

RELATIVELY HYPERBOLIC GROUPS WITH FIXED PERIPHERALS

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ABSTRACT

We build quasi-isometry invariants of relatively hyperbolic groups which detect the hyperbolic parts of the group; these are variations of the stable dimension constructions previously introduced by the authors.

We prove that, given any finite collection of finitely generated groups \mathcal{H} each of which either has finite stable dimension or is non-relatively hyperbolic, there exist infinitely many quasi-isometry types of one-ended groups which are hyperbolic relative to \mathcal{H} .

The groups are constructed using classical small cancellation theory over free products.

1. Introduction

In [CH17] we defined a new quasi-isometry invariant for geodesic metric spaces: the **stable asymptotic dimension**, asdim_s . The purpose of this paper is to construct related invariants, which are more naturally suited to distinguishing relatively hyperbolic groups with the same peripheral subgroups.

Gromov introduced relatively hyperbolic groups as a generalisation of hyperbolic groups, this exposition was substantially developed by Farb and Bowditch

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[Gro87, Far98, Bow12]. The class of relatively hyperbolic groups includes: hyperbolic groups, amalgamated products and HNN-extensions over finite subgroups, fully residually free (limit) groups [Dah03, Ali05]—which are key objects in solving the Tarski conjecture [Sel01, KM10], geometrically finite Kleinian groups and fundamental groups of non-geometric closed 3-manifolds with at least one hyperbolic component [Dah03].

Recall that a finitely generated group is relatively hyperbolic if it is hyperbolic relative to some finite collection of proper subgroups, and non-relatively hyperbolic (nRH) otherwise (see Definition 2.3).

Relative hyperbolicity is a quasi-isometry invariant [Dru09]. Moreover, hyperbolicity relative to non-relatively hyperbolic subgroups is quasi-isometrically rigid [BDM09]: that is, if G is hyperbolic relative to a collection of non-relatively hyperbolic groups \mathcal{G} and $q: G \rightarrow H$ is a quasi-isometry, then H is hyperbolic relative to a collection of non-relatively hyperbolic groups \mathcal{H} where each $H' \in \mathcal{H}$ is quasi-isometric to some $G' \in \mathcal{G}$, in particular the image under q of a coset of some $G' \in \mathcal{G}$ is contained in a uniform neighbourhood of some hH' with $H' \in \mathcal{H}$.

The goal of our paper is to give one approach to answering the following question which appears in [BDM09]:

Question: How may we distinguish non-quasi-isometric relatively hyperbolic groups with nRH peripheral subgroups when their peripheral subgroups are quasi-isometric?

One obvious invariant is the number of relative ends, which for an infinite group is either 2, when the group is virtually cyclic; ∞ , when the group splits as an amalgamated product or HNN extension over a finite subgroup; or 1, when no such splitting exists [Sta68, Sta71].

In the infinitely ended case, under some accessibility assumptions (for instance, finite presentability [Dun85]), the group admits a graph of groups decomposition with vertex groups which are finite or 1-ended and edge groups which are finite. These groups are quasi-isometric to free products, and quasi-isometries between free products are very well understood [PW02]. Our focus is therefore on 1-ended relatively hyperbolic groups.

Of course, two one-ended groups which are hyperbolic relative to the same nRH subgroups need not be quasi-isometric: examples of Schwartz provide

infinitely many non-quasi-isometric groups which are hyperbolic relative to \mathbb{Z}^2 [Sch95].

Another approach to the above question is highlighted by [Gro13, Theorem 6.3 and Corollary 6.5], where it is proved that the quasi-isometry type of the cusped (Bowditch) space is an invariant of relatively hyperbolic groups with nRH peripherals. Our invariants are different in character and have the advantage that they allow generalisations to other classes of peripherals. In general, one should expect the dimension of the Bowditch boundary and the boundary of the coned-off graph to be lower than the stable dimension, since stable geodesics in peripheral subgroups make no contribution to these boundaries. In the case of free products of groups with high stable dimension considered as hyperbolic relative to their free factors this is particularly noticeable.

Given a collection \mathcal{G} of finitely many finitely generated groups (possibly with repetitions) it is not clear that there even exist 1-ended groups which are hyperbolic relative to \mathcal{G} , so our first task is to provide a general construction using small cancellation theory over free products.

THEOREM 1: *Let \mathcal{G} be a collection of finitely many finitely generated groups (possibly with repetitions). There is a group G which is 1-ended and hyperbolic relative to \mathcal{G} .*

The proof of this theorem extends results in classical small cancellation theory by Champetier [Cha95]. There are other methods one may use to obtain 1-ended relatively hyperbolic groups, for instance, by taking amalgamated products over virtually malnormal cyclic subgroups when they exist [Dah03]. Moreover, by considering analogues of [DGP11, Theorem 1.5] and [Sil03, Theorem 2.16] in the case of quotients of free products, one could hope to construct relatively hyperbolic groups satisfying property FA and Kazhdan's property (T) respectively. We choose our approach for two reasons: firstly, along the way we will prove the appropriate analogue of Champetier's techniques for ensuring small cancellation groups are 1-ended; secondly, we will be able to give a presentation of the relatively hyperbolic group explicitly in terms of the groups in \mathcal{H} rather than arguing that such presentations exist with overwhelming probability or by adding sufficiently many relations.

Our main goal is to produce invariants which allow us to deduce statements of the following form: under certain hypotheses on \mathcal{G} , there are infinitely many non-quasi-isometric 1-ended groups which are hyperbolic relative to \mathcal{G} .

In [CH17, Theorem I] we proved that the **stable dimension** of a relatively hyperbolic group is finite if and only if its peripheral subgroups have finite stable dimension. Our next result in this paper shows that in this case, the stable dimension is an interesting quasi-isometry invariant of relatively hyperbolic groups.

THEOREM 2: *Let \mathcal{G} be a finite collection of finitely generated groups each of which has finite stable dimension. There is an infinite family of 1-ended groups $(G_n)_{n \in \mathbb{N}}$, with strictly increasing stable dimension, where each G_n is hyperbolic relative to \mathcal{G} .*

In particular, the groups G_n are pairwise non-quasi-isometric. Notice that without the 1-ended assumption, the result easily follows from the existence of an infinite family of hyperbolic groups of unbounded asymptotic dimension and [PW02] by considering free products.

The construction uses a family of hyperbolic groups of unbounded asymptotic dimension $\{H^n\}_{n \in \mathbb{N}}$ and an application of Theorem 1 to $\mathcal{G}^n = \mathcal{G} \cup \{H^n\}$.

Since relatively hyperbolic groups are finitely relatively presented [Osi06], there are only a countably infinite number of finitely generated groups which are hyperbolic relative to a specified collection of peripherals, so this result is optimal.

The class of finitely generated groups with finite stable dimension is very large: it includes all virtually solvable groups, all groups with finite asymptotic dimension and all groups which virtually split as a direct product of two infinite groups [CH17]. There is currently no example of a finitely generated amenable group with stable dimension > 1 .

However, this result does not exactly address the question raised in [BDM09], since there exist non-relatively hyperbolic groups with infinite stable dimension [Gru]. To deal with this, we will introduce the notion of **relative stable dimension**: the supremal asymptotic dimension of a stable subset of a space X which “avoids”, in some sense, a collection of subspaces \mathcal{Y} of X . We denote this $\text{asdim}_s(X; \mathcal{Y})$. This dimension is defined precisely in Section 3.2.

Under certain hypotheses the relative stable dimension is also a quasi-isometry invariant. To ease notation, given a group G and a collection of subgroups \mathcal{G} of G , we write $LC(\mathcal{G}) = \{gG' \mid g \in G, G' \in \mathcal{G}\}$.

THEOREM 3: *Let G be a group which is hyperbolic relative to a collection of non-relatively hyperbolic subgroups \mathcal{G} . If H is a group quasi-isometric to G , then for any collection of non-relatively hyperbolic subgroups \mathcal{H} such that H is hyperbolic relative to \mathcal{H} we have*

$$\text{asdim}_s(G; LC(\mathcal{G})) = \text{asdim}_s(H; LC(\mathcal{H})) < \infty.$$

The existence of such a collection of subgroups \mathcal{H} is guaranteed by [BDM09]. Since the stable subsets we consider avoid peripherals, they embed quasi-isometrically into a coned-off graph (cf. Definition 2.3). The fact that the relative stable dimension is finite in this case then follows from [Osi05, Theorem 17].

Again, we prove that this quasi-isometry invariant can be used to distinguish groups which are hyperbolic relative to quasi-isometric peripheral subgroups.

