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A symmetry of the descent algebra of a finite Coxeter group

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Abstract

The descent algebra $\mathscr{D}_W$ of a finite Coxeter group W, discovered by Solomon in 1976, is a subalgebra of the group algebra of W. Due to Solomon, it is intimately linked to the representation theory of W, by means of a homomorphism of algebras θ mapping $\mathscr{D}_W$ into the algebra of class functions of W. For W of type A, Jöllenbeck and Reutenauer derived the identity $\theta(X)(Y) = \theta(Y)(X)$ for all $X, Y \in \mathscr{D}_W$, where class functions of W have been extended to the group algebra of W linearly. They conjectured that this symmetry property of $\mathscr{D}_W$ holds for arbitrary finite Coxeter groups W. This conjecture—actually a combinatorial refinement—is proven here.

As a consequence, several properties of the characters of W afforded by the primitive idempotents of $\mathscr{D}_W$ may be derived at once, including a symmetry of the corresponding character table, and a combinatorial description of their intertwining numbers with the descent characters of W. This recovers and extends results of Gessel-Reutenauer and Scharf-Thibon on the symmetric group, and of Poirier on the hyperoctahedral group.

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1. Introduction

Let (W, S) be a finite Coxeter system and denote the length of w as a word in the elements of S by $\ell(w)$, for all $w \in W$. For each $I \subseteq S$, let W_I be the parabolic subgroup of W generated by I, then

$$W^I = \{ x \in W \mid \ell(x) < \ell(xs) \text{ for all } s \in I \}$$

is a cross-section of the left cosets of W_I in W, consisting of the unique representatives of minimal length (for general reference, see [5,17]).

Due to a remarkable result of Solomon [25], the linear span $\mathcal{D}_W$ of the elements

$$X^I = \sum_{x \in W^I} x \quad (I \subseteq S)$$

is a subalgebra of the integral group algebra $\mathbb{Z}W$, the *descent algebra* of W. Furthermore, if φ_I denotes the character of W induced by the principal character of W_I, then the linear map θ defined by

$$\theta(X^I) = \varphi_I$$

for all $I \subseteq S$ is a homomorphism of rings mapping the descent algebra into the ring of class functions of W.

Several surprising properties and generalizations of $\mathcal{D}_W$ have been achieved through the last 15 years, linking the descent algebra to such different fields as the theory of the free Lie algebra, the probabilistic theory of card shuffling and more general random walks, or the structure and representation theory of the underlying group W (see, for example, [1–4,6,11,12,14,19,23]).

The main goal of this paper is to further expose the interplay between the descent algebra and the representation theory of W, based on the following symmetry property of $\mathcal{D}_W$.

Main Theorem. *For all* $X, Y \in \mathcal{D}_W$,

$$\theta(X)(Y) = \theta(Y)(X),$$

where each class function of W has been extended to $\mathbb{Z}W$ *linearly.*

This result was conjectured by Jöllenbeck and Reutenauer, who provided a proof in case W is of type A, by means of an algebraic approach involving higher Lie representations of the symmetric group [18].

Note that, more explicitly, the Main Theorem implies the combinatorial identity

$$\sum_{y \in X^J} \varphi_I(y) = \sum_{x \in X^I} \varphi_J(x)$$

for all $I, J \subseteq S$ when applied to basis elements $X = X^I$ and $Y = X^J$ (and vice versa, by linearity).

The conjugacy classes of W may be lumped together to sets of λ-elements in W if, roughly speaking, they cannot be separated by parabolic subgroups [3]. Here λ is a Coxeter class of W, and consists of subsets of S generating conjugate parabolic subgroups. Any complete set of primitive idempotents of $\mathcal{D}_W$ may be indexed by the Coxeter classes λ of W in a natural way. An explicit construction of such idempotents was given by Bergeron et al. [3], generalizing and, to some extent, also clarifying the ideas developed by Garsia and Reutenauer for the symmetric group [12].

In Section 3, we show that the symmetry of θ implies a number of surprising results on the characters χ_λ of W afforded by the primitive idempotents of $\mathcal{D}_W$, which are referred to as *Solomon characters* of W here. To give an instance, recall that $\mathrm{Des}(w) = \{ s \in S \mid \ell(ws) < \ell(w) \}$ is the descent set of w, for all $w \in W$, and consider the basis of $\mathcal{D}_W$ consisting of the sums of descent classes

$$Y^J = \sum_{\mathrm{Des}(w)=J} w = \sum_{I \subseteq J} (-1)^{\# J - \# I} X^I \quad (J \subseteq S)$$

and the corresponding descent characters

$$\delta^J = \theta(Y^J)$$

of W [24,25]. The Main Theorem allows to give the following description of the intertwining numbers of these characters with the Solomon characters.

Theorem 1.1. *The number of λ-elements in W with descent set J is $(\chi_\lambda, \delta^J)_W$, for all Coxeter classes λ of W and all $J \subseteq S$.*

This result extends those of Gessel and Reutenauer on the symmetric group [15], and of Poirier on the hyperoctahedral group [20]. Further applications of the Main Theorem concern the values of the Solomon characters (see Theorem 3.3), and the dimension of the fixed space of a parabolic subgroup W_I in the corresponding W-modules (see Theorem 3.5). The latter includes, in particular, the result on the dimension of these modules of Bergeron et al. [3, Theorem 7.15].

The proof of the Main Theorem is closely adapted to its combinatorial nature, thus also sheds new light on the symmetry of $\mathcal{D}_W$ in the symmetric group case studied in [18]. In fact, our objective is the combinatorial *refinement* stated below (which was conjectured in [16] only recently). Some illustrations of our terminology in case W is of type A, can be found at the end of Section 2.

Let $I \subseteq S$, then each $w \in W$ has a unique factorization

$$w = w^I \, w_I$$

into parabolic components $w^I \in W^I$ and $w_I \in W_I$. If, additionally, $J \subseteq S$, then the elements b of $W^{I,J} := (W^I)^{-1} \cap W^J$ represent the (W_I, W_J)-double cosets in W [25, Lemma 1]. The sets

$$\mathcal{W}(I, J, b) := \{ w \in W \mid w^J b^{-1} = (wb^{-1})_I \}$$

play a key role in our approach. More precisely, here is what may be viewed as the (combinatorial) core of our efforts.

Theorem 1.2. *$\#\mathcal{W}(I, J, b) = \#\mathcal{W}(J, I, b^{-1})$, for all $I, J \subseteq S$ and $b \in W^{I,J}$.*

This result implies the Main Theorem (see Section 2). Its proof, given in the remaining Sections 4 to 6, builds on subtle combinatorics related to the sets W^I. In particular, in crucial Section 4, conjugates of minimal coset representatives $x \in W^I$ are studied, based on the notion of parabolic conjugator. This relates to work of Fleischmann [9] on the subject and could be of independent interest.

2. Symmetry of the descent algebra

This is a short recall of arguments already presented in [16], showing that Theorem 1.2 implies the Main Theorem.

For the proof of the symmetry property, it suffices to consider basis elements $X = X^J$ and $Y = X^I$, by linearity. In this case, the permutation character $\varphi_J = \theta(X^J)$ of W counts fixed points in W/W_J. In other words, the Main Theorem says that

$$\begin{aligned}
\theta(X^J)(X^I) &= \sum_{x \in W^I} \varphi_J(x) \\
&= \sum_{x \in W^I} \#\{ y \in W^J \mid xyW_J = yW_J \} \\
&= \#\{ (x, y) \in W^I \times W^J \mid xyW_J = yW_J \}
\end{aligned}$$

is invariant with respect to exchange of I and J. The latter set decomposes into the subsets

$$\mathcal{F}(I, J, b) := \{ (x, y) \in W^I \times (W^J \cap W_I b W_J) \mid xyW_J = yW_J \} \quad (b \in W^{I,J}),$$

according to the double coset $W_I b W_J$ containing the second component y. Write $bJb^{-1} = \{ bsb^{-1} \mid s \in J \}$ for all $b \in W$, $J \subseteq S$, then there is the following variant of a result due to Howlett [7, Proposition 2.7.5], which is useful here and later on.

Proposition 2.1. *For all $I, J \subseteq S$ and $b \in W^{I,J}$,*

$$W^J \cap W_I b W_J = W^J \cap W_I b = (W^{I \cap bJb^{-1}} \cap W_I)b.$$

(see [16, Proposition 1]) The (easier) first identity allows to link the sets $\mathcal{F}(I, J, b)$ to Theorem 1.2.

Corollary 2.2. $\#\mathcal{F}(I, J, b) = \#\mathcal{W}(I, J, b)$, *for all $I, J \subseteq S$ and $b \in W^{I,J}$.*

Proof. A bijection from $\mathcal{F}(I, J, b)$ onto $\mathcal{W}(I, J, b)$ is afforded by the product mapping $(x, y) \mapsto xy$. For, if $(x, y) \in \mathcal{F}(I, J, b)$ and $w = xy$, then $wW_J = yW_J$ implies $w^J = y$, since $y \in W^J$. Furthermore, $x \in W^I$ and $y \in W^J \cap W_I b W_J = W^J \cap W_I b$, by Proposition 2.1, hence $(wb^{-1})_I = (x(yb^{-1}))_I = yb^{-1} = w^J b^{-1}$. It follows that $w \in \mathcal{W}(I, J, b)$.

