Two-Dimensional Conformal Field Theories: the Irrational, the Rational and the Celestial



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A Pepe.
Sempre volverán a florecer as camelias.

In loving memory of my godfather José Jorge-Montenegro (1944-2023).

Abstract

The aim of this thesis is to explore the world of two-dimensional Conformal Field Theories via bootstrap techniques. The thesis is divided in three central chapters, where we will study Irrational, Rational and Celestial CFTs.

The holographic dual to a theory of pure higher-spin gravity in AdS_3 is an *Irrational* CFT with $\mathcal{W}_N$ algebra, where N corresponds to the highest-spin field present in the theory. Our aim in chapter 3 will be to study the spectrum of such a theory of gravity by constructing the density of states via the Poincaré and Rademacher constructions, which both exploit the modular invariance of the torus partition function. When studying the densities, we encounter exponentially large amounts of negative-norm states in the near-extremal region, which would render the theory non-unitary. We propose to cure these negativities by adding operators of low twist, which can be interpreted naturally from the gravity perspective as orbifolds with conical defects and can be added without losing unitarity.

The classification of *Rational* CFTs will occupy our attention in chapter 4. Our goal is to understand the spectral decomposition of the torus partition function of several RCFTs, and see how it can aid in the efforts to explore the space of such theories. We begin by introducing the basics of harmonic analysis on the fundamental domain of the modular group. Afterwards, we study the Rankin-Selberg transform of one-character RCFTs' partition function and find an interesting relation between different central charges. From there, we start investigating multi-character RCFTs, and make a proposal to build families of multi-character RCFTs from one-character theories via their Rankin-Selberg transforms.

The duality between scattering amplitudes of massless particles in $\mathbb{R}^{(2,2)}$ space and the correlation functions of a *Celestial* CFT living on the celestial torus will be the

subject of Chapter 5. The goal of this chapter is to provide a prescription that allows us to go from momentum-space amplitudes to the ambidextrous basis, where the celestial amplitudes take the form of a two-dimensional CFT correlation function, via an improved light transform. We explicitly build 2-, 3- and 4-point amplitudes of massless external particles, and show how the ambidextrous prescription maintains conformal covariance in all scattering channels whilst ridding the amplitude of its distributional factors. We then move onto an explicit tree-level four-gluon, where we will use alpha space decomposition, an inversion formula method specific to $SL(2,\mathbb{R})$ primaries, to obtain data from the Celestial CFT's spectrum.

Statement of Originality

This thesis contains material done in collaboration, all to which the author contributed substantially:

- Chapter 3 is based on [1], done in collaboration with Luis F. Alday, Jin-Beom Bae and Nathan Benjamin.

  [1] L.F. Alday, J.-B. Bae, N. Benjamin and C. Jorge-Diaz, *On the spectrum of Pure Higher Spin Gravity*, *JHEP* **12** (2020) 001 [2009.01830]

- Chapter 4 contains unpublished work done in collaboration with Jin-Beom Bae and Murat Kologlu [2],

  [2] J.-B. Bae, C. Jorge-Diaz and M. Kologlu, *Harmonic analysis of RCFTs*, *To appear*

- Chapter 5 is based on [3], done in collaboration with Sabrina Pasterski and Atul Sharma.

  C. Jorge-Diaz, S. Pasterski and A. Sharma, *Celestial amplitudes in an ambidextrous basis*, *JHEP* **02** (2023) 155 [2212.00962]

The Appendices also draw some material from work in collaboration. Appendix A.2 is based on work from [1]. Appendix B, containing material related to Chapter 4, is based on work from [2]. Appendix C.1 is based on work from [3], and Appendix C.2 presents some extra material from [3] that wasn't published in the main article.

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Chapter 1

Introduction

Conformal field Theories have been a field of interest for physicists for quite a few decades now. Interest originally sparked in the 70s in the context of thermodynamics and critical phenomena: certain kinds of second-order phase transitions can be described with a CFT. However, this is not the only field that brought CFTs to the interest of theoretical physicists: the early development of String Theory from the 60s onward, where the worldsheet of strings' dynamics are governed by a CFT, was key. For quite a while, CFTs were developed in parallel to String Theory in that realm, but interest in CFT from a fundamental physics perspective also came from the fact that the fixed point of renormalisation group flow is described by a CFT, and as such they are key in the study of QFTs.

How come CFTs can appear in so many different contexts? Well, it all boils down to a fundamental feature of conformal invariance: in general terms, we can see conformal symmetry as a local scale invariance. Whenever one finds a system where such a scale invariance is present, it can often be described with a CFT.

Furthermore, it was soon realised [4] that the structure of these theories and, more in particular, their operator algebra, had very interesting characteristics. Rather than exploring the features of CFTs via a Lagrangian, as it had been done customarily, one could constraint observables by imposing conditions due to self-consistency and conformal invariance, and extract information or even solve the theory: this is called a '*bootstrap*' approach.

These ideas were put to practice in the seminal work of Belavin, Polyakov and Zamolodchikov [5], commonly referred to as BPZ, where the authors studied two-dimensional theories using the

bootstrap approach and showed how to build the so-called minimal models, which are completely solvable. From here, the two-dimensional bootstrap flourished for a while, but later progress slowed off as people moved on to other problems.

However, things would change when the AdS/CFT Correspondence was introduced by Maldacena [6]: the possibility to solve theories of quantum gravity via a CFT duality made people more interested in exploring CFTs again, in two and in higher dimensions.

In fact, it was in 2008 when the bootstrap program was revived thanks to the seminal work of Ratazzi, Rychkov, Tonni and Vichi [7], which brought people's attention to solving CFTs via now both numerical and analytical methods, not only in two but also (and specially) in higher dimensions.

Nowadays, one can find CFTs being studied in a wide breadth of subjects: they can be used to study exotic materials, superconductors and quantum critical phenomena, but they also play a central role in fundamental research as they can be seen as a doorway to developing new techniques to solve strongly-correlated QFTs and even quantum gravity.

In this thesis, we will be focusing on employing a variety of bootstrap techniques to investigate various types of Conformal Field Theories in two dimensions. We will be using different *bootstrap* techniques for that, which will be explained in each chapter.

As the title suggests, we will be interested in exploring irrational, rational and celestial CFTs. What do these adjectives stand for in this context?

- **Irrational** CFTs are theories that have an infinite amount of primary fields of the full symmetry group, generally with $c > 1$. We are interested in irrational theories containing higher-spin fields that are dual to pure, higher spin gravity in three dimensions (in the context of the AdS/CFT correspondence).

- **Rational** CFTs are theories that have a finite amount of primary operators: this constrains both the central charge and conformal weights of the operators, which have to be rational numbers [8]. We will be concerned with those theories with $c > 1$, in the context of exploring the moduli space of RCFTs using harmonic analysis techniques.

- **Celestial** CFTs correspond to a recent and exotic family of Conformal Field Theories that are found when studying amplitudes in asymptotically flat spaces. When one takes

these amplitudes to null infinity, they can be seen to exhibit properties of CFT correlation functions and, furthermore, one can find how Lorentz transformations become global conformal transformations in this limit. We will be using bootstrap techniques to help build the dictionary between the amplitudes and the CFT, and to understand the structure of these highly exotic theories.

This thesis is organised as follows. Chapter 2 will be an overview of conformal field theories in two dimensions, covering their main properties and then introducing the reader to the conformal bootstrap in two dimensions. Chapter 3, containing material from [1], is dedicated to a family of Irrational CFTs with $\mathcal{W}_N$ symmetry dual to higher-spin gravity. Chapter 4, containing material from [2], studies how harmonic analysis can be a tool used in the classification of RCFTs. Chapter 5, containing material from [3], will deal with the dictionary between the Celestial CFT and scattering amplitudes of massless particles. Finally, Chapter 6 will provide a space for concluding remarks and outlook for possible future ideas to explore in the topics discussed earlier.

We will now present an overview of the motivation and content of each of the central chapters (3, 4, 5) in the thesis.

1.1 Irrational CFTs and pure higher spin gravity

Low-dimensional gravity ($d < 4$) has long been used as a toy model to explore properties of gravitational theories in higher dimensions. This is because both two and three-dimensional gravity theories have no dynamical degrees of freedom or, in other words, they don't have propagating gravitons in the bulk. These theories are topological, but not constrained enough to be trivial as a classical theory, and one wonders what happens when we try to quantise them: are these too constrained to exist as consistent theories of quantum gravity?

The case of Jackiw-Teitelboim gravity in AdS_2 was famously solved by Saad, Shenker and Stanford, who found that two-dimensional JT gravity is dual (via AdS/CFT correspondence) to not a single theory, but rather an ensemble of quantum systems, described by random matrix theory [9].

Three-dimensional gravity appears to be a bit more slippery when trying to quantise it in

a bottom-up approach. Even before the AdS/CFT correspondence was first proposed, Brown and Henneaux [10] found that the asymptotic symmetries of *pure* gravity in AdS_3 were two copies of the Virasoro algebra, and proposed that these could be described via a CFT: this is actually what AdS/CFT predicts in this scenario. The CFT in this scenario should be dual to semiclassical gravity for large central charge, and also irrational when considering pure gravity. However, when we try to study the spectrum of this theory [11–13] we find that it is continuous, and furthermore that there are an infinite amount of negative-norm states [14]. A continuous spectrum can be understood if we consider the possibility of ensemble averages, as in the two-dimensional case, but the negativity problem is much more worrisome. Multiple other proposals to understanding the duality more precisely, and (in those most recent ones) either solve or avoid the negative-norm states have appeared over the years [15–28].

In this spirit, Chapter 3 will explore similar questions in the context of pure higher spin gravity in AdS_3. These theories have a finite number of higher spin ($s > 2$) fields, a feature that is only allowed consistently in three dimensions, and they have a larger asymptotic symmetry group than Virasoro. This is $\mathcal{W}_N$ symmetry [29, 30], where N is the highest spin field added to the theory, and where $N = 2$ corresponds to Virasoro symmetry.

The goal of the chapter is to study the spectrum of said theories via AdS/CFT correspondence: the theories are dual to $\mathcal{W}_N$ CFTs, and we will study the partition funciton of said theories on the torus (which is dual to the one-loop partition function of the gravity theory).

We will build the density of states of the theory using modular invariance through two different constructions: Poincaré and Rademacher, from where we will find similar issues as those encountered in the lower-spin case: the spectrum is continuous and possesses an infinite amount of negative-norm states in the near-extremal limit.

We propose curing the negativities by introducing a minimal amount of low-twist operators, whose tori partition funciton can eliminate the negativities observed, to one-loop order. These can be beautifully interpreted from the gravitational side as massive states: they correspond to orbifolds with conical singularities.

Generally speaking, the prescription to compute a gravitational partition function involves a sum over smooth topologies, however we propose a minimal relaxation of this condition to allow for conical orbifolds, in the spirit of [21] for the Virasoro case.

We finalise by computing the partition function of higher spin gravity to one-loop order, via the heat kernel method. When computing the partition function of conical defect orbifold geometries via heat kernel, we find that it corresponds exactly with the partition function of a primary with the same weight as the proposed low-twist negativity-curing operators, thus putting more weight onto our proposal as the addition of these operators corresponds to a rather natural and minimal deviation from the traditional description of pure gravity.

1.2 Harmonic Analysis on Rational CFTs' partition functions

When studying CFTs, a big aim is to classify as many of them as possible, as well as identify the relevant features of the moduli space of theories. RCFTs, due to having a finite amount of primary states (relative to the largest symmetry group of the theory), seem to be a naturally good place to look. Of course, this route was initiated quite early on: already the remarkable BPZ paper [5] made a proposal for all the valid unitary, compact CFTs with c $<$ 1. The search for more theories, especially those with $c > 1$ was initiated, to quite large success: WZW models and coset constructions were discovered [31–35], as well as the extremely powerful MLDE method [36] of obtaining RCFTs starting only from the number of characters and some analytic properties.

However, we know of RCFTs that don't fall under any of these classifications (see, for instance [37, 38]) and, furthermore, most of them don't allow us to expand our search and thus somehow characterise the *moduli space*[1] of theories beyond certain features. In this line, in Chapter 4 we will develop the study of RCFTs partition function from a different method: harmonic analysis.

We begin by introducing the basics of harmonic analysis on the fundamental domain of $SL(2,\mathbb{Z})$, which can be applied to torus partition functions of RCFTs. This is akin to Fourier decompositions, but in the context of a hyperbolic background, and it allows us to decompose modular forms into a basis. There are two issues we need to overcome in order to apply the techniques of Harmonic Analysis

- The partition function of RCFTs on the torus is not square-integrable, which is a require-

[1] We are using the term 'moduli space' loosely here, not necessarily in a formal sense, but rather referring to a possible full classification of the possible (R)CFTs and the relations amongst different theories.

ment of harmonic analysis in this context. However, it can become of slow growth by multiplying it by a suitable modular-invariant factor, and then one can use an adapted harmonic analysis method developed by Zagier [39].

- The basis we are decomposing onto is not fully analytical: in particular, as we will see, the Maass cusp forms are only known numerically and, in our case, working out the numerics don't lead to enlightening results. This means we will only specialise on the scalar sector of the theory, which is decomposed into non-Holomorphic Eisenstein series.

After applying the decomposition in a toy model, the one-character RCFT with $c = 8$ central charge, we move onto studying a family of $c = 8k$ proposed modular invariants, which lead to the discovery of an interesting relation between the partition functions of a family of one-character RCFTs. Furthermore, we study a series of several-character theories that have an interesting relation with the family of $c = 8k$ RCFTs via their continuous spectral decomposition, and we make a proposal on a technique for exploring the space of RCFTs in this manner.

1.3 Celestial CFTs: ambidextrous light transforms

AdS/CFT is certainly one of the most important results in modern high-energy physics, and it also has acted as a '*seed*' for the development of other dualities that have a similar base principle: we aim to describe gravitational objects via a lower-dimensional field theory that is linked to the higher-dimensional gravity theory via a dictionary. In this line, flat holography proposes a duality between scattering amplitudes in four dimensions and a CFT living in the celestial sphere (or the celestial torus), the null infinity of the gravity theory.

The main dictionary in this context is that elements of the S-matrix (in the scattering side) are dual to conformal correlators living in the celestial sphere (or the celestial torus) [40–43]. The duality is realised via an integral transform, which is a Mellin transform, bringing us from amplitudes to objects that transform like CFT correlation functions. On a symmetry level, it is worth noting that the Lorentz group in four dimensions $SO(3,1)$ is isomorphic to the global conformal group $SL(2,\mathbb{C})/\mathbb{Z}_2$, but furthermore one can see explicitly [44] how Lorentz transformations become global conformal transformations on the Celestial sphere. This is no proof of the existence of a CFT, however, it can be seen as an encouraging sign.

Overall, the appproach we are taking in Chapter 5 is to find a suitable transformation to take us from massless scattering amplitudes in momentum space to a basis where the conformal correlators have a recognisable CFT-like shape. What we aim to achieve is not only to set up a suitable prescription to go between both sides of the dictionary for this scenario, but also to investigate how bootstrap techniques can help us better understand the spectrum of the CCFT.

Dealing with massless amplitudes comes with a small trade-off: in Minkowski space, the three-point functions vanish on-shell. This is somewhat unlucky from the CCFT perspective, as it would imply that all three-point functions vanish and we would have a free theory, not helping us untangle many mysteries. However, not everything is lost: we can Wick-rotate to split signature $(2,2)$ into Klein space. In this case, rather than in the Celestial Sphere, the CFT will live in Celestial Torus, which will be introduced in due manner.

In this signature, the Lorentz group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, and this means that we need to carefully evaluate how to realise the holographic dictionary in this context. One starts, as usual, with momentum-space eigenstates, which then are Mellin transformed to boost eigenstates. These boost eigenstates should behave as (quasi)-primary operators in the CCFT, however their transformation under conformal mapping isn't the appropriate one. This is why we introduce 'boost+$\mathbb{Z}_2$' eigenstates, which can bring us from momentum eigenstates to operators that transform in irreducible representations of the Lorentz group, as one demands of (quasi)-primary operators in a CFT.

The celestial amplitudes obtained from the previous mapping suffer from a small issue: they have distributional factors, which difficult our using CFT techniques on them as this is not a characteristic of CFT correlators. However, these can be eliminated via a Light transform procedure, and we present a prescription that allows us to put 2-,3- and 4-point functions in the functional shape expected from a CFT side.

After doing so, we focus on the example of a 4-point gluon scattering, where we are able to extract spectral data from performing an adequate inversion-like formula on the four-point correlator. This not only shows how powerful the light-transform procedure can be for CCFT exploration, but it also showcases the agreement of our findings with previous results in CCFT literature, and paves the way for furthering the understanding of the characteristics of the Celestial CFT.

Chapter 2

Two-dimensional Conformal Field Theories

We will now present the basics and introduce the reader to the central topic of this thesis: Conformal Field Theories in two dimensions. The main reference we want to highlight is, of course, *the yellow book* [45], but other good sources are [46–51].

2.1 A gentle introduction

2.1.1 The conformal group in two dimensions

In very broad terms, we can define a conformal transformation as one which preserves angles in a given d-dimensional space.

In somewhat more formal terms, a conformal transformation is an invertible and smooth map $z^\mu \to w^\mu(\boldsymbol{z})$ such that the transformed metric is proportional to the original one

$$\tilde{g}^{\mu\nu}(\boldsymbol{w}) = \Lambda(\boldsymbol{z}) g^{\mu\nu}(\boldsymbol{z}). \tag{2.1}$$

We then define a Conformal Field Theory as a Field Theory that is invariant under conformal transformations.

Interestingly enough, there is a rather crucial separation in the properties and behaviour of the conformal group when $d = 1$, $d = 2$ or $d \geq 3$.

In particular, it can be shown [45, Chapter 4] that any smooth transformation in one dimension is conformal by definition. But this is no longer the case when $d > 1$, however there is still a fundamental difference between the cases $d = 2$ and $d \geq 3$: in the former case, the conformal group is infinite dimensional, but not in the latter. Consequently, the properties and techniques used to investigate different Conformal Field Theories will be quite different depending on whether $d < 3$ or $d \geq 3$.

As the case that interests us is that of $d = 2$, from now onward it will always be assumed that any CFT we talk about is two-dimensional.

Let us then begin the study of CFTs in two dimensions. Suppose we have coordinates $z = (z^0, z^1)$ on the plane. An infinitesimal transformation $z^\mu \to w^\mu$ that obeys (2.1) will imply that $w(z)$ is either a holomorphic on an antiholomorphic function in an open set. This leads us to find more convenient coordinates

$$z = z^0 + iz^1 \,, \;\; \bar{z} = z^0 - iz^1 \,, \tag{2.2}$$

such that any holomorphic mapping $z \to w(z)$ is a conformal transformation. In general, z^0 and z^1 are extended to be complex variables, so that z and $\bar{z}$ are not related.

We can actually explicitly see from this characterisation, as pointed out in [45, Chapter 5], the infinite-dimensional nature of the conformal group in two dimensions: it is just the set of all analytic maps, with group multiplication defined as the composition of maps.

Can we see this more explicitly? Let us define an infinitesimal conformal transformation

$$w = z + \varepsilon(z) \,, \;\; \bar{w} = \bar{z} + \bar{\varepsilon}(\bar{z}) \,, \tag{2.3}$$

where $|\varepsilon| \ll |z|$, and similar for the barred counterparts. This allows us to express ε ($\bar{\varepsilon}$) in terms of a Laurent expansion around $z = 0$ ($\bar{z} = 0$),

$$\varepsilon(z) = \sum_{k=-\infty}^{\infty} A_k z^{k+1}, \;\; \bar{\varepsilon}(\bar{z}) = \sum_{k=-\infty}^{\infty} \bar{A}_k \bar{z}^{k+1} \,, \tag{2.4}$$

from where the infinite-dimensional nature of 2d conformal transformations can be seen explicitly.

Furthermore, from applying such infinitesimal transformations on an arbitrary function $f(z, \bar{z})$ defined on the complex plane, we can deduce that the generators of such infinitesimal transformations are

$$\ell_n = -z^{n+1}\partial_z \,, \quad \bar{\ell}_n = \bar{z}^{n+1}\partial_{\bar{z}} \,. \tag{2.5}$$

These define two copy of the Witt algebra, with commutation relations

$$\begin{aligned} [\ell_m, \ell_n] &= (m-n)\ell_{m+n} \,, \\ [\bar{\ell}_m, \bar{\ell}_n] &= (m-n)\bar{\ell}_{m+n} \,, \\ [\ell_m, \bar{\ell}_n] &= 0 \,. \end{aligned} \tag{2.6}$$

We should not end here though: the definition of a group requires us to have not just a local map, but a map that is both invertible and can map the whole Riemann sphere to itself. We can see from the definitions that neither ℓ_n nor $\bar{\ell}_n$ are globally well defined: in particular, the only ones that are non-singular everywhere in the Riemann sphere correspond to $n = -1, 0, 1$.

This leads us to the definition of *global* conformal transformations, generated by $\{\ell_{-1}, \ell_0, \ell_1\}$ and their antiholomorphic counterparts.

In coordinate terms,

- ℓ_{-1} generates translations.
- ℓ_0 generates rotations and dilations.
- ℓ_1 generates the so-called *special conformal transformations*. These can be understood simply as translations in $-1/z$.

From here, it can be seen that the complete set of globally defined conformal maps is

$$z \to \frac{a\,z + b}{c\,z + d} \,, \; a\,d - b\,c = 1 \,, \tag{2.7}$$

where $a, b, c, d \in \mathbb{C}$.

This allows us to see that the global conformal group is isomorphic to the Möbius group $SL(2, \mathbb{C})/\mathbb{Z}_2$, which is also isomorphic to $SO(3, 1)$, the Lorentz group.

2.1.2 The Virasoro Algebra

When we want to quantise a theory, we should take the algebra of the clasical theory (2.6) and realise it projectively. This is due to the physical states of a quantum mechanical theory being described by rays on the Hilbert space [1] (see [48] for details).

The central extension of the Witt algebra is the so-called Virasoro algebra. We will denote the Virasoro generators as $L_k, \bar{L}_k$. More in particular, L_n and $\bar{L}_n$ are the generators of the local conformal transformations of the quantum theory, and $\{L_{-1}, L_0, L_1\}$ generate the global ones[2], where the commutation relations are now

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{\mathrm{c}}{12} n(n^2-1)\delta_{n+m,\,0}\,, \\
[\bar{L}_n, \bar{L}_m] &= (n-m)\bar{L}_{n+m} + \frac{\mathrm{c}}{12} n(n^2-1)\delta_{n+m,\,0}\,, \\
[L_n, \bar{L}_m] &= 0\,.
\end{aligned} \tag{2.8}$$

The parameter c is the central charge of the theory. The Virasoro algebra can also be obtained very beautifully through the study of the OPE of the energy-momentum tensor with itself. For that proof we refer the reader to [45, Chapter 6].

2.1.3 Primary fields

Suppose we have a field $\phi(z, \bar{z})$ in our theory with scaling dimension Δ and spin s. If we now perform a local conformal transformation $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$ and our field transforms as

$$\phi(z, \bar{z}) \to \phi'(w, \bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}}, \tag{2.9}$$

we can say it is a *primary* field. If it transforms like this under global conformal transformations only, we can refer to it as a *quasi-primary* field. Fields that don't obey this transformation are called *secondary*.

In particular, we should also note that the exponents $(h, \bar{h})$ are referred to as *conformal*

[1] Furthermore, this projective representation is generated by the stress-energy tensor, and it can be obtained through the computation of the OPE of $T(z)$ with itself.

[2] One should note that the generators $\{L_{-1}, L_0, L_1\}$ remain unchanged after the central extension of the algebra.

weights, and they are related to the scaling dimension and spin of the (quasi)-primary field as

$$\Delta = h + \bar{h}, \quad s = h - \bar{h}. \tag{2.10}$$

This definition allows us to compute the Ward identity for a (quasi-)primary field. Let us have an infinitesimal variation $z \to z + \varepsilon(z)$, $\bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})$, the variation on the transformed (quasi-)primary becomes

$$\delta_{\varepsilon,\bar{\varepsilon}}\phi(z,\bar{z}) = -(h(\partial\varepsilon) + \varepsilon\,\partial)\phi(z,\bar{z}) - (\bar{h}(\bar{\partial}\bar{\varepsilon}) + \bar{\varepsilon}\,\bar{\partial})\phi(z,\bar{z})\,. \tag{2.11}$$

2.1.4 Correlation functions of primary fields

Let us consider now the correlation functions of several primary fields. In Quantum Field Theories, the presence of symmetries implies constraints in the correlation functions: in particular, the invariance of the action under a global conformal transformation $(z,\bar{z}) \to (w,\bar{w})$ implies that the correlator of N primaries must behave as

$$\langle \phi_1(w_1,\bar{w}_1)\dots\phi_N(w_N,\bar{w}_N)\rangle = \prod_{i=1}^{N}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h_i}_{w=w_i}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i}\langle \phi_1(z_1,\bar{z}_1)\dots\phi_N(z_N,\bar{z}_N)\rangle\,. \tag{2.12}$$

This has very powerful consequences for the low-point correlation functions: in particular, it implies that the two and three-point functions are completely fixed.

The two-point function of primaries with different conformal weights is strictly zero. When both primary fields have the same conformal weights $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$, the two-point function becomes

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_{12})^{2h}(\bar{z}_{12})^{2\bar{h}}}\,, \tag{2.13}$$

where we have introduced the standard notation $z_{ij} = z_i - z_j$, $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$. Furthermore, in general we can choose to normalise the operators such that $C_{12} = 1$, that is, this constant doesn't hold physical information.

The three-point correlator of primary fields is fixed to be

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_3+h_1-h_2}} \times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}\,, \tag{2.14}$$

where C_{123} is a structure constant.

The four-point function (and further higher-order functions) is not fully fixed by conformal invariance, however it will have a somewhat constrained shape. Let us begin by choosing cross-ratios[3]

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}\,. \tag{2.15}$$

With this, the four-point function can be expressed as

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle = g(z,\bar{z})\prod_{i<j} z_{ij}^{h/3-h_i-h_j}\bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j}\,, \tag{2.16}$$

where we have defined $h = \sum_{i=1}^4 h_i$, $\bar{h} = \sum_{i=1}^4 \bar{h}_i$. Note that all the theory-specific data is encoded in the modular-invariant part of the correlation function. We will later see how to extract information from here.

2.1.5 The energy-momentum tensor

The theory's behaviour under conformal transformations is encoded in the energy-momentum tensor. This is to be expected, due to the energy-momentum tensor precisely measuring variations of the theory under a small variation of the metric.

The nature of conformal transformations, more in particular, scale invariance, imply that the energy-momentum tensor is traceless (in the Minkowski signature/coordinates) $T^\mu_\mu = 0$. As we are generally analysing the CFT in Euclidean signature, we are interested in understanding how to translate the tracelessness condition into our complex coordinates $(z,\bar{z})$. The condition

[3] Of course, we are just choosing a particular cross-ratio to exemplify the behaviour of the four-point function, but we could have chosen a different one. However, it should be noted here that all other cross-ratios we can build are dependent on this one in two dimensions.

can be computed through standard chain rule, and it is:

$$T_{z,\bar{z}} = T_{\bar{z},z} = 0\,. \tag{2.17}$$

Furthermore, from imposing translation invariance, we can find that the non-zero components are chiral

$$T_{zz}(z,\bar{z}) = T(z)\,, \quad T_{\bar{z}\bar{z}}(z,\bar{z}) = \bar{T}(\bar{z})\,. \tag{2.18}$$

2.1.6 Radial Quantisation

Now, we are interested in CFTs as quantum theories, and for that we should study the quantisation of the theory in more detail, which is somewhat special due to the theory's symmetries.

Let us start by defining our CFT in an Euclidean flat space, with coordinates (x^0, x^1). We can always switch between Euclidean and Lorentzian signatures via a standard Wick rotation, and it turns out that 2d CFTs are easier to study starting from an Euclidean signature.

Now, Euclidean coordinates have the issue that there is no clear identification of space or time coordinate-wise: it is rather arbitrary from that setup to choose one to be the time coordinate. However, our end-goal is to build the right setup for a QFT, and that includes being able to specify time or normal orderings, in order to preserve the notion of causality in the system.

We choose x^0 to be the time coordinate, and x^1 to be the spatial one. In order to avoid certain infrared divergences, we should put the CFT on a cylinder, and for that we start by compactifying x^1 on a circle of radius $R = 1$, and leaving x^0 to have an infinite span.

We now want to map the cyclinder to the complex plane, which then allows us to radially quantise the theory. For that, we start by introducing a complex coordinate ω for the cylinder[4]

$$\omega = x^0 + ix^1\,, \quad \omega \sim \omega + 2\pi i\,, \tag{2.19}$$

and then we map the cylinder to the plane through

$$z = e^{i\omega} = e^{x^0} e^{ix^1}\,. \tag{2.20}$$

[4]This can be seen more naturally if we start from a cylinder in Minkowksi space (t, x) that is Wick rotated to (x^0, x^1).

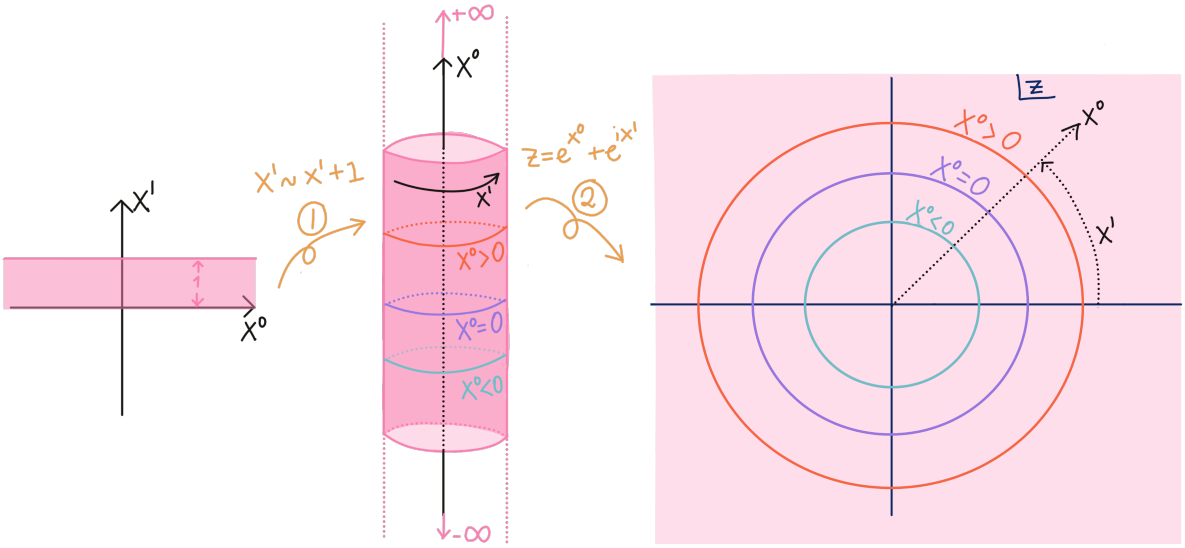


Figure 2.1: This figure shows the transformations taken from Euclidean two-dimensional space, on the left, to expressing the CFT on a cylinder via compactification of x^1 to the unit circle, in the middle, and to finally exploding the cylinder on the Riemann Sphere (complex plane + point at infinity), on the right.

We have represented graphically the different coordinate changes that allow us to reach the complex z-plane in figure 2.1. This mapping sets the remote past on the origin of the z plane and the remote future on the point at infinity on the Riemann sphere, mapping the whole cylinder onto the whole Riemann sphere (which allows us to identify some results from the previous subsections in this context). This is very useful, as it allows us to use complex analysis techniques to perform our computations, and we will refer in general to the z plane as the *conformal* plane.

The cylinder representation causes time translations to become complex dilations and space translations to become rotations. This allows us to identify some of the Virasoro generators (2.8) as the hamiltonian of the system

$$H = L_0 + \bar{L}_0 \,, \tag{2.21}$$

noting here that the terms with $L_{\pm 1,0}$ aren't affected by the central extension of the algebra.

2.1.7 The state-operator correspondence

So far, we have understood our theory in terms of fields, and from there we could define most of our theories through a path-integral formalism. However, radial quantisation will allow us to understand fields and, most importantly, correlation functions, from the perspective of a Hilbert space with a quantum mechanical evolution.

This perspective is fully based on the previous transformation of the Euclidean plane into the conformal plane via the cylinder prescription: we have now defined a foliation of space-time into circles of equal time, and hence in each circle we can define a Hilbert space.

Let us first define our operators. In QFTs, we often think about fields as distributions of operators on the spacetime. How do we specify this in our case?

Consider a field $\phi(z, \bar{z})$ of dimensions $(h, \bar{h})$ living on the conformal plane. Now, let us Taylor-expand it around the origin $(z, \bar{z}) = (0, 0)$ in *modes*

$$\phi(z, \bar{z}) = \sum_{n,m \in \mathbb{Z}} z^{-n-h} \bar{z}^{-m-\bar{h}} \phi_{n,m} \,, \tag{2.22}$$

where we can define the modes $\phi_{n,m}$ as

$$\phi_{n,m} = \left(\frac{1}{2\pi i}\right)^2 \oint \mathrm{d}z \, z^{n+h-1} \oint \mathrm{d}\bar{z} \, \bar{z}^{m+\bar{h}-1} \phi(z, \bar{z}) \,. \tag{2.23}$$

We can then quantise the field by promoting these modes to operators.

When we understand the conformal plane as foliated by constant-time slices, we can pick a particular time slice and insert the operators *inside* the particular circle. This can be understood as the creation of an 'in' state. An 'out' state can correspondingly be defined from operators inserted outside the circle.

We still don't have a vacuum! Well, in a free theory, the vacuum is defined as the state that can be annihilated by the positive-frequency sector of a free field. For an interacting theory, we are going to commit to the usual assumption that the interacting Hilbert space is the same as for the free theory, but with differing energy eigenstates. This allows us to define *asymptotic* fields and *asymptotic* states, assuming the interaction is negligible when $x^0 \to \pm\infty$. In particular, we

will define an asymptotic *in*-state at the infinite past, $x^0 \to -\infty$, equivalent to $z \to 0,\ \bar{z} \to 0$

$$|\psi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle\,. \tag{2.24}$$

This properly establishes the **state-operator** correspondence in the CFT, allowing us to use QFT formalism to deal with our theory, whilst exploiting the benefits of complex analysis for commutators and other calculations.

Note that, in particular, this imposes a condition on certain values of m, n of the field's modes, as to have a well-defined limit in $(z,\bar{z}) \to (0,0)$ we need that

$$\phi_{n,m}|0\rangle = 0, \quad \text{if } (n,m) > (-h,-\bar{h})\,. \tag{2.25}$$

Of course, an asymptotic *out*-state must also be defined, and it should simply be

$$\langle\phi_{out}| = |\phi_{in}\rangle^\dagger\,, \tag{2.26}$$

where Hermitian conjugation is defined in the surface $\bar{z} = z^*$

$$[\phi(z,\bar{z})]^\dagger = \bar{z}^{2h} z^{2\bar{h}} \phi(1/z, 1/\bar{z})\,, \text{ or } \phi^\dagger_{n,m} = \phi_{-n,-m}\,. \tag{2.27}$$

With these definitions, we can now define a proper inner product as $\langle\phi_{out}|\phi_{in}\rangle$.

Radial ordering

In QFTs, we define correlation functions with a particular time ordering. In the case of 2d CFTs, the mapping we have done to the conformal plane translates time ordering into *radial* ordering. From the definition of the coordinate changes, or from the depictions in figure 2.1, we can see that time ordering corresponds to ordering by the largest radius first. In other words, we define radial ordering as

$$\mathcal{R}\,\phi_1(z)\phi_2(w) = \begin{cases} \phi_1(z)\phi_2(w) & |z| > |w| \\ \phi_2(w)\phi_1(z) & |z| < |w| \end{cases}\,. \tag{2.28}$$

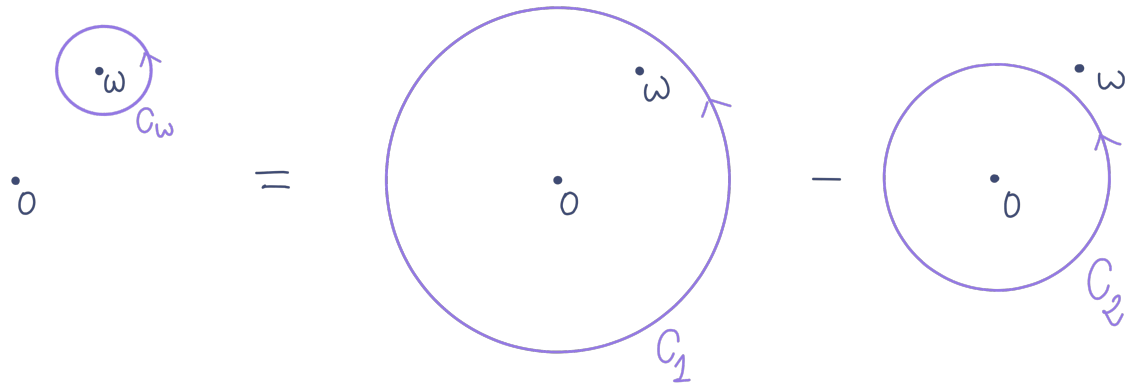


Figure 2.2: This figure shows the contour prescription for commutators and their relation to OPE in the context of radial quantisation. The contour C_1 has radius infinitesimally larger than $|w|$ and the contour C_2 has radius infinitesimally smaller than $|w|$.

This allows us to define commutators as OPEs in the theory. This works with a specific set of contour deformations that are shown in figure 2.2. Let us have two holomorphic fields $a(z)$, $b(z)$, the integral

$$\oint_{C_w} \mathrm{d}z\, a(z)b(w)\,, \tag{2.29}$$

performed counterclockwise surrounding the point $w \neq 0$ can be done through a convenient contour splitting

$$\oint_{C_w} \mathrm{d}z\, \mathcal{R}(a(z)b(w)) = \oint_{C_1} \mathrm{d}z\, a(z)b(w) - \oint_{C_2} \mathrm{d}z\, b(w)a(z) = [A, b(w)]\,, \tag{2.30}$$

where $A = \oint \mathrm{d}z\, a(z)$ is the integral at fixed time for the field $a(z)$. This allows us to formally define the commutator of two operators

$$[A, B] = \oint_{C_0} \mathrm{d}w \oint_{C_w} \mathrm{d}z\, a(z)b(w)\,, \tag{2.31}$$

where C_0 denotes a contour around the origin.

This is extremely important, as it allows us to translate the information of the theory contained in the OPEs into the operator language as commutation relations: after all, we are precisely studying the behaviour of operator a nearby b with $[A, b(w)]$. We will be following [45]'s convention that where we write $\oint$ without a specific contour we are integrating at fixed time around the origin.

OPEs with the stress-energy tensor

As we said before, the stress-energy tensor only has two non-zero components, which we denoted $T(z)$ and $\bar{T}(\bar{z})$ (2.18), and it acts as the generator of conformal transformations. Using only the holomorphic part for this section, as the antiholomophic can be done similarly, under an infinitesimal conformal transformation $z \to z + \varepsilon(z)$, we can define a conformal charge

$$Q_\varepsilon = \frac{1}{2\pi i} \oint \mathrm{d}z\, \varepsilon(z) T(z)\,, \tag{2.32}$$

which can then allow us to understand the conformal ward identity (2.11) as

$$\delta_\varepsilon \phi(w) = -[Q_\varepsilon, \phi(w)]\,. \tag{2.33}$$

Now recalling that the energy-momentum tensor has weights $(h, \bar{h}) = (2, 2)$, we can write it as

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n\,, \quad L_n = \frac{1}{2\pi i} \oint \mathrm{d}z\, z^{n+1} T(z)\,, \tag{2.34}$$

where L_n are the generators of the Virasoro algebra (2.8). This allows us to explicitly see how the Virasoro algebra is encoded in the radial ordering formalism and the stress-energy tensor.

In particular, this allows us to obtain the OPEs of the stress-energy tensor with any primary field, by comparing (2.11) with (2.33), which can give us an alternative definition[5] of primary operators as those fulfilling

$$\begin{aligned} T(z)\phi(w,\bar{w}) &= \frac{h}{(z-w)^2}\,\phi(w,\bar{w}) + \frac{1}{z-w}\,\partial_w\phi(w,\bar{w}) + \dots\,, \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &= \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\,\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\,\partial_{\bar{w}}\phi(w,\bar{w}) + \dots\,, \end{aligned} \tag{2.35}$$

where we are considering in general $|z| > |w|$.

Note also that the OPE of the stress-energy tensor with itself can be obtained in this formulation in a similar manner, and in fact it allows us to recover very nicely the Virasoro algebra

[5] Take into account that, following usual conventions, we won't make the radial ordering explicit in general, as to be able to introduce any expression in a correlator it needs to be radially ordered.

(2.8):

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \dots \tag{2.36}$$

The anti-holomorphic equivalent one gives us the *barred* sector of the Virasoro algebra, and we can get the zero correlators from the non-singular OPEs of the *cross*-holomorphic relations.

2.1.8 The structure of the spectrum of operators

We have now spent a fair amount of time setting up the conditions to define a proper Hilbert space, with a radial ordering, a notion of state-operator correspondence, and even alternative definitions for primary operators!

But we still haven't properly defined the vacuum and the structure of the spectrum of the theory. Let us do that now.

For our theory to be well-defined, we need the vacuum to remain invariant under transformations of the global conformal group. What that means, in terms of operators, is that

$$\begin{aligned} L_{-1}|0\rangle = L_0|0\rangle = L_1|0\rangle = 0\,, \\ \bar{L}_{-1}|0\rangle = \bar{L}_0|0\rangle = \bar{L}_1|0\rangle = 0\,. \end{aligned} \tag{2.37}$$

This sets the ground state energy to be zero, which is a common convention. But furthermore, we know that this condition is extended to more of the generators of the Virasoro group. As $T(z)$ and $\bar{T}(\bar{z})$ should be well-defined as $z \to 0$, $\bar{z} \to 0$, we obtain that

$$L_n|0\rangle = 0\,,\ n \geq -1\,, \tag{2.38}$$

$$\bar{L}_n|0\rangle = 0\ \ n \geq -1\,. \tag{2.39}$$

Our construction also means that primary fields create asymptotic states when acting on the vacuum, per our state-operator correspondence. From the OPE of $T(z)$ with a primary $\phi(z,\bar{z})$ of weights $(h,\bar{h})$, one can obtain that

$$\begin{aligned} L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle\,, \\ \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle\,, \end{aligned} \tag{2.40}$$

where we have defined $|h, \bar{h}\rangle = \phi(0,0)|0\rangle$ as the asymptotic state of the primary $\phi(z, \bar{z})$. Note that this means that the asymptotic state of a primary is an eigenvector of the Hamiltonian $H = L_0 + \bar{L}_0$ with eigenvalue $h + \bar{h}$.

We can also build excited states from primary operators, via ladder operators: these are the *descendants*. We can increase the weight of a state via generators L_{-m} with $m > 0$. Overall, given an asymptotic state $|h\rangle$, we may find its level N descendant via the operation

$$L_{-k_1} \dots L_{-k_m}|h\rangle \, (1 \leq k_1 \leq k_m) \, , \; \sum_i k_i = N \, , \tag{2.41}$$

which now is an L_0 eigenstate of eigenvalue $h + k_1 + \cdots + k_m = h + N$. Note that given a level N there will be $p(N)$ possible distinct descendant operators, where $p(N)$ represents the partitions of N.

Very importantly, the full set of a primary and its descendants is closed under the action of the Virasoro generators, and it is usually referred to as a *Verma module*.

2.1.9 OPE of primaries and conformal blocks

We earlier described the constraints put in the low-point correlation functions by the global conformal group, but armed with our operator picture, we will be able to formulate ways to extract the dynamical data.

CFTs are more constrained than other theories: we usually would need to know all correlators of a QFT to fully characterise it. However, in a CFT we *only* need to know two things: the spectrum of operators and the *operator algebra*. We define the latter as the collection of all OPEs between all primaries of the theory, and we will be using both terms interchangeably from now onwards.

The structure of the operator algebra is very powerful, so much so that it allows us to deconstruct any correlator of n-points into correlators of less points, all the way down to two-point functions.

We start with the two-point function, which for two primary fields takes the form

$$\langle \phi_a(w, \bar{w}) \phi_b(w, \bar{w}) \rangle = \frac{\delta_{ab}}{(w - z)^{2h} (\bar{w} - \bar{z})^{2\bar{h}}} \, , \tag{2.42}$$

where we have chosen the normalisation of $C_{ab} = \delta_{ab}$. Conformal transformations allow us to bring $w \to \infty$ and $z \to 0$, leading us to be able to consider the states as asymptotic, which lets us understand this as a product on the Hilbert space

$$\lim_{w,\bar{w}\to\infty} w^{2h}\bar{w}^{2\bar{h}}\langle\phi_a(w,\bar{w})\phi_b(0,0)\rangle = \langle h_a|h_b\rangle\langle\bar{h}_a|\bar{h}_b\rangle\,, \tag{2.43}$$

from where we see a clear orthogonality between different Verma modules[6]

Now, we know that this result must be invariant under scaling transformations. This means the operator algebra should have the form

$$\phi_1(z,\bar{z})\phi_b(0,0) = \sum_p \sum_{k,\bar{k}} C_{12}^{p,\{k,\bar{k}\}} z^{h_p-h_1-h_2+K}\bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2+\bar{K}}\phi_p^{k,\bar{k}}(0,0)\,, \tag{2.44}$$

where p is an index summing over primary fields in the theory, and $\phi_p^{k,\bar{k}}$ are the descendants of that Verma module. The indices $k,\bar{k}$ denote a precise descendant, with then $K,\bar{K}$ denoting the level of said descendent.

Let us take a correlation function of (2.44) with a third primary $\phi_r(w,\bar{w})$ of weights $(h_r,\bar{h}_r)$. Conformal invariance allows us to send $w,\bar{w}\to\infty$, obtaining

$$\langle\phi_r|\phi_1(z,\bar{z})|\phi_2\rangle = \lim_{w,\bar{w}\to\infty} w^{2h_r}\bar{w}^{2\bar{h}_r}\langle\phi_r(w,\bar{w})\phi_1(z,\bar{z})\phi_2(0,0)\rangle\,, \tag{2.45}$$

where now we can use (2.14) and obtain

$$\langle\phi_r|\phi_1(z,\bar{z})|\phi_2\rangle = \frac{C_{r12}}{z^{h_1+h_2-h_r}\bar{z}^{\bar{h}_1+\bar{h}_2-\bar{h}_r}}\,. \tag{2.46}$$

If we now compare (2.46) with (2.44), we see that only the term with $p=r$, $k=0$, $\bar{k}=0$ contributes to the operator algebra expansion in this three point function.

Therefore, we have found that

$$C_{12}^{r,\{0,0\}} = C_{r12}\,, \tag{2.47}$$

and so the structure constant of the three point function is an OPE coefficient! This is great

[6] At the end of the day, the primary field is just the highest-weight representative of the Verma module, and the one we choose by convention, but the results obtained here have consequences beyond primaries themselves, and concern the structure of Verma modules themselves.

news, and we see how the operator algebra can help us decode the information necessary to fully characterise a CFT.

This can be extended to descendent fields, and in general the coefficients of the operator algebra will have the form

$$C_{12}^{p,\{k,\bar{k}\}} = C_{p12}\beta_{12}^{p,k}\bar{\beta}_{12}^{p,\bar{k}}\,, \tag{2.48}$$

where we have separated the information coming from the Verma module from that coming from a specific element of it. It is customary to fix $\beta_{ij}^{p,0} = 1$.

The four-point function revisited

Let us pick a generic four-point function (2.16) $\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle$ and select the convenient locations $z_1 \to \infty$, $z_2 = 1$, $z_3 = z$, $z_4 = 0$, where note z is the cross-ratio picked earlier $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$. Armed with this, we see that our four-point correlation function can be written as some matrix element

$$\lim_{z_1,\bar{z}_1\to\infty} z_1^{2h_1}\bar{z}_1^{2\bar{h}_1}\langle\phi_1(z_1,\bar{z}_1)\phi_2(1,1)\phi_3(z,\bar{z})\phi_4(0,0)\rangle =: G_{34}^{21}(z,\bar{z})\,, \tag{2.49}$$

where one can define a matrix element between two asymptotic states

$$G_{34}^{21}(z,\bar{z}) = \langle h_1,\bar{h}_1|\phi_2(1,1)\phi_3(z,\bar{z})|h_4,\bar{h}_4\rangle\,. \tag{2.50}$$

Now, we can take the OPE between operators 3 and 4,

$$\phi_3(z,\bar{z})\phi_4(0,0) = \sum_p C_{34}^p z^{h_p-h_3-h_4}\bar{z}^{\bar{h}_p-\bar{h}_3-\bar{h}_4}\Psi_p(z,\bar{z}|0,0)\,, \tag{2.51}$$

where Ψ_p encompasses the descendant content of the expansion:

$$\Psi_p(z,\bar{z}|0,0) = \sum_{\{k,\bar{k}\}}\beta_{34}^{p,k}\bar{\beta}_{34}^{p,\bar{k}}z^K\bar{z}^{\bar{K}}\phi_p^{\{k,\bar{k}\}}(0,0)\,, \tag{2.52}$$

with the k's defined as previously.

