems natural to begin by validating our calculus and therefore our analytical formulae. We then proceed to a comparison between a Monte-Carlo simulation and closed forms of Section 4 for both Q^m-expectations and asking price. Obviously to do some implementations we need to specify all parameters introduced when building the model. We will essentially choose the same parameter values as Grasselli and Hurd [8], that is, we take: $b = 1.16$, $c = 0.2$, $\gamma = 1$, $\mu = 0.127$, $r = 0.04$ but for the value of a, we will choose a value such that our CIR process for the variance under P has a mean level of $\frac{a}{b} = 0.09$ which corresponds to a volatility of 30%. This leads us to take $a = 0.1044$ and will give a distribution of realized volatilities with lower ones around 10% and higher ones around 60%, which is more realistic than in [8]. We will then take $Y_0 = y$ equals to this mean level and choose a correlation of $\rho = 0.8$ and maturity of our variance swap: $T = 5$. It remains to us to verify the non-attainable boundary condition for the variance process mentionned in Proposition 17. This is the case as: $2(a - \rho c(\mu - r)) \geq c^2$ for the chosen parameters. And we will conserve these parameters along the whole numerical discussion.

We have then computed absolute errors between analytical formulae and Monte-Carlo simulations for different number of simulations M and number of steps in paths simulations N with the Matlab code displayed in Appendix A.1. The plot below gives this result for couple (N, M) where $N = M$ in which we use obvious denominations such that: expectation (71) and (72) respectively equals here for $t = 0$ to $E_y^{Q^m}[e^{\gamma\epsilon^2 B(V_T)}e^{-\int_0^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s}\mathrm{d}s}]$ and $E_y^{Q^m}[e^{-\int_0^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s}\mathrm{d}s}]$, are called numerator-expectation and denominator-expectation. And recalling that simulated Y-paths used to compute payoff and integral $\int_0^T \frac{1}{2}\epsilon^2 \frac{(\mu-r)^2}{Y_s}\mathrm{d}s$, follow the process under Q^m: $dY_t = [a - bY_t - \rho c(\mu - r)]dt + c\sqrt{Y_t}dW_t^\rho$, we obtain the following plot in Figure 1.

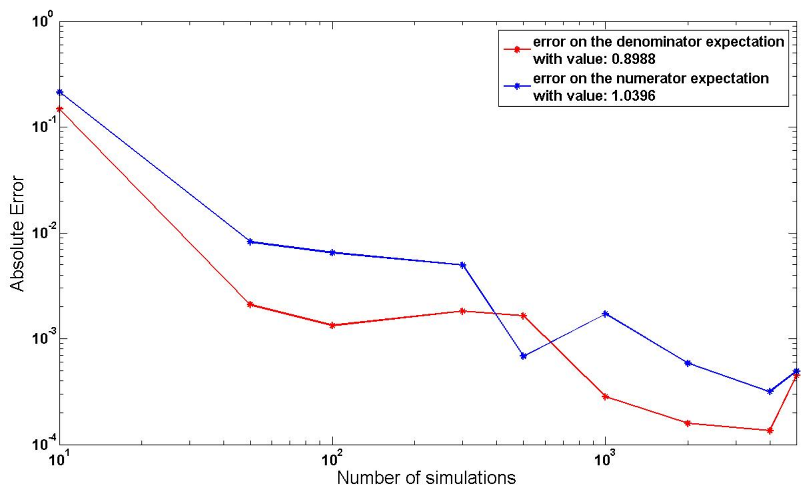

Figure 1: Verification of analytical formulae for Q^m expectations

We can see that MC-estimation actually converges to analytical formula, which verifies our calculus in Section 4. Another comment is that as we can expect, the error for the numerator-expectation is higher than the one for the denominator-expectation. This was expected as we need to approximate an additional term in computing the payoff in numerator-expectation and it implies an additional numerical error. We can also say that we rapidly reach an absolute error of approximately 10^{-3} for a number of simulations/steps around 1000. This gives a relative error around 0.1% for these values of expectations (respectively 1.0396 for the numerator-expectation and 0.8988 for the denominator-expectation), which seems acceptable.

We can do the same experiment for the price p^{ask} and with same parameters we get Figure 2.

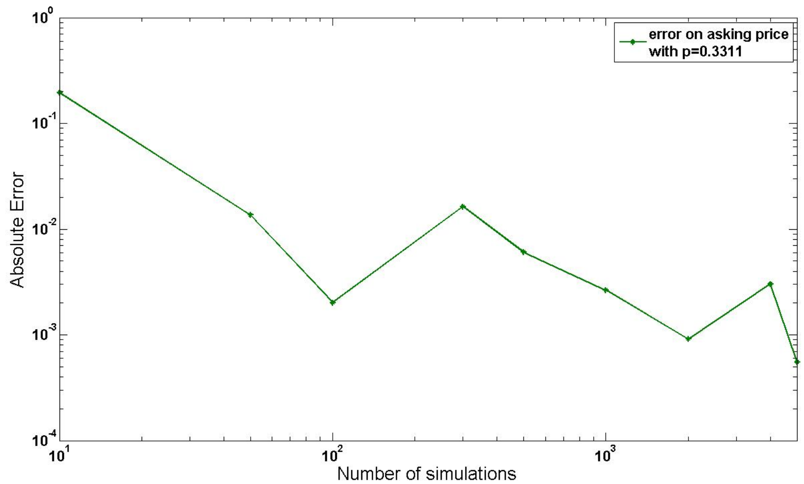

Figure 2: Verification of analytical formula for p^{ask}

Here the convergence is slower and in relative terms we only reach around 0.3% of relative error for several thousands of simulations/steps as the price value is $p^{ask} = 0.3311$. This can be justified by the following short calculus: if we denote by $err1$ the error made in approximating the numerator-expectation denoted $E1$ and by $err2$ the corresponding one for the denominator-expectation $E2$, we need to compute $log\frac{E1+err1}{E2+err2}$ to get the price. And in the worst case where $err1$ is positive and $err2$ is negative, p^{ask} is given by:

$$log\frac{E1(1+\frac{err1}{E1})}{E2(1+\frac{err2}{E2})} \simeq log\frac{E1}{E2} + \frac{err1}{E1} - \frac{err2}{E2}$$

so in the worst case we have two error terms $\frac{err1}{E1}$ and $-\frac{err2}{E2}$ which are both positive and for typical values of $E1$ and $E2$ which are around 1, or even less than 1, we can often reach a cumulative error $\geq |err1| + |err2|$, and then much bigger than the one attained on simple expectations. However all these numerical errors are sufficiently small to validate our analytical formulae of Section 4 for Q^m-expectations, p^{ask} and certainty equivalents.

