

Relaxation in Driven Integer Quantum Hall Edge States

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A highly nonthermal electron distribution is generated when quantum Hall edge states originating from sources at different potentials meet at a quantum point contact. The relaxation of this distribution to a stationary form as a function of distance downstream from the contact has been observed in recent experiments [C. Altimiras *et al.*, Phys. Rev. Lett. **105**, 056803 (2010)]. Here we present an exact treatment of a minimal model for the system at filling factor $\nu = 2$, with results that account well for the observations.

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Introduction.—The importance of understanding nonequilibrium dynamics and relaxation in many-body quantum systems has been recognised since the early years of quantum mechanics [1,2]. Settings in which such problems are of high current interest include, among others, cold atomic gases [2] and nanoscale electronic devices. [3–14] As a particular example, recent experiments [3] on quantum Hall (QH) edge states driven out of equilibrium at a quantum point contact (QPC) provide very detailed information on the approach to a steady state in an electron system that appears to be well isolated from other degrees of freedom. In this Letter we describe the exact solution of a simple model for these experiments and compare our results with the measurements.

In outline, the experiments [3] we are concerned with involve two sets of integer QH edge states, which meet at a QPC. When a bias voltage is applied to the QPC, tunneling between the edge states generates a nonequilibrium electron distribution. The form of this distribution in energy and its evolution as a function of distance downstream from the QPC are probed by monitoring the tunneling current from a point on the edge, through a quantum dot that has an isolated level of controllable energy. Close to the QPC, the measured distribution has two steps, reflecting the different energies of Fermi steps in each of the incident edges. With increasing distance from the QPC, the distribution relaxes to a single, broad step. The theoretical challenge presented by these observations is to understand and model this relaxation process.

The relationship between these edge state experiments and other recent work on many-body quantum dynamics far from equilibrium has several aspects worth emphasizing. First, in the context of QH edge states, these are the most recent of a series of striking observations of nonequilibrium effects in interferometers [6] and in thermal transport [7]. They are also the equivalent for a ballistic system of earlier studies [8] of local distributions in diffusive wires. Second, the measurements stand apart from earlier theory [13] and experiment [14] on nonequilibrium transport between

fractional QH edge states, because they probe local distribution functions, rather than the global nonlinear current-voltage characteristic. Third, and more broadly, the system studied is different in important ways from quantum impurity problems [9–12], as there is no impurity degree of freedom and interactions are not confined to an impurity site but instead operate in the ingoing and outgoing channels. Fourth, there is an analogy between edge state relaxation and cold atom experiments in the time domain [2], since distance from the QPC translates roughly as time, using the edge state velocity as a conversion factor. In that sense the experiment we consider, probing relaxation as electrons propagate, is equivalent to a quantum quench [15], in which time evolution is studied following a sudden change in the Hamiltonian. An important question in this context is whether a system thermalizes at long times. Since integrability is an obstacle to thermalization, it is noteworthy that the conventional model [16] of a QH edge state as a chiral Luttinger liquid is an integrable one.

Any attempt to model theoretically the experiments of Ref. [3] starting from a chiral Luttinger liquid description faces an obvious difficulty, since interactions are most naturally described in terms of collective modes using bosonization, but tunneling at the QPC is simple only in terms of fermionic variables. In pioneering work, two alternative approaches have been developed: one based on a Boltzmann-like equation for the electron distribution [17]; and the other using a phenomenological model for the plasmon distribution generated at the QPC [18]. From Ref. [18], and from subsequent discussion of a quantum quench in an isolated QH edge (with an initial state chosen to emulate the effects of a QPC) by the present authors [19], a physical picture has emerged, in which relaxation is seen as a consequence of plasmon dispersion or the presence at $\nu = 2$ of two plasmon modes with distinct velocities. In summary, an electron that tunnels at the QPC can be viewed as a superposition of plasmons. Dispersion or multiple plasmon velocities cause such a wave packet to broaden as it propagates. The length scale for relaxation of

the electron distribution downstream from the QPC is the distance at which the width of this wave packet is comparable to the characteristic separation between tunneling electrons. The argument identifies relevant scales but does not generate a prediction for the electron distribution and its dependence on distance from the QPC. One of our main aims here is to calculate this central quantity.