This allows us to write our matrix element as

$$G_{34}^{21}(z,\bar{z}) = \sum_p C_{12}^p C_{34}^p A_{34}^{21}(p|z,\bar{z})\,, \tag{2.53}$$

with

$$A_{34}^{21}(p|z,\bar{z}) = (C_{12}^p)^{-1} z^{h_p-h_3-h_4}\bar{z}^{\bar{h}_p-\bar{h}_3-\bar{h}_4}\langle h_1,\bar{h}_1|\phi_2(1,1)|\Psi_p(z,\bar{z}|0,0)\rangle\,. \tag{2.54}$$

All in all, we have written the four-point function as a sum over conformal families. Furthermore, we know that three-point functions are fixed by the algebra of the theory, and they factorise into a holomorphic and an antiholomorphic part

$$A_{34}^{21}(p|z,\bar{z}) = \mathcal{F}_{34}^{21}(p|z)\bar{\mathcal{F}}_{34}^{21}(p|\bar{z})\,, \tag{2.55}$$

which are known as conformal blocks, defined as

$$\mathcal{F}_{34}^{21}(p|x) = x^{h_p-h_3-h_4}\sum_k \beta_{34}^{p,k} z^K \frac{\langle h_1|\phi_2(1)L_{-k_1}\dots L_{-k_N}|h_p\rangle}{\langle h_1|\phi_2(1)|h_p\rangle}\,. \tag{2.56}$$

Overall, this allows us to write the four-point function very compactly

$$G_{34}^{21}(z,\bar{z}) = \sum_p C_{34}^p C_{12}^p \mathcal{F}_{34}^{21}(p|z)\bar{\mathcal{F}}_{34}^{21}(p|\bar{z})\,. \tag{2.57}$$

Virasoro blocks don't have an analytic formula, but they can be computed by means of a series expansion [52]. A similar procedure can be done with quasi-primaries, that would lead us to obtain the *global* conformal block, however the latter is more easily found via a study of the Sturm-Liouville problem for the quadratic Casimir. Its derivation can be found in [53], and we will see a special case in Chapter 5.

2.1.10 CFT on the torus

When studying CFTs on the conformal plane, we generally see a decoupling of holomorphic and anti-holomorphic sectors which isn't really natural: it has to do with the way we have decided to understand the theory on the conformal fixed point, which decouples both sectors. However, if we want to study the behaviour of the theory property, we need to study the coupling of both

parts of the theory. To do that without leaving the fixed point, we can use the geometry to link both the holomorphic and anti-holomorphic sectors. Generally speaking, this also relates to the fact that any local theory we build should not just be consistent when posed in flat space, but also on an arbitrary manifold.

From the cylinder to the torus

We previously put the CFT from the Euclidean plane on an infinite cylinder, see Figure 2.1. In particular, we start from coordinates (x^0, x^1) on the Euclidean plane and we compactify $x^1 \sim x^1 + 1$, which we can use to define a complex coordinate on the cylinder

$$w = x^0 + ix^1 \,. \tag{2.58}$$

This coordinate is related to the complex plane z via an exponential: $z = e^w$. We want to briefly note here that one needs to *translate* all the definitions previously made in the Riemann sphere to the cylinder. For that one should consider that we are simply doing a conformal mapping $z = e^w$, see [46, Chapter 4] for more details

- The mode expansion of a primary in w is now

$$\phi_{cyl}(w) = \sum_n \phi_n e^{-nw} \,. \tag{2.59}$$

- The energy-momentum tensor is related to the Riemann sphere one by

$$T_{cyl}(w) = z^2 T(z) - c/24 \,. \tag{2.60}$$

- The zero mode of the cylinder is shifted with respect to the complex plane:

$$(L_{cyl})_0 = L_0 - c/24 \,. \tag{2.61}$$

Now, we want to convert the cylinder into a torus: this is done by cutting the cylinder into a finite piece and then identifying the edges, or basically making x^0 periodic as well. However, there is a choice we can make at that point, which is how we glue the two ends together. In

other words, how much do we twist the ends before gluing them together? And which tori are equivalent to eachother?

In our parametrisation of the cylinder, we chose w to be a complex number. If we view this as a parameter over the complex plane, we can choose to parametrise a torus via an identification

$$w \sim w + m\,\alpha_1 + n\,\alpha_2\,, \tag{2.62}$$

where $m, n \in \mathbb{Z}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. These span a lattice that has as base cell the fundamental domain of the torus. The torus itself is identified via the *complex modulus*, defined as

$$\tau = \frac{\alpha_2}{\alpha_1} = x + iy\,, \tag{2.63}$$

where we are introducing notation that will be used throughout the thesis: $x = \Re(\tau)$, $y = \Im(\tau)$. To answer the question of which gluings give us the same torus, we should look at the definition of τ in terms of α_1, α_2. It can be seen quickly how different choices of α_1 and α_2 will lead to the same torus. In fact, the torus is invariant under any transformation

$$\tau \to \frac{a\tau + b}{s\tau - r}, \quad \begin{pmatrix} a & b \\ s & -r \end{pmatrix} \in SL(2,\mathbb{Z})/\mathbb{Z}_2\,. \tag{2.64}$$

In other words, we have found that the torus is invariant under the action of the modular group/$\mathbb{Z}_2$. The generators of the modular group are the T and S transformations:

$$T:\ \tau \to \tau + 1, \quad S:\ \tau \to -\frac{1}{\tau}\,. \tag{2.65}$$

These provide us with a template to study invariance under modular transformations.

The partition function

When we put the theory on the torus, we have to understand the new conditions our fields need to fulfil. In particular, the requirement that the action is invariant under translations on the periods w_1, w_2 imposes periodicity conditions on the local fields. In this context, the relevant observable to study is the partition function, $Z(\tau, \bar{\tau})$, understood to be the same object defined

on statistical mechanics, and which is key to study the spectrum of the theory.

To find an expression for the partition function, we begin by choosing the space direction to be $x = \Re(\tau)$ and the time direction to be $y = \Im(\tau)$, which is the opposite to what we chose previously - this doesn't really matter due to how S transformations swap the roles of both.

Now, we want to find the path integral from its definition, as a weighted sum over all possible configurations $Z = \text{Tr}\ _{\mathcal{H}}(e^{-\beta H})$. From a traditional QFT perspective, we can interpret this as an Euclidean theory with time compactified on a circle of radius $\beta = 1/T$. Due to the torus' geometry, a closed translation on time doesn't get you back to the original point in space, yielding a space translation too. This lets us define the partition function as

$$Z(x, y) = \text{Tr}\ _{\mathcal{H}}\left(e^{2\pi y H} e^{2\pi x P}\right) , \tag{2.66}$$

where P, H correspond to the momentum and hamiltonian operators, respectively, as they generate space and time translations.

One can find that the Hamiltonian operator on the cylinder of length L is

$$H = \left(L_0 + \bar{L}_0 - \frac{c + \bar{c}}{24}\right) , \tag{2.67}$$

and the momentum operator is

$$P = \frac{1}{L}\left(L_0 - \bar{L}_0\right) , \tag{2.68}$$

where L_0 and $\bar{L}_0$ are the Virasoro generators. Subbing this into (2.66), we obtain

$$\begin{aligned} Z(x, y) &= \text{Tr}\ _{\mathcal{H}}\left(e^{-2\pi y(L_0 + \bar{L}_0 - c/24 - \bar{c}/24)} e^{2\pi i x(L_0 - \bar{L}_0)}\right) \\ &= \text{Tr}\ _{\mathcal{H}}\left(e^{2\pi i(x+iy)(L_0 - c/24) - 2\pi i(x-iy)(\bar{L}_0 - \bar{c}/24)}\right) \\ &= \text{Tr}\ _{\mathcal{H}}\left(e^{2\pi i\tau(L_0 - c/24)} e^{-2\pi i\bar{\tau}(\bar{L}_0 - \bar{c}/24)}\right) , \\ Z(\tau, \bar{\tau}) &= \text{Tr}\ _{\mathcal{H}}\left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24}\right) , \end{aligned} \tag{2.69}$$

where we have now defined

$$q = e^{2\pi i\tau} , \quad \bar{q} = e^{2\pi i\bar{\tau}} . \tag{2.70}$$

More in general, the partition function can also be defined in terms of the corresponding characters of the algebra $\chi, \bar{\chi}$, which correspond to the individual terms of the trace over the Hilbert space of our CFT,

$$Z(\tau, \bar{\tau}) = \text{Tr}\ _{\mathcal{H}} \left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right) = \chi_{vac}(\tau, \bar{\tau}) + \sum_{h,\bar{h}} \chi_{h,\bar{h}}(\tau, \bar{\tau}) \,. \tag{2.71}$$

2.2 The conformal bootstrap in two dimensions

As we mentioned earlier, the conformal bootstrap program was first developed in the 80s [5], and then it was reignited in 2008 with the seminal work of [7].

The overarching goal of the bootstrap is to, rather than employ the path integral or other traditional QFT formalisms to compute observables in CFTs, exploit the constraints placed by the symmetries present in the CFTs on the observables, with the aim to obtain the two pieces of data needed in a CFT: the spectrum and the OPE coefficients.

There are two main strains of conformal bootstrap: numerical and analytical. The former tries to place constraints on the theory from numerically solving a set of differential equations, and the latter uses analytic techniques to the same end. This thesis' techniques are analytic in nature.

As it was highlighted in the introduction and the previous section, the conformal group in $d = 2$ is significantly different to its higher dimensional counterpart, for different reasons:

- There is a global conformal group, $SL(2, \mathbb{C})/\mathbb{Z}_2$, which is similar to the higher-dimensional one, but there is also a local algebra of transformations, the Virasoro algebra, which places completely different constraints on the theory.

- The group and algebra of the theory are infinite-dimensional.

- The fact that putting the theory on the torus leads to us studying modular forms makes the connection with an area of mathematics not always present in the higher dimensional counterparts: the study of modular forms.

- Finally, the organisation of the Verma modules themselves and the lack of an analytic formula for the Virasoro block causes certain difficulties when trying to shed light on the spectrum of the theory.

Whilst two-dimensional theories have advantadges coming from the fact that there are simply less degrees of freedom to complicate the theory, their key differences with larger-dimensional CFTs causes them to be studied separately.

Let us quickly summarise some of the general assumptions we make when trying to establish whether a given CFT can constitute a good candidate for a physical theory.

2.2.1 Assumptions on two-dimensional CFT behaviour

Part of the appeal of the bootstrap program is that one doesn't just solve a problem, but one also has to first *find* the problem to solve. To set up the search, we need to first understand what the main assumptions are, and when we can relax or change them.

- **Unitarity**: This is required at two levels:
  - We require that the Hilbert space has positive-normed vectors. This is important, as these allow us to count, for instance, the degeneracies in the spectrum of states.
  - We also require that only unitary representations of the symmetry group of the theory appear in the partition function. This sets bounds for the conformal weights of primaries appearing in the theory. For a generic 2d CFT, this is generally the requirement that $(h, \bar{h}) \geq 0$.

  Generally speaking, it is rare to find, in the context of *stringy* CFTs, non-unitary theories. A well-known example is the Liouville CFT with central charge $c < 1$, but non-unitary CFTs are more common in other fields, as they can be useful for instance in the study of certain crystalline lattices.

- **Compactness**: This is defined in different ways throughout the literature, but a nice summary of the most agreed-upon convention is found in [54]. The conditions Binder and Rychkov cite there are

  1. All scaling dimensions $\Delta \in \mathbb{R}$. This is related to unitarity, however in some instances we want to have states on the principal series, as for instance in Celestial CFTs as we will see later.

2. Given any Δ, there are a finite number of states with $\Delta' \leq \Delta$. This can imply two different conditions: discreteness and sparseness of the spectrum. Discreteness is usually required to define thermodynamic properties [55], so it is generally expected in traditional holography. Sparseness is usually required in most calculations trying to impose bounds on certain operator gaps and so on, but it can be broken.

3. There exists a unique vacuum with $\Delta = 0$, and there are no states below it. Uniqueness of the vacuum is crucial for constructing a consistent Hilbert space, and many of the assumptions made for, for instance, radial quantisation, would fail if this wasn't the case.

4. The dilatation operator can be diagonalised. Some theories, such as logarithmic CFTs can break this condition, but we will not be covering those.

5. The OPE expansion converges. This is again, a key assumption that we can't skip: otherwise, we can't formulate crossing equations for n-point functions, and neither can we compute n-point functions from smaller point functions through recursive OPE application, as proven in [56].

Generally speaking, the fifth assumption is always assumed, and so are 3,4. The first two can be broken in certain scenarios, but in that case they definitely present challenges to the treatment of CFTs. In much of the literature, 3-5 are always assumed by default.

Very overarchingly, compactness usually means that we expect the spectrum to be discrete, as for instance traditionally expected in the AdS/CFT correspondence. However, we know that this can be broken in low-dimensional examples, such as if we consider AdS_2, whose dual was found to be an ensemble of quantum mechanical theories [9].

- **Integrality**: This is probably one of the most difficult conditions to impose: we expect that the degeneracies of operators in the partition function are not only positive numbers, but also integers. This is due to the nature of these norms, they are counting states! In an ideal scenario, one would want to have an integer amount of primaries. Of course, this can be broken if we are considering the possibility of averaged theories, for instance.

So... what is the bootstrap then?

We have established a series of conditions we impose on our CFT, and on top of that we have to state the invariance of the theory under conformal transformations, and how this translates to observables.

The bootstrap is a very large topic covering a very wide range of sub-topics at this point in time, so we will be answering the question only for (certain parts) of the two-dimensional case.

On the one hand, there are constraints coming at the level of the torus partition function or other torus observables. As we pointed out previously, our theory has to be consistent when put on an arbitrary Riemann surface, and this includes of course the torus. One imposes invariance (usually of the partition function) under the action of the modular group, which brings very interesting consequences. This is usually referred to as the modular bootstrap, and Chapters 3 and 4 of this thesis deal with problems related to it. Solving these constraints via number-theoretic techniques is 'bootstrap'.

On the other hand, there are constraints more common to all CFTs, in particular the search of an inversion formula: a particular approach to tackling the Sturm-Liouville problem of the quadratic Casimir, which allows us to extract OPE coefficients and/or the spectrum present in the OPE of two primaries if we know the full four-point correlator. This is particularly useful in the context of Celestial CFTs, where we are trying to explore a highly exotic CFT that doesn't conform to traditional standards and of which we would like to assume as little as possible. This will be explored in Chapter 5. Performing these decompositions utilising a basis of eigenstates of the Casimir is 'bootstrap'.

Chapter 3

Irrational $\mathcal{W}_N$ CFTs and the spectrum of pure higher-spin AdS_3 gravity

A long-standing goal in the field of three-dimensional gravity has been to determine whether or not a non-trivial, consistent, theory of *pure* gravity can be constructed. Interestingly enough, despite not allowing for dynamical degrees of freedom in the bulk, these theories allow for the graviton to couple to a finite tower of higher spin particles [57] without becoming inconsistent.

Pure gravity in AdS_3, higher spin or not, is expected to be dual to an irrational[1] Conformal Field Theory in two dimensions, and it is precisely using CFT techniques that we will investigate the spectrum of pure higher spin gravity in three dimensions.

3.1 Introduction and background

Three-dimensional AdS gravity is certainly special: it doesn't have dynamical degrees of freedom in the bulk. In other words, there are no gravitational waves propagating in the bulk of this theory. This means the theory is topological, so one may wonder: is this theory just trivial?

Well, not classically: in the 90s, a black hole solution was found [58] by Bañados, Teitelboim and Zanelli. These are fundamentally different to the higher-dimensional notion of black holes,

[1]The expectation of irrationality comes from identifying semi-classical pure three-dimensional gravity with a large central charge 2d CFT where Virasoro is the only symmetry present.

mostly due to the implications arising from the lack of bulk gravitational waves.

There is more to the story: Brown and Henneaux showed [10] how the asymptotic symmetries of standard gravity on AdS_3 can be described by two copies of the Virasoro algebra living on the boundary. This means that we have dynamical degrees of freedom in the boundary, and in fact we can rephrase this statement following the AdS/CFT conjecture: pure AdS_3 gravity with radius ℓ and Newton constant G_N is dual to a boundary CFT_2 of central charge c_g related to the gravity constants as

$$c_g = \frac{3\ell}{2G_N}\,. \tag{3.1}$$

This statement is valid at the classical level, but can we build a consistent theory of Quantum Gravity in three dimensions, or is this theory too constrained to exist (other than being trivial)? Investigations on this topic have been done for a number of years, and the reader is encouraged to consult references [11–28]. We currently have a series of problems when trying to determine the spectrum of the gravity theory.

- We get a continuous spectrum, whereas a consistent theory of quantum gravity should in principle have a discrete spectrum.
- We have non-integer degeneracies.
- We get an exponential amount of negative norm states in the near-extremal limit. This is the limit where the mass and the spin of the black hole states become similar.

Now the first two questions are not as worrisome due to the recent developments in two-dimensional gravity showing how gravity is dual to an ensemble of CFTs, thus expecting continuous spectra and non-integer degeneracies in the partition function.

However, the third one, found in [14] implies a loss of unitarity in the gravity theory and, as it cannot be corrected naively without modifying the current definition of pure gravity, it is the most concerning.

Solutions for this problem have been searched in the context of non-higher-spin three-dimensional gravity. These range from adding conical-defect orbifolds [21] or non-classical topologies [18, 24] to the partition function in order to cure the negativities, to proposing seeing the duality between gravity and an ensemble of CFTs which generalises the JT gravity case [22, 23].

Now, we have done this small recap to understand what the starting point for us is a bit better. We are interested in a similar problem: finding the spectrum of three-dimensional pure gravity, but when there are (massless) higher spin fields in the theory as well.

We will consider higher spin gravity with massless particles of spin $s = 2, 3, \dots N$. In this setup, the asymptotic symmetries at the boundary can be described as two copies of the $\mathcal{W}_N$ algebra, with the same central charge relation as in (3.1), [29, 30], and an interesting spectrum of classical solutions [59]. The belief is that this algebra survives at the quantum level [60], however it is worth investigating whether or not they provide us with consistent theories of quantum gravity: it can be difficult to imagine a finite tower of massless higher-spin fields arising from limits of string theory [61]. From a CFT perspective, the problem can be framed as evaluating the existence of consistent unitary, irrational two-dimensional CFTs with extended symmetry $\mathcal{W}_N$.

To aid in such a quest, we will be studying the spectrum of massless higher-spin theories on thermal AdS_3 employing techniques of the modular bootstrap. We will find constraints on the partition function arising from unitarity, modular invariance and $\mathcal{W}_N$ symmetry, and then see how these affect the admissible spectrum of the theory. We will use two complementary approaches: first, from the two-dimensional CFT perspective and second from the three-dimensional gravity theory, which we will find to be in agreement.

3.2 Spectral densities in Higher-Spin Gravity

Let us consider an unitary, irrational two-dimensional CFT with $\mathcal{W}_N$ symmetry and central charge[2] $c > N - 1$. Its torus partition function is given by

$$Z(q, \bar{q}) = \sum_{h,\bar{h}} q^{h-c/24} \bar{q}^{\bar{h}-c/24} \,, \tag{3.2}$$

where $(h, \bar{h})$ correspond to the conformal weights of each of the states, and we are using the usual conventions for q, $\bar{q}$ (2.70).

This partition function is heavily constrained by the $\mathcal{W}_N$ symmetry (in the case where $N = 2$,

[2]Note that the constraint on the central charge for irrational CFTs gets uplifted from the Virasoro case ($c > 1$) to $c > N - 1$ as a consequence of the introduction of the $\mathcal{W}_N$ symmetry. See [62], for how this bound arises in the context of minimal models with $\mathcal{W}_N$ symmetry.

this will correspond to the Virasoro algebra), as this implies there exists a tower of conserved currents up to spin N, and we would ideally like to express the partition function in terms of the characters for the $\mathcal{W}_N$ algebra of generic N, as an extension of what is usually done for $N = 2$. In general circumstances, these are not known, however we can write them down when the chemical potential for the higher spin (> 2) currents is turned off.

Let us have a generic $\mathcal{W}_N$ algebra with only the lowest chemical potential turned on and with no other conserved currents present. We can express the characters of this theory as

$$\chi_{vac}^{(N)}(q) = \frac{q^{-\hat{c}_N}}{\eta(q)^{N-1}} \prod_{i=1}^{N-1} (1-q^i)^{N-i}\,, \quad \chi_h^{(N)}(q) = \frac{q^{h-\hat{c}_N}}{\eta(q)^{N-1}}\,, \tag{3.3}$$

where $\hat{c}_N = \frac{c-(N-1)}{24}$.

Armed with the characters, we can safely assume that the torus partition function of the theory can be decomposed as

$$Z(q,\bar{q}) = \chi_{vac}^{(N)}(q)\bar{\chi}_{vac}^{(N)}(\bar{q}) + \sum_{h,\bar{h}} d_{h,\bar{h}}\, \chi_h^{(N)}(q)\bar{\chi}_{\bar{h}}^{(N)}(\bar{q})\,. \tag{3.4}$$

Our focus will be on the numbers $d_{h,\bar{h}}$ as those represent the multiplicities of the $\mathcal{W}_N$ primaries: they should be integer and positive (due to the assumption of a discrete and unitary spectrum).

We have also assumed that only unitary representations appear in the character decomposition, and this is related to the values of the conformal weights $(h,\bar{h})$ of operators above the vacuum state. For the Virasoro case ($N = 2$), this is trivial, however when dealing with higher spin currents ($N > 2$) any non vacuum operator is bound to have conformal weights satisfying [63]

$$h \geq h_{crit} = \frac{c-(N-1)}{24}\left(1 - \frac{6\lfloor \frac{N}{2} \rfloor}{N(N^2-1)}\right), \tag{3.5}$$

with the same bound applying for $\bar{h}$. This unitarity requirement makes $\mathcal{W}_N$ symmetry stronger and more constraining than Virasoro, something expected[3].

We also assume modular invariance of the torus partition function. Under a modular trans-

[3]See Appendix A.2, also [1, Appendix B].

formation, the complex moduli of the torus is acted on by an an element $\gamma \in PSL(2,\mathbb{Z})$

$$\tau \to \gamma\,\tau = \frac{a\tau + b}{s\tau - r}\,, \quad \bar{\tau} \to \gamma\,\bar{\tau} = \frac{a\bar{\tau} + b}{s\bar{\tau} - r}\,, \tag{3.6}$$

where a, b, s, r are integers, s is taken to be positive, and $a\,r + b\,s = -1$. Note that this condition implies r, s are coprime: $(r, s) = 1$.

This transformation should leave the partition function as is:

$$Z(q, \bar{q}) = Z(\gamma q, \gamma \bar{q})\,. \tag{3.7}$$

In this chapter, we will choose the following parametrisation for convenience from here onward.

$$q = e^{-\beta + \frac{2\pi i r}{s}}\,, \qquad \gamma q = e^{-\frac{4\pi^2}{s^2\beta} + \frac{2\pi i a}{s}}\,, \tag{3.8}$$

$$\bar{q} = e^{-\bar{\beta} - \frac{2\pi i r}{s}}\,, \qquad \gamma \bar{q} = e^{-\frac{4\pi^2}{s^2\bar{\beta}} - \frac{2\pi i a}{s}}\,. \tag{3.9}$$

Lastly, we will be assuming a particular spectrum of light operators. In this context, an operator is deemed *light* when both its conformal weights lie underneath the BTZ black hole threshold $(h, \bar{h}) = (\hat{c}_N, \hat{c}_N)$.

With all of the assumptions previously mentioned, we would like to build a partition function of $\mathcal{W}_N$ primaries, or equivalently, find the density $\rho(h, \bar{h})$ of primaries. To facilitate that computation it is useful to define a new -still modular invariant- partition function, given by

$$Z^p(q, \bar{q}) = \left(y^{1/2}\eta(q)\eta(\bar{q})\right)^{N-1} Z(q, \bar{q})\,, \tag{3.10}$$

where $y = \Im(\tau)$. This can be easily decomposed in the quantities of interest as

$$Z^p(q, \bar{q}) = y^{\frac{N-1}{2}} \left(vac + \sum_{h,\bar{h}} q^{h - \hat{c}} \bar{q}^{\bar{h} - \hat{c}_N} \right) = y^{\frac{N-1}{2}} \left(vac + \int \mathrm{d}h\, \mathrm{d}\bar{h}\, \rho(h, \bar{h})\, q^{h - \hat{c}_N} \bar{q}^{\bar{h} - \hat{c}_N} \right), \tag{3.11}$$

where the sum runs over $\mathcal{W}_N$ primaries, and the spectral density $\rho(h, \bar{h})$ is defined so that the

following integral counts the number of states in the range $(h, \bar{h}) \in [h_0, h_0 + \Delta h] \times [\bar{h}_0, \bar{h}_0 + \Delta \bar{h}]$:

$$\int_{h_0}^{h_0+\Delta h} \int_{\bar{h}_0}^{\bar{h}_0+\Delta \bar{h}} \mathrm{d}h \, \mathrm{d}\bar{h} \, \rho(h, \bar{h}) = \text{number of states} \,. \tag{3.12}$$

On the other hand, the vacuum sector 'vac' of the partition function is defined as:

$$vac = q^{-\hat{c}} \bar{q}^{-\hat{c}} \left(\prod_{i=1}^{N-1} (1 - q^i)^{N-i} \right) \left(\prod_{i=1}^{N-1} (1 - \bar{q}^i)^{N-i} \right) . \tag{3.13}$$

More concretely, the density of states for a theory with a discrete spectrum should be a sum over delta functions.

Now that we have established the basics of constructing the density of states and the torus partition function of a $\mathcal{W}_N$ theory, let us introduce the procedure for imposing modular invariance as a constraint, and how to use it to recover the operator content of our theory.

Modular invariance at the level of the 'primary' partition function, Z^p, can be written as

$$vac + \int \mathrm{d}h \; \mathrm{d}\bar{h} \; \rho(h, \bar{h}) q^{h - \hat{c}_N} \bar{q}^{\bar{h} - \hat{c}_N} = \left(\frac{\gamma \, y}{y} \right)^{\frac{N-1}{2}} \left(\gamma \, vac + \int \mathrm{d}h \, \mathrm{d}\bar{h} \, \rho(h, \bar{h}) (\gamma q)^{h - \hat{c}} (\gamma \bar{q})^{\bar{h} - \hat{c}} \right) . \tag{3.14}$$

More in particular, we will implement a particular modular transformation as

$$\tau = x + iy = \frac{\beta}{2\pi} i + \frac{r}{s} \,, \qquad \bar{\tau} = x - iy = -\frac{\bar{\beta}}{2\pi} i + \frac{r}{s} \,, \tag{3.15}$$

so that $y = \frac{\beta + \bar{\beta}}{4\pi}$, while $\gamma y = \frac{\pi}{s^2} \left(\frac{1}{\beta} + \frac{1}{\beta} \right)$, which leads to

$$\frac{\gamma y}{y} = \frac{1}{s^2} \frac{4\pi^2}{\beta \bar{\beta}} = \frac{1}{s\tau - r} \frac{1}{s\bar{\tau} - r} \,. \tag{3.16}$$

Now, reconstructing the full spectrum/density of states can be tricky on a technical level due to the lack of a full analytic basis of non-holomorphic modular forms, leading to ambiguities in the heavy spectrum of the theory as we will see later. We will discuss two different approaches to constructing the desired density of states: the Poincaré and Rademacher constructions.

Before moving onto the density of states computation, a quick summary of the key assumptions we are making.

- The partition function can be decomposed in terms of $\mathcal{W}_N$ characters.
- Theory is unitary both in degeneracies and in representations.
- Partition function satisfies modular invariance.
- Particular spectrum of light operators (below BTZ threshold), containing only vacuum character of $\mathcal{W}_N$.

3.2.1 The Poincaré density of states

The Poincaré construction is a method for finding the full modular image of a particular 'seed' of light operators of weights $(h_0, \bar{h}_0)$, which in our context will be the vacuum state of the theory. It was introduced in this context in [12, 13]:

$$Z_{\text{Poincaré}}(\tau, \bar{\tau}) = \sum_{\gamma \in \mathbb{Z}\backslash PSL(2,\mathbb{Z})} \chi^{(N)}_{h_0}(\gamma\tau)\, \bar{\chi}^{(N)}_{\bar{h}_0}(\gamma\bar{\tau})\,. \tag{3.17}$$

Although this sum is divergent in principle, it can be regularised to obtain a convergent partition function as done in [12, 13]. We want to apply this methodology but starting from the vacuum of $\mathcal{W}_N$. We will use the method of kernels to obtain the density of states, as it was done in [21].

Starting from a particular seed, we will compute the contribution to the density of states for a particular $PSL(2,\mathbb{Z})$ image of the seed, and then perform the sum over all the images. For our convenience, we will work with the primary partition function $Z_p(q, \bar{q})$(3.11), and we will define the modular kernel $\mathbb{K}^{(\mathtt{w},\gamma)}_{h',h}$ as the solution to the density for the following problem:

$$\int_0^\infty \mathrm{d}h'\, \mathbb{K}^{(\mathtt{w},\gamma)}_{h',h} e^{2\pi i \tau h'} = (-i(s\tau - r))^{-\mathtt{w}} e^{2\pi i(\gamma\tau)h}\,. \tag{3.18}$$

The modular kernel allows us to express a transformed character (rescaled as (3.11)) as a linear combination of untransformed characters. We have an expression for general weight $\mathtt{w}$ to allow for the overall powers of y present in (3.11). For a $\mathcal{W}_N$ theory, $\mathtt{w} = \frac{N-1}{2}$. The kernel can be expressed as

$$\mathbb{K}^{(\mathtt{w},\gamma)}_{h'h} = \epsilon(\mathtt{w},\gamma)\left(\frac{2\pi}{s}\right)\left(\frac{h'}{h}\right)^{\frac{\mathtt{w}-1}{2}} e^{\frac{2\pi i}{s}(ah - rh')} I_{\mathtt{w}-1}\left(\frac{4\pi\sqrt{-h'h}}{s}\right), \tag{3.19}$$

where $\epsilon(\mathtt{w},\gamma)$ is a h,h'-independent phase (see (A.6) of [21]), and $I_k(x)$ is a modified Bessel function of the first kind. Let us consider a seed with arbitrary weights $(h_0,\bar{h}_0)$ and compute the resulting density for particular weights $(h,\bar{h})$. As in [13], it will be convenient to parametrise the seed and densities in terms of energies and spins, defined as

$$E = h_0 + \bar{h}_0 - 2\hat{c}_N\,, \quad J = h_0 - \bar{h}_0\,, \tag{3.20}$$

$$e = h + \bar{h} - 2\hat{c}_N\,, \quad j = h - \bar{h}\,, \tag{3.21}$$

where recall $\hat{c}_N = \frac{c-N+1}{24}$. The seed will then produce a term in the partition function proportional to

$$q^{\frac{E+J}{2}}\bar{q}^{\frac{E-J}{2}}. \tag{3.22}$$

The Poincaré sum of (3.22) is then given by

$$\begin{aligned}\rho_{E,J}(e,j) &= \frac{1}{2}\sum_{\gamma\in\mathbb{Z}\backslash SL(2,\mathbb{Z})} \mathbb{K}^{(\mathtt{w},\gamma)}_{\frac{e+j}{2},\frac{E+J}{2}}\,\mathbb{K}^{*(\mathtt{w},\gamma)}_{\frac{e-j}{2},\frac{E-J}{2}} \\ &= \frac{1}{2}\sum_{s=1}^{\infty}\sum_{r\in(\mathbb{Z}/s\mathbb{Z})^*}\sum_{n=-\infty}^{\infty} e^{\frac{2\pi i}{s}(aJ-rj)}e^{2\pi i n j}\,[\ldots] \\ &= \sum_{s=1}^{\infty}\frac{K(j,J;s)}{s^{2\mathtt{w}}}\,\rho^{s=1}_{E/s^2,J/s^2}(e,j)\,.\end{aligned} \tag{3.23}$$

Let us make some clarifications about (3.23). In the first line, the factor of $\frac{1}{2}$ is due to the Jacobian produced by the change of variables $(h,\,\bar{h})\to(e,j)$. In the second line, we expanded the sum over $\mathbb{Z}\backslash SL(2,\mathbb{Z})$, where we have taken $r'=r-ns$. The $[\ldots]$ refer to the remaining terms in the kernel (3.19), i.e. $\left(\frac{2\pi}{s}\right)\left(\frac{h'}{h}\right)^{\frac{\mathtt{w}-1}{2}} I_{\mathtt{w}-1}\left(\frac{4\pi\sqrt{-h'h}}{s}\right)$ which we suppress for brevity. The sum over n simply projects onto integer spin j. Finally in the last line, $K(j,J;s)$ is a Kloosterman sum, defined as

$$K(j,J;s) = \sum_{r\in(\mathbb{Z}/s\mathbb{Z})^*} e^{\frac{2\pi i}{s}(aJ-rj)}, \qquad -a\,r\equiv 1\ (\mathrm{mod}\ s)\,, \tag{3.24}$$

and $\rho^{s=1}_{E/s^2,J/s^2}(e,j)$ is defined as

$$\begin{aligned}\rho^{s=1}_{E,J}(e,j) &= 2\pi^2(-E+J)^{\frac{1-\mathtt{w}}{2}}(-E-J)^{\frac{1-\mathtt{w}}{2}}(e-j)^{\frac{\mathtt{w}-1}{2}}(e+j)^{\frac{\mathtt{w}-1}{2}} \\ &\quad\times I_{\mathtt{w}-1}\left(2\pi\sqrt{-E+J}\sqrt{e-j}\right) I_{\mathtt{w}-1}\left(2\pi\sqrt{-E-J}\sqrt{e+j}\right)\,,\end{aligned} \tag{3.25}$$

where we take the weight $\mathtt{w}$ to be $\frac{N-1}{2}$. We can summarise the $\mathcal{W}_N$ Poincaré density for a seed of quantum numbers (E, J) as

$$\rho^{(N)}_{E,J}(e,j) = \sum_{s=1,2,3,\cdots} \frac{K(j,J,s)}{s^{N-1}}\, \rho^{(N),s=1}_{E/s^2,J/s^2}(e,j)\,, \tag{3.26}$$

with [1]

$$\begin{aligned}\rho^{(N),s=1}_{E,J}(e,j) = 2\pi^2(-E+J)^{\frac{3-N}{4}}(-E-J)^{\frac{3-N}{4}}(e-j)^{\frac{N-3}{4}}(e+j)^{\frac{N-3}{4}} \\ \times I_{\frac{N-3}{2}}\left(2\pi\sqrt{-E+J}\sqrt{e-j}\right) I_{\frac{N-3}{2}}\left(2\pi\sqrt{-E-J}\sqrt{e+j}\right)\,.\end{aligned} \tag{3.27}$$

Interestingly enough, the convergence properties of the sum over images improves with larger values of N, where in particular if $N > 3$ the sum is absolutely convergent. The $N = 3$ sector is discussed in [1, Appendix B]

3.2.2 The Rademacher density of states

There is an alternative way of constructing a density of states for a seed (E, J) which respects modular invariance: the Rademacher procedure, introduced in [19] for the $\mathcal{W}_2$ case. The $\mathcal{W}_N$ case is a fairly straightforward generalisation of Virasoro, so the reader is encouraged to check [19] for further details. We describe the basics of the procedure in Appendix A.1.

The starting point is (3.14), which sets the modular invariance condition in general for the density of $\mathcal{W}_N$ primaries. If we rewrite it in x, y variables (recall $\tau = x + i\,y$ and $\bar{\tau} = x - i\,y$), assume discreteness of the spectrum (which allows us to replace the integrals by sums) and independence of (x, y) as complex variables, we find

$$vac + \sum_{j,e} \rho(e,j) e^{2\pi i j x} e^{-2\pi y e} = \left(\frac{1}{(sx-r)^2+s^2y^2}\right)^{\frac{N-1}{2}} \sum_{j',e'} e^{\frac{2\pi i a}{s} j'} e^{\frac{2\pi\, i(r-sx)\, j'}{s(r-sx)^2+s^3y^2}} e^{-\frac{2\pi\, y\, e'}{(r-sx)^2+s^2y^2}}\,. \tag{3.28}$$

This is a great starting point to study the implications of modularity in the (x, y) complex plane.

In particular, modular invariance, together with the condition that there are operators in

the light spectrum imply the existence of a tower of essential singularities:

$$x = \pm\, i\, y + \frac{r}{s}\,. \tag{3.29}$$

Let us analyse what happens if we have a single light operator of quantum numbers (E, J), producing such a singular tower:

$$vac + \sum_{j,e} \rho(e,j) e^{2\pi i j x} e^{-2\pi y e} \sim \left(\frac{1}{(sx-r)^2 + s^2 y^2}\right)^{\frac{N-1}{2}} e^{\frac{2\pi i a}{s} J} e^{\frac{2\pi i (r-sx) J}{s(r-sx)^2 + s^3 y^2}} e^{-\frac{2\pi y E}{(r-sx)^2 + s^2 y^2}}\,. \tag{3.30}$$

We know that, to produce such an essential singularity, the sum on the LHS must be infinite, but we also know that the spin is a discrete quantity. Hence, we can rewrite the LHS as

$$\sum_{j,e} \rho(e,j) e^{2\pi i j x} e^{-2\pi y e} = \sum_{j=-\infty}^{\infty} a_j(y) e^{2\pi i j x}, \tag{3.31}$$

where now we have defined $a_j(y)$ to encode the contribution of operators of a given spin to the partition function.

We now follow the procedure described in [19] to invert and find the Fourier coefficients $a_j(y)$. To illustrate the procedure, we can showcase the expression in the case of $s = 1$.

$$a_j^{s=1}(y) = -i \int_{\frac{1}{2\pi y} - i\infty}^{\frac{1}{2\pi y} + i\infty} \mathrm{d}w\, \frac{(2\pi)^{N-2} w^{N-3}}{(4\pi w y - 1)^{\frac{N-1}{2}}} e^{j(\frac{1}{w} - 2\pi\, y)} e^{\frac{2\pi^2 w}{4\pi w y - 1} \kappa_-} e^{2\pi^2 w\, \kappa_+}\,, \tag{3.32}$$

with $\kappa_\pm = -(E \pm J)$ and $w = -\frac{1}{2\pi i (x+iy)}$. The case $N = 2$ is precisely the case covered in [19]. For general N, we follow a similar approach. We will expand the expression first in powers of j and κ, and use the following identity term by term:

$$\frac{1}{2\pi i} \int_{\frac{1}{2\pi y} - i\infty}^{\frac{1}{2\pi y} + i\infty} \mathrm{d}w\, \frac{w^\alpha}{(4\pi w y - 1)^\beta} e^{2\pi^2 \kappa_+ w} = \frac{\kappa_+^{-\alpha+\beta-1}\, {}_1\tilde{F}_1\left(\beta; \beta - \alpha; \frac{\kappa_+ \pi}{2y}\right)}{2^{\alpha+\beta+1}\, \pi^{2\alpha-\beta+2}\, y^\beta}\,. \tag{3.33}$$

In this expression, ${}_1\tilde{F}_1$ denotes the regularised confluent hypergeometric function. Now that we have $a_j^{s=1}(y)$ in closed form, we can perform an inverse Laplace transform on it to recover the desired density $\rho(e, j)$.

Repeating the steps for all values of s, we get the following final result for the density [1]:

$$\rho^{(N)}_{E,J}(e,j) = \sum_{s=1,2,3,\cdots} \frac{K(j,J,s)}{s^{N-1}} \rho^{(N),s=1}_{E/s^2,J/s^2}(e,j)\,, \tag{3.34}$$

where now

$$\begin{aligned}\rho^{(N),s=1}_{E,J}(e,j) = 2\pi^2(-E+J)^{\frac{3-N}{4}}(-E-J)^{\frac{3-N}{4}}(e-j)^{\frac{N-3}{4}}(e+j)^{\frac{N-3}{4}} \\ \times I_{\frac{N-3}{2}}\left(2\pi\sqrt{-E+J}\sqrt{e-j}\right) I_{-\frac{N-3}{2}}\left(2\pi\sqrt{-E-J}\sqrt{e+j}\right)\,.\end{aligned} \tag{3.35}$$

This result is remarkably similar to the Poincaré construction (3.27). The only difference is the sign of the index of the second Bessel function. This only becomes relevant in the cases where N is even, as in the case of N odd, the two Bessel functions are actually identical. For N even, the difference between constructions isn't really relevant, as it is exponentially suppressed. The Cardy behaviour is the same in both constructions, otherwise we could use that to choose one over the other. This can be seen if we look at the large z expansion of a generic Bessel function of non-integer weight:

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}}\left(e^z\left(1+\frac{1-4\nu^2}{8z}+\cdots\right) - ie^{-i\pi\nu}e^{-z}\left(1-\frac{1-4\nu^2}{8z}+\cdots\right)\right)\,, \tag{3.36}$$

from where we can see explicitly how the density of states becomes, in the Cardy limit

$$\rho^{s=1}_{E,J}(e,j) \sim e^{2\pi\sqrt{e}(\sqrt{-E-J}+\sqrt{-E+J})}, \tag{3.37}$$

a result in agreement with [19]. This is true for all $\mathcal{W}_N$ CFTs, as the leading behaviour of the Bessel function doesn't depend on N.

3.2.3 Ambiguities in general $\mathcal{W}_N$

Both the densities of states we have found are compatible with a modular invariant partition function, however there are some 'ambiguities', manifest explicitly for even N as the densities are slightly different. This comes from an interesting observation for the Virasoro case in [19],

that we generalise here for any N. We can always add a term of the form

$$Z_{amb}(q,\bar{q}) = y^{\frac{N-1}{2}} \sum_{k,\bar{k}\geq 0} q^k \bar{q}^{\bar{k}} \tag{3.38}$$

to the partition function without breaking modular invariance.

The difference between the results for Poincaré and Rademacher (for even N) is a function of this kind. We really don't have any reason not to add a function of this form. Moreover, this modular form lives above the BTZ threshold, thus only adding some oscillations that don't affect Cardy behaviour. All of this makes the ambiguities unavoidable and not solvable in a naive manner.

All said above applies only if we add a finite number of ambiguities. If we had an exponentially large amount of ambiguous seed operators, the Cardy behaviour would be violated [19].

These issues are rooted in the core properties of non-holomorphic (or real) modular forms. In the space of holomorphic modular forms, we can prove that any bounded modular form of weight zero is a constant and, in particular, the Rademacher procedure is able to reconstruct entire modular forms of weight zero from their principal part unambiguously. However, this is no longer true for non-holomorphic modular forms and we are always left with these ambiguities.

3.3 Curing the negativities

We have a first proposal for the spectrum of an irrational $\mathcal{W}_N$ theory dual to a theory of higher spin gravity. So far, we have only imposed the condition of modular invariance on the partition function, and there are some issues when we try to implement discreteness and positivity:

- The spectra computed here are continuous, just as it happens for the MWK case [12, 13]. This is indicative of the proposed spectra not being that of a single, compact, unitary CFT. There is, however possible interpretation of such a spectrum: ensemble averages, such as the ones considered (albeit, for simpler settings) in [63, 64].

- Most importantly, at a certain range of spins and energies, the spectra are not positive-definite, thus breaking unitarity. If this was just a small number of states, it could be cured through the introduction of the aforementioned ambiguities, but we unfortunately

are in a similar scenario to the one found in the MKW partition function for the Virasoro case [14], and there is an arbitrarily large number of negative-norm states.

The densities of states constructed in the previous section were those generated by a single light state of arbitrary energy and spin. However, to study the partition function of a $\mathcal{W}_N$ theory we need to study the density generated[4] by the vacuum character of the theory. For a general N, the vacuum character is

$$\chi_0(\tau) = \frac{q^{-\hat{c}_N}}{\eta(\tau)^{N-1}} \prod_{i=1}^{N-1} (1-q^i)^{N-i} \,. \tag{3.39}$$

Now, defining

$$\prod_{i=1}^{N-1} (1-q^i)^{N-i} = \sum_{i=0}^{\frac{N^3-N}{6}} \mathrm{c}_N(i) q^i \,, \tag{3.40}$$

we get that the density of states is given by

$$\rho^{(N)}(e,j) = \sum_{i=0}^{\frac{N^3-N}{6}} \sum_{\bar{i}=0}^{\frac{N^3-N}{6}} \mathrm{c}_N(i)\mathrm{c}_N(\bar{i}) \rho^{(N)}_{-2\hat{c}_N+i+\bar{i},i-\bar{i}}(e,j) \,, \tag{3.41}$$

where

$$\rho^{(N)}_{E,J}(e,j) = \sum_{s=1}^{\infty} \frac{K(j,J;s)}{s^{N-1}} \rho^{(N),s=1}_{E/s^2,J/s^2}(e,j) \,, \tag{3.42}$$

and $\rho^{(N),s=1}_{E,J}(e,j)$ is defined by either (3.27) or (3.35), for the Poincaré and Rademacher sum respectively.

Let us note, before starting, that we will be considering $j \geq 0$ without loss of generality, as the density of states for negative j is the same as that for its absolute value.

The negativity present in the density of states can be found in the limit when the energy approaches the value of spin, which in gravity is interpreted as the near-extremal limit of black hole states. Note that this limit is taken before we consider large central charge (necessary for holography, this is the semiclassical limit).

When we take $e \to j$, we see a drastic simplification in the density of states (regardless if we are considering Poincaré or Rademacher). The scaling behaviour for both constructions in this limit is

[4] This is true for either the Poincaré or the Rademacher density of states

$$\rho_{E,J}^{(N),s=1}(e,j) \sim (e-j)^{\frac{N-3}{2}} \,. \tag{3.43}$$

Interestingly enough, this is a common scaling behaviour, independent of s in all terms. However, precisely because of that, the values of the $\mathrm{c}_N(i)$ combinations in (3.42) are going to determine the leading term.

- The term $s = N$ will dominate in this regime.
- The terms $s = 1, \ldots, N-1$ and $N+1, \ldots$ are going to be subdominant to $s = N$.

As the term $s = N$ of the expression is accompanied by the Kloosterman sum $K(j, J, N)$, it is no longer always positive. The requirement of positivity in the partition function is very strict: the density of states must be positive for all values of the spin. However, the Kloosterman sum of order N will be positive or negative depending on the spins present in each term. This will become more clear once we study concrete examples later.

To understand how harmful this negativity is, we should count how many states with negative norm there are. For that, we integrate over the density in the range where it is negative,

$$\text{negative states} \sim \int_{j}^{j+e^{-\frac{8\pi}{N^2}\sqrt{\hat{c}_N j}}} \mathrm{d}e\,(e-j)^{\frac{N-3}{2}} e^{\frac{4\pi}{N}\sqrt{\hat{c}_N j}} \sim e^{\frac{4\pi}{N^2}\sqrt{\hat{c}_N j}} \,. \tag{3.44}$$

We can appreciate here how bad this is: if we have either large central charge or large spin, there are an exponentially large amount of negative-norm operators in the theory.

How can we cure these negativities? Well, a natural way of curing them is to include a Poincaré or Rademacher sum of a low-twist operator, similarly to what was done in [21]. In this context, we define the twist of an operator as $\mathrm{t} = \min(h, \bar{h})$.

We shall consider two different regimes to remove the negativities.

1. $j \gg c \gg 1$:

   In this region, the density of states is of the form

$$\rho_{vac}(e,j) \approx -e^{\frac{4\pi}{s}\sqrt{\hat{c}_N j}}(e-j)^{\frac{N-3}{2}} \,, \tag{3.45}$$

which leads us to adding one operator with

$$\mathrm{t} \leq \frac{c-N+1}{24}\left(1-\frac{1}{N^2}\right). \tag{3.46}$$

2. $\underline{c \gg 1,\ j \ll c}$:

This region is a bit more complicated. The density of states in this limit behaves as

$$\rho_{vac}(e,j) \approx \sum_{i=0}^{\frac{N^3-N}{6}} \sum_{k=0}^{\frac{N^3-N}{6}} \frac{\mathrm{c}_N(i)\mathrm{c}_N(k)\,K(j,k-i,N)}{N^{N-1}} e^{\frac{4\pi}{N}\sqrt{j}\sqrt{\hat{c}_N-k}}\,(e-j)^{\frac{N-3}{2}}. \tag{3.47}$$

This is a rather complicated expression, but we can understand it as an expansion of the kind

$$\rho_{vac}(e,j) \sim \sum_{k=0}^{\frac{N^3-N}{6}} A_k e^{\frac{4\pi}{N}\sqrt{j}\sqrt{\hat{c}_N-k}}(e-j)^{\frac{N-3}{2}}, \tag{3.48}$$

where, depending on each specific combination of the $\mathrm{c}_N(i)$'s and the Kloosterman sum we will have an either positive or negative coefficient A_k.

To correct the negativity, need to add a certain number of states that scale as

$$\mathrm{t} \leq \frac{c}{24}\left(1-\frac{1}{N^2}\right)+\mathcal{O}(1). \tag{3.49}$$

To saturate this bound, we need to add a minimum of

$$\frac{N\phi(N)}{\gcd(\phi(N),N-1)} \tag{3.50}$$

states, where $\phi(N)$ is the Euler totient function.

In order to correct the negativity, we need to analyse the different spins $j=\{1,\ldots,N-1\}$ mod N and understand which one is the most negative[5]. Correcting the most negative case will be enough to ensure that all negativities are corrected.

If we add the minimum number of states (3.49) that cure the negativity, we find that their

[5]Due to the number-theoretic properties of the Kloosterman sum, the term with $j = 0 \mod N$ is always positive.

twist should average to

$$\mathrm{t}_{ave} \leq \frac{c-N+1}{24}\left(1-\frac{1}{N^2}\right)+\frac{N^2-1}{12N}=\frac{c+N+1}{24}\left(1-\frac{1}{N^2}\right)\,. \tag{3.51}$$

This correction is to one-loop order.