In this previous simulation, we have actually taken a particular form of the confluent hypergeometric function F_1. But as mentioned earlier in

Subsection 4.2, the form we choose for it depends on the value of y and $T-t$. And here since our calculus has been done for a fixed variance y at initial time, we have taken the only form that matches these values. However we are going to see in the next subsection that we need to change this representation for F_1 if we use varying values of variance and time to maturity.

5.2 Calibration of the Confluent Hypergeometric Function

We actually base this choice of the correct form of the confluent hypergeometric function on the discussion [13] of Hurd and Kuznetsov. They actually argue that $F_1(a,b,z)$ can be approximated by the expansion:

$$F_1(a,b,z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{2b(b+1)}z^2 + \bigcirc(z^3) \tag{83}$$

when the argument z is small (in absolute value) and/or time to maturity $T-t$ is large.
But when z is large (in absolute value) and/or $T-t$ is small, one should approximate $F_1(a,b,z)$ by the second form:

$$F_1(a,b,z) = (-z)^{-a}\frac{\Gamma(b)}{\Gamma(b-a)}\sum_{n=0}^{N}\frac{(a)_n(a-b+1)_n}{n!}(-z)^{-n} + \bigcirc(|z|^{-a-N-1}) \tag{84}$$

We raise then the problem of switching from a form of F_1 to another when $T-t$ is decreasing and so, for different values of y as it will be the case in the simulation of the residual risk process in Subsection 5.3. To illustrate this clearly, we plot the evolution of the numerator-expectation versus time to maturity $T-t$ for typical variance values. More precisely we will take six different values $(0.01, 0.04, 0.09, 0.16, 0.25, 0.36)$ corresponding to volatility values $(10\%, 20\%, 30\%, 40\%, 50\%, 60\%)$, which cover approximately the range of volatilities achieved with our CIR process for the variance. And to note for which time to maturity, we must change our formula for F_1, we compare the analytical formula (lines) for the numerator-expectation to the MC-approximation (stars) from Subsection 5.1. We can do this as we saw that this MC-estimation is sufficiently accurate. The following plot shows an overview of what happens when choosing the first form (83) for F_1:

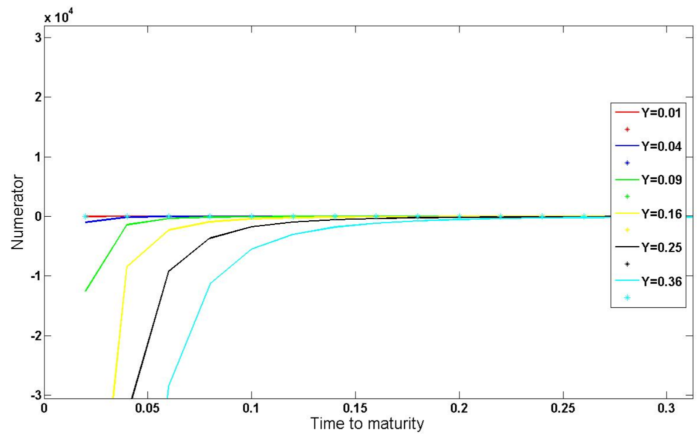

Figure 3: Overview of F_1 behaviour for the first form

We note from Figure 3, that as expected with form 1, the analytical fomula become less and less accurate when $T-t$ goes smaller (MC-estimation stars for the different values of y are superposed). And we also confirm the fact that for a given small time to maturity, the accuracy is much higher for smaller variance. Now we look at what happens with the second form of F_1. The corresponding overview is given in Figure 4:

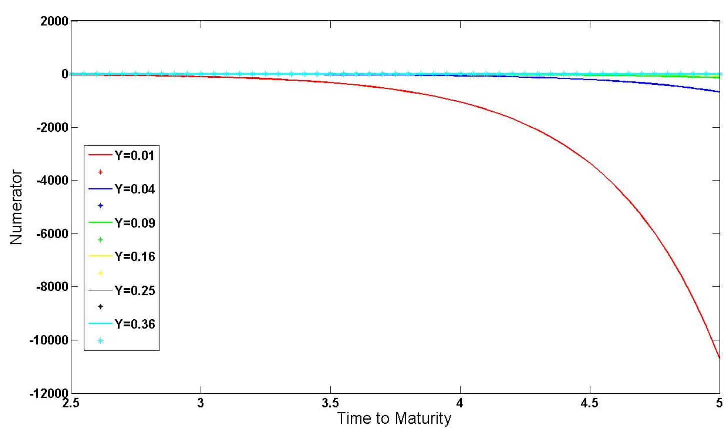

Figure 4: Overview of F_1 behaviour for the second form

Here, we observe the reverse phenomena and as expected the analytical formula materialized in the lines is less and less accurate when $T-t$ is increasing. In addition Figure 4 shows that for a given big $T-t$, this accuracy is bigger for large variance than for smaller one. Just look at the red line of $y = 0.01$ for illustration: the discrepancy between the analytical formula and the MC-estimation becomes incredibly high for big values of time to maturity. To conclude this overview we can remark that in Subsection 5.1 we have taken F_1 equals to the first form and respectively $T - 0 = 5$ and $y = 0.09$ (corresponding to the green line). We can see on Figure 3 that with these values there is no doubt about the choice of the form and we have then choose the correct , i.e the first form to conduct accurate implementation.