THEOREM 4: *Let \mathcal{H} be a finite collection of finitely generated groups which are non-relatively hyperbolic. There is an infinite family of 1-ended groups $(G_n)_{n \in \mathbb{N}}$, each of which is hyperbolic relative to \mathcal{H} , such that $\text{asdim}_s(G_n; LC(\mathcal{H}))$ is strictly increasing as a function of n .*

Combining the two constructions we obtain our most general statement.

THEOREM 5: *Let G and H be finitely generated groups which are hyperbolic relative to \mathcal{G} and \mathcal{H} respectively, such that each group in $\mathcal{G} \cup \mathcal{H}$ either has finite stable dimension, or has infinite stable dimension and is not relatively hyperbolic. If G and H are quasi-isometric, then*

$$\text{asdim}_s(G; LC(\mathcal{G}_\infty)) = \text{asdim}_s(H; LC(\mathcal{H}_\infty)) < \infty$$

where \mathcal{G}_∞ (resp. \mathcal{H}_∞) is the collection of $G' \in \mathcal{G}_\infty$ (resp. $H' \in \mathcal{H}_\infty$) with infinite stable dimension.

There is a subtlety here: in proving this theorem we obtain a quasi-isometry invariant on the class of groups which are hyperbolic relative to a collection of subgroups which are non-relatively hyperbolic or have finite stable dimension. There is no reason to suspect that this class of groups is quasi-isometrically rigid. Again we use Theorem 1 to build examples of groups which ensure that this invariant successfully distinguishes groups with the same peripherals.

THEOREM 6: *Let \mathcal{H} be a finite collection of finitely generated groups which have finite stable dimension or are non-relatively hyperbolic. There is an infinite family of 1-ended groups G_n , each of which is hyperbolic relative to \mathcal{H} , such that $\text{asdim}_s(G_n; LC(\mathcal{H}_\infty)) \rightarrow \infty$ as $n \rightarrow \infty$.*

PLAN OF THE PAPER. After establishing some preliminaries on relatively hyperbolic groups and stability, we give the construction of the relative stable dimension and prove Theorems 3 and 5 in Section 3. We dedicate Section 4 to classical small cancellation theory over free products and the proof of Theorem 1. Finally, in Section 5, we prove Theorems 2, 4 and 6.

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2. Preliminaries

Notation: we denote the A -neighbourhood of a subset Y of a metric space (X, d) by

$$[Y]_A = \{x \in X \mid d(x, Y) \leq A\}.$$

Given a set I , two functions $f, g : I \rightarrow \mathbb{R}$ and constants $K \geq 1, C \geq 0$ we write $f \approx_{K,C} g$ if the following holds for all $r \in I$:

$$(2.1) \quad \frac{1}{K}f(r) - C \leq g(r) \leq Kf(r) + C.$$

2.1. STABILITY.

Definition 2.1 (Morse geodesics): A geodesic γ in a metric space is said to be **Morse** if there exists a function $N = N(K, C)$ such that for any (K, C) -quasi-geodesic σ with endpoints on γ , we have that $\sigma \subset [\gamma]_N$. We call the function N a **Morse gauge** and say that γ is N -Morse.

Stable subsets were originally introduced by Durham–Taylor [DT15], here we use an equivalent definition from [CH17].

Definition 2.2 (Stable subsets): Let X be a geodesic metric space, let Y be a quasi-convex subset of X (every geodesic connecting a pair of points in Y stays uniformly close to Y) and let N be a Morse gauge. The subset Y is N -**stable** if, for every pair of points $x, y \in Y$ there is an N -Morse geodesic $[x, y] \subseteq X$. We say Y is **stable** if there exists some Morse gauge N such that Y is N -stable.

As defined in [CH17], the **stable dimension** of X is the supremal asymptotic dimension of a stable subset of X . This is denoted by $\text{asdim}_s(X)$. A particularly useful family of stable subsets is introduced in [CH17]: given a geodesic metric space X , some $e \in X$ and a Morse gauge N , we define $X_e^{(N)}$ to be the set of all points in X which can be connected to e by an N -Morse geodesic. These sets are stable, and universal in the sense that a quasi-convex subset Y of X is stable if and only if there is some N such that $Y \subseteq X_e^{(N)}$ [CH17, Theorem A.V].

2.2. RELATIVE HYPERBOLICITY. Let G be a finitely generated group and let \mathcal{G} be a collection of subgroups of G such that G is hyperbolic relative to \mathcal{G} . Let S be a finite symmetric generating set for G and let X be the Cayley graph of G with respect to S .

Definition 2.3: The **coned-off graph** \hat{X} of G with respect to \mathcal{G} (and S) is a graph obtained from X by attaching an additional vertex $v_{gG'}$ for every left coset of each $G' \in \mathcal{G}$ and adding an edge $(v_{gG'}, g')$ whenever $g' \in gG'$.

We say G is **hyperbolic relative to \mathcal{G}** if the following two conditions are satisfied:

- The coned-off graph of G is hyperbolic.
- (Bounded Coset Penetration Property) Let α, β be geodesics in $\hat{\Gamma}$ with the same endpoints and let $G' \in \mathcal{G}$. Then there exists a constant c such that:
 - (1) if $\alpha \cap gG' \neq \emptyset$ but $\beta \cap gG' = \emptyset$ for some $g \in G$, then the X -distance between the vertex at which α enters gH and the vertex at which α exits gG' is at most c , and

- (2) if $\alpha \cap gG' \neq \emptyset$ and $\beta \cap gG' \neq \emptyset$, and α (resp. β) first enters gG' at α_1 (resp. β_1) and last exits gG' at α_2 (resp. β_2), then $d_X(\alpha_j, \beta_j) \leq c$, for $j = 1, 2$.

We state here some important results in the theory of relatively hyperbolic groups which will be necessary in our paper.

THEOREM 2.4: [Osi06, Theorems 1.1, 1.5 and 4.20] *If G is finitely generated, then $|\mathcal{G}| < \infty$; every $G' \in \mathcal{G}$ is finitely generated; and if $G' \in \mathcal{G}$ is hyperbolic, then G is hyperbolic relative to $\mathcal{G} \setminus \{G'\}$.*

The following results all concern the geometry of X , and therefore all constants implicitly depend upon a choice of finite generating set S .

THEOREM 2.5: [DS05, Lemma 4.15] *For every $D > 0$, every $K \geq 1$ and every $C \geq 0$ there is some N such that every (K, C) -quasi-geodesic q whose endpoints are both contained in $[gG']_D$ satisfies $q \subset [gG']_N$ independent of the choice of $g \in G, G' \in \mathcal{G}$.*

This does not necessarily mean that the geodesic is N -Morse, but rather that it stays close to the coset gG' . In the particular case that $G' \in \mathcal{G}$ is hyperbolic, then G' is a stable subset of X .

It follows from [DS05, Theorem A.1] that X is **asymptotically tree-graded** with respect to $LC(\mathcal{G})$, the collection of left cosets of subgroups in \mathcal{G} . We will not give a definition of asymptotically tree-graded space here, but rather highlight the consequences of this we require. All constants appearing below will depend on the choice of Cayley graph X .

LEMMA 2.6: [DS05, Lemma 4.15] *There exists a constant $t \geq 1$ such that for any $D \geq 1$, any $A \in LC(\mathcal{G})$, and any geodesic q connecting two points in $[A]_D$, we have $q \subseteq [A]_{tD}$.*

LEMMA 2.7: [DS05, Theorem 4.1(α_2)] *There exists a constant $M(G)$ such that every geodesic q of length l with $q(0), q(l) \in [A]_{l/3}$ for some $A \in LC(\mathcal{G})$, we have $q \cap [A]_M \neq \emptyset$.*

We combine these in an obvious way.

LEMMA 2.8: *There exist constants $t \geq 1$ and $M \geq 1$ such that given any $D \geq 1$ and any geodesic q of length l connecting two points in $[A]_D$, for some*

$A \in LC(\mathcal{G})$, we have

$$q \setminus [A]_{tM} \subseteq q[0, 3tD] \cup q[l - 3tD, l].$$

Proof. By Lemma 2.6 $q \subseteq [A]_{tD}$ so any subgeodesic of length $3tD$ contains a point in $[A]_M$ by Lemma 2.7. In particular, there exist $0 \leq a, b \leq 3tD$ such that $q(a), q(l - b) \in [A]_M$. Applying Lemma 2.6 again we see that $q[a, l - b] \subseteq [A]_{tM}$. ■

LEMMA 2.9: *Given any $D, D' \geq tM$, any $L \geq 1$ and any geodesic q satisfying $\text{diam}(q \cap [A]_D) \leq L$ for some $A \in LC(\mathcal{G})$, we have*

$$\text{diam}(q \cap [A]_{D'}) \leq L + 6tD'.$$

Proof. If $D' \leq D$ then there is nothing to prove. Otherwise choose $a < b$ such that $q(a), q(b) \in [A]_{D'}$. From Lemma 2.8 we see that $q[a + 3tD', b - 3tD'] \subseteq [A]_{tM} \subseteq [A]_D$. It follows immediately that $b - a \leq L + 6tD'$. ■

Finally we require a version of Sisto's distance formula.