There are other proposals in the literature to cure the negativity in a modular-invariant way without the introduction of low-twist operators [18,24,28]. It would be interesting to either prove or disprove that introducing low-twist operators is the only way to eradicate the negativities.

We will now illustrate this procedure more explicitly for $\mathcal{W}_2$ (Virasoro) and $\mathcal{W}_3$, and then discuss $\mathcal{W}_N$ in more detail.

3.3.1 Curing the Virasoro negativities

Before we start with the Virasoro construction, let's set some notation.

In the near extremal and large central charge limit, we know that the density of states will behave as

$$\rho_{e,j} \sim \sum_{s=1}^{\infty}\sum_{i,k=0}^{\frac{N^3-N}{6}} \frac{\mathrm{c}_N(k)\mathrm{c}_N(i)K(j,k-i,s)}{s^{N-1}}(e-j)^{\frac{N-3}{2}}e^{\frac{4\pi}{s}\sqrt{(\hat{c}_N-k)j}}\,, \tag{3.52}$$

where we know that the presence of negativities is directly related to the sign of the factor

$$\sum_{i=0}^{\frac{N^3-N}{6}} \frac{\mathrm{c}_N(k)\mathrm{c}_N(i)K(j,k-i,s)}{s^{N-1}}(e-j)^{\frac{N-3}{2}}\,, \tag{3.53}$$

which we should analyse for every value of $k=0,\dots N-1$ and $j=\{1,\dots,N-1\} \mod N$, and for the dominant s in this regime. We consider terms with different k separately, and we denote it the *shift*, as k appears in the exponential factor shifting the classical central charge of each term [6].

In particular, we know that all of $s=1,\dots,N-1$ vanish in this limit (at leading order), and thus the leading behaviour, and the negativities, happen when we evaluate (3.52) for $s=N$.To ease the analysis of negativities, we will define a *degeneracy function*, that tells us the value of

[6]This name will make sense once we reach the gravity section of the chapter

the prefactor (3.53) (including its sign):

$$D(N,k,j) = \sum_{i=0}^{\frac{N^3-N}{6}} \frac{\mathrm{c}_N(k)\mathrm{c}_N(i)\,K(j,k-i,N)}{N^{N-1}}\,, \tag{3.54}$$

which represents is the degeneracy of each $\mathcal{W}_N$ primary at shift k.

Let's now properly start analysing the $\mathcal{W}_2$ case, Virasoro. The vacuum character of the Virasoro theory is

$$\chi_0(\tau) = \frac{e^{-2\pi i\tau\left(\frac{c-1}{24}\right)}}{\eta(\tau)}(1-q) \tag{3.55}$$

and, in terms of our previous notation (3.40), we have that

$$c_2(0) = 1,\ c_2(1) = -1. \tag{3.56}$$

As we mentioned, the negative norm states in the partition function were first found by Benjamin, Collier and Maloney [14], and a proposal to cure them by adding extra states was given by Alday and Bae [19], which were later re-interpreted as conical defect states by Benjamin et al [21].

Following the notation defined earlier, we know that the leading-order density of states for $\mathcal{W}_2$ is [1, 21]

$$\rho^{(2)}_{vac}(e,j) \approx \sum_{k=0}^{1} \frac{D(2,k,j)}{\sqrt{e-j}} e^{2\pi\sqrt{j}\sqrt{\hat{c}_2-k}}. \tag{3.57}$$

We now need to understand which values of k and j produce the negative norm states and, for that we present a summary of the different values of $D(2,k,j)$ in Table 3.1.

We can see the negativities arising from the terms with odd spin. In fact, the density of states in the near extremal limit and odd spin takes the form

$$\rho_{j\ \mathrm{odd}}(e) \sim -\frac{e^{2\pi\sqrt{\hat{c}_2 j}}}{\sqrt{e-j}} - \frac{e^{2\pi\sqrt{(\hat{c}_2-1)j}}}{\sqrt{e-j}}. \tag{3.58}$$

We can add the $SL(2,\mathbb{Z})$ sum of two primaries that have twist $\mathrm{t} = \frac{3}{4} + \frac{k_i}{4}$, and the overall density becomes

k	Multiplying what?	$D(2,k,\text{even } j)$	$D(2,k,\text{odd } j)$	Twist to add
0	$\exp\left(2\pi\sqrt{\hat{c}_2 j}\right)$	$+1$	-1	$\hat{c}_2\left(1-\frac{1}{2^2}\right)$
1	$\exp\left(2\pi\sqrt{(\hat{c}_2-1)\,j}\right)$	$+1$	-1	$\hat{c}_2\left(1-\frac{1}{2^2}\right)+\frac{1}{4}$
Average:		(positive)	$\frac{-1(0)-1(\frac{1}{4})}{2}=-\frac{1}{8}$	$\hat{c}_2\left(1-\frac{1}{2^2}\right)+\frac{1}{8}$ (two states)

Table 3.1: This table presents the information about the negative-norm states in the Virasoro vacuum character's modular sum. The entries in the third and fourth columns evaluate the degeneracy factor $D(2,k,j)$ for the values indicated. The second column indicates the factor multiplying the degeneracy in the partition function at large j or large c, and the last column indicates what twist a state should have so that its addition would counteract the negativity at each k,j combination. We see there are negative norm states for odd values of the spin j. In the limit of large c and finite j we can cure the negativities by introducing two states with $\mathrm{t}\leq\hat{c}_2\left(1-\frac{1}{2^2}\right)+\frac{1}{8}$.

$$\rho^{(2)}_{vac,\,2\,ops}(e,j)\sim-\frac{e^{2\pi\sqrt{\hat{c}_2\,j}}}{\sqrt{e-j}}-\frac{e^{2\pi\sqrt{(\hat{c}_2-1)j}}}{\sqrt{e-j}}+\sum_{i=1}^{2}\frac{e^{2\pi\sqrt{(\hat{c}_2-k_i)j}}}{\sqrt{e-j}}\,. \tag{3.59}$$

Now we want to understand which values of k_i are necessary in (3.59) to cancel out the negativities in the two limits of interest: large or finite spin (with respect to the central charge).

a) $j\gg c$ limit:

We should add at least one operator with $k_i\leq 0$.

b) $c\gg j$ limit:

This needs to be analysed a bit more carefully. Expression (3.59) in this limit becomes

$$\rho^{(2)}_{vac,\,2\,ops}(e,j)\sim\frac{e^{2\pi\sqrt{\hat{c}_2 j}}}{\sqrt{e-j}}\left(-\pi\sqrt{\frac{j}{\hat{c}_2}}(k_1+k_2-1)+\mathcal{O}(c^{-1})\right). \tag{3.60}$$

This expression is non-negative if the average of the k_i's is $k_{ave}\leq 1/2$ or, in terms of the twist

$$\mathrm{t}_{ave}\leq\hat{c}_2\left(1-\frac{1}{2^2}\right)+\frac{1}{8}. \tag{3.61}$$

Interestingly enough, this value for the twist has a very nice gravitational interpretation in terms of conical orbifolds [21], which will be reviewed later in the chapter and generalised to $\mathcal{W}_N$.

All of these results agree with what was found in [21] for the Virasoro case.

3.3.2 Curing the $\mathcal{W}_3$ negativities

W_3 has a more complex vacuum structure, which leads to somewhat more complex calculations, but we will see how nicely the results from the previous section generalise to this case.

The vacuum character for an irrational $\mathcal{W}_3$ theory is

$$\chi_0(\tau) = \frac{e^{-2\pi i\tau\hat{c}_3}}{\eta(\tau)^2}(1 - 2q + 2q^3 - q^4)\,, \tag{3.62}$$

which in our notation (3.40) translates to

$$c_3(0) = 1,\ c_3(1) = -2,\ c_3(2) = 0,\ c_3(3) = 2,\ c_3(4) = -1\,. \tag{3.63}$$

The degeneracy function for $\mathcal{W}_3$ becomes [1]

$$D(3,k,j) = \sum_{i=0}^{4} \frac{c_3(i)c_3(k)\,K(j,k-i,3)}{3^2}\,. \tag{3.64}$$

In this case, the dominant term in the Kloosterman sum expansion is $s = 3$ and hence we will want to study the degeneracies (3.64) for spin $j = 1, 2 \mod 3$ and shift $k = 0, 1, 2, 3, 4$ to understand where the negativities arise and how severe they are. Those results are summarised in Table 3.2.

Let us analyse the two relevant limits and the negativities we should cure.

a) $j \gg c$ limit:

We only need to add one operator with twist $\mathrm{t} \leq \hat{c}_3\left(1 - \frac{1}{3^2}\right)$, as in this limit the density is dominated by the largest exponential (which is $j = 1 \mod 3$).

b) $c \gg j$ limit:

We need to add three operators to cure the all the negativities (due to both $j = 1 \mod 3$ and $j = 2 \mod 3$).

We will now illustrate the case $j = 1 \mod 3$ for the reader. We start by adding three operators of twists

$$\mathrm{t}_i = \hat{c}_3\left(1 - \frac{1}{3^2}\right) + \frac{k_i}{9} + \mathcal{O}(c^{-1})\,. \tag{3.65}$$

k	Multiplying what?	$D(3,k,0 \bmod 3)$	$D(3,k,1 \bmod 3)$	$D(3,k,2 \bmod 3)$	Twist to add
0	$\exp\left(\frac{4\pi\sqrt{\hat{c}_3 j}}{3}\right)$	$+1$	-1	0	$\hat{c}_3\left(1-\frac{1}{3^2}\right)$
1	$\exp\left(\frac{4\pi\sqrt{(\hat{c}_3-1)j}}{3}\right)$	$+2$	0	-2	$\hat{c}_3\left(1-\frac{1}{3^2}\right)+\frac{1}{9}$
2	$\exp\left(\frac{4\pi\sqrt{(\hat{c}_3-2)j}}{3}\right)$	0	0	0	$\hat{c}_3\left(1-\frac{1}{3^2}\right)+\frac{2}{9}$
3	$\exp\left(\frac{4\pi\sqrt{(\hat{c}_3-3)j}}{3}\right)$	$+2$	-2	0	$\hat{c}_3\left(1-\frac{1}{3^2}\right)+\frac{3}{9}$
4	$\exp\left(\frac{4\pi\sqrt{(\hat{c}_3-4)j}}{3}\right)$	$+1$	0	-1	$\hat{c}_3\left(1-\frac{1}{3^2}\right)+\frac{4}{9}$
Average:		(positive)	$\frac{-1(0)-2(\frac{3}{9})}{3}=-\frac{2}{9}$	$\frac{-2(\frac{1}{9})-1(\frac{4}{9})}{3}=-\frac{2}{9}$	$\hat{c}_3\left(1-\frac{1}{3^2}\right)+\frac{2}{9}$ (three states)

Table 3.2: This table presents the information about the negative-norm states in the $\mathcal{W}_3$ vacuum character's modular sum. The entries in the third, fourth, and fifth columns evaluate the degeneracy factor $D(3,k,j)$ for the values indicated. The second column indicates the factor multiplying the degeneracy in the partition function at large j or large c, and the last column indicates what twist a state should have so that its addition would counteract the negativity at each k,j combination. We see there are negative norm states for spin $j \equiv 1,2 \pmod 3$. In the limit of large c and finite j we can cure the negativities by introducing three states with $\mathrm{t} \leq \hat{c}_3\left(1-\frac{1}{3^2}\right)+\frac{2}{9}$.

With these added, the overall density of states becomes

$$\rho^{(3)}_{vac}(e,j) \sim e^{\frac{4\pi}{3}\sqrt{\hat{c}_3 j}}\left((4\pi - \frac{2\pi(k_1+k_2+k_3)}{3})\sqrt{\frac{j}{\hat{c}_3}} + \mathcal{O}\left(c^{-1}\right)\right). \tag{3.66}$$

We see how we need to add three operators with an average shift $\frac{k_1+k_2+k_3}{3} \leq 2$ or, in terms of the twist

$$\mathrm{t}_{ave} \leq \hat{c}_2\left(1-\frac{1}{3^2}\right)+\frac{2}{9}. \tag{3.67}$$

Results are similar for the case $j = 2 \mod 3$.

There is a caveat, however: if we were to add less than three states, regardless of their twist, at sufficiently large central charge c the density of states would become negative.

3.3.3 Curing the $\mathcal{W}_N$ negativities

For general $\mathcal{W}_N$, in the regime where $j \gg c$, we will be required to add one operator of twist $\mathrm{t} \leq \hat{c}_N\left(1-\frac{1}{N^2}\right)$ to cure the negativities, generalising from what we have seen for the Virasoro and $\mathcal{W}_3$ cases[7]

[7] We actually looked in particular up to $N = 6$ for the general expressions in both limits.

In the case of $c \gg j$, we will be dealing with the degeneracies of the form

$$D(N,k,j) = \sum_{i=0}^{\frac{N^3-N}{6}} \frac{\mathrm{c}_N(k)\mathrm{c}_N(i)\, K(j,k-i,N)}{N^{N-1}}, \tag{3.68}$$

since for all of them the leading term is going to be the term $s = N$, as described before. With this, we can actually obtain the total amount of states we need to add at leading order to correct the negativities. For that, we minimise -since we are looking for the most negative value- the degeneracies over the spin:

$$\min_j \sum_{k=0}^{\frac{N^3-N}{6}} D(N,k,J) = -\frac{N\phi(N)}{GCD(\phi(N),N-1)}\,, \tag{3.69}$$

where $\phi(N)$ is the Euler totient function. Once we have this, we can actually compute the average twist necessary for curing the negativities. We compute it via the shift from the classical central charge, named k_{ave} , which can be found to be [1]

$$k_{ave} = \left(-\frac{N\phi(N)}{GCD(\phi(N),N-1)}\right)^{-1} \sum_{i=0}^{\frac{N^3-N}{6}} \sum_{k=0}^{\frac{N^3-N}{6}} \left(\frac{\mathrm{c}_N(k)\mathrm{c}_N(i)\, K(j,k-i,N)}{N^{N-1}}\right) \times \left(\frac{k}{N^2}\right) = \frac{N^2-1}{12N}\,. \tag{3.70}$$

With this shift, we find that we will need the added states to have an average twist of [1]

$$\mathrm{t}_{ave} \leq \hat{c}_N \left(1 - \frac{1}{N^2}\right) + \frac{N^2-1}{12N}\,, \tag{3.71}$$

which coincides with the result that will be found from the gravity side.

3.4 Conical defect orbifolds in Higher Spin gravity

We have just discussed in the previous sections how the partition functions of pure higher-spin gravity in AdS_3 suffer from negative-norm states, and argued we should add some extra operators to the partition function.

In particular, from a gravity viewpoint, those states lie below the BTZ threshold, and their

twist behaves as

$$\mathrm{t}_M = \hat{c}_N \left(1 - \frac{1}{M^2}\right) + \ldots, \quad M \geq N\,. \tag{3.72}$$

Can we interpret those from a gravity perspective? Actually, yes, and we wouldn't be the first people to do so.

Maloney and Witten, in their seminal paper on three-dimensional gravity [12], studied the possible topologies one should add to the gravity path integral. In particular, one looks for subsets of $\mathbb{CP}^1$ that can construct the confomal boundary of AdS_3. That manifold, denoted as $\mathcal{M}$, is isomorphic to $\mathcal{M} \cong AdS_3'/\Gamma$, where Γ is a discrete subgroup of $SO(3,1)$, and AdS_3' is the section of AdS_3 where Γ's action is discrete. If one then chooses $\Gamma \cong \mathbb{Z} \times \mathbb{Z}_M$, one finds that the resulting object is actually an orbifold with a conical defect.

Initially, those solutions were discarded as traditionally one would define *pure* gravity as only admitting manifold topologies, not orbifolds. However, progress over the years, in particular the finding of negative-normed states by Benjamin et al [14], corroborated by Alday and Bae using a different construction [19], led the field to the idea of challenging the previous definitions of pure gravity: should we maybe relax it a little bit?

In that line, Benjamin et al [21] proposed precisely for pure 'normal'-spin gravity in AdS_3 to include these conical orbifolds in the sum over topologies. They are, after all, easy to interpret from a gravitational viewpoint: they represent massive particles!

In particular, if our orbifold is $\mathbb{Z} \times \mathbb{Z}_M$, we find that there is a direct relation between the deficit angle and the mass of the particle (at the classical level) [21]:

$$\Delta\theta_M \approx 8\pi G_N m\,, \tag{3.73}$$

where the deficit angle is defined as $\Delta\theta_M = 2\pi\left(1 - \frac{1}{M}\right)$.

In this section we will consider conical singularities in the Euclidean partition function of higher-spin AdS_3 gravity. One should note that these exact conical defects are added to the partition function in *top-down* constructions of AdS_3 gravity. More precisely, the authors of [65] showed how conical defect geometries show up in the partition function of strings on $AdS_3 \times S^3 \times T^4$, and they are twisted sector ground states of the symmetric product orbifold. Furthermore, this identification of conical defect geometries was made in a more precise way for the tensionless

limit of string theory in [66].

Let us start analysing the higher spin case in our 'bottom-up' approach. The end-goal is to compute the one-loop Euclidean partition function when considering quadratic fluctuations around thermal AdS_3,

$$Z(q) = \int [\mathcal{D}\varphi] e^{-S[\varphi]} \,, \tag{3.74}$$

where we are using $q = e^{2\pi i \tau}$, where τ is the complex modulus of the boundary torus of thermal AdS_3.

We will start by summing only over smooth geometries. Smoothness in this context is interpreted a bit differently than in higher dimensions, due to three-dimensional gravity being topological: higher-spin AdS_3 gravity is described in term of $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$ connections, so smoothness of the metric is replaced by triviality of holonomies around the thermal circle, due to the former not being a gauge invariant concept.

In (3.74), $S[\varphi]$ represents the action of a tower of massless higher spin fields $\varphi(s)$ with spins ranging from 2 to N. In the simplest case, $S[\varphi_{(2)}]$ represents the usual gravity action on AdS_3, whilst $S[\varphi_{(s)}]$ is the Fronsdal action for a massless s-spin field in AdS_3. As a result, the overall partition function can be constructed as the product of the individual s-spin partition functions.

$$Z(q) = \prod_{s=2}^{N} Z^{(s)}(q) \,. \tag{3.75}$$

The one-loop partition function $Z^{(s)}(q)$ was computed for spin two in [67] and for generic spin s in [60]:

$$Z^{(s)}(q) = (q\bar{q})^{-\frac{1}{24}\epsilon(s)} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)(1-\bar{q}^n)} \,, \quad q = e^{2\pi i \tau} \,, \tag{3.76}$$

where $\epsilon(s)$ represents a one-loop shift on the central charge of the theory. This shift takes the value $\epsilon(2) = 13$ in the case of spin two [17, 67]. Values for larger spin cases will be given later.

Adding all of this information in the expression for the partition function, we find that it can be written as, including both tree-level and one-loop terms.

$$Z(q) = (q\bar{q})^{-\frac{c_g}{24} + \frac{shift}{24}} \prod_{s=2}^{N} \prod_{n=s}^{\infty} \frac{1}{(1-q^n)(1-\bar{q}^n)} \sim \chi^{(N)}_{vac}(q) \chi^{(N)}_{vac}(\bar{q}) \,, \tag{3.77}$$

where c_g is the *gravity* central charge arising from a Brown-Henneaux procedure.

This result agrees with the vacuum character of the $\mathcal{W}_N \times \bar{\mathcal{W}}_N$ algebra, up to a one-loop shift (denoted as 'shift'). We can then compute the one-loop partition function via the Heat Kernel method. In order to introduce this method, we will first reproduce (3.76), and then we will use this baseline and the method of images to extend the result obtained to the conical singularity orbifolds.

3.4.1 A gentle introduction to the heat kernel method

The main goal we have is finding the partition function for a quantum gravity theory embedded in an asymptotically AdS space-time, at a given temperature and angular potential. We will follow [67] and, rather than follow the traditional saddle-point approximation, we will compute the one-loop determinant $\det \Delta^{(2)}$ of the Laplacian operator for graviton fluctuations around the background metric.

Let us start by expressing a partition function we want to compute at one loop as

$$Z = \int \mathcal{D}\varphi e^{-1\frac{1}{\hbar^2}\int_{\mathcal{M}} \mathrm{d}^3x\sqrt{g}\varphi\Delta\varphi}\,, \tag{3.78}$$

where φ is a free scalar field and Δ the corresponding Laplacian operator on $\mathcal{M}$. As this is a Gaussian integral, we can give the result in terms of the determinant of Δ.

When one is working in a compact manifold, a complete, orthonormal basis of eigenfunctions for the Laplacian operator exists:

$$\Delta\varphi_n(x) = \lambda_n\,\varphi_n(x)\,, \tag{3.79}$$

such that

$$\sum_n \varphi_n(x)\varphi_n(y) = \delta(x,y) \qquad \int_{\mathcal{M}} \mathrm{d}^3x\sqrt{g}\varphi_m(x)\varphi_n(x) = \delta_{mn}\,, \tag{3.80}$$

with $\delta(x,y)$ the appropriate delta function on $\mathcal{M}$. We have also assumed that the basis is real for simplicity.

Using these, we can write the determinant as

$$S^{(1)} = -\frac{1}{2}\log\det\Delta = -\frac{1}{2}\sum_n \log\lambda_n\,. \tag{3.81}$$

Now we would like to use the heat kernel method to make the computation, as it is immensely powerful. One can find its computation for the 'normal'-spin gravity case in [67], and the general spin $s \geq 2$ case was studied in [60, 68].

The keat kernel can be defined as

$$K(t;x,y) = \langle y|e^{-t\Delta}|x\rangle = \sum_n \varphi_n(x)\varphi_n(y)e^{-\lambda_n t}\,. \tag{3.82}$$

If we now use the completeness relation of the basis, we can deduce that the determinant can be written in terms of the heat kernel.

$$S^{(1)} = \frac{1}{2}\lim_{\alpha\to 0}\partial_\alpha \int_0^\infty \mathrm{d}t \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_{\mathcal{M}} d^3x\sqrt{g}K(t;x,x)\,. \tag{3.83}$$

The reason we want to express the determinant directly in terms of the heat kernel is that this provides an easier way to compute the former. The heat kernel can be defined as a unique solution to a heat-like differential equation:

$$(\partial_t + \Delta_x)\,K(t;x,y) = 0, \quad K(0,x,y) = \delta(x,y)\,. \tag{3.84}$$

Before proceeding to studying the case of interest, there are a few considerations in place.

First, the heat kernel method applies to both spinless and spinful fields: in the latter case, we must supplement the heat kernel with spin indices, ie $K_{a,b}(t;x,y)$, where a,b are the labels for the $2s+1$ spin representation in use.

Next, we should note that H_3^+ (corresponding to AdS_3) can be understood as a coset homogeneous space $H_3^+ = SL(2,\mathbb{C})/SU(2)$, where we can identify a field of arbitrary spin s with a section of homogeneous vector bundles on H_3^+. This allows us to write the eigenfunction of the spin-s Laplacian as matrix elements of representations of $SL(2,\mathbb{C})$. But this poses a big problem! H_3^+ is a non-compact manifold, and we have previously assumed that the heat kernel was applied in a compact manifold. However, this is not fatal[8]: we can tackle this issue by allowing the basis of eigenfunctions of the Laplacian to be non-discrete. We will always have a divergent factor in the computation of the determinant, proportional to the volume of the

[8]Interestingly enough, one can also obtain the H_3^+ heat kernel through an analytic continuation from S^3 [68].

manifold in question. This divergence can be regularised through a standard procedure [17].

Lastly, we should note that we are starting with the baseline of AdS_3 and then move through to the cases of interest, as once we know the kernel for a manifold $\mathcal{M}$, it is pretty simple to determine it for any quotients $\mathcal{M}/\Gamma$, using the method of images.

The heat kernel in H_3^+

We are going to slowly build our heat kernel from the very starting point: H_3^+,and we will follow [68]. The general heat kernel for spin s in this background can be expressed as an integral (due to the continuous basis in a non-compact manifold):

$$K_{ab}^{(s)}(t;x,y) = \int_0^\infty \mathrm{d}\lambda \frac{1}{2\pi^2}(2-\delta_{s,0})\frac{\lambda^2+s^2}{2s+1}\phi_{\lambda,ab}^{(s)}(x,y)\, e^{-t(\lambda^2+s+1)}\,, \tag{3.85}$$

where the ϕ's are the matrix elements of the representation labelled by (λ, s) projected onto the spin s representation of $SU(2)$. There are a couple of simplifications that can help us: we are actually interested in the so-called *coincident, integrated* heat kernel, defined as follows

$$2S^{(1)} = \log\det(-\Delta^{(s)}) = \mathrm{Tr}\ \log(-\Delta^{(s)}) = -\int_0^\infty \frac{\mathrm{d}t}{t}\int \sqrt{g}\,\mathrm{d}^3x\, K_{aa}^{(s)}(x,x;t)\,. \tag{3.86}$$

It is precisely our target when looking for the determinant of the Laplacian [68].

Note that the *coincident*-only heat kernel is the same expression but without the integration over t. This can actually be computed from (3.85) and yields the following expression [68]:

$$K_{aa}^{(s)}(t) = \frac{1}{(4\pi t)^{\frac{3}{2}}}(2-\delta_{s,0})(1+2s^2t)e^{-t(s+1)}\,. \tag{3.87}$$

This is, as the authors of [68] remark, in agreement with the results of [67] for $s = 1, 2$. The authors of [67] followed a different approach, however their techniques are not easily extendable to arbitrary spin.

We will follow the procedures of [60, 68], which allow for a more straightforward generalisation: they rely on group theoretical properties of the Laplacian's eigenfunctions.

The heat kernel for Thermal AdS_3

We want to obtain the heat kernel for a field of spin s embedded in thermal Euclidean AdS_3, which corresponds to $H_3^+/\mathbb{Z}$. To do that, we will use the method of images, starting from the H_3^+ kernel [68] we just computed (3.87). To understand how we perform the method of images, we should before introduce the metric and action of $\mathbb{Z}$ on AdS_3.

We will start by presenting the metric of H_3^+

$$\mathrm{d}s^2 = \frac{\mathrm{d}z\mathrm{d}\bar{z} + \mathrm{d}y^2}{y^2}, \quad y > 0\,. \tag{3.88}$$

The generator of $\mathbb{Z}$, which we denote as γ acts on an arbitrary point as

$$\gamma \cdot (y,\, z,\, \bar{z}) = (|q|^{-1}y,\, q^{-1}z,\, \bar{q}^{-1}\bar{z})\,, \tag{3.89}$$

with $q = e^{2\pi i\tau}$, as before. This transformation leaves the metric unchanged, but it does correspond to a global identification leading to the boundary becoming a torus of modulus $\tau = \tau_1 + i\tau_2$.

Now, following the geometrical approach of [68], we find

$$K^{(s)}(t;\tau,\bar{\tau}) = \sum_{m\in\mathbb{Z}}\sum_{a}\int_{H_3^+/\mathbb{Z}} \mathrm{d}^3x\sqrt{g}K_{aa}^{(s)}(t;x,\gamma^m\cdot x)\,, \tag{3.90}$$

where the sum over m corresponds to the sum over images (related to the quotient H_3^+/γ). Again, following [68], we can use the relation between H_3^+ and $SL(2,\mathbb{C})$ and obtain the expression for the heat kernel in terms of $SL(2,\mathbb{C})$ characters $\chi_{j_1,j_2}(\alpha)$ of the diagonal group element $diag(\alpha,\alpha^{-1}) \in SL(2,\mathbb{C})$:

$$\chi_{j_1,j_2}(\alpha) = \frac{\alpha^{2j_1+1}\,\bar{\alpha}^{2j_2+1} + \alpha^{-2j_1-1}\,\bar{\alpha}^{-2j_2-1}}{(\alpha-\alpha^{-1})(\bar{\alpha}-\bar{\alpha}^{-1})}\,. \tag{3.91}$$

Taking these, we can rewrite the heat kernel as

$$K^{(s)}(t,\tau,\bar{\tau}) = K_0^{(s)}(t) + \tau_2\sum_{m\in\mathbb{Z}/\{0\}}\int_0^\infty \mathrm{d}\lambda\left(\chi_{j_1,j_2}(e^{m\pi i\tau}) + \chi_{j_2,j_1}(e^{m\pi i\tau})\right)e^{-t(\lambda^2+s+1)}\,, \tag{3.92}$$

with $2j_1 = s-1+i\lambda$ and $2j_2 = -s-1+i\lambda$. Note that the $m=0$ contribution has been separated: it is proportional to the volume of thermal AdS_3 and therefore divergent. This term

is the product of the coincident integrated traced heat kernel for H_3^+ (3.87) times the volume of $H_3^+/\mathbb{Z}$. One can regularise the volume following [17], where we find vol $\left(H_3^+/\mathbb{Z}\right) = -\tau_2\pi^2$. With all of that, the $m = 0$ contribution is [68]

$$K_0^{(s)}(t) = \frac{\text{vol}\left(H_3^+/\mathbb{Z}\right)}{(4\pi t)^{3/2}}(2-\delta_{s,0})(1+2s^2t)e^{-t(s+1)}\,. \tag{3.93}$$

We now integrate over λ for the $m > 0$ cases, and obtain the following result:

$$K^{(s)}(t,\tau,\bar{\tau}) = K_0^{(s)}(t) + \tau_2\sum_{m=1}\frac{\cos(2\pi sm\tau_1)}{\sin(m\pi\tau)\sin(m\pi\bar{\tau})}\frac{\sqrt{\pi}}{\sqrt{t}}e^{-\frac{\pi^2m^2\tau_2^2}{t}-t(s+1)}\,. \tag{3.94}$$

Armed with all the results, one can find the spin s contribution to the partition function in [60], including ghost contributions,

$$\begin{aligned}\log Z^{(s)}(\tau) &= -\frac{1}{2}\log\det\left(\Delta_s - s(s-3)\right) + \frac{1}{2}\log\det\left(\Delta_{s-1} - s(s-1)\right)\\ &= \frac{1}{2}\lim_{\alpha\to 0}\partial_\alpha\int_0^\infty \mathrm{d}t\,\frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(K^{(s)}(t,\tau,\bar{\tau})e^{-s(s-3)t} - K^{(s-1)}(t,\tau,\bar{\tau})e^{-s(s-1)t}\right)\\ &= -\tau_2\,\pi\frac{6s-6s^2-1}{6} + \sum_{m=1}^\infty\frac{1}{m}\left(\frac{q^{ms}}{1-q^m} + \frac{\bar{q}^{ms}}{1-\bar{q}^m}\right)\,.\end{aligned} \tag{3.95}$$

Undoing the logarithm, we find

$$Z^{(s)}_{1-loop}(\tau) = (q\bar{q})^{-\frac{1}{24}(6s^2-6s+1)}\prod_{n=s}^\infty\frac{1}{(1-q^n)(1-\bar{q}^n)}\,, \tag{3.96}$$

which is precisely the result that we expected from (3.76).

We can now move on to computing the overall partition function, by taking the product of (3.96) at spins $s = 2, 3, \ldots, N$:

$$Z(q) = (q\bar{q})^{-\frac{1}{24}(c_g-1-N+2N^3)}\prod_{s=2}^N\prod_{n=s}^\infty\frac{1}{(1-q^n)(1-\bar{q}^n)}\,. \tag{3.97}$$

This partition function is equivalent to what we would obtain if we consider the vacuum $\mathcal{W}_N$, but with a shift in the central charge:

$$c = c_g - 1 - N + 2N^3\,, \tag{3.98}$$

for c_g the classical central charge, which one can obtain through the Brown-Henneaux procedure.

One last interesting comment: supersymmetry will change this shift! For more details, we encourage the reader to consult [1, Appendix C], but the main results are that

- If we consider $\mathcal{N} = (1,1)$, the relationship between c and c_g is of order N^2.
- If we consider $\mathcal{N} = (2,2)$, the relationship between c and c_g is of order N.
- If we consider more supersymmetry, we will find $c = c_g$.

The aim of this SUSY computation is to understand how our ideas extended to SUSY, rather than to check for consistency between the CFTs and chiral algebras. As one can see in [69, 70], unitarity places nontrivial bounds on these with finite N and large c.

3.4.2 The heat kernel for AdS_3 with conical defect singularities

Let us extend the heat kernel to the case in which the space-time has conical singularities.

Before getting into details, we should talk about the precise conical singularities we are allowing here. As mentioned before, when adding higher spin fields to a gravity theory, the concept of smoothness gets redefined to that of trivial holonomy for the $SL(N,\mathbb{R})$ connection. In [71], the authors investigated conical defects in higher spin gravity, and those singularities had trivial holonomy thus resulting *smooth*. However, those theories are not admissible unitary theories (despite smooth), considering unitarity limits proposed in [63]. We are not working with those: our singularities don't have a trivial holonomy themselves, but rather it is their holonomy to the Mth power that is trivial; and furthermore they are unitary.

We are interested in a background given by $H_3^+/(\mathbb{Z} \times \mathbb{Z}_M)$, which means we now have two generators acting as

$$\begin{aligned} \gamma_1 \cdot (y, z, \bar{z}) &= (|q|^{-1/M} y, q^{-1/M} z, \bar{q}^{-1/M} \bar{z}) \,, \\ \gamma_2 \cdot (y, z, \bar{z}) &= (y, e^{-2\pi i/M} z, e^{2\pi i/M} \bar{z}) \,, \end{aligned} \tag{3.99}$$

with $\gamma_2^M = 0$.

We now want to apply the method of images for AdS_3 with a conical defect of angle $2\pi(1 -$

$1/M$), and this leads to an expression

$$K_M^{(s)}(t,\tau,\bar{\tau}) = K_{M,m=0}^{(s)}(t) + \frac{\tau_2}{M^2}\sum_{n=1}^{M}\sum_{m\in\mathbb{Z}/\{0\}}\int_0^\infty \mathrm{d}\lambda\Big(\chi_{j_1,j_2}(e^{\pi i\tau\frac{m}{M}}e^{\pi i\frac{n}{M}}) + \chi_{j_2,j_1}(e^{\pi i\tau\frac{m}{M}}e^{\pi i\frac{n}{M}})\Big)e^{-t(\lambda^2+s+1)}\,. \quad (3.100)$$

The values of $j_{1,2}$ are the same as before. The $m=0$ contribution has been separated due the need for regularisation of the volume of $\frac{H_3^+}{\mathbb{Z}\times\mathbb{Z}_M}$. We are going to expand this piece first:

$$\begin{aligned} K_{M,s=0}^{(s)}(t) &= \frac{\mathrm{vol}\left(\frac{H_3^+}{\mathbb{Z}\times\mathbb{Z}_M}\right)}{(4\pi t)^{3/2}}(2-\delta_{s,0})(1+2s^2t)e^{-t(s+1)} + \frac{\sqrt{\pi}\tau_2}{2\sqrt{t}M^2}\sum_{n=1}^{M-1}\frac{\cos\left(\frac{2\pi sn}{M}\right)}{\sin^2\left(\frac{n\pi}{M}\right)}e^{-t(s+1)} \\ &= \left(\frac{\mathrm{vol}\left(\frac{H_3^+}{\mathbb{Z}\times\mathbb{Z}_M}\right)}{(4\pi t)^{3/2}}(2-\delta_{s,0})(1+2s^2t) + \frac{\sqrt{\pi}\tau_2}{6\sqrt{t}M^2}(6s^2+M^2-6Ms-1)\right)e^{-t(s+1)}\,. \end{aligned} \quad (3.101)$$

The regularised volume in this case is

$$\mathrm{vol}\left(\frac{H_3^+}{\mathbb{Z}\times\mathbb{Z}_M}\right) = -\frac{\pi^2\tau_2}{M^2}\,. \quad (3.102)$$

We can now compute the $m=0$ piece,

$$K_{M,0}^{(s)}(t) = \frac{\sqrt{\pi}\tau_2\left(2t\left(M^2-6Ms+3s^2-1\right)-3\right)}{12M^2t^{3/2}}e^{-(s+1)t}\,, \quad (3.103)$$

and then move onto the λ integrals for the $m>0$ terms, which yield

$$K_M^{(s)}(t,\tau,\bar{\tau}) = K_{M,0}^{(s)}(t) + \frac{\sqrt{\pi}\tau_2}{2\sqrt{t}M^2}\sum_{n=1}^{M}\sum_{m\in\mathbb{Z}/\{0\}}\frac{\cos\left(\frac{2\pi s}{M}(n+m\tau_1)\right)}{\sin\left(\frac{\pi}{M}(n+m\tau)\right)\sin\left(\frac{\pi}{M}(n+m\bar{\tau})\right)}e^{-\frac{m^2\pi^2\tau_2^2}{M^2t}-(s+1)t}\,. \quad (3.104)$$

Armed with all these results, we can go back to the definition of the one-loop contribution to the partition function and obtain [1]

$$\begin{aligned} \log Z^{(s)}(\tau) &= \frac{1}{2}\lim_{\alpha\to0}\partial_\alpha\int_0^\infty \mathrm{d}t\,\frac{t^{\alpha-1}}{\Gamma(\alpha)}\left(K_M^{(s)}(t,\tau,\bar{\tau})e^{-s(s-3)t} - K_M^{(s-1)}(t,\tau,\bar{\tau})e^{-s(s-1)t}\right) \\ &= \frac{\pi}{6}\tau_2 - \sum_{n=1}^{M}\sum_{m=1}^{\infty}\sum_{\ell=0}^{\infty}\frac{1}{mM}\cos\left(\frac{2\pi n(\ell+s)}{M}\right)\left(q^{\frac{m}{M}(\ell+s)} + \bar{q}^{\frac{m}{M}(\ell+s)}\right)\,. \end{aligned} \quad (3.105)$$

The sum over m is straightforward to compute, and then the sum over n yields

$$\sum_{n=1}^{M} \cos\left(\frac{2\pi n(\ell+s)}{M}\right) = M\delta_{0,\ell+s \pmod M}\,. \tag{3.106}$$

With this result, we should note that $\ell + s = kM$ with $k \in \mathbb{Z}$, and in particular the lowest possible value for k is $\lfloor \frac{s-1}{M} \rfloor + 1$ (can be seen by observation of (3.105)). Taking all these considerations into account, one finally finds [1]

$$Z^{(s)}_{1-loop}(\tau) = (q\bar{q})^{-\frac{1}{24}} \prod_{k=\lfloor \frac{s-1}{M} \rfloor + 1} \frac{1}{(1-q^k)(1-\bar{q}^k)}\,. \tag{3.107}$$

Interestingly enough, the dependence on both M and the spin s is only through the floor function that indicates the starting value for k.

Now that we have the generic expression, let us find the total contribution from fields of spin $s = 2, 3, \ldots, N$ to the one-loop partition function [1].

$$Z^{(s),N}(\tau) = (q\bar{q})^{-\frac{c_g}{24M^2} - \frac{N-1}{24}} \prod_{s=2}^{N} \prod_{k=1}^{\infty} \frac{1}{|1-q^k|^2} = \frac{(q\bar{q})^{-\frac{c}{24M^2}}}{|\eta(\tau)|^{2(N-1)}}\,, \tag{3.108}$$

This partition function is exactly that of a $\mathcal{W}_N$ non-degenerate character of a scalar operator. If we call c the central charge for the CFT, we can compare it to c_g, which is the gravitational central charge obtained through the Brown-Henneaux procedure (3.98). From this comparison, we find that the conformal weight of the corresponding CFT operator is

$$h - \hat{c}_N = -\frac{c_g}{24M^2}\,. \tag{3.109}$$

This result is obtained from the comparison of (3.108) with the partition function of a $\mathcal{W}_N$ scalar seed expanded in terms of non-degenerate $\mathcal{W}_N$ characters. Let us now re-write (3.109) in terms of $\hat{c}_N$, obtaining

$$h = \hat{c}_N\left(1 - \frac{1}{M^2}\right) + \frac{N^3 - N}{12M^2}. \tag{3.110}$$

The first allowed conical defect used to correct negativities can be found by setting $M = N$ in this expression, which is in agreement with the CFT results in the $c \gg 1$, $c \gg j$ regime we found previously.

We have made an assumption before that we will only add terms with $M \geq N$, but is there a reason for that?

To answer that question, let us suppose we would like to add an orbifold term that has $M < N$. If we want to do that, we will need to add terms proportional to $(1-q)(1-\bar{q})$ in (3.108) to be able to write the overall partition function in terms of non-degenerate $\mathcal{W}_N$ characters. What that implies is that we are effectively adding two scalar operators with positive degeneracy, but also two operators of $J = \pm 1$ with negative degeneracy, thus violating unitarity.

As a result, we are forced from the gravity side to have $M \geq N$, which is in agreement with our results from Section 3.3.

Finally, we should make a comment regarding how far the *curing* of the negativities actually goes.

Let us use $\mathcal{W}_2$ as an example, and any results will extend in a similar fashion to general $\mathcal{W}_N$. We are considering the regime $c \gg j$ as well.

The first allowed conical defect operator in the gravity has a twist

$$\mathrm{t} = \hat{c}_N \left(1 - \frac{1}{N^2}\right) + \frac{N^2-1}{12N}\,, \tag{3.111}$$

in agreement with the average twist of operators found in the gravity side.

The density of states (with this first correction to cure negativities) in that scenario is

$$\rho_j(e) \sim -e^{2\pi\sqrt{\left(\frac{c-1}{24}\right)j}} - e^{2\pi\sqrt{\left(\frac{c-1}{24}-1\right)j}} + e^{2\pi\sqrt{\left(\frac{c-1}{24}-\frac{1}{2}+\delta\mathrm{t}_1\right)j}} + e^{2\pi\sqrt{\left(\frac{c-1}{24}-\frac{1}{2}+\delta\mathrm{t}_2\right)j}}\,, \tag{3.112}$$

where $\delta\mathrm{t}_1$ and $\delta\mathrm{t}_2$ are higher loop corrections to the twist of the conical defects we are adding. By corrections, we are referring to both any higher order corrections to the precise relation between c the CFT central charge and c_g the gravitational central charge; and also quantum corrections that affect the conical defects themselves.

We want to understand how the higher loop corrections can affect the overall negativity problem, depending on their explicit dependence on the central charge.

- $\underline{\delta\mathrm{t}_i \sim \frac{1}{c}}$:

  In this case, which is what one would naively expect, we can see that adding two conical defects above would not cure the negativity problem.

- $\underline{\delta \mathrm{t}_i > \frac{1}{c^{1/2}}}$:

  In this case, the quantum corrections become sufficiently important that the negativity could be cured.

Although it is not very common, there are plenty of reasons why a theory could have non-analytic terms in the $1/c$ expansion, for instance terms arising from higher dimensional embeddings that are necessary to make the theory UV complete, or that come from regularising divergences.

3.5 Conclusions

This chapter's goal was to construct the density of states for a $\mathcal{W}_N$ CFT dual to higher spin gravity. By using the Poincaré and Rademacher constructions, we found that both constructions uplift issues already present in the Virasoro theory: the spectrum is continuous and has negative norm states.

To cure the negativities, we proposed adding states of twist

$$\mathrm{t} \leq \frac{c}{24}\left(1 - \frac{1}{N^2}\right) + \mathcal{O}(1)\,, \tag{3.113}$$

which solve the issue in a rather natural manner from a CFT perspective, and are also consistent in the gravitational path integral provided one modifies the definition of *pure* gravity to allow conical orbifold topologies in the path integral.

Furthermore, the agreement between both sides goes beyond the classical level: we get the same one-loop order shift when demanding positivity of all spins (including $j \ll c$). This includes one-loop correction to the relation between the CFT and the gravity central charge

$$c = c_g + 2N^3 - N - 1 + o(1)\,. \tag{3.114}$$

However, all of this is not without its challenges, and in particular

- We don't know whether or not these Higher Spin theories are indeed too constrained to exist: we have imposed a (certainly non-trivial) constraint by demanding modular invariance of the partition function, but there are plenty of other ways in which further

constraints (mainly discreteness/integrality of the spectrum) could push these to an extent that the theory is indeed too constrained to exist.

- We have proposed here a particular method of solving the negativities, but it is worth mentioning that there are many other proposals for the Virasoro case, as mentioned before, and it is likely that one could extend some of those to some extent to the higher-spin case. At the moment of writing, there doesn't seem to be an argument that would let us decide firmly on the best method of those available.

- The issue of higher-loop corrections on the path integral: this could be crucial, as we have seen, to ensuring that this correction holds at all loops. However, it is remarkably complicated to access higher-order loops for this theory, due to (for instance) difficulties when trying to formulate the Chern-Simons action for gravity with higher-spin fields, as its gauge group is $SL(N,\mathbb{R}) \times SL(N,\mathbb{R})$, non-compact.

Chapter 4

Spectral decomposition of Rational CFTs' partition function

Rational Conformal Field Theories (RCFTs) are two-dimensional Conformal Field Theories which have an extended symmetry algebra, larger than Virasoro, under which they have a finite number of primaries. As a result, their partition function can be expressed with a finite number of blocks (each one being the product of a holomorphic and an antiholomorphic term). RCFTs may have a finite number of primaries under Virasoro (if their central charge $c < 1$), or an infinite number (if $c > 1$) [72]. The focus of this chapter, which showcases some of the results of [2], will be the latter.

A good set of introductory notes in the topic are those by Sen [73].

4.1 Introduction and background

A longstanding goal of theoretical high-energy physics has been to classify which theories we can form which are consistent not only mathematically but also physically. In that line, one usually wants to explore spaces of theories that are highly constrained. as that makes it easier to classify them. Rational Conformal Field Theories are of special interest precisely for that reason:

- They have a finite number of characters under which one can write the torus partition function: can we classify them in terms of the number of characters?

- They have a larger symmetry group than Virasoro, which is in many cases a Lie group: can we form a classification in terms of their affine algebra and the different relations between different algebras?

- Their characters are vector-valued modular forms, and their partition function are modular functions: how do those constraints translate to their explicit formulations?

If the central charge $c < 1$, we have a complete classification[1]. First introduced in BPZ [5], these theories are usually referred to as *minimal models*. They have a finite amount of primaries not only under the extended algebra coming from the larger symmetry group but also under the Virasoro algebra. Minimal models have also been fully classified through an ADE classification, conjectured in [74] and proven in [75, 76].

What happens when $c \geq 1$? Well, there are a set of theories where the extended algebra corresponds to an affine Lie group, usually referred to as WZW or WZNW models [31–35]. These theories are constructed via a coset procedure, but there are other alternatives, such as constructions from lattices [77–79].

However, not every RCFT constructed via the coset procedure is a WZW model [80], and there are also known theories that aren't constructed via either coset or lattice procedures [37,38].

How else can we build and classify RCFTs? Mathur, Mukhi and Sen [36] introduced a method based on the fact that the characters of RCFTs are vector-valued modular forms (VVMFs).

As we established before, the partition function of a Rational CFT consists of a finite number of characters N, and these all satisfy a modular-invariant differential equation of order N [36]: in fact, they are the N independent solutions of said equation.

The equation, widely known nowadays as the MLDE (for *Modular Linear Differential Equation*), can be written as:

$$\mathcal{D}_\tau^n f(\tau) + \sum_{k=0}^{n-1} \phi_k(\tau) \mathcal{D}_\tau^k f(\tau) = 0\,, \tag{4.1}$$

where $\phi_k(\tau)$ is a modular function of weight $2(n-k)$ and $\mathcal{D}$ is the covariant derivative on modular functions of weight r :

$$\mathcal{D} = \frac{\partial}{\partial \tau} - \frac{1}{6} i\pi r E_2(\tau)\,, \tag{4.2}$$

[1] We are assuming here that the theories are unitary and compact.

where $E_2(\tau)$ is the holomorphic Eisenstein series of weight 2. The holomorphic Eisenstein series can be defined as

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n \,, \tag{4.3}$$

where B_{2k} is the Bernoulli number.

The MLDE (4.1) gives rise to two characteristic numbers for a given RCFT: the number of characters n, which will be the same as the order of the equation, and the number of poles l that ϕ_k has. The latter is of crucial importance for uniqueness of solutions, as it controls the zeroes in the characters' Wronskian. More information on that can be found in [36].

Taking all of that into account, the procedure to procure RCFTs works as follows. We first identify how many characters we are looking for, and then we decide on how many poles we allow ϕ_k to have (we decide what Wronskian index l we want to study). Then, we can uniquely determine ϕ_k as any modular function can be expressed as a linear combination of terms such as $(E_4)^a(E_6)^b$, for $a, b \in \mathbb{Z}^+ \cup \{0\}$.

Now that we have set up the specific $MLDE$ we want to solve, constraints on VVMFs will ensure that the n independent solutions to the equation are the n characters of the theory. This allows us to expand the characters as a series in q, and impose that the coefficients should be non-negative integers. This condition comes from the physical interpretation of the coefficients as the degeneracies of each state in the theory. With all these constraints, one can identify uniquely the candidate characters, and then see if they can correspond to a CFT by performing more tests, such as examining the fusion rules of the candidate theory.

That the characters as solutions of the MLDE can be identified uniquely from series coefficients is due to them being vector-valued modular functions of weight zero: transform under a general modular transformation $\tau \to \gamma\tau = \frac{a\tau+b}{s\tau-r}$ with $ar + bs = -1$ and $\gamma \in SL(2,\mathbb{Z})$ as

$$f_i(\gamma\tau) = \mathcal{M}_{ij}(\gamma) f_j(\tau), \tag{4.4}$$

where $\mathcal{M}_{ij}$ is such that $M\,M^\dagger = I$. Then the partition function of a given RCFT of N characters will be modular invariant and have the form

$$Z(\tau,\bar{\tau}) = \sum_{i=0}^{N-1} f_i(\tau) f_i(\bar{\tau}). \tag{4.5}$$

This methodology has been successful for decades, providing many results and a wide classification of theories. Results up to 2019 are summarised in [81], but we want to highlight some more references where significant progress was made [82–97].