To define the inverse mapping, let $w \in \mathcal{W}(I, J, b)$ and put $y := w^J$ and $x := (wb^{-1})^I$. Then $x \in W^I$ and $y = (w^J b^{-1})b = (wb^{-1})_I b \in W^J \cap W_I b \subseteq W^J \cap W_I b W_J$. The observation that $w = (wb^{-1})^I (wb^{-1})_I \, b = xy$ and $xyW_J = wW_J = w^J W_J = yW_J$, completes the proof. $\square$

As a consequence of Corollary 2.2, another way of stating Theorem 1.2 is $\#\mathcal{F}(I, J, b) = \#\mathcal{F}(J, I, b^{-1})$, which implies

$$\theta(X^J)(X^I) = \sum_{b \in W^{I,J}} \#\mathcal{F}(I, J, b) = \sum_{b \in W^{I,J}} \#\mathcal{F}(J, I, b^{-1}) = \theta(X^I)(X^J)$$

and therefore the Main Theorem.

Consider the case where W is the symmetric group S_n on $[n] := \{1, \ldots, n\}$ and $S = \{s_1, \ldots, s_{n-1}\}$ consists of the transpositions s_i in S_n swapping i and $i+1$, for all $i \leqslant n-1$. When viewing elements $w \in W$ as bijections $[n] \to [n]$, the terminology introduced above has the following illustrative description.

A finite sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ of positive integers such that $\lambda_1 + \cdots + \lambda_l = n$ is a composition of n. Associated to each such λ, there is the set partition $P^\lambda = (P_1^\lambda, \ldots, P_l^\lambda)$ of $[n]$ which consists of the successive blocks in $[n]$ of order λ_1, λ_2, and so on. For example, $\lambda = (3, 2, 3)$ is a composition of $n = 8$ and $P^\lambda = (\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\})$.

The stabilizer of P^λ in W is the parabolic subgroup

$$W_I = W_\lambda = \{ w \in W \mid w(P_i^\lambda) = P_i^\lambda \text{ for all } i \leqslant l \}.$$

Here, I consists of all transpositions $s_i \in S$ such that i is not a partial sum of λ. For instance, if $\lambda = (3, 2, 3)$ as above, then $I = \{s_1, s_2, s_4, s_6, s_7\}$. Subsets of S as indices may thus be replaced by compositions of n. The set of minimal coset representatives of W_λ in W is

$$W^\lambda = \{ x \in W \mid x|_{P_i^\lambda} \text{ is non-decreasing for all } i \leqslant l \}.$$

In particular, the upper parabolic component w^λ of $w \in W$ is the unique permutation in W mapping P_i^λ onto $w(P_i^\lambda)$ in a non-decreasing fashion, for all $i \leqslant l$. For example, $(814\,75\,263)^{(3,2,3)} = 148\,57\,236$, where we used the notation $w = w(1)w(2)\cdots w(n)$ for $w \in W$.

Let $\mu = (\mu_1, \dots, \mu_k)$ be a second composition of n and $M = (m_{ij})$ be a $k \times l$ matrix of non-negative integers such that $m_{i1} + \cdots + m_{il} = \mu_i$ for all $i \leqslant k$ and $m_{1j} + \cdots + m_{kj} = \lambda_j$ for all $j \leqslant l$. The double coset representatives $b \in W^{\lambda,\mu} = (W^\lambda)^{-1} \cap W^\mu$ are in 1–1 correspondence to such matrices M. For example, if $\mu = (3, 5)$, $\lambda = (3, 2, 3)$ and

$$M = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix},$$

we may replace, columnwise from left-to-right, the entries m_{ij} of M by the consecutive subsets of $[n]$ of order m_{ij} to obtain

$$B = \begin{pmatrix} \{1,2\} & \emptyset & \{6\} \\ \{3\} & \{4,5\} & \{7,8\} \end{pmatrix}.$$

Arranging non-decreasingly the numbers occurring in each set and juxtaposing the resulting segments rowwise, from top to bottom, we get the corresponding double coset representative $b = b_M = 12\,6\,3\,45\,78 \in W^{\lambda,\mu}$. The matrix corresponding to $b_M^{-1} \in W^{\mu,\lambda}$ is M^t, the transpose of M. Using the simplified notation $\mathcal{F}(M)$ and $\mathcal{W}(M)$ for $\mathcal{F}(I, J, b_M)$ and $\mathcal{W}(I, J, b_M)$, respectively, Theorem 1.2 thus says $\mathcal{W}(M) = \mathcal{W}(M^t)$, or $\mathcal{F}(M) = \mathcal{F}(M^t)$, by Corollary 2.2.

We shall look at the set $\mathcal{F}(M)$ in more detail. An element $w \in W$ is contained in the double coset $W_\lambda b_M W_\mu$ if and only if $\#(w(P_i^\mu) \cap P_j^\lambda) = m_{ij}$ for all $i \leqslant k$, $j \leqslant l$. In particular, each $y \in W^\mu \cap W_\lambda b_M W_\mu$ can be visualized by an array $Y = (Y_{ij})$ of subsets of $[n]$ such that $\# Y_{ij} = m_{ij}$ for all i, j and P_j^λ is the disjoint union of $Y_{1j}, \dots, Y_{kj}$ for all $j \leqslant l$: simply assign to y the array defined by $Y_{ij} = y(P_i^\mu) \cap P_j^\lambda$ for all i, j. For example, with λ, μ and M as above, $y = 137\,24568 \in W^\mu \cap W_\lambda b_M W_\mu$ corresponds to

$$Y = \begin{pmatrix} \{1,3\} & \emptyset & \{7\} \\ \{2\} & \{4,5\} & \{6,8\} \end{pmatrix}.$$

Bearing in mind the definition of $\mathcal{F}(M)$, the elements $x \in W$ need to be determined such that $x \in W^\lambda$ and $xyW_\mu = yW_\mu$. This translates into the two more illustrative conditions that x is order-preserving on each "column" $P_j^\lambda = Y_{1j} \cup \cdots \cup Y_{kj}$ of Y, $j \leqslant l$, and stabilizes each "row" $y(P_i^\mu) = Y_{i1} \cup \cdots \cup Y_{il}$ of Y, $i \leqslant k$. For example, passing from Y to

$$\tilde{Y} = \begin{pmatrix} \{1,7\} & \emptyset & \{3\} \\ \{5\} & \{6,8\} & \{2,4\} \end{pmatrix}$$

(with the obvious understanding that $1 \mapsto 1$, $3 \mapsto 7$ in the upper left component, and so on), both conditions above are fulfilled, and we obtain $x = 157\,68\,234 \in W^{(3,2,3)}$ such that $xyW_\mu = yW_\mu$.

The corresponding element $w = xy$ of $\mathcal{W}(M)$ is obtained from $\tilde{Y}$ in the same way as b_M was obtained from B, yielding $w = 17\,3\,5\,68\,24$ in the example.

3. Applications to Solomon characters

Throughout this section, we work over a field K of characteristic 0. The group ring of W over K is denoted by KW, and $\mathcal{D}_W$ denotes the descent algebra in KW, that is, the K-linear span of $\{ X^I \mid I \subseteq S \}$ in KW.

A number of interesting consequences of the Main Theorem is obtained when idempotents in $\mathcal{D}_W$ are considered. The line of reasoning mimics the one presented in [18] for the symmetric group, and is based on the following result due to Frobenius (which actually holds for arbitrary finite groups G instead of W). As in the Main Theorem, class functions α of W are linearly extended to KW.

Lemma 3.1. *Let $E \in KW$ be idempotent and denote by χ_E the character of W afforded by the left ideal $(KW)E$ of KW. Then, for any class function α of W,*

$$(\alpha, \chi_E)_W = \alpha(E),$$

where $(\cdot,\cdot)_W$ is the usual scalar product on the ring of class functions of W.

([10, Section 5], see also [8, Section 9, Exercise 16]) In the case where $\alpha = \theta(X)$ for some $X \in \mathcal{D}_W$, and $E = Y \in \mathcal{D}_W$, this allows to give a representation theoretical interpretation of the term $\theta(X)(Y)$ occurring in the Main Theorem.

Some basic facts on idempotents in $\mathcal{D}_W$ are needed first. Denote the Jacobson radical of $\mathcal{D}_W$ by $\operatorname{rad}\mathcal{D}_W$. For $I, J \subseteq S$, write $I \sim J$ if $I = uJu^{-1}$ for some $u \in W$, then

$$\operatorname{rad}\mathcal{D}_W = \ker\theta = \langle \{ X^I - X^J \mid I \sim J \} \rangle_K \tag{3.1}$$

[25, Theorem 3]. The equivalence classes of subsets of S with respect to $\sim$ are the *Coxeter classes* of W. Denote the set of all Coxeter classes by $\Lambda(W)$, and the conjugacy class in W containing w by $C(w)$, for all $w \in W$. If $\lambda \in \Lambda(W)$, then $w \in W$ is a *λ-element* if there exists an element $I \in \lambda$ such that $C(w) \cap W_I \neq \emptyset$ and $C(w) \cap W_K = \emptyset$ for all $K \subseteq I$, $K \neq I$. The set of all λ-elements C_λ is a union of conjugacy classes of W. By [3, Section 6], the characteristic functions

$$\mathrm{char}_\lambda : W \longrightarrow K,\ w \longmapsto \begin{cases} 1 & \text{if } w \in C_\lambda \\ 0 & \text{if } w \notin C_\lambda \end{cases} \qquad (\lambda \in \Lambda(W))$$

form a linear basis and a complete set of mutually orthogonal idempotents of the image of $\mathcal{D}_W$ under θ. Consequently, for all $X \in \mathcal{D}_W$,

$$\theta(X) = \sum_{\lambda \in \Lambda(W)} \theta(X)(C_\lambda)\, \mathrm{char}_\lambda, \tag{3.2}$$

where $\theta(X)(C_\lambda)$ denotes the (unique) value of $\theta(X)$ on each λ-element in W.

