Improving Properties of Operators by Extensions and Reductions

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Abstract

This thesis presents and develops two tools which can be used to work with lower bounds of operators.

One tool in working with lower bounds is invertible extensions. They allow one to turn a lower bound of an operator into the norm of an inverse operator. Some results giving extensions are known for single operators and certain other semigroup representations. Chapter 3 includes some positive new results for operators on Hilbert space and also certain unbounded operators. An example shows that a uniform lower bound for the powers of an operator does not give an extension with power bounded inverse. Variations are given for generators of C_0-semigroups, and for operators on Hilbert space. A result by Read, which gives an extension with minimal spectrum raises the question how the lower bounds of the original operator are related to the resolvent bounds of the extension.

Another tool which is developed in this thesis is a reduction using seminorms. A seminorm can place a different emphasis on elements and even neglect some. In this way, we can *shape* a Banach space to attain properties that we impose. This idea is used to define *maximal parts* in Chapter 4. They are identified in the context of contractivity and expansiveness of a bounded operator, and in the context of dissipativity and accretivity for certain unbounded operators. Applications are an improvement of a theorem by Batty and Tomilov which characterises embeddings into hyperbolic C_0-semigroups, and a generalisation of a theorem by Goldberg and Smith leading to a characterisation of generators of C_0-semigroups which have an extending group with bounded inverses.

Contents

Chapter 1

Introduction

Similarity of operators is a simple concept which preserves spectral properties of the operators. If two operators are similar, then they have the same spectrum and comparable resolvent bounds, and we can relate their functional calculi, for example. We say two operators $T\colon X \to X$ and $S\colon Y \to Y$ on Banach spaces X and Y are similar if there is an invertible linear map $U\colon X \to Y$ such that $T = U^{-1}SU$. It is easy to characterise similarity to a linear contraction on a Banach space. An operator T is similar to a contraction S if and only if T is power bounded. Indeed, if T is similar to a contraction S then clearly $\|T^n\| \leq \|U^{-1}\|\|U\|$ for all $n \geq 1$. Conversely, if T is power bounded we can use an equivalent norm on X in which T becomes contractive, and use T on the renormed space as the similar contraction.

A property which is related to contractivity is expansiveness. We say that an operator $T\colon X \to X$ is *expansive* if $\|Tx\| \geq \|x\|$ for all $x \in X$. Expansiveness is related to contractivity in the following way, as has been shown by Arens in [Are58]. If T is expansive, then it has an invertible extension S to a larger Banach space such that S^{-1} is contractive. Conversely, any restriction of such an operator S to a closed invariant subspace is an expansive operator. We can use invertible extensions to transfer results between invertible operators and operators which are bounded below. This allows us to determine which operators T are similar to expansive operators S. We expect these operators T to take the place of the power bounded operators above. If $T = U^{-1}SU$ where S is an expansive operator, then T has a uniform lower bound in the sense that

$$\|T^n x\| \geq \left(\|U^{-1}\|\|U\|\right)^{-1}\|x\| \quad (x \in X, n \geq 1).$$

We will see later (in Example 3.2.7) that such a uniform lower bound does not imply similarity of T to an expansive operator. But there is a characterisation which makes use of invertible extensions. Badea and Müller found in [BM05] an intrinsic condition for the existence of an invertible extension S with $\sup_{n\geq 1}\|S^{-n}\| < \infty$ on a larger Banach space Y. This implies that T is similar to an expansive operator. Indeed, the expansive operator similar to T can then be obtained by renorming the extending space Y to make S^{-1} contractive and restricting this norm to X.

Consider a representation $T\colon \mathfrak{S} \to \mathcal{B}(X)$ of a semigroup $\mathfrak{S}$ on a Banach space X. For now, assume that $0 \in \mathfrak{S}$ and define $T(0) = I$, if necessary. If T is bounded with $\|T(s)\| \leq M$ for all $s \in \mathfrak{S}$ we can define an equivalent norm on X by taking

$$|||x||| = \sup_{s\in\mathfrak{S}} \|T(s)x\| \quad (x \in X).$$

In this norm, T is contractive, so we can use the norm $|||\cdot|||$ on X to define a Banach space Y. By equivalence of norms, the representation T on X is similar to the contractive representation T on Y.

To study semigroup representations with lower bounds in a similar way to expansive operators above, we need to make some restrictions. For some subsemigroups $\mathfrak{S} \subset \mathfrak{G}$ of abelian groups $\mathfrak{G}$, Batty and Yeates showed in [BY01] the following result which is comparable to Arens' result on expansive operators. If $\mathfrak{G} = \mathfrak{S} \cup (-\mathfrak{S})$ and if T is an expansive representation such that

$$\|T(s)x\| \geq \|x\| \quad (x \in X, s \in \mathfrak{S}),$$

then T has an extending representation S of the group $\mathfrak{G}$ to some Banach space $Y \supset X$ with $\|S(-s)\| \leq 1$ for all $s \in \mathfrak{S}$. More generally, if $\mathfrak{G} = \mathfrak{S} - \mathfrak{S}$ then T has such an extending representation S if T is super-expansive (we will define this in Section 3.2). This motivated the work [BM05] by Badea and Müller. Their result also applies to semigroup representations that were studied by Batty and Yeates. If we state it for semigroup representations, it characterises when a representation T of $\mathfrak{S}$ has an extending group representation S with $\|S(-s)\| \leq M$ for all $s \in \mathfrak{S}$.

A special case of semigroup representations are C_0-semigroups. The above extensions for C_0-semigroups in fact give C_0-groups. However, in this case we are also interested in characterising the similarity using the generator A of a C_0-semigroup T. The Lumer-Phillips theorem (Theorem 2.3.3) or the Hille-Yosida theorem (Theorem 2.4.1) characterise generators of contractive semigroups, while the Feller-Miyadera-Phillips theorem (Theorem 2.4.2), for example, characterises generators of bounded semigroups — these are similar to contractive semigroups on Banach space as shown above. A less well-known result by Goldberg and Smith ([GS78]) characterises generators of expansive semigroups. We will complement this by finding those generators which generate C_0-semigroups similar to expansive ones (in Theorem 4.7.4). In addition, we will show that the results giving invertible extensions for a single bounded operator remain valid for generators of C_0-semigroups.

A further topic is similarity problems on Hilbert space. Let $T\colon X \to X$ and $S\colon Y \to Y$ be similar operators on Hilbert spaces X and Y, and assume there is an invertible operator $U\colon X \to Y$ such that $T = U^{-1}SU$. Since X and Y are isomorphic Hilbert spaces, they are also isometrically isomorphic and there is a unitary map $V\colon Y \to X$ such that T and VSV^* are similar. Moreover, the operators S and VSV^* are indistinguishable. So on Hilbert space, it suffices to study similarity between operators on the same Hilbert space.

Let T be an operator on a Hilbert space X which is similar to a contraction. The following hold for T.

1. T is power bounded, that is, $\sup_{n\geq 1} \|T^n\| < \infty$.

2. T is polynomially bounded. There is $K \geq 1$ such that

$$\|p(T)\| \leq K\|p\|_\infty = K \sup\{|p(z)|\colon z \in \mathbb{C}, |z| = 1\}$$

   for all polynomials p.

3. T is completely polynomially bounded. This is a C^*-algebraic notion and means that there is $K \geq 1$ such that

$$\|(p_{i,j}(T))\| \leq K\|(p_{i,j})\| = K \sup\{\|(p_{i,j}(z))\|\colon z \in \mathbb{C}, |z| \leq 1\}$$

   for all matrices of polynomials $p_{i,j}$.

For any operator, $3 \Rightarrow 2 \Rightarrow 1$, but the reverse implications may fail. Foguel gave an example of an operator which is power bounded but not polynomially bounded ([Fog64] and [Leb68]). Paulsen showed that completely polynomially bounded operators are similar to contractions on Hilbert space ([Pau84]). Finally, Pisier gave an example of a polynomially bounded operator which is not completely polynomially bounded ([Pis97]).

Switching to expansive operators, the above means that we can expect it to be more difficult on Hilbert space to determine which operators are similar to expansive operators. We will see that we cannot follow the same line of reasoning as in the Banach space case. The situation is as follows. Let T be an expansive operator on a Hilbert space X. The invertible extension by Arens in general does not yield an extension on a Hilbert space, but we will find another (less natural) way of finding an invertible extension S on a Hilbert space Y such that S has a contractive inverse (Theorem 3.1.5). Using Hilbert space renorming techniques, this implies that an operator $T\colon X \to X$ which is similar to an expansive operator has an invertible extension $S\colon Y \to Y$ such that S^{-1} is similar to a contractive operator, which means S^{-1} is completely polynomially bounded. This is the only characterisation for similarity to an expansive operator on Hilbert space that we know. This leaves open the problem of finding an intrinsic characterisation for such a similarity.

Another concept that we will encounter in this thesis is continuous embeddings. They are naturally related to seminorms. Let $\rho\colon X \to Y$ be a continuous map between Banach spaces. By restricting Y to the closure of $\rho(X)$ we can assume that $\rho(X)$ is dense in Y. Then $\|\rho(\cdot)\|$ defines a seminorm on X and $\|\rho(x)\| \leq \|\rho\|\|x\|$ for all $x \in X$. Conversely, if $|||\cdot|||$ is a seminorm on X and there exists $C > 0$ such that $|||x||| \leq C\|x\|$ for all $x \in X$, we get a continuous map $\rho\colon X \to Y$ where Y is the completion of X modulo the null space of $|||\cdot|||$, and $\|\rho\| \leq C$. Another way to access the seminorm corresponding to a map $\rho\colon X \to Y$ is by considering its dual map $\rho'\colon Y' \to X'$. Since ρ has a dense image in Y, ρ' is injective. By the Hahn-Banach theorem we have

$$\|\rho(x)\| = \sup\{|(\rho' f)(x)|\colon f \in Y', \|f\| = 1\} \quad (x \in X)$$

and the seminorm is known when we identify the image of the unit ball of Y' under ρ'. We will look at continuous maps ρ which intertwine operators on X and Y in Chapter 4. These correspond to seminorms with additional properties.

We shall give two examples here, which we will discuss in more detail in Section 4.1. Let ρ intertwine $T\colon X \to X$ and some contractive $S\colon Y \to Y$. Then T is contractive in the seminorm $\|\rho(\cdot)\|$. We will see that there is a maximal seminorm with this property, and that it corresponds to a continuous map ρ, a Banach space Y and an operator S with a universal property. As another example, assume ρ intertwines $T\colon X \to X$ and an invertible operator $S\colon Y \to Y$, say with contractive inverse. In general, S is not invertible on the closure of $\rho(X)$. The property we can use instead here is expansiveness of S. Remember that expansiveness also implies the existence of an invertible extension by Arens' theorem. So we see that the properties we can consider for the seminorms are those which are invariant under restricting an operator to a closed invariant subspace.

A last word about Hilbert spaces. Since the methods here use suprema over some sets, it is not clear how they can be adapted to Hilbert spaces. Perhaps some

C^*-algebra methods can be used to obtain results for Hilbert spaces similar to the results we will see in Chapter 4, but this remains open.

Structure The thesis is organised into three main chapters. Chapter 2 contains preliminary information which should set the scene, that is, it should clarify the notation used and present some results which are essential for the thesis. Chapter 3 presents results on invertible extensions, where new findings are tied in with previously known results. Chapter 4 is about maximal parts, by which we mean continuous maps to spaces with universal properties for a given setting. For each of these latter two chapters, we now give an overview and a short outline.

1.1 Background to Chapter 3

Chapter 3 concerns invertible extensions of operators and operator families on Banach and Hilbert space. The existence of an invertible extension was first studied in the context of commutative Banach algebras by Arens. It is motivated by the study of an equation

$$ax = b$$

where a solution for x is sought for given a, b in a commutative Banach algebra $\mathcal{A}$. If a is invertible, the solution is simply given by $a^{-1}b$. In general, a is not invertible, and the equation does not have a solution for all b or the solution is not unique. If no solution exists, we can look for solutions of this equation in an algebra extension. For this purpose, Arens characterised in [Are58] the elements $a \in \mathcal{A}$ that have an inverse in some commutative Banach algebra $\mathcal{B}$ with $\mathcal{A} \subset \mathcal{B}$. For $a \in \mathcal{A}$ there is an inverse in some extension $\mathcal{B}$ if and only if

$$\|ax\| \geq c\|x\| \quad (x \in \mathcal{A})$$

for some $c > 0$. We call c a *lower bound* of a. Concerning the bound of the inverse a^{-1} in $\mathcal{B}$, given an element a with a lower bound c there is an extension $\mathcal{B}$ containing a^{-1} and $\|a^{-1}\|_{\mathcal{B}} \leq c^{-1}$, which also is the best bound we can expect for a^{-1}.

This result can be transferred to bounded linear operators on Banach spaces, either directly by adapting the proofs or by using a construction by Müller [Mül88]. To specify what the extension is in this case, let X be a Banach space and $T \in \mathcal{B}(X)$. An invertible extension for T is given by a Banach space Y, an isometric embedding

$X \subset Y$, and an invertible operator $S \in \mathcal{B}(Y)$ such that $Sx = Tx$ for all $x \in X$. Often, we also get $\|S\| = \|T\|$. Such an extension exists when there is $c > 0$ such that

$$\|Tx\| \geq c\|x\| \quad (x \in X).$$

Again, S can be chosen so that its inverse has norm $\|S^{-1}\| \leq c^{-1}$. We should note that, in general, not every operator $U \in \mathcal{B}(X)$ needs to extend to Y. However, if U commutes with T then this can be arranged. More specifically, we can choose Y such that there is an isometric algebra homomorphism φ from the commutant of T, the set

$$\{T\}' = \{\, U \in \mathcal{B}(X) \colon UT = TU \,\},$$

to $\mathcal{B}(Y)$ with $\varphi(T) = S$. If we use the aforementioned construction by Müller, we can obtain such a homomorphism from a commutative subalgebra of $\mathcal{B}(X)$ but not from the whole commutant. Throughout Chapter 3, we only work with unital algebra homomorphisms, so we always have that $\varphi(I) = I$. This implies that φ maps resolvent operators of T to resolvent operators of $S = \varphi(T)$, and that $\sigma(S) \subset \sigma(T)$.

The next immediate question is to ask what happens in the case of two commuting operators $T_1, T_2 \in \mathcal{B}(X)$. Assume for simplicity that

$$\|T_i x\| \geq \|x\| \quad (x \in X,\ i = 1, 2).$$

By the above, we know that each of the T_i has an invertible extension on some Banach space Y_i containing X. However, the spaces Y_i can be different. If we let $T = T_1 T_2$ then $\|Tx\| \geq \|x\|$ for all $x \in X$. So there is an invertible extension S of T with $\|S^{-1}\| \leq 1$. Since the operators T and T_i commute, we can extend T_i to some operator S_i using the algebra homomorphism φ. We then have $S = S_1 S_2 = S_2 S_1$ and we see that each S_i is invertible with inverse $S_i^{-1} = S^{-1} S_j$ $(i \neq j)$, but now we only have the norm estimate $\|S_i^{-1}\| \leq \|S_j\|$. Bollobas gave an example in [Bol73] which shows that, in general, we do not get the bounds $\|S_1^{-1}\| \leq 1$ and $\|S_2^{-1}\| \leq 1$ at the same time. We discuss next, when this is possible.

This brings us to semigroup representations. If two commuting operators $T_1, T_2 \in \mathcal{B}(X)$ are given, they form a representation of the semigroup $\mathbb{N} \times \mathbb{N}$ given by the map $T \colon (m, n) \mapsto T_1^m T_2^n$. Assume there are invertible extensions $S_1, S_2 \in \mathcal{B}(Y)$ for T_1 and T_2 on the same space Y. Then we can extend the representation of $\mathbb{N} \times \mathbb{N}$ to a representation of $\mathbb{Z} \times \mathbb{Z}$ on Y by mapping $S \colon (m, n) \mapsto S_1^m S_2^n$. Using this formulation we see that we have $\|S_i^{-1}\| \leq 1$ for $i = 1, 2$ if and only if $\|S(-(m, n))\| \leq 1$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$. In the following two citations, the authors work with semigroups $\mathfrak{S}$

which are subsemigroups of locally compact abelian groups $\mathfrak{G}$. Batty and Greenfield studied the problem of extending a semigroup representation to a group representation on the same Banach space for isometric representations ([BG94]) and Batty and Yeates studied when a semigroup representation allows an extending group representation with contractive inverses ([BY01]), assuming that $\mathfrak{G} = \mathfrak{S} - \mathfrak{S}$. They identify the semigroup representations $T\colon \mathfrak{S} \to \mathcal{B}(X)$ for which there is an extending group representation U of $\mathfrak{G}$ on a Banach space $Y \supseteq X$ with $\|U(-s)\| \leq 1$ for all $s \in \mathfrak{S}$ as the *super-expansive* representations. We say T is super-expansive if

$$\|x\| \leq \|x_1\| + \cdots + \|x_n\| \tag{1.1}$$

whenever $n \in \mathbb{N}$, $t, t_1, \ldots, t_n \in \mathfrak{S} \cup \{0\}$ and $x, x_1, \ldots, x_n \in X$ satisfy $t - t_i \in \mathfrak{S} \cup \{0\}$ for $i = 1, \ldots, n$ and

$$T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n. \tag{1.2}$$

It is easy to see that this is necessary. If T were a representation of a group $\mathfrak{G}$ such that $T(-s)$ is contractive for all $s \in \mathfrak{S}$, let t_i and x_i be elements that satisfy (1.2); we can simply apply $T(-t)$ on both sides of (1.2) to isolate x. The operators $T(t_i - t)$ are contractive and the triangle inequality gives (1.1). In the extension result by Batty and Yeates, there is again an isometric homomorphism from $\{T(s)\colon s \in \mathfrak{S}\}'$ to $\mathcal{B}(Y)$.

Let us apply what we have so far to C_0-semigroups. Let T be a C_0-semigroup on a Banach space X such that $\|T(1)x\| \geq \|x\|$ for all $x \in X$. Apply Arens' result to $T(1)$. Since the operators $T(t)$ for $t \geq 0$ commute with $T(1)$, there are extensions $S(t)$ for each $T(t)$ on some Banach space Y and $S(1)$ is invertible with $\|S(1)^{-1}\| \leq 1$. Since we extended the operator using a unital isometric algebra homomorphism, we have that $S(t_1)S(t_2) = S(t_1 + t_2)$ and $\|S(t)\| = \|T(t)\|$ for all $t, t_1, t_2 \in \geq 0$. Consider the restriction of S to the maximal subspace of Y on which S is strongly continuous. This subspace is closed, contains X and $S(1)$ is invertible on it. So without loss of generality we can assume that this subspace is equal to Y. Now, let $t > 0$ and choose $n \in \mathbb{N}$ such that $n - 1 \leq t < n$. The operator $S(1)^{-n}S(n - t)$ is an inverse of $S(t)$ and it is bounded by $\sup\{\|T(s)\|\colon 0 \leq s \leq 1\}$. So S is in fact a C_0-group on Y. However, although $\|S(t)\|$ is bounded for $t < 0$ we do not have that $S(t)$ is contractive for all $t < 0$. For the operators $S(t)$ for $t < 0$ to be contractive we have to assume that $\|T(t)x\| \geq \|x\|$ for all $x \in X$ and all $t \geq 0$. If T satisfies this we say that T is *expansive*. In fact, for C_0-semigroups this is equivalent to super-expansiveness, see [BY01, Proposition 2.2]. So if T is an expansive C_0-semigroup, the result by Batty and Yeates gives us an extending C_0-group $\tilde{S}$ with $\|\tilde{S}(-t)\| \leq 1$ for all $t \geq 0$.