However, when increasing both n and l we run into complications: the computations become very intensive, and it is more and more computing-costly and difficult to obtain possible candidate characters with integer coefficients. Having said that, advances in computing power have significantly aided in this pursuit in recent years.

A big goal in the pursuit of a classification of theories of some family, in some dimension is to establish - if possible - a moduli space of theories. In many cases, we have an interest in understanding how theories relate through to each other from their algebraic structure, and in a way this can be lost when working for complex cases of high n and l: especially as the number of characters is not necessarily related to the extended symmetry the theory possesses.

Interesting proposals attempting different ways of constructing new RCFTs include those based on the use of Hecke operators [98], quasi-characters [95, 96] or cosets of meromorphic CFTs [99], amongst others.

In this chapter we would like to present a proposal for a complementary approach to the question of classifying RCFTs by exploiting their modularity in a different way.

The partition function of RCFTs in the torus is a modular form of weight zero, or in other words it is invariant under any and all transformations of the modular group. Luckily for us, certain modular forms can be analysed using harmonic analysis in non-flat backgrounds. This has been well studied in the field of Mathematics for a while now, but only recently people started utilising those techniques for the study of partition functions of CFTs in two dimensions [25, 100–103]. We will study the partition function of known RCFTs using Harmonic Analysis on the fundamental domain of $PSL(2,\mathbb{Z})$, which will lead to interesting relations between them, as well as ideas for further exploration in this field.

4.2 Harmonic Analysis: spectral decomposition of slow growth modular forms

As we mentioned before, when dealing with holomorphic modular functions, we can uniquely describe them in terms of the basis Eisenstein series E_4 and E_6. However, the partition function of CFTs in the torus is a real (or non-holomorphic) modular form, and in that case we do not have access to a complete basis in closed form. We now will introduce the basics of harmonic analysis for symmetric hyperbolic spaces. Two good references for further information on harmonic analysis are [39, 104].

The partition function of two-dimensional CFTs lives in the Fundamental Domain of the Upper Half-Plane $\mathcal{F} = SL(2,\mathbb{Z}) \setminus \mathbb{H}$, where the UHP is defined as:

$$\mathbb{H} = \{x + iy \mid x, y \in \mathbb{R},\, y > 0\}\,. \tag{4.6}$$

From here onward, we use interchangeably the coordinates $(\tau, \bar{\tau}) = (x + iy, x - iy)$ to refer to points in $\mathbb{H}$.

As the partition function lives in $\mathcal{F}$, it can be decomposed in terms of the eigenfunctions of the hyperbolic Laplacian Δ. The goal of this method is to be able to replicate Fourier analysis in flat space for a hyperbolic space. The Laplacian in $\mathbb{H}$ is defined as:

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) . \tag{4.7}$$

This operator has two sets of eigenfunctions. The reason for this is that there are two main kinds of modular forms in $\mathbb{H}$: those that vanish at the cusp $y = i\infty$, known as cusp forms, and those that don't, which we usually just call 'forms'.

- **Maass cusp forms**: These forms have eigenvalue $(1/4 + R_n)$, and their Fourier decomposition[2] is

$$\nu_n(\tau, \bar{\tau}) = \sum_{j>0} a_n\, y^{1/2} K_{iR_n}(2\pi j y) \cos(2\pi j x)\,, \quad \nu_0(\tau, \bar{\tau}) = \sqrt{3/\pi}\,. \tag{4.8}$$

We should remark here that neither the Fourier coefficients a_n nor the eigenvalues R_n are

[2] In this context, Fourier decomposition means the usual flat Fourier series for the variable x.

known analytically, rather we only have numerical approximations for them. If the reader is interested, they can find a detailed discussion on this feature as well as conjectures for these numbers' general behaviour in Sections 3.5.2 - 3.5.5 of [104]. The most comprehensive list of both a_n and R_n can be found in [105], and there exist results with more significant digits up to $n = 5$ in [106].

These functions form an orthonormal basis for the discrete spectrum of the Laplacian.

- **Non-holomorphic Eisenstein series**: These have eigenvalue $s(1-s)$ with $s \in \mathbb{C}$, $\mathbb{R}(s) > 1$. They are defined as

$$\widehat{E}_s(\tau,\bar{\tau}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Im(\gamma\tau))^s\,, \qquad \Gamma_\infty = \pm\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\,. \tag{4.9}$$

In terms of the analytical structure in terms of s, $\widehat{E}_s$ has a pole at $s = 1$ with residue $3/\pi$, but it can be analytically continued to the whole $s-$*plane*.

Their Fourier expansion is given by

$$\widehat{E}_s(\tau,\bar{\tau}) = y^s + \frac{\Lambda(1-s)}{\Lambda(s)} + 2\sum_{j\neq 0} y^{1/2}\frac{|j|^{s-1/2}\sigma_{1-2s}(j)}{\Lambda(s)} K_{s-1/2}(2\pi|j|y)\cos(2\pi ijx)\,, \tag{4.10}$$

where $\sigma_s(j)$ is the divisor function and $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$.

We would want to highlight two further properties of these. Firstly, the set $\{\widehat{E}_s(\tau,\bar{\tau})$, $s = 1/2 + i\mathbb{R}\}$ forms an orthonomal basis for the continuous spectrum of the Laplacian. Secondly, for any value of s, the following functional equation is satisfied:

$$\Lambda(s)\widehat{E}_s(\tau,\bar{\tau}) = \Lambda(1-s)\widehat{E}_{1-s}(\tau,\bar{\tau})\,, \quad \Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)\,. \tag{4.11}$$

Now that we know what the eigenfunctions of the Laplacian look like, we need to define an inner product in $\mathcal{F}$ in order to be able to write the decomposition formula. This is known as the Petersson inner product.

$$(f,g) = (f(\tau,\bar{\tau}), g(\tau,\bar{\tau})) := \int_{\mathcal{F}} \mathrm{d}x\ \frac{\mathrm{d}y}{y^2}\, f(\tau,\bar{\tau})\overline{g(\tau,\bar{\tau})}\,. \tag{4.12}$$

With all the machinery at hand, we can define the Roelcke-Selberg Spectral Resolution of the Laplacian on $L^2(\mathcal{F})$. Let us have any $f(\tau,\bar{\tau}) \in L^2(\mathcal{F})$. We define the hyperbolic Fourier decomposition of f as

$$f(\tau,\bar{\tau}) = \sum_{n\geq 0}(f,\nu_n)\nu_n(\tau,\bar{\tau}) + \frac{1}{4\pi i}\int_{Re(s)=\frac{1}{2}} \mathrm{d}s(f,\widehat{E}_s)\widehat{E}_s(\tau,\bar{\tau})\,. \tag{4.13}$$

We see here why the Maass forms are referred to as the *discrete* part of the spectrum and the non-holomorphic Eisenstein series as the *continuous* part. This distinction is of relevance when f is a partition function: the discrete part of the decomposition corresponds to the contributions of the scalar spectrum of the theory, and the continuous part to the spinful spectrum's contributions.

Great! We now have an analogue to Fourier decomposition in a non-Euclidean space, $\mathcal{F}$, which will hopefully help us further our understanding of non-holomorphic modular functions. In this chapter we are mainly interested in understanding the continuous part of the spectrum, more in particular the product $(f,\widehat{E}_s)$. This is known as the Rankin-Selberg transformation of f [107, 108], as both authors independently developed a technique for calculating it. It involves a (nowadays standard) unfolding trick with the Eisenstein series.

$$\begin{aligned}(f,\widehat{E}_s) = \int_{\mathcal{F}} \mathrm{d}x\,\frac{\mathrm{d}y}{y^2}\, f(\tau,\bar{\tau})\widehat{E}_{\bar{s}}(\tau,\bar{\tau}) &= \int_{\mathcal{F}} \mathrm{d}x\,\frac{\mathrm{d}y}{y^2} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} f(\gamma\tau,\gamma\bar{\tau})\Im(\gamma\tau)^{\bar{s}} \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}} \mathrm{d}x\,\frac{\mathrm{d}y}{y^2}\, f(\tau,\bar{\tau})\Im(\tau)^{\bar{s}} \\ &= \int_0^\infty \frac{\mathrm{d}y}{y^2}\, y^{\bar{s}} \int_{-1/2}^{1/2} \mathrm{d}x\, f(x+iy,x-iy) \\ &= \int_0^\infty \mathrm{d}y\, f_0(y)\, y^{\bar{s}-2}\,,\end{aligned} \tag{4.14}$$

where we have defined $f_0(y)$ to be the zero-th Fourier coefficient of $f(\tau,\bar{\tau})$ with respect to x. This unfolding trick lets us see that the Rankin-Selberg transform of a function f is just the Mellin transform of $f_0(\tau)$. Furthermore, some of the analytic properties of $\widehat{E}_s$ will be inherited by the Rankin-Selberg transform:

- Holomorphic in the s plane for $\Re(s) > 1/2$ except for a pole at $s = 1$.

- The following identity is true for the Rankin-Selberg transform of $f(\tau,\bar{\tau})$:

$$\Lambda(\bar{s})(f,\widehat{E}_s)=\Lambda(1-\bar{s})(f,\widehat{E}_{1-s}). \tag{4.15}$$

This is all great, but the truth is that (4.14) can only attain absolute convergence for $\mathbb{R}(s)>1$ if $f(\tau,\bar{\tau})$ is of rapid decay at the cusp, meaning $f(\tau,\bar{\tau})\sim O(y^{-N})$ for any N as $y\to\infty$. That is why in our previous definitions we asked for $f\in L^2(\mathcal{F})$.

There are many modular functions of rapid decay, however it turns out that the best we can get for the partition functions of our interest is *slow growth* at the cusp. Luckily for us, Zagier developed [39] an alternative definition for the Rankin-Selberg transform for functions of slow growth.

Theorem 1. *Suppose $f(\tau,\bar{\tau})$ is a modular invariant function on $\mathbb{H}$ with slow growth at the cusp $y\to\infty$, namely*

$$f(\tau,\bar{\tau})=\varphi(y)+\mathcal{O}(y^{-N})\,,\quad \textit{at}\ \ y\to\infty\,, \tag{4.16}$$

where $\varphi(y)$ has the form of

$$\varphi(y)=\sum_{i=1}^{\ell}\frac{c_i}{\Gamma(n_i+1)}y^{\alpha_i}\log^{n_i}y\,,\quad c_i,\alpha_i\in\mathbb{C},\ n_i\in\mathbb{Z}_{\geq 0}\,. \tag{4.17}$$

The Rankin-Selberg transformation of $f(\tau,\bar{\tau})$ is defined as

$$(f,\widehat{E}_{\bar{s}})=\int_0^\infty \mathrm{d}y\,(f_0(y)-\varphi(y))\,y^{s-2}\,,\quad Re(s)\gg 0\,. \tag{4.18}$$

Then, $(f,\widehat{E}_{\bar{s}})$ is a meromorphic function with poles at $s=0$, 1, α_i, $1-\alpha_i$ and $\frac{\rho}{2}$, where ρ is the non-trivial zero of the Riemann zeta function. More precisely, the structure of $(f,\widehat{E}_{\bar{s}})$ is given by

$$(f,\widehat{E}_{\bar{s}})=\sum_{i=1}^{\ell}c_i\left(\frac{1}{(1-\alpha_i-s)^{n_i+1}}+\frac{\Lambda(1-s)}{\Lambda(s)}\frac{1}{(s-\alpha_i)^{n_i+1}}\right)+\frac{1}{\Lambda(s)}\frac{g(s)}{s(s-1)}\,, \tag{4.19}$$

where $g(s)$ is an entire function of s. Furthermore, $(f,\widehat{E}_s)$ satisfies the following functional

identity

$$(f,\widehat{E}_{\bar{s}}) = \frac{\Lambda(1-s)}{\Lambda(s)}(f,\widehat{E}_{1-\bar{s}})\,. \tag{4.20}$$

For completeness, let us do some further clarifications here.

First, can see that in (4.18) we subtract the slow growth part of the integrand - which is the problematic part in that integral. This can be done precisely because the Mellin transform of $\varphi(y)$ vanishes. This can be shown by considering a truncated fundamental domain $\mathcal{F}_T$

$$\mathcal{F}_T = \left\{\tau\in\mathbb{H}\middle||\tau|\geq 1, |x|\leq\frac{1}{2}, y\leq T\right\}, \tag{4.21}$$

where $T>0$ is some arbitrary cutoff. Let us now introduce the cutoff in the Mellin transform of $\varphi(y)$.

$$\varphi(y) = \int_0^\infty \mathrm{d}y\,\varphi(y)\,y^{s-2} = \int_0^T \mathrm{d}y\,\varphi(y)\,y^{s-2} + \int_T^\infty \mathrm{d}y\,\varphi(y)\,y^{s-2}\,, \tag{4.22}$$

where we can then meromorphically continue both integrals to all values of s, such that they end up cancelling eachother out.

Secondly, this truncated domain can be used to extract the pole structure of the renormalised Rankin-Selberg transform given in (4.19), following [39]:

$$\begin{aligned} &R.N.\left(\int_{\mathcal{F}}\mathrm{d}x\,\frac{\mathrm{d}y}{y^2}\,f(\tau,\bar{\tau})\widehat{E}_s(\tau,\bar{\tau})\right)\\ &\quad=\int_{\mathcal{F}_T}\mathrm{d}x\,\frac{\mathrm{d}y}{y^2}\,f(\tau,\bar{\tau})\widehat{E}_s(\tau,\bar{\tau}) - h_T(s) - \frac{\Lambda(1-s)}{\Lambda(s)}h_T(1-s)\\ &\quad+\int_{\mathcal{F}-\mathcal{F}_T}\mathrm{d}x\,\frac{\mathrm{d}y}{y^2}\left(f(\tau,\bar{\tau})\widehat{E}_s(\tau,\bar{\tau}) - \varphi(y)\big(y^s+\frac{\Lambda(1-s)}{\Lambda(s)}y^{1-s}\big)\right). \end{aligned} \tag{4.23}$$

This integral turns out to be identical to (4.18).The function $h_T(s)$ is defined as

$$h_T(s) = \int_0^T \mathrm{d}y\,y^{s-1}\,\varphi(y) = \sum_{i=1}^{\ell} c_i \sum_{m=0}^{n_i}\frac{(-1)^{n_i-m}}{\Gamma(m+1)}\frac{T^{s+\alpha_i-1}\log^m T}{(s+\alpha_i-1)^{n_i-m+1}} \tag{4.24}$$

and it is key to identifying the polar structure of the transform.

Let us now present some examples to familiarise the reader with this technique. First, a very simple case: the constant function $f(\tau,\bar{\tau})=1$. Following the renormalisation procedure, we can

see that $\varphi(y) = 1$ and the overall integral vanishes.

Second,let us do the Rankin-Selberg transform of the non-holomorphic Eisenstein series $\widehat{E}_r(\tau, \bar{\tau})$ with $r \neq s$.

In this case, the slow growth is tackled with

$$\varphi(s) = y^s + \frac{\Lambda(1-s)}{\Lambda(s)} y^{1-s} \,. \tag{4.25}$$

It we now identify this $\varphi(y)$ with (4.17), we shall make the following choice of parameters:

$$c_1 = 1,\ c_2 = \frac{\Lambda(1-r)}{\Lambda(r)},\ n_1 = 0,\ n_2 = 0,\ \alpha_1 = r,\ \alpha_2 = 1-r. \tag{4.26}$$

This implies equivalently that the Rankin-Selberg transformation $(\widehat{E}_r, \widehat{E}_s)$ is zero if $r \neq s$. That is consistent with the role of Eisenstein series as eigenfunctions of the Laplacian with different eigenvalues! See [39] for more details on this.

The fact that the renormalised inner product between two Eisenstein series of different weights is zero means that one can add or subtract a linear combination of them (or their derivatives[3]) to a modular form without modifying its Rankin-Selberg transform. That means that in this formula

$$\widetilde{f}(\tau, \bar{\tau}) = f(\tau, \bar{\tau}) - \sum_{\alpha_i \geq 1/2} \mathrm{c}_i \frac{\partial^{n_i}}{\partial \alpha_i^{n_i}} \widehat{E}_s(\tau, \bar{\tau}) \Big|_{s=\alpha_i} \tag{4.27}$$

both f and $\widetilde{f}$ have the same Rankin-Selberg transform, and the parameters c_i and α_i are as defined in Theorem 1. In particular, this is what allows us to, from a slow growth f, obtain a rapidly decaying $\widetilde{f}$ and analyse the continuous spectrum.

Given a slow growth $f(\tau, \bar{\tau})$, the complete spectral decomposition after renormalisation will be

$$\begin{aligned} f(\tau, \bar{\tau}) = \sum_{n=0}^{\infty} \frac{(f, \nu_n)}{(\nu_n, \nu_n)} \nu_n(\tau, \bar{\tau}) + \frac{1}{4\pi i} \int_{\Re(s)=\frac{1}{2}} \mathrm{d}s \, (\widetilde{f}, \widehat{E}_s) \widehat{E}_s(\tau, \bar{\tau}) \\ + \sum_{\alpha_i \geq 1/2} \mathrm{c}_i \frac{\partial^{n_i}}{\partial \alpha_i^{n_i}} \widehat{E}_s(\tau, \bar{\tau}) \Big|_{s=\alpha_i} . \end{aligned} \tag{4.28}$$

[3] Times some factors that guarantee modular invariance.

4.2.1 Setting up the partition function for spectral decomposition

In this chapter, we are only going to study theories that have an infinite number of primaries under the Virasoro algebra but are still Rational CFTs (and thus have an extended algebra under which there is a finite number of primaries): this means that their central charge $c \geq 1$ [72].

We refer to the partition function on the torus as $Z(\tau, \bar{\tau})$, which is a function living in $\mathcal{F}$. We will denote the characters of the theory under the extended algebra as $f_i(\tau)$ and $\bar{f}_i(\bar{\tau})$.

With all that, the full partition function for an N-character RCFT is

$$Z(\tau, \bar{\tau}) = \sum_{i=0}^{N-1} f_i(\tau) \bar{f}_i(\bar{\tau}) \,. \tag{4.29}$$

The partition function (4.29) has a fundamental problem when trying to perform its spectral decomposition as is: it is of rapid growth at the cusp. To overcome this, we can *prepare* $Z(\tau, \bar{\tau})$ for harmonic analysis by multiplying it by a modular-invariant factor that suppresses the exponential growth, in a similar fashion to what was done in [100] and, from here, we can investigate what properties can be unearthed with harmonic analysis.

The invariant piece is found to be $\left(y^{1/2}\eta(\tau)\bar{\eta}(\bar{\tau})\right)^c$. This defined what will be referred to as the *reduced partition function*:

$$Z_P(\tau, \bar{\tau}) = \left(y^{1/2}\eta(\tau)\bar{\eta}(\bar{\tau})\right)^c Z(\tau, \bar{\tau}) \,. \tag{4.30}$$

Interestingly enough, the reduced partition function has an interesting meaning/relation to algebraic properties of the theories at hand. Let us start with a CFT which has an extended algebra (not necessarily maximal) $U(1)^c \times U(1)^c$. Under this algebra the characters are

$$\chi_h(\tau) = \frac{q^h}{\eta(\tau)^c} \,, \tag{4.31}$$

from where we can build

$$Z(\tau, \bar{\tau}) = \chi_0(\tau)\bar{\chi}_0(\bar{\tau}) + \sum_{h,\bar{h}} d_{h,\bar{h}}\, \chi_h(\tau)\bar{\chi}_{\bar{h}}(\bar{\tau}) \,. \tag{4.32}$$

We call $d_{h,\bar{h}}$ degeneracy, and it counts the number of primary states under $U(1)^c \times U(1)^c$ having

conformal weight[4] h and $\bar{h}$.

With this understanding in mind, the reduced partition function (4.30) becomes

$$Z_P(\tau,\bar{\tau}) = y^{c/2} + y^{c/2}\sum_{h,\bar{h}} d_{h,\bar{h}} e^{-2\pi y(h+\bar{h})} e^{2\pi i x(h-\bar{h})}\,. \tag{4.33}$$

We will now move on to studying the Rankin-Selberg transform of several CFTs, starting with the one-character theories and then moving onto several-character examples. Our attempts at extracting useful information from the discrete spectrum were unsuccessful, however the scalar spectrum itself gives us plenty of insight, as we will see.

4.3 Harmonic analysis of One-Character RCFTs

We will begin by analysing partition functions that can be fully factorised into a holomorphic and an antiholomorphic part, and we will focus on their Rankin-Selberg transform.

One of the simplest CFTs we can study with a fully factorised partition function is the level one E_8 WZW model, which has $c = 8$. Afterwards, we will generalise the results obtained to a more general family of one-character CFTs.

4.3.1 One-character RCFT with $c = 8$

The partition function of the level one E_8 WZW model has a rather simple structure:

$$Z(\tau,\bar{\tau}) = \left(J(q)J(\bar{q})\right)^{1/3}\,, \tag{4.34}$$

where $J(q)$ is the so-called J-invariant. We now derive the reduced partition function as described in the previous section

$$Z_P(\tau,\bar{\tau}) = y^4 E_4(\tau)E_4(\bar{\tau})\,, \tag{4.35}$$

where $E_4(\tau)$ is the holomorphic Eisenstein series of weight 4.

As per our previous results, this reduced partition function is of slow growth, therefore we

[4] In this chapter, we restrict our interest to the two-dimensional CFTs with integer central charge to guarantee the positive integrality of the degeneracy in the decomposition (4.32).

can apply (4.18) and compute the Rankin-Selberg transformation of $Z_P(\tau,\bar{\tau})$. This turns out to be

$$(Z_P, E_{\bar{s}}) = \int_0^\infty \mathrm{d}y\, y^{s-2} \left(\sum_{n=1}^\infty 240^2\, \sigma_3(n)^2\, y^4\, e^{-4\pi n y} \right) = \sum_{n=1}^\infty 240^2\, \sigma_3(n)^2\, \frac{\Gamma(s+3)}{(4\pi n)^{s+3}}\,. \tag{4.36}$$

In this case, we find that the Rankin-Selberg transform of the reduced partition function yields an infinite sum involving divisor functions $\sigma_l(n)$

$$\sigma_z(n) = \sum_{d|n} d^z, \tag{4.37}$$

which means we should be careful about its convergence. As is, the sum in (4.36) diverges if $\Re(s) < -3$. However, the following identity by Ramanujan [109] can be applied to regularise the divergence:

$$I(a,b;s) \equiv \sum_{n=1}^\infty \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\,. \tag{4.38}$$

We can write the the Rankin-Selberg transform of (4.35) using (4.38) as

$$(Z_P, \widehat{E}_{\bar{s}}) = 240^2\, \frac{\zeta(s+3)\zeta(s)^2\zeta(s-3)}{\zeta(2s)} \frac{\Gamma(s+3)}{(4\pi)^{s+3}}\,. \tag{4.39}$$

To be able to write the full spectral decomposition of (4.35) in a coherent way, we can derive a relation from (4.20), by taking into account that in the s-integral's domain $\bar{s} = 1-s$:

$$\frac{\Gamma(s+3)}{(4\pi)^{s+3}} \frac{\zeta(s+3)\zeta(s)^2\zeta(s-3)}{\zeta(2s)} = \frac{\Lambda(1-s)}{\Lambda(s)} \frac{\Gamma(4-s)}{(4\pi)^{4-s}} \frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}\,. \tag{4.40}$$

This allows us to write the complete decomposition of the E_8 WZW model partition function as [2]

$$Z_P(\tau,\bar{\tau}) = \widehat{E}_4(\tau,\bar{\tau}) + Z_P^d(\tau,\bar{\tau}) + Z_P^c(\tau,\bar{\tau})\,, \tag{4.41}$$

with

$$Z_P^d(\tau,\bar{\tau}) = \sum_{n=0}^{\infty} \frac{(Z_P,\nu_n)}{(\nu_n,\nu_n)}\nu_n, \tag{4.42}$$

$$Z_P^c(\tau,\bar{\tau}) = \frac{240^2}{4\pi i}\int_{\Re(s)=\frac{1}{2}} \mathrm{d}s\, \frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}\frac{\Gamma(4-s)}{(4\pi)^{4-s}}\widehat{E}_s(\tau,\bar{\tau})\,, \tag{4.43}$$

and noting that the factor of $\widehat{E}_4$ appears due to the reduced partition function being of slow growth, following the procedure showcased in Theorem 1.

The discrete part of the spectrum is very difficult to compute, can only be done numerically and offers very little insight - so we will leave it be and focus our attention on the continuous sector. We can see how, as expected from Theorem 1, the Rankin-Selberg transform of $Z_P(\tau,\bar{\tau})$ in this case has poles at $s=-3,0,1,4$.

To see if we have done the decomposition consistently, we will recover the scalar part of the partition function by solving the relevant terms of (4.43).

We need to start by separating the real Eisenstein series into two sectors. This is due to divergences at $|s|\to\infty$ caused by the structure of the Bessel K-function. Let us recall the Fourier decomposition of the non-holomorphic Eisenstein series:

$$\widehat{E}_s(\tau,\bar{\tau}) = y^s + \frac{\Lambda(1-s)}{\Lambda(s)} + 2\sum_{j\neq 0} y^{1/2}\,\frac{|j|^{s-1/2}\sigma_{1-2s}(j)}{\Lambda(s)}\,K_{s-1/2}(2\pi|j|y)\,\cos(2\pi i j x)\,. \tag{4.44}$$

We can apply the following identity between first and second-kind modified Bessel functions:

$$K_s(x) = \frac{\pi}{2}\frac{I_{-s}(x)-I_s(x)}{\sin(s\pi)}\,, \tag{4.45}$$

which now allows us to write $\widehat{E}_s(\tau,\bar{\tau})$ as

$$\widehat{E}_s(\tau,\bar{\tau}) = y^s + \sum_{j=1}^{\infty}(-2\pi)\cos(2\pi jx)\frac{\sigma_{2s-1}(j)}{j^{s-1/2}\Lambda(s)}y^{1/2}\left(\frac{I_{s-\frac{1}{2}}(2\pi jy)}{\sin\left((s-\frac{1}{2})\pi\right)}\right) \tag{4.46}$$

$$+\frac{\Lambda(1-s)}{\Lambda(s)}y^{1-s} + \sum_{j=1}^{\infty}2\pi\cos(2\pi jx)\frac{\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}\Lambda(s)}y^{\frac{1}{2}}\left(\frac{I_{\frac{1}{2}-s}(2\pi jy)}{\sin\left((s-\frac{1}{2})\pi\right)}\right), \tag{4.47}$$

where we have used the green and blue colours in order to keep track of what goes where through-

out. Note that we assume throughout that $|2\pi jy| \lesssim c$. We can now define, for convenience,

$$\widehat{E}^{+}_{s,j}(\tau,\bar{\tau}) = -2\pi\cos(2\pi jx)\frac{\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}\Lambda(s)}y^{\frac{1}{2}}\left(\frac{I_{s-\frac{1}{2}}(2\pi jy)}{\sin\left((s-\frac{1}{2})\pi\right)}\right), \tag{4.48}$$

$$\widehat{E}^{-}_{s,j}(\tau,\bar{\tau}) = 2\pi\cos(2\pi jx)\frac{\sigma_{2s-1}(j)}{j^{s-\frac{1}{2}}\Lambda(s)}y^{\frac{1}{2}}\left(\frac{I_{\frac{1}{2}-s}(2\pi jy)}{\sin\left((s-\frac{1}{2})\pi\right)}\right). \tag{4.49}$$

This allows us to rewrite the decomposition of the non-holomorphic Eisenstein series more compactly:

$$\widehat{E}_s(\tau,\bar{\tau}) = y^s + \sum_{j=1}^{\infty}\widehat{E}^{+}_{s,j}(\tau,\bar{\tau}) + \frac{\Lambda(1-s)}{\Lambda(s)}y^{1-s} + \sum_{j=1}^{\infty}\widehat{E}^{-}_{s,j}(\tau,\bar{\tau}) \tag{4.50}$$

$$\tag{4.51}$$

$$= \widehat{E}^{+}_{s}(\tau,\bar{\tau}) + \widehat{E}^{-}_{s}(\tau,\bar{\tau})\,. \tag{4.52}$$

Furthermore, we can find the following relation from the Fourier decomposition of $\widehat{E}_s(\tau,\bar{\tau})$:

$$\widehat{E}^{-}_{s}(\tau,\bar{\tau}) = \frac{\Lambda(1-s)}{\Lambda(s)}\widehat{E}^{+}_{1-s}(\tau,\bar{\tau})\,. \tag{4.53}$$

From this relation, one can compactly write any given Eisenstein series as

$$\widehat{E}_s(\tau,\bar{\tau}) = \widehat{E}^{+}_{s}(\tau,\bar{\tau}) + \frac{\Lambda(1-s)}{\Lambda(s)}\widehat{E}^{+}_{1-s}(\tau,\bar{\tau}). \tag{4.54}$$

Let us now use these identities to solve the integral (4.43). We begin by writing the Rankin Selberg transform through (4.40):

$$(Z_P,\widehat{E}_{\bar{s}}) = 240^2\frac{\Lambda(1-s)}{\Lambda(s)}\frac{\Gamma(4-s)}{(4\pi)^{4-s}}\frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}. \tag{4.55}$$

Now, using (4.20), this[5] leads to us being able to write the continuous part of the spectrum

[5] Again, recall that on the support of the s-integral, $\bar{s} = 1-s$.

as

$$Z_P^c(\tau,\bar\tau) = \frac{240^2}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \mathrm{d}s \left(\frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}\frac{\Gamma(4-s)}{(4\pi)^{4-s}}\right) y^s \\ + \sum_j \frac{240^2}{2\pi i}\int_{\Re(s)=\frac{1}{2}} \mathrm{d}s \left(\frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}\frac{\Gamma(4-s)}{(4\pi)^{4-s}}\right) \widehat{E}^+_{s,j}(\tau,\bar\tau)\,. \tag{4.56}$$

Our current goal is to recover the continuous part of the spectrum of the partition function. For that we will need three things:

- The zero modes of the discrete spectrum.
- The scalar ($j=0$) part of the non-holomorphic Eisenstein series:

$$\widehat{E}_4(\tau,\bar\tau)\,|_{j=0} = y^4 + \frac{\Lambda(-3)}{\Lambda(4)} y^{-3}. \tag{4.57}$$

- The first line of (4.56).

We first will solve the integral in the first line of (4.56), starting at an arbitrary value of the real part of s:

$$\begin{aligned}\frac{240^2}{2\pi i}\int_{\Re(s)=\sigma} \mathrm{d}s \left(\frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}\frac{\Gamma(4-s)}{(4\pi)^{4-s}}\right) y^s \\ = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathrm{d}s \left(\sum_{n=1}^{\infty}\frac{240^2\sigma_3(n)^2}{n^{4-s}}\right)\frac{\Gamma(4-s)}{(4\pi)^{4-s}} y^s \\ = 240^2 y^4 \sum_{n=1}^{\infty}\sigma_3(n)^2 e^{-4\pi n y}\,,\end{aligned} \tag{4.58}$$

where we first applied an inverse Mellin transform and then chose a straight line contour along $\sigma < -3$ so that we can use the corresponding Ramanujan identity (4.38).

Now we should move the contour from a general $\sigma < -3$ to $\sigma = 1/2$. The integrand has poles at $s=-3$ and $s=0$, so we have to include their residues.

- Residue at $s=0$: $\frac{15\zeta(3)}{2\pi^3}\nu_0^2 = -(Z_P,\nu_0^2)\,.$
- Residue at $s=-3$: $-\frac{23625\zeta(7)}{8\pi^7 y^3} = -\frac{\Lambda(-3)}{\Lambda(4)} y^{-3}\,.$

This is great! The two residues cancel precisely the terms coming from the Maass cusp forms and the non-holomorphic Eisenstein series that we cannot find in the partition function. Let us

recapitulate. The scalar sector of the continuous spectrum is:

$$Z_P^c(\tau,\bar{\tau})\big|_{j=0} = -(Z_P,\nu_0^2) - \frac{\Lambda(-3)}{\Lambda(4)}y^{-3} + 240^2 y^4 \sum_{n=1}^{\infty}\sigma_3(n)^2 e^{-4\pi n y}. \tag{4.59}$$

The full scalar part of the partition function reconstructed from harmonic analysis as [2]

$$\begin{aligned}
&\big(\widehat{E}_4(\tau,\bar{\tau}) + Z_P^d(\tau,\bar{\tau}) + Z_P^c(\tau,\bar{\tau})\big)|_{j=0} && (4.60)\\
&\quad = y^4 + \frac{\Lambda(-3)}{\Lambda(4)}y^{-3} + (Z_P,\nu_0) + \left(-(Z_P,\nu_0^2) - \frac{\Lambda(-3)}{\Lambda(4)}y^{-3} + 240^2 y^4 \sum_{n=1}^{\infty}\sigma_3(n)^2 e^{-4\pi n y}\right) && (4.61)\\
&\quad = y^4 + 240^2 y^4 \sum_{n=1}^{\infty}\sigma_3(n)^2 e^{-4\pi n y} = Z_P(\tau,\bar{\tau})\,|_{j=0}\,, && (4.62)
\end{aligned}$$

which as we see is in full agreement with what we obtain by looking at the scalar part of the partition function. Hence, we have proven explicitly how, with enough care, the techniques for harmonic analysis of slow growth modular forms can be used to study partition functions of RCFTs, in particular their scalar spectrum. The spin sector integral (the cusp Maass form decomposition) can be completed, however, without enlightening results from the Maass form sector, we cannot derive interesting results and relations between modular forms. We will only consider the scalar spectrum in the next calculations, as this issue persists throughout.

4.3.2 Toy models with $c = 8k$

Let us now consider a hypothetical modular invariant object

$$Z_P(\tau,\bar{\tau}) = y^{c/2}E_{c/2}(\tau)E_{c/2}(\bar{\tau})\,, \qquad c = 8k\,,\ k\in\mathbb{N}\,, \tag{4.63}$$

This corresponds to an actual reduced partition function when $c = 8$, the case we just analysed, and when $c = 16$. The latter corresponds to the level-one $E_8 \times E_8$ WZW model. When $c \geq 24$, we can observe that it doesn't correspond to a proper partition function as the coefficients in the q-expansion aren't integers. To obtain the full partition functions, one needs to add cusp forms of the relevant weight. For instance, the holomorphic sector of the $c = 24$ CFT would

actually involve the full J-invariant:

$$J(\tau) = E_{12}(\tau) - \frac{82104}{691}\Delta(\tau)\,, \tag{4.64}$$

where $\Delta(\tau) = \eta(q)^{24}$ is commonly known as the discriminant function, first studied by Ramanujan [110]. Having said that, in current studies of two-dimensional CFTs, there are various interpretations related to the averaging of CFTs and the need (or lack of) for requiring integral coefficients.

We will disregard the requirement for integrality of Fourier coefficients and investigate further the spectral decomposition of (4.63), as it will uneart a rather interesting relation between some RCFTs.

We find a beautiful generalisation for general $c = 8k$ of the results presented in section 4.3.1. The Rankin-Selberg transform of (4.63) for general c can be compactly written as [2]

$$(Z_P, \widehat{E}_{\bar{s}}) = \left(\frac{c}{B_{c/2}}\right)^2 \frac{\Gamma(s+\frac{c}{2}-1)}{(4\pi)^{s+\frac{c}{2}-1}} \frac{\zeta(s+\frac{c}{2}-1)\zeta(s)^2\zeta(s-\frac{c}{2}+1)}{\zeta(2s)}\,. \tag{4.65}$$

Furthermore, the identity (4.40) can also be generalised to

$$\begin{aligned} &\frac{\Gamma(s+\frac{c}{2}-1)}{(4\pi)^{s+\frac{c}{2}-1}} \frac{\zeta(s+\frac{c}{2}-1)\zeta(s)^2\zeta(s-\frac{c}{2}+1)}{\zeta(2s)} \\ &= \frac{\Lambda(1-s)}{\Lambda(s)} \frac{\Gamma(\frac{c}{2}-s)}{(4\pi)^{\frac{c}{2}-s}} \frac{\zeta(\frac{c}{2}-s)\zeta(1-s)^2\zeta(2-\frac{c}{2}-s)}{\zeta(2-2s)}\,. \end{aligned} \tag{4.66}$$

This allows us to write the decomposition of the reduced partition function as [2]

$$\begin{aligned} Z_P(\tau,\bar{\tau}) &= \widehat{E}_{c/2}(\tau,\bar{\tau}) + \sum_{n=0}^{\infty} \frac{(Z_P,\nu_n)}{(\nu_n,\nu_n)}\nu_n \\ &+ \left(\frac{c}{B_{c/2}}\right)^2 \frac{1}{4\pi i}\int_{\Re(s)=\frac{1}{2}} \mathrm{d}s \left(\frac{\Gamma(\frac{c}{2}-s)}{(4\pi)^{\frac{c}{2}-s}} \frac{\zeta(\frac{c}{2}-s)\zeta(1-s)^2\zeta(2-\frac{c}{2}-s)}{\zeta(2-2s)}\right) \widehat{E}_s(\tau,\bar{\tau})\,, \end{aligned} \tag{4.67}$$

from where we can find the scalar sector of the continuous spectrum to be again a generalisation of the result for $c = 8$:

$$Z_P^c(\tau,\bar{\tau})\big|_{j=0} = -(Z_P,\nu_0^2) - \frac{\Lambda(1-c/2)}{\Lambda(c/2)}\, y^{1-c/2} + \left(\frac{c}{B_{c/2}}\right)^2 y^{c/2} \sum_{n=1}^{\infty} \sigma_{c/2-1}(n)^2 e^{-4\pi n y}\,. \tag{4.68}$$

4.3.3 A note on $c \geq 24$ CFTs

We mentioned earlier that the modular invariant object (4.63) is not the full partition function of one-character RCFTs for $c \geq 24$, as it doesn't have integer Fourier coefficients. To attain those, one needs to add to the holomorphic and antiholomorphic part a cusp form of weight $c/2$. These cusp forms lead to L-functions in the Rankin-Selberg transform and unfortunately, we do not know of or were able to find a compact Ramanujan-like relation for these. Not being able to write them compactly other than as an L-function means that we can't really find relations that would be useful for analysing possible relations between RCFTs in the moduli space of theories.

On a different note, if we take the limit of $c \to \infty$ in the modular invariant $y^{c/2} E_{c/2}(\tau) E_{c/2}(\bar{\tau})$, we can see that it tends naturally towards the large central charge limit of the non-holomorphic Eisenstein series $y^{c/2} \widehat{E}_{c/2}$ which, is actually the partition function of the $U(1)^c \times U(1)^c$ vacuum state. This is interesting as not only does the cusp form contributions become negligible at the large c limit, so we could disregard their contribution, but we also may be open to studying theories with non-integer Fourier coefficients. The reason for it is that, in that limit, (4.63) could be seen as the partition function of an ensemble of CFTs dual to a chiral theory of gravity, and the requirement of integer degeneracies would be lifted. As discussed in Chapter 3, ensemble average interpretations are not uncommon in low-dimensional AdS/CFT.

4.4 Harmonic analysis of multi-character partition functions

We have previously analysed theories (or prospective theories) whose partition function fully factorises: in other words, theories with only one character. However, to understand the moduli space of RCFTs better, we should dig deeper into it and analyse multi-character RCFTs.

Just as before, we will start with a particular example that will highlight a surprising family of RCFTs that relate to the ones studied before.

In what follows, we will denote Z_P from (4.63) as $\widetilde{Z}_P$ and it will be referred to as the *seed invariant*. We will find that dressing the Rankin-Selberg transform $(\widetilde{Z}_P, E_s)$ with $\widetilde{Z}_P$ as in (4.63) with a specific analytic function $\phi(s)$ generates new modular invariants Z_P that can be identified as multi-character reduced partition functions of other RCFTs.

More in particular, we will find a two-way relation between both [2]

$$\widetilde{Z}_P \underset{\text{orbifold w/ discrete symmetry}}{\overset{\text{dress RS transform with } \phi(s)}{\rightleftharpoons}} Z_P \tag{4.69}$$

4.4.1 SO(16) WZW model of level one

We will start by studying the Wess-Zumino-Witten model with affine Lie algebra $\hat{G} = \widehat{SO}(16)$. The group theory data of $\widehat{SO}(16)$ uniquely determines both the central charge and the conformal weight of primary states. This data can be found from the dual Coxeter number and the affine Weyl vector, as well as the affine Dynkin label for each primary.

This theory has $c = 8$ and three primaries of weight $h = 1/2,\, 1,\, 1$. Therefore, this theory will have four characters (one for the vacuum state and three for the primaries). We can write the reduced partition function of this CFT as

$$Z_P(\tau,\bar{\tau}) = y^4 \eta(\tau)^8 \eta(\bar{\tau})^8 \left(|f_0(\tau)|^2 + |f_1(\tau)|^2 + 2|f_2(\tau)|^2 \right) , \tag{4.70}$$

where the reduced conformal characters are

$$\begin{aligned}
\eta(\tau)^8 f_0(\tau) &= \frac{7}{15} E_4(\tau) + \frac{8}{15} E_4(2\tau) = 1 + 112 \sum_{n=1}^{\infty} \sigma_3(n) q^n + 128 \sum_{\ell=1}^{\infty} \sigma_3(\ell) q^{2\ell} , \\
\eta(\tau)^8 f_1(\tau) &= \frac{1}{30} E_4\left(\frac{\tau}{2}\right) - \frac{1}{30} E_4\left(\frac{\tau+1}{2}\right) = 16 \sum_{n=odd} \sigma_3(n) q^{\frac{n}{2}} , \\
\eta(\tau)^8 f_2(\tau) &= \frac{8}{15} E_4(\tau) - \frac{8}{15} E_4(2\tau) = 128 \sum_{n=1}^{\infty} \sigma_3(n) q^n - 128 \sum_{\ell=1}^{\infty} \sigma_3(\ell) q^{2\ell} .
\end{aligned} \tag{4.71}$$

The scalar spectrum of the reduced partition function in this case is

$$\begin{aligned}
Z_p(\tau,\bar{\tau})\big|_{j=0} = y^4 \Bigg[& 1 + 45312 \sum_{n=1}^{\infty} \sigma_3(n)^2 e^{-4\pi n y} - 36864 \sum_{n=1}^{\infty} \sigma_3(2n)\sigma_3(n) e^{-8\pi n y} \\
& + 49152 \sum_{n=1}^{\infty} \sigma_3(n)^2 e^{-8\pi n y} + 256 e^{-2\pi y} + 256 e^{-2\pi y} \sum_{n=1}^{\infty} \sigma_3(2n+1)^2 e^{-4\pi n y} \Bigg] ,
\end{aligned} \tag{4.72}$$

and its Rankin-Selberg transformation has the form

$$\begin{aligned}(Z_p, E_s) = 28928 \sum_{n=1}^{\infty} \sigma_3(n)^2 \frac{\Gamma(4-s)}{(4\pi n)^{4-s}} - 4096 \sum_{n=1}^{\infty} \sigma_3(2n)\sigma_3(n) \frac{\Gamma(4-s)}{(8\pi n)^{4-s}} \\ + 32768 \sum_{n=1}^{\infty} \sigma_3(n)^2 \frac{\Gamma(4-s)}{(8\pi n)^{4-s}} + 256 \sum_{n \text{ odd}}^{\infty} \sigma_3(n)^2 \frac{\Gamma(4-s)}{(2\pi n)^{4-s}} .\end{aligned} \tag{4.73}$$

Here, we can see some sums that cannot be computed with the Ramanujan identity (4.38) used in the previous part. However, these sums can be performed using (4.38) as a stepping stone. We will present and prove the relevant identities now.

Proposition 1 *Let us consider Dirichlet series with double divisor functions of the form*

$$I_1(a,b;s) = \sum_{n=odd}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s}, \quad I_2(a,b;s) = \sum_{n=1}^{\infty} \frac{\sigma_a(2n)\sigma_b(n)}{n^s}. \tag{4.74}$$

The above Dirichlet series satisfy the following identities

$$I_1(a,b;s) = \left(\frac{(2^s-1)(2^s-2^a)(2^s-2^b)(2^s-2^{a+b})}{2^{2s}(2^{2s}-2^{a+b})} \right) \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}, \tag{4.75}$$

$$I_2(a,b;s) = \left(\frac{2^s(2^a+1)-2^a(2^b+1)}{2^s-2^{a+b-s}} \right) \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}, \tag{4.76}$$

for $\Re(s) > max(1, a+1, b+1, a+b+1)$.

We can start the proof for the first identity (4.75) from (4.38):

$$\begin{aligned} I(a,b;s) &= \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} \\ &= \sum_{m=0}^{\infty} \sum_{\ell \text{odd}} \frac{\sigma_a(2^m\ell)\sigma_b(2^m\ell)}{2^{ms}\ell^s}, \end{aligned} \tag{4.77}$$

where we have relabelled the summation variable $n \to 2^m\ell$, for ℓ odd and $m \geq 0$.

We can then use the following identity

$$\sigma_a(mn) = \sigma_a(n)\sigma_a(n) \quad \text{for} \quad (m,n) = 1, \tag{4.78}$$

to rewrite (4.77) as

$$\begin{aligned} I(a,b;s) &= \sum_{m=0}^{\infty} \frac{\sigma_a(2^m)\sigma_b(2^m)}{2^{ms}} I_1(a,b;s) \\ &= \left(\frac{2^{2s}(2^{2s}-2^{a+b})}{(2^s-1)(2^s-2^a)(2^s-2^b)(2^s-2^{a+b})} \right) I_1(a,b;s)\,, \end{aligned} \tag{4.79}$$

where $I_1(a,b;s)$ corresponds to (4.75), thus proving that identity.

To prove (4.76), we start from $I_2(a,b;s)$ and work our way backwards: one can recast the summation variable of $I_2(a,b;s)$ as $2^m\ell$ where ℓ is odd and $m \geq 0$. This yields

$$\begin{aligned} I_2(a,b;s) &= \sum_{n=1}^{\infty} \frac{\sigma_a(2n)\sigma_b(n)}{n^s} \\ &= \sum_{m=0}^{\infty} \sum_{\ell \text{ odd}} \frac{\sigma_a(2^{m+1}\ell)\sigma_b(2^m\ell)}{2^{ms}\, n^s}\,, \end{aligned} \tag{4.80}$$

where we have now made the substitution $n \to 2^m\ell$ with ℓ odd and $m \geq 0$. We can now use the same identity (4.78) that allows us to recognise

$$I_2(a,b;s) = \frac{2^s(2^a+1)-2^a(2^b+1)}{2^s-2^{a+b-s}} I(a,b;s)\,, \tag{4.81}$$

with again $I(a,b;s)$ from (4.38). We have now proven (4.76).

At this stage, we present a follow-up proposition that generalises this formula to general prime numbers, as it will be useful later on.

Proposition 2 *For p prime, the sum of two series involving double divisor function satisfies the following identities.*

$$\begin{aligned} &\sum_{\ell=1}^{p-1} \left(\sum_{n=\ell \bmod p}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} \right) \\ &\qquad = \frac{(p^s-1)(p^s-p^a)(p^s-p^b)(p^s-p^{a+b})}{p^{2s}(p^{2s}-p^{a+b})} \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\,, \end{aligned} \tag{4.82}$$

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(pn)}{n^s} = \frac{p^s(1+p^b)-p^b(1+p^a)}{p^s-p^{a+b-s}} \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}\,. \tag{4.83}$$

This can be proven in a similar fashion to the previous identities.

Now that we know how to perform the sums present in (4.73) by using (4.75) and (4.76), we find that the Rankin-Selberg transformation of (4.70) has a rather interesting form [2]:

$$(Z_P, E_s) = \frac{2048(2^s + 2^{1-s} + 12)}{240^2}(\widetilde{Z}_P, E_s)\,, \tag{4.84}$$

where we should recall that

$$(\widetilde{Z}_P, E_s) \equiv 240^2 \frac{\Gamma(4-s)}{(4\pi)^{4-s}} \frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}\,, \tag{4.85}$$

is the Rankin-Selberg transform for the WZW $(E_8)_1$ model (4.39). Hence, we see that the RS transform of the reduced partition function of the $SO(16)_1$ and $(E_8)_1$ models are related in a simple manner. One may wonder at this stage if this implies the existence of some kind of relation between the theories. And one would be right: if we take an orbifold of the $SO(16)_1$ model with the $\mathbb{Z}_2$ subgroup of the center of $SO(16)$ we will obtain precisely the partition function of the $(E_8)_1$ model. Indeed, the relationship between both can be anticipated from the fact that we can use the characters of the $SO(16)_1$ model to build the partition function of the $(E_8)_1$ model:

$$f_0(\tau) + f_2(\tau) = J(\tau)^{1/3}\,. \tag{4.86}$$

One may see this and wonder if we can generalise the statement further. Can we relate some subset of multi-character RCFTs to the space of one-character CFT partition functions we have explored?