This overview of what changes using either the first form for F_1 or the second one, leads us naturally to the problem of calibrating this function according to varying values of y and $T - t$ in order to choose the correct one. We then conduct this calibration via graphical estimation and then more accurate calculus. And for a given variance, we look for the time to maturity when the analytical formula no longer matches the MC-estimation. Of course, as we saw in Subsection 5.1 the MC-estimation cannot be reliable further than 10^{-3} in absolute error, so we will consider that the form for F_1 remains correct as long as absolute errors between analytical formulae and MC-estimations are smaller than 5.10^{-3}. We will use formulae and simulation of the numerator-expectation to conduct the calibration. We give an example of graphical calibration for $y = 0.04$ (blue line) and $y = 0.09$ (green line) for the second form of F_1 in Figure 5:

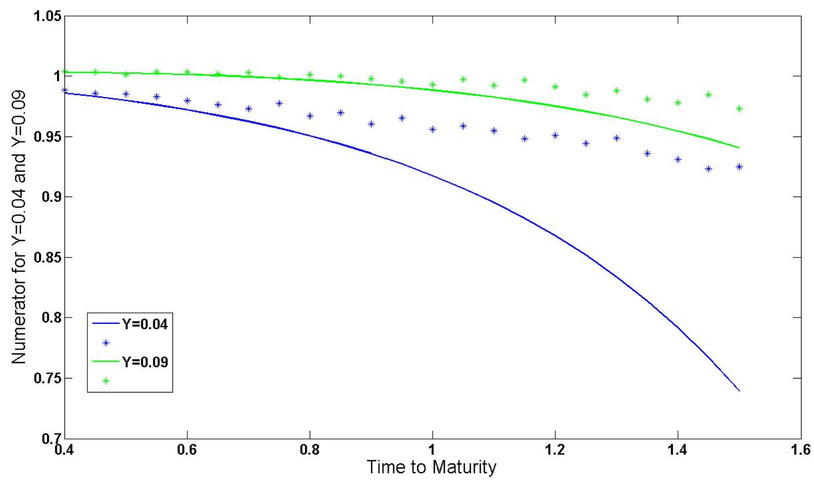

Figure 5: Calibration of the seconf form of F_1 for two chosen variances

We will consider here for instance that the analytical formula is correct until $T-t = 0.7$ for $y = 0.04$ and $T-t = 1.15$ for $y = 0.09$ (approximately). From this first graphical calibration, we conduct more accurate calculus of the discrepancy around the area of times where we have to change the form for the confluent hypergeometric function. This is easily done from algorithm in Appendix A.1 and Appendix A.2 respectively for the first form and the second form. It leads us to the following results for the calibration:

Variance	$y \leq 0.025$	$0.025 < y \leq 0.065$	$0.065 < y \leq 0.125$	$0.125 < y \leq 0.205$	$0.205 < y \leq 0.305$	$0.305 < y$
Switch at T-t	0.25	0.6	1.05	1.4	1.75	2

Table 1: Calibration of F_1 results

Commenting Table 1, we can say that the first form of F_1 is convenient for a bigger part of the variance swap lifetime than the second form as all changement in F_1 are done for time to maturity smaller than 2. These results are identical to what we get by calibration of the denominator-expectation. We also argue that we do not go further in variance than $y \simeq 0.36$ which corresponds to 60% of volatility, which is the maximum attained in the next simulation of Subsection 5.3 (one can refer to Figure 8). We can now take into account this calibration for simulating our residual risk process in the following subsection.

5.3 Hedging Performance for the Optimal Strategy

Our contribution is then to implement the residual risk process R derived in Proposition 14 in order to compute the terminal hedging error distribution. This error being obviously given for our variance swap by $R_T = X_T^* - \int_0^T Y_s ds$, i.e by the final value of the residual risk at maturity T. To achieve this aim we use our analytical and calibrated formulae for p_y^{ask} given by (81) and for cer_y^0 given by corollary 20. The algorithm displayed in Appendix A.3 gives the implementation of: $dR_t = rR_t dt - p_y^{ask} c\sqrt{Y_t}\epsilon dW_t^2 + \frac{1}{2}\gamma\epsilon^2 b(t,T)c^2 Y_t((p_y^{ask})^2 + 2p_y^{ask} cer_y^0)dt$ taking into account the calibration of Subsection 5.2 and using parameter values specified in Subsection 5.1 and summarized in the next table:

a	b	c	ρ	γ	μ	r	y	T
0.1044	1.16	0.2	0.8	1	0.127	0.04	$\frac{a}{b}$	5

Table 2: Implemented parameters

We simulate a large number of Y-paths, let say $10,000$ as suggested by Monoyios in [16], such that we will rebalance the hedge 200 times during the variance swap's lifetime according to optimal strategy π^{Δ}. This implementation is done according to Matlab code displayed in Appendix A.3 and to illustrate it, we give the following examples of a simulated particular Y-path and R-path respectively in Figure 6 and Figure 7:

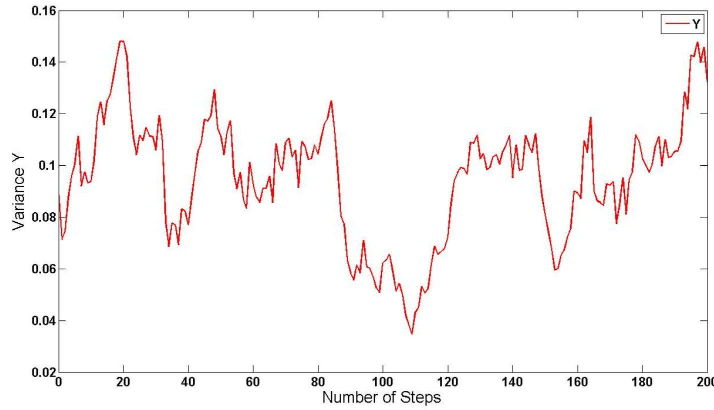

Figure 6: One of the Y-path simulated

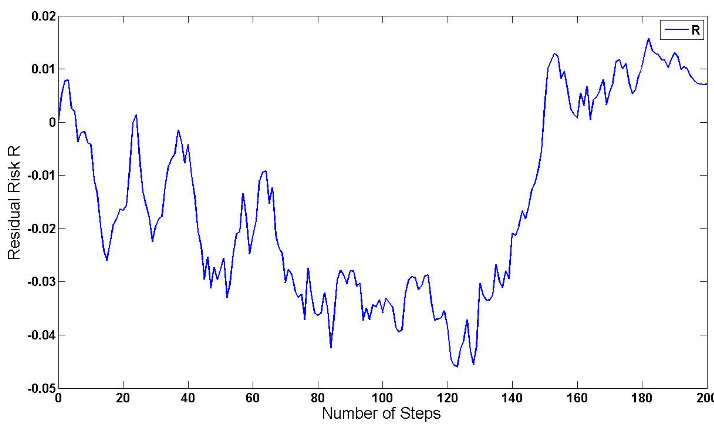

Figure 7: The corresponding R-path

We observe as expected that our residual risk process R tends to increase when the variance is going under its mean level. We can financially interpret this as the fact that when we sell the variance swap for a given price $p^{ask}(0, Y_0 = y)$ corresponding to the given level of variance $y = 0.09$ at initial time, then if the realized variance is much higher than y it means that we have sold our claim too cheap and in addition we have to invest more money in the stock to cover our short position. The result of such a situation is consequently a loss on our whole position and we will have a negative residual risk. Conversely if the realized variance remains small, R will tend to increase and end at a positive level. Another graphical investigation is to see how the simulated variance is distributed for our 200 steps over the 10,000 simulations. We get the following histogram for our 2,000,000 realized variance values:

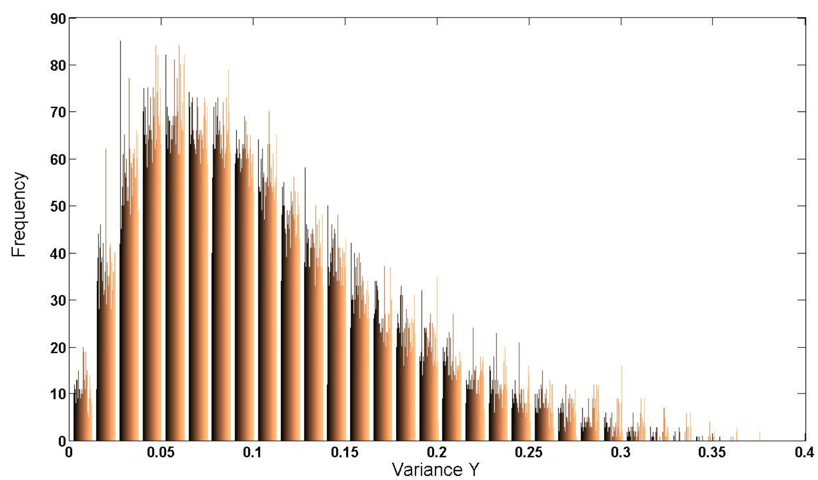

Figure 8: Distribution of realized variance

As we discussed in Subsection 5.1 we choosed parameters in order to get no greater volatilities than approximately 60% i.e a variance $y \simeq 0.36$, which is verified in Figure 8. We also observe that variance around $0.05-0.1$ are the most frequent and correspond to a high frequency of $20\% - 30\%$ of volatilities. And this is the kind of value we can expect from a given stock in standard market conditions. Now if we look finally at the histogram of terminal hedging error, we obtain Figure 9:

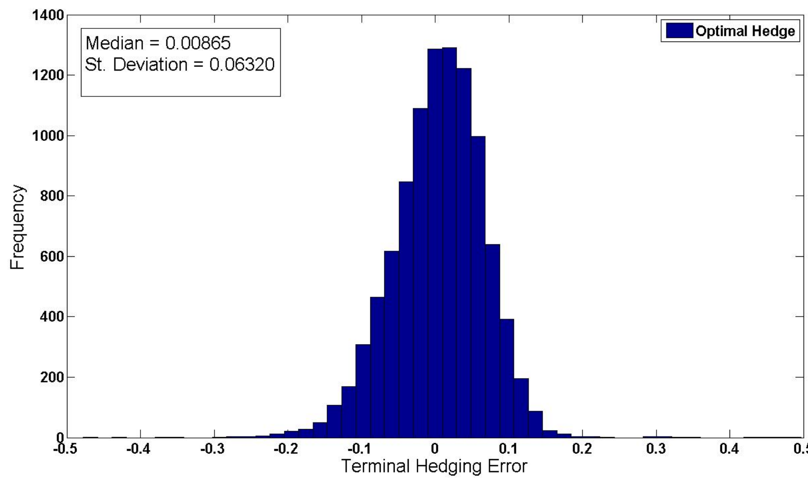

Figure 9: Distribution of Terminal Hedging Error with Optimal Hedge

And we give more precisely below values of important statistical quantities of this distribution:

Median	St.Deviation	Mean	Maximum	Minimu
0.00865	0.06320	0.00514	0.49676	-0.47771

Table 3: Statistical Results for the Distribution of Terminal Hedging Error with Optimal Hedge

What can be inferred from this simulation is that we get a positive median, which is here the really significant statistics for the distribution in Table 3 and means that on average our hedged position will result in a positive hedging error. But to be meaningful this study should be compared to another strategy to see how really interpret these statistical values. This is the point of next subsection where we will complete the hedge by allowing trading in a particular straddle of vanilla options.

5.4 Improvement via Vanilla Derivatives

As mentioned in the introduction we can perfectly replicate a variance swap if we are allowed to trade in a large number (and ideally an infinity) of vanilla calls and puts. To be more accurate, we give here a short proof of this, as did by Carr, Ellis and Gupta [3] and which is commonly quoted in the literature [2, 7, 4].