Super-expansiveness appears with a small variation in a more general result. Instead of restricting our attention to contractive inverses or those that we obtain from rescaling the representations, we can look at other growth rates for the inverse. In the case of an operator T with a lower bound $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$, rescaling the operator to $c^{-1}T$ only gives us an extension S of T with an inverse bounded by $\|S^{-1}\| \leq c^{-1}$. For a C_0-semigroup T and $\omega \in \mathbb{R}$, we would similarly get an extending C_0-group S with $\|S(-t)\| \leq e^{-\omega t}$ for all $t \geq 0$ if we assume that $\|T(t)x\| \geq e^{\omega t}\|x\|$ for all $x \in X$ and $t \geq 0$. Going back to an operator T, these results do not tell us when we can have an invertible extension S with $\|S^{-n}\| \leq c_n$ $(n \geq 1)$ for a given sequence (c_n). A sensible class for such (c_n) are sub-multiplicative sequences. If S is invertible we have that $\|S^{-(m+n)}\| \leq \|S^{-n}\|\|S^{-m}\|$ and we say that (c_n) is sub-multiplicative if $c_{m+n} \leq c_m c_n$ holds for all $m, n \geq 1$. Such bounds (c_n) were studied by Badea and Müller in [BM05]. Their results include the following. Let (c_n) be a sub-multiplicative sequence. A bounded operator T has an extension S with $\|S^{-n}\| \leq c_n$ for all $n \geq 1$ if

$$\|x\| \leq c_n\|x_0\| + c_{n-1}\|x_1\| + \cdots + c_1\|x_{n-1}\|$$

whenever $n \geq 1$, and $T^n x = x_0 + Tx_1 + \cdots + T^{n-1}x_{n-1}$ for some $x, x_i \in X$.

Outline of Chapter 3 The chapter begins with a section on invertible extensions for a single operator. After stating and proving Arens' theorem for Banach space operators [Are58] and a similar statement with worse norm estimates for Hilbert space operators by Badea and Müller [BM05], we give a new construction for Hilbert space operators (Theorem 3.1.5) which allows improved bounds as in Arens' theorem. The next section contains a result by Batty and Yeates [BY01] and another one by Badea and Müller [BM05], which are about extensions for semigroup representations and give control of the norm of the inverses. A new example (Example 3.2.5) shows that it is not possible to control the norm using lower bounds of an operator alone. This implies that an operator with a uniform lower bound is not equivalent to an expansive operator. For C_0-semigroups on Hilbert space, we use a result by Zwart [Zwa13] to find an extending C_0-group on a Hilbert space (Theorem 3.2.14). Sections 3.3 and 3.4 contain generalisations to unbounded operators of Arens' theorem, our Hilbert space extension, and Badea and Müller's growth control for the inverse. This will be done mainly for generators of C_0-semigroups. For generators of integrated semigroups, we can make similar statements, which only allow a continuous embedding. In Section 3.5 we consider dissipative operators and left inverses on Hilbert space. We can extend the Cayley transform of a dissipative operator to a larger Hilbert space to find an extending C_0-semigroup generator. Zwart showed that a left inverse of a C_0-semigroup implies the existence of a left inverse C_0-semigroup [Zwa13]. We show how the maximal dissipative extension gives a bound for the left inverse C_0-semigroup if we assume that the C_0-semigroup is expansive. This is also related to a result by Goldberg and Smith [GS78]. Finally, Section 3.6 contains a result by Read [Rea88] which is related to invertible extensions. It gives an extension with the smallest possible spectrum. Here, we will show that this result also holds for generators of C_0-semigroups. However, in this case we would be interested in a stronger, quantitative version which also gives resolvent bounds — this remains an open problem.

1.2 Background to Chapter 4

Chapter 4 concerns a concept which we shall call maximal parts here. In a broad sense, it describes quotient spaces with certain universal properties. The ideas in this chapter are motivated by hyperbolic semigroups (see [EN00, Section V.1 c]). Before we introduce what we mean by a maximal part, let us first take a look at these semigroups. A C_0-semigroup T on a Banach space X is hyperbolic if $X = X_{\mathrm{s}} \oplus X_{\mathrm{u}}$ for closed, T-invariant subspaces X_{s} and X_{u} which we call the *stable* and *unstable part*, respectively, such that, for some $t > 0$, $T(t)$ restricted to the unstable part X_{u} is invertible and

$$\begin{aligned} \|T(t)x\| &\le \frac{1}{2}\|x\| \quad (x \in X_{\mathrm{s}}), \\ \|(T(t)\restriction_{X_{\mathrm{u}}})^{-1}x\| &\le \frac{1}{2}\|x\| \quad (x \in X_{\mathrm{u}}). \end{aligned}$$

This means that $\|T(t)x\|$ decays exponentially as $t \to \infty$ and uniformly in $x \in X_{\mathrm{s}}$, while $\|T(t)x\|$ decays exponentially as $t \to -\infty$ and uniformly in $x \in X_{\mathrm{u}}$. Here, if $t < 0$ and $x \in X_{\mathrm{u}}$ we mean $(T(-t)\restriction_{X_{\mathrm{u}}})^{-1}x$ when writing $T(t)x$. These semigroups and their generators have been studied extensively ([CL95, CL99, Kat70, Kat73, Kat11, LS99]). A spectral characterisation that should be mentioned here is that a C_0-semigroup T is hyperbolic if and only if

$$\mathbb{T} \cap \sigma(T(1)) = \emptyset,$$

where $\mathbb{T}$ is the unit circle. If this spectral condition is met, the (Riesz) spectral projection associated with $\{\lambda \in \sigma(T(1))\colon |\lambda| < 1\}$ gives the required splitting of the underlying Banach space into the stable and unstable part. The class of hyperbolic operators is not closed under restriction of a semigroup to a closed invariant subspace

$Y \subset X$. On the level of the splitting this means that the subspace Y might lie "diagonally" with respect to the splitting $X = X_{\mathrm{s}} \oplus X_{\mathrm{u}}$. In terms of the spectrum, it might happen that $\sigma(T(1)\restriction_Y) \not\subseteq \sigma(T(1))$. However, for the approximate point spectrum

$$\sigma_{\mathrm{ap}}(T(1)) := \{ \lambda \in \mathbb{C} \colon (\lambda - T(1)) \text{ is not bounded below} \}$$

we have that $\sigma_{\mathrm{ap}}(T(1)\restriction_Y) \subset \sigma_{\mathrm{ap}}(T(1))$. The more general class of restrictions of hyperbolic semigroups is called *quasi-hyperbolic* semigroups and has been studied by Batty and Tomilov in [BT10]. Quasi-hyperbolic semigroups are characterised by the spectral condition

$$\sigma_{\mathrm{ap}}(T(1)) \cap \mathbb{T} = \emptyset.$$

Now, let Y be a closed subspace of X which is invariant under a hyperbolic semigroup T. If $P \in \mathcal{B}(X)$ is the projection onto X_{s} along X_{u}, we can define a seminorm on Y by

$$\|y\|_{\mathrm{s}} := \|Py\| \quad (y \in Y).$$

With respect to this seminorm, the quasi-hyperbolic semigroup $T\restriction_Y$ satisfies the same uniform exponential decay as the hyperbolic semigroup on X_{s}. Similarly, we can define $\|y\|_{\mathrm{u}} := \|(I-P)y\|$ for which T has a lower bound that comes from the norm bound on the inverse of T on X_{u}.

Our intention for defining the maximal parts for some property is to mimic the seminorms $\|\cdot\|_{\mathrm{s}}$ and $\|\cdot\|_{\mathrm{u}}$ in a more general setting and to understand their roles. The exponential type of the stable part of a hyperbolic semigroup is negative, but that is all we can say about it. When we talk about the maximal seminorm for a range of exponential decays, the range has to be closed for maximality to make sense. That is why we will look at contractive parts of semigroups, corresponding to the exponential type 0. Let T now be a general C_0-semigroup on a Banach space X. The contractive part of the semigroup will be determined by a seminorm $\|\cdot\|_{\mathrm{s}}$ on X for which

$$\|T(t)x\|_{\mathrm{s}} \leq \|x\|_{\mathrm{s}} \quad (x \in X, t \geq 0).$$

We will see that we can give an abstract definition for the maximal seminorm with this property, and a direct one using the semigroup T explicitly. From generation theorems we know that a semigroup is contractive if and only if its generator A is dissipative. Dissipativity can also be characterised using the norm, and we can define the maximal dissipative part of an operator A using a seminorm $|||\cdot|||$ with the property

$$|||(\lambda - A)x||| \geq \lambda |||x||| \quad (x \in D(A), \lambda > 0)$$

Again, there is an abstract definition and another one using A explicitly, but this time it is not clear if the two agree.

The dissipative part and the contractive part are related as one would wish. If A is the generator of a C_0-semigroup T then the contractive part of T is generated by the dissipative part of A. In the case when the corresponding seminorms are norms, this brings us back to the generation theorem by Feller, Miyadera and Phillips which extends the Hille-Yosida generation theorem for contraction semigroups. We will also establish a similar connection between the expansive part of a semigroup and the accretive part of its generator. This will make use of a result by Goldberg and Smith from [GS78] which characterises generators of expansive semigroups.

Outline of Chapter 4 In Section 4.1 we introduce maximal parts by looking at the contractivity and expansiveness of a bounded operator. We then apply the same strategy to dissipativity and accretivity of an unbounded operator where we reach a satisfactory description for generators of C_0-semigroups. For this description, we use ideas from invertible extensions in Chapter 3. In Sections 4.4 and 4.5 we find the contractive and expansive part of a C_0-semigroup and relate it to the dissipative and accretive part of the generator. After a short discussion of other examples of maximal parts in Section 4.6, we apply the dissipative and accretive parts to quasi-hyperbolic semigroups, and to generation theorems in Section 4.7.

Chapter 2

Preliminaries

Throughout the thesis, we will generally assume that (normed) vector spaces are defined over the complex numbers and typically denote these spaces by X, Y and Z. We denote the set of bounded linear operators on a normed vector space X by $\mathcal{B}(X)$, and the operators usually by T and S, although sometimes we will use other alphabetically nearby letters due to the presence of more than two operator. To avoid confusion later on, note that we will also denote C_0-semigroups (and other semigroup representations) by T and S. It should always be clear from the context what the letters T and S refer to. We say that a linear map $\pi\colon X \to Y$ *intertwines* two operators $T\colon X \to X$ and $S\colon Y \to Y$ if $\pi T = S\pi$.

We denote the real numbers by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$. The following subsets of $\mathbb{C}$ will be used repeatedly:

$$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\} \qquad \text{(the open unit disc)},$$
$$\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\} \qquad \text{(the unit circle)}.$$

The closure of a subset M of a topological space is denoted by $\overline{M}$. We denote the set of integers by $\mathbb{Z}$ and define the natural numbers as the non-negative integers $\mathbb{N} = \{0, 1, \dots\}$. The characteristic function of a subset $A \subset \Omega$ is the map

$$\chi_A\colon \Omega \to \mathbb{R}, \qquad \chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \quad (x \in \Omega).$$

Section 2.5 on k-times integrated semigroups contains a discussion involving quotient spaces which cannot be found in the literature in this way and to this extent. The reason for this is that the nature of some results in Chapter 3 seems to depend on quotient spaces.

2.1 Unbounded operators and resolvents

An unbounded operator on a Banach space X is a linear map $A\colon D(A) \subset X \to X$. The linear subspace $D(A) \subset X$ is the domain of A. We will denote unbounded operators by A, B and C. We will use two types of spectra, the spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$ where

$$\rho(A) = \{\lambda \in \mathbb{C} \colon \lambda - A \text{ has an inverse in } \mathcal{B}(X)\}$$

is the resolvent set of A, and the approximate point spectrum

$$\sigma_{\mathrm{ap}}(A) = \{\lambda \in \mathbb{C} \colon \lambda - A \text{ is not bounded below}\}.$$

The resolvent of an operator A is the map $R(\cdot, A)\colon \rho(A) \to \mathcal{B}(X)$ where $R(\lambda, A)$ is the (bounded) inverse of $(\lambda - A)$. A resolvent satisfies the resolvent equation

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A) \tag{2.1}$$

for all $\lambda, \mu \in \rho(A)$. Let $\Lambda \subset \mathbb{C}$. A general map $R\colon \Lambda \to \mathcal{B}(X)$ which satisfies Equation (2.1) for $\lambda, \mu \in \Lambda$ is called a pseudo-resolvent. The following can be found in [ABHN01, Appendix B].

Proposition 2.1.1 ([ABHN01, Proposition B.6]). *Let $R\colon \Lambda \to \mathcal{B}(X)$ be a pseudo-resolvent. Then*

1. *The kernel* $\ker R(\lambda)$ *and the image* $\operatorname{ran} R(\lambda)$ *are independent of* $\lambda \in \Lambda$.
2. *There is an operator A on X such that $R(\lambda) = R(\lambda, A)$ for all $\lambda \in \Lambda$ if and only if* $\ker R(\lambda) = \{0\}$.

The spectrum of an operator is related to the spectrum of its resolvent in the following way.

Lemma 2.1.2 ([ABHN01, Proposition B.2]). *Let A be an operator with non-empty resolvent set $\rho(A)$ and let $\mu \in \rho(A)$. Let $\lambda \in \mathbb{C}, \lambda \neq \mu$. Then*

1. *$\lambda \in \rho(A)$ if and only if $(\mu - \lambda)^{-1} \in \rho(R(\mu, A))$,*
2. *$\lambda \in \sigma_{ap}(A)$ if and only if $(\mu - \lambda)^{-1} \in \sigma_{ap}(R(\mu, A))$.*

Let us introduce a few more notions for an operator A. We say that

- A is *closed* if its graph $\mathcal{G}(A) = \{(x, Ax) \colon x \in D(A)\}$ is closed in $X \times X$.
- A is *densely defined* if $D(A)$ is dense in X.

- A is *sectorial* if $(-\infty, 0) \subset \rho(A)$ and $\sup_{\lambda<0} \|\lambda R(\lambda, A)\| < \infty$.

If A has non-empty resolvent set $\rho(A)$, then A is closed. For more information on sectorial operators, see [Haa06]. Let A be a sectorial operator and $x \in D(A)$. Since $x = R(\lambda, A)(\lambda - A)x$ for $\lambda \in \rho(A)$ we have that

$$\|x - \lambda R(\lambda, A)x\| \leq \|R(\lambda, A)\|\|Ax\| \to 0 \quad \text{as } \lambda \to -\infty \ (\lambda < 0). \tag{2.2}$$

By uniform boundedness of the operators $\lambda R(\lambda, A)$ $(\lambda < 0)$, the convergence in (2.2) holds for all $x \in \overline{D(A)}$. Conversely, assume that $(-\infty, 0) \subset \rho(A)$ and $\lambda R(\lambda, A)x \to x$ as $\lambda \to -\infty$ $(\lambda < 0)$ for all $x \in X$. Then A is densely defined and by the uniform boundedness theorem, $\sup_{\lambda<0} \|\lambda R(\lambda, A)\| < \infty$. Hence A is densely defined and sectorial. We will use the convergence in (2.2) in Section 4.2.2 for densely defined operators.

Let A be a densely defined operator with non-empty resolvent set. The inclusion $\iota\colon D(A^{n+1}) \to D(A^n)$ is continuous in the graph norm $\|x\|_{A^n} = \|x\| + \|A^n x\|$ on $D(A^n)$ for $n \geq 0$. Moreover, ι has dense range. This follows since each $x \in D(A^n)$ can be written as $x = R(\lambda, A)^n y$ for some $y \in X$ and $\lambda \in \rho(A)$, and any sequence $y_k \in D(A)$ with

$$\|y - y_k\| \to 0 \quad \text{as } k \to \infty$$

gives a sequence $x_k = R(\lambda, A)^n y_k \in D(A^{n+1})$ with

$$\|x - x_k\|_{A^n} \to 0 \quad \text{as } k \to \infty.$$

This implies by the Mittag-Leffler Theorem (see [Run05, Theorem 2.4.14], for example) that

$$D(A^\infty) := \bigcap_{n \geq 0} D(A^n)$$

is dense in X.

If A is an operator on a Banach space X and $Y \subset X$ is a subspace we define the part of A in Y to be the operator

$$A_Y y = Ay, \qquad D(A_Y) = \{\, y \in Y \cap D(A)\colon Ay \in Y \,\}.$$

2.2 C_0-semigroups

Strongly continuous semigroups, or C_0-semigroups, are a central element in this thesis. Two standard references are [EN00] or [ABHN01]. Consult these for more information.

A C_0-semigroup T on a Banach space X is a strongly continuous representation of $\mathbb{R}_+ = [0, \infty)$. That means, it is a map $T\colon \mathbb{R}_+ \to \mathcal{B}(X)$ such that

$$\begin{aligned} T(0) &= I, \\ T(s)T(t) &= T(s+t) \quad (t, s \geq 0) \end{aligned}$$

and such that $t \mapsto T(t)x$ is continuous for each $x \in X$. We shall sometimes say semigroup where we mean C_0-semigroup, without creating ambiguity. A semigroup T has a *generator* by which it is uniquely determined. Its generator is the operator A given by

$$\begin{aligned} D(A) &:= \{x \in X \colon t \mapsto T(t)x \text{ is right-differentiable at } t = 0\}, \\ Ax &:= \lim_{t \downarrow 0} \frac{1}{t}(T(t)x - x). \end{aligned}$$

The generator of a C_0-semigroup is closed and densely defined. A C_0-semigroup T satisfies the growth estimate $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ for some $\omega \in \mathbb{R}, M \geq 1$ and the infimum of these ω is the *growth bound* $\omega_0(T)$. The resolvent $R(\lambda, A)$ of A is given by the Laplace transform

$$R(\lambda, A)x = \mathcal{L}Tx(\lambda) := \int_0^\infty e^{-\lambda t} T(t)x \,\mathrm{d}t \quad (x \in X, \operatorname{Re}\lambda > \omega_0(T)).$$

The set $D(A^\infty)$ is a *core* for A, that is it is dense in $D(A)$ in the graph norm $\|\cdot\|_A$. We say that an operator U commutes with a semigroup T if $UT(t) = T(t)U$ for all $t \geq 0$. For commuting operators, we have the following easy lemma.

Lemma 2.2.1. *Let A be the generator of a C_0-semigroup T on a Banach space X. An operator $U \in \mathcal{B}(X)$ commutes with T if and only if it commutes with the resolvent of A.*

Proof. We have $UT(t) = T(t)U$ for all $t \geq 0$. Take the Laplace transform on both sides of this equation to see that it holds if and only if $UR(\lambda, A) = R(\lambda, A)U$ for $\lambda > \omega_0(T)$. From the resolvent identity it follows that

$$R(\lambda, A) = (I + (\mu - \lambda)R(\lambda, A))R(\mu, A)$$

for all $\lambda, \mu \in \rho(A)$. If U commutes with $R(\lambda, A)$ we get that

$$(I + (\mu - \lambda)R(\lambda, A))UR(\mu, A) = (I + (\mu - \lambda)R(\lambda, A))R(\mu, A)U.$$

The first factors on both sides are equal and injective on $D(A)$. Thus, we get $UR(\mu, A) = R(\mu, A)U$ for all $\mu \in \rho(A)$. So an operator commutes with one resolvent operator if and only if it commutes with all resolvent operators. □

For a generator A, the following lemma expresses the density of $D(A)$ in X in terms of the operator norm. Note that $\operatorname{ran} R(\lambda, A) = D(A)$ for $\lambda \in \rho(A)$.