Suppose that there is a modular invariant invariant object Z_P^* whose Rankin-Selberg transformation takes the following form [2]:

$$(Z_P^*, \widehat{E}_s) = \left(\frac{B_{c/2}}{c}\right)^2 \phi(s)(\widetilde{Z}_P^{(c)}, \widehat{E}_s)\,, \tag{4.87}$$

where in the RHS we find the Rankin-Selberg transform corresponding to the proposed one-character theories studied beforehand, (4.65),

$$(\widetilde{Z}_P^{(c)}, \widehat{E}_s) = \left(\frac{c}{B_{c/2}}\right)^2 \frac{\Gamma(\frac{c}{2}-s)}{(4\pi)^{\frac{c}{2}-s}} \frac{\zeta(\frac{c}{2}-s)\zeta(1-s)^2\zeta(2-\frac{c}{2}-s)}{\zeta(2-2s)}\,. \tag{4.88}$$

Assuming that $\phi(s)$ is an analytic function in s, we wish to understand what constraints it must follow in order to have the role observed in the Rankin-Selberg transformation of Z_P^* (4.87). The constraint (4.20) on the reflection properties of the Rankin-Selberg transform implies that

$$\phi(s) = \phi(1-s)\,. \tag{4.89}$$

Furthermore, any function $f(\tau,\bar{\tau})$ of which we would like to perform a Rankin-Selberg transform needs to be of slow growth. This will further constraint $\phi(s)$

Let us observe (4.12). Regardless of the function we are investigating, one can safely compute the contour integral on the line $\Re(s) < 1 - \frac{c}{2}$ (assuming that the reduced partition function is holomorphic in $\tau \to i\infty$). Next, we shift the contour to the line $\Re(s) = \frac{1}{2}$, which means we need to add the contributions of two poles at $s = 0, 1 - \frac{c}{2}$. The pole at $s = 0$ simply cancels the contribution of the zero-th cusp Maass form, and is not relevant for constraining $\phi(s)$.

The second pole at $s = 1 - \frac{c}{2}$ is, however, relevant for $\phi(s)$. We know that the partition function must be of slow growth at the cusp $y \to \infty$, so as to counteract the polynomial growth of the Eisenstein series' scalar term at the cusp. This can be controlled through the residue of the Rankin-Selberg transform at $s = 1 - \frac{c}{2}$, which requires the following constraint:

$$\mathrm{Res}_{s=1-\frac{c}{2}}(Z_P, E_s) = -\frac{\Lambda(1-\frac{c}{2})}{\Lambda(\frac{c}{2})} y^{1-\frac{c}{2}}\,, \tag{4.90}$$

or, in terms of ϕ,

$$\phi\left(1-\frac{c}{2}\right) = \left(\frac{c}{B_{c/2}}\right)^2\,. \tag{4.91}$$

We now propose that the two constraints (4.89) and (4.91) constrain $\phi(s)$ to have the following analytic form [2]:

$$\phi(s) = \alpha_1(p^s + p^{1-s}) + \alpha_2\,, \tag{4.92}$$

where p is a prime and α_1 and α_2 are rational numbers. One can now go ahead and plug in this equation into (4.87) to find the scalar spectrum of a new candidate RCFT partition function Z_P^*.

4.4.2 The space of partition functions generated by the $c = 8$ one-character RCFT.

We are now going to apply our proposition (4.92) for $\phi(s)$ and study the space of scalar spectra that can be generated by starting from the *seed* invariant (4.35), the $(E_8)_1$ WZW model.

In general terms, we are interested in a Rankin-Selberg transform of the form

$$(Z_P^*, \widehat{E}_s) = 240^2 \phi(s) \frac{\Gamma(4-s)}{(4\pi)^{4-s}} \frac{\zeta(4-s)\zeta(1-s)^2\zeta(-2-s)}{\zeta(2-2s)}. \tag{4.93}$$

$\phi(s)$ with $p = 2$

We start by the smallest prime number, 2. Following our ansatz, let us take $\phi(s)$ as

$$\phi(s) = \alpha_1(2^s + 2^{1-s}) + \alpha_2\,, \tag{4.94}$$

where α_1 and α_2 are rational numbers. We now perform an inverse Mellin transform on $\phi(s)$ in order to reverse-engineer the characters, and we find that the new modular invariant has primaries of conformal weights $h = \frac{1}{2}$ or $h = 1$, and that the constants α_1 and α_2 are fixed by an integer M as follows:

$$\alpha_1 = 8M, \quad \alpha_2 = 57600 - 129M\,. \tag{4.95}$$

With this in mind, we present possible[6] new modular invariant candidates to partition functions of the form

$$Z_P^*(\tau, \bar{\tau}) = y^4 \left(|\eta(\tau)^4 f_0(\tau)|^2 + M|\eta(\tau)^4 f_1(\tau)|^2 + \frac{1}{4} M(768 - M)|\eta(\tau)^4 f_2(\tau)|^2 \right)\,, \tag{4.96}$$

[6]Note that this method, although powerful, would not allow us to recover information that is fully orthogonal to the continuous spectrum of the theory, so those presented here are not the only theories that could exist with this scalar spectral decomposition, but rather a natural candidate from previous observations in this context.

where the characters can be written in terms of the holomorphic Eisenstein series of weight 4 and a certain sum over divisor functions of weight 3.

$$\begin{aligned}\eta(\tau)^4 f_0(\tau) &= \left(1 - \frac{M}{480}\right) E_4(\tau) + \frac{M}{480} E_4(2\tau)\,,\\ \eta(\tau)^4 f_1(\tau) &= \sum_{n\,\text{odd}}^{\infty} \sigma_3(n) q^{\frac{n}{2}}\,,\\ \eta(\tau)^4 f_2(\tau) &= \frac{1}{240} E_4(\tau) - \frac{1}{240} E_4(2\tau)\,.\end{aligned} \tag{4.97}$$

Great! We have the constraint of modular invariance, but what else do we need to have prospective partition functions? Positive integer Fourier coefficients in the reduced partition function, as those ensure positive norms and integer degeneracies of the energy eigenstates. This yields the constraint on M [2]

$$M = 2n, \quad n \in \mathbb{Z}^+\,, \quad 1 \leq n \leq 120\,. \tag{4.98}$$

In particular, $M = 256$ recovers (4.84) and $M = 384$ recovers the reduced partition function of the $SO(8)_1 \times SO(8)_1$ WZW model.

$\phi(s)$ with $p = 3$

We consider the analytic function $\phi(s)$ with $p = 3$, as per our ansatz this looks like

$$\phi(s) = \alpha_1(3^s + 3^{1-s}) + \alpha_2\,, \tag{4.99}$$

where α_1, α_2 are undetermined constants. The modular invariant candidates in this case have conformal characters of weight $h = 0, \frac{1}{3}, \frac{2}{3}, 1$. The constants α_1 and α_2 are determined through an integer M as before

$$\alpha_1 = 27M\,, \qquad \alpha_2 = 57600 - 2188M\,. \tag{4.100}$$

We obtain the following modular invariant candidate to partition function:

$$Z_P^*(\tau, \bar{\tau}) = y^4 \left(|\eta(\tau)^4 f_0|^2 + M|\eta(\tau)^4 f_1|^2 + M|\eta(\tau)^4 f_2|^2 + 81M(36 - M)|\eta(\tau)^4 f_3|^2\right)\,. \tag{4.101}$$

The conformal characters in this case have the following form:

$$\begin{aligned}
\eta(\tau)^4 f_0(\tau) &= \left(1 - \frac{3}{80}M\right) E_4(\tau) + \left(\frac{3}{80}M\right) E_4(3\tau)\,, \\
\eta(\tau)^4 f_1(\tau) &= \sum_{n=1 \bmod 3}^{\infty} \sigma_3(n) q^{\frac{n}{3}}\,, \\
\eta(\tau)^4 f_2(\tau) &= \sum_{n=2 \bmod 3}^{\infty} \sigma_3(n) q^{\frac{n}{3}}, \\
\eta(\tau)^4 f_3(\tau) &= \frac{1}{240} E_4(\tau) - \frac{1}{240} E_4(3\tau)\,.
\end{aligned} \tag{4.102}$$

The further requirement of positive, integral Fourier cofficients constrains M to be [2]

$$M \in \mathbb{Z}^+, \quad 1 \leq M \leq 26\,. \tag{4.103}$$

In particular, we find that $M = 18$ recovers the reduced partition function of the $SU(3)_1 \times (E_6)_1$ WZW model.

Interestingly enough, the product theory $\mathcal{B} \equiv SU(3)_1 \times (E_6)_1$ is closely related to the $\tilde{\mathcal{B}} \equiv (E_8)_1$ WZW model through a biyective transformation.

If we start from $\mathcal{B}$, we know that the non-anomalous $\mathbb{Z}_3$ symmetry of $\mathcal{B}$ can be chosen to be a center symmetry of either $SU(3)$ or E_6. If we gauge said symmetry, we actually obtain the partition function of $\tilde{\mathcal{B}}$ (regardless of the choice of center symmetry).

On the other hand, starting from the partition function of $\tilde{\mathcal{B}}$ we can get to $\mathcal{B}$ through a defect operator inside $\tilde{\mathcal{B}}$ [111], which is considered as the centre of $SU(3)$ or of E_6. One can define the twisted partition function with this defect operator $\mathcal{L}$ as follows:

$$Z^{\mathcal{L}}(\tau, \bar{\tau}) \equiv \mathrm{Tr}_{\mathcal{H}}\left[\mathcal{L} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}\right]. \tag{4.104}$$

We get $Z_{\mathcal{L}}(\tau, \bar{\tau})$ by applying the S-transformation to $Z^{\mathcal{L}}(\tau, \bar{\tau})$, and $Z^{\mathcal{L}}_{\mathcal{L}}(\tau, \bar{\tau})$ by applying the T-transformation to $Z_{\mathcal{L}}(\tau, \bar{\tau})$. We can then obten the target $Z_{\mathcal{B}}(\tau, \bar{\tau})$ through the following

sum,

$$Z_{\mathcal{B}}(\tau,\bar{\tau}) = \frac{1}{2}\Big[Z_{\tilde{\mathcal{B}}}(\tau,\bar{\tau}) + Z^{\mathcal{L}}(\tau,\bar{\tau}) + Z_{\mathcal{L}}(\tau,\bar{\tau}) + Z^{\mathcal{L}}_{\mathcal{L}}(\tau,\bar{\tau})\Big]\,, \quad Z_{\tilde{\mathcal{B}}}(\tau,\bar{\tau}) = J(q)^{1/3}J(\bar{q})^{1/3}\,, \tag{4.105}$$

in agreement with (4.101), $M = 18$.

$\phi(s)$ with $p = 4$

When presenting our ansatz, we mentioned that it only applied when p is prime. However, can we extend it to non-prime numbers? Let us see what can be done for $p = 4$.

We present a slightly different ansatz for $\phi(s)$ [2]:

$$\phi(s) = \alpha_1(4^s + 4^{1-s}) + \alpha_2(2^s + 2^{1-s}) + \alpha_3\,, \tag{4.106}$$

which satisfies the constraints (4.89) and (4.91). Following the same Mellin-inverse procedure as before, we find that the modular invariant has primaries of weights $h = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. By plugging in the ansatz (4.106) into (4.87), we find the following candidate reduced partition function:

$$Z^*_P(\tau,\bar{\tau}) = y^4\left(|\eta^4 g_0|^2 + \mathcal{M}_1|\eta^4 g_1|^2 + \mathcal{M}_2|\eta^4 g_2|^2 + \mathcal{M}_1|\eta^4 g_3|^2 + \mathcal{M}_3|\eta^4 g_4|^2\right)\,, \tag{4.107}$$

where $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ are constants that will be fixed later, once we impose that the Fourier coefficients be positive integers. This leads to a candidate five-character theory where the five characters are given as,

$$\begin{aligned}
\eta(\tau)^4 g_0(\tau) &= \alpha E_4(\tau) + \beta E_4(2\tau) + (1-\alpha-\beta)E_4(4\tau)\,,\\
\eta(\tau)^4 g_1(\tau) &= \sum_{n=1 \bmod 4}^{\infty} \sigma_3(n) q^{\frac{n}{4}}\,,\\
\eta(\tau)^4 g_2(\tau) &= \sum_{n=1 \bmod 2}^{\infty} \sigma_3(n) q^{\frac{n}{2}}\,,\\
\eta(\tau)^4 g_3(\tau) &= \sum_{n=3 \mod 4}^{\infty} \sigma_3(n) q^{\frac{n}{4}}\,,\\
\eta(\tau)^4 g_4(\tau) &= \frac{1}{240}E_4(\tau) + \gamma E_4(2\tau) + \left(-\frac{1}{240} - \gamma\right)E_4(4\tau)\,.
\end{aligned} \tag{4.108}$$

In particular, the following choice of parameters [2]

$$\begin{aligned}
&\alpha = \frac{8}{15}, \quad \beta = -\frac{3}{5}, \quad \gamma = \frac{3}{560}, \\
&\mathcal{M}_1 = 4, \quad \mathcal{M}_2 = 0, \quad \mathcal{M}_3 = 4, \\
&\alpha_1 = 256, \quad \alpha_2 = -2592, \quad \alpha_3 = 33856\,,
\end{aligned} \tag{4.109}$$

correspond to the $SU(2)_1 \times (E_7)_1$ theory. In order to evaluate their expressions explicitly, one should introduce the following proposition:

Proposition 3 *The Dirichlet series of double divisor functions satisfies the following identities.*

$$\begin{aligned}
&\sum_{n=1 \ mod\ 4}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} + \sum_{n=3 \ mod\ 4}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} \\
&\qquad = \left(\frac{(2^s-1)\,(2^s-2^a)\,\left(2^s-2^b\right)\,\left(2^s-2^{a+b}\right)}{2^{4s}-2^{a+b+2s}} \right) I(a,b;s)\,,
\end{aligned} \tag{4.110}$$

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(4n)}{n^s} = \left(\frac{(2^a+1)\,\left(2^b+1\right)\,\left(-2^{b+s}\right)+2^{a+2b}+\left(2^b+2^{2b}+1\right)\,2^{2s}}{2^{2s}-2^{a+b}} \right) I(a,b;s)\,, \tag{4.111}$$

where $I(a,b;s)$ corresponds to (4.38).

To prove (4.110), we start from (4.38) and relabel the summation variable n as follows:

$$\begin{aligned}
I(a,b;s) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \sum_{m=0}^{\infty} \sum_{\ell=1,3 \ \mathrm{mod}\ 4}^{\infty} \frac{\sigma_a(4^m\ell)\sigma_b(4^m\ell)}{4^{ms}\ell^s} + \\
\sum_{m=0}^{\infty} \sum_{\ell=2 \ \mathrm{mod}\ 4}^{\infty} \frac{\sigma_a(2\cdot 4^m(\frac{\ell}{2}))\sigma_b(2\cdot 4^m(\frac{\ell}{2}))}{4^{ms}2^s(\ell/2)^s}\,.
\end{aligned} \tag{4.112}$$

Next, we apply the identity (4.78) of the divisor function to express $I(a,b;s)$ as

$$\begin{aligned}
I(a,b;s) = &\left(\sum_{m=0}^{\infty} \frac{\sigma_a(4^m)\sigma_b(4^m)}{4^{ms}} \right) \left(\sum_{\ell=1,3 \ \mathrm{mod}\ 4}^{\infty} \frac{\sigma_a(\ell)\sigma_b(\ell)}{\ell^s} \right) \\
&+ \left(\sum_{m=0}^{\infty} \frac{\sigma_a(2\cdot 4^m)\sigma_b(2\cdot 4^m)}{4^{ms}2^s} \right) \left(\sum_{\ell=1 \ \mathrm{mod}\ 2}^{\infty} \frac{\sigma_a(\ell)\sigma_b(\ell)}{\ell^s} \right).
\end{aligned} \tag{4.113}$$

Then, we also need to use the identities

$$\sigma_a(4^m) = \frac{2^{(2m+1)a}-1}{2^a-1}, \quad \sigma_a(2\cdot 4^m) = \frac{2^{(2m+2)a}-1}{2^a-1}, \tag{4.114}$$

together with proposition 1 to finalise the proof of (4.82).

To prove (4.111), we expand that expression by a suitable change of summation variable

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sigma_a(n)\sigma_b(4n)}{n^s} = &\left(\sum_{m=0}^{\infty}\frac{\sigma_a(4^m)\sigma_b(4^{m+1})}{4^{ms}}\right)\left(\sum_{\ell=1,3 \bmod 4}^{\infty}\frac{\sigma_a(\ell)\sigma_b(\ell)}{\ell^s}\right)\\ &+\left(\sum_{m=0}^{\infty}\frac{\sigma_a(2\cdot 4^m)\sigma_b(2\cdot 4^{m+1})}{4^{ms}2^s}\right)\left(\sum_{\ell=1 \bmod 2}^{\infty}\frac{\sigma_a(\ell)\sigma_b(\ell)}{\ell^s}\right),\end{aligned} \tag{4.115}$$

and then use (4.110).

Let us now compare the product theory $\mathcal{B}' = SU(2)_1 \times (E_7)_1$ to the one-character $c = 8$ RCFT $\tilde{\mathcal{B}}$. These two theories are related in a similar fashion to the relation between the $(E_8)_1$ and the $(SU(3))_1 \times (E_6)_1$ theories we saw before.

Starting from $\mathcal{B}'$, we first consider the $\mathbb{Z}_2$ center symmetry of $SU(2)$ or E_7 and consider it to be non-anomalous. If now we take the quotient $\mathcal{B}'/\mathbb{Z}_2$, we will see that its partition function matches precisely that of $\tilde{\mathcal{B}}$.

The other direction can be recasted, as previously, through the introduction of a defect operator in $\tilde{\mathcal{B}}$.

4.5 Conclusions

The goal of this chapter is to study the spectral decomposition of partition functions of RCFTs, with the aim to unearth new links between different theories. By looking at a family of fully factorisable invariants (almost one-character partition functions), we found that their Rankin-Selberg transform can be expressed rather compactly

$$(\widetilde{Z}_P^{(c)}, \widehat{E}_s) = \left(\frac{c}{B_{c/2}}\right)^2 \frac{\Gamma(\frac{c}{2}-s)}{(4\pi)^{\frac{c}{2}-s}} \frac{\zeta(\frac{c}{2}-s)\zeta(1-s)^2\zeta(2-\frac{c}{2}-s)}{\zeta(2-2s)}\,. \tag{4.116}$$

Furthermore, from there we proposed a technique for generating candidate multi-character

theories Z_P^* from the RS transform of the one-character quasi-partition functions $(\widetilde{Z}_P^{(c)}, \widehat{E}_s)$

$$(Z_P^*, \widehat{E}_s) = \left(\frac{B_{c/2}}{c}\right)^2 \phi(s)(\widetilde{Z}_P^{(c)}, \widehat{E}_s)\,, \tag{4.117}$$

where $\phi(s)$ is an analytic function that we propose looks like

$$\phi(s) = \alpha_1(p^s + p^{1-s}) + \alpha_2\,. \tag{4.118}$$

After looking at several examples that are based on the $c = 8$ case, we see that this procedure is not only effective at generating RS transforms, but in particular it allows us to obtain the full partition function of multi-character candidate theories from just the scalar spectrum. Some further theories found for other values of c can be found in B.

This is a rather interesting result. In general CFTs, Benjamin et al [100] conjectured that we need both the scalar and one value of the spinful spectrum to fully constrain the partition function of a two-dimensional CFT. However, from these results it seems that one can generate candidate theories from just the scalar spectrum. Whilst this is true, these are not uniquely determined, an important caveat comes from the fact that one could in principle add a modular-invariant term that is completely orthogonal to the continuous spectrum and it would not be recoverable through the process presented in this chapter. This means that our results are in accordance with [100] and one needs input from the discrete spectrum to uniquely identify a theory. Having said that, it would be interesting to understand further how physical input can constrain the form of any additional terms that only overlap with the discrete spectrum. This is not an easy task to do, and it probably would require individual numerical study on known theories.

During our search for generating multi-character RCFTS, we also found multiple examples that link our precise functional form for $\phi(s)$ to orbifold operations on CFTs. It would be interesting to study whether for other possible families of theories we can find a different $\phi(s)$ relating two theories by a different kind of transformation.

A limitation of our methods comes from the fact that we can't always find a closed-form expression for the Rankin-Selberg transform, and one can often encounter L-functions that don't have a closed form in simpler functions. We are thus limited to theories with characters that

have a holomorphic Eisenstein series-like appearance so far. It would be interesting to see if we can find other families of theories with characters whose L-functions can be resummed. For instance, we looked at families of theories with Lie Algebras A_n, D_n, E_6, E_7 , E_8, which have associated integral Euclidean lattices (their root lattices) . Because of this, the reduced characters of some of these theories can be written in terms of polynomial combinations of Jacobi theta functions, which is certainly interesting, however we didn't manage to find enlightening structures Rankin-Selberg transforms, but they seem like a natural place to look next.

Finally, we have worked on the partition function of the RCFTs as it's a modular form of the full modular group. However, the characters of the RCFT are vector-valued modular forms, and in particular as they have integral Fourier coefficients they are invariant[7] under a principal congruence subgroup of $PSL(2,\mathbb{Z})$. It would be very interesting to study characters separately with congruence subgroup harmonic analysis techniques, and see what interesting relations between theories could be derived from such an analysis.

[7]This is a general conjecture for VVMFs with integer Fourier coefficients [112], but it has been proven for RCFT characters [113].

Chapter 5

Shedding light (transforms) on the Celestial CFT

Proposing a consistent theory of quantum gravity has been a long-standing goal of theoretical physicists for many decades. Besides from the now *traditional* holography (AdS/CFT), one can also consider other kinds of dualities between a gravity theory and a lower-dimensional, non-gravitational identification of the features we would like to describe in the higher dimensional theory. In this line, celestial or flat-space holography poises a duality between scattering in asymptotically flat spaces and a Conformal Field Theory living in the celestial sphere.

It will be our goal in this chapter to establish and investigate the celestial CFT in the context of massless scattering in four-dimensional asymptotically flat space, by means of constructing an appropriate light transform that allows us to understand these amplitudes as conformal correlators.

5.1 Introduction and background

The AdS/CFT dictionary is built on identifications over the boundary of AdS, however in flat-space holography there isn't a naive boundary to perform said identifications [1]. This means that the holographic dictionary will be built fundamentally differently: we will exploit the kinematics of the scattering processes we want to study. We identify the generators of four-dimensional asymptotic symmetries as the currents of a two-dimensional CFT, such that we can prepare

[1] There is finally a recent construction for this boundary [114].

scattering states to transform in a manner amenable to the 2d theory. Using this approach will allow one to recast S-matrix elements of the amplitudes into correlators of the 2d CFT [40], via the study of studying external wavepackets that are boost eigenstates [41–43].

Our focus here will be massless scattering of four or less particles, from where we aim to shed light on the operator content of the Celestial CFT via conformal block decomposition. However, massless scattering imposes a special constraint: we need to work in $(2,2)$ signature rather than the standard $(1,3)$. The reason for this is that momentum conservation has no on-shell support for three-point amplitudes of massless external fields in $(1,3)$ signature. Due to the properties of the operator algebra of CFTs, if the three-point correlator is zero, so are all higher-order ones: this wouldn't shed a lot of light on the correspondence, as the dual would be a free CFT. However, in $(2,2)$ signature this is no longer the case and we can construct a meaningful duality, considering that now the Celestial CFT (CCFT) lives in the celestial *torus* rather than in the celestial sphere, and that the Lorentz group is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

We approach the problem by exploiting the flexibility of the celestial dictionary and reorganise the CCFT data in a way such that we can apply standard CFT techniques to extract data about the operator content of the CCFT. But how do we actually go from one side to the other of the duality?

Let us start with a $(2,2)$ amplitude $A(p_i)$ and explain what the standard dictionary would look like [41, 115, 116]. We first perform a Mellin transform in the energy variables, and we obtain what is referred to in the literature as a *celestial amplitude*. We can then observe that this amplitude transforms under the action of the Lorentz group as a CFT correlator of weights $h = 1/2(\Delta_i + J_i)$, $\bar{h}_i = (\Delta_i - J_i)$, where Δ_i is the Rindler energy and J_i is the helicity of the external state[2].

Further from the expected covariance under the global conformal group (which can be identified with the Lorentz group), this object has to obey constraints coming from translation invariance when we have 4 or less external particles. In practical terms, these amplitudes have to be distributional, a characteristic not present in traditional CFT correlators.

We expect that the CCFT is somewhat an exotic theory: for instance, we don't expect some of the usual constraints of unitarity of representations present in the spectrum to hold.

[2] We perform an analytical continuation from $\mathbb{R}^{(1,3)}$ to obtain these quantities in $\mathbb{R}^{(2,2)}$.

However, we need to be able to extract the data, in particular, we would like to be able to exploit techniques coming from CFT, such as analytical bootstrap techniques. How can we go about this?

Well, there are certain further transformations one can make to the celestial amplitude to get rid of the distributional factor. We will later propose to establish a prescription where we perform a set of improved light transforms, which preserve Lorentz covariance whilst allowing us to have an object more similar-looking to a CFT correlation function. The overall process takes the plane-wave scattering amplitude through the traditional Mellin transform, then the light transform, and we end up with an object in the so-called ambidextrous basis [117].

Let us now move onto understanding *where* the CCFT lives: how do we reach the celestial torus from flat space?

5.1.1 The Celestial Torus

Our starting point is flat space-time $\mathbb{R}^{2,2}$, equipped with a metric of ultra-hyperbolic signature.

$$\mathrm{d}s^2 = -(\mathrm{d}X^0)^2 + (\mathrm{d}X^1)^2 - (\mathrm{d}X^2)^2 + (\mathrm{d}X^3)^2\,, \tag{5.1}$$

commonly referred to as *Klein Space* [118]. The asymptotic boundary of this space at null infinity is foliated by tori [118, 119], which are what is commonly referred to as the celestial torus. This structure is also the analytic continuation of the $\mathbb{R}^{(1,3)}$ celestial sphere into the $(2,2)$ signature [119].

We should note at this stage that the bilinear form (5.1) is also present in momentum space, and as such any discussion here carries over there by replacing null vectors X^μ by null momenta p^μ.

Let us reach the celestial torus from scratch and also set the notation for the rest of the chapter. We start by building the light cone of the origin of Klein space

$$(X^0)^2 + (X^2)^2 = (X^1)^2 + (X^3)^2\,, \tag{5.2}$$

and we then find its cross sections from quotienting out the dilatations[3]:

$$(X^0)^2 + (X^2)^2 = (X^1)^2 + (X^3)^2 = 1\,, \tag{5.3}$$

thus obtaining the geometry of a torus $S^1 \times S^1$: the celestial torus, which has a single connected component.

The next step is to find coordinates such that the action of the Lorentz group acts on them in a simple, or rather, recognisable way. It turns out that a convenient choice is a pair $(z, \bar{z})$

$$z = \frac{X^1 + X^2}{X^0 + X^3}\,, \qquad \bar{z} = \frac{X^1 - X^2}{X^0 + X^3}\,. \tag{5.4}$$

These are real, independent and invariant under projective rescalings of X^μ. The induced metric in these coordinates is

$$ds^2 = -dz\, d\bar{z}\,, \tag{5.5}$$

from where we can see that they span a Poincaré patch $\mathbb{R}^{(1,1)} \subset S^1 \times S^1$.

Moreover, in these coordinates the action of the Lorentz group becomes a real Möbius transformation, independent in $z, \bar{z}$.

$$z \mapsto \frac{az + b}{cz + d}\,, \qquad \bar{z} \mapsto \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}}\,, \tag{5.6}$$

where $ad - bc = 1 = \bar{a}\bar{d} - \bar{b}\bar{c}$.

This is phenomenal! The coordinates are very simple! Unfortunately, there is a price to pay for their simplicity. These coordinates aren't global on the celestial torus: in particular, they can only be used as coordinates on half of the celestial torus.

How can we solve this issue? Well, we can start by cutting the torus along the hyperplane $X^0 + X^3 = 0$, which results in a subspace $\mathbb{Z}_2 \times R^{1,1} \subset S^1 \times S^1$ with two Poincaré patches:

- A *future* Poincaré patch where $X^0 + X^3 > 0$.
- A *past* Poincaré patch where $X^0 + X^3 < 0$.

These patches allow us to introduce some notion of incoming and outgoing states, so that

[3]These are the simple notion of rescaling the null vector $X^\mu \to \lambda X^\mu$, $\lambda > 0$.

we can draw better parallels with Lorentzian signature amplitudes. Furthermore, both patches can be exchanged through the antipodal map $X^\mu \to -X^\mu$, but we will use the same coordinates $z, \bar{z}$ for both.

We should note that there is, unfortunately, a further issue: $\text{sgn}(X^0 + X^3)$ is not invariant under Lorentz transformations. As a result, neither is the choice of Poincaré patch.

In principle, we define null momenta states living on the past (future) Poincaré patch as incoming (outgoing), from an analytical continuation of the celestial sphere [119]. However, as the choice of patch is not Lorentz invariant, neither is the notion of incoming and outgoing states on the celestial torus! This is an issue, as in Minkowski space there is no mixing under Lorentz transformations.

There is a solution though: we can use representation theory of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ to build operators that live in a single patch of the celestial torus, thus being able to set up a description that seems like a more natural extension of $(1,3)$ signature.

5.2 A Lorentz-invariant basis of boost eigenstates

Now that we have set up the background where our CCFT will live, we should want to set up a basis for on-shell scattering amplitudes. For that, we start with a review of the basis constructed in [42], and we use that as a stepping stone to build a basis of $\mathbb{Z}_2$ eigenstates living in a single Poincaré patch of the celestial torus. These will also transform as the conformal primaries of a two-dimensional CFT living in said patch.

Note that we are restricting our attention to integer spin massless particles.

5.2.1 Boost eigenstates

Let us start with a massless particle of helicity $J \in \mathbb{Z}$ that carries null momentum

$$p^\mu = E\, q^\mu\,, \quad q^\mu(z,\bar{z}) = \frac{1}{\sqrt{2}}\,(1 + z\bar{z}, z + \bar{z}, \bar{z} - z, 1 - z\bar{z})\,, \quad E \in \mathbb{R}^*\,,\ z, \bar{z} \in \mathbb{R}\,, \tag{5.7}$$

where $z, \bar{z}$ are coordinates on a Poincaré patch. The specific Poincaré patch is labelled by the scaling E. We then use these parameters to denote such a momentum eigenstate as $|E, z, \bar{z}, J\rangle$.

Often one would decompose $E = \varepsilon\omega$ as

$$\varepsilon = \operatorname{sgn} E = \operatorname{sgn}(p^0 + p^3)\,, \qquad \omega = |E|\,. \tag{5.8}$$

As we have established before, we use $\operatorname{sgn}(p^0+p^3)$ to assign a Poincaré patch of the celestial torus to a particular eigenstate. We can thus treat $(\varepsilon, z, \bar{z}) \in \mathbb{Z}_2 \times \mathbb{R}^{1,1}$ as coordinates spanning the whole torus (minus a codimension 1 locus $p^0 + p^3 = 0$).

Now, let us start trying to understand how a Lorentz transformation affects these coordinates. In Minkowski signature, we would define a state as incoming (outgoing) if $\varepsilon = -1$ ($\varepsilon = 1$). However, in $(2,2)$ signature, we can no longer make this distinction due to the fact that ε transforms under the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ group as

$$\begin{aligned} &z \mapsto \frac{az+b}{cz+d}\,, \qquad \bar{z} \mapsto \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\,, \\ &\omega \mapsto |cz+d|\,|\bar{c}\bar{z}+\bar{d}|\,\omega\,, \qquad \varepsilon \mapsto \operatorname{sgn}\big[(cz+d)(\bar{c}\bar{z}+\bar{d})\big]\,\varepsilon\,. \end{aligned} \tag{5.9}$$

Under such a transformation, a momentum eigenstate becomes (again, assuming that J is an integer)

$$|\varepsilon\omega, z, \bar{z}, J\rangle \mapsto |cz+d|^J\,|\bar{c}\bar{z}+\bar{d}|^{-J}\,|\varepsilon\omega, z, \bar{z}, J\rangle\,. \tag{5.10}$$

Now, if we want to build the holographic dictionary, we should define a *boost eigenstate* as a Mellin transform of a momentum eigenstate

$$|\varepsilon, z, \bar{z}, h, \bar{h}\rangle = \int_0^\infty \frac{d\omega}{\omega}\,\omega^\Delta\,|\varepsilon\omega, z, \bar{z}, J\rangle\,, \tag{5.11}$$

where now

$$h = \frac{\Delta + J}{2}\,, \qquad \bar{h} = \frac{\Delta - J}{2}\,, \tag{5.12}$$

are the conformal weights and Δ is simply the conformal dimension.

We can find a complete basis of states that live in the principal series[4], where $\Delta = 1 + i\mathbb{R}$. We will also label those CFT operators that are dual to a given boost eigenstate as $\mathcal{O}^{\varepsilon}_{h,\bar{h}}(z,\bar{z})$.

[4]Here we start noticing differences with a traditional CFT, where usually we wouldn't allow for the conformal dimension to be a complex number

5.2.2 Lorentz-invariant boost+$\mathbb{Z}_2$ eigenstates

We have defined a basis, however we still have the issue that our operators will change Poincaré patch under Lorentz transformations:they do not transform in an irreducible representation of the conformal group, so they can't be identified as (quasi)-primary operators

A solution to this issue can (finally!) be achieved by diagonalising the action of the $\mathbb{Z}_2$ symmetry. We define the following *symmetric* and *antisymmetric* operators, respectively [3]:

$$\mathcal{O}^{S}_{h,\bar{h}}(z,\bar{z}) := \mathcal{O}^{+}_{h,\bar{h}}(z,\bar{z}) + \mathcal{O}^{-}_{h,\bar{h}}(z,\bar{z})\,, \tag{5.13}$$

$$\mathcal{O}^{A}_{h,\bar{h}}(z,\bar{z}) := \mathcal{O}^{+}_{h,\bar{h}}(z,\bar{z}) - \mathcal{O}^{-}_{h,\bar{h}}(z,\bar{z})\,, \tag{5.14}$$

which now transform in irreducible representations of the conformal group, hence now we can say that $\mathcal{O}^{S}_{h,\bar{h}}$ and $\mathcal{O}^{A}_{h,\bar{h}}$ act as conformal primaries of a Lorentzian CFT that lives in a single Poincaré patch of the celestial torus.

We will refer to these as *symmetric* and *antisymmetric* primaries. We can see how they transform under the action of the conformal group,

$$\mathcal{O}^{S}_{h,\bar{h}}\left(\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = |cz+d|^{2h}\,|\bar{c}\bar{z}+\bar{d}|^{2\bar{h}}\,\mathcal{O}^{S}_{h,\bar{h}}(z,\bar{z})\,, \tag{5.15}$$

$$\mathcal{O}^{A}_{h,\bar{h}}\left(\frac{az+b}{cz+d},\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) = \mathrm{sgn}\big[(cz+d)(\bar{c}\bar{z}+\bar{d})\big]\,|cz+d|^{2h}\,|\bar{c}\bar{z}+\bar{d}|^{2\bar{h}}\,\mathcal{O}^{A}_{h,\bar{h}}(z,\bar{z})\,, \tag{5.16}$$

which is analogue to the representations of $SL(2,\mathbb{R})$ that are discussed in [120, Chapter VII], and they also appear when studying harmonic analysis on $SL(2,\mathbb{R})$ [121].

We should also note that we can distinguish between both kinds of primaries as they are symmetric or antisymetric, respectively, under the antipodal map.

In terms of the duality, we map them to be dual to $\mathbb{Z}_2$ eigenstates in the bulk.

$$\mathcal{O}^{S}_{h,\bar{h}}(z,\bar{z}) \quad\longleftrightarrow\quad |z,\bar{z},h,\bar{h}\rangle_S := |+,z,\bar{z},h,\bar{h}\rangle + |-,z,\bar{z},h,\bar{h}\rangle\,, \tag{5.17}$$

$$\mathcal{O}^{A}_{h,\bar{h}}(z,\bar{z}) \quad\longleftrightarrow\quad |z,\bar{z},h,\bar{h}\rangle_A := |+,z,\bar{z},h,\bar{h}\rangle - |-,z,\bar{z},h,\bar{h}\rangle\,. \tag{5.18}$$

Note that these $\mathbb{Z}_2$ eigenstates still span the same spectrum as the original boost eigenstates, however they are fully invariant under any Lorentz transformations. They also remain well-

defined as operators on a single copy of the celestial sphere when we perform a Wick rotation to Lorentzian signature.

On a more practical note, using $\mathbb{Z}_2$ eigenstates will change the structure of the step functions arising with momentum conservation (which are actually absent when we have a symmetric operator). Overall, this simplifies the calculation of light and shadow transforms, see [122, 123]. However, it also obscures to an extent the study of other properties like crossing symmetry, as studied in Minkowski spaces from boost eigenstates in [124]. It could be useful to understand better how studying crossing in this context, although more complicated from a technical perspective, could teach us new properties of CCFTs.

5.2.3 Direct transformations from momentum to boost+$\mathbb{Z}_2$ eigenstates

In general, we can find a direct integral transform mapping momentum eigenstates to such boost + $\mathbb{Z}_2$ eigenstates by combining various Mellin integrals. For instance, one can express the symmetric primary state as

$$\begin{aligned} |z,\bar{z},h,\bar{h}\rangle_S &= \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\,\omega^\Delta \left(|\omega,z,\bar{z},J\rangle + |-\omega,z,\bar{z},J\rangle\right) \\ &= \int_{-\infty}^\infty \frac{\mathrm{d}\omega}{|\omega|}\,|\omega|^\Delta\,|\omega,z,\bar{z},J\rangle\,. \end{aligned} \tag{5.19}$$

And similarly the antisymmetric primary becomes

$$|z,\bar{z},h,\bar{h}\rangle_A = \int_{-\infty}^\infty \frac{\mathrm{d}\omega}{\omega}\,|\omega|^\Delta\,|\omega,z,\bar{z},J\rangle\,. \tag{5.20}$$

These can be used to compute amplitudes of our $\mathbb{Z}_2$ eigenstates directly from momentum space amplitudes. The fact that the integration range of the ω integrals is now the whole real line instead of $(0,\infty)$ will rid celestial amplitudes of their step function support.

5.3 Towards finding the CCFT data: light transforms

We previously mentioned how amplitudes for four or less external massless particles have a distributional factor due to momentum conservation. Furthermore, they come with step functions that specify the channels for a particular scattering process.

When we swap from boost to boost+$\mathbb{Z}_2$ eigenstates, the step functions become signs, which are somewhat easier to handle, but the distributional factors are still present. To eliminate those and be able to attain a CFT-recognisable object, we turn to certain kinds of integral transforms: the so-called light or shadow transforms.

5.3.1 Light-transformed boost eigenstates

Due to how we have defined our dual operators $\mathcal{O}^{\varepsilon}_{h,\bar{h}}(z,\bar{z})$, we need to be rather careful in defining a prescription for the light transform. In fact, it will have to differ slightly to previous literature [117, 125] to account for the fact that the sign ε will change when we move along the light cone of a point on the celestial torus and cross from one Poincaré patch to the other.

We need to start by understanding where the future light cone of a point lies within the celestial torus: light-like lines have constant values for z and $\bar{z}$, and as such we can construct the light cone from two light rays emanating from a particular point $(\varepsilon, z, \bar{z})$ and re-converging at the antipodal point:

- $\{(-\mathrm{sgn}(\alpha)\,\varepsilon\,,\ z-\alpha^{-1}\,,\ \bar{z}) \in \mathbb{Z}_2 \times \mathbb{R}^{1,1} \mid \alpha \in \mathbb{R}\}\,,$ (5.21)
- $\{(-\mathrm{sgn}(\alpha)\,\varepsilon\,,\ z\,,\ \bar{z}-\alpha^{-1}) \in \mathbb{Z}_2 \times \mathbb{R}^{1,1} \mid \alpha \in \mathbb{R}\}\,.$ (5.22)

To be able to construct a consistent light transform, we need to explore how ε changes when flowing through these light rays.

As an example, let us choose the first geodesic, (5.21), where we see that we need to consider the sign of α to study any change of Poincaré patch. We also depict the contour studied in Figure 5.1

1. $\underline{-\infty < \alpha < 0}$:

   Whilst in this range, we see that $z < z - \alpha^{-1}$ and there is no change of Poincaré patch.

2. $\underline{0 < \alpha < \infty}$:

   In this range, we cross to the antipodal Poincaré patch and hence $\varepsilon \to -\varepsilon$. We cover the values $-\infty < z - \alpha^{-1} < z$, and we reach the antipodal point $(-\varepsilon, z, \bar{z})$ when $\alpha \to \infty$.

We can deal with the other geodesic (5.22) in a very similar fashion.

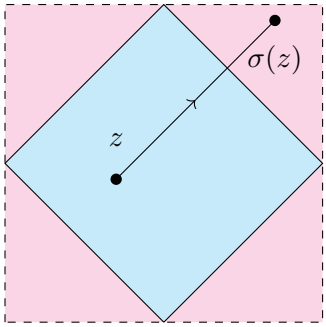


Figure 5.1: This picture showcases the contour for light transforms that extend to the next Poincaré patch. The celestial torus is built by identifying along the dashed lines. $\sigma(z)$ denotes the point antipodal to z. For simplicity, we have suppressed $\varepsilon, \bar{z}$.

How do we define light transforms of an operator along these null geodesics? Well, we smear said operator along the geodesics (5.21) and (5.22) using the kernels [126], and obtain [3]

$$\mathbf{L}[\mathcal{O}^{\varepsilon}_{h,\bar{h}}](z,\bar{z}) := \int_{-\infty}^{\infty} \mathrm{d}\alpha\, |\alpha|^{-2h}\, \mathcal{O}^{-\mathrm{sgn}(\alpha)\varepsilon}_{h,\bar{h}}(z-\alpha^{-1},\bar{z})\,, \tag{5.23}$$

$$\overline{\mathbf{L}}[\mathcal{O}^{\varepsilon}_{h,\bar{h}}](z,\bar{z}) := \int_{-\infty}^{\infty} \mathrm{d}\alpha\, |\alpha|^{-2\bar{h}}\, \mathcal{O}^{-\mathrm{sgn}(\alpha)\varepsilon}_{h,\bar{h}}(z,\bar{z}-\alpha^{-1})\,. \tag{5.24}$$

Note that there is a difference between our definitions and those in [126], namely the absolute values on the α factors. This is done to be consistent with the absolute values that appear in the definition we gave of conformal primaries of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, and in particular one can see [3, Appendix A] that this appears naturally if we consider the higher-dimensional prescription reducing to two dimensions, due to the operators gaining an extra phase whilst travelling through the Poincaré patches. This prescription also ensures that any light-transformed amplitudes are single-valued.

Light transforms are used to eliminate the distributional factors in amplitudes we want to understand from a CCFT perspective. However, they do more than that, and indeed they generate the action of the Weyl group of the Lorentz group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ on the conformal primaries [126]. In other words, the light transform also changes the conformal weights from the original operator. These changes are noted in table 5.1.

These are the definitions of light transforms we are going to be using for the rest of the chapter.

$\mathcal{O}$	Δ	J	h	$\bar{h}$
$\mathbf{L}\mathcal{O}$	$1-J$	$1-\Delta$	$1-h$	$\bar{h}$
$\bar{\mathbf{L}}\mathcal{O}$	$1+J$	$-1+\Delta$	h	$1-\bar{h}$
$\mathbf{L}\bar{\mathbf{L}}\mathcal{O}$	$2-\Delta$	$-J$	$1-h$	$1-\bar{h}$

Table 5.1: Weyl reflections induced by light transforms.

5.3.2 Light-transformed boost+$\mathbb{Z}_2$ eigenstates

As we have established before, the actual CFT primaries are linear combinations of the boost eigenstates (5.13), (5.14), so we should apply the light transforms to the boost+$\mathbb{Z}_2$ eigenstates. Let's do an example and perform the holomorphic light-transform (5.23) of a symmetric operator (5.13):

$$\begin{aligned}\mathbf{L}[\mathcal{O}^S_{h,\bar{h}}](z,\bar{z}) &= \mathbf{L}[\mathcal{O}^+_{h,\bar{h}}](z,\bar{z}) + \mathbf{L}[\mathcal{O}^-_{h,\bar{h}}](z,\bar{z}) \\ &= \int_z^\infty \frac{\mathrm{d}w}{|w-z|^{2-2h}} \left(\mathcal{O}^+_{h,\bar{h}}(w,\bar{z}) + \mathcal{O}^-_{h,\bar{h}}(w,\bar{z})\right) \\ &\qquad + \int_{-\infty}^z \frac{\mathrm{d}w}{|w-z|^{2-2h}} \left(\mathcal{O}^+_{h,\bar{h}}(w,\bar{z}) + \mathcal{O}^-_{h,\bar{h}}(w,\bar{z})\right) . \end{aligned} \tag{5.25}$$

Note that the integrals are broken into two sectors due to the absolute value of $w-z$ present in the integrand. We are simply breaking it into the sectors where that expression has a definite sign.

Overall, we obtain [3]

$$\mathbf{L}[\mathcal{O}^S_{h,\bar{h}}](z,\bar{z}) = \int_{-\infty}^\infty \frac{\mathrm{d}w}{|w-z|^{2-2h}}\, \mathcal{O}^S_{h,\bar{h}}(w,\bar{z})\,, \tag{5.26}$$

and for the antiholomorphic light transform

$$\bar{\mathbf{L}}[\mathcal{O}^S_{h,\bar{h}}](z,\bar{z}) = \int_{-\infty}^\infty \frac{\mathrm{d}\bar{w}}{|\bar{w}-\bar{z}|^{2-2\bar{h}}}\, \mathcal{O}^S_{h,\bar{h}}(z,\bar{w})\,. \tag{5.27}$$

These light transforms were also considered in [123], and they are much easier to deal with than the light transforms of pure boost eigenstates (5.23) and (5.24).

We should also summarise here the results for an antisymmetric primary operator [3]. The

holomorphic light transform is

$$\mathbf{L}[\mathcal{O}^{A}_{h,\bar{h}}](z,\bar{z}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}w\ \mathrm{sgn}(w-z)}{|w-z|^{2-2h}}\, \mathcal{O}^{A}_{h,\bar{h}}(w,\bar{z})\,, \tag{5.28}$$

and the antiholomorphic one is

$$\overline{\mathbf{L}}[\mathcal{O}^{A}_{h,\bar{h}}](z,\bar{z}) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\bar{w}\ \mathrm{sgn}(\bar{w}-\bar{z})}{|\bar{w}-\bar{z}|^{2-2\bar{h}}}\, \mathcal{O}^{A}_{h,\bar{h}}(z,\bar{w})\,. \tag{5.29}$$

Now that we have all the transformations of interest, we can observe how they transform under Lorentz transformations (see [3] for more detail), and we can see how these results agree with specific results relating representations of $SL(2,\mathbb{R})$ to each other in [120, Chapter VII]. In particular, light transforms preserve (anti)symmetry of the conformal primaries, taking (anti)symmetric primaries into (anti)symmetric primaries.

5.3.3 We have a proposed light transform, but is it invertible?

So far, we have dedicated our efforts to studying that the light transforms have the right covariance under a Lorentz transformation, to ensure that we still have conformal primary states after applying them.

However, there remains another very important characteristic that we need to study: are these transformations invertible?

One can start by doing a naive check using the weights from table 5.1, where we can see without much effort that $\mathbf{L}^2$ and $\overline{\mathbf{L}}^2$ take us back to conformal primaries with the same dimensions that we started with. Furthermore, we know that Weyl reflections square to one, and thus that gives more support to the (for now) hypothesis that we have invertible light transforms: they should square to a constant function of the data that just labels the particular irreducible representation in question.

In fact, in somewhat different conditions [118, 126], one ends up with either having to deal specifically with operator images on each Poincaré patch, or with a restriction in the spectrum of possible operators. In our case, we went back a bit and set the light transformed operators to be precisely defined on the $\mathbb{Z}_2$ quotient of the celestial torus, and as a result our conformal primaries (and their light transforms) to be irreducible representations of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

on its conformal compactification: all that is left is to judiciously normalise the light-transforms to be invertible[5]. More details of this justification can be found in [3, Section 4. 4].

5.4 Celestial scattering amplitudes: shedding light on the CFT properties

One of the main results presented in this chapter is a prescription to, starting from a celestial amplitude coming from the scattering of $n \leq 4$ particles, transform it into a new basis where massless particle amplitudes take the form of standard CFT correlators, i.e. eliminate the distributional factors that appear in the usual Mellin-transform celestial dictionary [41, 128] completely regardless of the crossing channel, whilst maintaining the right Lorentz covariance.

The proposed prescription [3] is that one $\mathbf{L}$-transforms all positive helicity boost eigenstates and $\overline{\mathbf{L}}$-transforms all negative helicity boost eigenstates, and as a result all amplitudes of massless external particles become non-distributional, which is our end-goal. In fact, this prescription was already introduced in [117], where evidence was shown for the simplest crossing channels for 2 and 3-point amplitudes. In this chapter, we will present the expansion of all that to all crossing channels [3]. We will take the spins to be nonzero and integral, for simplicity.