THEOREM 2.10: *There exist constants $D \geq tM$ and L_0 such that for all $L \geq L_0$ there exist constants $K \geq 1$, $C \geq 0$ satisfying the following:*

For any $x, y \in X$ and any geodesic γ from x to y in X

$$(2.2) \quad d_X(x, y) \approx_{K,C} d_{\hat{X}}(x, y) + \sum_{B \in LC(\mathcal{G})} \{\{\text{diam}(\gamma \cap [B]_D)\}\}_L,$$

where $\{\{x\}\}_D = x$ if $x \geq D$ and 0 otherwise.

The distance formula above is a combination of [Sis13, Theorem 0.1 and Lemma 1.15]. The purpose of Lemma 1.15 is to show that (for $D = R(1, 0)$) the values $\text{diam}(\gamma \cap [B]_D)$ and $\text{diam}(\pi_B(x) \cup \pi_B(y))$, where $\pi_B : X \rightarrow B$ is a (choice of) closest point projection, differ by at most a fixed constant. Using Lemma 2.9 we can fix $D = tM$ by increasing L_0 .

3. Stable subsets of relatively hyperbolic groups

In this section we outline three constructions of stable subsets of a relatively hyperbolic group, depending upon the peripherals, which can be used to produce quasi-isometry invariants.

The three situations we will consider are as follows: G is hyperbolic relative to \mathcal{G} and

- (1) each $G' \in \mathcal{G}$ has finite stable dimension,
- (2) each $G' \in \mathcal{G}$ is not relatively hyperbolic,
- (3) each $G' \in \mathcal{G}$ has finite stable dimension or is not relatively hyperbolic.

The first two will be treated separately and then we will combine them in order to tackle the third (most general) situation.

In each case the goal is the following. To each triple (G, \mathcal{G}, S) where G is hyperbolic relative to \mathcal{G} and S is a finite symmetric generating set of G , we will produce a family of stable subsets $\mathcal{X}(G)$ of the Cayley graph of G with respect to S with the following properties:

- $\sup \{\text{asdim}(Y) \mid Y \in \mathcal{X}(G)\} < \infty$,
- if $q: G \rightarrow H$ is a quasi-isometry, then for each $X \in \mathcal{X}(G)$ there is some $Y \in \mathcal{X}(H)$ such that $q(X) \subseteq Y$.

It follows immediately from this condition that the maximum of the asymptotic dimensions of the sets $X \in \mathcal{X}(G)$ is a quasi-isometry invariant.

We split this into three parts, corresponding to the three possible types of peripheral subgroups. When each $G' \in \mathcal{G}$ has finite stable dimension, the collection of spaces $\mathcal{X}(G)$ is exactly the “universal” collection of stable subsets constructed in [CH17, Theorem A].

3.1. STABLE APPROXIMATIONS. Given a geodesic metric space X and a point $e \in X$ we define the following collection of stable subspaces of X indexed by Morse gauges N : $X_e^{(N)}$ is the set of all points $y \in X$ such that there exists an N -Morse geodesic $[e, y]$ in X . From [CH17, Theorem A] it follows that the $X_e^{(N)}$ are hyperbolic, stable, and that if $q: X \rightarrow Y$ is a quasi-isometry, then for all N there exists some N' such that $q(X_e^{(N)})$ is a quasi-convex subset of $Y_{q(e)}^{(N')}$.

The **stable dimension** of X is defined to be the supremum of the asymptotic dimension of $X_e^{(N)}$ over all Morse gauges N . This value is a quasi-isometry invariant. Moreover, from [CH17, Theorem I] we know that if G is hyperbolic relative to \mathcal{G} and each $G' \in \mathcal{G}$ has finite stable dimension, then so does G . In particular, if $G' \in \mathcal{G}$ is hyperbolic, then it is a stable subgroup of G (a subgroup which is stable as a subset), so $\text{asdim}(G') \leq \text{asdim}_s(G)$.

In case (1) we take $\mathcal{X}(G) = \{G_e^{(N)}\}$ where N ranges over all Morse gauges.

3.2. AVOIDING NON-RELATIVELY HYPERBOLIC PERIPHERALS. Let us fix notation. We first deal with the case where G is hyperbolic relative to \mathcal{G} , and every $G' \in \mathcal{G}$ is non-relatively hyperbolic. Fix a finite symmetric generating set S

of G , define $X = \text{Cay}(G, S)$, and let \hat{X} be the coned-off graph of X . We will identify G with the vertex set VX of the Cayley graph X . Fix $D(G) = tM$, where t, M are the constants appearing in Lemmas 2.6 and 2.7 respectively.

Let N be a Morse gauge and let $L \geq 1$. Define $(G; LC(\mathcal{G}))_L^{(N)}$ to be the set of all vertices $x \in VX$ which can be connected to the identity e by an N -Morse geodesic γ_x such that, for any $g \in G$ and $G' \in \mathcal{G}$, $\text{diam}(\gamma_x \cap [gG']_{D(G)}) \leq L$. We will set

$$\mathcal{X}(G) = \left\{ (G; LC(\mathcal{G}))_L^{(N)} \mid N : [0, \infty)^2 \rightarrow [0, \infty), L \geq 1 \right\}.$$

PROPOSITION 3.1: *Let G be a finitely generated group which is hyperbolic relative to a collection of non-relatively hyperbolic groups \mathcal{G} . Let H be a finitely generated group. If $q : G \rightarrow H$ is a quasi-isometry, then for any collection \mathcal{H} of non-relatively hyperbolic subgroups of H such that H is hyperbolic relative to \mathcal{H} and every $N, L \geq 1$ there exist $N', L' \geq 1$ such that $q \left((G; LC(\mathcal{G}))_L^{(N)} \right) \subseteq (H; LC(\mathcal{H}))_{L'}^{(N')}$.*

The fact that such a collection \mathcal{H} of subgroups of H exists is proved in [BDM09].

Proof. Choose metrics d_G and d_H on G and H coming from word metrics on finite symmetric generating sets of G and H respectively. Replacing q by $q'(g) = q(e_G)^{-1}q(g)$ we may assume $q(e_G) = e_H$. Let r be a quasi-inverse of q . We fix $K \geq 1$, $C \geq 0$ such that q, r are (K, C) -quasi-isometries and $d_X(x, q \circ r(x)) \leq C$ for all $x \in G$. Now fix N, L .

It is immediate from [CH17, Theorem A] that for every N there is some N' such that $q \left(G_{e_G}^{(N)} \right) \subseteq H_{e_H}^{(N')}$, so it remains to show that coset intersections are controlled. Let $x \in G_{e_G}^{(N)}$, let γ_x be an N -Morse geodesic from e_G to x , set $y = q(x)$ and let γ_y be an N' -Morse geodesic from e_H to y .