Lemma 2.2.2. *Let T be a C_0-semigroup on X and let $\lambda > \omega_0(T)$. Let A be the generator of T. Then for $t \geq 0$*

$$\|e^{-\lambda t}T(t)R(\lambda, A) - R(\lambda, A)\| \leq \int_0^t \|e^{-\lambda s}T(s)\| \, \mathrm{d}s \tag{2.3}$$

and

$$\left\| \frac{e^{-\lambda t}T(t)+I}{t} R(\lambda, A)^2 - R(\lambda, A) \right\| \leq \frac{1}{t} \int_0^t (t-s)\|e^{-\lambda s}T(s)\| \, \mathrm{d}s. \tag{2.4}$$

Proof. We can rescale the semigroup T and use $e^{-\lambda t}T(t)$ to assume that $\lambda = 0$. Then $0 \in \rho(A)$ and $R(0, A) = -A^{-1}$. Let $x \in X$. The derivative of $T(t)A^{-1}x$ in t is $T(t)x$, so

$$\begin{aligned} \|\big(T(t)A^{-1} - A^{-1}\big)x\| &= \left\| \int_0^t T(s)x \, \mathrm{d}s \right\| \\ &\leq \int_0^t \|T(s)\| \, \mathrm{d}s \cdot \|x\| \end{aligned}$$

for all $x \in X$ and $t \geq 0$. This proves (2.3). In the same way, we get

$$\begin{aligned} \left\| \Big(\frac{T(t)A^{-1} - A^{-1}}{t} A^{-1} - A^{-1} \Big) x \right\| &= \left\| \frac{1}{t} \int_0^t T(s)A^{-1}x \, \mathrm{d}s - A^{-1}x \right\| \\ &\leq \frac{1}{t} \int_0^t \|(T(s)A^{-1} - A^{-1})x\| \, \mathrm{d}s \\ &\leq \frac{1}{t} \int_0^t \left(\int_0^s \|T(u)x\| \, \mathrm{d}u \right) \mathrm{d}s \\ &= \frac{1}{t} \int_0^t (t-s)\|T(s)x\| \, \mathrm{d}s. \end{aligned}$$

This shows (2.4). □

2.3 Dissipative and accretive operators

Dissipative operators are of interest in the theory of C_0-semigroups. We will give two equivalent definitions, present a generation theorem for contractive C_0-semigroups and introduce the Cayley transform of an operator. For more detailed information see [ABHN01, p.140] and [EN00, II.3.23 and III.2.7].

Definition 2.3.1. An operator A on a Banach space X is *dissipative* if

$$\|(\lambda - A)x\| \geq \lambda \|x\| \quad (x \in D(A), \lambda > 0).$$

We say that A is *accretive* if $-A$ is dissipative.

The following proposition gives an alternative formulation for dissipative operators. It makes use of norming functionals of a vector $x \in X$ which make up the set

$$\mathcal{J}(x) = \{ \varphi \in X' \colon \varphi(x) = \|x\|^2 = \|\varphi\|^2 \}.$$

Proposition 2.3.2 ([EN00, Proposition II.3.23])**.** *An operator A is dissipative if and only if for each $x \in D(A)$ there is $f \in \mathcal{J}(x)$ such that*

$$\operatorname{Re} f(Ax) \leq 0.$$

If A is the generator of a C_0-semigroup of contractions, this holds for all $f \in \mathcal{J}(x)$.

This proposition shows that the generator a C_0-semigroup of contractions is dissipative. With an additional assumption on a dissipative operator, the converse holds as well. The following theorem is known as the Lumer-Phillips theorem, see [EN00, Theorem II.3.15].

Theorem 2.3.3 (Lumer-Phillips)**.** *Let A be a densely defined operator on X. Then A generates a C_0-semigroup of contractions on X if and only if A is dissipative and $(\lambda - A)D(A) = X$ for some $\lambda > 0$.*

For an operator A for which $(I - A)$ is injective, we can define the *Cayley transform.* It is given by the linear fractional transformation

$$C = (I + A)(I - A)^{-1}, \quad D(C) = \operatorname{ran}(I - A).$$

In particular, C exists if A is dissipative. Moreover, if $1 \in \rho(A)$ then C is a bounded operator with $D(C) = X$. For Cayley transforms, not much information can be found in the books cited above. Instead refer to [EZ08] and [Dav07, Section 10.4], which contain also the following standard fact.

Proposition 2.3.4. *Let A be an operator on a Banach space such that $(I - A)$ is injective. Then A is dissipative if and only if its Cayley transform C is contractive.*

2.4 Generation theorems

We already saw one generation theorem for C_0-semigroups of contractions. There is another one, kown as the Hille-Yosida theorem, which states the following (see for example [EN00, Theorem II.3.5]).

Theorem 2.4.1 (Hille-Yosida). *An operator A is the generator of a C_0-semigroup of contractions if and only if A is densely defined, $(0, \infty) \subset \rho(A)$ and*

$$\|\lambda R(\lambda, A)\| \leq 1$$

for all $\lambda > 0$.

The proof of this theorem is not relevant here. Instead we are more interested in its relation to the next result. The Hille-Yosida theorem has the following consequence, the proof being taken from [EN00, Theorem II.3.8]. In Section 4.2.2, we will use the idea of its proof in a new construction.

Theorem 2.4.2 (Feller-Miyadera-Phillips). *An operator A is the generator of a C_0-semigroup with $\|T(t)\| \leq M$ for all $t \geq 0$ if and only if A is densely defined, $(0, \infty) \subset \rho(A)$ and*

$$\|\lambda^n R(\lambda, A)^n\| \leq M \tag{2.5}$$

for all $\lambda > 0$ and $n \geq 1$.

For this theorem, we only sketch the proof to show the important non-technical steps. We shall only look at the statement that an operator A which satisfies the resolvent bounds generates a bounded C_0-semigroup.

Sketch of Proof. Here, we shall only show that an operator A satisfying (2.5) generates a C_0-semigroup. Assume that A is an operator that satisfies $\|\lambda^n R(\lambda, A)^n\| \leq M$ for all $\lambda > 0$ and $n \geq 1$. For each $\mu > 0$, define the new norm

$$\|x\|_\mu := \sup_{n \geq 0} \|\mu^n R(\mu, A)^n x\| \quad (x \in X).$$

These norms are equivalent with $\|\cdot\| \leq \|\cdot\|_\mu \leq M\|\cdot\|$. It can be shown that the operators $\lambda R(\lambda, A)$ are contractive in $\|\cdot\|_\mu$ for $0 < \lambda \leq \mu$ and the norms increase in μ, so $\|x\|_\lambda \leq \|x\|_\mu$ for $0 < \lambda \leq \mu$ and $x \in X$. The supremum

$$|||x||| := \sup_{\mu > 0} \|x\|_\mu \quad (x \in X)$$

is equal to the limit $\lim_{\mu \to \infty} \|x\|_\mu$ and is a norm for which $\|x\| \leq |||x||| \leq M\|x\|$ and $|||\lambda R(\lambda, A)x||| \leq |||x|||$ for all $x \in X, \lambda > 0$. So Theorem 2.4.1 applies. □

The Hille-Yosida theorem can also be used to characterise C_0-group generators. An operator A generates a C_0-group if and only if A and $-A$ generate a C_0-semigroup. On Hilbert space, Zwart showed in [Zwa01] that it is sufficient to show that A is a C_0-semigroup generator and in addition a (weaker) resolvent estimate is satisfied. The result says the following.

Theorem 2.4.3. *Let A be the generator of a C_0-semigroup on a Hilbert space X. Then A generates a C_0-group if and only if there is $r < 0$ such that $\{\lambda\colon \operatorname{Re}\lambda < r\} \subset \rho(A)$ and for all $\alpha > 0$ there is $r_\alpha < r$ such that*

$$\sup_{\operatorname{Re}\lambda < r_\alpha} e^{\alpha \operatorname{Re}\lambda}\|R(\lambda, A)\| < \infty.$$

The following result is a theorem by Goldberg and Smith which describes generators of C_0-semigroups which have a lower bound ([GS78]).

Theorem 2.4.4 (Goldberg-Smith)**.** *Let A be the generator of a C_0-semigroup T on a Banach space X. The following are equivalent.*

1. $\|T(t)x\| \geq \|x\|$ *for all* $x \in X, t \geq 0$.
2. $\|(\lambda - A)x\| \geq -\lambda\|x\|$ *for all* $x \in D(A), \lambda < 0$.

2.5 Integrated semigroups

Some of the results in Sections 3.3 and 3.4 are for generators of C_0-semigroups. The statements in the results show no obvious reason why they should not hold for a wider class of operators, such as operators with non-empty resolvent set. We try to obtain these results for a class of operators which is more general than that of C_0-semigroup generators, and choose to work with generators of integrated semigroups. This class contains C_0-semigroup generators, and is contained in, for example, the class of generators of distribution semigroups. The key to our results for semigroup generators is Proposition 2.6.2, which allows us to pass to a quotient space. We try to substitute it by something similar. Although the results obtained here do not seem to work in the proofs of Chapter 3, they demonstrate which difficulties have to be overcome when trying to give a similar proof for this wider class.

An interesting result relating integrated semigroups with C_0-semigroups is the sandwich theorem which is cited as Proposition 2.5.11 below, and for which we verify some additional properties here. It can be used in combination with results for C_0-semigroups. The theorem itself gives a continuous embedding from an integrated semigroup to a C_0-semigroup. However, there is no obvious way of extracting the integrated semigroup from the C_0-semigroup, say by fixing the space on which it should be defined.

The motivation behind integrated semigroups is the following (see [ABHN01, Sections 3.1 and 3.2]). A C_0-semigroup solves the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t) \quad (t \geq 0) \\ u(0) = x \end{cases}$$

and $u(t) = T(t)x$ is a weak solution for all $x \in X$, while integrated semigroups solve the integrated version

$$\begin{cases} v'(t) = Av(t) + \frac{t^{k-1}}{k-1!}x \quad (t \geq 0) \\ v(0) = 0, \end{cases}$$

of the Cauchy problem, where $k \geq 1$. If T is a C_0-semigroup,

$$S(t)x := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} T(s)x \,\mathrm{d}s \quad (x \in X, t \geq 0)$$

defines a k-times integrated semigroup. However, there are k-times integrated semigroups S for which there is no semigroup T such that this relation holds (see [ABHN01, Corollary 3.9.14]). A detailed discussion of k-times integrated semigroups and their properties can be found in [ABHN01, Section 3.2]. Note that we also allow the degenerate case in the definition here.

Definition 2.5.1. Let $S\colon \mathbb{R}_+ \to \mathcal{B}(X)$ be a strongly continuous function such that

$$\left\| \int_0^t S(s)x \,\mathrm{d}s \right\| \leq Me^{\omega t}\|x\| \quad (x \in X, t \geq 0) \tag{2.6}$$

for some $M \geq 1$ and $\omega \geq 0$. Let $k \geq 1$. We say S is a *(degenerate) k-times integrated semigroup* if $S(0) = 0$ and

$$R(\lambda)x = \lambda^k \int_0^\infty e^{-\lambda t} S(t)x \,\mathrm{d}t \quad (x \in X, \lambda > \omega) \tag{2.7}$$

is a pseudo-resolvent. Moreover, the pseudo-resolvent is a resolvent of an operator A if and only if $S(t)x = 0\,(t \geq 0)$ implies $x = 0$. In that case, S is *non-degenerate* and A is its generator.

The next proposition is a helpful characterisation of integrated semigroups.

Proposition 2.5.2 ([ABHN01, Proposition 3.2.4]). *Let $S\colon \mathbb{R}_+ \to \mathcal{B}(X)$ be strongly continuous and such that (2.6) holds for some $M \geq 1$ and $\omega \geq 0$. Let $k \geq 1$. Then S is a k-times integrated semigroups if and only if*

$$S(t)S(s) = \frac{1}{k!}\left[\int_t^{t+s} (s+t-r)^{k-1} S(r)\,dr - \int_0^s (s+t-r)^{k-1} S(r)\,dr\right] \qquad (2.8)$$

and it is non-degenerate if and only if $S(t)x = 0$ for all $t \geq 0$ implies $x = 0$.

Remark 2.5.3. Some definitions of k-times integrated semigroups allow the case $k = 0$. This only makes sense when we do not assume that $S(0) = 0$. Otherwise, $S(t) = S(t)S(0) = 0$ $(t \geq 0)$ as can be seen from the proof of [ABHN01, Theorem 3.1.7]. On the other hand, if we only work with non-degenerate S and assume that the function R in Definition 2.5.1 is the resolvent $R(\lambda, A)$ for some operator A and $k \geq 1$, then we get $S(0) = 0$. This argument can be found in [ABHN01, Lemma 3.2.2].

Let us consider the following question. Let $k \geq 1$ and let S be a non-degenerate k-times integrated semigroup on a Banach space X. For which closed subspaces $Y \subset X$ is $\hat{S}$ defined by

$$\hat{S}(t)(x+Y) := S(t)x + Y \quad (x \in X, t \geq 0) \qquad (2.9)$$

a non-degenerate k-times integrated semigroup on $X\,/\,Y$? To have that $\hat{S}$ is well-defined we need at least that Y is invariant under $S(t)$ for all $t \geq 0$. This is also sufficient to obtain a degenerate k-times integrated semigroup on the quotient space, but we require more to obtain a non-degenerate k-times integrated semigroup as we will see in the next proposition. Here, $q\colon X \to X\,/\,Y$ is the quotient map.

Proposition 2.5.4. *Let A be the generator of a k-times integrated semigroup S on a Banach space X for some $k \geq 1$. Let $Y \subset X$ be a closed subspace that is invariant under $S(t)$ for all $t \geq 0$. Assume further that $S(t)x \in Y$ $(t \geq 0)$ implies $x \in Y$.*
Then the strongly continuous function $\hat{S}$ given by (2.9) on the quotient $X\,/\,Y$ is again a k-times integrated semigroup and has generator $\hat{A}$ with $\hat{A}q(x) = q(Ax)$ and $D(\hat{A}) = qD(A)$.

Proof. Strong continuity of $\hat{S}$ on the quotient $Z = X\,/\,Y$ follows from contractivity of q. Indeed, for an element $q(x) \in Z$ in the quotient, we have

$$\left\|\hat{S}(t)q(x) - \hat{S}(s)q(x)\right\| = \|q(S(t)x - S(s)x)\| \leq \|S(t)x - S(s)x\| \quad (t, s \geq 0),$$

and S is known to be strongly continuous on X.

To see that $\hat{S}$ is a k-times integrated semigroup on Z, the characterisation from Proposition 2.5.2 can be used. We have that S satisfies (2.8) for some $M \geq 1, \omega \geq 0$, since S is an integrated semigroup. Now S satisfies (2.8). Since q is contractive, $\hat{S}$ satisfies the estimate $\|\int_0^t \hat{S}(s)\,\mathrm{d}s\| \leq Me^{\omega t}$ $(t \geq 0)$ and $\hat{S}$ also satisfies (2.8). The property that $\hat{S}(t)q(x) = 0$ $(t \geq 0)$ implies $q(x) = 0$ follows from the additional assumption in the proposition. The last assertion about $\hat{A}$ follows from the integral representation (2.7) of its resolvent. □

We note the following about this proposition. If $\hat{S}$ is non-degenerate, we necessarily get the implication $S(t)x \in Y$ $(t \geq 0) \Rightarrow x \in Y$. For a C_0-semigroup T, this is always satisfied if the closed subspace Y is invariant under T, since then $x = \lim_{t\to 0} T(t)x \in Y$. In the next example, we use Proposition 2.5.4 to identify the relevant subspaces for dual integrated semigroups of C_0-semigroups.

Example 2.5.5. Let X be a Banach space and X' be its dual space. Let T be a C_0-semigroup on X with generator A. If X is reflexive, A' is densely defined and generates the C_0-semigroup $T(t)'$ on X'. In general, we have that A' generates the once integrated semigroup S given by $S(t)f = \int_0^t T'(s)f\,\mathrm{d}s$ $(t \geq 0, f \in X')$. For a closed subspace $Y \subset X$ we have the isomorphisms $Y' \simeq X' \,/\, Y^\perp$ and $(X \,/\, Y)' \simeq Y^\perp$. We find a similar relation between the invariant subspaces of S and T.

We can relate the invariant subspaces for S and for T as follows. If a subspace $W \subset X'$ is invariant under S in the sense of Proposition 2.5.4 then

$$W_\perp = \{x \in X \colon f(x) = 0 \text{ for all } f \in W\}$$

is invariant under T. On the other hand, the subspaces

$$Y^\perp = \{f \in X' \colon f(x) = 0 \text{ for all } x \in Y\}$$

are S-invariant in this sense if and only if $Y \subset X$ is T-invariant. To see this, let $t > 0, x \in X$ and $f \in X'$. Then

$$f\left(\int_0^t T(s)x\,\mathrm{d}s\right) = \int_0^t f(T(s)x)\,\mathrm{d}s = \int_0^t (T'(s)f)(x)\,\mathrm{d}s = (S(t)f)(x).$$

So if Y is invariant under T, this shows that $Y^\perp$ is invariant under S. Assume now that $x \in W_\perp$ and $f \in W$ for some S-invariant subspace W. Note that $\frac{1}{t}\int_0^t T(s)x\,\mathrm{d}s \to x$ as $t \to 0$. We see that

$$0 = \frac{1}{s}(S(t+s)f(x) - S(t)f(x)) = f\left(\frac{1}{s}\int_t^{t+s} T(r)x\,\mathrm{d}r\right) \to f(T(t)x)$$

as $s \to 0$. This shows that $W_\perp$ is invariant under T. It is not known whether $W = (W_\perp)^\perp$ for all such invariant W.

We saw in Proposition 2.5.4 that we need more than just invariance of the subspace under the integrated semigroup, in order to get a non-degenerate integrated semigroup after taking the quotient with respect to that subspace. But we would still get a degenerate integrated semigroup on the quotient if we just assume that the subspace is invariant. Let us consider degenerate integrated semigroups. The *degeneration space* is defined in [Thi90] as the subspace

$$N = \{x \in X : S(t)x = 0 \text{ for all } t \geq 0\}.$$

Example 2.5.6 (degenerate integrated semigroups)**.** Let $k \geq 2$ and define S as the matrix valued function

$$S(t) = \begin{pmatrix} 0 & t^{k-1} \\ 0 & 0 \end{pmatrix} \quad (t \geq 0)$$

Then $S(0) = 0$. The function $R(\lambda) = \lambda^k (\mathcal{L}S)(\lambda)$ $(\operatorname{Re} \lambda > 0)$ is the pseudo-resolvent with entire extension

$$R(\lambda) = \begin{pmatrix} 0 & (k-1)! \\ 0 & 0 \end{pmatrix} \quad (\lambda \in \mathbb{C}).$$

Thus S is a degenerate k-times integrated semigroup. Its degeneration space is $N = \mathbb{C} \cdot \binom{1}{0}$ and $\hat{S}(t) = 0$ on the (non-trivial) quotient space $\mathbb{C}^2 \,/\, N$. On $\mathbb{C}^n$, we have the following pseudo-resolvent with its degenerate twice integrated semigroup.

$$R_n(\lambda) = \begin{pmatrix} 0 & \lambda & -\lambda^2 & \cdots & (-\lambda)^{n-1} \\ 0 & 0 & \lambda & \cdots & (-\lambda)^{n-2} \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & \lambda \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad S_n(t) = \begin{pmatrix} 0 & t & -\frac{t^2}{2} & \cdots & \frac{(-t)^{n-1}}{(n-1)!} \\ 0 & 0 & t & \cdots & \frac{(-t)^{n-2}}{(n-2)!} \\ \vdots & \vdots & & \ddots & \vdots \\ & & & & t \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The integrated semigroup $\hat{S}_n$ on the quotient $\mathbb{C}^n \,/\, N$ is given by S_{n-1} for $n \geq 1$.