Let us establish some notation. The (color-stripped, when working with gluons) n-point momentum space amplitude will be

$$A(1^{J_1}2^{J_2}\cdots n^{J_n}) \equiv A(p_1, J_1; p_2, J_2; \cdots; p_n, J_n), \tag{5.30}$$

where the momenta p_i were defined in (5.7). We will then denote the delta-stripped amplitude as $A[1^{J_1}2^{J_2}\cdots n^{J_n}]$:

$$A(1^{J_1}2^{J_2}\cdots n^{J_n}) = A[1^{J_1}2^{J_2}\cdots n^{J_n}]\,\delta^4\left(\sum_{i=1}^{n} p_i\right). \tag{5.31}$$

Now, we know which particular light transform to apply on a given state by looking at the signs of their helicities. We will then use the following short-handed notation for light-

[5]There are a few special *resonant* cases when $h, \bar{h} \in \mathbb{Z} + 1/2$ or $h\,bh \in \mathbb{Z}$, whose treatment can be found in [120, 127].

transformed celestial amplitudes:

$$L(1^- \cdots n^+) \equiv \langle \overline{\mathbf{L}}[\mathcal{O}^{\varepsilon_1}_{h_1,\bar{h}_1}](z_1,\bar{z}_1) \cdots \mathbf{L}[\mathcal{O}^{\varepsilon_n}_{h_n,\bar{h}_n}](z_n,\bar{z}_n)\rangle \,. \tag{5.32}$$

When working with symmetric and antisymmetric primaries, we will also simpify the notation to

$$L(1^-_S \cdots n^+_A) \equiv \langle \overline{\mathbf{L}}[\mathcal{O}^{S}_{h_1,\bar{h}_1}](z_1,\bar{z}_1) \cdots \mathbf{L}[\mathcal{O}^{A}_{h_n,\bar{h}_n}](z_n,\bar{z}_n)\rangle \,. \tag{5.33}$$

5.4.1 Conventions for spinor-helicity

We can express momentum space amplitudes in quite a convenient way through spinor-helicity variables[6], and we will introduce the conventions we will use throughout. In Klein space, we can rearrange the components of a given massless particle's momentum p^μ into a matrix of determinant zero

$$p^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} p^0+p^3 & p^1+p^2 \\ p^1-p^2 & p^0-p^3 \end{pmatrix} = \varepsilon\,\omega \begin{pmatrix} 1 & \bar{z} \\ z & z\bar{z} \end{pmatrix} = \varepsilon\,\omega \begin{pmatrix} 1 \\ z \end{pmatrix} \otimes \begin{pmatrix} 1 & \bar{z} \end{pmatrix} , \tag{5.34}$$

where $\alpha = 0, 1$, $\dot{\alpha} = \dot{0}, \dot{1}$ are spinor indices of two components that, under a Lorentz transformation, transform in the fundamental representation of $SL(2,\mathbb{R})$, and can be raised and lowered by 2×2 Levi Civita symbols.

From this decomposition, we can define spinor-helicity variables $\lambda^a, \bar{\lambda}^{\dot{\alpha}} \in \mathbb{R}^2$ as independent spinors of two components such that

$$p^{\alpha\dot{\alpha}} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}} \,. \tag{5.35}$$

This allows us to define $\lambda^\alpha, \bar{\lambda}^{\dot{\alpha}}$ only up to to little group scaling

$$\lambda^\alpha \sim s\,\lambda^\alpha \,, \qquad \bar{\lambda}^{\dot{\alpha}} \sim s^{-1}\bar{\lambda}^{\dot{\alpha}} \,, \qquad s \in \mathbb{R}^* \,. \tag{5.36}$$

We can define the corresponding spinor-helicity eigenstate with these variables as $|\lambda, \bar{\lambda}, J\rangle$. These

[6] A textbook reference can be found in [129].

states obey the following transformation law under little group scalings.

$$|s\lambda, s^{-1}\bar{\lambda}, J\rangle = s^{-2J}\,|\lambda, \bar{\lambda}, J\rangle \qquad \forall\, s \in \mathbb{R}^*. \tag{5.37}$$

From here we can build scattering amplitudes of states $|\lambda_i, \bar{\lambda}_i, J_i\rangle$, $i = 1, \dots, n$: these are linear functionals on these states, and as such they will transform in the same way under little group scalings:

$$\begin{aligned} A(p_i, J_i) &\equiv A(\lambda_i, \bar{\lambda}_i, J_i)\,, \\ A(s_i\lambda_i, s_i^{-1}\bar{\lambda}_i, J_i) &= \prod_{k=1}^{n} s_k^{-2J_k}\, A(\lambda_i, \bar{\lambda}_i, J_i)\,. \end{aligned} \tag{5.38}$$

Interestingly enough, this is very helpful to impose constraints on the structure of the amplitude, especially in the case of three external operators. In particular, Lorentz invariance forces these amplitudes to be functions of the $SL(2,\mathbb{R})$ invariant contractions

$$\langle i\, j\rangle = \epsilon_{\beta\alpha}\lambda_i^\alpha\lambda_j^\beta\,, \qquad [i\, j] = \epsilon_{\dot\beta\dot\alpha}\bar{\lambda}_i^{\dot\alpha}\bar{\lambda}_j^{\dot\beta}\,. \tag{5.39}$$

A final note on convention is that we will, most of the time, fix little group scalings as

$$\lambda_i^\alpha = \sqrt{\omega_i}\begin{pmatrix} 1 \\ z_i \end{pmatrix}\,, \qquad \bar{\lambda}_i^{\dot\alpha} = \varepsilon_i\sqrt{\omega_i}\begin{pmatrix} 1 \\ \bar{z}_i \end{pmatrix}\,. \tag{5.40}$$

5.4.2 Light-transformed two-point amplitude

Let us start by considering two-point amplitudes in the bulk of Minkowski space. In that context, these represent free particles propagating through the bulk, and the setup is clear. In order to write the amplitude in $(2,2)$ signature we need to use the extrapolate dictionary [130]. In particular, the S-matrix in $(1,3)$ signature is prepared through the smearing of in (out) states on the past (future) conformal boundary, however in $(2,2)$ signature we only have one boundary component, thus turning the S-matrix into an S-vector [118]. This agrees with our boundary-correlator interpretation.

We decide to choose operator normalisation such that the only invariant content of the two-point function is the particle species and the corresponding irreducible representation of Poincaré

($m = 0$ and helicity $l = \varepsilon J$).

If we Mellin transform to boost eigenstates, we can find the following formula for two-point correlator of particles of the same species[7]:

$$\langle \mathcal{O}^{\varepsilon_1}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}^{\varepsilon_2}_{h_2,\bar{h}_2}(z_2,\bar{z}_2)\rangle = (2\pi)^3\,\Theta\left(-\frac{\varepsilon_1}{\varepsilon_2}\right)\boldsymbol{\delta}(\mathrm{i}(\Delta_1+\Delta_2-2))\,\delta_{J_1,-J_2}\,\delta^2(z_1-z_2)\,. \tag{5.41}$$

It can seem that using Heavyside step functions Θ in this context is a bit of an overkill, however they appear naturally in the context of light transforms as we will see later. Note also that the $\delta_{J_1,-J_2}$ is a byproduct of labelling celestial amplitudes through their outgoing helicity. Finally, one can find the definition of $\boldsymbol{\delta}$ in [131] and its purpose is to allow an analytical continuation off the Principal series. This is important in the context of studying conformal soft modes [132], but it reduces to a Dirac delta function if we are in the principal series.

Let us now apply our prescription for light-transforming into a more 'CFT'-like shape. We will z-smear the $+$ helicity and $\bar{z}$-smear the $-$ helicity. We will be using the shorthand notation (5.32).

We introduce here some notation that will be maintained for the rest of the chapter: we will use sans serif font h_i for the light-transformed weights under our prescription[8], and serif font h_i for the non-transformed weights.

$$(\mathsf{h}_i,\bar{\mathsf{h}}_i) = \begin{cases} (1-h_i,\bar{h}_i) & \text{if } J_i = h_i-\bar{h}_i > 0 \\ (h_i,1-\bar{h}_i) & \text{if } J_i = h_i-\bar{h}_i < 0. \end{cases} \tag{5.42}$$

With that in mind, the light-transformed amplitude becomes [3]

$$L(1^-2^+) = \Theta\left(\frac{\varepsilon_1\bar{z}_{12}}{\varepsilon_2 z_{12}}\right)\frac{(2\pi)^3\boldsymbol{\delta}(2\mathrm{i}\mathsf{h}_{12})}{|z_{12}|^{2\mathsf{h}_1}|\bar{z}_{12}|^{2\bar{\mathsf{h}}_1}}, \tag{5.43}$$

where $1^-, 2^+$ in this case are particles with the same spin but opposite helicity (recall that otherwise the two-point function would be zero). We also have introduced the standard abbreviations

$$\mathsf{h}_{ij} \equiv \mathsf{h}_i - \mathsf{h}_j\,, \qquad z_{ij} \equiv z_i - z_j\,, \qquad \bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j\,, \tag{5.44}$$

[7] The two-point function of particles of different species is zero.
[8] One can quickly check their relation to un-transformed weights using 5.1.

and finally we have used the fact that $J_1 + J_2 = 0 \Rightarrow \Delta_1 + \Delta_2 - 2 = 2(\mathsf{h}_1 - \mathsf{h}_2)$ to make simplifications.

This is great! We have managed to put the amplitude into a power law form that agrees with what we would expect from a 2-point CFT correlator.

However, we would ideally want to diagonalise two point functions into the symmetric and antisymmetric combinations, which allow us to obtain all correlators for a free bulk theory. [3]

$$L(1_S^- 2_A^+) = L(1_A^- 2_S^-) = 0\,, \tag{5.45}$$

$$L(1_S^- 2_S^+) = \frac{(2\pi)^3\,\boldsymbol{\delta}(\mathsf{i}\mathsf{h}_{12})}{|z_{12}|^{2\mathsf{h}_1}|\bar{z}_{12}|^{2\bar{\mathsf{h}}_1}}\,, \tag{5.46}$$

$$L(1_A^- 2_A^+) = \mathrm{sgn}(z_{12}\bar{z}_{12})\,\frac{(2\pi)^3\,\boldsymbol{\delta}(\mathsf{i}\mathsf{h}_{12})}{|z_{12}|^{2\mathsf{h}_1}|\bar{z}_{12}|^{2\bar{\mathsf{h}}_1}}\,. \tag{5.47}$$

This is rather interesting overall: the light transformation eliminates the distributional delta functions, and furthermore transforming to boost+$\mathbb{Z}_2$ eigenstates allows us to remove the step functions, which leads to a final result that looks and behaves much more like a standard CFT correlator.

5.4.3 Light-transformed three-point amplitude

We can find the delta-stripped three-point amplitude by imposing Poincaré invariance, little group covariance and locality, obtaining [129]

$$A[1^{J_1}2^{J_2}3^{J_3}] = \begin{cases} c_{123}\,\langle 1\,2\rangle^{J_3-J_1-J_2}\langle 3\,2\rangle^{J_1-J_2-J_3}\langle 1\,3\rangle^{J_2-J_3-J_1}, & J_1+J_2+J_3<0 \\ \tilde{c}_{123}\,[1\,2]^{J_1+J_2-J_3}[3\,2]^{J_2+J_3-J_1}[1\,3]^{J_3+J_1-J_2}, & J_1+J_2+J_3>0, \end{cases} \tag{5.48}$$

with J_i the outgoing helicities.

Note also that following our conventions (5.40) we have that

$$\langle i\,j\rangle = \sqrt{\omega_i\omega_j}\,z_{ij}\,, \qquad [i\,j] = \varepsilon_i\varepsilon_j\sqrt{\omega_i\omega_j}\,\bar{z}_{ij}\,. \tag{5.49}$$

This causes the conditions on the spins from (5.48) to make the amplitude vanish on the z_{ij} or $\bar{z}_{ij} = 0$ locus, and to have non-trivial support on the other locus.

We perform the Mellin transform following [41]. The integral splits into two factors: a

scale integral that will give a delta function over the weights, and a simplex integral over the $\sigma_i := \omega_i/(\sum_i \omega_i)$. At this point, we need to perform the analytic continuation to $\mathbb{R}^{2,2}$ because otherwise the integral vanishes when the delta function that conserves momentum is restored. This gives us two loci of support where the amplitude is of interest: $z_{ij} \neq 0$ or $\bar{z}_{ij} \neq 0$. Furthermore, the simplex integrals lead us to deduce that different kinematical crossings will have disjoint support, which the reader can consult in more detail in [3].

We want to, without loss of generality, focus on the MHV case, for which one can obtain the following 3-point Mellin amplitude:

$$\langle \mathcal{O}^{\varepsilon_1}_{h_1,\bar{h}_1}(z_1,\bar{z}_1)\mathcal{O}^{\varepsilon_2}_{h_2,\bar{h}_2}(z_2,\bar{z}_2)\mathcal{O}^{\varepsilon_3}_{h_3,\bar{h}_3}(z_3,\bar{z}_3)\rangle = \pi c_{123}\,\mathrm{sgn}(z_{12}z_{13}z_{23})^{s_1+s_2+s_3} \times \frac{\boldsymbol{\delta}(\mathrm{i}(\bar{h}-2))\delta(\bar{z}_{12})\delta(\bar{z}_{13})\Theta\left(\frac{\varepsilon_3 z_{23}}{\varepsilon_1 z_{12}}\right)\Theta\left(\frac{\varepsilon_3 z_{31}}{\varepsilon_2 z_{12}}\right)}{|z_{12}|^{h_1+h_2-h_3}|z_{23}|^{h_2+h_3-h_1}|z_{31}|^{h_1+h_3-h_2}}\,, \quad (5.50)$$

where $\bar{h} = \sum_i \bar{h}_i$ and we have assumed that all of the J_i are integers so that $(-1)^{\pm J_i} = (-1)^{s_i}$ for $s_i = |J_i|$. One should also note at this stage that $\mathrm{sgn}(z_{12}z_{13}z_{23})$ is $SL(2,\mathbb{R})$ invariant. All the theory-dependent content is c_{123}, the three-point coupling.

Now, let us consider the MHV amplitude of (5.50) when particles 1 and 2 have negative helicity and 3 has positive helicity and apply our prescription to render it non-distributional.

The ambidextrous prescription as proposed tells us we should light transform in $\bar{z}_1, \bar{z}_2, z_3$, as it can be illustrated in figure 5.2.

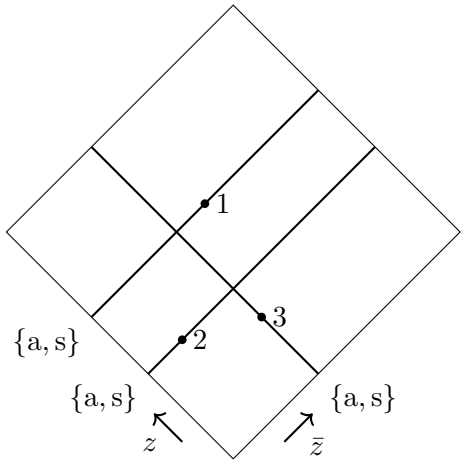


Figure 5.2: We link the null rays over which the light transforms are performed, thus controlling the causal dependence of the 3-point function. Helicities' signs will determine the direction of the lightcone one integrates over, and as a result different assignments of 'A' and 'S' causes OPE coefficients that aren't symmetric under the exchange of labels.

We can see how the delta functions in $\bar{z}_{ij}$ present in (5.50) localise the $\overline{\mathbf{L}}$ transforms, so we only really need to 'compute' the z_3 integral, which has the effect of performing the Weyl reflection $h_3 \to \mathsf{h}_3$, and we have a CFT-looking object[9]!

Using again $\mathsf{h}_i, \bar{\mathsf{h}}_i$ to refer to the light-transformed weights, we find that the correlator can be written as [3]

$$\begin{aligned} L(1^-2^-3^+) = \pi c_{123}\, \boldsymbol{\delta}(\mathrm{i}(\bar{\mathsf{h}}_1 + \bar{\mathsf{h}}_2 - \bar{\mathsf{h}}_3))|\bar{z}_{12}|^{-\bar{\mathsf{h}}_1-\bar{\mathsf{h}}_2+\bar{\mathsf{h}}_3}|\bar{z}_{13}|^{-\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2-\bar{\mathsf{h}}_3}|\bar{z}_{23}|^{\bar{\mathsf{h}}_1-\bar{\mathsf{h}}_2-\bar{\mathsf{h}}_3} \\ \times |z_{12}|^{-\mathsf{h}_1-\mathsf{h}_2+\mathsf{h}_3}|z_{13}|^{-\mathsf{h}_1+\mathsf{h}_2-\mathsf{h}_3}|z_{23}|^{\mathsf{h}_1-\mathsf{h}_2-\mathsf{h}_3}\mathrm{sgn}(z_{12}z_{23}z_{13})^{s_1+s_2+s_3}\mathcal{N}. \end{aligned} \tag{5.51}$$

We have split the correlator into two factors: a conformally invariant factor $\mathcal{N}$, and a conformally covariant part (the rest of the correlator). One should note here that the conformally covariant part behaves as a three-point correlator would under a conformal transformation.

The conformally invariant factor $\mathcal{N}$ is of especial interest to us as, together with c_{123}, it contains the OPE data. It can be computed by solving the integral

$$\begin{aligned} \mathcal{N} = \int \mathrm{d}z_3' \, \frac{|z_{12}|^{-2\mathsf{h}_3+1}|z_{13}|^{\mathsf{h}_1-\mathsf{h}_2+\mathsf{h}_3}|z_{23}|^{-\mathsf{h}_1+\mathsf{h}_2+\mathsf{h}_3}}{|z_{3'3}|^{2\mathsf{h}_3}|z_{23'}|^{\mathsf{h}_2+1-\mathsf{h}_3-\mathsf{h}_1}|z_{3'1}|^{\mathsf{h}_1+1-\mathsf{h}_3-\mathsf{h}_2}} \\ \times \mathrm{sgn}\left(\frac{z_{23'}z_{13'}}{z_{23}z_{13}}\right)^{\sum_i s_i} \Theta\left(\frac{\varepsilon_3 z_{23'}\bar{z}_{31}}{\varepsilon_1 z_{12}z_{3'3}}\right)\Theta\left(\frac{\varepsilon_3 z_{3'1}\bar{z}_{32}}{\varepsilon_2 z_{12}z_{3'3}}\right), \end{aligned} \tag{5.52}$$

where $z_{i'j} = -z_{ji'} \equiv z_i' - z_j$.

This integral can be evaluated[10] by introducing a cross-ratio u (where we treat z_1, z_2, z_3, z_3' as indepent points) and conformally invariant signs η_1, η_2:

$$u = \frac{z_{31}z_{23'}}{z_{12}z_{3'3}}, \quad \eta_1 = \mathrm{sgn}\left(\frac{\varepsilon_3\bar{z}_{13}}{\varepsilon_1 z_{13}}\right), \quad \eta_2 = \mathrm{sgn}\left(\frac{\varepsilon_3\bar{z}_{23}}{\varepsilon_2 z_{23}}\right), \tag{5.53}$$

after which one obtains an integral in u that can be solved by breaking the domain into steps. The final result is [3]

$$\begin{aligned} \mathcal{N}(\mathsf{h}_i, \eta_i) = \Big[&(-1)^{s_1+s_2+s_3}\Theta(\eta_1)\Theta(\eta_2)B(\mathsf{h}_3+\mathsf{h}_1-\mathsf{h}_2, \mathsf{h}_2+\mathsf{h}_3-\mathsf{h}_1) \\ &+ \Theta(\eta_1)\Theta(-\eta_2)B(1-2\mathsf{h}_3, \mathsf{h}_2+\mathsf{h}_3-\mathsf{h}_1) + \Theta(-\eta_1)\Theta(\eta_2)B(1-2\mathsf{h}_3, \mathsf{h}_3+\mathsf{h}_1-\mathsf{h}_2)\Big] \end{aligned} \tag{5.54}$$

[9] Admittedly, after some tricks with the Heavyside Theta functions, see [3] for details.
[10] Again, for details, we encourage the reader to consult [3].

where $B(x,y) = \Gamma(x)\,\Gamma(y)/\Gamma(x+y)$ is the Euler beta function.

Not only is this overall result conformally covariant and non-distributional, but it more importantly contains the beta function OPE coefficients that are known to be present for the celestial OPE of two boost eigenstates of the same helicity [133].

Since we expect the Celestial CFT to be, well, a CFT, we should understand well the symmetries that the three-point function has, as they should determine higher-point correlators.

- If we have an MHV vertex, there is a conservation law for the $\bar{\mathsf{h}}$ weights.
- If we have an anti-MHV vertex, there is a conservation law for the h weights.
- The two point function preserves h and $\bar{\mathsf{h}}$, which allows to glue vertices in CSW-like expansions.
- There is a discrete parity symmetry under $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \to (-\varepsilon_1, -\varepsilon_2, -\varepsilon_3)$.

This last symmetry is rather interesting, and it is related to the fact that the conformally invariant signs defined in (5.53) depend only on $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ through the *sign* vector

$$\vec{s} = (\varepsilon_1\varepsilon_3, \varepsilon_2\varepsilon_3)\,. \tag{5.55}$$

Note that this is invariant under the discrete symmetry we referred to: one can build a one-to-one map between the entries of $\vec{s}$ and the different crossing channels.

It is this symmetry that influences greatly the structure of light-transformed amplitudes for the symmetric and antisymmetric primaries, as the sum over channels corresponds to including all chambers of the step functions.

Let us step back for a moment here and set some notation for a general amplitude. We are going to write a formal label for the (anti)symmetry of the i-th particle

$$\mathfrak{p}_i \in \{S, A\}. \tag{5.56}$$

For a symmetric operator we weigh the two possible channels equally, and for an antisymmetric one we weigh the different channels with a relative phase. Taking this into account, we can obtain the light-transformed celestial correlator of a combination of antisymmetric and symmetric

primaries as

$$L(1_{\mathfrak{p}_1}^{J_1} 2_{\mathfrak{p}_2}^{J_2} \cdots n_{\mathfrak{p}_n}^{J_n}) = \sum_{\varepsilon_i=\pm 1} \prod_{\mathfrak{p}_j=A} \varepsilon_j \, L(1^{J_1} 2^{J_2} \cdots n^{J_n}) \,, \tag{5.57}$$

where $L(1^{J_1} 2^{J_2} \cdots n^{J_n})$ implicitly depends on the ε_j.

What does this mean for a three-point result?

Well, in a general case we obtain a result that can still be separated in a covariant and invariant parts [3]:

$$\begin{aligned} L(1_{\mathfrak{p}_1}^{-} 2_{\mathfrak{p}_2}^{-} 3_{\mathfrak{p}_3}^{+}) = \mathcal{N}^{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3} \pi c_{123} \, \boldsymbol{\delta}(\mathrm{i}(\bar{\mathsf{h}}_1 + \bar{\mathsf{h}}_2 - \bar{\mathsf{h}}_3)) |\bar{z}_{12}|^{-\bar{\mathsf{h}}_1 - \bar{\mathsf{h}}_2 + \bar{\mathsf{h}}_3} |\bar{z}_{13}|^{-\bar{\mathsf{h}}_1 + \bar{\mathsf{h}}_2 - \bar{\mathsf{h}}_3} |\bar{z}_{23}|^{\bar{\mathsf{h}}_1 - \bar{\mathsf{h}}_2 - \bar{\mathsf{h}}_3} \\ \times |z_{12}|^{-\mathsf{h}_1 - \mathsf{h}_2 + \mathsf{h}_3} |z_{13}|^{-\mathsf{h}_1 + \mathsf{h}_2 - \mathsf{h}_3} |z_{23}|^{\mathsf{h}_1 - \mathsf{h}_2 - \mathsf{h}_3} \mathrm{sgn}(z_{12} z_{23} z_{13})^{s_1+s_2+s_3} . \end{aligned} \tag{5.58}$$

As before, the conformally invariant part corresponds to $\mathcal{N}^{\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3}$. Before we give an explicit expression for the possible nonzero $\mathcal{N}$'s, we should note that $\mathbb{Z}_2$ symmetry adds an extra factor of 2 in each possible combination of $\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$, as well as the fact that the step functions one would expect to make an appearance are absorbed by the sum over channels [123].

$$\begin{aligned} \mathcal{N}^{\bar{\mathrm{s}}\bar{\mathrm{s}}\mathrm{s}} = 2\Big[& (-1)^{s_1+s_2+s_3} B(\mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) \\ & + B(1 - 2\mathsf{h}_3, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) + B(1 - 2\mathsf{h}_3, \mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2) \Big], \end{aligned} \tag{5.59}$$

$$\begin{aligned} \mathcal{N}^{\bar{\mathrm{s}}\bar{\mathrm{a}}\mathrm{a}} = 2\,\mathrm{sgn}(z_{23}\bar{z}_{23}) \Big[& (-1)^{s_1+s_2+s_3} B(\mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) \\ & + B(1 - 2\mathsf{h}_3, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) - B(1 - 2\mathsf{h}_3, \mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2) \Big], \end{aligned} \tag{5.60}$$

$$\begin{aligned} \mathcal{N}^{\bar{\mathrm{a}}\bar{\mathrm{s}}\mathrm{a}} = 2\,\mathrm{sgn}(z_{13}\bar{z}_{13}) \Big[& (-1)^{s_1+s_2+s_3} B(\mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) \\ & - B(1 - 2\mathsf{h}_3, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) + B(1 - 2\mathsf{h}_3, \mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2) \Big], \end{aligned} \tag{5.61}$$

$$\begin{aligned} \mathcal{N}^{\bar{\mathrm{a}}\bar{\mathrm{a}}\mathrm{s}} = 2\,\mathrm{sgn}(z_{13} z_{23} \bar{z}_{13} \bar{z}_{23}) \Big[& (-1)^{s_1+s_2+s_3} B(\mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) \\ & - B(1 - 2\mathsf{h}_3, \mathsf{h}_2 + \mathsf{h}_3 - \mathsf{h}_1) - B(1 - 2\mathsf{h}_3, \mathsf{h}_3 + \mathsf{h}_1 - \mathsf{h}_2) \Big], \end{aligned} \tag{5.62}$$

and finally

$$\mathcal{N}^{\bar{\mathrm{s}}\bar{\mathrm{s}}\mathrm{a}} = \mathcal{N}^{\bar{\mathrm{s}}\bar{\mathrm{a}}\mathrm{s}} = \mathcal{N}^{\bar{\mathrm{a}}\bar{\mathrm{s}}\mathrm{s}} = \mathcal{N}^{\bar{\mathrm{a}}\bar{\mathrm{a}}\mathrm{a}} = 0. \tag{5.63}$$

The last set of vanishing results is expected as the discrete symmetry makes any correlators with

an odd number of A operators vanish identically.

5.4.4 Light-transformed four point amplitude

The four-point momentum-conservation-stripped amplitude is quite more complicated than its three-point counterpart, but we can still factor out a relevant part capturing the little group covariance [134, Appendix A].

$$A[1^{J_1}2^{J_2}3^{J_3}4^{J_4}] = H(\lambda_i,\bar{\lambda}_i)M(s,t), \quad H(\lambda_i,\bar{\lambda}_i) = \prod_{i<j}\left(\frac{\langle i\, j\rangle}{[i\, j]}\right)^{\frac{1}{2}(\frac{1}{3}\sum_k J_k - J_i - J_j)}, \tag{5.64}$$

with $s = (p_1+p_2)^2$ and $t = (p_1+p_3)^2$ the usual Mandelstam invariants. One can obtain the locus of momentum conservation following again [41].

Again, more details of the calculation can be found in [3], but we want to highlight some similarities with the three-point functions, as well as some relevant particulars.

When localising the simplex integrals, one finds again that there are different kinematical crossings with different regions of support of cross ratios. In particular, a key difference that appears when working in $(2,2)$ signature with respect to $(1,3)$ is that there are additional channels that have non-trivial support [123].

Furthermore, it is useful to work with the usual conformal cross ratios

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}, \tag{5.65}$$

which are restricted to be real as a result of the support of momentum conservation.

We should also define some *lightcone-indicator* variables

$$\eta_{ij} = \mathrm{sgn}\left(\varepsilon_i\varepsilon_j z_{ij}\bar{z}_{ij}\right), \tag{5.66}$$

that allow us to generalise the sign-vectors that were introduced in the three-point case to a labelling of the relevant channels. They are indeed Lorentz invariant, but they are not all independent from each other.

Once these have been introduced to express the relevant indicator functions, we can look into the Mellin transformed amplitude. As we are in the four-point function, we would like to

factorise it into a covariant leg factor and an invariant piece. The key is that we want the leg factor to be kinematical and take into account the external weights, however there is not a single way to define it and each specific context will find some definitions more useful than others. We used as a starting point (5.64), which is in the conventions of [41,134], however for our purposes it will be more useful to utilise the following correlator decomposition [135,136], where we will be using the support $z = \bar{z}$ throughout to simplify expressions.

$$\left\langle \prod_{i=1}^{4} \mathcal{O}^{\varepsilon_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i) \right\rangle = \delta(z-\bar{z})\, Y(z_j,\bar{z}_j)\, G(z) \prod_{i=1}^{4} \mathbf{1}_{[0,1]}(\sigma_{*i}) \prod_{j<k} (\eta_{jk})^{\frac{1}{2}(\frac{1}{3}\sum_\ell J_\ell - J_j - J_k)}\,, \quad (5.67)$$

where the indicator function can be expanded as

$$\prod_{i=1}^{4} \mathbf{1}_{[0,1]}(\sigma_{*i}) = \Theta\left(-\frac{\varepsilon_4}{\varepsilon_1}\frac{z_{24}\bar{z}_{34}}{z_{12}\bar{z}_{13}}\right)\Theta\left(\frac{\varepsilon_4}{\varepsilon_2}\frac{z_{34}\bar{z}_{14}}{z_{23}\bar{z}_{12}}\right)\Theta\left(-\frac{\varepsilon_4}{\varepsilon_3}\frac{z_{24}\bar{z}_{14}}{z_{23}\bar{z}_{13}}\right)\,, \quad (5.68)$$

and the sign factor is

$$\prod_{j<k} (\eta_{jk})^{\frac{1}{2}(\frac{1}{3}\sum_\ell J_\ell - J_j - J_k)} = \eta_{12}^{\frac{1}{2}(J_3+J_4-J_1-J_2)} \eta_{13}^{\frac{1}{2}(J_2+J_4-J_1-J_3)} \eta_{14}^{\frac{1}{2}(J_2+J_3-J_1-J_4)}\,. \quad (5.69)$$

Furthermore, the leg factor is

$$\begin{aligned} Y(z_i,\bar{z}_i) &= \left|\frac{z_{24}}{z_{14}}\right|^{h_{12}} \left|\frac{z_{14}}{z_{13}}\right|^{h_{34}} \left|\frac{\bar{z}_{24}}{\bar{z}_{14}}\right|^{\bar{h}_{12}} \left|\frac{\bar{z}_{14}}{\bar{z}_{13}}\right|^{\bar{h}_{34}} \frac{|1-z|^{\frac{1}{2}(h_{12}-h_{34})}}{|z_{12}|^{h_1+h_2}|z_{34}|^{h_3+h_4}} \frac{|1-\bar{z}|^{\frac{1}{2}(\bar{h}_{12}-\bar{h}_{34})}}{|\bar{z}_{12}|^{\bar{h}_1+\bar{h}_2}|\bar{z}_{34}|^{\bar{h}_3+\bar{h}_4}} \\ &= \left|\frac{1-z}{z^2}\right|^{\frac{1}{6}\sum_k h_k} \left|\frac{1-\bar{z}}{\bar{z}^2}\right|^{\frac{1}{6}\sum_k \bar{h}_k} \prod_{i<j} |z_{ij}|^{\frac{1}{3}\sum_k h_k - h_i - h_j} |\bar{z}_{ij}|^{\frac{1}{3}\sum_k \bar{h}_k - \bar{h}_i - \bar{h}_j}\,, \end{aligned} \quad (5.70)$$

with $h_{ij} = h_i - h_j$, $\bar{h}_{ij} = \bar{h}_i - \bar{h}_j$ as usual.

Finally, the conformally invariant data of the 4-point amplitude is the part is theory-dependent, and as such we cannot write it explicitly without specifying our amplitude's theoretical framework.

$$G(z) = \frac{z^2}{2^{\beta/2}} \int_0^\infty \frac{\mathrm{d}\omega}{\omega}\, \omega^\beta\, M\big(\eta_{12}\omega^2, -\eta_{12}\omega^2 z^{-1}\big)\,, \quad (5.71)$$

with $\beta = \sum_{i=1}^4 (\Delta_i - 1)$ the net boost weight, and $M(s,t)$ the little group invariant part of the momentum space amplitude (5.64).

It is rather fascinating to see how one can draw already at this stage a correspondence

between the invariant CFT data and the invariant amplitude data.

But our work here isn't done: we need to light-transform this amplitude. We will consider, without loss of generality, $1^-2^-3^+4^+$, and denote the ambidextrous light transform as

$$L(1^-2^-3^+4^+) = \int_{\mathbb{R}^4} \frac{\mathrm{d}\bar{z}'_1}{|\bar{z}_{1'1}|^{2-2\bar{\mathsf{h}}_1}} \frac{\mathrm{d}\bar{z}'_2}{|\bar{z}_{2'2}|^{2-2\bar{\mathsf{h}}_2}} \frac{\mathrm{d}z'_3}{|z_{3'3}|^{2-2\mathsf{h}_3}} \frac{\mathrm{d}z'_4}{|z_{4'4}|^{2-2\mathsf{h}_4}} \times \left\langle \prod_{\ell=1}^{2} \mathcal{O}^{\varepsilon'_\ell}_{h_\ell,\bar{h}_\ell}(z_\ell,\bar{z}'_\ell) \prod_{r=3}^{4} \mathcal{O}^{\varepsilon'_r}_{h_r,\bar{h}_r}(z'_r,\bar{z}_r) \right\rangle \tag{5.72}$$

with $z_{i'j} = z'_i - z_j$, $\bar{z}_{i'j} = \bar{z}'_i - \bar{z}_j$. The helicities are $h_\ell - \bar{h}_\ell < 0$ for $\ell = 1,2$ and $h_r - \bar{h}_r > 0$ for $r = 3,4$, and the light-transformed signs ε'_i are

$$(\varepsilon'_1,\varepsilon'_2,\varepsilon'_3,\varepsilon'_4) = (\mathrm{sgn}(\bar{z}_{1'1})\varepsilon_1, \mathrm{sgn}(\bar{z}_{2'2})\varepsilon_2, \mathrm{sgn}(z_{3'3})\varepsilon_3, \mathrm{sgn}(z_{4'4})\varepsilon_4)\,. \tag{5.73}$$

We want to compute the integrals whilst decomposing the final result again into a covariant leg factor and an invariant dynamical piece.

To be able to perform the integrals, one needs to make some smart choices as integration variables (see also parallel work on this in [137]) in the form of cross-ratios between the puncture and integrated points, and then re-use the cross-ratio $z, \bar{z}$ (5.65) and the conformally invariant signs η_{ij} (5.66) with $i = 1,2$ and $j = 3,4$ but for the light-transformed operators. For some more detail, we encourage the reader to consult [3].

What do we obtain at the end of this road? The overall ambidextrous light-transformed 4-point amplitude can be expressed as [3]

$$L(1^-2^-3^+4^+) = K(z_i,\bar{z}_i)\,\mathscr{L}(z,\bar{z})\,, \tag{5.74}$$

where $K(z_i,\bar{z}_i)$ is a leg factor extracted in such a way as to facilitate conformal block expansions and alpha space transforms,

$$K(z_i,\bar{z}_i) = \left|\frac{z_{24}}{z_{14}}\right|^{\mathsf{h}_{12}} \left|\frac{z_{14}}{z_{13}}\right|^{\mathsf{h}_{34}} \left|\frac{\bar{z}_{24}}{\bar{z}_{14}}\right|^{\bar{\mathsf{h}}_{12}} \left|\frac{\bar{z}_{14}}{\bar{z}_{13}}\right|^{\bar{\mathsf{h}}_{34}} \frac{1}{|z_{12}|^{\mathsf{h}_1+\mathsf{h}_2}|z_{34}|^{\mathsf{h}_3+\mathsf{h}_4}} \frac{1}{|\bar{z}_{12}|^{\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2}|\bar{z}_{34}|^{\bar{\mathsf{h}}_3+\bar{\mathsf{h}}_4}}, \tag{5.75}$$

and $\mathscr{L}(z,\bar{z})$ is the conformally invariant, dynamical data, which will be theory-dependent

$$\begin{aligned}\mathscr{L}(z,\bar{z}) = 2^{-\frac{\beta}{2}}\,\frac{|z|^{\mathsf{h}_3+\mathsf{h}_4}|\bar{z}|^{\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2}}{|1-z|^{\mathsf{h}_{34}-\mathsf{h}_{12}}}\int_{-\infty}^{\infty}\mathrm{d}x\,|x|^{\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2+\mathsf{h}_3+\mathsf{h}_4-2}|1-x|^{\frac{1}{2}(J_{12}-J_{34})}\\ \times\int_{-\infty}^{\infty}\mathrm{d}t\,|1-t|^{\mathsf{h}_3+\mathsf{h}_4-\mathsf{h}_{12}-1}|x-t|^{-2\mathsf{h}_3}|z-t|^{-2\mathsf{h}_4}\\ \times\int_{-\infty}^{\infty}\mathrm{d}\bar{t}\,|1-\bar{t}|^{\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2-\bar{\mathsf{h}}_{34}-1}|x-\bar{t}|^{-2\bar{\mathsf{h}}_1}|\bar{z}-\bar{t}|^{-2\bar{\mathsf{h}}_2}\\ \times\Theta_4(\eta_{ij},z,\bar{z},x,t,\bar{t})\,\mathfrak{s}_4(\eta_{ij},z,\bar{z},x,t,\bar{t})\int_0^{\infty}\frac{\mathrm{d}\omega}{\omega}\,\omega^{\beta}\,M\big(\eta_{1'2'}\omega^2,\eta_{1'2'}\omega^2x^{-1}\big)\,.\end{aligned}\tag{5.76}$$

Here $J_{ij}=J_i-J_j$, and t, $\bar{t}$, x are cross-ratios defined to aide the computation

$$x=\frac{z_{12}z_{3'4'}}{z_{13'}z_{24'}}=\frac{\bar{z}_{1'2'}\bar{z}_{34}}{\bar{z}_{1'3}\bar{z}_{2'4}}\,,\qquad t=\frac{z_{12}z_{34'}}{z_{13}z_{24'}}\,,\qquad \bar{t}=\frac{\bar{z}_{12'}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{2'4}}\,,\tag{5.77}$$

where the intermediate steps lead to that double definition of x.

The transformed indicator function Θ_4 is

$$\Theta_4(\eta_{ij},z,\bar{z},x,t,\bar{t})\equiv\Theta\bigg(\eta_{13}\,\frac{x-t}{x-\bar{t}}\bigg)\Theta\bigg(\eta_{14}\,\frac{(z-1)(x-\bar{t})}{(z-t)(1-\bar{t})}\bigg)\Theta\bigg(\eta_{24}\,\frac{(1-t)(\bar{z}-\bar{t})}{(z-t)(1-\bar{t})}\bigg)\,,\tag{5.78}$$

and the sign factor is

$$\begin{aligned}\mathfrak{s}_4(\eta_{ij},z,\bar{z},x,t,\bar{t})\equiv\Big(\eta_{12}\,\mathrm{sgn}[\bar{z}(1-\bar{t})(x-\bar{t})(\bar{z}-\bar{t})]\Big)^{\frac{1}{2}\sum_i J_i}\\ \times(\mathrm{sgn}\,x)^{\frac{1}{2}(J_3+J_4-J_1-J_2)}\,(\mathrm{sgn}\,(1-x))^{\frac{1}{2}(J_2+J_3-J_4-J_1)}\,.\end{aligned}\tag{5.79}$$

Interestingly enough, the integrals over $t,\bar{t}$ are of the form of Euler hypergeometric integrals, but restricted to the integration ranges determined by the step functions: this gives us already a hint of the CFT-like structure of the four-point correlator.

One could go forward and build explicit linear combinations that give us the light transformed amplitudes of (anti)symmetric operators, however that is rather cumbersome to list for a general 4-point amplitude due to all the possible combinations that arise. Rather than doing that, we will illustrate how to obtain the MHV amplitude for the gluon and extract CFT data directly from it.

5.5 Celestial gluon amplitudes

Let us now focus more on a precise problem: tree-level gluon amplitudes. In this case, we can find efficient expressions that allow us to go directly from momentum amplitudes to light-transformed celestial amplitudes. Using these together with Grassmannian formulae for tree-level gluon amplitudes, we can derive expressions for light-transformed amplitudes of four gluon symmetric primaries. Afterwards, we extract spectral data on the dual CCFT.

5.5.1 From momentum eigenstates to light transforms

The main goal we have is to find a map from momentum space amplitudes directly to ligh-transformed symmetric and antisymmetric primaries. To this end, we will be using spinor-helicity variables.

We will only be discussing massless states with integral helicity $J \in \mathbb{Z}$, whose null momenta decompose as $p^{\alpha\dot{\alpha}} = \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$ in spinor-helicity variables. We can define scaled fixed spinors as

$$\zeta^\alpha = (1, z)\,, \qquad \bar{\zeta}^{\dot{\alpha}} = (1, \bar{z})\,, \tag{5.80}$$

and fix little group scaling through

$$\lambda^\alpha = \sqrt{\omega}\,\zeta^\alpha\,, \qquad \bar{\lambda}^{\dot{\alpha}} = \varepsilon\sqrt{\omega}\,\bar{\zeta}^{\dot{\alpha}}\,. \tag{5.81}$$

We can now extract $\varepsilon, \omega, z, \bar{z}$ through

$$\varepsilon = \mathrm{sgn}(\lambda^0 \bar{\lambda}^{\dot{0}})\,, \quad \omega = |\lambda^0 \bar{\lambda}^{\dot{0}}|\,, \quad z = \frac{\lambda^1}{\lambda^0}\,, \quad \bar{z} = \frac{\bar{\lambda}^{\dot{1}}}{\bar{\lambda}^{\dot{0}}}\,. \tag{5.82}$$

We earlier (section 5.2) denoted the momentum eigenstate of a helicity particle as $|\varepsilon\omega, z, \bar{z}, J\rangle$, but now we want to use (5.82) as the definition of $\varepsilon, \omega, z, \bar{z}$, we can define a more general spinor-helicity eigenstate

$$|\lambda, \bar{\lambda}, J\rangle = \mathrm{sgn}(\lambda^0)^{-2J} \left|\frac{\bar{\lambda}^{\dot{0}}}{\lambda^0}\right|^J |\varepsilon\omega, z, \bar{z}, J\rangle \tag{5.83}$$

labeled by spinor-helicity variables without making a choice of little group fixing. One can check

that they transform like (5.37) under little group scalings.

Now with these new states (5.83) and their little group transformations, the symmetric primary state (5.19) can be rewritten as

$$
\begin{aligned}
|z,\bar{z},h,\bar{h}\rangle_S &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{|\omega|}\,|\omega|^{\Delta}\,\Big|\,|\omega|^{\frac{1}{2}}\zeta\,,\,\mathrm{sgn}(\omega)|\omega|^{\frac{1}{2}}\bar{\zeta}\,,\,J\Big\rangle \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{|\omega|}\,|\omega|^{2\bar{h}}\,|\zeta,\omega\bar{\zeta},J\rangle \qquad (5.84)\\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{|\omega|}\,|\omega|^{2h}\,|\omega\zeta,\bar{\zeta},J\rangle\,, \qquad (5.85)
\end{aligned}
$$

and the antisymmetric (5.20) becomes

$$
\begin{aligned}
|z,\bar{z},h,\bar{h}\rangle_A &= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\omega}\,|\omega|^{2h}\,|\omega\zeta,\bar{\zeta},J\rangle \qquad (5.86)\\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{\omega}\,|\omega|^{2\bar{h}}\,|\zeta,\omega\bar{\zeta},J\rangle\,. \qquad (5.87)
\end{aligned}
$$

Now, the light transform $\mathbf{L}$ of a symmetric primary is

$$
\begin{aligned}
\mathbf{L}|z,\bar{z},h,\bar{h}\rangle_{\mathrm{s}} &= \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{|w-z|^{2-2h}}\,|w,\bar{z},h,\bar{h}\rangle_{\mathrm{s}} \\
&= \int_{-\infty}^{\infty}\mathrm{d}w\int_{-\infty}^{\infty}|\omega|\,\mathrm{d}\omega\,|\omega(w-z)|^{2h-2}\,|\omega\zeta,\bar{\zeta},J\rangle\,. \qquad (5.88)
\end{aligned}
$$

As this notation is a bit cumbersome, we can introduce a new integration variable of two component $\kappa^{\alpha}=\omega\,(1,w)$, which allows us to write the above light transform more compactly

$$
\mathbf{L}|z,\bar{z},h,\bar{h}\rangle_{\mathrm{s}} = \int_{\mathbb{R}^2}\mathrm{d}^2\kappa\,|\langle\kappa\,\zeta\rangle|^{2h-2}\,|\kappa,\bar{\zeta},J\rangle\,. \qquad (5.89)
$$

Note that we have used the notation $\langle\kappa\,\zeta\rangle\equiv\epsilon_{\beta\alpha}\kappa^{\alpha}\zeta^{\beta}=\omega(w-z)$ for spinor contractions.

We now list the other possible light transforms

$$
\begin{aligned}
\overline{\mathbf{L}}|z,\bar{z},h,\bar{h}\rangle_S &= \int_{\mathbb{R}^2}\mathrm{d}^2\bar{\kappa}\,|[\bar{\kappa}\,\bar{\zeta}]|^{2\bar{h}-2}\,|\zeta,\bar{\kappa},J\rangle\,, \qquad (5.90)\\
\mathbf{L}|z,\bar{z},h,\bar{h}\rangle_A &= \int_{\mathbb{R}^2}\mathrm{d}^2\kappa\,|\langle\kappa\,\zeta\rangle|^{2h-2}\,\mathrm{sgn}(\langle\kappa\,\zeta\rangle)\,|\kappa,\bar{\zeta},J\rangle\,, \qquad (5.91)\\
\overline{\mathbf{L}}|z,\bar{z},h,\bar{h}\rangle_A &= \int_{\mathbb{R}^2}\mathrm{d}^2\bar{\kappa}\,|[\bar{\kappa}\,\bar{\zeta}]|^{2\bar{h}-2}\,\mathrm{sgn}([\bar{\kappa}\,\bar{\zeta}])\,|\zeta,\bar{\kappa},J\rangle\,, \qquad (5.92)
\end{aligned}
$$

with $[\bar{\kappa}\,\bar{\zeta}] \equiv \epsilon_{\dot{\beta}\dot{\alpha}}\bar{\kappa}^{\dot{\alpha}}\bar{\zeta}^{\dot{\beta}}$. These light transforms hold some similarities to Witten's half-Fourier transforms to twistor/dual twistor space [138] which were crucial to investigate mathematical structures within scattering amplitudes. Indeed, a preliminary study between light transforms and the half-Fourier transforms can be found in [117]. We will move onto using equations (5.89)-(5.92) in the study of gluon amplitudes.

5.5.2 4-gluon amplitude

We are going to study colour-stripped, tree-level four-gluon amplitudes. There are many descriptions of those in the literature, but we find that the one introduced in [128] is the most useful for our light-transformed prescription. It is based on Grassmannian formulae.

We can define the 4-gluon momentum space amplitude through the Parke-Taylor formula

$$A(1^-\,2^-\,3^+\,4^+) = \frac{\langle 1\,2\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}\,\delta^4\Big(\sum_{i=1}^{4}\lambda_i\bar{\lambda}_i\Big)\,, \tag{5.93}$$

which can be represented through an integral over a Grassmannian $\mathrm{Gr}(2,4)$ [139, 140]

$$A(1^-\,2^-\,3^+\,4^+) = \int_{\mathbb{R}^4}\frac{\mathrm{d}c_{13}\,\mathrm{d}c_{14}\,\mathrm{d}c_{23}\,\mathrm{d}c_{24}}{c_{13}c_{24}(c_{13}c_{24}-c_{23}c_{14})}\,\delta^2(\bar{\lambda}_1+c_{13}\bar{\lambda}_3+c_{14}\bar{\lambda}_4)\,\delta^2(\bar{\lambda}_2+c_{23}\bar{\lambda}_3+c_{24}\bar{\lambda}_4)$$
$$\times\,\delta^2(\lambda_3-c_{13}\lambda_1-c_{23}\lambda_2)\,\delta^2(\lambda_4-c_{14}\lambda_1-c_{24}\lambda_2)\,. \tag{5.94}$$

The integration variables c_{13}, c_{14}, etc. are known as *link variables* and act as affine coordinates on $\mathrm{Gr}(2,4)$ when we arrange them into a 2×4 matrix

$$C = \begin{pmatrix} 1 & 0 & c_{13} & c_{14} \\ 0 & 1 & c_{23} & c_{24} \end{pmatrix}\,. \tag{5.95}$$

Integrating out the last four delta functions of (5.94) fixes

$$c_{13} = \frac{\langle 3\,2\rangle}{\langle 1\,2\rangle}\,,\quad c_{14} = \frac{\langle 4\,2\rangle}{\langle 1\,2\rangle}\,,\quad c_{23} = \frac{\langle 1\,3\rangle}{\langle 1\,2\rangle}\,,\quad c_{23} = \frac{\langle 1\,4\rangle}{\langle 1\,2\rangle} \tag{5.96}$$

and this results in a Jacobian of $\langle 1\,2\rangle^{-2}$. For these values of the link variables, the remaining integrand of (5.94) readily reduces to (5.93).