Recalling that a variance swap pays out the quadratic variation of $logS_t$ at maturity T, which is equals on our model to $\int_0^T Y_s ds$ and if we now apply the Ito's Lemma to this integral we find the following equality:

$$\int_0^T Y_s ds = 2(\int_0^T \frac{dS_t}{S_t} - log\frac{S_T}{S_0}) \tag{85}$$

So we raise the problem of perfect replication of a log contact. This can be solved using the work of Breeden and Litzenberger in [1] who prove that for f being a twice continuously differentiable function which represents the payoff of a European derivative:

$$f(S_T) = f(x)+f'(x)(S_T-x)+\int_x^\infty f''(K)(S_T-K)^+dK+\int_0^x f''(K)(K-S_T)^+dK \tag{86}$$

Then applying (86) to the log contract, Carr, Ellis and Gupta derive a perfect replication of payoff $log\frac{S_T}{S_0}$ via:

$$log\frac{S_T}{S_0} = \frac{S_T - S_0}{S_0} - \int_{S_0}^\infty \frac{1}{K^2}(S_T - K)^+dK - \int_0^{S_0} \frac{1}{K^2}(K - S_T)^+dK \tag{87}$$

Discounting now (87) and taking the expectation, gives the price of the variance swap at time 0, such that:

$$\begin{aligned} VarSwap(0,T) = 2\Big(&E[e^{-rT}(\int_0^T \frac{dS_t}{S_t} - \frac{S_T - S_0}{S_0})] + \int_{S_0}^\infty \frac{1}{K^2}C(K,T)dK \\ &+ \int_0^{S_0} \frac{1}{K^2}P(K,T)dK]\Big) \end{aligned} \tag{88}$$

Obviously this static perfect hedge of the variance swap implies that one can trade in an infinity of calls $C(K,T)$ and puts $P(K,T)$, which is of course an idealization and which is the opposite of our set-up where we can only trade in the underlying stock. Financial practice is then to choose a small number of call-put pairs with same T and K available on the market, by an optimization technique. One can refer to [2] for further comments on this point.

Then, we observe that the first couple of calls and puts involved in the variance swap replication is: $\frac{C(S_0,T)+P(S_0,T)}{S_0^2}$. This kind of position is well known and combines positions in the same sense on one call and one put with same maturites and strikes, it is then a straddle. From (88), we see that if you are in a short position on the variance swap you can partly replicate it being short on this static straddle. In order to hedge the sale of our variance swap we will consequently take a static long position on this straddle and hedge dynamically with the optimal hedge as in Subsection 5.3. It is actually very intuitive to use this static position to cover the risk associated with the variance as the straddle will result in a gain if the realized variance is high and that S_T ends far from its initial point S_0. And we know that in this case without this straddle we will loose (on average) with optimal hedging as we will have sold the variance swap too cheap compared to the high realized variance. To illustrate how our implemented straddle behaves we draw its payoff in Figure 10 (we take $S_0 = 100$ and $T = 5$):

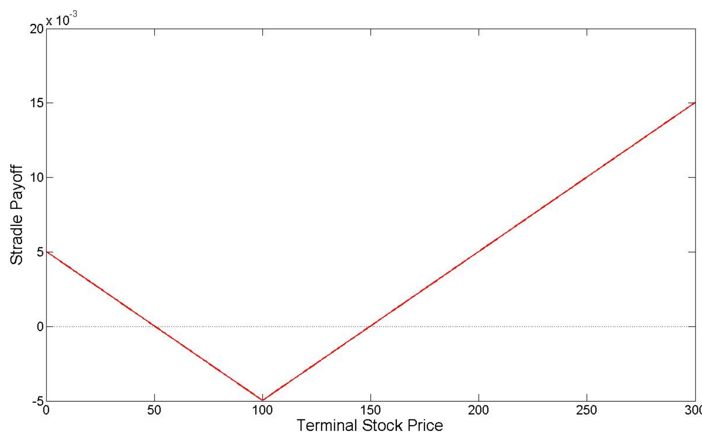

Figure 10: Our Straddle Payoff

We note that it leads to a gain only if $S_T \leq 50$ or $S_T \geq 150$ and so if we reache sufficiently high realized volatility. It will initially cost $\frac{C(0,S_0,S_0,T)+P(0,S_0,S_0,T)}{S_0^2}$ to enter the straddle and we need to take this cost into account for the following implementation.

As in Subsection 5.3, we illustrate the simulation by giving two graphical examples of simulated Y-path and S-path as we need here to simulate S_t:

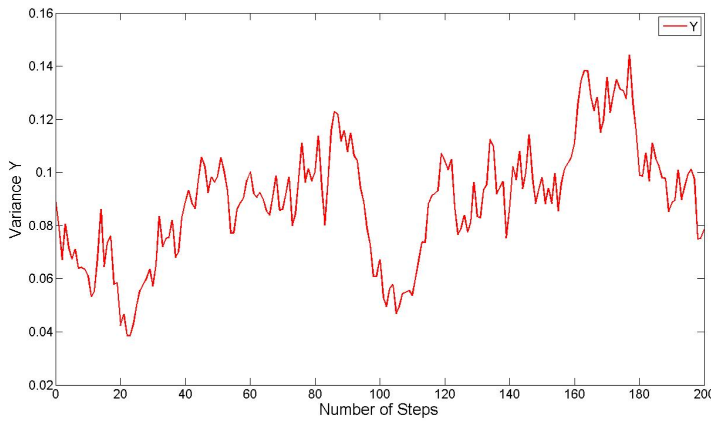

Figure 11: Y-path simulated with the static position in the straddle

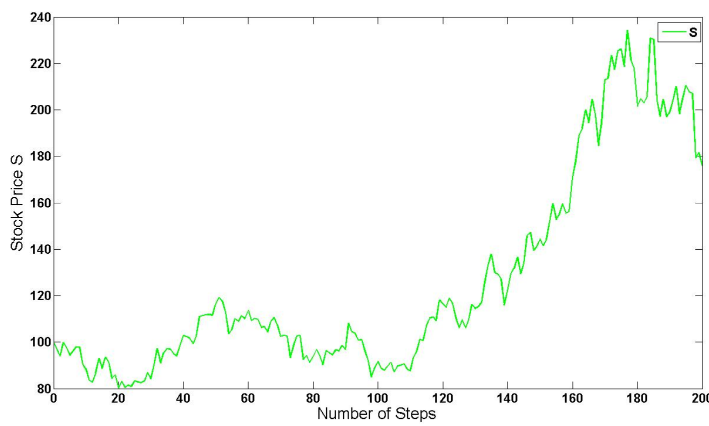

Figure 12: Corresponding S-path simulated with the static position

And finally we get the following terminal hedging error distribution when adding the static long position in the straddle:

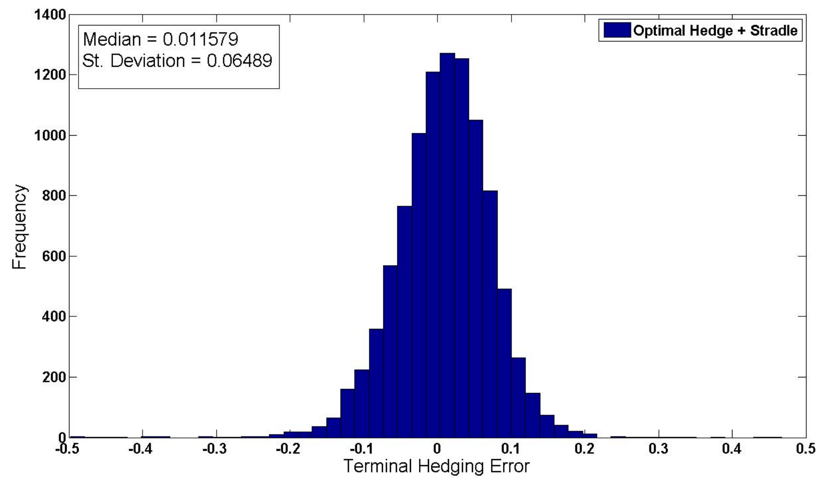

Figure 13: Terminal Hedging Error Distribution with the Straddle

The corresponding main statistics are reported in Table 4:

Median	St.Deviation	Mean	Maximum	Minimu
0.011579	0.06489	0.00872	0.46722	-0.49806

Table 4: Statistical Results for the Distribution of Terminal Hedging Error with Optimal Hedge

Compared with results of simple optimal hedge in Table 3, we get a much higher median. Actually it is even 34% higer here i.e the adding static position in the straddle will result in a more frequent positive hedging error than for the unique optimal hedge. We improve then substantially the performance of our hedge with longing the straddle. We can observe a similar behaviour for means even if there are less meaningful. Ranges of Table 3 and 4 are similar and do not tell anything. In addition the standard deviation is comparable to the one achieved for the optimal hedge as it is here higher of around 3%. The peak of the distribution is just a little bit less sharp than in Subsection 5.3, which financially means that if we are lucky and gain on the terminal hedging error, this error will be a little bit higher here and then we will achieve better performance but conversely if we are not lucky we will loose more money than with the simple optimal hedge. But this is a very small advantage for the simple optimal strategy compared to the much higher median achieved by the improved strategy with the straddle.

6 Conclusion

This dissertation has recalled the primal and dual formulations of indifference pricing and optimal hedging in a utility based framework where we worked with a variance swap. One of the main result we have proved is the derivation of the pricing PDE for p^{ask} in Proposition 7 from the primal problem in terms of the certainty equivalent cer^0. This PDE has allowed us then to derive the SDE satisfied by the residual risk process R in Proposition 14. We have at the same time extended Grasselli and Hurd formulae of asking price and certainty equivalent in terms of Q^m-expectations to non zero r model. Both have been then used to get analytical formulae by adapting calculus of Hurd and Kuznetsov to our particular set-up, which outcomes in the analytical formula for p^{ask} (Theorem 20) and our additional contribution for π^{Δ} (Theorem 21). Our main work has been then to carry on with the extended study of Grasselli and Hurd and contribute to it by conducting numerical experimentations using this material. So we have first verified and validated the accuracy of these analytical formulae and then calibrated the confluent hypergeometric function to be able to implement them. This has allowed us to simulate the residual risk process and then derive a terminal hedging error distribution when hedging optimally the sale of one variance swap. To be relevant, we have finally compared this main result with a different strategy taking into account the perfect replication of a variance swap in a complete model by adding a static position in the discussed straddle. We have here concluded that this additional static hedge improves the performance of the optimal hedge even if it leads to a less sharp distribution. Another possibility could be now to compute again this terminal hedging error but setting the variance swap in a different model. It will be also interesting to see what happens if we change the correlation for a lower one and how the static position reacts to this. Another further investigation concerns the use of the exponential utility and it would be a good idea to look at different utility functions or even to replace the asking price $p^{ask} = p^{n=-1}$ by the marginal price $\lim_{n\to 0} p^n$ which would give an insight for results with quadratic uitility function. Finally another idea would be to compute and implement only the optimal wealth process X_t^* without computing the whole residual risk process. Then we would simply substract the payoff: $\int_0^T Y_s ds$ to the terminal wealth X_T^* and get like that the terminal hedging error, which could be compared to the one obtained in Subsection 5.3 by computing the residual risk process.

A Appendix

We report in the Appendix most of the code used along Section 5. We remark that we have voluntarily not displayed all input paramters and commands such as plot. Consequently this code will not run independently but all relevant functions and files are reported.