This example shows that dividing out the degeneration space of a k-times integrated semigroup does not give us a non-degenerate integrated semigroup if $k \geq 2$. Nor does it help if we repeat this procedure a (k-dependent) fixed number of times. However, the space N can be (sensibly) divided out for once-integrated semigroups. The following results show that Example 2.5.6 summarizes well what is missing when $k \geq 2$.

Lemma 2.5.7. *Let $f\colon \mathbb{R}_+ \to \mathbb{C}$ be a continuous function. Assume there is $k \geq 1$ such that*

$$\int_t^{t+s} (s+t-r)^{k-1} f(r)\,\mathrm{d}r = \int_0^s (s+t-r)^{k-1} f(r)\,\mathrm{d}r \tag{2.10}$$

for all $t, s \geq 0$. Then $f(t) = ct^{k-1}$ for some constant $c \in \mathbb{C}$.

Proof. The integrands in (2.10) are continuous in (r, s) and differentiable in s, so we can differentiate both sides with respect to s. For $k \geq 2$ this gives

$$(k-1)\left(\int_t^{t+s}(s+t-r)^{k-2}f(r)\,\mathrm{d}r - \int_0^s (s+t-r)^{k-2}f(r)\,\mathrm{d}r\right) = t^{k-1}f(s).$$

In particular, f is continuously differentiable. The left hand side again has the form of (2.10) and we can differentiate another $k-2$-times with respect to s. This results in

$$\begin{aligned}&(k-1)!\left(\int_t^{t+s} f(r)\,\mathrm{d}r - \int_0^s f(r)\,\mathrm{d}r\right)\\ &= t^{k-1}f^{(k-2)}(s) + (k-1)t^{k-2}f^{(k-3)}(s) + \cdots + (k-1)\ldots 3t^2 f'(s) + (k-1)!tf(s).\end{aligned}$$

Differentiating once more and rearranging the terms finally shows that f is a polynomial of degree $k-1$.

$$f(s+t) = t^{k-1}\frac{f^{(k-1)}(s)}{(k-1)!} + t^{k-2}\frac{f^{(k-2)}(s)}{(k-2)!} + \cdots + t^2\frac{f''(s)}{2} + tf'(s) + f(s).$$

Using this with $s=0$ we can write $f(r) = \sum_{i=0}^{k-1} a_i r^i$ with $a_i = \frac{f^{(i)}(s)}{i!}$. Inserting in (2.10) leads to

$$\sum_{i=0}^{k-1} a_i \int_t^{t+s}(s+t-r)^{k-1}r^i\,\mathrm{d}r = \sum_{i=0}^{k-1} a_i \int_0^s (s+t-r)^{k-1}r^i\mathrm{d}r.$$

Substitute $u = t+s$ to rewrite the equation as

$$\int_{u-s}^{u}\sum_{i=0}^{k-1} a_i(u-r)^{k-1}r^i\,\mathrm{d}r = \int_0^s \sum_{i=0}^{k-1} a_i(u-r)^{k-1}r^i\,\mathrm{d}r.$$

Since this holds for all $s \in [0, u]$, the integrand has to be invariant under the substitution $r \mapsto u-r$. This shows that $a_i = 0$ for $i = 0, \ldots, k-2$, and $a_{k-1} \in \mathbb{C}$ is arbitrary. Hence, the result follows. □

The following Proposition gives insight into the nature of solutions with higher regularity. If a function comes from the k-times integrated version of a C_0-semigroup, then the first $k-1$ derivatives have to vanish at $t=0$. This is true for non-degenerate semigroups in general as the following Proposition shows, and it motivates Proposition 2.5.9.

Proposition 2.5.8. *Let S be a non-degenerate k-times integrated semigroup on a Banach space X for some $k \geq 1$. Let $x \in X$ be such that $t \mapsto S(t)x$ is $(k-1)$-times differentiable. Then $S^{(n)}(0)x = 0$ for $n = 0, 1, \ldots, k-1$.*

Proof. Let A be the generator of S. By [ABHN01, Lemma 3.2.2.d)] we have that $A\int_0^t S(s)x\,\mathrm{d}s = S(t)x - \frac{t^k}{k!}x$ for all $x \in X$. Using that A is closed we get that

$$AS^{(n-1)}(t)x = S^{(n)}(t)x - \frac{t^{k-n}}{(k-n)!}x$$

for $n = 1, 2, \dots, k-1$. Evaluating at $t = 0$ gives

$$AS^{(n-1)}(0)x = S^{(n)}(0)x.$$

Now we can use that $S(0) = 0$ by definition and get $S^{(n)}(0)x = 0$ recursively. □

Proposition 2.5.9. *Let S be a (degenerate) k-times integrated semigroup on a Banach space X for some $k \geq 1$. Assume that $S^{(k-1)}(0)x = 0$ whenever $t \mapsto S(t)x$ is $(k-1)$-times continuously differentiable. Let N be the degeneration space of S. Then the operators $\hat{S}(t)$ defined in* (2.9) *form a non-degenerate k-times integrated semigroup on $X \,/\, N$.*

Proof. The operators $\hat{S}(t)$ clearly still satisfy the composition property (2.8) of a k-times integrated semigroup. So we only have to show that $\hat{S}(t)(x + N) = N$ for all $t \geq 0$ implies $x \in N$. Pick an $x \in X$ with $S(t)x \in N$ for all $t \geq 0$. This means that $S(t)S(s)x = 0$ for all $t, s \geq 0$. From (2.8) we see that $f(S(\cdot)x)$ satisfies the assumptions of Lemma 2.5.7 for any $f \in X'$. So we have $f(S(t)x) = c_f t^{k-1}$ $(t \geq 0, f \in X')$ for some $c_f \in \mathbb{C}$ which in turn gives $S(t)x = t^{k-1}y$ $(t \geq 0)$ with $y = S(1)x$. The assumption gives $0 = S^{(k-1)}(0)x = (k-1)!y$ so that $S(t)x = 0$ $(t \geq 0)$, or $x \in N$. □

Corollary 2.5.10. *Let S be a once integrated semigroup on a Banach space X. Let N be its degeneration space. Then $\hat{S}$ defined by* (2.9) *is a non-degenerate once integrated semigroup on $X \,/\, N$.*

Proof. Since $S(t)x$ is continuous for all $x \in X$ and $S(0) = 0$ by definition, the assumptions of Proposition 2.5.9 are satisfied. Hence $\hat{S}$ is non-degenerate. □

Let S be a non-degenerate once integrated semigroup on a Banach space X. If $Y \subset X$ is a closed, invariant subspace then $\hat{S}$ on $X \,/\, Y$ is a degenerate once integrated semigroup with degeneration space N. By Corollary 2.5.10, $\hat{\hat{S}}$ on $Z = (X \,/\, Y) \,/\, N$ is a non-degenerate once integrated semigroup. It would be interesting to know if we can have that $N = X \,/\, Y$ for some $Y \neq X$.

Finally, let us state some properties of the embedding from the Sandwich Theorem [ABHN01, Theorem 3.10.4].

Proposition 2.5.11. *Let A be the generator of a (non-degenerate) k-times integrated semigroup S on a Banach space X with $\|S(t)\| \leq Me^{\omega t}$ for some $M \geq 1, \omega \geq 0$. There exists a Banach space Y and a C_0-semigroup generator B on Y such that*

1. $D(B^k) \subset X \hookrightarrow Y$,
2. $R(\lambda, B)X \subset X$ *for some* $\lambda \in \rho(B)$,
3. $A = B_X$.

Further, we have $\rho(B) \supset \rho(A)$. If A satisfies $\|Ax\|_X \geq c\|x\|_X$ $(x \in D(A))$ then $\|By\|_Y \geq c\|y\|_Y$ $(y \in D(B))$.

Proof. Statements 1, 2, 3 are shown in [ABHN01]. Define

$$T(t)x := S(t)A^k x + \frac{t^{k-1}}{(k-1)!}A^{k-1}x + \cdots + tAx + x$$

for $x \in D(A^k)$ and $t \geq 0$. The space Y is the completion of X under the new norm

$$\|x\|_Y := \sup_{t \geq 0} \|e^{-bt}T(t)R(\mu_0, A)^k x\| \quad (x \in X)$$

where $\mu_0 > b > \omega$ are fixed. The resolvent operators $R(\lambda, A)$ can be extended to operators $R(\lambda)$ on the space Y since they are bounded in the new norm. This holds, since for $x \in X, t \geq 0$ and $\lambda \in \rho(A)$,

$$\|e^{-bt}T(t)R(\mu_0, A)^k R(\lambda, A)x\| \leq \|R(\lambda, A)\| \|e^{-bt}T(t)R(\mu_0, A)^k x\|.$$

Moreover, these operators form a pseudo-resolvent. It was shown in [ABHN01, Theorem 3.10.4] that $(R(\lambda))_{\lambda > b}$ is the resolvent of B, so we must have $\rho(A) \subset \rho(B)$. Now assume that A has a lower bound $c > 0$. For $\lambda \in \rho(A)$ we have

$$c\|R(\lambda, A)x\| \leq \|AR(\lambda, A)x\| = \|\lambda R(\lambda, A)x - x\| \quad (x \in X)$$

and thus

$$c\|R(\lambda, B)x\|_Y \leq \|\lambda R(\lambda, B)x - x\|_Y = \|BR(\lambda, B)x\|_Y \quad (x \in X).$$

By density of X in Y this holds for all $x \in Y$. Since $R(\lambda, B)$ maps Y onto $D(B)$ we thus see

$$c\|y\|_Y = c\|R(\lambda, B)(\lambda - B)y\|_Y \leq \|By\|_Y \quad (y \in D(B)).$$

□

2.6 Sums, quotients and sequence spaces

Let us recall some elementary facts about operators on direct sums and quotients of Banach spaces. For a closed subspace Y of a Banach space X, denote the quotient map by $q\colon X \to X\,/\,Y$.

Proposition 2.6.1. *Let T be a bounded operator on a Banach space X and assume $Y \subset X$ is a closed invariant subspace. Then $\hat{T}q(x) := q(Tx)$ $(x \in X)$ is well-defined on the quotient $X\,/\,Y$ and $\|\hat{T}\| \leq \|T\|$.*

This proposition does not generalise to all unbounded operators, however, we get the following for generators of semigroups. We will need it in Sections 3.3 and 3.4.

Proposition 2.6.2 ([EN00, I.5.13 and II.2.4])**.** *Let T be a C_0-semigroup on a Banach space X with generator A. Assume $Y \subset X$ is a closed, T-invariant subspace, that is Y is invariant under each $T(t)$ $(t \geq 0)$. Then*

$$\hat{T}(t)q(x) := q(T(t)x) \quad (x \in X, t \geq 0)$$

is a C_0-semigroup on $X\,/\,Y$ and its generator $\hat{A}$ is given by $\hat{A}q(x) = q(Ax)$ with $qD(A) = D(\hat{A})$. In particular, $\hat{R}(\lambda, A) = R(\lambda, \hat{A})$ $(\lambda \in \rho(A))$ is a resolvent.

The next example illustrates that, in general, taking the quotient with respect to a closed invariant subspace does not preserve resolvents. If we have a resolvent on X and Y is a closed subspace invariant under the resolvent, we might only get a pseudo-resolvent on $X\,/\,Y$. This happens for the resolvent of a bounded operator T, for example, if Y is not invariant under T.

Example 2.6.3. Let T be the right shift on $X = \ell_1(\mathbb{Z})$. Then T is invertible, its inverse is given by the left shift $T^{-1} = L$. The subspace $Y = \ell_1(\mathbb{Z}_{<0})$ is invariant under L but not under T. The operator $\hat{L}$ on the quotient $X\,/\,Y$ is isometrically equivalent to the (left) shift on $\ell_1(\mathbb{N})$, thus it is not invertible. Moreover, the resolvent of T becomes a pseudo-resolvent on $X\,/\,Y$ with non-trivial (1-dimensional) kernel $\ker \hat{L}$.

We use the shorthand notation $\ell_p = \ell_p(\mathbb{N})$ and for a Banach space X we write $\ell_p(X) = \ell_p(\mathbb{N}, X)$ for $1 \leq p \leq \infty$.

Proposition 2.6.4. *Let A be an (unbounded) operator on a Banach space X. For $1 \leq p < \infty$, define B on $\ell_p(X)$ by $B(x_n) := (Ax_n)$ with domain $D(B) = \{(x_n) \in \ell_p(X)\colon x_n \in D(A), (Ax_n) \in \ell_p(X)\}$. If $D(A)$ is dense in X, $D(B)$ is dense in $\ell_p(X)$.*

Proof. The space of finite sequences with values in $D(A)$, $c_{00}(D(A))$, is dense in $\ell_p(X)$. Also, $c_{00}(D(A))$ is contained in $D(B)$. Hence, $D(B)$ is dense in $\ell_p(X)$. $\square$

In some of the examples in later chapters, we will work with weighted shift operators on sequence spaces. There are two common ways to introduce a weight to a shift on ℓ_p $(1 \leq p \leq \infty)$. We can either multiply each coordinate by a weight w after shifting it, or we can put a weight m into the norm. For a sequence of positive numbers (m_n) we define the weighted ℓ_p-spaces

$$\ell_p(m) = \Big\{(x_n)\colon \sum_{n\in\mathbb{N}} m_n^p |x_n|^p < \infty\Big\},$$
$$\|x\|_{p,m} = \Big(\sum_{n\in\mathbb{N}} m_n^p |x_n|^p\Big)^{1/p}$$

for $1 \leq p < \infty$ and

$$\ell_\infty(m) = \{(x_n)\colon \sup_{n\in\mathbb{N}} m_n|x_n| < \infty\},$$
$$\|x\|_{\infty,m} = \sup_{n\in\mathbb{N}} m_n|x_n|.$$

We will illustrate here for the right shift that both methods of introducing weights are equivalent. Let

$$S_w(x_0, x_1, x_2, \dots) := (0, w_0x_0, w_1x_1, \dots)$$

on $X = \ell_p$ for a positive sequence (w_n). We see that S_w is bounded if and only if $\sup_{n\in\mathbb{N}} w_n < \infty$. Let

$$T_m(x_0, x_1, x_2, \dots) := (0, x_0, x_1, \dots)$$

on $Y = \ell_p(m)$. Then T_m is bounded if and only if $\sup_{n\in\mathbb{N}} (\frac{m_{n+1}}{m_n}) < \infty$. The map

$$\pi(x_0, x_1, x_2, \dots) := (x_0m_0^{-1}, x_1m_1^{-1}, \dots)$$

is isometric from X onto Y and intertwines S_w and T_m if w and m are related by

$$w_n = \frac{m_{n+1}}{m_n} \quad (n \in \mathbb{N}).$$

A similar correspondence holds between weighted shifts S_w on $\ell_p(\mathbb{Z})$ and shifts T_m on a weighted space $\ell_p(\mathbb{Z}, m)$. Depending on the context, we will work with weighted shifts, and with the shift on weighted spaces.

The spectrum for weighted shifts S_w on $\ell_p(\mathbb{Z})$ was first characterised for $p = 2$ in [Rid70]. The same characterisation has later been extended to the cases $1 \leq p < \infty$.

We follow the presentation in [BT10, Example 2.4 and Proposition 2.5]. It is easier to formulate the results using the isometrically equivalent right shift T_m on a weighted $\ell_p(\mathbb{Z}, m)$ space. Define

$$\begin{aligned} i^+(T_m) &= \lim_{n\to\infty} \left(\inf_{k>0} \frac{m_{n+k}}{m_k} \right)^{1/n}, \quad i^-(T_m) = \lim_{n\to\infty} \left(\inf_{k<0} \frac{m_k}{m_{k-n}} \right)^{1/n}, \\ r^+(T_m) &= \lim_{n\to\infty} \left(\sup_{k>0} \frac{m_{n+k}}{m_k} \right)^{1/n}, \quad r^-(T_m) = \lim_{n\to\infty} \left(\sup_{k<0} \frac{m_k}{m_{k-n}} \right)^{1/n}. \end{aligned} \tag{2.11}$$

Proposition 2.6.5. *Let T_m be the right shift on the weighted space $\ell_p(\mathbb{Z}, m)$, for some $1 \le p < \infty$. Then*

$$\sigma(T_m) = \big\{ \lambda \in \mathbb{C} \colon \min\{i^+(T_m), i^-(T_m)\} \le |\lambda| \le \max\{r^+(T_m), r^-(T_m)\} \big\}.$$

If $r^-(T_m) < i^+(T_m)$ then

$$\sigma_{ap}(T_m) = \big\{ \lambda \in \mathbb{C} \colon i^-(T_m) \le |\lambda| \le r^-(T_m) \big\} \cup \big\{ \lambda \in \mathbb{C} \colon i^+(T_m) \le |\lambda| \le r^+(T_m) \big\}.$$

Otherwise,

$$\sigma_{ap}(T_m) = \sigma(T_m).$$

Chapter 3

Inverses of Extensions

Sections 3.1 and 3.2 contain previously known results about invertible extensions for bounded operators and for semigroup representations. We also give a new result for bounded operators on Hilbert spaces (Theorem 3.1.5). Example 3.2.7 will show that lower bounds are not sufficient to describe fully the growth of inverses in extensions. We then transfer the results for bounded operators to unbounded operators in Sections 3.3 and 3.4. In Section 3.5, we look at dissipative operators and left inverse semigroups on Hilbert spaces. Finally, we will state a related theorem by Read, which allows one to reduce the spectrum of a bounded operator to the approximate point spectrum by extending the operator. This result transfers to generators of C_0-semigroups.

3.1 The inverse of a bounded operator

Let us start by looking at a result by Arens, which is stated here for a bounded operator T on a Banach space X. It gives a sufficient condition for when there is an invertible extension of T. The proof is adapted from [Are58, Theorem 3.5].

Theorem 3.1.1 (see [Are58])**.** *Let T be a bounded operator on a Banach space X and assume that*

$$\|Tx\| \geq c\|x\| \quad (x \in X)$$

for some $c > 0$. Then there exists a larger Banach space $Y \supseteq X$ and a bounded operator $S \in \mathcal{B}(Y)$ with $\|S\| = \|T\|$ which restricts on X to $S\restriction_X = T$ and which has a bounded inverse with $\|S^{-1}\| \leq c^{-1}$.

Furthermore, there is a unital isometric algebra homomorphism

$$\varphi\colon \{T\}' \to \mathcal{B}(Y)$$

with $\varphi(T) = S$.

Proof. First replace T by $c^{-1}T$, so that we can assume $c = 1$. We will construct the space Y as a quotient of $\ell_1(X)$ and use the right shift R on this space. Let J be the closure of the subspace

$$\{(I - TR)f\colon f \in \ell_1(X)\},$$

$Y := \ell_1(X) \,/\, J$ and define $\rho\colon X \to Y$ by $\rho(x) = x\,\mathbf{e}_0 + J$ where $\mathbf{e}_0$ is the sequence $(1, 0, 0, \dots)$. We have that $\|\rho(x)\| \leq \|x\|$ for all $x \in X$.