Considering inverse images of the char_λ's under θ and lifting these modulo $\mathrm{rad}\,\mathcal{D}_W = \ker\theta$, a complete set $\{E_\lambda \mid \lambda \in \Lambda(W)\}$ of mutually orthogonal primitive idempotents of $\mathcal{D}_W$ is obtained such that $\theta(E_\lambda) = \mathrm{char}_\lambda$ for all λ (for an explicit construction, see [3]). The characters of W afforded by the W-left module $(KW)E_\lambda$ are called *Solomon characters* of W, and denoted by χ_λ. Not surprisingly, these characters do not depend on the actual choice of the E_λ's, as the following, slightly more general result shows.

Proposition 3.2. *Let $E \in \mathcal{D}_W$ be an idempotent, then the character afforded by the W-left module $(KW)E$ is given by*

$$\chi_E = \sum_\lambda \chi_\lambda,$$

where the sum ranges over all $\lambda \in \Lambda(W)$ such that $\theta(E)(C_\lambda) \neq 0$.

Proof. Let Λ_E be the set of all $\lambda \in \Lambda(W)$ such that $\theta(E)(C_\lambda) \neq 0$. Since the char_λ's are mutually orthogonal idempotents and $\theta(E)^2 = \theta(E^2) = \theta(E)$, (3.2) implies

$$\theta(E) = \sum_{\lambda \in \Lambda_E} \mathrm{char}_\lambda.$$

In particular, there exists an element $r \in \ker\theta$ such that $E = \sum_{\lambda \in \Lambda_E} E_\lambda + r$, and $F := \sum_{\lambda \in \Lambda_E} E_\lambda = E - r$ is an idempotent with $\chi_F = \sum_{\lambda \in \Lambda_E} \chi_\lambda$. Thus $\chi_E = \chi_F$ remains. But r is nilpotent, by (3.1), hence $1 - r$ and $1 + r$ are invertible in KW. It follows that right multiplication with E yields a surjective homomorphism of W-left modules from $(KW)F$ onto $(KW)FE = (KW)(1-r)E = (KW)E$, while right multiplication with F yields a surjective homomorphism from $(KW)E$ onto $(KW)EF = (KW)(1+r)F = (KW)F$. This implies $(KW)E \cong (KW)F$ and $\chi_E = \chi_F$. $\square$

If W is of type A, then the modules $(KW)E_\lambda$ are isomorphic to certain modules of W arising from the Poincaré–Birkhoff–Witt basis of the free associative algebra, the *higher Lie modules* of the symmetric group (see [21, Chapter 8]).

Now, as a first consequence of the Main Theorem, there is the following symmetry of the characters χ_λ.

Theorem 3.3. *For all* $\lambda, \mu \in \Lambda(W)$,

$$\sum_{w \in C_\mu} \chi_\lambda(w) = \sum_{v \in C_\lambda} \chi_\mu(v).$$

Proof. Each $w \in W$ is conjugate to its inverse [13, Corollary 3.2.14]. Thus, by definition of the scalar product and Lemma 3.1,

$$\frac{1}{\#W} \sum_{w \in C_\mu} \chi_\lambda(w) = (\chi_\lambda, \mathrm{char}_\mu)_W = \mathrm{char}_\mu(E_\lambda) = \theta(E_\mu)(E_\lambda).$$

The latter term is symmetric in μ and λ, by the Main Theorem. $\square$

For the symmetric group, this yields the symmetry of the higher Lie characters discovered by Scharf and Thibon [22, Remark 3.11] (see also [26, Example 7.89g]), since the C_λ's are simply the conjugacy classes of W in this case. This symmetry was indeed the starting point of [18], and was proven directly by means of an elegant and amazingly short symmetric-function argument, while the symmetry of θ was derived as a consequence. Here, for arbitrary W, the line of reasoning has been reversed. However, Theorem 3.3 states that $\theta(E_\mu)(E_\lambda) = \theta(E_\lambda)(E_\mu)$ for all $\lambda, \mu \in \Lambda(W)$, which in return implies the Main Theorem as in the symmetric group case. For, the idempotents E_λ span a complement of $\mathrm{rad}\,\mathcal{D}_W = \ker\,\theta$ in $\mathcal{D}_W$, and each class function of W vanishes on the (nilpotent) elements of $\mathrm{rad}\,\mathcal{D}_W$.

Notice that, in general, the Solomon characters of W are *not* constant on the sets C_λ (and thus not contained in the image of θ). Here is an illustrating example.

Example 3.4. Let (W, S) be the Coxeter system of type $I_2(p)$, that is, W is the dihedral group of order $2p$ with generating set $S = \{r, s\}$ and defining relations

$$r^2 = s^2 = (rs)^p = e.$$

The descent algebra $\mathcal{D}_W$ is then generated by the elements

$$\begin{aligned} X^S &= e, \\ X^{\{r\}} &= e + s + rs + srs + rsrs + \cdots \quad (p \text{ summands}), \\ X^{\{s\}} &= e + r + sr + rsr + srsr + \cdots \quad (p \text{ summands}), \\ X^{\emptyset} &= \textstyle\sum_{w \in W} w. \end{aligned}$$

We restrict ourselves to the (more interesting) case where p is even, then

$$\Lambda(W) = \{\emptyset, \{s\}, \{r\}, S\}$$

with corresponding sets of λ-elements $C_\emptyset = C(e)$, $C_{\{r\}} = C(r)$, $C_{\{s\}} = C(s)$, and $C_S = \bigcup_{k=1}^{p/2} C((sr)^k)$. A complete set of primitive idempotents of $\mathcal{D}_W$ is

$$\begin{aligned} E_\emptyset &= \tfrac{1}{2p} X^{\emptyset}, \\ E_{\{r\}} &= \tfrac{1}{2}(X^{\{r\}} - \tfrac{1}{2} X^{\emptyset}), \\ E_{\{s\}} &= \tfrac{1}{2}(X^{\{s\}} - \tfrac{1}{2} X^{\emptyset}), \\ E_S &= e - \tfrac{1}{2} X^{\{r\}} - \tfrac{1}{2} X^{\{s\}} + \tfrac{p-1}{2p} X^{\emptyset} \end{aligned}$$

(see [3, (5.1)]). The values of the corresponding Solomon characters χ_λ may be computed by means of the formula

$$\chi_\lambda(w) = \frac{\# W}{\# C(w)} \sum_{I \subseteq S} \#(W^I \cap C(w))\, E_{\lambda, I}$$

for all $w \in W$, where $E_\lambda = \sum_{I \subseteq S} E_{\lambda,I} X^I$. This follows from Lemma 3.1, applied to the characteristic function of $C(w)$ and $E = E_\lambda$. It turns out that the Solomon characters slightly depend on the parity of $m := p/2$. In the table below, their values are displayed in case m is even, while those for odd m are added in brackets (if different).

	$C(e)$	$C(r)$	$C(s)$	$C(sr)$	$\cdots$	$C((sr)^{m-1})$	$C((sr)^m)$
$\chi_\emptyset$	1	1	1	1	$\cdots$	1	1
$\chi_{\{s\}}$	$\frac{p}{2}$	0 (1)	0 (-1)	0	$\cdots$	0	$-\frac{p}{2}$
$\chi_{\{r\}}$	$\frac{p}{2}$	0 (-1)	0 (1)	0	$\cdots$	0	$-\frac{p}{2}$
χ_S	$p-1$	-1	-1	-1	$\cdots$	-1	$p-1$

In particular, $\chi_{\{r\}}$ and $\chi_{\{s\}}$ are not constant on C_S. However,

$$\sum_{w \in C_S} \chi_{\{r\}}(w) = \chi_{\{r\}}((sr)^m) = -p/2 = p/2\,\chi_S(r) = \sum_{w \in C_{\{r\}}} \chi_S(w),$$

verifying Theorem 3.3 in this particular case. As another immediate consequence, observe that the Solomon characters are linearly dependent if $p/2$ is even.

We proceed with a second application of the Main Theorem.

Theorem 3.5. *Let* $\lambda \in \Lambda(W)$ *and* $I \subseteq S$, *then* $(\chi_\lambda, \varphi_I)_W = \#\,(C_\lambda \cap W^I)$.