We can now combine the Grassmannian formula (5.94) with the integrals (5.89)-(5.92) to compute the corresponding ambidextrously light transformed celestial amplitudes. For simplicity, we take all four gluon states to be symmetric primaries with null momenta decomposed as

$$\begin{aligned} p_i^{\alpha\dot\alpha} &= \zeta_i^\alpha \bar\kappa_i^{\dot\alpha}\,, \qquad \zeta_i^\alpha = (1, z_i)\,, \qquad i = 1, 2\,, \\ p_i^{\alpha\dot\alpha} &= \kappa_i^\alpha \bar\zeta_i^{\dot\alpha}\,, \qquad \bar\zeta_i^{\dot\alpha} = (1, \bar z_i)\,, \qquad i = 3, 4\,. \end{aligned} \tag{5.97}$$

We thus need to perform the following integral

$$\begin{aligned} L(1_S^- 2_S^- 3_S^+ 4_S^+) = \prod_{i=1}^{2} \int_{\mathbb{R}^2} \mathrm{d}^2\bar\kappa_i \, |[\bar\kappa_i \, \bar\zeta_i]|^{\Delta_i - 1} \prod_{j=3}^{4} \int_{\mathbb{R}^2} \mathrm{d}^2\kappa_j \, |\langle \kappa_j \, \zeta_j\rangle|^{\Delta_j - 1} \\ \times \int_{\mathbb{R}^4} \frac{\mathrm{d}c_{13}\,\mathrm{d}c_{14}\,\mathrm{d}c_{23}\,\mathrm{d}c_{24}}{c_{13}c_{24}(c_{13}c_{24} - c_{23}c_{14})} \prod_{i=1}^{2} \delta^2\Big(\bar\kappa_i + \sum_{k=3}^{4} c_{ik}\bar\zeta_k\Big) \prod_{j=3}^{4} \delta^2\Big(\kappa_j - \sum_{l=1}^{2} c_{lj}\zeta_l\Big)\,. \end{aligned} \tag{5.98}$$

We choose to integrate out $\bar\kappa_1, \bar\kappa_2$ and κ_3, κ_4 first, rather than the c_{ij}, and we immediately land on the integral formula

$$\begin{aligned} L(1_S^- 2_S^- 3_S^+ 4_S^+) = \int_{\mathbb{R}^4} \frac{\mathrm{d}c_{13}\,\mathrm{d}c_{14}\,\mathrm{d}c_{23}\,\mathrm{d}c_{24}}{c_{13}c_{24}(c_{13}c_{24} - c_{23}c_{14})} \, |c_{13}\bar z_{13} + c_{14}\bar z_{14}|^{\Delta_1 - 1} \, |c_{23}\bar z_{23} + c_{24}\bar z_{24}|^{\Delta_2 - 1} \\ \times |c_{13}z_{13} + c_{23}z_{23}|^{\Delta_3 - 1} \, |c_{14}z_{14} + c_{24}z_{24}|^{\Delta_4 - 1}\,. \end{aligned} \tag{5.99}$$

Similar calculations yield amplitudes containing anti-symmetric primaries without extra effort. We will focus on (5.99) fpr the rest of this section.

To continue the integration of (5.99), we make the following judicious substitutions

$$\begin{aligned} c_{13} &\mapsto \frac{1}{z_{14}}\frac{1}{\bar z_{23}}\, c_{13}\,, \qquad & c_{14} &\mapsto \frac{1}{z_{14}}\frac{\bar z_{13}}{\bar z_{14}\bar z_{23}}\, c_{14}\,, \\ c_{23} &\mapsto \frac{z_{13}}{z_{14}z_{23}}\frac{1}{\bar z_{23}}\, c_{23}\,, \qquad & c_{24} &\mapsto \frac{z_{13}}{z_{14}z_{23}}\frac{\bar z_{13}}{\bar z_{14}\bar z_{23}}\, c_{24}\,. \end{aligned} \tag{5.100}$$

These changes turn (5.99) into

$$L(1_S^- 2_S^- 3_S^+ 4_S^+) = \left|\frac{z_{13}}{z_{14}}\right|^{\Delta_2+\Delta_3-2} \left|\frac{\bar{z}_{13}}{\bar{z}_{23}}\right|^{\Delta_1+\Delta_4-2} \frac{|z_{14}|^{1-\Delta_1}\,|z_{23}|^{1-\Delta_2}}{|\bar{z}_{23}|^{\Delta_3-1}\,|\bar{z}_{14}|^{\Delta_4-1}} \int \frac{\mathrm{d}c_{13}\,\mathrm{d}c_{14}\,\mathrm{d}c_{23}\,\mathrm{d}c_{24}}{c_{13}c_{24}(c_{13}c_{24}-c_{23}c_{14})}$$
$$\times\, |c_{13}+c_{14}|^{\Delta_1-1} \left|c_{23}+\frac{c_{24}}{1-\bar{z}}\right|^{\Delta_2-1} |c_{13}+c_{23}|^{\Delta_3-1} \left|c_{14}+\frac{c_{24}}{1-z}\right|^{\Delta_4-1}, \quad (5.101)$$

where we have defined the cross ratios

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}. \quad (5.102)$$

To perform the integrals, one should find that the following choice of integration variables t_1, t_2, x defined by

$$c_{14} = \frac{1-\bar{z}}{t_1-1}\,c_{13}, \qquad c_{23} = \frac{t_2-1}{1-x}\,c_{13}, \qquad c_{24} = \frac{(1-\bar{z})(t_2-1)}{t_1-1}\,c_{13} \quad (5.103)$$

allows us to separate out the c_{13} integral to then give

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}c_{13}}{|c_{13}|}\,|c_{13}|^{\sum_{i=1}^4(\Delta_i-1)} = 4\pi\,\delta(\mathrm{i}\beta) \quad (5.104)$$

where $\beta = \Delta_1+\Delta_2+\Delta_3+\Delta_4-4$ is the net boost weight.

The resulting correlator decomposes in a CFT-like manner: a conformally covariant prefactor times a conformally invariant function of the cross ratios [3]:

$$L(1_{\mathrm{s}}^- 2_{\mathrm{s}}^- 3_{\mathrm{s}}^+ 4_{\mathrm{s}}^+) = 4\pi\,\delta(\mathrm{i}\beta)\,K(z_i,\bar{z}_i)\,z^{\mathsf{h}_{12}}\bar{z}^{\bar{\mathsf{h}}_{12}}\,\mathscr{L}(z,\bar{z}), \quad (5.105)$$

where we have chosen the covariant kinematic prefactor K to be symmetric in $1,2$ and $3,4$ separately,

$$K(z_i,\bar{z}_i) = \left|\frac{z_{24}\bar{z}_{14}}{z_{14}\bar{z}_{24}}\right|^{\mathsf{h}_{12}} \left|\frac{z_{14}\bar{z}_{13}}{z_{13}\bar{z}_{14}}\right|^{\mathsf{h}_{34}} \left|\frac{\bar{z}_{12}}{z_{12}}\right|^{\mathsf{h}_1+\mathsf{h}_2} \left|\frac{\bar{z}_{34}}{z_{34}}\right|^{\mathsf{h}_3+\mathsf{h}_4}, \quad (5.106)$$

note that, as before, h_i are the left-moving conformal weights of the light-transformed primaries:

$$\mathsf{h}_1 = \frac{\Delta_1-1}{2}, \quad \mathsf{h}_2 = \frac{\Delta_2-1}{2}, \quad \mathsf{h}_3 = \frac{1-\Delta_3}{2}, \quad \mathsf{h}_4 = \frac{1-\Delta_4}{2}, \quad (5.107)$$

and $\mathsf{h}_{ij} = \mathsf{h}_i - \mathsf{h}_j$. The right-moving weights will be correspondingly $\bar{\mathsf{h}}_i$, so that $\bar{\mathsf{h}}_{ij} = \bar{\mathsf{h}}_i - \bar{\mathsf{h}}_j$, though they are not shown explicitly in (5.106) as they are related to the left-moving weights due to gluons having spin 1: $\bar{\mathsf{h}}_i = -\mathsf{h}_i$.

The dynamical data $\mathscr{L}(z,\bar{z})$ encodes the conformal block expansion, and so is our focus. We find it to be [3]

$$\begin{aligned}\mathscr{L}(z,\bar{z}) = \left|\frac{z}{\bar{z}}\right|^{\Delta_2-1}\left|\frac{1-z}{1-\bar{z}}\right|^{\Delta_1+\Delta_3-2} \\ \times\int_{-\infty}^{\infty}\frac{|1-x|^{\Delta_1+\Delta_4-2}}{x\,(x-1)}\,\mathscr{J}\,(\Delta_2,\Delta_3,\Delta_4\,|\,x,z)\,\mathscr{J}\,(\Delta_3,\Delta_2,\Delta_1\,|\,x,\bar{z})\,\mathrm{d}x\,,\end{aligned} \tag{5.108}$$

where $\mathscr{J}$ denotes the integral

$$\mathscr{J}\,(a,b,c\,|\,u,v) := \int_{-\infty}^{\infty}\mathrm{d}t\,|1-t|^{a-2}\,|u-t|^{b-1}\,|v-t|^{c-1}\,. \tag{5.109}$$

The result (5.108) factorises rather nicely into chiral and anti-chiral halves brought together through the x integral: this is coming from the underlying Grassmannian geometry (5.94).

We would like to note here that (5.108) matches the result obtained in the parallel work of [137] by applying to $\mathscr{J}\,(\Delta_2,\Delta_3,\Delta_4\,|\,x,z)$ the identity

$$\mathscr{J}\,(a,b,c\,|\,u,v) = |1-u|^{a+b-2}|1-v|^{a+c-2}\,\mathscr{J}\,(4-a-b-c,c,b\,|\,u,v)\,. \tag{5.110}$$

This identity can be derived by substituting $t \mapsto 1-(1-u)(1-v)/(1-t)$ in (5.109). The integral (5.109) was referred to as a *four marked point integral* in [123], where it was evaluated by breaking the integration domain into regions on which $1-t, u-t, v-t$ take definite signs. After quite a lot of algebra, one ends up with

$$\mathscr{J}\,(a,b,c\,|\,u,v) = \Theta\left(\frac{u-1}{1-v}\right)\,\mathscr{J}_+(a,b,c\,|\,u,v) + \Theta\left(\frac{1-u}{1-v}\right)\,\mathscr{J}_-(a,b,c\,|\,u,v)\,. \tag{5.111}$$

The two pieces $\mathscr{J}_\pm$ can be written as linear combinations of Gauss hypergeometric functions

$$\begin{aligned}\mathscr{J}_+(a,b,c\,|\,u,v) = \frac{\mathscr{C}\,(a-1,\,b)}{|1-u|^{2-a-b}\,|1-v|^{1-c}}\;{}_2F_1\left(a-1,\,1-c\,;\,a+b-1\,;\,\frac{1-u}{1-v}\right) \\ +\frac{\mathscr{C}\,(a+b-2,\,c)}{|1-v|^{3-a-b-c}}\;{}_2F_1\left(3-a-b-c,\,1-b\,;\,3-a-b\,;\,\frac{1-u}{1-v}\right),\end{aligned} \tag{5.112}$$

$$\mathcal{J}_{-}(a,b,c\,|\,u,v) = \frac{\mathscr{C}(b,\,c)}{|u-v|^{1-b-c}\,|1-u|^{2-a}}\;{}_2F_1\left(2-a,\,b\,;\,b+c\,;\,\frac{u-v}{u-1}\right) + \frac{\mathscr{C}(b+c-1,\,a-1)}{|1-u|^{3-a-b-c}}\;{}_2F_1\left(3-a-b-c,\,1-c\,;\,2-b-c\,;\,\frac{u-v}{u-1}\right), \tag{5.113}$$

where we have defined some coefficient functions built out of Euler Beta functions,

$$\mathscr{C}(a,b) = B(a,b) + B(a,1-a-b) + B(b,1-a-b)\,. \tag{5.114}$$

As probed in more detail in [123], some information on OPE data can be found from studying the coefficient functions $\mathscr{C}(a,b)$.

One would now be tempted to perform the x integral directly, but this leads to a generalised multi-variable hypergeometric mess that doesn't shed light on its structure. However, at the end of the day we want to obtain spectral and if possible OPE data, so we will instead use an appropriate *inversion* formula. We will use the so-called *alpha*-space transform on (5.108), as it was done for the four-scalar celestial amplitude in [125].

5.5.3 Alpha space decomposition

Bootstrap techniques allow us to obtain otherwise inaccessible data about a CFT by imposing constraints coming from the invariance or covariance of certain observables under conformal transformations. In particular, conformal covariance allows us to decompose any 4-point correlation function into a *conformal partial wave expansion*: we can express the conformally invariant part of the correlator as an expansion. This expansion is done over the orthornomal basis of solutions of the two-particle conformal Casimir equation.

Conformal partial waves are the starting point for obtaining information about OPE coefficients in a theory from a known four-point function. In more *traditional* settings, one would want to use the Euclidean [141, 142] or the Lorentzian [143, 144] inversion formulae, both cases in which the conformal waves are supported on the complex plane or its analytic continuation. However, we are dealing with primary operators that transform in irreducible representations of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, intrinsically defined in $\mathbb{R}^{1,1}$, so a different formalism will be better suited: the *alpha space decomposition*, introduced and formalised in [145, 146]. The four-point function, as a result of the Lorentz group in this signature being $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$, should be decom-

posed in $SL(2,\mathbb{R})$ conformal blocks and we will be studying the Sturm-Liouville problem of the $SL(2,\mathbb{R})$ Casimir operator. In this instance, the conformal partial waves have support for real $z, \bar{z} \in [0,1] \times [0,1]$, so we can see how they are natural for correlators that live on the celestial torus.

One last comment is in order that further supports our wanting to use alpha-space formalism: the Lorentzian inversion formula gives us OPE coefficients for some dimension and spin, but it doesn't give information about which primaries are present in the spectrum of the theory. However, the alpha-space formalism will tell us, about the spectrum as well, as we will shortly see. In particular, we don't want to assume that our spectrum follows what one can deduce from single particle spectrum of the free theory [42], but rather use this inversion formula to further our understanding of the operator content of the theory.

The basics of alpha space

Two-dimensional alpha space decomposition [146] is heavily based on the one-dimensional case [145], and so we will begin by briefly reviewing the latter.

Consider a four-point correlation function in the s-channel, in a one-dimensional CFT. From general CFT theory, we know that the correlator can be separated into the product of a conformally covariant and a conformally invariant part.

$$F^s_{1234}(z_1,\ldots,z_4) = K^s(z_1,z_2,z_3,z_4)\mathcal{F}^s(z)\,, \qquad z = \frac{z_{12}z_{34}}{z_{13}z_{24}}\,. \tag{5.115}$$

We will mostly be interested in the conformally invariant factor, $\mathcal{F}^s(z)$. This can be decomposed into an infinite sum of $SL(2,\mathbb{R})$ blocks, $k^s_h(z)$, times the appropriate OPE coefficients,

$$\mathcal{F}^s(z) = \sum_{\mathcal{O}} C_{12\mathcal{O}} C_{34\mathcal{O}} k^s_{h_{\mathcal{O}}}(z), \quad k^s_h = z^{h+a}\,{}_2F_1(h+a, h+b; 2h; z), \tag{5.116}$$

where $a = -h_{12},\ b = h_{34}$. We are going to set $0 < 1 < z$ to guarantee OPE convergence in the crossing equation.

Conformal blocks are eigenfunctions of the quadratic Casimir of $SL(2,\mathbb{R})$, but they don't form a complete basis. However, a complete basis exists, and it is formed by *conformal partial waves*.

The Casimir operator of $SL(2,\mathbb{R})$, D is defined as [145]

$$D \cdot f(z) = z^2 \frac{\partial}{\partial z}\left((1-z)f'(z)\right). \tag{5.117}$$

This means we can define an inner product that D is self-adjoint with respect to [145] as

$$\langle f, g\rangle_s = \int_0^1 \mathrm{d}z\, w_s(z)\overline{f(z)}g(z)\,, \qquad w_s(z) = \frac{(1-z)^{a+b}}{z^{2+2a}}\,. \tag{5.118}$$

Studying the Sturm-Liouville problem, we can reach the following orthonormal basis of eigenfunctions [145]

$$\begin{aligned}\Psi^s_\alpha(z) &= \frac{1}{2}\left(Q(\alpha)k^s_{1/2+\alpha}(z) + Q(-\alpha)k^s_{1/2-\alpha}(z)\right)\\ &= {}_2F_1\left(\frac{1}{2}+a+\alpha, \frac{1}{2}+a-\alpha; 1+a+b; \frac{z-1}{z}\right)\,,\end{aligned} \tag{5.119}$$

where

$$Q(\alpha) = \frac{2\Gamma(-2\alpha)\Gamma(1+a+b)}{\Gamma(1/2+a-\alpha)\Gamma(1/2+b-\alpha)}\,. \tag{5.120}$$

These allow us to define the alpha space transform and inverse transform of $\mathcal{F}^s(z)$ as follows:

$$\widehat{\mathcal{F}}^s(\alpha) = \int_0^1 \mathrm{d}z\, w_s(z)\,\mathcal{F}^s(z)\,\Psi^s_\alpha(z)\,, \qquad \mathcal{F}^s(z) = \int \frac{[\mathrm{d}\alpha]}{N_s(\alpha)}\,\widehat{\mathcal{F}}^s(\alpha)\,\Psi^s_\alpha(z)\,, \tag{5.121}$$

with

$$N_s(\alpha) = \frac{2\Gamma(\pm 2\alpha)\Gamma^2(1+a+b)}{\Gamma(\frac{1}{2}+a\pm\alpha)\Gamma(\frac{1}{2}+b\pm\alpha)}\,, \qquad \Gamma(p\pm q) := \Gamma(p+q)\Gamma(p-q). \tag{5.122}$$

As said before, the two-dimensional case builds on the one-dimensional construction. In particular, let us have a two-dimensional function $f(z,\bar{z})$ defined on the Loretnzian square. We can define its alpha space transform as [146]

$$\widehat{f}(\alpha,\bar{\alpha}) = \int_0^1 \mathrm{d}z\, w_s(z)\Psi_\alpha(z)\int_0^1 \mathrm{d}\bar{z}\,\bar{w}_s(\bar{z})\Psi_{\bar{\alpha}}(\bar{z})f(z,\bar{z})\,, \tag{5.123}$$

$$f(z,\bar{z}) = \int [d\alpha]\int [d\bar{\alpha}]\,\frac{4\Psi_\alpha(z)\Psi_{\bar{\alpha}}(\bar{z})}{N_s(\alpha)\bar{N}_s(\bar{\alpha})}\,\widehat{f}(\alpha,\bar{\alpha})\,, \tag{5.124}$$

where $\bar{w}_s$ and $\bar{N}_s$ are obtained from w_s and N_s by replacing $h_i \mapsto \bar{h}_i$.

This construction allows us [146] to find contributions to the conformal block expansion by

finding and studying twin poles in the alpha space density:

$$\hat{f}(\alpha,\bar{\alpha}) \supset \frac{R}{(\alpha-h+1/2)(\bar{\alpha}-\bar{h}+1/2)} + (\alpha \leftrightarrow \bar{\alpha}) \tag{5.125}$$

where R denotes the residues of these twin poles. These functions should be invariant under Weyl reflections, so we actually have a set of eight twin poles on alpha space for every conformal block (from all independent exchanges of $\alpha \leftrightarrow -\alpha$ and $\bar{\alpha} \leftrightarrow -\bar{\alpha}$).

These poles give rise to a corresponding conformal block in position space, and thus we can read off the OPE coefficients as

$$C_{12\mathcal{O}}C_{34\mathcal{O}} = \frac{4R}{Q(-h+1/2)\bar{Q}(-\bar{h}+1/2)}\,. \tag{5.126}$$

Note here that we only consider one pair of the eight Weyl reflections possible, as they all correspond to a single operator.

4-gluon amplitude in alpha space

We now have a formalism that we would like to use on the amplitude (5.105) in order to extract information about the spectrum of the celestial CFT it encodes.

Let us start then. We want to apply the alpha transform on the dynamical data $\mathscr{L}(z,\bar{z})$ of (5.108). This expression includes an integral over the variable x, and it has singularities at $x = 0, 1$ and, as we are integrating $z, \bar{z}$ over the Lorentzian square, we need to divide the x-integral into two sectors:

- A first sector where $x \in (-\infty, 1)$ and the factor $\mathscr{J}(z,\bar{z})$ from (5.109) is simply $\mathscr{J}_{-}(z,\bar{z})$ given by (5.113). The corresponding contribution to $\mathscr{L}(z,\bar{z})$ will be denoted $\mathscr{L}_{-}(z,\bar{z})$. This sector will be further divided into two smaller sectors due to the presence of absolute values in the integral: $(-\infty, 0)$ and $(0, 1)$.

- A second sector where $x \in (1, \infty)$ and the factor $\mathscr{J}(z,\bar{z})$ from (5.109) is simply $\mathscr{J}_{+}(z,\bar{z})$ given by (5.112). The corresponding contribution to $\mathscr{L}(z,\bar{z})$ will be denoted $\mathscr{L}_{+}(z,\bar{z})$.

Overall, this separation is equivalent to defining the x integral via principal value prescription.

Having said that, there is an issue: the structure of the x integral leads to mixing of the z

and $\bar{z}$ sectors, without having an analytic solution that then allows us to read off the poles and residues we want from an alpha space transform of $\mathscr{L}$.

We need to take a different route. We will assume convergence of the x integral (after the addition of the three sectors we study) and commute this integral with the alpha-space one. Performing the alpha-space transform isn't easy either due to the particular analytic structure of the $\mathscr{J}_+$ and $\mathscr{J}_-$ in terms of $x, z, \bar{z}$: it leads to objects that structurally resemble some (very) generalised hypergeometric functions of several variables, for which we don't have good analytic expressions. Having said that, we can overcome this and extract the pole structure through Mellin-Barnes representations and series expansions. We will illustrate this with an example.

An example: a term of $\mathscr{J}_+$. We will show the explicit computation we perform for one of the terms of $\mathscr{J}_+$, with all others performed in a similar way. The procedure one should follow is also very similar for $\mathscr{J}_-$ in the range $x \in (-\infty, 0)$. The range $x \in (0,1)$ is more difficult to alpha-transform, but the pole structure can be obtained after carefully analysing the integrals, an example is provided in Appendix C.2.

The overall integral we want to solve is

$$\begin{aligned}
&\int_0^1 \mathrm{d}z\, w_s(z) \int_0^1 \mathrm{d}\bar{z}\, \bar{w}_s(\bar{z})\, \mathscr{L}_+(z,\bar{z})\, \Psi_\alpha(z)\, \Psi_{\bar{\alpha}}(\bar{z}) \\
&= \int_1^\infty \mathrm{d}x\, \frac{(x-1)^{\Delta_1+\Delta_4-2}}{x(x-1)} \int_0^1 \mathrm{d}z\, w_s(z)\, \Psi_\alpha(z)\, \frac{\mathscr{J}_+(\Delta_2,\Delta_3,\Delta_4\,|\,x,z)}{z^{1-\Delta_2}(1-z)^{2-\Delta_1-\Delta_3}} \\
&\qquad\qquad \times \int_0^1 \mathrm{d}\bar{z}\, \bar{w}_s(\bar{z})\, \Psi_{\bar{\alpha}}(\bar{z})\, \frac{\mathscr{J}_+(\Delta_3,\Delta_2,\Delta_1\,|\,x,\bar{z})}{\bar{z}^{\Delta_2-1}(1-\bar{z})^{\Delta_1+\Delta_3-2}}\,.
\end{aligned} \tag{5.127}$$

Note that we evaluate the kernels $w_s(z)$, $\bar{w}_s(\bar{z})$ at the light transformed weights $\mathsf{h}_i, \bar{\mathsf{h}}_i$. The factor of $(1-z)^{-\mathsf{h}_{12}+\mathsf{h}_{34}}(1-\bar{z})^{-\bar{\mathsf{h}}_{12}+\bar{\mathsf{h}}_{34}}$ in $w_s(z)\bar{w}_s(\bar{z})$ cancels out completely against the prefactor $(\frac{1-z}{1-\bar{z}})^{\Delta_1+\Delta_3-2}$ on the support of $\mathsf{h}_1+\mathsf{h}_2 = \mathsf{h}_3+\mathsf{h}_4$.

We will now, as we prefaced, leave the integral x alone (assuming our integrals are all convergent) and focus on solving the z integral.

$$\int_0^1 \mathrm{d}z\, w_s(z)\, \Psi_\alpha(z)\, \frac{\mathscr{J}_+(\Delta_2,\Delta_3,\Delta_4\,|\,x,z)}{z^{1-\Delta_2}(1-z)^{2-\Delta_1-\Delta_3}}. \tag{5.128}$$

The $\bar{z}$ integral is analogous. We will explicitly show only one term in the $\mathscr{J}_+$ integral, for the

sake of brevity, as the procedure is quite similar for all of the terms:

$$\begin{aligned}\widehat{\mathscr{J}}_{+}(\alpha,x)=\int_0^1 \mathrm{d}z\,(1-z)^{-2\mathsf{h}_4}z^{2\mathsf{h}_1-2}(x-1)^{2\mathsf{h}_2-2\mathsf{h}_3}\\ \times\,{}_2F_1\left(2\mathsf{h}_2,2\mathsf{h}_4;1-2\mathsf{h}_{23};\frac{1-x}{1-z}\right)\\ \times\,{}_2F_1\left(\frac{1}{2}-\alpha-\mathsf{h}_{12},\frac{1}{2}+\alpha-\mathsf{h}_{12};1-\mathsf{h}_{12}+\mathsf{h}_{34};\frac{z-1}{z}\right).\end{aligned}\tag{5.129}$$

We will now devise a procedure that allows us to separate the x-dependence from the z factors. The reason for this is that we are looking to read off the spectrum from the position of the poles in z, and we can thus read them off without needing to explicitly solve the x-integral.

To do this, we first expand as a series on its argument the ${}_2F_1$ coming from the amplitude (in the first line), and we then express the partial wave (in the second line) as a Mellin-Barnes integral

$$\begin{aligned}\hat{\mathscr{J}}_{+}(\alpha,x)=\sum_{q=0}^{\infty}(-1)^q\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}t}{2\pi\mathrm{i}}\int_0^1\mathrm{d}z\,z^{2\mathsf{h}_1-2}(x-1)^{2\mathsf{h}_{23}+q}\\ \times(1-z)^{-2\mathsf{h}_4-q}\frac{\Gamma\left(1+2\mathsf{h}_{23}\right)\Gamma\left(2\mathsf{h}_2+q\right)\Gamma\left(2\mathsf{h}_4+q\right)}{\Gamma\left(2\mathsf{h}_2\right)\Gamma\left(2\mathsf{h}_4\right)\Gamma(1+q)\Gamma\left(1+2\mathsf{h}_{23}+q\right)}\\ \times\left(\frac{1-z}{z}\right)^t\frac{\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}\right)\Gamma(-t)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}+t\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}+t\right)}{\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}+t\right)}\\ =\sum_{q=0}^{\infty}(-1)^q\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}t}{2\pi\mathrm{i}}\,(x-1)^{2\mathsf{h}_{23}+q}\end{aligned}\tag{5.130}$$

$$\begin{aligned}\times\frac{\Gamma\left(1+2\mathsf{h}_{23}\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}\right)\Gamma\left(2\mathsf{h}_2+q\right)\Gamma\left(2\mathsf{h}_4+q\right)}{\Gamma\left(2\mathsf{h}_2\right)\Gamma\left(2\mathsf{h}_4\right)\Gamma(q+1)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)\Gamma\left(1+2\mathsf{h}_{23}+q\right)\Gamma\left(2\mathsf{h}_{14}-1\right)}\\ \times\frac{\Gamma(-t)\Gamma\left(2\mathsf{h}_1-1-t\right)\Gamma\left(1-2\mathsf{h}_4-q+t\right)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}+t\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}+t\right)}{\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}+t\right)}.\end{aligned}\tag{5.131}$$

We are left with a slightly more complicated integral, however it can be identified as the Mellin-

Barnes representation of a Meijer-G fucntion:

$$\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\mathrm{d}t}{2\pi\mathrm{i}}\,\frac{\Gamma(-t)\Gamma\left(2\mathsf{h}_1-1-t\right)\Gamma\left(1-2\mathsf{h}_4-q+t\right)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}+t\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}+t\right)}{\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}+t\right)}$$
$$=G^{2,3}_{3,3}\left(1\,\middle|\,\begin{matrix}\alpha+\mathsf{h}_{12}+\frac{1}{2},-\alpha+\mathsf{h}_{12}+\frac{1}{2},2\mathsf{h}_4+q\\ 2\mathsf{h}_1-1,0,\mathsf{h}_{12}-\mathsf{h}_{34}\end{matrix}\right). \quad (5.132)$$

The Meijer G function we obtained can be expressed as a sum over two simpler hypergeometrics, after which we obtain the following expression overall [3]

$$\widehat{\mathscr{J}}_+(\alpha,x)=\sum_{q=0}^{\infty}(x-1)^{2\mathsf{h}_{23}+q}(-1)^q$$
$$\times\frac{\Gamma\left(1+2\mathsf{h}_{23}\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}\right)\Gamma\left(2\mathsf{h}_2+q\right)\Gamma\left(2\mathsf{h}_4+q\right)}{\Gamma\left(2\mathsf{h}_2\right)\Gamma\left(2\mathsf{h}_4\right)\Gamma(q+1)\Gamma\left(1+2\mathsf{h}_{23}+q\right)\Gamma\left(2\mathsf{h}_{14}-1\right)}$$
$$\times\Bigg[\frac{\Gamma\left(1-2\mathsf{h}_1\right)\,\Gamma\left(-\frac{1}{2}-\alpha+\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(-\frac{1}{2}+\alpha+\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(2\mathsf{h}_{14}-q\right)}{\Gamma\left(2\mathsf{h}_3\right)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)}$$
$$\times\,{}_3F_2\left(-\frac{1}{2}-\alpha+\mathsf{h}_1+\mathsf{h}_2,-\frac{1}{2}+\alpha+\mathsf{h}_1+\mathsf{h}_2,2\mathsf{h}_{14}-q;2\mathsf{h}_1,2\mathsf{h}_3;1\right)$$
$$+\frac{\Gamma\left(2\mathsf{h}_1-1\right)\Gamma\left(-q-2\mathsf{h}_4+1\right)}{\Gamma\left(-\mathsf{h}_{12}+\mathsf{h}_{34}+1\right)}$$
$$\times\,{}_3F_2\left(\frac{1}{2}-\alpha-\mathsf{h}_{12},\frac{1}{2}+\alpha-\mathsf{h}_{12},1-q-2\mathsf{h}_4;2-2\mathsf{h}_1,1-\mathsf{h}_{12}+\mathsf{h}_{34};1\right)\Bigg]. \quad (5.133)$$

The terms that are highlighted in light grey are those that yield poles: they are two gamma functions. As we know, the location of poles helps us determine the spectrum of operators appearing in the conformal block expression. To obtain those, we will study the poles in α for the first highlighted Gamma, the other will yield the Weyl reflections of the operator. Hence, the poles are at the values $\alpha=-1/2+\mathsf{h}_1+\mathsf{h}_2+\mathbb{N}$. These poles indicate the presence of an infinite tower of states with left-moving dimension $h=\mathsf{h}_1+\mathsf{h}_2+\mathbb{N}$.

We have the operators present in the conformal block expansion from the location of the poles, so one may wonder if there is a net gain from computing the x integral explicitly, as this would allow us to determine the OPE coefficients. We can technically do it, but we end up

with rather obscure expressions, as one can see in Appendix C.1 (see also [3, Appendix C]. We compute there the contribution of (5.133) and one of its antiholomorphic counterparts to the OPE coefficients.

Overall, we can see that the result coming from any term of the $\mathscr{L}$ integral is functionally compatible with a generalised hypergeometric-like function of four variables (more akin to a Meijer-G), but with the same polar structure we can find without performing the x-integral (as expected).

The extracted spectrum of operators

Our integrals yield the following sets of poles coming from the full alpha transform (considering all regions of x) of $\mathscr{L}(z,\bar{z})$ as defined in (5.108):

- Poles at $\alpha = -\frac{1}{2} + \mathsf{h}_1 + \mathsf{h}_2 + \mathbb{N}$
- Poles at $\bar{\alpha} = -\frac{1}{2} + \bar{\mathsf{h}}_1 + \bar{\mathsf{h}}_2 + \mathbb{N}$
- Poles at $\alpha = \frac{\mathbb{Z}^+}{2}$
- Poles at $\bar{\alpha} = \frac{\mathbb{Z}^+}{2}$

There are also poles present for the Weyl reflections of these values, but they are all representing the same operators. One should note here that the $\bar{\alpha}$ poles come from performing the $\bar{z}$ integral in (5.127) in the same way we performed the z one (but independently of it).

The first two sets of poles can be found in the two larger sectors of the x integral $x \in (-\infty, 0) \cup (1, \infty)$, as all the different terms direct us to the same polar structure in α. Maybe more interesting to the reader are, at this stage, the poles for half-integer α or $\bar{\alpha}$, which can only be found when $x \in (0,1)$, due to the presence of a $|x-z|$ factor in the integrand, which changes signs in the regime of integration (when $x \notin (0,1)$ this factor is also present but it holds a fixed sign). We present an example of where one finds said poles in Appendix C.2. Interestingly enough, this sign-switching critically affects the analytic structure of the α-transform and leads to poles in new locations, still coming from Gamma functions, but with more complicated expressions at play.

Poles in α, $\bar{\alpha}$ relate to the dimensions of exchanged operators by a shift of $\frac{1}{2}$ (5.125). Thus, the poles we found correspond to two towers of operators [3]

1. $h = \mathsf{h}_1 + \mathsf{h}_2 + \mathbb{N} = h_1 + h_2 + \mathbb{N}\,,$
   $\bar{h} = \bar{\mathsf{h}}_1 + \bar{\mathsf{h}}_2 + \mathbb{N} = 2 - \bar{h}_1 - \bar{h}_2 + \mathbb{N}\,,$

2. $h = (1 + \mathbb{Z}^+)/2\,,\ \bar{h} = (1 + \mathbb{Z}^+)/2\,.$

We can also identify these operators to have dimensions and spins

- $(\Delta, J) = (K, \Delta_1 + \Delta_2 - 2 + L)\,, \qquad K \geq 1\,,\ \ 1 - K \leq L \leq K - 1\,,$ (5.134)
- $(\Delta, J) = \left(\dfrac{3+K}{2}, \dfrac{L}{2}\right), \qquad K \geq 1\,,\ \ 1 - K \leq L \leq K - 1\,,$ (5.135)

with K and L integer numbers.

Let us now compare these results with those obtained in the literature from different methods.

In the first set, we can identify the operator with $K = L = 0$ to the $\overline{\mathbf{L}}$ light transform of a negative-helicity gluon of weight $\Delta_1 + \Delta_2 - 1$, as expected from celestial OPE [133]. The other operators found are consistent, to different extents, with exchanges studied in other conformal block decompositions [122, 123, 147].

In the second set we would like to highlight those with $L = K - 1$. When expanding scalar celestial amplitudes in t-channel blocks [125], one can find a discrete set of operator exchanges that are very similar in nature. One can think of a possible relation from soft gluon exchanges in the t channel, as the condition $1 - K \leq L \leq K - 1$ is similar to the generators of the S algebra [148], but further study would be necessary to properly understand how both are related.

5.6 Conclusions

This chapter aimed to guide the reader through a prescription that allows one to study massless celestial amplitudes using CFT techniques by performing integral transforms in the amplitudes that make them *look* like a correlation function of a two-dimensional CFT.

This is key to allowing us to obtain crucial information about the spectrum of the CFT, as in this form we can apply bootstrap techniques to extract information about the celestial CFT. We don't want to make assumptions about the composition of the CFT beforehand and indeed the Weyl reflections appearing due to the light transforms lead us to a theory with

operators of continuous spin and discrete dimensions, something rather exotic from a CFT perspective. Furthermore, as we discussed earlier, the x-integral glues together the holomorphic and antiholomorphic sectors even in individual contributions to the conformal block expansion, which we wouldn't expect from a *normal* CFT. This is very bizarre, however these correlators are really a two-dimensional projection of a higher-dimensional object [149].

On the topic of the integrals, we saw throughout Section 5.5.3 that they are extremely difficult to perform analytically; however this kind of integrals can be performed numerically for particular amplitudes of interest. After all, this prescription is presented with the hopes that it can further the exploration and understanding of Celestial CFTs in more explicit cases. It is not trivial to know which particular amplitudes one should test with these techniques, but key work is being done by studying the symmetries of celestial amplitudes in more detail [134, 150, 151].

Another topic we want to touch as a conclusion is the basis for local operators in the dual CFT. This has been a topic of discussion in the celestial literature [42, 116, 152–154], especially as the choice of basis heavily influences which symmetries are manifest in the CFT construction. Our treatment of the CFT deals mostly with the global conformal group, and in that regard there is no clear preference for whether it is the operators in the Mellin distributional correlation functions or those in the ambidextrous basis that correspond to the local operators of the CFT. It could be either, and it would definitely be interesting to dwell further in this topic, as determining which are the local operators would make it possible to apply more powerful results coming from the *traditional* CFT literature.

At the end of the day, this is a prescription that introduces new points of view from different representations of the Lorentz group: this can hide some results but bring others closer to the surface. It very well might be that there is a *better* basis where we can find simpler partial wave expansions, however the ambidextrous basis is strongly connected to twistorial studies of the same physical problems [117, 155–162], and it would definitely be interesting to look into this connection further.

Chapter 6

Outlook

Two-dimensional Conformal Field Theories... As we have seen, two-dimensional CFTs come in many shapes and forms, despite the strong constraints imposed by conformal invariance. Throughout this thesis we have explored different ways to exploit modular invariance to constrain the torus partition function of both rational and irrational CFTs, and we have also investigated how the global conformal group can help us extract OPE data from the more exotic Celestial CFT.

This further showcases the flexibility and strength of bootstrap techniques: they can adapt to a multitude of highly constrained scenarios, as they can *drink* from different supporting areas of Mathematics (number theory, complex analysis, geometry...) to further the scope of application of certain techniques.

... the Irrational,... It is certainly enriching to work out higher spin theories in the context of AdS_3, as it offers us a unique perspective into the constraints they can impose in the theory. As we saw, they indeed are more constraining than having just Virasoro symmetry, but not in a radically different way. Of course, there are stronger constraints for unitary representations, however the overall behaviour of the spectrum is remarkably similar: continuous and with negativities.

As mentioned earlier, the positive aspects of curing the negativities include that it is a pretty minimal cure, it is modular invariant by construction and it is also relatively natural from a gravitational point of view. However, there is some risk as we currently don't know if there are

higher order corrections that could potentially undo some of our work.

More interesting in terms of outlook is to think about the *issue* of a continuous density of states. As mentioned earlier, there could be an explanation for this: maybe the AdS/CFT correspondence isn't one-to-one in 3-vs-2 dimensions, and a single AdS spacetime is dual to an ensemble of CFTs. In order to justify this, we would need to investigate spectral form factors and wormhole-like solutions, which are present in higher geni structures. This question has been studied in the non-higher-spin case, [22, 23], and some authors [20, 163] relate it with an issue of quantum chaos and the semi-classical gravity theory only being able to access quantities like the OPE operators as if they were random variables.

The main question that remains is: what kind of averaging and over what ensemble? It is true that if we manage to bring our theory to an appropriate AdS_2-like limit, we could potentially see how to extrapolate some of the results, but it is certainly tricky to imagine how to extend those in general. Most importantly, as discussed for a significant portion of the thesis, we don't know what the moduli space of 2d CFTs looks like, especially in the irrational sector, so it is rather unclear what we would average over. Some very interesting discussions on the interpretations of ensemble averaging can be found in [164] and in the (very!) recent RMT vs AdS_3 framework presented in [103].

... the Rational... Although we have been studying RCFTs for many decades now, results don't cease to appear, and new techniques and approaches are used to push the boundaries of our knowledge.

In using harmonic analysis, a tool developed in part to provide us with a more complete understanding of non-holomorphic modular forms, which don't have a *proper basis*, we uncovered some interesting connections between theories that go beyond previous points of view: the spectrum (and not all of it, just the scalar sector!) of different theories is related in a rather non-trivial manner, with this generating function linking RCFTs of different character number, central charge...

During this chapter we have focused on theories that have reduced characters that are somehow related to the holomorphic Eisenstein series: this is key in finding closed-form expressions for the RS transforms of the partition function. When looking beyond that, even in theories whose characters had recognisable forms (for instance, in terms of Jacobi theta functions), we

found performing the resummations that allow us to write compact expressions such as (4.66) impossible to do, and this obscures the possibility of finding relations between theories.

When investigating one-character theories, we gave up the requirement of integrality of the degeneracies of primary states in order to have a closed-form RS transform of the original reduced partition function. In practical terms, this means that we remove the cusp form contribution from the characters of these theories, leaving us with the Eisenstein-series contribution (4.63). What is very interesting is that, in the limit of large central charge, the cusp forms are exponentially suppressed with respect to the Eisenstein series contribution. This is the limit where the CFT could become holographic, and it would be dual to a chiral theory of gravity. It would be very interesting to understand how harmonic analysis' techniques could be extended to cover such gravity theories.

Finally, a big question remains: in [100], it was conjectured that the scalar and one spin spectrum is enough to fully characterise a CFT's partition function, due to the properties of spectral decomposition. Are there any CFTs for which the scalar sector is enough? Can we see a hint of that in this analysis? Maybe there can be some limits where the larger group of symmetries present on an RCFT is constraining enough to limit the spectrum further. On that note, it would be incredibly useful to explore the numerical integrals for the spinful section in more detail, as they could hold key information. We tried but weren't able to extract fruitful conclusions from them, but attacking them in a different way, with a tailored numerical integration algorithm could maybe give us more insight into the spinful sector.

... and the Celestial. Celestial CFTs are the most exotic theory we have dealt with in the thesis, and we have treated their properties with no other assumptions than the presence of conformal symmetry.

Being able to take the massless amplitudes to a basis where the CFT structure becomes apparent via a consistent prescription is very nice, but we are confronted with a rather arduous task to extract the CFT data from this basis. In particular, getting to obtaining the poles from the alpha space transformations was extremely difficult. The integrals involve multi-variable Meijer-G-like structures, with the integration bounds leading us to regions of a complicated analytic structure in terms of the convergence of said Meijer-G functions. If the reader who has made it here is interested, the author is happy to provide more insight into a calculation

of this sort. Some useful references when dealing with integrals of generalised hypergeometric functions are [165–167]. The OPE coefficients could technically be reconstructed, but it involved several multiple-variable sums that seemed in-expressible as an analytic formula. Although the prescription presented shows clearly the right functional dependence for CFT correlation functions, one wonders if there is a simpler way to extract CFT data in this case.

The calculations simplify in certain limits, but they are still not small, pretty formulas [3, Appendix B]. However, we can still learn a lot from these results: the operators appearing in the OPE expansions are expected from across the literature, and we see how there is consistency over all the integral.

The potential of this kind of formalism is, in any case, seemingly good. We currently don't know that much about the CCFT, although top-down constructions from String Theory have begun to emerge in recent times (see, for instance [114, 168]) that could maybe guide our vision more. The aim of developing techniques such as those presented here at this stage is that, once we understand the CCFT better, these techniques can be applied to obtain information about gravity amplitudes we can't currently compute. In other words, use the duality in the other direction.

Furthermore, from a *CFT-er* perspective, studying CFTs of which we know very little about is rather exciting: we don't know if the theory is unitary or not, rational or irrational, compact or non-compact... We tried to assume as little as possible about it from the get-go, which is something we tend to not do when studying other kinds of CFTs (mostly because we can't). This raises a question: what characteristics that we take for granted on CFTs are a consequence of our own bias from studying other known theories, and which ones are truly crucial for obtaining a consistent theory? How do certain usual characteristics constrain *common* CFTs? It is precisely by looking outside of the *traditional* box that we can further our knowledge of what is inside of it.

Overall, two-dimensional CFTs are a rich and intricate subject with still many unanswered questions, but with a plethora of options, backgrounds, symmetries and integrals to explore and to learn from. The author hopes that this small dip into the pond does justice to a field that she has so much enjoyed over the last four years.

Appendix A

Appendices to Chapter 3

A.1 The Rademacher procedure in detail

In this Appendix, we provide a review of how to use the Rademacher method, developed in [169–171] to recover modular forms from their polar terms. We mostly follow the explanation provided in [19]. We start with a general modular form with a finite number of polar terms

$$Z(q) = \sum_{\mu>0} \frac{a_{-\mu}}{q^{\mu}} + \sum_{n=0}^{\infty} a_n q^n \,, \tag{A.1}$$

where we recall $q = e^{2\pi i \tau} = e^{2\pi i (x+iy)}$ and we have separated the principal (polar) part from the regular part at the cusp $y \to \infty$. We will now use the same parametrisation as in the main text (3.8) for

$$q = e^{-\beta + \frac{2\pi i\, r}{s}}, \quad \gamma\, q = e^{-\frac{4\pi^2}{s^2\beta} + \frac{2\pi a}{s}} \,, \tag{A.2}$$

where note we have defined $\beta = -2\pi i \tau$. In a generic modular transformation $\tau \to \frac{a\tau+b}{s\tau-r}$ with $a\, r + b\mathrm{s} = -1$ and a, b, r, $s \in \mathbb{Z}$, we should note that $(r, s) = 1$ (so r and s are coprime); and we are free to take $0 < a, r \leq s$, from where we constrain $a = a_{r,s} = (-r)^{\phi(s)-1} \mod s$, with $\phi(s)$ the Euler totient function.

Using this parametrisation for $Z(q)$, we assume that this form is at most covariant under modular transformations, which translated to an expression means

$$Z\left(e^{-\beta + \frac{2\pi i r}{s}}\right) = f_{r,s}(\beta)\; Z\left(e^{-\frac{4\pi^2}{s^2\beta} + \frac{2\pi i a}{s}}\right), \tag{A.3}$$

where $f_{r,s}(\beta)$ is the factor absorbing covariance, and for the cases we are interested in it behaves as a power law as $\beta \to 0$ [19]. We now can sub in (A.1)

$$\sum_{\mu>0} a_{-\mu} e^{-\frac{2\pi i r\mu}{s}} e^{\beta\mu} \; + \; \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i r n}{s}} e^{-\beta n} = f_{r,s}(\beta) \left(\sum_{\mu>0} a_{-\mu} e^{\mu \frac{4\pi^2}{s^2\beta} + \mu \frac{2\pi i a}{s}} + \sum_{n=0}^{\infty} a_n e^{-\frac{4\pi^2 n}{s^2\beta} + \frac{2\pi a n}{s}} \right) . \tag{A.4}$$

We are going to be interested in the lightcone limit, which in this expression corresponds to taking $\beta \to 0$. In this limit, the polar terms in the RHS diverge exponentially! In fact, the infinite sum over n is negligible with respect to the exponential terms.

For us to have a divergence at $\beta \to 0$ on the LHS, we see that it has to come from the regular part of the sum, as the polar sum is finite. Only an infinite sum of terms can diverge, so the divergence will be caused by the a_n terms.

We can get the Fourier coefficients a_n through the Cauchy residue theorem in the q plane and then translate it to the τ plane:

$$a_n = \oint \frac{\mathrm{d}q}{q} \frac{Z(q)}{q^n} = \int_\Gamma \mathrm{d}\tau \, Z(q) e^{-2\pi i \tau n} , \tag{A.5}$$

where the contour of the first integral has radius smaller than 1 and encloses the origin. The contour of the second integral, in the τ plane, is a path between $\tau = i$ and $\tau = 1 + i$.

Let us analyse the integral in the τ plane with a bit more detail. As said before, we will only include the terms coming from the principal part of the partition function in the RHS, as those coming from the regular terms are exponentially suppressed in the limit of small β.

$$a_n = \int_\Gamma \mathrm{d}\tau \, f_{r,s}(\beta) \sum_{\mu>0} \left(a_{-\mu} e^{\mu \frac{4\pi^2}{s^2\beta} + \mu \frac{2\pi i a}{s}} \right) . \tag{A.6}$$

As the amount of polar terms is finite, we can exchange the sum with the integral without any issues. We will now continue the calculation considering just one polar term, as all others would be treated in the same way. As mentioned earlier, the covariant factor should behave as a power law, so we take without loss of generality $f_{r,s}(\beta) = \beta^\alpha$ for $\alpha > 0$. Hence, we now have

$$a_n = \int_\Gamma \mathrm{d}\tau \, \beta^\alpha a_{-\mu} e^{\mu \frac{4\pi^2}{s^2\beta} + \mu \frac{2\pi i a}{s}} . \tag{A.7}$$

This is all great, however, we need to be very mindful of the contour. As we can see from looking at (A.3) and (A.4), in the limit of $\beta \to 0$, or $\tau \to 0$, $Z(e^{2\pi i\tau})$ has essential singularities at all points $\tau = r/s$ where r, s are coprime integers. Rademacher proposed a method to change the contour Γ into a series of smaller contours $\Gamma_{r,s}$, such that each of them enclose only one singularity, the one at $\tau = r/s$.

These contours are built on the so-called Ford circles. A Ford circle $C_{r,s}$ with r, s two coprime integers is a circle of radius $\frac{1}{2s^2}$ and center $(x, y) = \left(\frac{r}{s}, \frac{1}{2s^2}\right)$.

We will know which Ford circles to use for the contour through a Farey sequence. A Farey sequence of order N the set of all possible possible rational numbers r/s between 0 and 1 that satisfy two conditions: $0 \leq r \leq s \leq n$ and $(r, s) = 1$. So our contour at order N will be composed of the Ford circles $C_{r,s}$ such that $r/s \in \{\text{Farey sequence of order } N\}$.

Let us do an example. Take the Farey sequence of order 4:

$$\left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\} \tag{A.8}$$

The contour we should draw would contain the corresponding Farey circles C_{01}, C_{14}, C_{13}, C_{12}, C_{23}, C_{34}, 1, and these are represented in Figure A.1

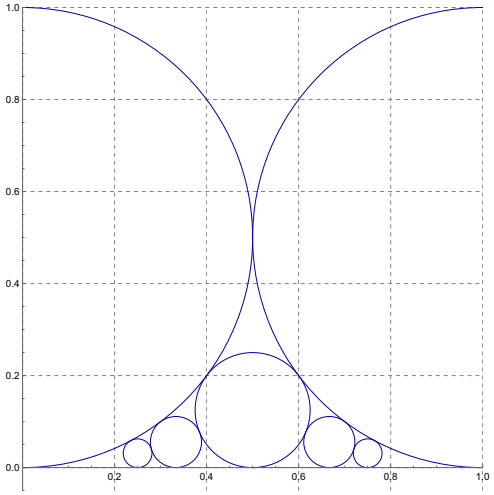


Figure A.1: Ford circles of a Farey sequence of order four.