To show that the map $\rho\colon X \to Y$ is an isometric embedding we need to show that $\|\rho(x)\| \geq \|x\|$ $(x \in X)$. Let $x \in X$ and let $f \in \ell_1(X)$ be a sequence of finite support. Then

$$\|x\,\mathbf{e}_0 - (I - TR)f\| = \|x - f(0)\| + \sum_{n\geq 1} \|f(n) - Tf(n-1)\|$$

where the sum is finite. We use the triangle inequality and the lower bound of T to see that

$$\begin{aligned} \|x\| &\leq \|x - f(0)\| + \|f(0)\| \\ &\leq \|x - f(0)\| + \|Tf(0)\|. \end{aligned} \tag{3.1}$$

Repeating this step now on $Tf(0)$ in place of x we get

$$\|x\| \leq \|x - f(0)\| + \|Tf(0) - f(1)\| + \|Tf(1)\|.$$

Some further iterations of this step show $\|x\| \leq \|x\,\mathbf{e}_0 - (I - TR)f\|$. Since sequences of finite support are dense in $\ell_1(X)$ and as $(I - TR)$ is bounded, the estimate

$$\|x\| \leq \|x\,\mathbf{e}_0 - (I - TR)f\| \tag{3.2}$$

holds for all $f \in \ell_1(X)$. By definition of the norm on Y, it follows that $\|\rho(x)\| \geq \|x\|$ for all $x \in X$. This gives that ρ is an isometric embedding.

Define S on Y by

$$S(f + J) := \big(Tf(n)\big)_n + J \quad (f \in \ell_1(X)),$$

so that it comes from the coordinatewise multiplication by T on $\ell_1(X)$. It is clear from this definition that S restricts to T on $\rho(X)$, that is $\rho T = S\rho$. Since the closed subspace J is invariant under the multiplicative action of T, we see that S is well-defined and $\|S\| \leq \|T\|$. Since X is isometrically embedded in Y, we thus must have $\|S\| = \|T\|$.

We still have to show that S is invertible on Y. Define V on Y by $V(f + J) := Rf + J$. Since J is invariant under R, the operator V is well-defined and bounded by $\|V\| \leq \|R\| = 1$. For $f \in \ell_1(X)$ we have that

$$f + J = (f - TRf) + TRf + J = TRf + J$$

so that $SV = VS = I$. So S is invertible and $S^{-1} = V$.

In order to define the homomorphism φ, let $U \in \{T\}'$. Define

$$(M_U f)(n) := Uf(n) \quad (f \in \ell_1(X)).$$

Then M_U has the same norm as U and J is invariant under M_U. So the operator

$$\varphi(U)(f + J) := M_U f + J \quad (f \in \ell_1(X))$$

is well-defined and has norm $\|\varphi(U)\| \leq \|U\|$, since Y is equipped with the quotient norm. Also, ρ intertwines U and $\varphi(U)$. Since the embedding ρ is isometric, we get that φ is isometric. Moreover, it follows from the definition that $\varphi(U_1U_2) = \varphi(U_1)\varphi(U_2)$ for $U_1, U_2 \in \{T\}'$, that is, φ is a homomorphism. Moreover, $\varphi(I) = I$. □

The theorem can be applied to the right shift $T \in \mathcal{B}(\ell_p)$ for $1 \leq p \leq \infty$ where

$$T(x_0, x_1, x_2, \dots) = (0, x_0, x_1, \dots).$$

It is clear that T is isometric, so it satisfies $\|Tx\| \geq \|x\|$ for all $x \in \ell_p$. For these operators T, we have an obvious candidate for the extending space Y and the invertible operator S. We can take $Y = \ell_p(\mathbb{Z})$ and let S be the right shift on Y. The commutant of shift operators on Banach spaces was studied by Crownover in [Cro72]. If T is the shift on ℓ_1 or ℓ_∞, then the commutant can be identified with ℓ_1. A sequence $f \in \ell_1$ defines an operator

$$U_f(x) := \sum_{n=0}^{\infty} f(n)T^n x \quad (f \in X),$$

which clearly commutes with T. Every U commuting with T is given by such an $f \in \ell_1$ by [Cro72]. If we go to the extending bilateral shift S on $\ell_1(\mathbb{Z})$ or $\ell_\infty(\mathbb{Z})$, the sequence still defines an operator $V_f = \sum_{n=0}^{\infty} f(n)S^n$ which commutes with S and restricts to U_f. It is easy to see that $\|V_f\| = \|U_f\| = \|f\|$. For the shift on ℓ_p ($1 \leq p < \infty$), the construction in the proof of Theorem 3.1.1 gives in fact the bilateral shift on $\ell_p(\mathbb{Z})$. However, the same construction applied to the shift on ℓ_∞ gives a strict subspace of $\ell_\infty(\mathbb{Z})$. Both is shown in the next example.

Example 3.1.2. Let T be an isometry on a Banach space X. Douglas showed that an isometric representation of a commutative semigroup can be extended to yield invertible isometries, and there is a minimal extension which is unique up to isometry ([Dou69]). Let us denote this extension by U. Another extension S on a Banach space Y can be obtained from Theorem 3.1.1 (with $c = 1$), which is also an invertible isometry. The subspaces $S^{-n}X$ form an increasing sequence (with respect to set inclusion). The subspace

$$\bigcup_{n\geq 1} S^{-n}X$$

is dense in the extending space Y which shows that S is a minimal extension of T, too. By uniqueness S and U must be equal up to an intertwining isometry. To compare the extension directly, note that for an operator T Douglas' construction uses the space

$$Y_0 = \{(x_n)\colon \exists k \geq 0 \text{ with } x_{n+k} = T^n x_k \,\forall n \in \mathbb{N}\} \subset \ell_\infty(X)$$

with the norm $\|(x_n)\| := \lim_{n\to\infty}\|x_n\|$. The extending space is the completion Y_1 of the quotient

$$Y_0 \,/\, \{(x_n) \in Y_0 \colon \|(x_n)\| = 0\}$$

and X is embedded via $x \mapsto (T^n x)_n$. The isometric map between the space Y with S and Y_1 with U is given by the correspondence of $U^{-k}x$ and $S^{-k}x$, or by their representatives $(T^{n+k}x)_n$ in Y_0 and $x\,\mathbf{e}_k$ in $\ell_1(X)$ $(x \in X, k \geq 0)$.

For the right shift on ℓ_p, an invertible isometric extension is given by the right shift on $\ell_p(\mathbb{Z})$ if $1 \leq p \leq \infty$. This extension is minimal if $1 \leq p < \infty$, while for $p = \infty$ the restriction onto the closed subspace $\{x \in \ell_\infty(\mathbb{Z})\colon \lim_{n\to-\infty} x_n = 0\}$ is minimal.

The next example shows that, in general, the extending space Y constructed in the proof of Theorem 3.1.1 is not a Hilbert space if X is one.

Example 3.1.3. Let T_0 be the right shift on ℓ_2 and

$$T = 2T_0 - I.$$

Then $\|Tx\| \geq 2\|T_0x\| - \|x\| \geq \|x\|$ $(x \in \ell_2)$ and T satisfies the assumption of Theorem 3.1.1 with $c = 1$. There is an obvious invertible extension for T on $\ell_2(\mathbb{Z})$. We also get this extension if we take the extension of the isometry T_0 and use the homomorphism φ to extend the commuting operator T (see the previous example). However, when applying the construction in the proof of Theorem 3.1.1 to T, we get a different extension which is not on a Hilbert space. Let Y be the extending space which is constructed as a quotient of $\ell_1(\ell_2)$. To see that Y is not a Hilbert space, we exhibit two elements $x, y \in \ell_2$ such that the parallelogram law does not hold over $x\mathbf{e}_0 + J$ and $y\mathbf{e}_1 + J$ in Y. Let us calculate the norm of the sum of two such elements. Firstly,

$$\begin{aligned} \|x\mathbf{e}_0 + y\mathbf{e}_1 + J\| &= \inf_{g_n\in\ell_2}\left(\|x+g_0\| + \|y+g_1-Tg_0\| + \sum_{n\geq 2}\|g_n - Tg_{n-1}\|\right) \\ &\geq \inf_{g\in\ell_2}\left(\|x+g\| + \|y-Tg\|\right). \end{aligned} \tag{3.3}$$

Here, we apply the estimate (3.2) to $y - Tg_0$, and we get equality in (3.3) by choosing $g_n = 0$ for $n \geq 1$. Take $y \in \ker T^* = (\operatorname{ran} T)^\perp$ and $x = 0$. Since $\|y - Tg\|^2 \geq \|y\|^2$ for all $g \in \ell_2$ (as $\langle y, Tg\rangle = 0$ by choice of y) we get

$$\|y\mathbf{e}_1 + J\| = \inf_{g\in\ell_2} (\|g\| + \|y - Tg\|) = \|y\|.$$

For $y = 0$ and $x = (1, 0, 0, \dots)$ we get

$$\|x\mathbf{e}_0 + J\| = \inf_{g\in\ell_2} (\|x + g\| + \|Tg\|) = \|x\|.$$

So for $y \in \ker T^*$ with $\|y\| = 1$ and $x = (1, 0, 0, \dots)$ we have

$$2\|x\mathbf{e}_0 + J\|^2 + 2\|y\mathbf{e}_0 + J\|^2 = 4.$$

On the other hand, making use of the fact that $y \in (\operatorname{ran} T)^\perp$ we have

$$\begin{aligned}&\|x\mathbf{e}_0 - y\mathbf{e}_1 + J\| = \|x\mathbf{e}_0 + y\mathbf{e}_1 + J\|\\ &= \inf_{(t_n)\in\ell_2} \Big(|1 + t_0|^2 + \sum_{n=1}^{\infty}|t_n|^2\Big)^{1/2} + \Big(\|y\|^2 + |t_0|^2 + \sum_{n=1}^{\infty}|t_n - 2t_{n-1}|^2\Big)^{1/2}.\end{aligned}$$

We can estimate this from above by choosing $(t_n) = (t, 0, 0, \dots)$ which gives

$$\|x\mathbf{e}_0 \pm y\mathbf{e}_1 + J\| \leq |1 + t| + \sqrt{1 + 5|t|^2} < 2$$

for $-\frac{1}{2} < t < 0$. Hence,

$$2\|x\mathbf{e}_0 + J\|^2 + 2\|y\mathbf{e}_1 + J\|^2 \neq \|x\mathbf{e}_0 + y\mathbf{e}_1 + J\|^2 + \|x\mathbf{e}_0 - y\mathbf{e}_1 + J\|^2$$

and Y is not a Hilbert space.

The example shows that Theorem 3.1.1 does not preserve Hilbert space structure. The construction in Theorem 3.1.1 can be adapted to Hilbert space operators, as shown by Badea and Müller (see [BM05, Corollary 4.8] — their proof gives weaker estimates than Theorem 3.1.1 on the bounds of the extensions). They noticed that the same construction can be made using the X-valued ℓ_2-sequences. We give the proof here for illustration.

Theorem 3.1.4 (see [BM05, Corollary 4.8]). *Let T be a bounded operator on a Hilbert space X and assume that*

$$\|Tx\| \geq c\|x\| \quad (x \in X)$$

for some $c > 0$. Then there exists a Hilbert space $Y \supseteq X$ and a bounded invertible operator S on Y such that $Sx = Tx$ for all $x \in X$ and $\|S^{-k}\| \le c^{-k}2^{(k+1)/2}$ for all $k \ge 1$.

Furthermore, there is a unital homomorphism

$$\varphi\colon \{T\}' \to \mathcal{B}(Y)$$

with $\varphi(T) = S$ and $\|\varphi(U)\| \le \sqrt{2}\|U\|$ for all $U \in \{T\}'$.

Proof. Replace T by $c^{-1}T$ if necessary to assume that $c = 1$. We will give a construction similar to the one for Theorem 3.1.1, but this time on $\ell_2(X)$. Let J be the closure of

$$\{(I - \sqrt{2}TR)f\colon f \in \ell_2(X)\},$$

and let Y be the quotient space $\ell_2(X) \,/\, J$.

Consider the embedding $\rho\colon X \to Y$ and let $x \in X$. Note that $(a+b)^2 \le 2(a^2+b^2)$ for all real numbers a, b. Using this and (3.1) we have

$$\|x\|^2 \le 2\|x - f(0)\|^2 + \|\sqrt{2}Tf(0)\|^2,$$

where f is a sequence with values in X. Repeating the argument yields

$$\|x\|^2 \le 2\|x - f(0)\|^2 + 2\|\sqrt{2}Tf(0) - f(1)\|^2 + \|\sqrt{2}Tf(1)\|^2.$$

Iterating further shows that

$$\|x\|^2 \le 2\|x - f(0)\|^2 + \sum_{n\ge 1} 2\|f(n) - \sqrt{2}Tf(n-1)\|^2 = 2\|x\mathbf{e}_0 - f + T\sqrt{2}Rf\|^2$$

for all sequences $f \in \ell_2(X)$ of finite support.

Since the norm on Y is the quotient norm with respect to the subspace J, this shows that $\|x\| \le \sqrt{2}\|\rho(x)\|$. It is clear that the inequality $\|\rho(x)\| \le \|x\|$ holds. So the embedding ρ is isomorphic. The operators $S((x_n) + J) := (Tx_n)_n + J$ and $V(f + J) := (\sqrt{2}Rf_n)_n + J$, are well-defined and satisfy $\|S\| \le \|T\|$ and $\|V\| \le \sqrt{2}$. Again, we have that $SV = VS = I$ as can be verified by a simple calculation.

To get an isometric embedding, we have to renorm the space Y. This is easily achieved, as the closed subspace $\rho(X)$ has an orthogonal complement in Y. We can use the Hilbert norm

$$|\rho(x) + y| := \left(\|x\|_X^2 + \|y\|_Y^2\right)^{1/2} \quad (x \in X, y \in \rho(X)^\perp).$$

This satisfies $\|y\| \le |y| \le \sqrt{2}\|y\|$ for $y \in Y$ and $|\rho(x)| = \|x\|_X$ for $x \in X$.

For an operator $U \in \mathcal{B}(Y)$ we have

$$|Uy| \leq \sqrt{2}\|Uy\| \leq \sqrt{2}\|U\||y| \quad (y \in Y)$$

so that $|V^k| \leq 2^{(k+1)/2}$ for the operator norm of V^k with respect to the norm $|\cdot|$. Finally, note that each $U \in \{T\}'$ defines an operator

$$\varphi(U)((x_n) + J) := (Ux_n) + J \quad ((x_n) \in \ell_2)$$

with $\varphi(U)\rho = \rho U$, and $|\varphi(U)| \leq \sqrt{2}\|\varphi(U)\| \leq \sqrt{2}\|U\|$. $\square$

The difference between Theorem 3.1.4 and Theorem 3.1.1, apart from them treating different types of spaces, is quite evidently in the norm estimates on the homomorphism φ and on the inverse S^{-1}. The next theorem, Theorem 3.1.5, gives the optimal bound for the extension S and its inverse while still preserving the Hilbert space structure. However, the extension of commuting operators is no longer guaranteed, that is we do not obtain the homomorphism φ for commuting operators. The idea we will use in the proof resembles a construction for some subnormal operators, namely a construction that gives a normal extension for quasinormal operators (see [Con81, Chapter III, Proposition 1.7] for example).

The proof of the next theorem makes use of the polar decomposition of an operator T on a Hilbert space X. It allows us to write an operator T as the product $T = UP$ of a partial isometry U and a positive operator P. We give a brief description of how the polar decomposition is obtained for a closed, densely defined operator T (see [Kat76, Chapter VI, Section 2.7]). Let $P = (T^*T)^{\frac{1}{2}}$ be the positive square root of T^*T with $D(P) = D(T)$. The map $\operatorname{ran} P \to \operatorname{ran} T$ given by $Px \mapsto Tx$ is isometric, since $\|Px\|^2 = \langle P^2x, x\rangle = \langle T^*Tx, x\rangle = \|Tx\|^2$ for x in the dense set $x \in D(P^2)$. So the map can be extended to an isometry U between the closures of the ranges. Extend U on $(\operatorname{ran} P)^\perp$ by 0, then U is a partial isometry. The operators T and P have the same domain and $T = UP$.

Although the following theorem does not include an isometric homomorphism from the commutant of the operator as Theorems 3.1.1 and 3.1.4 do, we still have a weaker property. Note that if a (unitary) homomorphism φ maps $\varphi(T) = S$ then $\varphi(R(\lambda, T)) = R(\lambda, S)$ for $\lambda \in \rho(T)$, and in particular $\sigma(S) \subset \sigma(T)$. This spectral inclusion still holds in the next theorem.

Theorem 3.1.5. *Let T be a bounded linear operator on a Hilbert space X. Assume that*

$$\|Tx\| \geq c\|x\| \quad (x \in X)$$

for some $c > 0$. Then there exists a Hilbert space $Y \supseteq X$ and an invertible operator $S \in \mathcal{B}(Y)$ such that $Sx = Tx$ for all $x \in X$, $\|S\| = \|T\|$ and $\|S^{-1}\| \leq c^{-1}$. Moreover, we have that $\sigma(S) \subset \sigma(T)$.

Proof. If T is invertible, the statement becomes trivial. So let us assume that T is not invertible. Substitute T by $c^{-1}T$ so we can further assume that $c = 1$.

Let $T = UP$ be the polar decomposition of T into a partial isometry U and a positive operator P. Let us show that U is an isometry and that P is invertible. Since T is expansive we have

$$\|x\|^2 \leq \langle T^*Tx, x\rangle \leq \|T^*Tx\|\|x\| \quad (x \in X).$$

The first inequality implies that T^*T has dense range. From the inequality $\|x\| \leq \|T^*Tx\|$ we see that T^*T is injective and has closed range. So T^*T must be invertible and P is invertible, too. The partial isometry U has kernel $\ker U = (\operatorname{ran} P)^\perp = \{0\}$ which shows that U is an isometry.

Take the ℓ_2-direct sum $Y = X \oplus X$ and let

$$S := \begin{pmatrix} U & I - UU^* \\ 0 & U^* \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} T & I - UU^* \\ 0 & U^* \end{pmatrix}$$

Since U is an isometric operator, the first matrix is a unitary operator, and we note that the unitary dilation of U is the restriction of this matrix to a subspace of Y. The image of the orthogonal projection $I - UU^*$ is $(\operatorname{ran} T)^\perp = (\operatorname{ran} U)^\perp$ so that

$$\|S(x,y)\|^2 = \|Tx\|^2 + \|(I - UU^*)y\|^2 + \|UU^*y\|^2 = \|Tx\|^2 + \|y\|^2 \quad (x, y \in X)$$

which is bounded from below by $\|(x,y)\|^2$ and bounded from above by $\|T\|^2\|(x,y)\|^2$. The following operator is an inverse of S.

$$S^{-1} = \begin{pmatrix} P^{-1} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} U^* & 0 \\ I - UU^* & U \end{pmatrix} = \begin{pmatrix} P^{-1}U^* & 0 \\ I - UU^* & U \end{pmatrix}$$

We get that $\|S^{-1}\| \leq 1$ since $\|S(x,y)\| \geq \|(x,y)\|$. So S is an invertible operator with $\|S\| = \|T\|$ and $\|S^{-1}\| \leq 1$.