An overview of our treatment is as follows: we make key use of the fact that, since experiments are at Landau level filling factor $\nu = 2$, each QH edge carries two copropagating channels. Interactions mix excitations in different channels, and in the minimal model eigenmodes are characterized using two velocities. Our approach combines re-fermionization [20,21] with nonequilibrium bosonization methods [22–26]: by first bosonizing, then recombining bosons to form a new set of fermions, we transform the Hamiltonian for the interacting system into one for free particles. Under this transformation, the measured distribution (or more accurately, the spectral function probed by the tunneling conductance) is expressed as a free-fermion determinant. This determinant has a form similar to that appearing in the theory [27,28] of full counting statistics [FCS], and related quantities appear in a variety of other nonequilibrium problems [29]. Its numerical evaluation can be done accurately and efficiently, yielding the results we present below. In addition, simple analytical expressions can be found for some quantities, and we give one for the total energy in each channel as a function of distance from the QPC, thus providing a Hamiltonian derivation of a result obtained previously in the phenomenological treatment of Ref. [18].

The work we set out here is related in a variety of ways to other studies of devices built from integer QH edge states. In particular, experiments that revealed striking nonequilibrium effects in Mach Zehnder interferometers [6] have stimulated extensive theoretical research, [25,30–37] including calculations [33] of interferometer dephasing based on the same bosonized model [16] for edge states at $\nu = 2$ with contact interactions that we adopt in the following. In the context of edge state relaxation a parallel development to the present Letter described in Ref. [26], is based on an approximation (involving a factorization of bosonic correlators) that is expected to be accurate sufficiently far from the QPC. Our solution recovers such a factorization, but only at large distances compared to the relaxation length.

Model.—The experimental system is illustrated in the upper panels of Figs. 1 and 2: two alternative geometries arise, according to whether the tunneling conductance is measured in the *same* or the *opposite* channel to that coupled by the QPC.

We take a Hamiltonian with kinetic, interaction and tunneling terms, in the form $\hat{H} = \hat{H}_{\text{kin}} + \hat{H}_{\text{int}} + \hat{H}_{\text{tun}}$. Using the labels $\eta = 1, 2$ to distinguish edges according to their source, and $s = \uparrow, \downarrow$ to differentiate between the

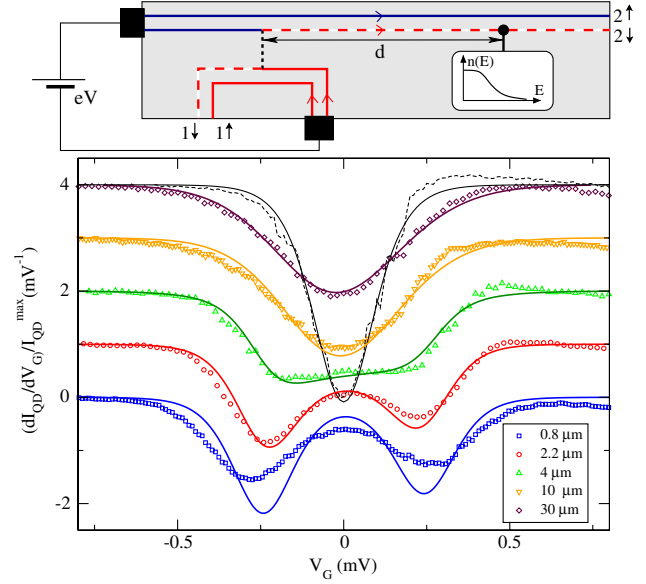


FIG. 1 (color online). Top: sketch of the experimental geometry in which the tunneling conductance is measured in the *same* channel as that coupled by the QPC. Bottom: differential conductance calculated (lines) and measured [3] (symbols) in this geometry at indicated distances from the QPC. Fits use $T = 44$ mK, $V = 30.3$ μ V ($\beta eV = 8$), and $p = 0.545$. Black lines: calculations (solid line) and data (dashed line) for $V = 0$.