Once we have the circles, we can draw a contour going from $\tau = i$ to $\tau = 1 + i$ whilst avoiding all the singularities. We start at C_{01} and we follow along until it touches the circle C_{14}, then turn to C_{13}, and continue until we reach C_{11} again. This leads to the contour $\Gamma_{01} \cup \Gamma_{14} \cup \Gamma_{13} \cup$

$\Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{34} \cup \Gamma_{11}$ presented in figure A.2

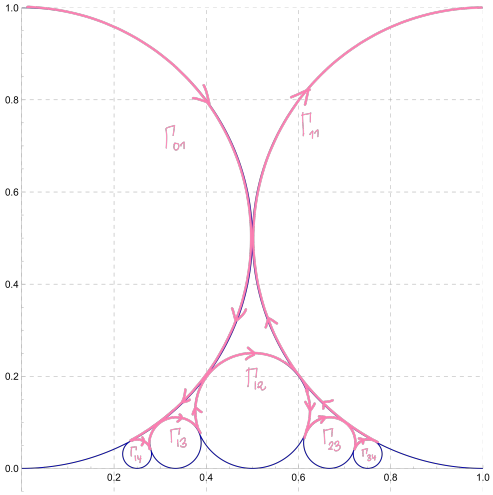


Figure A.2: Rademacher contour $\Gamma = \Gamma_{01} \cup \Gamma_{14} \cup \Gamma_{13} \cup \Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{34} \cup \Gamma_{11}$ to order 4 on a Farey sequence of Ford circles.

The Rademacher procedure iterates these circles up to $N \to \infty$, ensuring all the singularities can be enveloped in one circle each.

To exemplify, we follow again [19] and compute the contribution corresponding to $s = 1$, which consists of the paths Γ_{01} which envelop the singularity at 0 and 1 (by symmetry). The path to consider then is $\Gamma_{s=1} = \Gamma_{01} \cup \Gamma_{11}$. In practice, one shifts the contour Γ_{11} to the left, so that the path tends to the simple circle $C'_{01} = C_{0,1} \setminus (0,0)$, only the origin is excluded if we take $N \to \infty$.

With this in mind, the contribution of $s = 1$ to the coefficient a_n from (A.7) is

$$a_n^{s=1} = \int_{C'_{0,1}} \beta^\alpha e^{\kappa/\beta} e^{-2\pi i \tau n} d\tau \,, \tag{A.9}$$

where $\kappa = 4\pi^2 \mu$.

We now change variable $\tau \to \omega = \frac{1}{\beta}$, and the integral becomes a modified Bessel function of the first kind,

$$a_n^{s=1} = \frac{1}{2\pi i} \int_{\frac{1}{2\pi} - i\infty}^{\frac{1}{2\pi} + i\infty} \omega^{-\alpha} e^{\kappa\omega} e^{\frac{n}{\omega}} \frac{d\omega}{\omega^2} = \left(\frac{\kappa}{n}\right)^{\frac{1+\alpha}{2}} I_{\alpha+1}(2\sqrt{\kappa n}). \tag{A.10}$$

This allows us to define the Rademacher inversion function [19] as

$$\mathcal{R}^{-1}\left(\beta^\alpha e^{\kappa/\beta}\right) = \left(\frac{\kappa}{n}\right)^{\frac{1+\alpha}{2}} I_{\alpha+1}\left(2\sqrt{\kappa n}\right) \,. \tag{A.11}$$

If we reproduce this expression for every value of (r,s), we obtain the expression for a_n for all singularities and all polar terms:

$$a_n = \sum_{s=1,2,\dots} \sum_{0\leq r<s,\,(r,s)=1} e^{-\frac{2\pi i r}{s}} \mathcal{R}^{-1}\left[f_{r,s}(\beta) \sum_{\mu>0} a_{-\mu} e^{\mu \frac{4\pi^2}{s^2\beta} - \mu \frac{2\pi i a}{s}} \right] . \tag{A.12}$$

A.2 Positive degeneracies and unitarity in $\mathcal{W}_N$ CFTs

As presented in Section 2.2.1, when we refer to a CFT as *unitary*, this can have two meanings: it can be referring to the possible representations of the algebra appearing in the partition function being unitary, or it can refer to the states themselves having a positive norm or, in practical terms, to have a positive degeneracy in the partition function.

In this appendix, we are concerned with the latter. We would like to know how strong the constraint of being able to decompose a partition function into $\mathcal{W}_N$ characters is, versus decomposing said partition function into Virasoro or a $\mathcal{W}_M$ theory with $M < N$. The expectation is that the more higher-spin fields that are added to the partition function, the more constrained the theory is.

To understand this issue, we will be comparing $\mathcal{W}_2$ and $\mathcal{W}_3$ theories.

Let us have have a non-vacuum $\mathcal{W}_3$ character $\chi_h^{(3)}(q)$ and decompose it into $\mathcal{W}_2$ characters $\chi_h^{(2)}(q)$. This is an infinite expansion, of the form

$$\chi_h^{(3)}(q) = \chi_h^{(2)}(q) + \chi_{h+1}^{(2)}(q) + 2\chi_{h+2}^{(2)}(q) + 3\chi_{h+3}^{(2)}(q) + \cdots = \sum_{n=0} a(n)\chi_{h+n}^{(2)}(q), \tag{A.13}$$

with $a(n)$ representing the number of partitions of n, which is always a positive number. If we do the same for the vacuum character of $\mathcal{W}_3$, we can find a similar expansion: it can be decomposed as the vacuum character of $\mathcal{W}_2$ plus an infinite tower of characters of unitary representations with positive multiplicities.

What happens if we try to do the opposite operation? If we try to decompose a non-vacuum $\mathcal{W}_2$ character into $\mathcal{W}_3$ characters, we obtain something of the form

$$\chi_h^{(2)}(q) = \chi_h^{(3)}(q) - \chi_{h+1}^{(3)}(q) - \chi_{h+2}^{(3)}(q) + \chi_{h+5}^{(3)}(q) + \chi_{h+7}^{(3)}(q) - \chi_{h+12}^{(3)}(q) + \cdots \tag{A.14}$$

which clearly has negative multiplicities. The same happens for the vacuum character of $\mathcal{W}_2$ if we try to express it in terms of $\mathcal{W}_3$ characters.

This means that if we have a partition function that can be decomposed into $\mathcal{W}_3$ characters with unitary representations and positive multiplicities, it will also admit a decomposition into $\mathcal{W}_2$ characters with unitary representations (since the unitary representation bound is stronger for $\mathcal{W}_3$ than for $\mathcal{W}_2$) and positive multiplicities: we can rewrite this partition function of $\mathcal{W}_3$ in terms of $\mathcal{W}_2$. However, we can't always decompose a partition function that can be decomposed in unitary representations of $\mathcal{W}_2$ with positive multiplicities into unitary representations of $\mathcal{W}_3$ with positive multiplicities.

This was somewhat to be expected, as we can think of $\mathcal{W}_2$ as contained in $\mathcal{W}_3$, but the converse is not true. However, it is important that this consistency check was done explicitly.

We can hence see how partition functions are indeed constrained by the biggest $\mathcal{W}_N$ in which they can be decomposed into with unitary representations and positive multiplicities. The bigger the N, the stronger the constraints!

Appendix B

Appendix to Chapter 4

B.1 RCFTs generated by the invariant $y^{c/2}E_{c/2}(\tau)E_{c/2}(\bar{\tau})$

In Chapter 4, we explored the spaces of CFTs generated from the modular invariant (4.63)

$$\widetilde{Z}_P^{(c)} = y^{c/2}E_{c/2}(\tau)E_{c/2}(\bar{\tau}) \tag{B.1}$$

in the case of $c = 8$. However, during our exploration of the space of RCFTs through the RS ansatz (4.87) we found some more theories for this and other central charges, and we wanted to include them in an appendix. In fact, although there may not exist a consistent one-character CFT at $c = 4$, we can study the space of functions generated by $E_2(\tau)$, as appropriate combinations will yield modular-invariant objects.

We will organise the theories by the prime number p appearing in the functional $\phi(s)$ (4.92).

$$\phi(s) = \alpha_1(p^s + p^{1-s}) + \alpha_2\,. \tag{B.2}$$

B.1.1 Theories with $p = 2$

The RS transform of these theories is dressed with the factor $\phi(s) = \alpha_1(2^s + 2^{1-s}) + \alpha_2$.

Taking that into account, we can obtain a partition function of the form

$$Z_P^* = y^{\frac{c}{2}}\left(|\eta^{\frac{c}{2}}f_0|^2 + \mathcal{M}_1|\eta^{\frac{c}{2}}f_1|^2 + \mathcal{M}_2|\eta^{\frac{c}{2}}f_2|^2\right), \tag{B.3}$$

, where the $\mathcal{M}_{1,2}$ represent the degeneracies of the conformal primaries in the theory. The conformal characters of the theory are given by

$$\begin{aligned}
&\eta^{\frac{c}{2}} f_0(\tau) = \alpha E_{\frac{c}{2}}(\tau) + (1-\alpha) E_{\frac{c}{2}}(2\tau), \quad \eta^{\frac{c}{2}} f_1(\tau) = \sum_{n=1 \bmod 2}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{2}}, \\
&\eta^{\frac{c}{2}} f_2(\tau) = -\frac{B_{\frac{c}{2}}}{c} E_{\frac{c}{2}}(\tau) + \frac{B_{\frac{c}{2}}}{c} E_{\frac{c}{2}}(2\tau).
\end{aligned} \tag{B.4}$$

The list of possible candidate theories that have positive, integer Fourier coefficients, that we found [2] is presented in Table B.1. We also present in Table B.2 some specific values of the parameters for which the generated theory is a known one.

c	α_0	α_1	$\mathcal{M}_2$	α	Degeneracies $\in \mathbb{Z}^+$
4	$\frac{64-\mathcal{M}_1}{64}$	$\frac{\mathcal{M}_1}{288}$	$\frac{\mathcal{M}_1(192-\mathcal{M}_1}{16}$	$\frac{96-\mathcal{M}_1}{96}$	$\mathcal{M}_1 = 192$
8	$1-\frac{43\mathcal{M}_1}{19200}$	$\frac{\mathcal{M}_1}{7200}$	$\frac{\mathcal{M}_1(\mathcal{M}_1-768)}{4}$	$1-\frac{\mathcal{M}_1}{480}$	$\mathcal{M}_1 \in 2\mathbb{Z}^+,\ 2 \leq \mathcal{M}_1 \leq 240$
12	$1-\frac{683\mathcal{M}_1}{84672}$	$\frac{\mathcal{M}_1}{7938}$	$16\mathcal{M}_1(\mathcal{M}_1-192)$	$1-\frac{\mathcal{M}_1}{126}$	$\mathcal{M}_1 \in \mathbb{Z}^+,\ 131 < \mathcal{M}_1 < 192$
16	$1-\frac{3641\mathcal{M}_1}{25600}$	$\frac{\mathcal{M}_1}{1800}$	$16\mathcal{M}_1(3072-289\mathcal{M}_1)$	$1-\frac{17\mathcal{M}_1}{120}$	$\mathcal{M}_1 \in \mathbb{Z}^+,\ 1 \leq \mathcal{M}_1 \leq 7$

Table B.1: In this table, we are summarising the possible values of the parameters $\alpha_0, \alpha_1, \mathcal{M}_2$ and α, defined in (4.92) and (B.3), in terms of $\mathcal{M}_1$ for candidate theories that are built from the RS functional (4.92) with $p = 2$. The first column represents the central charge of the candidate theories, the next four represent the parameters, and the last one indicates the values of $\mathcal{M}_1$ for which we can find integral, positive values of the Fourier coefficients in the partition function. For $c > 16$, we weren't able to find combinations with integer Fourier coefficients.

c	$\mathcal{M}_1$	Known theories
4	192	$SO(8)_1$
8	256	$SO(16)_1$
8	384	$SO(8)_1 \times SO(8)_1$
12	192	$(E_8)_1 \times SO(8)_1$

Table B.2: In this table, we present candidate theories found from the functional (4.92) for $p = 2$ which correspond to known theories. The first column represents the central charge, the second is the parameter $\mathcal{M}_1$ from (B.3), and the third are the corresponding identified theories.

B.1.2 Theories with $p = 3$

The RS transform of these theories is dressed with the factor $\phi(s) = \alpha_1(2^s + 2^{1-s}) + \alpha_2$.

Taking that into account, we can obtain a partition function of the form

$$Z_P^* = y^{\frac{c}{2}} \left(|\eta^{\frac{c}{2}} f_0|^2 + \mathcal{M}_1 |\eta^{\frac{c}{2}} f_1|^2 + \mathcal{M}_1 |\eta^{\frac{c}{2}} f_2|^2 + \mathcal{M}_2 |\eta^{\frac{c}{2}} f_3|^2 \right). \tag{B.5}$$

with the following conformal characters

$$\begin{aligned} \eta^{\frac{c}{2}} f_0(\tau) &= \alpha E_{\frac{c}{2}}(\tau) + (1-\alpha) E_{\frac{c}{2}}(3\tau), \quad \eta^{\frac{c}{2}} f_1(\tau) = \sum_{n=1 \bmod 3}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{3}}, \\ \eta^{\frac{c}{2}} f_2(\tau) &= \sum_{n=2 \bmod 3}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{3}}, \quad \eta^{\frac{c}{2}} f_3(\tau) = -\frac{B_{\frac{c}{2}}}{c} E_{\frac{c}{2}}(\tau) + \frac{B_{\frac{c}{2}}}{c} E_{\frac{c}{2}}(3\tau). \end{aligned} \tag{B.6}$$

The list of possible candidate theories that have positive, integer Fourier coefficients, that we found [2] is presented in Table B.3. We also present in B.4 some specific values of the parameters for which the generated theory is a known one.

c	α_0	α_1	$\mathcal{M}_2$	α	Degeneracies $\in \mathbb{Z}^+$
4	$1 - \frac{7\mathcal{M}_1}{144}$	$\frac{\mathcal{M}_1}{192}$	$\mathcal{M}_1(36 - \mathcal{M}_1)$	$1 - \frac{\mathcal{M}_1}{24}$	$\mathcal{M}_1 = 36$
8	$1 - \frac{547\mathcal{M}_1}{14400}$	$\frac{3\mathcal{M}_1}{6400}$	$81\mathcal{M}_1(36 - \mathcal{M}_1)$	$1 - \frac{3\mathcal{M}_1}{80}$	$\mathcal{M}_1 \in \mathbb{Z}^+,\ 1 \leq \mathcal{M}_1 \leq 26$

Table B.3: In this table, we are summarising the possible values of the parameters $\alpha_0, \alpha_1, \mathcal{M}_2$ and α, defined in (4.92) and (B.5), in terms of $\mathcal{M}_1$ for candidate theories that are built from the RS functional (4.92) with $p = 3$. The first column represents the central charge of the candidate theories, the next four represent the parameters, and the last one indicates the values of $\mathcal{M}_1$ for which we can find integral, positive values of the Fourier coefficients in the partition function. For $c > 8$, we weren't able to find combinations with integer Fourier coefficients.

c	$\mathcal{M}_1$	Known theories
4	36	$SU(3)_1 \times SU(3)_1$
8	18	$SU(3)_1 \times (E_6)_1$

Table B.4: In this table, we present candidate theories found from the functional (4.92) for $p = 3$ which correspond to known theories. The first column represents the central charge, the second is the parameter $\mathcal{M}_1$ from (B.5), and the third are the corresponding identified theories.

B.1.3 Theories with $p = 4$

As discussed in the main text, 4 is not a prime number, however we can still do some calculations with it. In fact, with the same modified $\phi(s)$ function as in (4.106),

$$\phi(s) = \alpha_1(4^s + 4^{1-s}) + \alpha_2(2^s + 2^{1-s}) + \alpha_3\,, \tag{B.7}$$

we can now set the proposed partition function

$$Z_P^* = y^{\frac{c}{2}}\left(|\eta^{\frac{c}{2}} f_0|^2 + \mathcal{M}_1|\eta^{\frac{c}{2}} f_1|^2 + \mathcal{M}_2|\eta^{\frac{c}{2}} f_2|^2 + \mathcal{M}_1|\eta^{\frac{c}{2}} f_3|^2 + \mathcal{M}_3|\eta^{\frac{c}{2}} f_4|^2\right), \tag{B.8}$$

and characters

$$\begin{aligned}
&\eta^{\frac{c}{2}} f_0(\tau) = \alpha E_{\frac{c}{2}}(\tau) + \beta E_{\frac{c}{2}}(2\tau) + (1-\alpha-\beta)E_{\frac{c}{2}}(4\tau), \quad \eta^{\frac{c}{2}} f_1(\tau) = \sum_{n=1 \bmod 4}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{4}},\\
&\eta^{\frac{c}{2}} f_2(\tau) = \sum_{n=1 \bmod 2}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{2}}, \quad \eta^{\frac{c}{2}} f_3(\tau) = \sum_{n=3 \bmod 4}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{4}},\\
&\eta^{\frac{c}{2}} f_4(\tau) = -\frac{B_{\frac{c}{2}}}{c} E_{\frac{c}{2}}(\tau) + \gamma E_{\frac{c}{2}}(2\tau) + \left(\frac{B_{\frac{c}{2}}}{c} - \gamma\right) E_{\frac{c}{2}}(4\tau)\,.
\end{aligned} \tag{B.9}$$

Finding constraints on the parameters of this construction is rather difficult, but we found [2] two explicit known theories.

In the main text we discussed the $c = 8$ $SU(2)_1 \times (E_7)_1$, which had parameters

$$\begin{aligned}
&\alpha = \frac{8}{15}, \quad \beta = -\frac{3}{5}, \quad \gamma = \frac{3}{560},\\
&\mathcal{M}_1 = 4, \quad \mathcal{M}_2 = 0, \quad \mathcal{M}_3 = 4,\\
&\alpha_1 = 256, \quad \alpha_2 = -2592, \quad \alpha_3 = 33856\,.
\end{aligned} \tag{B.10}$$

It turns out that we can also find for $c = 4$ the $SU(2)_1 \times SU(2)_1 \times SU(2)_1 \times SU(2)_1$ theory, from the parameters $\alpha_1 = 64$, $\alpha_2 = -96$, $\alpha_3 = -32$ with parameters

$$\begin{aligned}
&\alpha = -\frac{1}{3}, \quad \beta = 0, \quad \gamma = \frac{1}{8},\\
&\mathcal{M}_1 = 4, \quad \mathcal{M}_2 = 6, \quad \mathcal{M}_3 = 1,\\
&\alpha_1 = 64, \quad \alpha_2 = -96, \quad \alpha_3 = -32\,.
\end{aligned} \tag{B.11}$$

B.1.4 Theories with $p = 5$

For $p = 5$, (4.92) becomes

$$\phi(s) = \alpha_1(5^s + 5^{1-s})\,. \tag{B.12}$$

The proposed partition function of this kind is a six-character RCFT

$$Z_P^* = |\eta^{\frac{c}{2}} f_0|^2 + \mathcal{M}_1|\eta^{\frac{c}{2}} f_1|^2 + \mathcal{M}_1|\eta^{\frac{c}{2}} f_2|^2 + \mathcal{M}_1|\eta^{\frac{c}{2}} f_3|^2 + \mathcal{M}_1|\eta^{\frac{c}{2}} f_4|^2 + \mathcal{M}_2|\eta^{\frac{c}{2}} f_5|^2, \tag{B.13}$$

with characters of the form

$$\begin{aligned}
&\eta^{\frac{c}{2}} f_0(\tau) = \alpha E_{\frac{c}{2}}(\tau) + (1-\alpha)E_{\frac{c}{2}}(5\tau), \quad \eta^{\frac{c}{2}} f_1(\tau) = \sum_{n=1 \bmod 5}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{5}}, \\
&\eta^{\frac{c}{2}} f_2(\tau) = \sum_{n=2 \bmod 5}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{5}}, \quad \eta^{\frac{c}{2}} f_3(\tau) = \sum_{n=3 \bmod 5}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{5}}, \\
&\eta^{\frac{c}{2}} f_4(\tau) = \sum_{n=4 \bmod 5}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{5}}, \quad \eta^{\frac{c}{2}} f_5(\tau) = -\frac{c}{B_{\frac{c}{2}}} E_{\frac{c}{2}}(\tau) + \frac{c}{B_{\frac{c}{2}}} E_{\frac{c}{2}}(5\tau).
\end{aligned} \tag{B.14}$$

This case becomes rather complex, and we only found some constraints for $c = 4$ but no larger value of the central charge [2]. We can see that all parameters of the coefficient function of $p = 5$ can be written in terms of the degeneracies of the characters

$$a_0 = \frac{32 - 7\mathcal{M}_1}{32}, \quad a_1 = \frac{5\mathcal{M}_1}{576}, \quad \alpha = -\frac{1}{24}\sqrt{576 - 90\mathcal{M}_1 - \mathcal{M}_2}\,. \tag{B.15}$$

Having dependence in both $\mathcal{M}_1$ and $\mathcal{M}_2$ allows us to have more possibilities for positive integer Fourier coefficients, and we provide some examples of those here

$$(\mathcal{M}_1, \mathcal{M}_2) = (1,2),\ (1,45),\ (1,86),\ (1,125),\ (1,162),\ (1,197),\ (1,230), \tag{B.16}$$

$$(1,261),\ (1,290),\ (1,317),\ (1,342),\ (1,365),\ (1,386),\ (1,405), \tag{B.17}$$

$$(1,422),\ (1,437),\ (1,450),\ (2,35),\ (2,72),\ (2,107),\ (2,140), \tag{B.18}$$

$$(2,171),\ (2,200),\ (2,227),\ (2,252),\ (2,275),\ (2,296),\ (2,315), \tag{B.19}$$

$$(2,332),\ (2,347),\ (2,360),\ (3,17),\ (3,50),\ (3,81),\ (3,110), \tag{B.20}$$

$$(3,137),\ (3,162),\ (3,185),\ (3,206),\ (3,225),\ (3,242),\ (3,257), \tag{B.21}$$

$$(3,270),\ (4,20),\ (4,47),\ (4,72),\ (4,95),\ (4,116),\ (4,135), \tag{B.22}$$

$$(4,152),\ (4,167),\ (4,180),\ (5,5),\ (5,26),\ (5,45),\ (5,62),\ (5,77),\ (5,90)\,. \tag{B.23}$$

It is unclear at the time of writing if any of these corresponds to a known theory.

B.1.5 Theories with general p

From the previous results, can conjecture that, for a generic prime number $p¿5$, we can build the RS transform of a theory (4.87) associated with (4.92) and generate a reduced partition function [2]

$$Z_P^* = y^{\frac{c}{2}} \left(|\eta^{\frac{c}{2}} f_0|^2 + \mathcal{M}_1 \sum_{i=1}^{p-1} |\eta^{\frac{c}{2}} f_i|^2 + \mathcal{M}_2 |\eta^{\frac{c}{2}} f_p|^2 \right). \tag{B.24}$$

with the following conformal characters

$$\begin{aligned}
\eta^{\frac{c}{2}} f_0(\tau) &= \sum_{p|k, p\neq k} \alpha_k E_{\frac{c}{2}}(k\tau) + (1 - \sum_{p|k, p\neq k} \alpha_p) E_{\frac{c}{2}}(D\tau), \\
\eta^{\frac{c}{2}} f_i(\tau) &= \sum_{n=i \bmod D}^{\infty} \sigma_{\frac{c}{2}-1}(n) q^{\frac{n}{D}}, \quad i = 1, 2, \cdots, D-1 \\
\eta^{\frac{c}{2}} f_D(\tau) &= -\frac{B_{\frac{c}{2}}}{c} E_{\frac{c}{2}}(\tau) + \sum_{\substack{D|p \\ p\neq D, p\neq 1}} \gamma_p E_{\frac{c}{2}}(p\tau) + \left(\frac{B_{\frac{c}{2}}}{c} - \sum_{\substack{D|p \\ p\neq D, p\neq 1}} \gamma_p \right) E_{\frac{c}{2}}(D\tau).
\end{aligned} \tag{B.25}$$

Appendix C

Appendices to Chapter 5

C.1 Extracting contributions to the OPE coefficients

Our goal in this appendix is to compute a contribution of the operators $(h,\bar{h}) = (\mathbb{N}+\mathsf{h}_1+\mathsf{h}_2, \mathbb{N}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2)$ exchanged in the alpha space decomposition to the OPE coefficients. In particular, we are going to present the contribution coming from a product of one holomorphic and one antiholomorphic term from the $\mathcal{J}_+$ portion of (5.108). The term from which we aim to extract polar contributions is:

$$\begin{aligned}\mathscr{L}_+^{(1,1)}(\alpha,\bar{\alpha}) = &\int_0^1 \mathrm{d}z\, w_s(z)\, \Psi_\alpha^s(z) \int_0^1 \mathrm{d}\bar{z}\, \bar{w}_s(\bar{z})\, \Psi_{\bar{\alpha}}^s(\bar{z}) \ \left(\frac{z}{\bar{z}}\right)^{\Delta_2-1} \left(\frac{1-z}{1-\bar{z}}\right)^{\Delta_1+\Delta_3-2} \\ &\times \int_1^\infty \mathrm{d}x \frac{(x-1)^{\Delta_1+\Delta_4-2}}{x\,(x-1)} (x-1)^{2\mathsf{h}_2-2\mathsf{h}_3} (1-z)^{-2\mathsf{h}_4}\, {}_2F_1\left(2\mathsf{h}_2, 2\mathsf{h}_4; 1+2\mathsf{h}_{23}; \frac{1-x}{1-z}\right) \\ &\times (x-1)^{2\bar{\mathsf{h}}_3-2\bar{\mathsf{h}}_2} (1-\bar{z})^{-2\bar{\mathsf{h}}_1}\, {}_2F_1\left(2\bar{\mathsf{h}}_1, 2\bar{\mathsf{h}}_3; 1-2\bar{\mathsf{h}}_{23}; \frac{1-x}{1-\bar{z}}\right). \end{aligned} \tag{C.1}$$

We perform the $\bar{\alpha}$ decomposition following the same steps as in section 5.5.3. For the sake of simplicity, we will only include in the following expressions the terms that give rise to the precise towers of operators $(h,\bar{h}) = (\mathbb{N}+\mathsf{h}_1+\mathsf{h}_2, \mathbb{N}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2)$, but not their Weyl reflections,[1] as these are the ones we will use to compute this terms' contribution to the conformal block

[1] We are using a slightly different convention to the one used in [145,146] due to some features of our integrals it makes sense to partition the α space decomposition in this way. We are, at the end of the day, still only counting one family of Weyl reflections, and constructing the tower of operators accordingly, which is all that amtters.

expansion,

$$\mathcal{L}_+^{(1,1)}(\alpha,\bar{\alpha}) = \sum_{q,\bar{q}=0}^{\infty}\int_1^{\infty} \mathrm{d}x \frac{(x-1)^{2\mathsf{h}_1-2\mathsf{h}_4}}{x\,(x-1)}(x-1)^{2\mathsf{h}_{23}}(1-x)^q(x-1)^{-2\bar{\mathsf{h}}_{23}}(1-x)^{\bar{q}}$$

$$\times\Bigg[\frac{\Gamma\left(1-2\mathsf{h}_1\right)\Gamma\left(1+2\mathsf{h}_{23}\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}\right)\Gamma\left(2\mathsf{h}_2+q\right)\Gamma\left(2\mathsf{h}_{14}-q\right)\Gamma\left(2\mathsf{h}_4+q\right)}{\Gamma\left(2\mathsf{h}_2\right)\Gamma\left(2\mathsf{h}_3\right)\Gamma\left(2\mathsf{h}_4\right)\Gamma(1+q)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)\Gamma\left(1+2\mathsf{h}_{23}+q\right)\Gamma\left(2\mathsf{h}_{14}-q\right)}$$

$$\times\,\Gamma\left(-\frac{1}{2}-\alpha+\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(-\frac{1}{2}+\alpha+\mathsf{h}_1+\mathsf{h}_2\right)$$

$$\times\,{}_3F_2\left(-\frac{1}{2}-\alpha+\mathsf{h}_1+\mathsf{h}_2,-\frac{1}{2}\alpha+\mathsf{h}_1+\mathsf{h}_2,2\mathsf{h}_{14}-q;2\mathsf{h}_1,2\mathsf{h}_3;1\right)\Bigg]$$

$$\times\Bigg[\frac{\Gamma\left(1-2\bar{\mathsf{h}}_1\right)\Gamma\left(1-2\bar{\mathsf{h}}_{23}\right)\Gamma\left(1-\bar{\mathsf{h}}_{12}+\bar{\mathsf{h}}_{34}\right)\Gamma\left(2\bar{\mathsf{h}}_1+\bar{q}\right)\Gamma\left(2\bar{\mathsf{h}}_3+\bar{q}\right)}{\Gamma\left(2\bar{\mathsf{h}}_1\right)\Gamma\left(2\bar{\mathsf{h}}_3\right)\Gamma\left(2\bar{\mathsf{h}}_3\right)\Gamma(1+\bar{q})\Gamma\left(\frac{1}{2}-\bar{\alpha}-\bar{\mathsf{h}}_{12}\right)\Gamma\left(\frac{1}{2}+\bar{\alpha}-\bar{\mathsf{h}}_{12}\right)\Gamma\left(1-2\bar{\mathsf{h}}_{23}+\bar{q}\right)}$$

$$\times\,\Gamma\left(-\frac{1}{2}-\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2\right)\Gamma\left(-\frac{1}{2}+\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2\right)$$

$$\times\,{}_3F_2\left(-\frac{1}{2}-\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2,-\frac{1}{2}+\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2,-\bar{q};2\bar{\mathsf{h}}_1,2\bar{\mathsf{h}}_3;1\right)\Bigg]. \tag{C.2}$$

We now compute the x integral, obtaining the following result:

$$\mathcal{L}_+^{(1,1)}(\alpha,\bar{\alpha}) = \Gamma\left(-\frac{1}{2}-\alpha+\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(-\frac{1}{2}-\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2\right) \tag{C.3}$$

$$\times\,\frac{\Gamma\left(-\frac{1}{2}+\alpha+\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(-\frac{1}{2}+\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2\right)}{\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}-\bar{\alpha}-\bar{\mathsf{h}}_{12}\right)\Gamma\left(\frac{1}{2}+\bar{\alpha}-\bar{\mathsf{h}}_{12}\right)}$$

$$\times\,\frac{\Gamma\left(1-2\mathsf{h}_1\right)\Gamma\left(1+2\mathsf{h}_{23}\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}\right)\Gamma\left(1-2\bar{\mathsf{h}}_1\right)\Gamma\left(1-2\bar{\mathsf{h}}_{23}\right)\Gamma\left(1-\bar{\mathsf{h}}_{12}+\bar{\mathsf{h}}_{34}\right)}{\Gamma\left(2\mathsf{h}_2\right)\Gamma\left(2\mathsf{h}_3\right)\Gamma\left(2\mathsf{h}_4\right)\Gamma\left(2\bar{\mathsf{h}}_1\right)\Gamma\left(2\bar{\mathsf{h}}_3\right)^2}$$

$$\times\sum_{q,\bar{q}=0}^{\infty}\frac{(-1)^{(q+\bar{q})}\Gamma\left(2\mathsf{h}_2+q\right)\Gamma\left(2\mathsf{h}_4+q\right)\Gamma\left(2\bar{\mathsf{h}}_1+\bar{q}\right)\Gamma\left(2\bar{\mathsf{h}}_3+\bar{q}\right)\pi\csc\left(\pi\left(q+\bar{q}-2\bar{\mathsf{h}}_{23}\right)\right)}{\Gamma(1+q)\Gamma(1+\bar{q})\Gamma\left(1+2\mathsf{h}_{23}+q\right)\Gamma\left(1-2\bar{\mathsf{h}}_{23}+q\right)}$$

$$\times\,{}_3F_2\left(-\frac{1}{2}-\alpha+\mathsf{h}_1+\mathsf{h}_2,-\frac{1}{2}\alpha+\mathsf{h}_1+\mathsf{h}_2,2\mathsf{h}_{14}-q;2\mathsf{h}_1,2\mathsf{h}_3;1\right)$$

$$\times\,{}_3F_2\left(-\frac{1}{2}-\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2,-\frac{1}{2}+\bar{\alpha}+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2,-\bar{q};2\bar{\mathsf{h}}_1,2\bar{\mathsf{h}}_3;1\right). \tag{C.4}$$

Using (5.126), we see that, to find the contribution to the OPE coefficient of the poles

$(h_n, \bar{h}_m) = (1/2 + n + \mathsf{h}_1 + \mathsf{h}_2, 1/2 + m + \bar{\mathsf{h}}_1 + \bar{\mathsf{h}}_2)$, we should obtain the residue (C.4). We find that the residue is

$$R_+^{(1,1)}(n,m) = \frac{(-1)^{m+n}\Gamma\left(n + 2\mathsf{h}_1 + 2\mathsf{h}_2 - 1\right)\Gamma\left(m + 2\bar{\mathsf{h}}_1 + 2\bar{\mathsf{h}}_2 - 1\right)}{m!\, n!\,\Gamma\left(-n - 2\mathsf{h}_1 + 1\right)\Gamma\left(n + 2\mathsf{h}_2\right)\Gamma\left(-m - 2\bar{\mathsf{h}}_1 + 1\right)\Gamma\left(m + 2\bar{\mathsf{h}}_2\right)}$$

$$\times \sum_{q,\bar{q}=0}^{\infty} \frac{(-1)^{(q+\bar{q})}\Gamma\left(2\mathsf{h}_2 + q\right)\Gamma\left(2\mathsf{h}_4 + q\right)\Gamma\left(2\bar{\mathsf{h}}_1 + \bar{q}\right)\Gamma\left(2\bar{\mathsf{h}}_3 + \bar{q}\right)\pi\csc\left(\pi\left(q + \bar{q} - 2\bar{\mathsf{h}}_{23}\right)\right)}{\Gamma(1+q)\Gamma(1+\bar{q})\Gamma\left(1 + 2\mathsf{h}_{23} + q\right)\Gamma\left(1 - 2\bar{\mathsf{h}}_{23} + q\right)}$$

$${}_3F_2\left(-n, n + 2\mathsf{h}_1 + 2\mathsf{h}_2 - 1, -q + 2\mathsf{h}_1 - 2\mathsf{h}_4; 2\mathsf{h}_1, 2\mathsf{h}_3; 1\right)$$

$${}_3F_2\left(-m, m + 2\bar{\mathsf{h}}_1 + 2\bar{\mathsf{h}}_2 - 1, -\bar{q}; 2\bar{\mathsf{h}}_1, 2\bar{\mathsf{h}}_3; 1\right) . \tag{C.5}$$

Very interestingly, for each pair n, m, the hypergeometric series terminates! So those functions are actually polynomials, and this behaviour agrees with results found in [145].

Now that we have the residue, we can present the contribution of this term to the overall conformal block decomposition [3]:

$$\mathscr{L}_+^{(1,1)}(z,\bar{z}) = \sum_{n,m=0}^{\infty} \frac{4R_+^{(1,1)}(n,m)\, k_{n+\mathsf{h}_1+\mathsf{h}_2}(z) k_{m+\bar{\mathsf{h}}_1+\bar{\mathsf{h}}_2}(\bar{z})}{Q(1/2 - n - \mathsf{h}_1 - \mathsf{h}_2)Q(1/2 - m - \bar{\mathsf{h}}_1 - \bar{\mathsf{h}}_2)}. \tag{C.6}$$

If we wanted to obtain the full OPE coefficient, we would have to identify the residues for all the terms present in both $\mathscr{J}_+$ and $\mathscr{J}_-$ and in all the regions of x. One would then go onto performing the x integral and then summing over all the regions (i.e. in the previous formula we would substitute $R_+^{(1,1)}$ by $R_+^{(\text{all terms } \mathscr{J}_+)} + R_-^{(\text{all terms } \mathscr{J}_-)}$). One can already see from the calculation above that this will come with serious technical challenges, especially when examining the more complicated structure of the $\mathscr{J}_-$ integrals. In fact, for each of the terms we would be dealing with quadruple sums, and then one would need a regularisation prescription due to partitioning the x-integral in three regions. Furthermore, one has to include both these poles and the additional poles found in the $(0,1)$ region: $(h, \bar{h}) = (\mathbb{Z}^+/2, \mathbb{Z}^+/2)$.

Most of these complications are directly related to the generalised-Meijer-G-type expressions that appear in the α-space integrals, however a future goal would be finding a way to verify if and how the residues of these expressions factorise into coefficients of 3-point amplitudes.

C.2 Obtaining the poles in $\alpha = \mathbb{Z}^+/2$

We are going to, as an example, show the integration on the z variable for one of the terms that provides us with the second kind of poles at $\alpha = \frac{\mathbb{Z}^+}{2}$. These poles are only present in the section of the integral where $x \in (0,1)$.

Following a similar spirit than section 5.5.3, we will only show one specific term of the holomorphic integral where we can explicitly see these poles arising.

We are focusing on the integral

$$\int_0^1 \mathrm{d}z\, w_s(z)\, \Psi_\alpha(z)\, \frac{\mathcal{J}_-(\Delta_2,\Delta_3,\Delta_4\,|\,x,z)}{z^{1-\Delta_2}(1-z)^{2-\Delta_1-\Delta_3}}, \tag{C.7}$$

where now $x \in (0,1)$. We will pick one of the terms of $\mathcal{J}_-$ to find these poles. This integral is another sector of (5.127)

Now, in order to perform this integral properly, we need to separate it into two sectors, one where $0 < z < x$ and one where $z < x < 1$. Very interestingly, the sector $z < x < 1$ doesn't produce any poles, they are all concentrated in the first sector.

Thus, we will actually perform the integral

$$\begin{aligned}\widehat{\mathcal{J}}_-(\alpha,x) = \int_0^x \mathrm{d}z\, (1-z)^{-2\mathsf{h}_4} z^{2\mathsf{h}_1-2}(x-1)^{-1+2\mathsf{h}_2}(x-z)^{1-2\mathsf{h}_3-2\mathsf{h}_4} \\ \times\, {}_2F_1\left(1-2\mathsf{h}_2, 1-2\mathsf{h}_3\; 2-2\mathsf{h}_3-2\mathsf{h}_4; \frac{x-z}{x-1}\right) \\ \times\, {}_2F_1\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}, \frac{1}{2}+\alpha-\mathsf{h}_{12}; 1-\mathsf{h}_{12}+\mathsf{h}_{34}; \frac{z-1}{z}\right).\end{aligned} \tag{C.8}$$

We now, following what we did for the $\widehat{\mathcal{J}_+}$ integral, expand the first ${}_2F_1$ as a series on its argument, and the partial wave as a Mellin-Barnes integral.

This results in

$$\hat{\mathcal{J}}_-(\alpha,x) = \sum_{q=0}^{\infty} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}t}{2\pi\mathrm{i}} \int_0^1 \mathrm{d}z\, z^{2\mathsf{h}_1-2}(x-1)^{-1+2sh_2-q}$$

$$\times (x-z)^{1-2\mathsf{h}_3-2\mathsf{h}_4+q} \frac{\Gamma(2-2\mathsf{h}_3-2\mathsf{h}_4)\,\Gamma(1-2\mathsf{h}_2+q)\,\Gamma(1-2\mathsf{h}_3+q)}{\Gamma(1-2\mathsf{h}_2)\,\Gamma(1-2\mathsf{h}_3)\,\Gamma(1+q)\Gamma(2-2\mathsf{h}_3-2\mathsf{h}_4+q)}$$

$$\times \left(\frac{1-z}{z}\right)^t \frac{\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}\right)\Gamma(-t)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}+t\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}+t\right)}{\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}+t\right)}$$

$$= \sum_{q=0}^{\infty} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}t}{2\pi\mathrm{i}}\, \pi e^{i\pi(1+t-2\mathsf{h}_1)}(-1)^{q+1}(1-x)^{2\mathsf{h}_2-1-q}x^{2\mathsf{h}_1-2\mathsf{h}_3-2\mathsf{h}_4+q-t}\left(\cot\left(\pi\left(t-2\mathsf{h}_1\right)\right)-i\right) \tag{C.9}$$

$$\times \frac{\Gamma\left(-\mathsf{h}_{12}+\mathsf{h}_{34}+1\right)\Gamma\left(-2\left(\mathsf{h}_3+\mathsf{h}_4-1\right)\right)\Gamma\left(1-2\mathsf{h}_2+q\right)\Gamma\left(1-2\mathsf{h}_3+q\right)}{\Gamma\left(1-2\mathsf{h}_2\right)\Gamma\left(1-2\mathsf{h}_3\right)\Gamma(1+q)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)}$$

$$\times \frac{\Gamma(-t)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}+t\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}+t\right)}{\Gamma\left(2-2\mathsf{h}_1+t\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}+t\right)}$$

$$\times {}_2\widetilde{\mathcal{F}}_1\left(-t, 2\mathsf{h}_1-1-t; 1+2\mathsf{h}_1-2\mathsf{h}_3-2\mathsf{h}_4+q-t; x\right), \tag{C.10}$$

where ${}_2\widetilde{\mathcal{F}}_1$ represents a regularised hypergeometric function.

We have a factor that separates this expression further into two separate terms:

$$\left(\cot\left(\pi\left(t-2\mathsf{h}_1\right)\right)-i\right). \tag{C.11}$$

Both terms contain the 'extra' poles so, for simplicity's sake, we will move on with just the cotangent term, and we will denote the expression now as $\hat{\mathscr{J}}_{-}^{cot}(\alpha, x)$ to note that. To be able to identify the correct function in the Mellin Barnes integral, we expand the hypergeometric function that appeared when integrating over z, as a power series. We will also expand the cotangents into gamma functions.

$$\hat{\mathscr{J}}_{-}^{cot}(\alpha, x) = \sum_{q=0}^{\infty}\sum_{p=0}^{\infty} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\mathrm{d}t}{2\pi\mathrm{i}} \pi e^{i\pi(1+t-2\mathsf{h}_1)}(-1)^{q+1}(1-x)^{2\mathsf{h}_2-1-q}x^{2\mathsf{h}_1-2\mathsf{h}_3-2\mathsf{h}_4+p+q-t}$$

$$\times \frac{\Gamma\left(-\mathsf{h}_{12}+\mathsf{h}_{34}+1\right)\Gamma\left(-2\left(\mathsf{h}_3+\mathsf{h}_4-1\right)\right)\Gamma\left(1-2\mathsf{h}_2+q\right)\Gamma\left(1-2\mathsf{h}_3+q\right)}{\Gamma\left(1-2h_2\right)\Gamma\left(1-2h_3\right)\Gamma(1+p)\Gamma(1+q)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_1+\mathsf{h}_2\right)} \tag{C.12}$$

$$\times \frac{\Gamma\left(-2\mathsf{h}_1+t\right)\Gamma\left(1+2\mathsf{h}_1-t\right)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}+t\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}+t\right)}{\Gamma\left(\frac{1}{2}-2\mathsf{h}_1+t\right)\Gamma\left(2-2\mathsf{h}_1+t\right)\Gamma\left(-1+2\mathsf{h}_1-t\right)\Gamma\left(\frac{1}{2}+2\mathsf{h}_1-t\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}+t\right)}$$

$$\times \frac{\Gamma(p-t)\Gamma\left(-1+2\mathsf{h}_1+p-t\right)}{\Gamma\left(1+2\mathsf{h}_1-2\mathsf{h}_3-2\mathsf{h}_4+p+q-t\right)}. \tag{C.13}$$

Via the Mellin-Barnes definition of the Meijer G function, we can solve the t-integral, which leads to the following expression

$$\hat{\mathscr{J}}_{-}^{cot}(\alpha,x)=\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}\pi e^{i\pi(1-2\mathsf{h}_1)}(-1)^{q+1}(1-x)^{2\mathsf{h}_2-1-q}x^{2\mathsf{h}_1-2\mathsf{h}_3-2\mathsf{h}_4+p+q}$$

$$\times\frac{\Gamma\left(-\mathsf{h}_{12}+\mathsf{h}_{34}+1\right)\Gamma\left(-2\left(\mathsf{h}_3+\mathsf{h}_4-1\right)\right)\Gamma\left(1-2\mathsf{h}_2+q\right)\Gamma\left(1-2\mathsf{h}_3+q\right)}{\Gamma\left(1-2h_2\right)\Gamma\left(1-2h_3\right)\Gamma(1+p)\Gamma(1+q)\Gamma\left(\frac{1}{2}-\alpha-\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(\frac{1}{2}+\alpha-\mathsf{h}_1+\mathsf{h}_2\right)} \tag{C.14}$$

$$G^{3,3}_{6,6}\left(-\frac{1}{x}\,\middle|\,\begin{matrix}2\mathsf{h}_1+1,-\alpha+\mathsf{h}_{12}+\frac{1}{2},\alpha+\mathsf{h}_{12}+\frac{1}{2},2\mathsf{h}_1-1,2\mathsf{h}_1+\frac{1}{2},-2\mathsf{h}_2+p+q+1\\ p,2\mathsf{h}_1+1,2\mathsf{h}_1+p-1,2\mathsf{h}_1+\frac{1}{2},2\mathsf{h}_1-1,\mathsf{h}_{12}-\mathsf{h}_{34}\end{matrix}\right). \tag{C.15}$$

This Meijer G can be expressed as a sum over two hypergeometric functions, from where we will obtain the long-awaited poles.

$$\hat{\mathscr{J}}_{-}^{cot}(\alpha,x)=\sum_{q=0}^{\infty}\sum_{p=0}^{\infty}e^{i\pi(1-2\mathsf{h}_1)}(-1)^{q+1}(1-x)^{2\mathsf{h}_2-1-q}x^{2\mathsf{h}_1-2\mathsf{h}_3-2\mathsf{h}_4+p+q}(-1/x)^{\mathsf{h}_{12}-1/2}$$

$$\frac{\Gamma\left(2-2\mathsf{h}_3-2\mathsf{h}_4\right)\Gamma\left(1-\mathsf{h}_{12}+\mathsf{h}_{34}\right)\Gamma\left(1-2\mathsf{h}_2+q\right)\Gamma\left(1-2\mathsf{h}_3+q\right)}{\Gamma\left(1-2\mathsf{h}_2\right)\Gamma\left(1-2\mathsf{h}_3\right)\Gamma(1+p)\Gamma(1+q)} \tag{C.16}$$

$$\times\left(\left(-\frac{1}{x}\right)^{\alpha}\frac{\Gamma(2\alpha)\ \Gamma\left(-\frac{1}{2}-\alpha+\mathsf{h}_1+\mathsf{h}_2+p\right)\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}\right)_p}{\Gamma\left(\alpha+\mathsf{h}_{34}+\frac{1}{2}\right)\Gamma\left(\alpha-\mathsf{h}_1-\mathsf{h}_2\right)\Gamma\left(\alpha-\mathsf{h}_{12}+\frac{1}{2}\right)\Gamma\left(-\alpha+\mathsf{h}_1+\mathsf{h}_2+1\right)\Gamma\left(\frac{3}{2}-\alpha-\mathsf{h}_1-\mathsf{h}_2+p+q\right)}\right.$$

$$\left.\times\ {}_3F_2\left(\frac{1}{2}-\alpha-\mathsf{h}_{12}+p,-\alpha+\mathsf{h}_1+\mathsf{h}_2+p-\frac{1}{2},\frac{1}{2}-\alpha-\mathsf{h}_{34};1-2\alpha,-\alpha-\mathsf{h}_1-\mathsf{h}_2+p+q+\frac{3}{2};-x\right)\right)$$

$$+(-x)^{\alpha}\frac{\Gamma(1-2\alpha)\ \Gamma\left(-\frac{1}{2}+\alpha+\mathsf{h}_1+\mathsf{h}_2+p\right)\left(\frac{1}{2}+\alpha-\mathsf{h}_{12}\right)_p}{(2\alpha)_1\Gamma\left(-\alpha-\mathsf{h}_1-\mathsf{h}_2+1\right)\Gamma\left(-\alpha-\mathsf{h}_{12}+\frac{1}{2}\right)\Gamma\left(\alpha+\mathsf{h}_1+\mathsf{h}_2\right)\Gamma\left(-\alpha+\mathsf{h}_{34}+\frac{1}{2}\right)\Gamma\left(p+q+\alpha-\mathsf{h}_1-\mathsf{h}_2+\frac{3}{2}\right)}$$

$$\left.\times\ {}_3F_2\left(\alpha-\mathsf{h}_{12}+p+\frac{1}{2},\alpha+\mathsf{h}_1+\mathsf{h}_2+p-\frac{1}{2},\alpha-\mathsf{h}_{34}+\frac{1}{2};2\alpha+1,\alpha-\mathsf{h}_1-\mathsf{h}_2+p+q+\frac{3}{2};-x\right)\right), \tag{C.17}$$

where we have used Pochhammer symbols to simplify the expressions $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$. The terms in grey present the poles that are found throughout the whole span of the x-integral, whilst the terms in pink present those poles that are only found in the regime $x\in(0,1)$. We see that the Weyl reflections are present in this case as well, as expected.

Even though we see that α enters the x-integral in this sector, due to the particular mixing of holomorphic and antiholomorphic sectors that performing that integral provokes, it shouldn't produce twin poles of the form of (5.125). Only the highlighted terms produce poles that will enter conformal block decomposition.

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