Let us now deal with the spectral inclusion. Let us first show the inclusion $\sigma(U^*) \subset \sigma(T)$. We will do this by showing that $\sigma(U^*) \subset \overline{\mathbb{D}} \subset \sigma(T)$ where $\overline{\mathbb{D}}$ is the closed unit disk. Since $\|U^*\| \leq 1$, its spectrum is contained in $\overline{\mathbb{D}}$. On the other hand, $0 \in \sigma(T)$ since T is not invertible. Since

$$\|(T - \lambda)x\| \geq \|Tx\| - |\lambda|\|x\| \geq (1 - |\lambda|)\|x\| \quad (x \in X, \lambda \in \mathbb{C}),$$

it is clear that $\sigma_{\text{ap}}(T) \subset \{\lambda \in \mathbb{C}\colon |\lambda| \geq 1\}$. The boundary of the spectrum is included in the approximate point spectrum of an operator, and since $0 \in \sigma(T)$ by assumption, we have $\overline{\mathbb{D}} \subset \sigma(T)$. Consequently, if $\lambda \in \rho(T)$ then $\lambda \in \rho(U^*)$ and the matrix

$$R(\lambda, S) = \begin{pmatrix} R(\lambda, T) & R(\lambda, T)(I - UU^*)R(\lambda, U^*) \\ 0 & R(\lambda, U^*) \end{pmatrix}$$

is an inverse to $\lambda - S$ so that $\lambda \in \rho(S)$. □

In Theorem 3.1.1 we constructed an extending space which in fact is minimal. This follows since the elements $x\mathbf{e}_n$ for $x \in X$ and $n \geq 0$ are dense in the quotient space which acts as the extending space in the proof of Theorem 3.1.1. For the extension S on Y from Theorem 3.1.5 we can find the minimal extending space Z, that is the smallest closed subspace $Z \subset Y$ which contains X and on which S is invertible. Clearly, Z is the closure of

$$\bigcup_{n \geq 0} S^{-n} X.$$

We can give a more explicit description of Z in terms of the isometry U from the polar decomposition $T = UP$. We have

$$Z = X \oplus \overline{\operatorname{span}}\{U^k(I - UU^*)x\colon x \in X, k \geq 0\},$$

or

$$Z = X \oplus \bigoplus_{k=0}^{\infty} Z_k.$$

where $Z_k := \operatorname{ran}(U^k(I - UU^*))$ $(k \geq 0)$. The subspaces Z_k are pairwise orthogonal and also orthogonal to $X \times \{0\} \subset Y$. In this decomposition, $S\restriction_X = T$, S maps Z_0 isometrically onto $(\operatorname{ran} T)^\perp$ in X, and Z_k onto Z_{k-1} for $k \geq 1$. Note that for invertible T, U is unitary and $(I - UU^*) = 0$ so that $Z = X$ as expected.

Suppose there is another extension of T given by $\tilde{S}$ on a Hilbert space

$$\tilde{Z} = X \oplus \bigoplus_{k=0}^{\infty} \tilde{Z}_k$$

where the $\tilde{Z}_k$ are pairwise orthogonal subspaces. Suppose further that $\tilde{S}\restriction_X = T$ and that $\tilde{S}$ maps $\tilde{Z}_0$ expansively onto $\operatorname{ran} T^\perp$ and $\tilde{Z}_k$ onto $\tilde{Z}_{k-1}$. It can be shown that the map $\pi\colon Z \to \tilde{Z}$ defined by

$$\pi(S^{-n}x) := \tilde{S}^{-n}x \quad (n \geq 0, x \in X)$$

intertwines S and $\tilde{S}$ and that π is contractive. The fact that such an intertwining map π exists for all pairs $(\tilde{S}, \tilde{Z})$ as described above can be seen as a universal property of the pair (S, Z). If $\tilde{S}\restriction_{\tilde{Z}_k}$ is not only expansive for $k \geq 0$ but isometric, then π is unitary. So S is uniquely determined by the decomposition of Z and by acting as an isometry when restricted to the subspaces Z_k.

Any extension $\tilde{S}$ (with $\tilde{S}\restriction_{\tilde{Z}_k}$ expansive) has a polar decomposition and it can be shown that in matrix form, they can be expressed as

$$\tilde{S} = \begin{pmatrix} U & A \\ 0 & B \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

Here, $T = UP$ is the polar decomposition of T, $\left(\begin{smallmatrix} U & A \\ 0 & B \end{smallmatrix}\right)$ is unitary, Q is positive and $Q\restriction_{Z_k}$ is an expansive isomorphism of $\tilde{Z}_k$ for each $k \geq 0$.

We have seen two results which allow us to find an invertible extension S on a Hilbert space Y for a given operator T on a Hilbert space X. In Theorem 3.1.4, the extension allows a unital homomorphism φ by which we can extend operators commuting with T. Unlike in Theorem 3.1.1, φ is not isometric and the norm estimates on S and S^{-1} are not optimal. That is, we do not have $\|S\| = \|T\|$ and $\|S^{-1}\| \leq 1$ when we assume that $\|Tx\| \geq \|x\|$ for all $x \in X$. On the other hand, in Theorem 3.1.5 we do get the optimal bounds $\|S\| = \|T\|$ and $\|S^{-1}\| \leq 1$, but we possibly do not have $\|S^n\| \leq \|T^n\|$ for $n \geq 2$. Also, we do not have a unital homomorphism φ defined on the commutant of T. This leaves a gap between the two results. We can ask whether a result similar to Theorem 3.1.1 exists.

Open Question 3.1.6. For which operators T on a Hilbert space X with $\|Tx\| \geq \|x\|$ for all $x \in X$ exist an invertible extension S of T on a Hilbert space Y containing X isometrically and an isometric unital homomorphism $\varphi\colon \{T\}' \to \mathcal{B}(Y)$ such that $\varphi(T) = S$?

This is trivially possible for invertible operators T, and by Douglas also for isometric operators T (see Example 3.1.2). No example is known where the answer is negative.

Let us consider operators which are similar to expansive operators on Hilbert space. Let $T \in \mathcal{B}(X)$ be an operator on a Hilbert space X and assume there is an equivalent Hilbert norm $\|\cdot\|'$ such that $\|Tx\|' \geq \|x\|'$ for all $x \in X$. We can apply Theorem 3.1.5 to T on $(X, \|\cdot\|')$ and get an invertible extension S on a Hilbert space $(Y, \|\cdot\|'_Y)$ such that $\|S^{-1}\|' \leq 1$. Let P be the $\|\cdot\|'_Y$-orthogonal projection onto X and renorm Y by

$$\|y\|_Y^2 = \|Py\|^2 + \|(I - P)y\|_Y'^{\,2} \quad (y \in Y).$$

This gives an isometric embedding of $(X, \|\cdot\|)$ in the Hilbert space $(Y, \|\cdot\|_Y)$ and we see that S is an extension of T. Moreover, S^{-1} on $(Y, \|\cdot\|_Y)$ is similar to a contraction.

If we assume that T has an extension S such that S^{-1} is similar to a contraction, we see that T is similar to an expansive operator. Similarity to contractive operators on Hilbert space is characterised by complete polynomial boundedness (see [Pau84]). So an operator T on a Hilbert space is similar to an expansive operator if and only if it has an invertible extension S such that S^{-1} is completely polynomially bounded.

Let us look at the weighted shift on ℓ_2,

$$T(x_0, x_1, x_2, \dots) = (0, w_0 x_0, w_1 x_1, \dots)$$

where $w\colon \mathbb{N} \to (0, \infty)$ is the weight of T. Since we want T to be bounded we require that $\sup\{\, w_n \colon n \in \mathbb{N} \,\} < \infty$. Also, we assume that $w_n \geq 1$ for all $n \in \mathbb{N}$ which is equivalent to $\|Tx\| \geq \|x\|$ for all $x \in \ell_2$. In the case of the (unweighted) unilateral shift we would have $w_n = 1$ for all $n \in \mathbb{N}$, and the extension S on the space Y constructed in the proof of Theorem 3.1.5 is isometrically isomorphic to the (unweighted) bilateral shift on $\ell_2(\mathbb{Z})$. If the weight w is different and we were looking for an invertible extension amongst the weighted shifts on $\ell_2(\mathbb{Z})$, there is no longer a natural choice for the weight of the extending shift. In this case, Theorem 3.1.5 gives a weighted shift on $\ell_2(\mathbb{Z})$ and it simply extends the weight w of the ℓ_2-shift T to the negative integers by $w_n = 1$ for $n < 0$.

Since the idea used for the construction of the extension in Theorem 3.1.5 leans on the construction of normal extensions for quasinormal operators, one might wonder whether the extension S of T is subnormal, for example, if T is a subnormal operator on a Hilbert space X such that $\|Tx\| \geq \|x\|$ for all $x \in X$. The following example shows that the extension S need not be subnormal when T is.

Example 3.1.7. Let T be the weighted shift on ℓ_2 with weight $w\colon \mathbb{N} \to (0, \infty)$. We saw that T satisfies the assumptions of Theorem 3.1.5 with $c = 1$ if there is $M \geq 1$ such that $1 \leq w_n \leq M$ for all $n \in \mathbb{N}$. In fact,

$$\|T\| = \sup_{n \geq 0} w_n.$$

The extension S of T is given by the weighted bilateral shift on $\ell_2(\mathbb{Z})$ with weight

$$\tilde{w}(n) = \begin{cases} w(n) & n \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}$$

If T is subnormal, it must be hyponormal which means that $T^*T \geq TT^*$. For the shift T this is the case if and only if the weight w is increasing (see [Con81, Chapter III, Propositions 4.2 and 8.6]). Not all increasing, bounded weights w give subnormal operators, as can be seen from [Con81, Chapter III, Proposition 8.18].

Take

$$w(n) = \begin{cases} 1 & n = 0 \\ 2 & n \geq 1. \end{cases}$$

This gives a subnormal weighted shift (see [Con81, Chapter III, Example 11.4]). The invertible extension S from Theorem 3.1.5 is the shift with weight

$$\tilde{w}(n) = \begin{cases} 1 & n \leq 0 \\ 2 & n \geq 1. \end{cases}$$

We can show that S is not subnormal. We will use condition (e) from [Con81, Chapter III, Theorem 1.9]. It says that S is subnormal if and only if

$$\sum_{j,k=0}^{n} \langle S^{j+k} f_j, S^{j+k} f_k \rangle \geq 0$$

for all $n \geq 0$ and $f_0, \ldots, f_n \in \ell_2$. We can rewrite this sum as

$$\sum_{j,k} \sum_{r \in \mathbb{Z}} f_j(r)\overline{f_k(r)}\tilde{w}(r)^2 \ldots \tilde{w}(r+j+k-1)^2 = \sum_{r \in \mathbb{Z}} \langle A(r)f(r), f(r) \rangle_{\mathbb{C}^{n+1}}$$

where $A(r)$ is the $(n+1) \times (n+1)$ matrix $(a_{j,k})_{j,k=0}^n$ with

$$a_{j,k} = \tilde{w}(r)^2 \ldots \tilde{w}(r+j+k-1)^2 \quad (j, k = 0, \ldots, n),$$

and $f(r)$ is the vector $(f_k(r))_{k=0}^n$. So if we can show that for some n and r the matrix $A(r)$ is not positive semi-definite, we get that S is not subnormal. For $n = 2$ and $r = -1$ we get

$$A(-1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 4 \\ 1 & 4 & 16 \end{pmatrix}$$

and we see that this matrix has a negative eigenvalue. In fact, its characteristic polynomial is $x^3 - 18x^2 + 15x + 9$, a continuous function which tends to $-\infty$ as $x \to -\infty$ and takes a positive value at $x = 0$. So S is not subnormal.

For this example, the minimal normal extension N of T is identified in [JJS13, Remark 4.3] as a shift on the directed tree $\mathbb{Z} \cup \{\omega\}$ where $\omega \notin \mathbb{Z}$ is a single point that branches off at 0. So N has a one-dimensional kernel (the functions that are supported on ω) and cannot have an invertible extension.

3.2 The inverse of a semigroup representation

The results in the previous section give an invertible extension for a single operator T on a Banach space or a Hilbert space X. When we try to find the inverses to a family of operators, this brings us to semigroup representations. The simplest example is the extension of two commuting operators T_1 and T_2. They define a representation of $\mathbb{N}^2$ by $(m,n) \mapsto T_1^m T_2^n$. If there are invertible commuting extensions S_1 and S_2 on a Banach space Y then they extend the representation of $\mathbb{N}^2$ on X to a representation of $\mathbb{Z}^2$ on Y. Similarly, one can look at more general abstract semigroup representations. We assume that $\mathfrak{S}$ is a semigroup contained in a locally compact abelian group $\mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{S} - \mathfrak{S}$. This setting was studied in [BY01]. To state an extension result, we need the following definition.

Definition 3.2.1. Let T be a representation of a semigroup $\mathfrak{S}$ on a Banach space X. We say that T is *super-expansive* if

$$\|x\| \leq \|x_1\| + \cdots + \|x_n\|$$

whenever

$$T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n$$

with $t, t_i \in \mathfrak{S} \cup \{0\}$ such that $t - t_i \in \mathfrak{S} \cup \{0\}$ and $x, x_i \in X$.

Theorem 3.2.2 ([BY01, Theorem 3.3])**.** *Let T be a representation of a semigroup $\mathfrak{S}$ on a Banach space X. Assume that T is super-expansive. Then there is a representation U of $\mathfrak{G}$ on a Banach space $Y \supset X$ such that $U(t)x = T(t)x$ for all $x \in X, t \in \mathfrak{S}$ and $\|U(-t)\| \leq 1$ for all $t \in \mathfrak{S}$. Moreover, there is a unital isometric algebra homomorphism*

$$\varphi\colon \{T\}' \to \mathcal{B}(Y)$$

such that for every $V \in \{T(t)\colon t \in \mathfrak{S}\}'$ we have $\varphi(V)x = Vx$ for all $x \in X$.

For semigroups $\mathfrak{S}$ with $\mathfrak{G} = \mathfrak{S} \cup (-\mathfrak{S})$, super-expansiveness is equivalent to expansiveness ([BY01, Proposition 2.2]). For example, a representation of $\mathbb{N}$ given by $n \mapsto T^n$ is super-expansive if $\|Tx\| \geq \|x\|$ for all $x \in X$, or a C_0-semigroup T is super-expansive if $\|T(t)x\| \geq \|x\|$ for all $x \in X, t > 0$.

Motivated by the construction used in the proof of Theorem 3.2.2 in [BY01], Badea and Müller found a way to describe the existence of invertible extensions for operators T such that the powers of the inverse S satisfy $\|S^{-n}\| \leq c_n$ $(n \geq 1)$ for some sequence (c_n) (see [BM05]). We assume the sequence (c_n) is submultiplicative, that is, it satisfies $c_{m+n} \leq c_m c_n$ $(m, n \geq 1)$. In [BM05], extensions S which allow an isomorphic embedding of the Banach space are considered. We state the following variant of their main result for isometric embeddigs.

Theorem 3.2.3 (variant of [BM05, Theorem 3.1 (ii)]). *Let $(c_n)_{n\geq 1}$ be a positive, submultiplicative sequence. Let T be a bounded linear operator on a Banach space X and suppose that whenever*

$$T^n x = x_0 + T x_1 + \cdots + T^{n-1} x_{n-1}$$

we have that

$$\|x\| \leq c_n \|x_0\| + c_{n-1}\|x_1\| + \cdots + c_1 \|x_{n-1}\|.$$

Then there is an invertible bounded operator S on a Banach space $Y \supset X$ such that $Sx = Tx$ for all $x \in X$, $\|S\| = \|T\|$ and $\|S^{-n}\| \leq c_n$ for all $n \geq 1$.

Moreover, there is a unital isometric algebra homomorphism

$$\{T\}' \to \mathcal{B}(Y)$$

which maps T to S.

This statement also generalises to representations of abelian semigroups $\mathfrak{S}$, as in Theorem 3.2.2. We will only need the following version for C_0-semigroups later in Section 4.7. Similarly to submultiplicative sequences, we say that a funcion $c\colon \mathbb{R}_+ \to (0, \infty)$ is submultiplicative if $c(t_1 t_2) \leq c(t_1)c(t_2)$ for all $t_1, t_2 \geq 0$.

Theorem 3.2.4 ([BM05, Remarks 3.2.(iv)]). *Let $c\colon \mathbb{R}_+ \to (0, \infty)$ be a submultiplicative function. Let T be a C_0-semigroup on a Banach space X such that*

$$\|x\| \leq c(t_n)\|x_0\| + c(t_{n-1})\|x_1\| + \cdots + c(t_1)\|x_n\|$$

whenever $T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq t$. Then there is an isometric embedding $X \subset Y$ and a C_0-group S on Y with $S(t)x = T(t)x$ and $\|S(-t)\| \leq c(t)$ for all $x \in X, t \geq 0$.

Simple examples of multiplicative sequences are constant sequences (c_n) with $c_n = M$ $(n \geq 0)$ for some $M \geq 1$. For representations of $\mathbb{N}$, that is the powers of an operator T, we saw the following in Theorem 3.1.1. An operator T has an invertible extension S with $\|S^{-1}\| \leq 1$ if and only if $\|Tx\| \geq \|x\|$ for all $x \in X$. If T has an extension S with $\|S^{-n}\| \leq M$ for all $n \geq 0$ then T satisfies $\|T^n x\| \geq \frac{1}{M}\|x\|$ for all

$x \in X, n \geq 0$. Example 3.2.7, which is based on the next example, shows that an operator T can satisfy

$$\|T^n x\| \geq c\|x\| \quad (x \in X, n \geq 0)$$

for some $c > 0$ without having an extension S with power bounded inverse.

In the following example, we use the shift operator on a weighted L_∞ space over a specially chosen (directed) tree to construct an operator with a lower bound c, but for which no invertible extension has power bound smaller than ϵ^{-1} for some chosen $\epsilon < c$. In the construction, we aim to violate the condition from Theorem 3.2.3 to say that no such invertible extension exists. We achieve this by taking a function f which is supported on the root of the tree. The shift will 'split' the function at those vertices, where the tree has two branches. This allows us to rewrite an image of the function f under the shift in terms of functions f_n with connected support. We can choose the functions f_n in such a way that the norm condition of Theorem 3.2.3 is violated for the submultiplicative sequence $c_n = c^{-1}$ where c is the lower bound of the shift operator.