two channels on a given edge, the fermion creation operator at point x for channel η, s is $\hat{\psi}_{\eta s}^\dagger(x)$. It obeys the standard anticommutation relation $\{\hat{\psi}_{\eta s}^\dagger(x), \hat{\psi}_{\eta' s'}(x')\} = \delta_{ss'}\delta_{\eta\eta'}\delta(x-x')$. The density operator is $\hat{\rho}_{\eta s}(x) = \hat{\psi}_{\eta s}^\dagger(x)\hat{\psi}_{\eta s}(x)$. Taking all four channels η, s to have the same bare velocity v , and assuming a contact interaction of strength g between electrons in different channels on the same edge, we have [16]

$$\hat{H}_{\text{kin}} = -i\hbar v \sum_{\eta, s} \int \hat{\psi}_{\eta s}^\dagger(x) \partial_x \hat{\psi}_{\eta s}(x) dx \quad (1)$$

and

$$\hat{H}_{\text{int}} = 2\pi\hbar g \sum_{\eta} \int \hat{\rho}_{\eta\uparrow}(x) \hat{\rho}_{\eta\downarrow}(x) dx. \quad (2)$$

Note that short-range intrachannel interactions can simply be absorbed into the value of v . Tunneling with amplitude t_{QPC} at the QPC between channels $1\downarrow$ and $2\downarrow$ is described by

$$\hat{H}_{\text{tun}} = t_{\text{QPC}} \hat{\psi}_{1\downarrow}^\dagger(0) \hat{\psi}_{2\downarrow}(0) + \text{H.c.} \quad (3)$$

A bias voltage V generates a chemical potential difference eV between incident electrons on edge 1 and those on edge 2.

The observable of interest is the tunneling conductance as a function of energy E and distance $d > 0$ from the QPC, in channel 2, s . This is the Fourier transform of the correlator

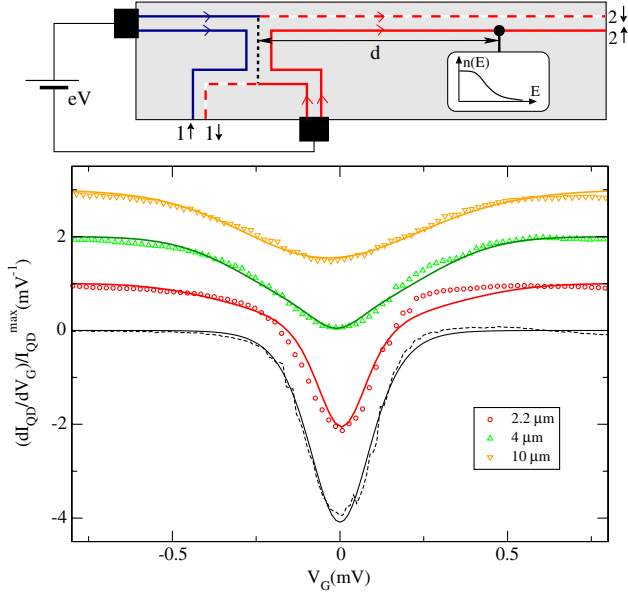


FIG. 2 (color online). Top: sketch of the experimental geometry in which the tunneling conductance is measured in the *opposite* channel to that coupled by the QPC. Bottom: differential conductance calculated (lines) and measured [3] (symbols) in this geometry at indicated distances from the QPC. Fits use $T = 44$ mK, $V = 45.5$ μ V, and $p = 0.545$. Black lines: calculations (solid line) and data (dashed line) for $V = 0$.

$$G_s(d, \tau) = \langle e^{i\hat{H}\tau/\hbar} \hat{\psi}_{2s}^\dagger(d) e^{-i\hat{H}\tau/\hbar} \hat{\psi}_{2s}(d) \rangle, \quad (4)$$

where the average is taken in the nonequilibrium steady state, with $s = \downarrow$ in the geometry of Fig. 1 and $s = \uparrow$ in that of Fig. 2. The tunneling conductance is determined by a measurement of the current $I_{\text{QD}}(E)$ through a quantum dot with a single level at energy E weakly coupled to the channel. With tunneling amplitude t_D to the dot, this is [19]

$$I_{\text{QD}}(E) = \frac{e|t_D|^2}{\hbar^2} \int G_s(d, \tau) e^{-iE\tau/\hbar} d\tau. \quad (5)$$