Suppose there are $y_1, y_2 \in \gamma_y$, $h \in H$ and $H' \in \mathcal{H}$ such that $y_1, y_2 \in [hH']_{D(H)}$, then there are $y'_1, y'_2 \in q(\gamma_x)$ satisfying $y'_1, y'_2 \in [hH']_{D(H)+N'(K,C)}$ and $x_1, x_2 \in \gamma_x$ satisfying

$$x_1, x_2 \in r \left([hH']_{D(H)+N'(K,C)+C} \right) \subseteq [r(hH')]_{KD(H)+KN'(K,C)+KC+C} \quad \text{and}$$

$$(3.1) \quad d_H(y_1, y_2) \leq Kd_G(x_1, x_2) + 2N'(K, C) + C.$$

Since peripheral subgroups are not relatively hyperbolic, by [BDM09, Theorem 4.1] there exists some left coset gG' of a subgroup in \mathcal{G} such that $r(hH') \subseteq [gG']_\lambda$

for some λ which may be chosen independently of the coset hH' . Therefore

$$x_1, x_2 \in [(gG')]_{KD(H)+KN'(K,C)+KC+C+\lambda}.$$

Fix $E = KD(H) + KN'(K, C) + KC + C + \lambda$. By Lemma 2.9, a geodesic γ in G which satisfies $\text{diam}(\gamma \cap [gG']_E) > L + 2tE$ for some $g \in G$, $G' \in \mathcal{G}$ must also satisfy $\text{diam}(\gamma \cap [gG']_{D(G)}) > L$. Therefore, if $x \in (G; LC(\mathcal{G}))_L^{(N)}$ and γ_x was chosen accordingly, then $d_G(x_1, x_2) \leq L + 2tE$, and thus by (3.1)

$$d_H(y_1, y_2) \leq K(L + 2tE + 2N'(K, C) + C) =: L'.$$

It follows that $y \in (H; LC(\mathcal{H}))_{L'}^{(N')}$. ■

Definition 3.2: Let G be a finitely generated group and let \mathcal{G} be a collection of subgroups of G . The **stable dimension of G relative to $LC(\mathcal{G})$** $\text{asdim}_s(G; LC(\mathcal{G}))$ is defined to be the supremum of the asymptotic dimensions of the sets $(G; LC(\mathcal{G}))_L^{(N)}$.

We state one immediate corollary of Proposition 3.1.

COROLLARY 3.3: *Let G be a finitely generated group which is hyperbolic relative to a collection of non-relatively hyperbolic groups \mathcal{G} . If H is a finitely generated group and $g : G \rightarrow H$ is a quasi-isometry then for any collection \mathcal{H} of non-relatively hyperbolic subgroups of H such that H is hyperbolic relative to \mathcal{H} we have $\text{asdim}_s(G; LC(\mathcal{G})) = \text{asdim}_s(H; LC(\mathcal{H}))$.*

PROPOSITION 3.4: *For each Morse gauge N and each L there is a quasi-isometric embedding $(G; LC(\mathcal{G}))_L^{(N)} \rightarrow \hat{X}$.*

Proof. Notice that if $x \in (G; LC(\mathcal{G}))_L^{(N)}$ then there is a geodesic γ_x connecting e to x such that all vertices of γ_x are contained in $(G; LC(\mathcal{G}))_L^{(N)}$. Using the distance formula for relatively hyperbolic groups – with L the maximum of the original L and the L_0 in the hypothesis of Theorem 2.10 – we see that there exist $K = K(L)$, $C = C(L)$ such that the image of γ in \hat{X} is a (K, C) -quasi-geodesic with K, C independent of x . ■

COROLLARY 3.5: *Let G be a finitely generated group which is hyperbolic relative to a collection of nRH groups \mathcal{G} . Then $\text{asdim}_s(G; LC(\mathcal{G})) < \infty$.*

Proof. This follows immediately from the fact that $\text{asdim}(\hat{X}) < \infty$ [Osi05, Theorem 17] and Proposition 3.4. ■

Proof of Theorem 3. This follows immediately from Corollaries 3.3 and 3.5.

■

3.3. MIXED CONSTRUCTIONS. We now make a related construction under the assumption that G is hyperbolic relative to $\mathcal{G}_\infty \cup \mathcal{G}_F$ where each $G_\infty \in \mathcal{G}_\infty$ has infinite stable dimension but is non-relatively hyperbolic, and each $G_f \in \mathcal{G}_F$ has finite stable dimension. In this case we will take

$$\mathcal{X}(G) = \left\{ (G; LC(\mathcal{G}_\infty))_L^{(N)} \mid N : [0, \infty)^2 \rightarrow [0, \infty), L \geq 1 \right\}.$$

Here we are only avoiding the cosets of peripheral subgroups with infinite stable dimension.

PROPOSITION 3.6: *Let G, H be finitely generated groups which are hyperbolic relative to $\mathcal{G} = \mathcal{G}_\infty \cup \mathcal{G}_F$ and $\mathcal{H} = \mathcal{H}_\infty \cup \mathcal{H}_F$ respectively, where each group in $\mathcal{G}_\infty \cup \mathcal{H}_\infty$ has infinite stable dimension and is non-relatively hyperbolic, and each group in $\mathcal{G}_F \cup \mathcal{H}_F$ has finite stable dimension.*

If $q : G \rightarrow H$ is a quasi-isometry, then for every N, L there exist N', L' such that $q(G; LC(\mathcal{G}_\infty))_L^{(N)} \subseteq (H; LC(\mathcal{H}_\infty))_{L'}^{(N')}$.

Proof. If $\Gamma \in \mathcal{G}_\infty$, then $q(\Gamma)$ is contained in a uniform neighbourhood of some peripheral coset $\Lambda \in LC(\mathcal{H})$ [BDM09]. As quasi-isometries map stable subsets to stable subsets, and preserve their asymptotic dimensions, such Λ has infinite stable dimension. By assumption $\Lambda = hH'$ for some $H' \in \mathcal{H}_\infty$. Let r be a quasi-isometric inverse of q , by the same logic we deduce that $r(\Lambda)$ is contained in a uniform neighbourhood of some peripheral coset gG' , so q maps cosets of subgroups in \mathcal{G}_∞ to (uniform neighbourhoods of) cosets of subgroups in \mathcal{H}_∞ .

We may now use exactly the same argument as in the proof of Proposition 3.1. ■

COROLLARY 3.7: *Let G, H be finitely generated groups which are hyperbolic relative to $\mathcal{G} = \mathcal{G}_\infty \cup \mathcal{G}_F$ and $\mathcal{H} = \mathcal{H}_\infty \cup \mathcal{H}_F$ respectively, where each group in $\mathcal{G}_\infty \cup \mathcal{H}_\infty$ has infinite stable dimension and is non-relatively hyperbolic, and each group in $\mathcal{G}_F \cup \mathcal{H}_F$ has finite stable dimension.*

Then $\text{asdim}_s(G; LC(\mathcal{G}_\infty)) = \text{asdim}_s(H; LC(\mathcal{H}_\infty))$.

PROPOSITION 3.8: *Let G be as above. Then $\text{asdim}_s(G; LC(\mathcal{G}_\infty)) < \infty$.*

Proof. We immediately deduce the following version of the distance formula (cf. Theorem 2.10) for pairs of points in $(G; LC(\mathcal{G}_\infty))_L^{(N)}$. There exists a constant

$M_0 = M_0(L)$ such that for all $M \geq M_0$ there exist constants $K \geq 1$, $C \geq 0$ satisfying the following:

For any $x, y \in (G; LC(\mathcal{G}_\infty))_L^{(N)}$ and any geodesic γ from x to y in G

$$d_G(x, y) \approx_{K, C} d_{\hat{X}}(x, y) + \sum_{B \in LC(\mathcal{G}_F)} \{\{\text{diam}(\gamma \cap [B]_{D(G)})\}\}_M,$$

where $\{\{r\}\}_M = r$ if $r \geq M$ and 0 otherwise. (The notation $\approx_{K, C}$ is given in (2.1)).

Any subset $\mathcal{Y} \subseteq LC(\mathcal{G})$ (considered as a collection of subsets of X) satisfies the axioms of [BBF15, §4.1] so we may build a **quasi-tree of spaces** $\mathcal{C}(\mathcal{Y})$. In [MS13, §4] it is implicitly proved (and made more explicit in [Hum17, §5.3]) that the distance formula (2.2) and an application of [BBF15, §4.1] to $LC(\mathcal{G})$ yields a quasi-isometric embedding

$$G \rightarrow \hat{X} \times \mathcal{C}(LC(\mathcal{G})).$$

Applying the same argument with $\mathcal{C}(LC(\mathcal{G}_F))$ instead of $\mathcal{C}(LC(\mathcal{G}))$ yields a map

$$\psi : G \rightarrow \hat{X} \times \mathcal{C}(LC(\mathcal{G}_F)).$$

where for all M sufficiently large there exist constants K', C' such that for any $x, y \in G$,

$$(3.2) \quad d(\psi(x), \psi(y)) \approx_{K', C'} d_{\hat{X}}(x, y) + \sum_{B \in LC(\mathcal{G}_F)} \{\{\text{diam}(\gamma \cap [B]_{D(G)})\}\}_M.$$

It follows that ψ quasi-isometrically embeds each $(G; LC(\mathcal{G}_\infty))_L^{(N)}$ into $\hat{X} \times \mathcal{C}(LC(\mathcal{G}_F))$ (with quasi-isometry constants which will depend on L). Theorem 1 of [Hum17] allows us to replace $\mathcal{C}(LC(\mathcal{G}_F))$ by a tree-graded graph \mathcal{T} with respect to pieces which are either single vertices or uniformly quasi-isometric to groups in \mathcal{G}_F and deduce that the new $\psi : G \rightarrow \hat{X} \times \mathcal{T}$ still satisfies (3.2). We recall that, by definition, a connected graph Γ is tree-graded with respect to a collection of connected subgraphs \mathcal{P} if $V\Gamma = \bigcup_{P \in \mathcal{P}} VP$, the intersection of any two subgraphs in \mathcal{P} is at most a single vertex, and every simple loop in Γ is contained in some $P \in \mathcal{P}$.