Proof. By Lemma 3.1 and the Main Theorem,

$$(\chi_\lambda, \varphi_I)_W \;=\; \varphi_I(E_\lambda) = \theta(X^I)(E_\lambda) = \theta(E_\lambda)(X^I) = \mathrm{char}_\lambda(X^I) = \#\,(C_\lambda \cap W^I). \qquad \square$$

Notice that $(\chi_\lambda, \varphi_I)_W = \dim\, e_I(KW)E_\lambda$, where $e_I = \frac{1}{\#W_I}\sum_{w \in W_I} w$ is the trivial idempotent in KW_I affording the permutation character φ_I of W. In other words, $(\chi_\lambda, \varphi_I)_W$ is the dimension of the fixed space of W_I in $(KW)E_\lambda$. In particular, for $I = \emptyset$, it follows that the dimension of $(KW)E_\lambda$ is equal to $\#\,C_\lambda$. This latter result is due to Bergeron et al. [3, Theorem 7.15].

The descent characters $\delta^J = \theta(Y^J)$, $J \subseteq S$, of W mentioned in the introduction decompose the regular character of W and have been studied in [24] already. The result on their intertwining numbers with the Solomon characters is proved along the same lines as Theorem 3.5 above:

Proof of Theorem 1.1. By Lemma 3.1 and the Main Theorem,

$$(\chi_\lambda, \delta^J)_W \;=\; \theta(Y^J)(E_\lambda) = \theta(E_\lambda)(Y^J) = \mathrm{char}_\lambda(Y^J) = \#\{\, w \in C_\lambda \mid \mathrm{Des}(w) = J\,\}.$$

This, of course, also follows from Theorem 3.5, by triangularity. $\square$

The ultimate significance of the Solomon characters for the representation theory of W is yet to be discovered, and we propose a more detailed study of their properties.

4. Parabolic conjugators

The goal of this section is to construct, for certain elements $w \in W$ (which will be called I-positive), an element $c \in W_I$ such that $cwc^{-1} \in W^I$. Up to some extent, this recovers results of [9]. However, the study of the conjugator c is emphasized here.

In what follows, W is considered as a reflection group acting faithfully on an Euclidean $\mathbb{R}$-vector space $(V, \langle\cdot,\cdot\rangle)$, which has a linear basis $\Delta = \{\alpha_s \mid s \in S\}$ in bijective correspondence with S such that $\langle \alpha_s, \alpha_s \rangle = 1$ for all $s \in S$. The action of W is given by

$$sv = v - 2\langle v, \alpha_s \rangle \alpha_s$$

for all $s \in S$, $v \in V$. Thus s acts as the reflection in the hyperplane orthogonal to α_s. We recall some basic terminology on reflection groups (see [17, Chapter 1]). The set $\Phi = \{ w\alpha \mid \alpha \in \Delta,\ w \in W \}$ is the root system of (W, S), and the elements of Δ are the simple roots. Those roots contained in $\Phi^+ := \langle \Delta \rangle_{\mathbb{R}_{\geqslant 0}} \cap \Phi$ are positive, while those contained in $\Phi^- = -\Phi^+$ are negative. Each root $\alpha \in \Phi$ is either positive or negative, that is, Φ is the disjoint union of Φ^+ and Φ^-.

Definition 4.1. If $A \subseteq B \subseteq V$, then A is *closed* (respectively, *completely closed*) in B if $\alpha + \beta \in A$ for all $\alpha, \beta \in A$ such that $\alpha + \beta \in B$ (respectively, if $a\alpha + b\beta \in A$ for all $\alpha, \beta \in A$ and $a, b \in \mathbb{R}_{>0}$ such that $a\alpha + b\beta \in B$). Furthermore, A is *(completely) biclosed* in B if both A and $B \backslash A$ are (completely) closed in B.

For example, for all $w \in W$, the set

$$N(w) := \Phi^+ \cap w^{-1}\Phi^-$$

of inversions of w is readily seen to be (completely) biclosed in Φ^+. If W is a Weyl group, with a root system in the terminology of [5] (i.e., a crystallographic root system with normalisation conditions in the terminology of [17]), then the mapping $w \mapsto N(w)$ is a bijection from W onto the set of all biclosed subsets in Φ^+ [5, p. 225, Exercise 16]. However, some caution is advised when generalizing this result to *arbitrary* finite Coxeter groups, with an arbitrary root system in the terminology of [17, Chapter 1] (such as those of type H_3 and H_4, in particular). In fact, this extension requires the restriction to *completely* biclosed subsets in Φ^+ (although, by mistake, stated and proven without this restriction in [9, Proposition 3.2]):

Proposition 4.2. *The map $w \mapsto N(w)$ is a bijection from W onto the set of all completely biclosed subsets in Φ^+.*

We would like to thank Cédric Bonnafé and Pierre Baumann for valuable discussions on the topic. A proof of the surjectivity part follows, for the reader's convenience; concerning the proof of the injectivity part, see [9, Proposition 3.2].

Proof of the surjectivity part of Proposition 4.2. As a first step, observe that

($*$) If $A \subseteq \Phi^+$ is completely closed in Φ^+ and $\Delta \subseteq A$, then $A = \Phi^+$.

To prove ($*$), let $\alpha \in \Phi^+$, then there exist $w \in W$ and $\beta \in \Delta$ such that $\alpha = w\beta$, by [17, Corollary 1.5]. Proceed by induction on $k := \ell(w)$ to show that $\alpha \in A$. If $k = 0$, then $\alpha = \beta \in \Delta \subseteq A$. Suppose $\alpha \notin \Delta$, then $k > 0$. Choose $s_1, \ldots, s_k \in S$ such that

$w = s_1 \cdots s_k$ and set $\alpha_1 := \alpha_{s_1}$, then

$$\gamma := s_1\alpha \in s_1(\Phi^+ \backslash \Delta) \subseteq s_1(\Phi^+ \backslash \{\alpha_1\}) \subseteq \Phi^+ \backslash \{\alpha_1\},$$

by [17, Proposition 1.4], hence $\gamma = s_2 \cdots s_k \beta \in A$, by induction. But $\alpha = s_1\gamma = \gamma - 2\langle \gamma, \alpha_1 \rangle \alpha_1 \in \Phi^+$ and $\gamma, \alpha_1 \in \Phi^+$ imply $\langle \gamma, \alpha_1 \rangle < 0$. It follows that $\alpha \in A$ as desired, since A is completely closed in Φ^+, and the proof of $(*)$ is complete.

Now let $A \subseteq \Phi^+$ be completely biclosed in Φ^+. We show by induction on $n := \#A$ that there exists an element $w \in W$ such that $N(w) = A$.

If $n = 0$, then $A = \emptyset = N(e)$. Let $n > 0$, then $A \neq \emptyset$ implies $\Phi^+ \neq \Phi^+ \backslash A$. It thus follows from $(*)$, applied to $\Phi^+ \backslash A$, that $\Delta \not\subseteq \Phi^+ \backslash A$. Choose $s \in S$ such that $\alpha := \alpha_s \in \Delta \cap A$, then $\tilde{A} := s(A \setminus \{\alpha\}) = s(A) \setminus \{-\alpha\} \subseteq \Phi^+$ and $\Phi^+ \backslash \tilde{A} = \Phi^+ \backslash s(A) = s(\Phi^+ \setminus A) \cup \{\alpha\} \subseteq \Phi^+$, by [17, Proposition 1.4] again.

Both $\tilde{A}$ and $\Phi^+ \backslash \tilde{A}$ are completely closed in Φ^+: for $\tilde{A}$, this is readily seen; to show that $\Phi^+ \backslash \tilde{A}$ is also completely closed in Φ^+, it suffices to consider $v \in \Phi^+ \setminus A$ and $a, b \in \mathbb{R}_{>0}$ such that $asv + b\alpha \in \Phi^+$. Observe that $\pi = av - b\alpha \in \Phi^+$. If π was contained in A, then $v = a^{-1}(\pi + b\alpha) \in \Phi^+$ would imply $v \in A$, since $\pi, \alpha \in A$, a contradiction. Therefore, $\pi \in \Phi^+ \setminus A$ and $a(sv) + b\alpha = s\pi \in \Phi^+ \backslash \tilde{A}$.

Consequently, by induction, there exists an element $u \in W$ such that $N(u) = \tilde{A}$, since $\#\tilde{A} < n$. Observe that $\alpha \notin \tilde{A}$, that is, $u\alpha \in \Phi^+$. Setting $w = us$, it follows that $A = sN(u) \cup \{\alpha\} = N(us) = N(w)$. $\square$

For the remainder of this section, $I \subseteq S$ is fixed. The set $\Delta_I = \{ \alpha_s \mid s \in I \}$ then consists of simple roots for the Coxeter system (W_I, I), and $\Phi_I = \langle \Delta_I \rangle_{\mathbb{R}} \cap \Phi$ is the corresponding root system. The distinguished coset representatives of W_I in W may be described as

$$W^I = \{ x \in W \mid x\Delta_I \subseteq \Phi^+ \} = \{ x \in W \mid x{\Phi_I}^+ \subseteq \Phi^+ \}.$$

Furthermore, if $w \in W$, then

$$w \in W_I \text{ if and only if } N(w) \subseteq \Phi_I^+. \tag{4.1}$$

Let $\mathbb{N}$ denote the set of positive integers, and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $w \in W$, define

$$N_I(w) := \{ \alpha \in \Phi^+ \mid \exists\, k \in \mathbb{N}\colon\ \alpha,\ w\alpha,\ \ldots,\ w^{k-1}\alpha \in \Phi_I,\ w^k\alpha \in \Phi^- \backslash \Phi_I \}.$$

These sets will be of major importance for all what follows. We start with some routine calculations, which allow to apply Proposition 4.2 in order to define an element $c \in W$ by the property $N(c) = N_I(w)$.