Example 3.2.5. Let M, c, ϵ be such that $M > 1 > c > \epsilon > 0$. We find an operator T such that

$$\|T^n x\| \geq c\|x\| \quad (x \in X, n \geq 0)$$

and $\|T\| \leq M$, and if S is any invertible extension of T then

$$\sup_{n \geq 1} \|S^{-n}\| \geq \epsilon^{-1} > c^{-1}.$$

Find a positive integer k such that $c^k < \frac{\epsilon}{2}$ and $M^k \frac{\epsilon}{2} > \frac{M}{1-c}$ and find some ϵ' such that $\frac{\epsilon}{2} \leq \frac{M}{1-c}\epsilon' < \epsilon - c^k$. Then $M^k \epsilon' > 1$. Let

$$\Omega = \{\, (m, n) \in \mathbb{N}^2 \colon n = 0 \text{ or } k \mid m \,\}$$

and let X be the weighted ℓ_∞-space (or c_0-space) on this set with the norm

$$\|f\| = \sup \{w_{m,n} |f_{m,n}| \colon (m, n) \in \Omega\}$$

where

$$w_{m,n} = \begin{cases} c^{\lfloor m/k \rfloor} & n = 0, \\ M^n \epsilon' c^{m/k} & n > 0. \end{cases}$$

So X consists of all functions $f\colon \Omega \to \mathbb{C}$ such that $\|f\| < \infty$ (or of the norm closure of the finitely supported functions for the separable c_0-space). Define T on X by

$$(Tf)(m,n) = \begin{cases} 0 & m=n=0 \\ f(m,n-1) & n>0 \\ f(m-1,0) & \text{otherwise.} \end{cases}$$

This definition gives a bounded operator T. To get the bound of T, we have to estimate $\|Tf\| = \sup\{w_{m,n}|(Tf)(m,n)|\}$ in terms of $\|f\|$. So it is sufficient to estimate each term $w_{m,n}|(Tf)(m,n)|$ separately. For $(m,n) \neq (0,0)$ we get

$$w_{m,n}|(Tf)(m,n)| = \begin{cases} \dfrac{w_{m,0}}{w_{m-1,0}} w_{m-1,0}|f(m-1,0)| & n=0, \\ \dfrac{w_{m,n}}{w_{m,n-1}} w_{m,n-1}|f(m,n-1)| & n>0. \end{cases}$$

So the norm of T is bounded by (and in fact equal to) the supremum of the fractions $\dfrac{w_{m,n}}{w_{m,n-1}} \leq M$ and $\dfrac{w_{m,0}}{w_{m-1,0}} \leq 1$, that is

$$\|T\| \leq M.$$

Let us show that $\|T^i f\| \geq c\|f\|$ for all $f \in X$ and $i \geq 1$. Let $f \in X$ and $i \geq 1$. To calculate the lower bound, we can estimate $\|f\|$ in terms of $\|T^i f\|$. We first find that

$$(T^i f)(m,n) = \begin{cases} 0 & m+n<i \\ f(m,n-i) & n \geq i \\ f(m+(n-i),0) & \text{otherwise.} \end{cases}$$

The values of $T^i f$ at two different points can be defined by the same value $f(m,n)$. We can make suitable choices and rearrange these equations to get

$$f(m,n) = \begin{cases} (T^i f)(m,n+i) & n>0 \\ (T^i f)(m+i,0) & n=0, i<k+d \\ (T^i f)(m+d,i-d) & n=0, i \geq k+d \end{cases}$$

where $d = k\lceil m/k \rceil - m$ is such that $m+d$ is the least multiple of k which is at least as big as m. Multiply this by $w_{m,n}$ and estimate the following fractions:

$$\begin{aligned} \frac{w_{m,n}}{w_{m,n+i}} &= \frac{M^n c^{\lfloor m/k \rfloor}\epsilon'}{M^{n+i}c^{\lfloor m/k \rfloor}\epsilon'} \leq 1 & n>0, \\ \frac{w_{m,0}}{w_{m+i,0}} &= \frac{c^{\lfloor m/k \rfloor}}{c^{\lfloor (m+i)/k \rfloor}} \leq c^{-1} & n=0, i<k+d, \\ \frac{w_{m,0}}{w_{m+d,i-d}} &= \frac{c^{\lfloor m/k \rfloor}}{M^{i-d}\epsilon' c^{\lfloor (m+d)/k \rfloor}} \leq c^{-1} & n=0, i \geq k+d. \end{aligned}$$

This shows that

$$\|f\| \leq c^{-1}\|T^i f\| \quad (f \in X, i \geq 1).$$

Now we will estimate the power bound for the inverse of an invertible extension S of T from below. Denote by $\mathbf{e}_{m,n}$ the function which is supported on (m,n) and has the value 1 at this point. Note that

$$T^{k^2+1}\mathbf{e}_{0,0} = \mathbf{e}_{k^2+1,0} + \sum_{i=0}^{k} \mathbf{e}_{ki,k(k-i)+1} = \mathbf{e}_{k^2+1,0} + \sum_{i=0}^{k} T^{k(k-i)}\mathbf{e}_{ki,1}$$

and

$$\|\mathbf{e}_{k^2+1,0}\| + \sum_{i=0}^{k}\|\mathbf{e}_{ki,1}\| = c^k + \sum_{i=0}^{k} M\epsilon' c^i < c^k + \epsilon'\frac{M}{1-c} < \epsilon.$$

Let S be any extension of T. If we had $\|S^{-n}\| \leq \epsilon^{-1}$ for all $n > 0$ then

$$1 = \|\mathbf{e}_{0,0}\| = \|S^{-k^2-1}T^{k^2+1}\mathbf{e}_{0,0}\| < 1$$

and we see that this cannot be true. So T has the stated properties.

Remark 3.2.6. In the example, we could also arrange that T is power bounded. For this we would have to change the weight, say to

$$w_{m,n} = \begin{cases} c^{\lfloor m/k \rfloor} & n = 0, \\ M^n \epsilon' c^{m/k} & 0 < n \leq k, \\ M^k \epsilon' c^{m/k} & k \leq n. \end{cases}$$

We note that the power bound is M^k, which can be chosen close to $\frac{2}{\epsilon(1-c)}$ by varying M.

Given $M > 1 > c > 0$, we can find an example of an operator T with

$$\|T^n x\| \geq c\|x\| \quad (x \in X, n \geq 0)$$

such that for any invertible extension S of T, S^{-1} is not power bounded. One way of doing this is to vary Example 3.2.5 by introducing a dependence of ϵ' in the definition of $w_{m,n}$ on the coordinate m to make ϵ' decreasing in m. This also has the effect of making k depend on m. We choose an alternative way and take a direct sum of operators from the previous example for different values of ϵ.

Example 3.2.7. Let $M > 1 > c > 0$. For $n > c^{-1}$, let $\epsilon = \frac{1}{n}$ and let T_n be the operator from Example 3.2.5, on a Banach space X_n. For $n > c^{-1}$, we have

$$\|T_n\| \leq M,$$
$$\|T_n^k x\| \geq c\|x\| \quad (k \geq 0, x \in X_n)$$

and if S_n is an invertible extension of T_n then $\sup_{k\geq 0} \|S_n^{-k}\| \geq n$. Define T on the ℓ_∞ direct sum $X = \bigoplus_{n>c^{-1}} X_n$ by $T(x_n) = (T_n x_n)$. Then T is bounded, as

$$\|T(x_n)\| = \sup_{n>c^{-1}} \|T_n x_n\| \leq M\|(x_n)\| \quad ((x_n) \in X)$$

and T has the uniform lower bound

$$\|T^k(x_n)\| \geq c\|(x_n)\| \quad (k \geq 0, (x_n) \in X).$$

If S is an invertible extension of T, then S is also an invertible extension of each T_n. Thus $\sup_{k\geq 0} \|S^{-k}\| \geq n$ for every $n > c^{-1}$ so that S^{-1} cannot be power-bounded.

Remark 3.2.8. In this example, we can use the power bounded version of Example 3.2.5 which was mentioned in Remark 3.2.6. We could then construct T in Example 3.2.7 such that $\|T^l\|$ grows less than exponentially fast in $l \geq 0$. That is, given $\alpha > 1$, we would have $\sup_{l\geq 0} \alpha^l \|T^l\| < \infty$. For this, we should fix $c > 0$, and choose decreasing bounds M_n, say $M_n = M_1^{1/2^n}$, to find T_n (we still assume $\epsilon = \frac{1}{n}$).

We can adapt Example 3.2.5 to C_0-semigroups by changing from the discrete domain Ω to a continuous domain.

Example 3.2.9. Let $M > 1 > c > \epsilon > 0$ be given. Find $k > 0$ such that $c^k < \epsilon$ and $M^k \frac{\epsilon - c^k}{2} > \frac{Mc}{1-c}$ and ϵ' such that $\frac{\epsilon-c^k}{2} < \frac{M}{1-c}\epsilon' < \epsilon - c^k$. Then $M^k\epsilon' > M^{k-1}(1-c)\frac{\epsilon-c^k}{2} > c$. Let

$$\Omega = \{(x,y) \in \mathbb{R}_+^2 : x = 0 \text{ or } y \in k\mathbb{N}\}$$

and define maps $T(t)$ by

$$(T(t)f)(x,y) = \begin{cases} f(x-t,y) & t < x \\ f(0,y-t+x) & x \leq t < x+y \\ f(0,0) & \text{otherwise} \end{cases}$$

for $t \geq 0$ and for functions $f\colon \Omega \to \mathbb{C}$. The operators satisfy $T(s)T(t) = T(s+t)$ for $t, s \geq 0$. Let μ denote the one-dimensional Lebesgue measure, which we can use on Ω. For a weight $w\colon \Omega \to (0,\infty)$ we can define the weighted L_p norms

$$\|f\|_p = \begin{cases} (\int_\Omega |f|^p w^p \,\mathrm{d}\mu)^{1/p} & 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_\Omega |f| w & p = \infty. \end{cases}$$

We will work with the ∞-norm. Before we specify the space on which we are working, let us define a weight w. Define

$$w(x,y) = \begin{cases} 1 & x = y = 0, \\ c^{\lceil y/k \rceil - 1} & x = 0, y > 0 \\ c^{y/k} M^x \epsilon' & x > 0. \end{cases}$$

Let $\tilde{X}$ be the Banach space

$$\tilde{X} := \{\, f\colon \Omega \to \mathbb{C}\colon \|f\|_{\infty,w} < \infty, f \text{ is continuous} \,\}$$

and let X be the closed subspace

$$X = \{\, f \in \tilde{X}\colon f(0,0) = 0, f(x,y) \to 0 \text{ as } (x,y) \to \infty \,\}.$$

Continuous functions on Ω with compact support are dense in X. Moreover, the operators $T(t)$ are bounded on X for $t \geq 0$. Indeed, observe that

$$\begin{aligned} w(x+t,y) &\leq M^t w(x,y) \\ w(0,y+t) &\leq w(0,y) \end{aligned} \qquad ((x,y) \in \Omega, t \geq 0),$$

which implies that $\|T(t)\| \leq M^t$ for all $t \geq 0$. It is clear that X is invariant under $T(t)$ $(t \geq 0)$. Strong continuity of T follows easily for continuous functions with compact support, as T acts as a translation. Since these functions are dense in X, and since $\|T(t)\| \leq M$ for $0 \leq t \leq 1$, we have $\|T(t)f - f\| \to 0$ as $t \to 0$ for all $f \in X$. Hence, T is a C_0-semigroup. We show next that

$$\|T(t)f\| \geq c\|f\| \quad (f \in X, t \geq 0).$$

Let $f \in X$ and $t \geq 0$. We shall show that $\|f\| \leq c^{-1}\|T(t)f\|$. We can do this pointwise. If $(x,y) \in \Omega$ with $x > 0$ then clearly

$$w(x,y)|f(x,y)| \leq w(x+t,y)|f(x,y)| = w(x+t,y)|T(t)f(x+t,y)|. \tag{3.4}$$

If $x = 0$ choose $n = \lceil y/k \rceil$. Then either $0 \leq t < (nk - y) + k$ and

$$w(0,y)|f(0,y)| \leq c^{-1}w(0,y+t)|T(t)f(0,y+t)|, \tag{3.5}$$

or $t \geq (nk - y) + k$ and

$$w(0,y)|f(0,y)| \leq c^{-1}w(t-(nk-y),nk)|T(t)f(t-(nk-y),nk)|. \tag{3.6}$$

The terms on the right hand side in (3.4), (3.5) and (3.6) are bounded by $c^{-1}\|T(t)f\|$. We can take the supremum over (x,y) in (3.4)-(3.6) to get that $c\|f\| \leq \|T(t)f\|$.

Next, we will show that for any extending group S of T we have

$$\sup_{t \geq 0} \|S(-t)\| \geq \epsilon^{-1}.$$

Let $0 < \delta < k$ and take a continuous function $\varphi\colon \mathbb{R} \to [0,1]$ with support in $(0,\delta)$ and $\sup_{x\in\mathbb{R}} \varphi(x) = 1$. Let

$$f(x,y) = \begin{cases} \varphi(y) & x = 0, y \geq 0, \\ 0 & x > 0. \end{cases}$$

Then $f \in X$ and $\|f\| = 1$. For $i \geq 1$, let

$$f_i(x,ik) = \begin{cases} \varphi(x) & y = ik, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_i \in X$ and $\|f_i\| \leq M^\delta c^{i-1}\epsilon'$. Finally, let

$$g(x,y) = \begin{cases} \varphi(y-k^2) & x = 0 \\ 0 & x > 0, \end{cases}$$

so that $g \in X$ and $\|g\| = c^k$. We have

$$T(k^2)f = g + \sum_{i=1}^{k} T(k(k-i))f_i$$

and

$$\begin{aligned} \|g\| + \sum_{i=1}^{k} \|f_i\| &\leq c^k + \epsilon' M^\delta \frac{1}{1-c} \\ &\leq c^k + (\epsilon - c^k) \leq \epsilon. \end{aligned}$$

If S is an extending group of T then

$$1 = \|S(-k^2)T(k^2)f\| \leq \|S(-k^2)\| \left(\|g\| + \sum_{i=1}^{k} \|f_i\| \right) \leq \sup_{t\geq 0} \|S(-t)\|\epsilon.$$

We see that $\sup_{t\geq 0} \|S(-t)\| \geq \epsilon^{-1}$ must hold.

Remark 3.2.10. The generator of T in Example 3.2.9 is the operator $Af = -f'$ with maximal domain

$$D(A) = \{\, f \in X \colon f \text{ is differentiable and } f' \in X \,\}.$$

Example 3.2.11. We can repeat the direct sum construction from Example 3.2.7, this time for C_0-semigroups T_n from Example 3.2.9. What we get is the following. Let $M > 1 > c > 0$. There exists a C_0-semigroup T on a Banach space X such that

$$\begin{aligned} \|T(t)\| &\leq M \\ \|T(t)x\| &\geq c\|x\| \end{aligned} \qquad (x \in X, t \geq 0),$$

and if S is any extending C_0-group of T, then $\sup_{t\geq 0} \|S(t)\| = \infty$. Note that this in particular implies that T is not similar to an expansive C_0-semigroup.

Let us now turn to C_0-semigroups on Hilbert spaces.

Example 3.2.12. We can use the shift operator from Example 3.2.9 on the Hilbert space $X = L_2(\Omega, w, \mu)$ with norm

$$\|f\|^2 = \int_\Omega |f|^2 w^2 \,\mathrm{d}\mu.$$

It is easy to see that $T(t)$ is bounded for $t \geq 0$ and that T is a C_0-semigroup on X. Indeed, we can estimate $\|T(t)\|$ by

$$\begin{aligned}\|T(t)f\|^2 &= \int_\Omega |T(t)f|^2 w^2 \,\mathrm{d}\mu \\ &\leq 2M^{2t} \int_\Omega w^2 \,\mathrm{d}\mu \qquad (f \in X, 0 \leq t \leq k). \\ &= 2M^{2t} \|f\|^2\end{aligned}$$

This implies $\|T(t)\| \leq \sqrt{2}M^t$ for $0 \leq t \leq k$. Hence, we get the trivial estimate

$$\|T(t)\| \leq \sqrt{2}\Big(\sqrt{2}^{1/k} M\Big)^t \quad (t \geq 0).$$

Let us verify the inequality

$$\|T(t)f\| \geq c\|f\| \quad (f \in X, t \geq 0).$$

For this, let $f \in X$, $(x, y) \in \Omega$ and $t \geq 0$. In (3.4)-(3.6) in Example 3.2.9, we found

$$w(x,y)f(x,y) \leq c^{-1} \begin{cases} w(x+t,y)|T(t)f(x+t,y)| & x > 0, \\ w(0,y+t)|T(t)f(0,y+t)| & x = 0, t < k - y', \\ w(t+y',nk)|T(t)f(t+y',nk)| & x = 0, t \geq k - y', \end{cases} \tag{3.7}$$

where $y' = y - nk$ and $n = \lceil y/k \rceil$. To calculate $\|f\|$, integrate $|T(t)f|^2 w^2$ over $(x, y) \in \Omega$. Split the integral into contributions from the disjoint subsets

$$\{(x,y) \colon x > 0\}, \quad \{(0,y) \colon t < k - y + \lceil y/k \rceil\}, \quad \{(0,y) \colon t \geq k - y + \lceil y//k \rceil\}.$$

We can estimate each contribution pointwise by (3.7). Now take the sum of these inequalities and use the triangle inequality to get $\|f\| \leq c^{-1}\|T(t)f\|$.

To see that no extending group S of T can be bounded by ϵ^{-1}, we can use the same method as in Example 3.2.9. This time, we can choose $\varphi \geq 0$ with $\operatorname{supp} \varphi \subset (0, \delta)$ and $\int_\mathbb{R} \varphi(x)^2 \mathrm{d}x = 1$.

Consider a C_0-semigroup T on a Hilbert space X. We shall try to find an extending C_0-group on a Hilbert space. The results we obtain do not resemble Theorem 3.2.2 or Theorem 3.2.4. Remember that in Theorem 3.1.4, we get an invertible extension for a single operator on a Hilbert space. We can use the extending homomorphism to obtain the following result for C_0-semigroups on Hilbert space.

Proposition 3.2.13. *Let T be a C_0-semigroup on a Hilbert space X. Assume that*

$$\|T(t)x\| \geq c\|x\| \quad (x \in X, t > 0).$$

for some $c > 0$. For each $\omega > 0$, there exist $M \geq 1$, a Hilbert space $Y \supset X$ and a C_0-group S on Y such that $S(t)x = T(t)x$ and $\|S(-t)\| \leq Me^{\omega t}$ for all $x \in X, t > 0$.

Proof. Let $\alpha > 0$. We can apply Theorem 3.1.4 to $T(\alpha)$ to find an extending group S of T on a Hilbert space $Y \supset X$. To find S, we can define it by

$$S(t) := \varphi(T(t)) \quad (t \geq 0),$$

where φ is the homomorphism defined on the commutant of $T(\alpha)$. Since $\|\varphi\| \leq \sqrt{2}$, the C_0-semigroup S has the same exponential growth as T. Moreover, $S(\alpha) = \varphi(T(\alpha))$ is invertible. Restricting to the largest subspace on which S is strongly continuous, if necessary, we can assume that S is strongly continuous on Y. Hence, S is a C_0-group on Y and S extends T. For the powers of $S(\alpha)^{-1}$, Theorem 3.1.4 gives the estimate

$$\|S(-\alpha n)\| \leq \frac{\sqrt{2}^{n+1}}{c^n} \quad (n \geq 1).$$

This implies

$$\|S(-t)\| \leq Me^{\omega t} \quad (t > 0)$$

where $\omega = \frac{\log\sqrt{2} - \log c}{\alpha}$ and $M \geq 1$ is chosen suitably. So for any given $\omega > 0$ we can choose $\alpha = \frac{\omega}{\log\sqrt{2} - \log c} > 0$ to get an extension S such that $\omega_0(S^-) \leq \omega$, where S^- is the semigroup $S^-(t) = S(-t)$ $(t > 0)$. □

There is another construction of an extending C_0-group on a Hilbert space, which is based on work by Zwart (see [Zwa13, Theorem 3]). In this result, an estimate on the norm of $\|S(t)\|$ for $t \in \mathbb{R}$ is not immediate. That is, we do not know how $\|S(t)\|$ and $\|T(t)\|$ can be related for $t > 0$. Also, if we assume further that $\|T(t)x\| \geq \|x\|$ for $t \geq 0$, for example, the theorem does not say anything about $\|S(-t)\|$ for $t > 0$.