Results.—We show in Ref. [38] that the Hamiltonian \hat{H} can be brought into a free particle form by introducing transformed fermion operators $\hat{\Psi}_\alpha(x)$ with index $\alpha = A_\pm, S_\pm$ and eigenmode velocities $v_\alpha \equiv v_\pm = v \pm g$. Using these new operators

$$\begin{aligned} \hat{H} = & -i\hbar \sum_\alpha v_\alpha \int \hat{\Psi}_\alpha^\dagger \partial_x \hat{\Psi}_\alpha dx \\ & - [t_{\text{QPC}} \hat{\Psi}_{A+}^\dagger(0) \hat{\Psi}_{A-}(0) + \text{H.c.}] \end{aligned} \quad (6)$$

Moreover the tunneling conductance is obtained from

$$G_s(d, \tau) \propto \langle e^{-i\pi[\pm \hat{\mathcal{N}}_{A-}(d, \tau) + \hat{\mathcal{N}}_{A+}(d, \tau)]} \rangle, \quad (7)$$

where the upper sign is taken in the first term of the exponent for $s = \uparrow$ and the lower sign for $s = \downarrow$. This expectation value is taken in the stationary scattering state

of \hat{H} specified by the temperature and chemical potentials of incident channels, and the operators

$$\hat{\mathcal{N}}_\alpha(d, \tau) = \int_d^{d+v_\pm\tau} \hat{\Psi}_\alpha^\dagger(y) \hat{\Psi}_\alpha(y) dy \quad (8)$$

count electrons within the spatial intervals $d \leq y \leq d + v_\pm\tau$ on the channels A_\pm .

Equations (6)–(8) form the central results of this Letter: they express the observable of interest in an interacting, nonequilibrium system, in terms of the expectation value of a single-particle operator, evaluated as an average in a stationary scattering state of a single-particle Hamiltonian. We now use these equations to discuss the physics of relaxation in this system and to calculate the tunneling conductance.

A simple physical picture of the relaxation process, and an identification of relevant length scales, follows from the form of Eq. (7). This picture has the same content as our discussion (above and in Ref. [19]) of relaxation arising from two plasmon velocities, but is phrased in terms of the transformed fermions. Note first that the operators $\hat{\mathcal{N}}_\pm(d, \tau)$ count fermions that pass through the QPC in two time windows, both of duration τ , but with a relative delay of d/v_{eff} , where

$$1/v_{\text{eff}} = 1/v^- - 1/v^+ = 2g/(v^2 + g^2). \quad (9)$$

At $d = 0$ the two windows exactly overlap. Then, for example, $G_\uparrow(0, \tau)$ acquires the same contribution from each fermion inside the window, regardless of tunneling. In consequence, and as expected, the electron distribution at $d = 0$ in the \uparrow channel is unaffected by tunneling, since the QPC couples \downarrow channels. (Conversely, it is clear that $G_\downarrow(d, \tau)$ is affected by tunneling even for $d \rightarrow 0$, since it depends on the difference and not the sum of the fermion numbers in the two windows.) Far downstream from the QPC, by contrast, $d/v_{\text{eff}} \gg \tau$, the two windows are widely separated in time, and the fermion numbers in each are uncorrelated. In this case, the right side of Eq. (7) is the product of independent factors, each with the form of a FCS generating function [27,28]. At large d the tunneling conductance is the same in both channels. It exactly matches the result obtained previously [19] by considering a quantum quench, with the important consequence that even the limiting distribution far from the QPC is non-thermal, although deviations from a Fermi-Dirac form are small [19]. The velocity v_{eff} can be combined with voltage V or temperature T to define two lengths:

$$l_V = \hbar v_{\text{eff}}/e|V| \quad \text{and} \quad l_T = \hbar v_{\text{eff}}/2\pi k_B T. \quad (10)$$

These set the scale for relaxation, and diverge in the non-interacting limit.