Proposition 4.3. *$N_I(w)$ is completely biclosed in Φ^+, for each $w \in W$.*

Proof. This is essentially a consequence of the following general observation:

($*$) If $a, b \in \mathbb{R}_{>0}$, $\gamma \in \Phi_I \cup \Phi^- \setminus \Phi_I$ and $\delta \in \Phi^- \setminus \Phi_I$ such that $a\gamma + b\delta \in \Phi$, then $a\gamma + b\delta \in \Phi^- \setminus \Phi_I$.

To see this, let $\gamma = \sum_{v \in \Delta} g_v v$ and $\delta = \sum_{v \in \Delta} d_v v$, then $d_\psi < 0$ for some $\psi \in \Delta \backslash \Delta_I$. But the coefficient of v in $a\gamma + b\delta$ is $ag_v + bd_v$, for each $v \in \Delta$. As $\gamma \in \Phi_I \cup \Phi^- \setminus \Phi_I$ implies $g_\psi \leqslant 0$, it follows that $ag_\psi + bd_\psi < 0$, hence $a\gamma + b\delta \in \Phi^- \setminus \Phi_I$.

Suppose now $a, b \in \mathbb{R}_{>0}$ and $\alpha, \beta \in N_I(w)$ such that $a\alpha + b\beta \in \Phi^+$. Choose $k, l \in \mathbb{N}$ minimal such that $w^k\alpha \notin \Phi_I$, respectively $w^l\beta \notin \Phi_I$. We may assume that $k \leqslant l$. Then $w^m(a\alpha + b\beta) = aw^m\alpha + bw^m\beta \in \Phi_I$ for all $m < k$, since $\Phi_I = \langle \Delta_I \rangle_{\mathbb{R}} \cap \Phi$ is completely closed in Φ. Furthermore, $w^k\beta \in \Phi_I \cup \Phi^- \setminus \Phi_I$ and $w^k\alpha \in \Phi^- \setminus \Phi_I$ imply $w^k(a\alpha + b\beta) \in \Phi^- \setminus \Phi_I$, by ($*$). Thus, $a\alpha + b\beta \in N_I(w)$ as asserted.

Let $a, b \in \mathbb{R}_{>0}$ and $\alpha, \beta \in \Phi^+ \setminus N_I(w)$ such that $a\alpha + b\beta \in \Phi^+$. If $w^m\alpha, w^m\beta \in \Phi_I$ for all $m \in \mathbb{N}_0$, then also $w^m(a\alpha + b\beta) \in \Phi_I$ for all $m \in \mathbb{N}_0$. In this case, $a\alpha + b\beta \notin N_I(w)$ follows directly.

Assume that $w^k\beta \notin \Phi_I$, or $w^k\alpha \notin \Phi_I$ for some $k \in \mathbb{N}_0$, and choose k minimal with this property. Exchanging α and β, if necessary, we may assume that $w^k\beta \notin \Phi_I$. Then $w^m(a\alpha + b\beta) = aw^m\alpha + bw^m\beta \in \Phi_I$ for all $m < k$, by minimality of k. Furthermore, $\alpha, \beta \notin N_I(w)$ implies that $w^k\alpha \in \Phi_I \cup \Phi^+ \setminus \Phi_I$ and $w^k\beta \in \Phi^+ \setminus \Phi_I$. Applying ($*$) to $\gamma = -w^k\alpha$ and $\delta = -w^k\beta$, yields $w^k(a\alpha + b\beta) \in \Phi^+ \setminus \Phi_I$, hence $a\alpha + b\beta \notin N_I(w)$. □

Corollary 4.4. *Let $w \in W$, then there exists a unique element $c \in W$ such that $N(c) = N_I(w)$, called the I-*conjugator *of w. Furthermore, $c \in W_I$.*

Proof. Existence and uniqueness of c follow from Propositions 4.2 and 4.3, while $c \in W_I$ follows from (4.1) and $N_I(w) \subseteq {\Phi_I}^+$. □

We now introduce the notion of *I*-positivity of $w \in W$, which allows to derive $cwc^{-1} \in W^I$ for the *I*-conjugator c of w.

Definition 4.5. An element $w \in W$ is *I-positive* if $w\Phi_I^+ \cap \Phi_I \subseteq \Phi^+$.

Certainly, each $x \in W^I$ is *I*-positive. More important examples of *I*-positive elements are these.

Example 4.6. (i) If $x, y \in W^I$, then $x^{-1}y$ is *I*-positive.
(ii) If $b \in W^{I,J}$ and $a \in W_J \cap W^{J \cap b^{-1}Ib}$, then bab^{-1} is *I*-positive.

Proof. Let $\alpha \in {\Phi_I}^+$ such that $\beta := x^{-1}y\alpha \in \Phi_I$. Then $x\beta = y\alpha \in y{\Phi_I}^+ \subseteq \Phi^+$, since $y \in W^I$, hence also $\beta \in {\Phi_I}^+$, since $\beta \in \Phi_I$ and $x \in W^I$. This proves the first part.

The second part follows from the first, as $x = b^{-1} \in W^I$, by the definition of $W^{I,J}$, and $y = ab^{-1} \in (W_J \cap W^{J \cap b^{-1}Ib})b^{-1} = W^I \cap W_J b^{-1} \subseteq W^I$, by Proposition 2.1. □

Some of the impact of $N(w)$ on $N_I(w)$ for I-positive w is illustrated by the next observation, which will be useful in the sections that follow (see Propositions 5.4 and 6.3).

Proposition 4.7. *Let $w \in W$ be I-positive and $A \subseteq \Phi$ such that $wA \cap \Phi_I^+ \subseteq A$, then $A \cap N(w) = \emptyset$ implies $A \cap N_I(w) = \emptyset$.*

Proof. Let $\alpha \in A \cap N_I(w)$ and choose $k \in \mathbb{N}$ minimal such that $w^k\alpha \notin \Phi_I$. Then $w^k\alpha \in \Phi^-$, and I-positivity of w implies $w^m\alpha \in wA \cap \Phi_I^+ \subseteq A$ for all $0 < m < k$, by induction. It follows that $w^{k-1}\alpha \in A \cap N(w)$. In other words, $A \cap N_I(w) \neq \emptyset$ implies $A \cap N(w) \neq \emptyset$, and we are done. □

Theorem 4.8. *Let $w \in W$ be I-positive and $c \in W_I$ be the I-conjugator of w, then $cwc^{-1} \in W^I$.*

Proof. Observe first that $\Phi_I{}^+ \cap w^{-1}\Phi_I{}^+ = \Phi_I{}^+ \cap w^{-1}\Phi_I$, since w is I-positive, and that

$(*)$ $\beta \in N(c)$ if and only if $w\beta \in N(c)$, for all $\beta \in \Phi_I{}^+ \cap w^{-1}\Phi_I$,

as is immediate from the definition of $N(c) = N_I(w)$.

Let $\alpha \in \Delta_I$. We set $\beta := c^{-1}\alpha \in \Phi_I$ and show $cwc^{-1}\alpha = cw\beta \in \Phi^+$.

Suppose that $w\beta \in \Phi_I$. If, on the one hand, $\beta \in \Phi_I{}^+$, then $c\beta = \alpha \in \Phi^+$ implies $\beta \in \Phi^+ \setminus N(c)$. By $(*)$, it follows that $w\beta \in \Phi^+ \setminus N(c)$, hence $cw\beta \in \Phi^+$ as desired. If, on the other hand, $-\beta \in \Phi_I{}^+$, then $c(-\beta) = -\alpha \in \Phi^-$ implies $-\beta \in N(c)$. Again by $(*)$, it follows that $w(-\beta) \in N(c)$, hence $cw\beta = -cw(-\beta) \in -\Phi^- = \Phi^+$.

Suppose now that $w\beta \notin \Phi_I$. If $\beta \in \Phi_I{}^+$, then $\beta \notin N(c) = N_I(w)$ as above, hence $w\beta \in \Phi^+ \setminus \Phi_I$ by the definition of $N_I(w)$; while in case $\beta \in \Phi_I{}^-$, it follows that $-\beta \in N(c) = N_I(w)$, hence $w(-\beta) \in \Phi^- \setminus \Phi_I$, again by the definition of $N_I(w)$. But $c \in W_I$ implies $N(c) \subseteq \Phi_I$, thus $w\beta \in \Phi^+ \setminus \Phi_I \subseteq \Phi^+ \setminus N(c)$ in either case. As a consequence, $cw\beta \in \Phi^+$, and the proof is complete. □

To conclude this section, we mention the following related result due to Fleischmann which will be needed later.

Lemma 4.9. *Let $x, y \in W^I$ and $u \in W_I$ such that $y = uxu^{-1}$, then $x = y$.*

([9, Proposition 1.1], see also [16, Lemma 1]).