Let ϕ denote the map from $(G; LC(\mathcal{G}_\infty))_L^{(N)}$ to \mathcal{T} . For every N there exists some N' such that $\phi\left((G; LC(\mathcal{G}_\infty))_L^{(N)}\right) \subseteq \mathcal{T}_{\phi(e)}^{(N')}$, this follows exactly the same argument used in the last three paragraphs of the proof of [CH17, Theorem 8.3].

It is easy to see that the intersection of a piece P with $\mathcal{T}_{\phi(e)}^{(N')}$ is contained in $P_p^{(N')}$ where p is the unique closest vertex to ϕ_e in p . By assumption, the asymptotic dimension of the $P_p^{(N')}$ is bounded independent of N' and p , so the spaces $\mathcal{T}_{\phi(e)}^{(N')}$ have uniformly bounded asymptotic dimension (this is easily deduced from the proof of [BDK04, Theorem 2] which bounds the asymptotic dimension of free products of groups, but local finiteness does not play a role). Since the coned-off graph \hat{X} has finite asymptotic dimension [BF08], and the asymptotic dimension of a product is at most the sum of the dimensions of the factors (see for instance [BD08, Theorem 32]), we have a uniform bound on the asymptotic dimension of the $(G; LC(\mathcal{G}_\infty))_L^{(N)}$ so $\text{asdim}_s(G; LC(\mathcal{G}_\infty)) < \infty$ as required. ■

Proof of Theorem 5. This follows from Corollary 3.7 and Proposition 3.8. ■

4. Small cancellation over free products

Our next task is to prove Theorem 1: given any finite collection of finitely generated groups \mathcal{G} there exists a 1-ended group G which is hyperbolic relative to \mathcal{G} . For this we will use small cancellation theory over free products.

The strategy is to generalise [Mac12, Lemmas 5.3-5.5] (of which, 5.3 and 5.4 are essentially Lemmas 4.19 and 4.20 of [Cha95] respectively) to the context of small cancellation over free products, as described in [LS01, Section V.9].

4.1. SMALL CANCELLATION GROUNDWORK. Let F be a free product of non-trivial groups G_j . Each non-trivial element of F has a unique normal form $w = y_1 y_2 \dots y_n$ where each y_i is a non-trivial element of some $G_{j(i)}$ and for all $1 \leq i \leq n$, $j(i-1) \neq j(i)$. We call n the **length** of w , and write this $|w|$.

Given two words $u, v \in F$ with normal forms $u = x_1 \dots x_k$ and $v = y_1 \dots y_l$ we say that the word $w =_F uv$ has **semi-reduced form** uv if $x_k \neq (y_1)^{-1}$ and w has **reduced form** uv if $x_1 \dots x_k y_1 \dots y_l$ is the normal form of w ; that is $w = uv$ and x_k and y_1 are elements of different G_j . In the semi-reduced but not reduced case w has normal form $x_1 \dots x_{k-1} a y_2 \dots y_l$ where $a = x_k (y_1)^{-1}$. If w does not have semi-reduced form uv , then there is some $m \geq 1$ such that $x_{k-(n-1)} = (y_n)^{-1}$ for all $1 \leq n \leq m$, but $x_{k-m} \neq (y_{m+1})^{-1}$. Note that in this case w has semi-reduced form $x_1 \dots x_{k-m} y_{m+1} \dots y_l$.

An element $w \in F$ with normal form $w = y_1 \dots y_n$ is said to be **cyclically reduced** if $|w| \leq 1$, or y_1 and y_n lie in different G_j . We say w is **weakly cyclically reduced** if $|w| \leq 1$, or $y_n \neq (y_1)^{-1}$.

A subset R of F is called **symmetrized** if every $r \in R$ is weakly cyclically reduced, and R contains all weakly cyclically reduced conjugates (by elements of F) of r and r^{-1} . A word $b \in F$ is called a **piece** if R contains distinct elements r_1, r_2 with semi-reduced forms bc_1 and bc_2 respectively. Given $\lambda > 0$, we say that a symmetrized subset $R \subset F$ satisfies the $C'_*(\lambda)$ **condition** if, given any $r \in R$ with semi-reduced form bc , where b is a piece, we have $|b| < \lambda|r|$. We will also require that $|r| > \lambda^{-1}$ to avoid pathological cases. When writing R satisfies $C'_*(\lambda)$ we implicitly assume that R is symmetrized.

LEMMA 4.1: [LS01, Lemma V.9.1] *Let R satisfy $C'_*(\lambda)$ and suppose $r \in R$ has semi-reduced form $r = b_1 \dots b_j c$ where the b_i are pieces. Let c' be of maximal length such that the following hold: c has reduced form $yc'z$ for some $y, z \in F$ and there is some cyclically reduced conjugate r' of r with reduced form $c'w$ for some $w \in F$. Then $|c'| > (1 - j\lambda)|r'|$.*

We now move to the construction of diagrams. Let $F = *_j G_j$, and let R be a symmetrized subset of F .

Definition 4.2: Let w be a product of conjugates of elements of R . A diagram D for w is a connected, contractible, finite, planar 2-complex D with a marked vertex $O \in \partial D$ which satisfies the following conditions:

- (1) Each edge of D is oriented and labelled by a non-trivial word in F .
- (2) For each closed 2-cell (hereafter called faces) $B \subset D$, reading the edge labels around the boundary of B yields a semi-reduced form of a word $r \in R$.
- (3) Reading the edge labels of ∂D counterclockwise from O yields a semi-reduced form of w .

We say D is reduced if there are never two distinct faces B_1, B_2 which intersect in at least one edge such that the labels of ∂B_1 and ∂B_2 read from this edge clockwise and counterclockwise respectively, are equal as elements of F . Given a diagram, we may remove any vertex v of degree two, replacing the two edges incident at v by a single edge labelled by the product of the labels.

We introduce the following notation for a diagram D : let $d(v)$ denote the degree of a vertex v , let $d(B)$ denote the degree of a face B (the number of other faces it shares at least one edge with), and let $e(B)$ (resp. $i(B)$) denote the number of exterior (resp. interior) edges of B . An **arc** in a diagram is an embedded subpath all of whose interior vertices have degree 2. An **interior arc** is an arc which intersects ∂D in a subset of its end vertices.

LEMMA 4.3: [LS01, Lemma V.9.2] *Let F be a free product and let $R \subset F$ satisfy $C'_*(\frac{1}{p})$ for some $p \geq 6$. Then for any $w \in \langle\langle R \rangle\rangle$ there is a reduced diagram for w . Moreover, in any reduced diagram, the label on any interior arc is a piece, and for any face $B \subset D$, $d(B) > p$ whenever $e(B) = 0$.*

The statement $d(B) > p$ requires the additional hypothesis made above that $|r| \geq p$.

LEMMA 4.4: [Str90] *Suppose D is a reduced diagram which is homeomorphic to a disc. Then*

$$(4.1) \quad 6 = 2 \sum_v (3 - d(v)) + \sum_B (6 - 2e(B) - i(B)).$$

Lemmas 4.1, 4.3 and 4.4 are precisely the ingredients needed to deduce that Strebel's classification of geodesic bigons and triangles extends to the $C'_*(\frac{1}{6})$ setting. We require only the following:

LEMMA 4.5: (cf. [Mac12, Lemma 3.12]) *Suppose D is a reduced diagram whose boundary is labelled by, in order: a geodesic $[p, u]$, part of the boundary of a face $B \subset D$, and a geodesic $[v, p]$, and $[p, u] \cap [v, p] = \{p\}$. Then either B is the only face in D ; or there is a unique face $B' \neq B$ whose boundary contains p , $e(B) = e(B') = i(B) = i(B') = 1$ and for all other faces $B'' \subset D$, $e(B'') = i(B'') = 2$.*

Proof. We begin by removing all degree 2 vertices from D . By Lemma 4.3, every internal face has interior degree at least 7, and every face with one exterior edge (other than possibly B and a face B' which originally contained p in its boundary) has at least 4 interior edges. To see this, notice that the label of the boundary of this face B'' can be written in semi-reduced form as $r = b_1 \dots b_j c$ where $j = i(B'')$, the b_i are pieces and $|c| \leq \frac{1}{2}|r|$. It follows immediately from Lemma 4.1 that $j \geq 4$.