Detailed comparison of our results with the measurements of Ref. [3] requires an evaluation of the tunneling

conductance at general d . For this we compute $G_s(d, \tau)$ from Eq. (7) as summarized in Ref. [38]. The outcome is shown for the two alternative geometries in Figs. 1 and 2. The theoretical curves match the data well, except for the smallest value of d in Fig. 1. Four parameters enter the calculations: v_{eff} , T , V , and the tunneling probability p at the QPC. In addition, one further parameter is needed to set the energy scale for the data: the gate voltage-to-energy lever arm η_G (defined in Ref. [3], Supplemental Material). Of these parameters, only v_{eff} is unconstrained by independent measurements. We arrive at the fits in Figs. 1 and 2 as follows. We fix $\eta_G = 0.062$ and $p \simeq 0.5$ (consistent with the values determined in Refs. [3,4]) and obtain the bath temperature $T = 44$ mK (close to the experimental value of $T = 40$ mK) by fitting data at zero bias voltage to a thermal Fermi-Dirac distribution. We then compare numerical results at large d with data measured in the \downarrow channel at the maximum distance from the QPC ($30 \mu\text{m}$). We obtain a good fit at $\beta eV = 8$, and use this value for computations shown in Fig. 1 at all other d . Finally, we fix the interaction scale v_{eff} by fitting data at one intermediate distance ($4 \mu\text{m}$). The theoretical predictions at other distances are then fully determined. (Our specific value for p [0.545 rather than 0.500] is significant for the fit only at intermediate d and small energy.) Data for the geometry of Fig. 2 were taken using a different bias voltage from that in Fig. 1 and we calculate the theoretical curves shown in Fig. 2 using the same values of η_G , T and v_{eff} as for Fig. 1, but scaling the value of βeV by the ratio of experimental voltages. Both values of βeV are 25% lower than the experimental ones, and this discrepancy is consistent with the “missing energy” reported in Ref. [3]. Possible interpretations of this discrepancy include loss of energy at the QPC or the existence of additional, neutral edge modes [3,17,18], or invoke [19,26] the distinction in an interacting system between the measured spectral function and the electron distribution in energy. The fact that we obtain good fits to all data except that in Fig. 1 for the smallest d suggests that, if energy is lost from the edge modes in the experimental system, this probably happens very close to the QPC. The most significant outcome from this fitting process is a value for the interaction parameter: $v_{\text{eff}} = 6.5 \times 10^4 \text{ ms}^{-1}$. A roughly comparable result ($v_{\text{eff}} = 10^5 \text{ ms}^{-1}$) was obtained from the same data in Ref. [18], but we believe our fitting procedure is more precise. Both determinations are in the range measured for edge state velocities in gated samples [3].

An additional quantity of interest is the energy density $\varepsilon^s(d)$ in channel s : it characterizes with a single value the electron distribution [3]. Moreover, we can evaluate it explicitly from Eq. (7) because (see Ref. [19]) it is proportional to $\partial_\tau G^\pm(d, \tau)|_{\tau=0}$ and $G^\pm(d, \tau)$ has an expansion for small τ in cumulants of $\hat{N}_\pm(d, \tau)$. Interactions cause energy exchange between channels, so that $\varepsilon^s(d)$ evolves with d . We obtain

$$\varepsilon^\pm(d) = \varepsilon_T + \frac{\varepsilon_V}{2} \left[1 \mp \frac{l_V^2}{l_T^2} \frac{\sin^2 d/2l_V}{\sinh^2 d/2l_T} \right]. \quad (11)$$

Here $\varepsilon_V \equiv p(1-p)eV/4\pi\hbar v$ is the excess energy in a single channel due to a double-step electron distribution generated by a QPC with tunneling probability p . Equation (11), derived using a different approach in Ref. [18], shows that the energy densities interpolate between values just after the QPC of $\varepsilon^-(0) = \varepsilon_T + \varepsilon_V$ and $\varepsilon^+(0) = \varepsilon_T$, and values at large d of $\varepsilon^\pm(d) = \varepsilon_T + \varepsilon_V/2$. The relaxation length is set by the smaller of l_T and l_V . This relaxation is oscillatory, but oscillations are strongly suppressed if $eV \ll k_B T$.

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