5. Lower parabolic components

Throughout this section, $I, J \subseteq S$ and $b \in W^{I,J}$ are fixed. Recall that

$$\mathcal{W}(I, J, b) = \{ w \in W \mid w^J b^{-1} = (wb^{-1})_I \}$$

and denote the I-conjugator of bab^{-1} by c_a, for all $a \in W_J$. A crucial step in the proof of the equality $\#\mathcal{W}(I, J, b) = \#\mathcal{W}(J, I, b^{-1})$ is the characterization of the lower parabolic components w_J and $(wb^{-1})_I$ associated with $w \in \mathcal{W}(I, J, b)$, which results in a bijection from $\mathcal{W}(I, J, b)$ onto

$$\mathcal{X}(I, J, b) := \{ (a, g) \in (W_J \cap W^{J \cap b^{-1}Ib}) \times C_{W_I}(bab^{-1}) \mid c_a g \in W^{I \cap bJb^{-1}} \}.$$

(see Lemma 5.6) Here $C_{W_I}(bab^{-1})$ denotes the centralizer of bab^{-1} in W_I. A bijection from $\mathcal{X}(I, J, b)$ onto $\mathcal{X}(J, I, b^{-1})$ is then constructed in final Section 6, completing the proof of Theorem 1.2.

To start with, here are some well-known results on minimal coset representatives taken from [3, Lemmata 2.1–2.4] (see also [25, Lemma 2]).

Proposition 5.1. (i) $W_I \cap bW_Jb^{-1} = W_{I \cap bJb^{-1}}$;
(ii) $W^{I \cap bJb^{-1}}b = W^{J \cap b^{-1}Ib}$;
(iii) $W^I(W_I \cap W^{I \cap bJb^{-1}}) = W^{I \cap bJb^{-1}}$.

(In (iii)—and throughout—the notation $UV = \{uv \mid u \in U,\ v \in V\}$ is used for U, $V \subseteq W$.) Note that (i) may be restated as

$$\Phi_I \cap b\Phi_J = \Phi_{I \cap bJb^{-1}} \tag{5.1}$$

in terms of roots. Applying this identity twice, yields

$$\Phi_I \cap bab^{-1}\Phi_{I \cap bJb^{-1}} \subseteq \Phi_I \cap b\Phi_J = \Phi_{I \cap bJb^{-1}} \tag{5.2}$$

for all $a \in W_J$. Furthermore, if $a \in W_J \cap W^{J \cap b^{-1}Ib}$, then

$$bab^{-1} \in W^J(W_J \cap W^{J \cap b^{-1}Ib})b^{-1} = W^{I \cap bJb^{-1}}, \tag{5.3}$$

by Proposition 5.1(ii) and (iii). More importantly, Propositions 2.1 and 5.1 allow to derive the following necessary conditions on the lower parabolic components associated with $w \in \mathcal{W}(I, J, b)$.

Proposition 5.2. *Let* $w \in \mathcal{W}(I, J, b)$, *then*:
(i) $(wb^{-1})_I \in W_I \cap W^{I \cap bJb^{-1}}$;
(ii) $w_J \in W_J \cap W^{J \cap b^{-1}Ib}$.

Proof. By Proposition 2.1, $(wb^{-1})_Ib = w^J \in W^J \cap W_Ib = (W_I \cap W^{I \cap bJb^{-1}})b$, since $w \in \mathcal{W}(I, J, b)$. This implies the first claim, and furthermore

$$wb^{-1} = (wb^{-1})^I(wb^{-1})_I \in W^I(W_I \cap W^{I \cap bJb^{-1}}) = W^{I \cap bJb^{-1}},$$

by Proposition 5.1(iii). It follows that

$$w^J w_J = w \in W^{I\cap bJb^{-1}} b = W^{J\cap b^{-1}Ib} = W^J (W_J \cap W^{J\cap b^{-1}Ib}),$$

by Proposition 5.1(ii) and (iii), hence the second claim. □

Our next aim is to prove the converse of (ii) in the above Proposition:

Lemma 5.3. *Let $a \in W_J \cap W^{J\cap b^{-1}Ib}$, then $v = c_a ba \in \mathcal{W}(I, J, b)$ and $v_J = a$.*

Combined with Proposition 5.2(ii), this amounts to saying that the lower *J*-component $w \mapsto w_J$ maps $\mathcal{W}(I, J, b)$ onto $W_J \cap W^{J\cap b^{-1}Ib}$. For the proof of Lemma 5.3, an additional property of the *I*-conjugator c_a is needed.

Proposition 5.4. *Let $a \in W_J \cap W^{J\cap b^{-1}Ib}$, then $c_a \in W^{I\cap bJb^{-1}}$.*

Proof. Set $A = \Phi_{I\cap bJb^{-1}}$ and $w = bab^{-1}$. We need to show that $\Phi^+_{I\cap bJb^{-1}} \cap N(c_a) = \emptyset$ or, equivalently, $A \cap N_I(w) = \emptyset$. The latter identity can be derived using Proposition 4.7, as follows.

Recall first that *w* is *I*-positive, by Example 4.6(ii). Furthermore, by (5.2),

$$wA \cap \Phi_I^+ = (bab^{-1}\Phi_{I\cap bJb^{-1}} \cap \Phi_I) \cap \Phi^+ \subseteq \Phi_{I\cap bJb^{-1}} \cap \Phi^+ \subseteq A.$$

Finally, (5.3) yields $w \in W^{I\cap bJb^{-1}}$, hence

$$w(A \cap N(w)) = w\left(\Phi^+_{I\cap bJb^{-1}} \cap w^{-1}\Phi^-\right) \subseteq \Phi^+ \cap \Phi^- = \emptyset.$$

This shows $A \cap N(w) = \emptyset$ and thus also $A \cap N_I(w) = \emptyset$, by Proposition 4.7. □

Proof of Lemma 5.3. Proposition 5.4 and the second part of Corollary 4.4 imply $c_a \in W_I \cap W^{I\cap bJb^{-1}}$. It follows that $c_a b \in (W_I \cap W^{I\cap bJb^{-1}})b = W^J \cap W_I b W_J$, by Proposition 2.1. In particular, $v = (c_a b)a$ has parabolic components $v^J = c_a b$ and $v_J = a$. Furthermore, $c_a(bab^{-1})c_a^{-1} \in W^I$, by Theorem 4.8. As a consequence,

$$(vb^{-1})_I = (c_a bab^{-1})_I = \left((c_a bab^{-1}c_a^{-1})c_a\right)_I = c_a = v^J b^{-1},$$

hence $v \in \mathcal{W}(I, J, b)$. □

Concerning the lower parabolic components $(wb^{-1})_I$, there is the following consequence of Lemma 4.9.

Proposition 5.5. *Let $a \in W_J$ and $v, w \in \mathcal{W}(I, J, b)$ such that $v_J = a = w_J$, then*

$$(vb^{-1})_I^{-1} (wb^{-1})_I \in C_{W_I}(bab^{-1}).$$

Proof. Let $d_w := (wb^{-1})_I = w^J b^{-1}$ and $d_v := (vb^{-1})_I = v^J b^{-1}$, then

$$d_w^{-1}(wb^{-1})^I d_w = b(w^J)^{-1}wb^{-1} = bab^{-1} = b(v^J)^{-1}vb^{-1} = d_v^{-1}(vb^{-1})^I d_v.$$

Thus Lemma 4.9 implies that $(wb^{-1})^I = (vb^{-1})^I$, and that $d_w d_v^{-1}$ is contained in the centralizer of $(vb^{-1})^I = d_v bab^{-1} d_v^{-1}$ in W_I. This is another way of stating the claim. $\square$

Lemma 5.6. *The mapping*

$$B : \mathcal{W}(I, J, b) \longrightarrow \mathcal{X}(I, J, b), \quad w \longmapsto \left(w_J,\, c_{w_J}^{-1}(wb^{-1})_I\right)$$

is a bijection.

Proof. Let $w \in \mathcal{W}(I, J, b)$ and $(a, g) = B(w)$, that is, $a = w_J$ and $g = c_a^{-1}(wb^{-1})_I$.

To see that $(a, g) \in \mathcal{X}(I, J, b)$, observe first that $a \in W_J \cap W^{J \cap b^{-1}Ib}$ follows from Proposition 5.2(ii). As a consequence, $v = c_a ba$ is contained in $\mathcal{W}(I, J, b)$ and $v_J = a$, by Lemma 5.3. In particular, $c_a = v^J b^{-1} = (vb^{-1})_I$. An application of Proposition 5.5 thus yields $g = (vb^{-1})_I^{-1}(wb^{-1})_I \in C_{W_I}(bab^{-1})$. Finally, $c_a g = (wb^{-1})_I \in W^{I \cap bJb^{-1}}$, by Proposition 5.2(i).

The map $C : \mathcal{X}(I, J, b) \to \mathcal{W}(I, J, b)$, $(a, g) \mapsto c_a gba$ is the inverse of B, as will be shown now to complete the proof.

Let $(a, g) \in \mathcal{X}(I, J, b)$ and set $w = C(a, g) = c_a gba$. Then $c_a gb \in (W_I \cap W^{I \cap bJb^{-1}})b \subseteq W^J$, by Proposition 2.1, thus $w^J = c_a gb$ and $w_J = a$. Furthermore,

$$wb^{-1} = c_a g(bab^{-1}) = c_a(bab^{-1})g = vb^{-1}g,$$

where $v := c_a ba$ as in Lemma 5.3. This implies $(wb^{-1})_I = (vb^{-1})_I g$ and $(wb^{-1})^I = (vb^{-1})^I$, since $g \in W_I$. Now by Lemma 5.3, $w^J b^{-1} = c_a g = (vb^{-1})_I g = (wb^{-1})_I$, hence $w \in \mathcal{W}(I, J, b)$ and $B(C(a, g)) = B(w) = (a, g)$.