Applying this to (4.1), the only possible positive contribution to the right hand side is from the faces B and B' , and a contribution of at least 6 is possible if and only if both these faces have interior degree 1. There are no internal faces (as these contribute negatively), and every other face either satisfies $e(B'') = i(B'') = 2$ or $e(B'') = 1$ and $i(B'') = 4$. The faces neighbouring B and B' must both satisfy the first of these, and continuing, we see that every face in D besides B and B' satisfies $e(B'') = i(B'') = 2$, as required. ■

One can apply the above result directly to diagrams D whose boundary is a union of two geodesics γ from p to v and γ' from v to p which intersect only at p and v , by replacing γ' by part of the boundary of the face B containing v and a geodesic $[u, p]$.

LEMMA 4.6: (cf. [Mac12, Lemma 4.6]) *If D is a reduced diagram containing a face B and a geodesic γ , then $\gamma \cap B$ is connected.*

Proof. Suppose not. Let γ' be a positive length subgeodesic of γ which intersects ∂B only at its end vertices, let D_0 be the subdiagram of D enclosed by B and γ' , and remove all degree two vertices from D_0 . Every internal face B' of D_0 satisfies $i(B) \geq 7$ (by Lemma 4.3), and every face B' with an external edge, except possibly B , satisfies $i(B') \geq 4$ (by Lemma 4.1). Applying (4.1), we see that $6 \leq 6 - e(B) - 2i(B) \leq 3$, which is a contradiction. ■

The techniques given above are sufficient to deduce a version of Greendlinger's lemma in the free product setting (cf. [LS01, Theorem 9.3]): we summarise the consequences of this required in the theorem below.

THEOREM 4.7: *Let F be a free product of non-trivial groups $\{G_j \mid j \in I\}$ and let $R \subset F$ satisfy $C'_*(\lambda)$ for some $\lambda \leq \frac{1}{6}$. Set $N = \langle\langle R \rangle\rangle$ and $G = F/N$.*

- *For each j , the natural map $G_j \rightarrow G$ is a monomorphism.*
- *[Osi06, Theorem 1.5] If $N = \langle\langle R' \rangle\rangle$ for some finite subset $R' \subset F$, then the relative presentation $G = \langle G_j \mid R' \rangle$ has linear relative Dehn function, so G is hyperbolic relative to $\{G_j \mid j \in I\}$.*

4.2. THE WRONG CAYLEY GRAPH IS 1-ENDED. Let F be a free product of groups $\{G_j \mid j \in I\}$ and let R satisfy $C'_*(\lambda)$. Set $G = F/\langle\langle R \rangle\rangle$. In this section we will prove that $X = \text{Cay}(G, Y)$ is 1-ended, where $Y = \bigcup_j G_j \setminus \{1\}$.

Our primary tools (as in the construction of [Mac12, Section 5]) are two lemmas of Champetier [Cha95, Lemmas 4.19-20]. We start with a useful lemma.

LEMMA 4.8: *Suppose $\lambda \leq \frac{1}{6}$. Let $r \in R$ have semi-reduced form $y_1 \dots y_k$. The cycle C_r in X starting at 1 with edges e_1, \dots, e_k labelled by y_1, \dots, y_k respectively is embedded in X .*

Proof. Suppose not, then there is some proper subword $w = y_i \dots y_j$ $1 \leq i < j \leq k$ which is trivial in G , choose one of minimal length. Let D be a reduced diagram with boundary word w . Removing vertices of degree 2 we see that every face in D (except possibly for one, B' , whose label is a weakly reduced conjugate of r) satisfies $e(B) = 1$ and $i(B) \geq 6$ or $i(B) \geq 7$. This contradicts (4.1) unless B' is the only face of D , but this is also not possible as the label of the boundary of D is not weakly conjugate to r . ■

LEMMA 4.9: (cf. [Cha95, Lemma 4.19] and [Mac12, Lemma 5.3]) *If $\lambda \leq \frac{1}{6}$, then for every vertex $a \in X$, the set of $y \in Y$ such that $d_X(ay, 1) \leq d_X(a, 1)$ is contained in a union of at most two G_j .*

Proof. Let γ_1 be a geodesic from a to 1 and let the vertex $b \in \gamma_1$ satisfy $d_X(a, b) = 1$. Suppose the label of the edge $[a, b]$ is in G_j . Suppose there is some vertex $c \in X$ such that $d_X(1, c) \leq d_X(1, a)$, $d_X(c, a) = 1$ and the edge $[c, a]$ is labelled by an element of G_k with $k \neq j$. Let γ_2 be any geodesic from c to 1. Notice that $a, b \notin \gamma_2$. Let D be a reduced diagram whose boundary consists of γ_1, γ_2 and the edge $[c, a]$, let B be the face in D whose boundary contains the edge $[c, a]$, and let r be the (cyclically reduced) label of the face Π starting at a and ending with the edge $[c, a]$. Applying Lemma 4.5, we see that the boundary of Π consists of subgeodesics of γ_1 and γ_2 , the edge $[c, a]$ and at most one piece. It follows that $|\gamma_1 \cap \partial\Pi| > |r| - \frac{1}{2}|r| - \frac{1}{6}|r| - 1 > \frac{1}{6}|r|$.

Now suppose for a contradiction that there is some $c' \in VX$ such that $d_X(1, c') \leq d_X(1, a)$, $d_X(c', a) = 1$ and the edge $[c', a]$ is labelled by an element of $G_{k'}$ with $k' \notin \{j, k\}$. Applying the same reasoning as above we obtain a relation $r' \in R$ and a word $u \in F$ of length $\geq \frac{1}{6} \min\{|r|, |r'|\}$ such that r, r' have reduced forms uv and uv' respectively. By the small cancellation hypothesis, $r = r'$ but this is impossible as their normal forms end with words in different factors, G_k and $G_{k'}$ respectively. ■

LEMMA 4.10: (cf. [Cha95, Lemma 4.20] and [Mac12, Lemma 5.4]) *If $\lambda \leq \frac{1}{8}$, then for every vertex $u' \in X$ the set of all $u \in X$ such that $d_X(1, u) = d_X(1, u') + d_X(u', u) = d_X(1, u') + 3$, and so that some geodesic $\gamma_u = [u, 1]$ starts with a subword of a relator $r \in R$ of length greater than $|r|/4 + 3$, is contained in a single coset of some G_j .*

Proof. Fix such a point u and such a geodesic γ_u . We first explain why we may assume that $u' \in \gamma_u$. If this is not the case then the two geodesics γ_u and $\gamma_{u'} = [u, u'] \cup [u', 1]$ define a geodesic bigon which splits at a vertex of $[u, u']$ and hence, in a diagram D whose boundary is this bigon there is a face Π with boundary label r' containing the point closest to u' at which the bigon splits and containing subgeodesics of both γ_u and $\gamma_{u'}$ of length at least $\frac{1}{4}|r'|$. By the small cancellation hypothesis $r = r'$, so the bigon splits at u and $\gamma_{u'}$ also starts with a subword of a relator $r' \in R$ of length greater than $|r'|/4 + 3$. We may now replace γ_u by $\gamma_{u'}$ and assume in what follows that $u' \in \gamma_u$.

Now suppose we have two points u, v with corresponding geodesics γ_u, γ_v and relators r, r' . We may assume $u' \in \gamma_u \cap \gamma_v$. Define $\gamma'_u = \gamma_u \cap r$ and $\gamma'_v = \gamma_v \cap r'$, and let q be the maximal length subgeodesic of $\gamma'_u \cap \gamma'_v$ containing u' .