A.1 Matlab Code for Validation of Analytical Formulae

We display here the code used to compute analytical formulae for Q^m expectations and compare them to MC-estimations. We note here that the code should be optimized by avoiding loops:

```
function G=hurd(T,y,d1,d2,a,b,c)
alpha=((2*a)/(c^2))-1;
beta=(2*b)/(c^2);
v1=0.5*(-beta+sqrt(beta^2+((8*d1)/(c^2))));
v2=0.5*(-alpha+sqrt(alpha^2+((8*d2)/(c^2))));
k=(beta+2*v1)*(1-exp(-(b+v1*c^2)*T))^(-1);

G=exp(-y*(v1+((k*(-v1))/(k-v1))*exp(-(b+v1*c^2)*T)))*exp(-(a*v1+b*v2+c^2*v1*v2)*T)...
*(y^v2)*(k-v1)^(-alpha-v2-1)*k^(alpha+2*v2+1)*(gamma(alpha+v2+1)/gamma(alpha+2*v2+1))...
*hyperg(v2,alpha+2*v2+1,-((k^2*y*exp(-(b+v1*c^2)*T))/(k-v1)));
end

function F=hyperg(a,b,z)
F=1+(a/b)*z+0.5*((a*(a+1))/(b*(b+1)))*z^2+(1/6)*((a*(a+1)*(a+2))/(b*(b+1)*(b+2)))*z^3;
end

function num= numerat(M,N,a,b,c,rho,gam,mu,r,y,T)
eps=sqrt(1-rho^2);
h=T/N;
U=zeros(N,M);
Z=zeros(N,M);
Y=zeros(N+1,M);
val=zeros(1,M);

for j=1:M
    U(:,j)=rand(1,N);
    Z(:,j)=ncfinv(U(:,j));
    Y(1,j)=y;
    for i=1:N
        if Y(i,j)<0
            Y(i,j)=0;
```

```
        end
        Y(i+1,j)=Y(i,j)+(a-rho*c*(mu-r)-b*Y(i,j))*h+c*sqrt(max(Y(i,j),0))*sqrt(h)*Z(i,j);
    end


    S=0;
    for k=2:N
        S=S+(1/Y(k,j));
    end

    S2=0;
    for l=2:N
        S2=S2+Y(l,j);
    end

    payoff=h*(S2+(Y(1,j)+Y(N+1,j))/2);
    lamb=((mu-r)^2)*h*(S+(1/Y(1,j)+1/Y(N+1,j))/2);
    val(j)=exp(gam*eps^2*payoff)*exp(-0.5*eps^2*lamb);
end
num=(1/M)*sum(val);
end

function den= denom(M,N,a,b,c,rho,mu,r,y,T)
eps=sqrt(1-rho^2);
h=T/N;
U=zeros(N,M);
Z=zeros(N,M);
val=zeros(1,M);
Y=zeros(N+1,M);

for j=1:M

    U(:,j)=rand(1,N);
    Z(:,j)=ncfinv(U(:,j));

    Y(1,j)=y;
    for i=1:N
        if Y(i,j)<0
            Y(i,j)=0;
        end
        Y(i+1,j)=Y(i,j)+(a-rho*c*(mu-r)-b*Y(i,j))*h+c*sqrt(max(Y(i,j),0))*sqrt(h)*Z(i,j);
    end


    S=0;
```

```
    for k=2:N
        S=S+(1/Y(k,j));
    end
    lamb=((mu-r)^2)*h*(S+(1/Y(1,j)+1/Y(N+1,j))/2);
    val(j)=exp(-0.5*eps^2*lamb);

end

den=(1/M)*sum(val);
end
```

A.2 Matlab Code for Calibration of F_1

Here we find the same code as in the first Appendix A.1 but with the implementation of the second form of the confluent hypergeometric function in order to calibrate it.

```
function G=hurd2(T,y,d1,d2,a,b,c)
alpha=((2*a)/(c^2))-1;
beta=(2*b)/(c^2);
v1=0.5*(-beta+sqrt(beta^2+((8*d1)/(c^2))));
v2=0.5*(-alpha+sqrt(alpha^2+((8*d2)/(c^2))));
k=(beta+2*v1)*(1-exp(-(b+v1*c^2)*T))^(-1);

G=exp(-y*(v1+((k*(-v1))/(k-v1))*exp(-(b+v1*c^2)*T)))*exp(-(a*v1+b*v2+c^2*v1*v2)*T)...
*(y^v2)*(k-v1)^(-alpha-v2-1)*k^(alpha+2*v2+1)*(gamma(alpha+v2+1)/gamma(alpha+2*v2+1))...
*hyperg2(v2,alpha+2*v2+1,-((k^2*y*exp(-(b+v1*c^2)*T))/(k-v1)));
end

function F=hyperg2(a,b,z)
F=(-z)^(-a)*(gamma(b)/gamma(b-a))*(1+a*(a-b+1)*(-z)^(-1)+((a+1)*(a-b+2)/2)*(-z)^(-2));
end
```