Conversely, let $w \in \mathcal{W}(I, J, b)$ and set $(a, g) = B(w)$, then

$$C(B(w)) = C(a, g) = c_a gba = (wb^{-1})_I ba = (w^J b^{-1})ba = w^J w_J = w. \qquad \square$$

6. Bijection

Throughout, $I, J \subseteq S$ and $b \in W^{I,J}$ are fixed. We define a mapping

$$T_{I,J,b} : \mathcal{X}(I, J, b) \longrightarrow W \times W$$

as follows. Let $(a, g) \in \mathcal{X}(I, J, b)$ and denote the $(I \cap bJb^{-1})$-conjugator of g by n. Now set $\hat{n} := b^{-1}nb$ and

$$T_{I,J,b}(a, g) := (ngn^{-1}, \hat{n}a\hat{n}^{-1}).$$

The following concluding result implies Theorem 1.2 (and therefore the Main Theorem), since $\#\mathcal{X}(I, J, b) = \#\mathcal{W}(I, J, b)$, by Lemma 5.6.

Theorem 6.1. *$T_{I,J,b}$ is a bijection from $\mathcal{X}(I, J, b)$ onto $\mathcal{X}(J, I, b^{-1})$.*

Two auxiliary results on the second component of $\mathcal{X}$-sets are needed for the proof of Theorem 6.1. For all $w \in W$, set

$$\Phi_I^{\infty}(w) := \{\, \alpha \in \Phi_I \mid w^k\alpha \in \Phi_I \text{ for all } k \in \mathbb{N} \,\}.$$

Proposition 6.2. *Let $a \in W_J \cap W^{J \cap b^{-1}Ib}$ and $g \in C_{W_I}(bab^{-1})$, then the following conditions are equivalent*:

(i) $(a, g) \in \mathcal{X}(I, J, b)$;
(ii) $g(\Phi^+_{I \cap bJb^{-1}} \cap \Phi_I^{\infty}(bab^{-1})) \subseteq \Phi^+$.

Proof. Let $w := bab^{-1}$ and set $c := c_a$, the I-conjugator of w. By definition of $\mathcal{X}(I, J, b)$, (i) is equivalent to $cg \in W^{I \cap bJb^{-1}}$.

Let $cg \in W^{I \cap bJb^{-1}}$ and choose $\alpha \in \Phi^+_{I \cap bJb^{-1}} \cap \Phi_I^{\infty}(w)$, then also $g\alpha \in \Phi_I^{\infty}(w)$, since $g \in C_{W_I}(w)$. Assume, for a contradiction, that $g\alpha \in \Phi^-$, then $-g\alpha$ is contained in

$$(*) \quad \Phi^+ \cap \Phi_I^{\infty}(w) \subseteq \Phi^+ \setminus N_I(w) = \Phi^+ \setminus N(c).$$

This implies $c(-g\alpha) \in \Phi^+$, hence $cg\alpha \in \Phi^-$ contradicting $cg \in W^{I \cap bJb^{-1}}$. Thus (i) implies (ii).

Now assume that (ii) holds and choose $\alpha \in \Delta_{I \cap bJb^{-1}}$. We need to show $cg\alpha \in \Phi^+$, for this implies $cg \in W^{I \cap bJb^{-1}}$, hence (i).

If $\alpha \in \Phi_I^{\infty}(w)$, then $g\alpha \in \Phi^+$, by (ii), and also $g\alpha \in \Phi_I^{\infty}(w)$, since $g \in C_{W_I}(w)$. Therefore $(*)$ implies $cg\alpha \in \Phi^+$ as desired.

Consider now the case that $w^k\alpha \notin \Phi_I$ for some $k \in \mathbb{N}$, and choose k minimal. Recall from (5.3) that $w \in W^{I \cap bJb^{-1}}$. It thus follows that $w^k\alpha \in \Phi^+$, since $w^m\alpha \in \Phi_{I \cap bJb^{-1}}$ for all $m < k$, by (5.2). Furthermore, $g \in C_{W_I}(w)$ implies that $w^m(g\alpha) = g(w^m\alpha) \in \Phi_I$ for all $m < k$ and $w^k(g\alpha) = g(w^k\alpha) \in g(\Phi^+ \setminus \Phi_I) \subseteq \Phi^+$, by (4.1). If $g\alpha \in \Phi^+$, we may conclude that $g\alpha \in \Phi^+ \setminus N_I(w) = \Phi^+ \setminus N(c)$, hence $cg\alpha \in \Phi^+$. If $g\alpha \in \Phi^-$, then $-g\alpha \in N_I(w) = N(c)$, hence $c(-g\alpha) \in \Phi^-$ and $cg\alpha \in \Phi^+$. This completes the proof. $\square$

Proposition 6.3. *Let $(a, g) \in \mathcal{X}(I, J, b)$, then g is $(I \cap bJb^{-1})$-positive.*

Furthermore, if $n \in W_{I\cap bJb^{-1}}$ *denotes the* $(I \cap bJb^{-1})$*-conjugator of* g*, then* $n(\Phi^+_{I\cap bJb^{-1}} \cap \Phi^\infty_I(bab^{-1})) \subseteq \Phi^+$.

Proof. There exists an element $w \in \mathcal{W}(I, J, b)$ such that $a = w_J$ and $g = c_a^{-1}(wb^{-1})_I$, by Lemma 5.6. In particular, $(wb^{-1})_I \in W^{I\cap bJb^{-1}}$ and $c_a \in W^{I\cap bJb^{-1}}$, by Propositions 5.2(i) and 5.4, respectively. The first claim thus follows from Example 4.6(i).

Combined with Proposition 6.2, this allows to apply Proposition 4.7 to g (instead of w), $I \cap bJb^{-1}$ (instead of I), and $A = \Phi^+_{I\cap bJb^{-1}} \cap \Phi^\infty_I(bab^{-1})$. As a result, $nA \subseteq \Phi^+$ as asserted. □

We are now ready to give the

Proof of Theorem 6.1. It suffices to prove that $T_{I,J,b}$ maps into $\mathcal{X}(J, I, b^{-1})$ and is 1-1, since both sets $\mathcal{X}(I, J, b)$ and $\mathcal{X}(J, I, b^{-1})$ are finite and the arguments also apply to $T_{J,I,b^{-1}}$ instead of $T_{I,J,b}$.

We first show that $T_{I,J,b}$ maps into $\mathcal{X}(J, I, b^{-1})$. Let $(a, g) \in \mathcal{X}(I, J, b)$. Denote the $(I \cap bJb^{-1})$-conjugator of g by n, and set $\hat{g} := ngn^{-1}$, $\hat{n} := b^{-1}nb$ and $\hat{a} := \hat{n}a\hat{n}^{-1}$. Then $T_{I,J,b}(a, g) = (\hat{g}, \hat{a})$. By Proposition 6.3 and Theorem 4.8, $\hat{g} \in W^{I\cap bJb^{-1}}$, while $g \in W_I$ and $n \in W_{I\cap bJb^{-1}} \subseteq W_I$ imply $\hat{g} \in W_I$. Furthermore, $\hat{n} \in W_{J\cap b^{-1}Ib} \subseteq W_J$, by Proposition 5.1(i), hence $\hat{a} := \hat{n}a\hat{n}^{-1} \in W_J$ follows from $a \in W_J$. We also observe that, simply by definition,

$$\begin{aligned}\hat{a}\,(b^{-1}\hat{g}b) &= b^{-1}nb\,a\,b^{-1}n^{-1}b\,(b^{-1}ngn^{-1}b)\\ &= b^{-1}n\,(bab^{-1})g\,n^{-1}b\\ &= b^{-1}n\,g(bab^{-1})\,n^{-1}b \quad (\text{since } g \in C_{W_I}(bab^{-1}))\\ &= (b^{-1}\hat{g}b)\,\hat{a},\end{aligned}$$

that is, $\hat{a} \in C_{W_J}(b^{-1}\hat{g}b)$. For the proof of $(\hat{g}, \hat{a}) \in \mathcal{X}(J, I, b^{-1})$, the technical condition in Proposition 6.2(ii) (for $\hat{a}$ and $b^{-1}\hat{g}b$ instead of g and bab^{-1}) remains. First two helpful observations:

(a) If $\alpha \in \Phi$ and $\hat{\alpha} := n^{-1}b\alpha$, then

$$(b^{-1}\hat{g}b)^k\,\hat{a}^m\,\alpha \in \Phi_{J\cap b^{-1}Ib} \text{ if and only if } g^k\,(bab^{-1})^m\,\hat{\alpha} \in \Phi_{I\cap bJb^{-1}},$$

for all $k, m \in \mathbb{N}_0$.

(b) $\Phi_{J\cap b^{-1}Ib} \cap \Phi^\infty_J(b^{-1}\hat{g}b) \subseteq \Phi^\infty_{J\cap b^{-1}Ib}(b^{-1}\hat{g}b)$.