If $|q| \geq \frac{r}{8}$ then $r = r'$ by the small cancellation hypothesis. Otherwise, let w be the vertex in q closest to 1 and let r'' be the relator containing w in a reduced diagram whose boundary is the bigon consisting of subgeodesics of γ_u and γ_v from w to 1. Now $|r'' \cap \gamma_u|, |r'' \cap \gamma_v| \geq \frac{3|r''|}{8}$, so

$$|r \cap r''| \geq \min \left\{ \frac{3|r''|}{8}, \frac{|r|}{4} \right\}$$

and $r = r''$ by the small cancellation hypothesis. Similarly, $r' = r''$ and hence $r = r'$. Since the words r, r' are equal in F and all relators have weakly reduced boundary labels, it follows that u, v are in a common coset of some G_j - if this is not the case then it is clear that r, r' have labels which are different in F . ■

Let us now give a general condition to ensure that X is 1-ended. We say a subset $R \subset F$ is **7-full** if there exist symmetric generating sets S_j of G_j satisfying the following: for every word w of length 7 in F whose normal form $y_1 \dots y_7$ is such that each y_i is contained in $S = \bigcup_j S_j$, there is some cyclically reduced $r \in R$ and some $x \in F$ such that r has reduced form wx . The rest of this section is devoted to the proof of the proposition below.

PROPOSITION 4.11: *Let $|I| \geq 5$, and let F be a free product of non-trivial groups $\{G_j \mid j \in I\}$. If R is a 7-full, $C'_*(\frac{1}{8})$ subset of F then X is 1-ended.*

Proof. Recall that $X = \text{Cay}(F/\langle\langle R \rangle\rangle, \bigcup_{j \in I} G_j)$. By Lemma 4.9, X has infinite diameter. To prove X is 1-ended, we will inductively (on l) construct paths connecting any pair of points a, b in X with $d_X(1, a) = d_X(1, b) = l$, which contain no vertex in the closed ball of radius $l - 1$ centred at 1.

Initial step: Choose vertices $a, b \in X$ so that $[1, a]$ and $[1, b]$ are two distinct edges in X . If $a, b \in G_j$ for some j then there is a single edge connecting them and we are done. Otherwise, there exist distinct $j, k \in I$ such that $a \in G_j$, $b \in G_k$. Replace $[1, a]$ (resp. $[1, b]$) by a pair of edges $[1, s_a]$ and $[s_a, a]$ (resp. $[1, s_b]$ and $[s_b, b]$) where $s_a \in S_j$ ($s_b \in S_k$). Let $r \in R$ be cyclically reduced with reduced form $s_b x s_a^{-1}$ for some $x \in F$. Since r embeds into X , the path from a to b consisting of, in order, the edge $[a, s_a]$, the path of length $|r| - 2$ in r from s_a to s_b and the edge $[s_b, b]$ does not contain the vertex 1.

Inductive step: Let $a, b \in X$ satisfy $d_X(1, a) = d_X(1, b) = l$ let a', b' be points on geodesics $[1, a]$ and $[1, b]$ respectively so that $d_X(1, a') = d_X(1, b') = l - 1$, and, via the inductive hypothesis, let P' be a path in X from a' to b' such that $d_X(p, 1) \geq l - 1$ for every vertex $p \in P$. Define P'' to be the path obtained from P' by adding the edge $[a, a']$ at the start and $[b', b]$ at the end, and define P to be the path obtained from P'' by replacing each edge labelled by a non-trivial word in some G_j by a path with the same end vertices where every edge is labelled by an element of S_j . Since cosets of the G_j have diameter 1 in X , every vertex v in P satisfies $d_X(1, v) \geq l - 2$. Enumerate the vertices on P , in order, by $a = v_0, \dots, v_m = b$. Let $t_i \in S$ be the label of the edge $[v_{i-1}, v_i]$.

For each $0 \leq i \leq m$ choose $v'_i \in X$ so that

- $d_X(1, v'_i) = d_X(1, v_i) + d_X(v_i, v'_i) = d_X(1, v_i) + 3$,
- the label of $[v'_i, v_i]$ has normal form $s_i^0 s_i^1 s_i^2$, each $s_i^k \in S$, and s_i^2 is not in the same S_j as the labels of the edges $[v_{i-1}, v_i]$ or $[v_i, v_{i+1}]$,
- no geodesic $[v_i, 1]$ starts with a subword of some relator $r \in R$ of length greater than $\frac{1}{4}|r| + 3$.

This is possible as there are at most 4 choices of j from which we cannot pick s_i^2 , at most two coming from the labels of the edges $[v_{i-1}, v_i]$ or $[v_i, v_{i+1}]$, and at most 2 which do not extend a geodesic $[1, v_i]$ by Lemma 4.9. For each s_i^1 there are at most two choices of j such that choosing $s_i^1 \in S_j$ need not extend the geodesic, and at most one more which does, but can lead to a situation in

which the third point above is not satisfied by Lemma 4.10. Choosing s_i^1 in one of the remaining S_j , and s_i^2 extending the geodesic defines a suitable v'_i . For each i define $h_i = s_i^0 s_i^1 s_i^2$.

For each $0 \leq i \leq m$ let $r_i \in R$ be cyclically reduced with reduced form $t_i h_i x_i h_{i-1}^{-1}$ for some $x_i \in F$. We claim that the path from a to b formed by, in order, a geodesic $[a, v'_0]$, for each $1 \leq i \leq m$ the path P_i of length $|x_i^{-1}|$ with label x_i^{-1} from v'_{i-1} to v'_i , and a geodesic $[v'_m, b]$, contains no vertex in the closed ball of radius $l - 1$ centred at 1 in X . The claim is verified by the following lemma.

LEMMA 4.12: (cf. [Mac12, Lemma 5.5]) *Let p be a vertex on one of the paths P_i described above. Then $d_X(1, p) \geq l$.*

Proof. For each i fix a geodesic $[v'_i, 1]$ containing v_i . Suppose there is a geodesic $[1, p]$ which does not contain either of v_{i-1}, v_i . We may assume the first edge of this geodesic is not in P_i , and we may also assume (switching the roles of a and b in the above construction if necessary), that the path from p to v'_i along P_i has length $\leq \frac{1}{2} |r_i|$.

Fix x to be the vertex closest to p which is contained in $[1, p] \cap [v'_i, 1]$. Consider the diagram D whose boundary is composed of the subgeodesic $[x, p]$ of $[1, p]$, the subpath of P_i from p to v'_i and the subgeodesic $[v'_i, x]$ of $[v'_i, 1]$. Note that v_i is contained in $[v'_i, x]$.

We claim that D cannot contain the face B whose boundary contains v'_i and has the label $h_i x_i h_{i-1}^{-1} t_i^{-1}$ (or its inverse) when reading counterclockwise from v'_i . If this were the case, the boundary of D would satisfy the hypotheses of Lemma 4.5, as it consists of the geodesics $[x, p]$ (contained in $[1, p]$), $[v'_i, x]$ and part of the boundary of $B \subset D$. Now $[x, p] \cap \partial B = \{p\}$ by Lemma 4.6 and the fact that the edge of $[x, p]$ containing p is not in the boundary of the face B . By assumption, the path in ∂B from p to v'_i - which is contained in the boundary of D - has length $\leq \frac{1}{2} |B|$, so $B \cap [v_i, 1]$ is a geodesic containing v_i of length at least $|B| - \frac{|B|}{2} - \frac{|B|}{8} \geq \frac{|F|}{4}$. Hence $\partial B \cap [v'_i, 1]$ is a geodesic of length at least $\frac{1}{4} |r| + 3$ containing v'_i contradicting the choice of v'_i .

Now let us add the face B to D and call the resulting diagram D' . This is possible as the intersection of B with D (using Lemma 4.6) is a path consisting of a geodesic in $[x, p]$ containing p , the subpath of P_i connecting p to v'_i and a subgeodesic of $[v'_i, 1]$ containing $[v'_i, v_i]$. Applying Lemma 4.5 to D' we see that

the boundary of the unique face B' neighbouring B (there must be one, since $v_i \notin [1, p]$) consists of two geodesics and at most 2 pieces. The piece which is the intersection with B contains $[v'_i, v_i]$. Since at most half the edges in the boundary of B are contained in any geodesic, it follows that the boundary of B' intersects $[v'_i, 1]$ in a geodesic of length at least $\frac{r}{4} + 3$, again contradicting the choice of v'_i . In the last step we are crucially using the fact that, by construction, the two edges in $[v'_i, 1]$ next to the vertex v_i have labels from different G_j .