A.3 Matlab Code to compute Terminal Hedging Error

We give here the code used to simulate the residual risk process taking into account the calibration of Subsection 5.2. We voluntarily use loops as we want to store all paths and particularly output the whole Y-distribution. We also give preliminary codes to compute p_y^{ask} and cer_y^0. (We will not that we will not give the code for the hedge with straddle here because it is mainly the same as below with the simulation of S-paths and adjusting R with the straddle).

```
function dG= hurdder(T,y,d1,d2,a,b,c)
alpha=((2*a)/(c^2))-1;
beta=(2*b)/(c^2);
v1=0.5*(-beta+sqrt(beta^2+((8*d1)/(c^2))));
```

```
v2=0.5*(-alpha+sqrt(alpha^2+((8*d2)/(c^2))));
k=(beta+2*v1)*(1-exp(-(b+v1*c^2)*T))^(-1);

dG=-(v1+((k*(-v1))/(k-v1))*exp(-(b+v1*c^2)*T))+v2/y+(v2*hyperg(v2+1,alpha+2*v2+2,...
-((k^2*y*exp(-(b+v1*c^2)*T))/(k-v1))))/((alpha+2*v2+1)*hyperg(v2,alpha+2*v2+1,...
-((k^2*y*exp(-(b+v1*c^2)*T))/(k-v1))));
end

function dG= hurdder2(T,y,d1,d2,a,b,c)
alpha=((2*a)/(c^2))-1;
beta=(2*b)/(c^2);
v1=0.5*(-beta+sqrt(beta^2+((8*d1)/(c^2))));
v2=0.5*(-alpha+sqrt(alpha^2+((8*d2)/(c^2))));
k=(beta+2*v1)*(1-exp(-(b+v1*c^2)*T))^(-1);
dG=-(v1+((k*(-v1))/(k-v1))*exp(-(b+v1*c^2)*T))+v2/y+(v2*hyperg2(v2+1,alpha+2*v2+2,...
-((k^2*y*exp(-(b+v1*c^2)*T))/(k-v1))))/((alpha+2*v2+1)*hyperg2(v2,alpha+2*v2+1,...
-((k^2*y*exp(-(b+v1*c^2)*T))/(k-v1))));
end

function dp=hurdderiv(T,y,d1,d2,a,b,c,gam,eps,r)
dp=exp(-r*T)*(1/(gam*(eps)^2))*(hurdder(T,y,d1,d2,a,b,c)-hurdder(T,y,0,d2,a,b,c));
end


function dp=hurdderiv2(T,y,d1,d2,a,b,c,gam,eps,r)
dp=exp(-r*T)*(1/(gam*(eps)^2))*(hurdder2(T,y,d1,d2,a,b,c)-hurdder2(T,y,0,d2,a,b,c));
end

function dp=cerder(T,y,d2,a,b,c,gam,eps,r)
dp=exp(-r*T)*(1/(gam*(eps)^2))*(hurdder(T,y,0,d2,a,b,c));
end

function dp=cerder2(T,y,d2,a,b,c,gam,eps,r)
dp=exp(-r*T)*(1/(gam*(eps)^2))*(hurdder2(T,y,0,d2,a,b,c));
end

a=0.1044;
b=1.16;
c=0.2;
rho=0.8;
gam=1;
mu=0.127;
r=0.04;
y=a/b;
T=5;
```

```
eps=sqrt(1-rho^2);
R0=0;
d2=0.5*(eps^2)*(mu-r)^2;
d1=-gam*(eps^2);

M=10000;
N=200;
h=T/N;
L=sqrt(h)*[1 0;eps rho];

Y=zeros(N+1,M);
R=zeros(N+1,M);
W2=zeros(N,M);
Wrho=zeros(N,M);

for j=1:M

    W2(:,j)=randn(1,N);
    Wrho(:,j)=randn(1,N);

    Y(1,j)=y;
    R(1,j)=R0;
    for i=1:N
        W=L*[W2(i,j);Wrho(i,j)];
        Y(i+1,j)=Y(i,j)+(a-b*Y(i,j))*h+c*sqrt(max(Y(i,j),0))*W(2);
        if Y(i,j)<0
            Y(i,j)=0.001;
            R(i+1,j)=R(i,j)*(1+r*h);
        elseif Y(i,j)<= 0.025
            if T<=0.25
                      R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
                      (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv2(T-(i-1)*h,Y(i,j)...
                      ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2 ...
                          +2*hurdderiv2(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                          ,b,c,gam,eps,r)*cerder2(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                          (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv2(T-(i-1)*h,..
                          Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                          .*c*sqrt(max(Y(i,j),0)).*W(1);
            else
                          R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
                          (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv(T-(i-1)*h,Y(i,j)...
                          ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2...
                          +2*hurdderiv(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                          ,b,c,gam,eps,r)*cerder(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                          (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv(T-(i-1)*h...
```

```
                ,Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
               .*c*sqrt(max(Y(i,j),0)).*W(1);
    end
elseif 0.025<Y(i,j)<=0.065
    if T<=0.6
             R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
             (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv2(T-(i-1)*h,Y(i,j)...
             ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2 ...
                 +2*hurdderiv2(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                 ,b,c,gam,eps,r)*cerder2(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                 (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv2(T-(i-1)*h,..
                 Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                 .*c*sqrt(max(Y(i,j),0)).*W(1);
    else
                 R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
                 (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv(T-(i-1)*h,Y(i,j)...
                 ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2...
                 +2*hurdderiv(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                 ,b,c,gam,eps,r)*cerder(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                 (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv(T-(i-1)*h...
                 ,Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                .*c*sqrt(max(Y(i,j),0)).*W(1);
    end

elseif 0.065<Y(i,j)<=0.125
     R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
             (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv2(T-(i-1)*h,Y(i,j)...
             ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2 ...
                 +2*hurdderiv2(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                 ,b,c,gam,eps,r)*cerder2(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                 (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv2(T-(i-1)*h,..
                 Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                 .*c*sqrt(max(Y(i,j),0)).*W(1);
    else
                 R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
                 (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv(T-(i-1)*h,Y(i,j)...
                 ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2...
                 +2*hurdderiv(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                 ,b,c,gam,eps,r)*cerder(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                 (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv(T-(i-1)*h...
                 ,Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                .*c*sqrt(max(Y(i,j),0)).*W(1);
    end
elseif 0.125<Y(i,j)<=0.205
    if T<=1.4
```

```
        R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
              (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv2(T-(i-1)*h,Y(i,j)...
              ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2 ...
                  +2*hurdderiv2(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                  ,b,c,gam,eps,r)*cerder2(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                  (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv2(T-(i-1)*h,..
                  Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                  .*c*sqrt(max(Y(i,j),0)).*W(1);
    else
                  R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
                  (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv(T-(i-1)*h,Y(i,j)...
                  ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2...
                  +2*hurdderiv(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                  ,b,c,gam,eps,r)*cerder(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                  (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv(T-(i-1)*h...
                  ,Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                 .*c*sqrt(max(Y(i,j),0)).*W(1);
    end
elseif 0.205<Y(i,j)<=0.305
    if T<=1.75
        R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
              (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv2(T-(i-1)*h,Y(i,j)...
              ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2 ...
                  +2*hurdderiv2(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                  ,b,c,gam,eps,r)*cerder2(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                  (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv2(T-(i-1)*h,..
                  Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                  .*c*sqrt(max(Y(i,j),0)).*W(1);
    else
                  R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
                  (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv(T-(i-1)*h,Y(i,j)...
                  ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2...
                  +2*hurdderiv(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                  ,b,c,gam,eps,r)*cerder(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                  (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv(T-(i-1)*h...
                  ,Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                 .*c*sqrt(max(Y(i,j),0)).*W(1);
    end
else
    if T<=2
        R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
              (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv2(T-(i-1)*h,Y(i,j)...
              ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2 ...
                  +2*hurdderiv2(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                  ,b,c,gam,eps,r)*cerder2(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
```

```
                         (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv2(T-(i-1)*h,..
                         Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                         .*c*sqrt(max(Y(i,j),0)).*W(1);
            else
                         R(i+1,j)=R(i,j)*(1+r*h)+0.5*c^2*gam*eps^2*exp...
                         (r*(T-(i-1)*h))*Y(i,j)*((hurdderiv(T-(i-1)*h,Y(i,j)...
                         ,d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r))^2...
                         +2*hurdderiv(T-(i-1)*h,Y(i,j),d1,d2,a-rho*c*(mu-r)...
                         ,b,c,gam,eps,r)*cerder(T-(i-1)*h,Y(i,j),d2,a-rho*c*...
                         (mu-r),b,c,gam,eps,r))*h-eps*hurdderiv(T-(i-1)*h...
                         ,Y(i,j),d1,d2,a-rho*c*(mu-r),b,c,gam,eps,r)...
                        .*c*sqrt(max(Y(i,j),0)).*W(1);
            end
        end
     end
   if Y(N+1,j)<0
      Y(N+1,j)=0.001;
   end
end
```

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