Let $\beta = (b^{-1}\hat{g}b)^k\,\hat{a}^m\,\alpha$ and $\gamma = g^k\,(bab^{-1})^m\,\hat{\alpha}$, then $\beta = b^{-1}n\gamma$, hence (a) follows from $n \in W_{I\cap bJb^{-1}}$ and (5.1). Claim (b) is immediate from (5.2) (applied to $b^{-1}\hat{g}b$, J and I instead of bab^{-1}, I and J).

Now let $\alpha \in \Phi^+_{J\cap b^{-1}Ib} \cap \Phi^\infty_J(b^{-1}\hat{g}b)$. We need to show that $\hat{a}\alpha \in \Phi^+$.

Start with observing that

(c) $\hat{\alpha} := n^{-1}b\alpha \in \Phi^+_{I\cap bJb^{-1}}$ (hence also $bab^{-1}\hat{\alpha} \in \Phi^+$, by (5.3)).

Indeed, (b) yields $\alpha \in \Phi^{\infty}_{J\cap b^{-1}Ib}(b^{-1}\hat{g}b)$, hence (a) implies $\hat{\alpha} \in \Phi^{\infty}_{I\cap bJb^{-1}}(g)$ (with $m = 0$). In particular, $\hat{\alpha} \notin N_{I\cap bJb^{-1}}(g) = N(n)$, thus $n\hat{\alpha} = b\alpha \in b\Phi^{+}_{J} \subseteq \Phi^{+}$ implies $\hat{\alpha} \in \Phi^{+}$, and (c) is proven.

For a contradiction, assume that $\hat{a}\alpha \in \Phi^{-}$ (hence $\hat{a}\alpha \in \Phi^{-}_{J}$), then $n(bab^{-1}\hat{\alpha}) = b\hat{a}\alpha \in \Phi^{-}$, as $b \in W^{J}$. It follows that $bab^{-1}\hat{\alpha} \in N(n) \subseteq \Phi_{I\cap bJb^{-1}}$, by (c), and thus $\hat{a}\alpha \in \Phi_{J\cap b^{-1}Ib}$, by (a) (with $k = 0$, $m = 1$).

But $\alpha \in \Phi^{\infty}_{J}(b^{-1}\hat{g}b)$ and $\hat{a} \in C_{W_J}(b^{-1}\hat{g}b)$ imply $\hat{a}\alpha \in \Phi^{\infty}_{J}(b^{-1}\hat{g}b)$, hence even $\hat{a}\alpha \in \Phi^{\infty}_{J\cap b^{-1}Ib}(b^{-1}\hat{g}b)$, by (b), and finally $bab^{-1}\hat{\alpha} \in \Phi^{\infty}_{I\cap bJb^{-1}}(g)$, by (a) (with $m = 1$). This yields $bab^{-1}\hat{\alpha} \notin N_{I\cap bJb^{-1}}(g) = N(n)$, a contradiction.

We have thus proven that $T_{I,J,b}(a, g) = (\hat{g}, \hat{a}) \in \mathcal{X}(J, I, b^{-1})$.

Injectivity of $T_{I,J,b}$ remains. Let $(a_1, g_1), (a_2, g_2) \in \mathcal{X}(I, J, b)$ such that

$$(\hat{g}, \hat{a}) := T_{I,J,b}(a_1, g_1) = T_{I,J,b}(a_2, g_2),$$

and define n_i and $\hat{n}_i$ according to the definition of $T_{I,J,b}(a_i, g_i)$, for $i \in \{1, 2\}$. Then

$$\hat{n}_1 a_1 \hat{n}_1^{-1} = \hat{a} = \hat{n}_2 a_2 \hat{n}_2^{-1}.$$

But $a_1, a_2 \in W^{J\cap b^{-1}Ib}$ and $\hat{n}_1, \hat{n}_2 \in W_{J\cap b^{-1}Ib}$, so that Lemma 4.9 implies $a_1 = a_2$. As another consequence, $\hat{n}_1^{-1}\hat{n}_2 = b^{-1}n_1^{-1}n_2 b$ centralizes $a := a_1$. Equivalently, $d := n_1^{-1}n_2 \in W_{I\cap bJb^{-1}}$ centralizes bab^{-1}. As $n_1 g_1 n_1^{-1} = \hat{g} = n_2 g_2 n_2^{-1}$, it remains to show that d is the identity of W.

Assume that $N(d) \neq \emptyset$, and let $\alpha \in N(d)$, then $\alpha \in \Phi^{+}_{I\cap bJb^{-1}}$. It follows that $bab^{-1}\alpha \in \Phi^{+}$ and $d(bab^{-1}\alpha) = bab^{-1}(d\alpha) \in \Phi^{-}$, by (5.3). This yields $bab^{-1}\alpha \in N(d)$ and, inductively,

$$ba^k b^{-1}\alpha \in N(d) \subseteq \Phi^{+}_{I\cap bJb^{-1}} \subseteq \Phi_I$$

for all $k \in \mathbb{N}$. In particular, $n_1 d\alpha = n_2\alpha \in \Phi^{+}$, by the second part of Proposition 6.3 (applied to n_2). But $-d\alpha \in \Phi^{+}_{I\cap bJb^{-1}}$ and

$$ba^k b^{-1}(-d\alpha) = d(-ba^k b^{-1}\alpha) \in \Phi_{I\cap bJb^{-1}} \subseteq \Phi_I$$

for all $k \in \mathbb{N}$. The second part of Proposition 6.3 (now applied to n_1) yields $n_1(-d\alpha) \in \Phi^{+}$—a contradiction. This shows $N(d) = \emptyset$, and $T_{I,J,b}$ is injective. □

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References

[1] M.D. Atkinson, G. Pfeiffer, S. van Willigenburg, The *p*-modular descent algebra, Algebra Represent. Theory 5 (2002) 101–113.

[2] F. Bergeron, N. Bergeron, A decomposition of the descent algebra of the hyperoctahedral group I, J. Algebra 148 (1992) 86–97.

[3] F. Bergeron, N. Bergeron, R.B. Howlett, D.E. Taylor, A decomposition of the descent algebra of a finite Coxeter group, J. Algebraic Combin. 1 (1992) 23–44.

[4] N. Bergeron, A decomposition of the descent algebra of the hyperoctahedral group II, J. Algebra 148 (1992) 98–122.

[5] N. Bourbaki, Groupes et Algèbres De Lie, Masson, France, 1981 (Chapitre 4,5,6).

[6] K.S. Brown, Semigroups, rings, and Markov chains, J. Theoret. Probab. 13 (2000) 871–938.

[7] R.W. Carter, Finite groups of Lie type: conjugacy classes and complex characters, Wiley-Interscience, New York, 1985.

[8] C.W. Curtis, I. Reiner, Methods of representation theory with applications to finite groups and orders, vol. I, Wiley-Interscience, New York, 1981.

[9] P. Fleischmann, On pointwise conjugacy of distinguished coset representatives in Coxeter groups, J. Group Theory 5 (2002) 269–283.

[10] F.G. Frobenius, Über die Darstellung der endlichen Gruppen durch lineare Substitutionen II, Sitzungsber. der Kgl. Pr. Akad. Wiss. Berlin (1899) 482–500.

[11] J. Fulman, Descent algebras, hyperplane arrangements, and shuffling cards, Proc. Amer. Math. Soc. 129 (2001) 965–973.

[12] A.M. Garsia, C. Reutenauer, A decomposition of Solomon's descent algebra, Adv. Math. 77 (1989) 189–262.

[13] M. Geck, G. Pfeiffer, Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras, London Mathematical Society Monographs, New Series, vol. 21, Oxford University Press, Oxford, 2000.

[14] I.M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. Retakh, J.-Y. Thibon, Noncommutative symmetric functions, Adv. Math. 112 (1995) 218–348.

[15] I.M. Gessel, C. Reutenauer, Counting permutations with given descent set and cycle structure, J. Combin. Theory Ser. (A) 64 (1993) 189–215.

[16] C. Hohlweg, M. Schocker, On a parabolic symmetry of finite Coxeter groups, Bull. London Math. Soc. 36 (2004) 289–293.

[17] J.E. Humphreys, Reflection groups and Coxeter groups, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[18] A. Jöllenbeck, C. Reutenauer, Eine Symmetrieeigenschaft von Solomons Algebra und der höheren Lie-Charaktere, Abh. Math. Sem. Univ. Hamburg 71 (2001) 105–111.

[19] F. Patras, L'algèbre des descentes d'une bigèbre graduée, J. Algebra 170 (1994) 547–566.

[20] S. Poirier, Cycle type and descent set in wreath products, Discrete Math. 180 (1998) 315–343.

[21] C. Reutenauer, Free Lie Algebras, London Mathematical Society Monographs, New Series, vol. 7, Oxford University Press, Oxford, 1993.

[22] T. Scharf, J.-Y. Thibon, A Hopf algebra approach to inner plethysm, Adv. Math. 104 (1994) 30–58.

[23] M. Schocker, Lie idempotent algebras, Adv. Math. 175 (2003) 243–270.

[24] L. Solomon, A decomposition of the group algebra of a finite Coxeter group, J. Algebra 9 (1968) 220–239.

[25] L. Solomon, A Mackey formula in the group ring of a Coxeter group, J. Algebra 41 (1976) 255–268.

[26] R.P. Stanley, Enumerative Combinatorics II, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.