Finally, $[1, p] \cap \partial B$ is connected and contains p and v_i , so $[v_i, p] \cap \partial B$ is a geodesic of length at least 3, hence

$$d_X(1, p) \geq d_X(1, v_i) + 3 \geq (l - 2) + 3 \geq l. \quad \blacksquare$$

This completes the proof of Proposition 4.11. \blacksquare

4.3. THE RIGHT CAYLEY GRAPH IS 1-ENDED. Let F be a free product of finitely many groups G_j , $j = 1, \dots, n \geq 4$ and assume each G_j admits a finite symmetric generating set S_j . Let R satisfy $C'_*(\frac{1}{8})$ and set $G = F / \langle\langle R \rangle\rangle$. In this section we will prove that $X = \text{Cay}(G, S)$ is 1-ended where $S = \bigcup_j S_j$. To clarify notation, let us now denote the Cayley graph $\text{Cay}(G, Y)$ considered in §4.2 by \overline{X} . Note that since $S \subseteq Y$, the obvious map $X \rightarrow \overline{X}$ is 1-Lipschitz, so $d_{\overline{X}}(a, b) \leq d_X(a, b)$ for all vertices a, b .

LEMMA 4.13: *If \overline{X} is 1-ended, then X is 1-ended.*

Proof. Since \overline{X} has infinite diameter, X is either 1-ended or there is some $l \geq 2$ such that removing the closed ball of radius l centred at 1 from X leaves at least 2 connected components C_1, C_2 of infinite diameter. We claim that there are vertices x_i in C_i satisfying $d_X(x_i, 1) \geq d_{\overline{X}}(x_i, 1) \geq 2l$ – to see this take any points in C_i at distance $4l$ from 1, and extend the geodesic from 1 to this point in \overline{X} $2l$ times using appropriate elements of S and Lemma 4.9. By Proposition 4.11 there is a path P from x_1 to x_2 in \overline{X} containing no vertices in the closed ball of radius $2l - 1$ in \overline{X} . Expanding the label of each edge using edges with labels in S we obtain a path P' from x_1 to x_2 in \overline{X} , all of whose edges are labelled by elements of S and containing no vertices in the ball of radius $2l - 2 \geq l$. This path determines a path from x_1 to x_2 in X containing no vertices in the closed ball of radius l , contradicting the assumption that no such path from C_1 to C_2 exists. \blacksquare

4.4. CONSTRUCTING PRESENTATIONS. To complete the proof of Theorem 1 it suffices to prove that, given any finite collection of finitely generated groups $\{G_j \mid j \in I\}$ with $|I| \geq 5$ admitting finite symmetric generating sets $S_j \not\cong 1$, there is a finite 7-full subset R of $F = *_{j \in I} G_j$ such that the collection of weakly cyclically reduced conjugates of elements of R and their inverses satisfies $C'_*(\frac{1}{8})$. Define $S = \bigsqcup S_j$.

Fix $s_1 \in S_1$, $s_2 \in S_2$ and $s_3 \in S_3$. Enumerate all words in F (w^1, w^2, \dots, w^n) which can be written in reduced form as $w^i = s_1^{i_1} \dots s_{k_i}^{i_{k_i}}$ where all $s_j^i \in S$, $s_1^i, s_{k_i}^i \in S_1$, and $7 \leq |w^i| = k_i \leq 9$.

Define $r = \prod_{l=1}^n (s_2 s_3)^{10l} w^l$, and set R to be the set of weakly cyclically reduced conjugates of $r^{\pm 1}$. Notice that since $|I| \geq 5$ and the G_j are non-trivial, $n \geq 5 \cdot 4^6$.

LEMMA 4.14: R is a 7-full, $C'_*(\frac{1}{8})$ subset of F .

Proof. For every word $w \in F$ of length 7 with normal form $y_1 \dots y_7$ and each $y_i \in S$, there is some w^l whose normal form contains $y_1 \dots y_7$ as a (not necessarily proper) subword. Therefore, there is a cyclically reduced conjugate of r with normal form $y_1 \dots y_7 x$ for some $x \in F$. Hence R is 7-full.

Suppose t, t' are distinct weakly cyclically reduced conjugates of $r^{\pm 1}$, which have weakly reduced forms uv and uv' respectively, and let u have normal form $y_1 \dots y_p$. Consider the sequence $a = (a_i)_{i=1}^p \in \{0, 1\}^p$ defined by $a_i = 1$ if $y_i \in S_1$ and 0 otherwise.

If $p = |u| > 60n + 20$, then there is some $k \in \{1, \dots, n\}$ such that a contains a subsequence of the form 1 followed by $20k$ 0's followed by a sequence of length between 7 and 9 beginning and ending with a 1 followed by $20(k \pm 1)$ 0's followed by a 1. (A worst-case scenario is when a normal form of u begins $(s_2 s_3)^{10(n-2)} w^{n-2} (s_2 s_3)^{10(n-1)} w^{n-1} (s_2 s_3)^{10n} w^n \dots$, and $|w^{n-2}| = |w^{n-1}| = |w^n| = 9$).

In the case where the second sequence of 0's is longer we see that the normal form of u must contain the subword $(s_2 s_3)^{10k} w^k (s_2 s_3)^{10k+1}$ starting at some y_k . Together with y_1 this is sufficient to determine t and therefore $t = t'$ which is a contradiction. Hence any piece has length $\leq 60n + 20$. Now r is cyclically reduced, so

$$|r| \geq \sum_{i=1}^n (20n + 7) = 10n(n + 1) + 7n = 10n^2 + 18n.$$

If R is not $C'_*(\frac{1}{8})$ then $8(60n + 20) \geq 10n^2 + 18n$, so $n \leq 49$, but this is a contradiction as we have already established that $n \geq 5 \cdot 4^6$. ■

Proof of Theorem 1. Let $\mathcal{G} = \{G_j \mid j \in I\}$ be a finite collection of finitely generated groups. Define \mathcal{G}' to be \mathcal{G} plus $\max\{0, 5 - |I|\}$ copies of \mathbb{Z} . Fix finite symmetric generating sets S_j for the groups in \mathcal{G}' and define the relation $r \in F = *_{j \in I} G_j$ as in §4.4. Now by Lemma 4.14, Proposition 4.11 and Lemma 4.13 the Cayley graph of $G = F / \langle\langle r \rangle\rangle$ with respect to the generating set $S = \bigsqcup_i S_i$ is 1-ended. By Theorem 4.7 G is hyperbolic relative to \mathcal{G}' so by Theorem 2.4, G is hyperbolic relative to \mathcal{G} . ■

5. Main Theorems

Here we combine the above results to prove Theorems 2, 4 and 6.

The final key ingredient is a family of hyperbolic groups with unbounded asymptotic dimension. All hyperbolic groups have finite asymptotic dimension [Roe05], however it is in general very difficult to construct a hyperbolic group of high dimension, examples come from cocompact arithmetic lattices in $SO^+(1, n)$, see [BH78].

Let \mathcal{G} be a non-empty finite collection of finitely generated groups, possibly with repetitions. We apply the construction in Theorem 1 to the collections $\mathcal{G}_n = \mathcal{G} \cup \{H^n\}$ where the H^n are hyperbolic groups of unbounded asymptotic dimension. From this we obtain a collection of 1-ended groups G^n which are hyperbolic relative to \mathcal{G}_n . By Theorem 2.4 each G^n is hyperbolic relative to \mathcal{G} .

Proof of Theorem 2. If each $G' \in \mathcal{G}$ has finite stable dimension, then the G^n have unbounded but finite stable dimension. It follows from [CH17, Corollary B] that there are infinitely many non-quasi isometric 1-ended groups which are hyperbolic relative to \mathcal{G} completing the proof of Theorem 2. ■

Proof of Theorem 4. If every $G' \in \mathcal{G}$ is non-relatively hyperbolic then considering each G^n as a group hyperbolic relative to \mathcal{G} , we see that for each n there exists some N, L such that $H^n \subset (G^n; LC(\mathcal{G}))_L^{(N)}$, so $\text{asdim}_s(G; LC(\mathcal{G})) \geq n$. By Corollary 3.5, $\text{asdim}_s(G; LC(\mathcal{G})) < \infty$. Therefore, Corollary 3.3 implies that the collection of groups G^n exhibit infinitely many different quasi-isometry types. ■

Proof of Theorem 6. Set $\mathcal{G} \cup \{H^n\} = \mathcal{G}_F \cup \mathcal{G}_\infty$, where each $G' \in \mathcal{G}_F$ has finite stable dimension and each $G' \in \mathcal{G}_\infty$ has infinite stable dimension and is non-relatively hyperbolic.

Since H^n has finite stable dimension, it belongs to \mathcal{G}_F and therefore we have $\text{asdim}_s(G; LC(\mathcal{G})) \geq n$ as in the proof of Theorem 4. By Proposition 3.8, $\text{asdim}_s(G; LC(\mathcal{G})_\infty) < \infty$. Therefore, Corollary 3.7 implies that the collection of groups G^n exhibit infinitely many different quasi-isometry types. ■

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