Theorem 3.2.14. *Let T be a C_0-semigroup on a Hilbert space X. Assume that*

$$\|T(t)x\| \geq c(t)\|x\| \quad (x \in X, t \geq 0)$$

for some $c(t) > 0$. There exist a Hilbert space $Y \supset X$ and a C_0-group S on Y such that $S(t)x = T(t)x$ for all $t \geq 0, x \in X$.

Proof. Let A be the generator of T. By [Zwa13, Theorem 3], there exist a bounded operator Q and an equivalent Hilbert space norm $\|\cdot\|'$ on X such that $A+Q$ generates an isometric semigroup $\tilde{T}$.

As an isometric semigroup, $\tilde{T}$ has a unitary extension $\tilde{S}$ to some Hilbert space Y with a norm $\|\cdot\|'_Y$. We now renorm the space Y. Let P be the orthogonal projection in Y onto X, orthogonal with respect to $\|\cdot\|'_Y$, and let

$$\|y\|_Y^2 := \|Py\|^2 + \|(I-P)y\|_Y'^{\,2} \quad (y \in Y).$$

This defines a Hilbert space norm on Y which is equivalent to $\|\cdot\|'$ and is equal to $\|\cdot\|$ on X. Let $\tilde{B}$ be the generator of $\tilde{S}$ and let S be the C_0-group generated by the bounded perturbation $B = \tilde{B} - QP$. Then B extends A and S extends T. ☐

3.3 The inverse of an unbounded operator

The results presented in Sections 3.1 and 3.2 applied only to bounded operators. We can also try to find an extension with a bounded inverse for an unbounded operator. The technique used in Theorems 3.1.1 was originally formulated for Banach algebras and could be easily reformulated for bounded operators. But the results from Section 3.1 cannot be directly generalised to unbounded operators. However, we will find some positive results for generators of C_0-semigroups. In the case of Theorem 3.1.5, remember that the proof relies on the polar decomposition of the operator T in question. The polar decomposition exists if T is closed and has dense domain. Indeed, we will find that the statement of Theorem 3.1.5 holds for those operators.

The following result is a version of Theorem 3.1.1 for generators of C_0-semigroups.

Theorem 3.3.1. *Let A be the generator of a C_0-semigroup on a Banach space X. Assume that*

$$\|Ax\| \geq c\|x\| \quad (x \in D(A))$$

for some $c > 0$. Then there exist a Banach space $Y \supseteq X$ and an operator B on Y which is the generator of a C_0-semigroup, has a bounded inverse with $\|B^{-1}\| \leq c^{-1}$ and satisfies $D(A) = D(B) \cap X$ and $Bx = Ax$ for all $x \in D(A)$.

Furthermore, there is a unital isometric algebra homomorphism

$$\varphi\colon \{\, R(\lambda, A)\colon \lambda \in \rho(A) \,\}' \to \mathcal{B}(Y)$$

such that $\varphi(U)x = Ux$ $(x \in X)$ and $\varphi(R(\lambda, A)) = R(\lambda, B)$ whenever $\lambda \in \rho(A)$.

Proof. By rescaling we can assume without loss of generality that $c = 1$. Let T be the C_0-semigroup generated by A. As in Proposition 2.6.4, define operators

$$\tilde{T}(t)(x_n) := (T(t)x_n) \quad ((x_n) \in \ell_1(X), t \geq 0)$$

on $\ell_1(X)$. Then $\tilde{T}$ is a C_0-semigroup on $\ell_1(X)$ and its generator C is given by $C(x_n) = (Ax_n)$ with domain

$$D(C) = \{(x_n) \in \ell_1(X)\colon x_n \in D(A)(n \in \mathbb{N}),\ (Ax_n) \in \ell_1(X)\}.$$

The resolvent of C is given by

$$R(\lambda, C)(x_n) = (R(\lambda, A)x_n) \quad (\lambda \in \rho(A),\ (x_n) \in \ell_1(X)).$$

Let J be the closure of $\{f - CRf\colon f \in D(C)\}$ in $\ell_1(X)$, where R denotes the right shift on $\ell_1(X)$. Then J is invariant under $\tilde{T}$ and R.

Let $Y = \ell_1(X) \,/\, J$ and embed X in Y using the inclusion $\rho(x) = x\,\mathbf{e}_0 + J$. By definition of the quotient norm, it is clear that $\|\rho(x)\| \leq \|x\|$ for all $x \in X$. Using the triangle inequality and the lower bound of A repeatedly (see (3.1)), we get for $x \in X$ and finite sequences $f = (x_n)$ with $x_n \in D(A)$

$$\begin{aligned}\|x\| &\leq \|x - x_0\| + \|Ax_0\| \leq \|x - x_0\| + \|x_1 - Ax_0\| + \|Ax_1\| \\ &\leq \|x - x_0\| + \sum_{n=1}^{\infty} \|x_n - Ax_{n-1}\|.\end{aligned}$$

Note that the series converges since it is in fact a finite sum. Such sequences f are in $D(C)$ and the inequality reads as

$$\|x\| \leq \|x\mathbf{e}_0 - (1 - CR)f\|. \tag{3.8}$$

Let $f \in D(C)$. Define g^k by

$$g^k_n = \begin{cases} f_n & n \leq k \\ 0 & n > k \end{cases} \quad (n \in \mathbb{N}).$$

Then $g^k \in D(C)$ and (3.8) holds with f replaced by g^k. We show that $g^k \to f$ as $k \to \infty$ in the graph norm of C. It is clear that $\|g^k - f\| \to 0$ as $k \to \infty$. Since $(Cf)_n = Ag_n^k$ when $n \le k$ we also get $\|Cg^k - Cf\| \to 0$ as $k \to \infty$. This means $\|g^k - f\|_C \to 0$ as $k \to \infty$. So we can take the limit in

$$\|x\| \le \|x\mathbf{e}_0 - (1 - CR)g^k\|$$

as $k \to \infty$ and see that f satisfies (3.8). Hence $\|x\| \le \|\rho(x)\|$ for all $x \in X$ and ρ is isometric.

Let $\pi\colon \ell_1(X) \to Y$ be the quotient map $\pi(f) = f + J$. On Y, define operators

$$S(t)\pi(f) := \pi(\tilde{T}(t)f) \quad (f \in \ell_1(X), t \ge 0)$$

as in Proposition 2.6.2. Then S is a C_0-semigroup and it has a generator B with domain $D(B) = \pi D(C)$. We have that

$$S(t)\rho(x) = \rho(T(t)x) \quad (x \in X, t \ge 0),$$

so that $\rho(x) \in D(B)$ and $B\rho(x) = \rho(Ax)$ for all $x \in D(A)$. Since ρ is an isometry onto its image, we see that if $\rho(x) \in D(B)$ then $x \in D(A)$. This shows $D(A) = D(B) \cap X$.

We will show next that B has a bounded inverse. Define V on Y by

$$V\pi(f) := \pi(Rf) \quad (f \in \ell_1(X)).$$

Since J is invariant under R, the operator V is well-defined and bounded with $\|V\| \le \|R\| = 1$. To see that V is the inverse of B, let $y \in D(B)$ and choose $f \in D(C)$ with $y = \pi(f)$. Then

$$BVy = \pi(CRf) = \pi(RCf) = VBy$$

and

$$BVy = \pi(CRf) = \pi(f) - \pi(f - CRf) = y.$$

This shows that $VBy = y$ for all $y \in D(B)$ and $BVy = y$ for all $y \in Y$, since B is closed, V is bounded and $D(B)$ is dense in Y.

Now, assume that $U \in \mathcal{B}(X)$ commutes with $R(\lambda, A)$ for some (or equivalently all) $\lambda \in \rho(A)$. The operator

$$\varphi(U)\pi(f) := \pi((Uf_n)) \quad (f \in \ell_1(X))$$

is well-defined and satisfies $\|\varphi(U)\| \leq \|U\|$. It is clear that φ is a unital algebra homomorphism and for $U = R(\lambda, A)$ this implies that $\{\varphi(R(\lambda, A)) \colon \lambda \in \rho(A)\}$ is a pseudo-resolvent. For all $f \in \ell_1(X)$ we have that

$$\varphi(R(\lambda, A))\pi(f) = \pi((R(\lambda, A)f_n)) = \pi(R(\lambda, C)f) = R(\lambda, B)\pi(f).$$

So $\varphi(R(\lambda, A)) = R(\lambda, B)$ holds for all $\lambda \in \rho(A)$. □

Remark 3.3.2. It seems plausible that Theorem 3.3.1 holds for operators A with non-empty resolvent set, or at least for such A which are densely defined. We can try to proceed as in the proof, given an operator A with non-empty resolvent set $\rho(A)$. It is possible to define an operator C on $\ell_1(X)$ with domain

$$D(C) = \{(x_n) \in \ell_1(X) \colon (Ax_n) \in \ell_1(X)\}$$

and $\rho(C) = \rho(A)$ (see Proposition 2.6.4). We can define a closed subspace J as in the proof of Theorem 3.3.5, which is invariant under $R(\lambda, C)$ $(\lambda \in \rho(C))$. However, the operators

$$R_\lambda \pi(f) := \pi(R(\lambda, C)f) \quad (f \in \ell_1(X), \lambda \in \rho(C))$$

on the quotient space $\ell_1(X) \,/\, J$ are only a pseudo-resolvent and possibly not injective. We have to analyse J further to see if R_λ is injective and thus the resolvent of an operator B on $\ell_1(X) \,/\, J$.

The same construction allows us to get a weaker result for generators of once integrated semigroups where the embedding is no longer isometric but merely continuous.

Proposition 3.3.3. *Let A be the generator of a once integrated semigroup on a Banach space X. Assume that*

$$\|Ax\| \geq c\|x\| \quad (x \in D(A))$$

for some $c > 0$. Then there exist a Banach space Y, a continuous embedding $\pi \colon X \to Y$ and an operator B on Y which is the generator of a once integrated semigroup and has a bounded inverse. Furthermore, $\|B^{-1}\| \leq c^{-1}$ and π satisfies $\pi(D(A)) \subset D(B)$ and $B\pi(x) = \pi(Ax)$ for all $x \in D(A)$.

Proof. Without loss of generality, assume that $c = 1$. Let T be the once integrated semigroup generated by A with

$$\left\| \int_0^t T(s)x \,\mathrm{d}s \right\| \leq Me^{\omega t}\|x\| \quad (x \in X, t \geq 0)$$

for some $\omega \geq 0$ and $M \geq 1$. Pass to the quotient $\tilde{Y} = \ell_1(X) \,/\, J$. The embedding of X into Y is still isometric by the same argument as in the proof of Theorem 3.3.1. On $\tilde{Y}$, there is a degenerate once integrated semigroup given by

$$\tilde{S}(t)((x_n) + J) := ((T(t)x_n) + J) \quad ((x_n) \in \ell_1(X), t \geq 0).$$

By Corollary 2.5.10 we can pass to a quotient space $Y = \tilde{Y} \,/\, N$ where N is the degeneration space of $\tilde{S}$. We get a non-degenerate once integrated semigroup S on Y. The composition $\pi \colon X \to Y$ of the quotient maps intertwines T and S. It remains to show that π is injective and the generator B of S is invertible with $\|B^{-1}\| \leq 1$.

Note that the map $\rho \colon X \to \ell_1(X) \,/\, J$ with $\rho(x) = x\mathbf{e}_0 + J$ is isometric (see the proof of Theorem 3.3.1). To see that π is injective, it is sufficient to show that $\rho(x) \in N$ implies $x = 0$. Let $x \in X$ such that $\rho(x) \in N$. By definition of N, we have $\tilde{S}(t)\rho(x) = 0$ for all $t \geq 0$. Since ρ is injective and intertwines T and $\tilde{S}$, we have $T(t)x = 0$ for all $t \geq 0$. But T is non-degenerate so this implies $x = 0$, as required.

Let $\tilde{T}$ be the non-degenerate once integrated semigroup

$$\tilde{T}(t)(x_n) := (T(t)x_n) \quad ((x_n) \in \ell_1(X), t \geq 0)$$

on $\ell_1(X)$ with generator C. Let $V((f + J) + N) := (Rf + J) + N$ so that $\|V\| \leq 1$. Let $y \in D(B) \subset Y$. Then

$$y = R(\lambda, B)z = \lambda \int_0^\infty e^{-\lambda t} S(t)z \,\mathrm{d}t$$

for some $z \in Y$ and $\lambda > \omega$. Take $g \in \ell_1(X)$ such that $z = (g + J) + N$ and let $f = R(\lambda, C)g$. Then $y = (f + J) + N$ and

$$VBy = (RCf + J) + N = (f + J) + N = y.$$

We conclude that $B^{-1} = V$ as in the proof of Theorem 3.3.1. □

Open Question 3.3.4. Does Proposition 3.3.3 hold for generators of k-times integrated semigroups?

We get the following analogue of Theorem 3.1.5 for unbounded operators.

Theorem 3.3.5. *Let A be a closed, densely defined operator on a Hilbert space X. Assume that*

$$\|Ax\| \geq c\|x\| \quad (x \in D(A))$$

for some $c > 0$. Then X embeds isometrically into a Hilbert space Y and there is an invertible, densely defined operator B on Y such that $D(A) = D(B) \cap X, Ax = Bx$ for all $x \in D(A)$ and $\|B^{-1}\| \leq c^{-1}$. Moreover, we have that $\sigma(B) \subset \sigma(A)$.

Moreover, if A generates a C_0-semigroup, then so does B.

Proof. Without loss of generality we assume that $c = 1$ and that A is not invertible. The operator has a polar decomposition $A = UP$ where $P = (A^*A)^{\frac{1}{2}}$ has domain $D(P) = D(A)$. The operator A^*A has dense range so that U is an isometry from X onto the closed subspace $\operatorname{ran} A$. The construction of Y and B works as in the proof of Theorem 3.1.5, where B takes the role of S. So we take $Y = X \oplus X$ and define B as the operator

$$B = \begin{pmatrix} A & 0 \\ 0 & U^* \end{pmatrix} + \begin{pmatrix} 0 & I - UU^* \\ 0 & 0 \end{pmatrix}.$$

with domain $D(B) = D(A) \times X$. Invertibility of B with $\|B^{-1}\|$ and the inclusion $\sigma(B) \subset \sigma(A)$ follow also as in the proof of Theorem 3.1.5.

If A generates a C_0-semigroup T, we see that B is a bounded perturbation of the generator of the C_0-semigroup

$$\begin{pmatrix} T(t) & 0 \\ 0 & \exp(tU^*) \end{pmatrix} \quad (t \geq 0).$$

Thus B itself generates a C_0-semigroup. □

In the proof of the theorem, we saw that the extension B of A generates a C_0-semigroup if A generates one. Let T be the C_0-semigroup generated by A. The bounded perturbation in the definition of B acts like a shift operator. We can give an explicit description of the C_0-semigroup S generated by B on a dense subset of the minimal part of Y. The minimal part of Y is the closure Z of the set

$$\bigcup_{n \geq 0} B^{-n} X.$$

It is given by

$$Z = X \oplus \overline{\operatorname{span}}\{U^n(I - UU^*)x \colon x \in X, n \geq 0\},$$

as in the case of a bounded operator. To describe the C_0-semigroup S generated by B on Z, we will use the Dyson-Phillips variation of parameters formula (see [EN00, Corollary III.3.15]). The C_0-semigroup S satisfies

$$\begin{aligned} S(t)z = &\begin{pmatrix} T(t) & 0 \\ 0 & \exp(tU^*) \end{pmatrix} z \\ &+ \int_0^t S(s) \begin{pmatrix} 0 & (I - UU^*) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T(t-s) & 0 \\ 0 & \exp((t-s)U^*) \end{pmatrix} z \,\mathrm{d}s \end{aligned}$$

for $z \in Z \cap (D(A) \times X)$ and $t \geq 0$. We can solve this equation and find

$$\begin{aligned} S(t) &= \begin{pmatrix} T(t) & \int_0^t T(s)(I - UU^*) \exp((t-s)U^*) \,\mathrm{d}s \\ 0 & \exp(tU^*) \end{pmatrix} \\ &= \begin{pmatrix} T(t) & \sum_{k=0}^\infty S_{k+1}(t)(I - UU^*)(U^*)^k \\ 0 & \exp(tU^*) \end{pmatrix} \end{aligned} \quad (t \geq 0), \tag{3.9}$$

where the integral and the series exist in the strong operator topology, and where

$$S_k(t)x := \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} T(s)x \,\mathrm{d}s \qquad (x \in X, k \geq 1)$$

is the k-times integrated semigroup of T. To check that this formula defines the C_0-semigroup generated by B we show that it satisfies the semigroup property and that its right-derivative at 0 is the operator B. For the semigroup property, we first note that

$$T(s)S_{k+1}(t)x = \int_s^{s+t} \frac{((s+t)-u)^k}{k!} T(u)x \,\mathrm{d}u \qquad (x \in X,\ t,s \geq 0)$$

and

$$\sum_{n=0}^k \frac{t^n}{n!} S_{k+1-n}(s)x = \int_0^t \sum_{n=0}^k \frac{t^n (s-u)^{k-n}}{n!(k-n)!} T(u)x \,\mathrm{d}u = \int_0^s \frac{((s+t)-u)^k}{k!} T(u)x \,\mathrm{d}u \tag{3.10}$$

for $x \in X$ and $t, s \geq 0$. The top right corner in the product $S(s)S(t)$ is

$$\begin{aligned} & T(s)\sum_{k=0}^\infty S_{k+1}(t)(I-UU^*)(U^*)^k + \sum_{k=0}^\infty S_{k+1}(s)(I-UU^*)(U^*)^k \exp(tU^*) \\ &= \sum_{k=0}^\infty T(s)S_{k+1}(t)(I-UU^*)(U^*)^k + \sum_{k,m=0}^\infty \frac{t^m}{m!} S_{k+1}(s)(I-UU^*)(U^*)^{k+m} \\ &= \sum_{k=0}^\infty \left(T(s)S_{k+1}(t) + \sum_{m=0}^k \frac{t^m}{m!} S_{k+1}(s) \right) (I-UU^*)(U^*)^k, \end{aligned}$$

which is equal to the top right corner of the matrix representation (3.9) of $S(s+t)$. The remaining entries of $S(s)S(t)$ and $S(s+t)$ trivially agree. This shows that $S(s)S(t) = S(s+t)$ for all $s, t \geq 0$. We can calculate the derivative of S formally, noting that $\frac{1}{t}S_k(t)x \to 0$ as $t \to 0$ for $x \in X$ when $k > 1$, and

$$\frac{1}{t}S_1(t)x = \frac{1}{t}\int_0^t T(u)x \,\mathrm{d}u \to x \qquad (x \in X)$$

as $t \to 0$. Using this in (3.9) shows that S is generated by B.

3.4 Growth rates for the inverse of an unbounded operator

Let us now turn to invertible extensions of an unbounded operator which allow control of the growth of the inverse. The next proposition is a version of Theorem 3.2.3 for generators of C_0-semigroups.

Proposition 3.4.1. *Let A be the generator of a C_0-semigroup and let (c_n) be a positive submultiplicative sequence. Let $\mathcal{D} \subset D(A^\infty)$ be a core of A. Assume that*

$$\|x\| \le c_n\|x_0\| + c_{n-1}\|x_1\| + \cdots + c_1\|x_{n-1}\|$$

whenever $x, x_i \in \mathcal{D}$ and $A^n x = x_0 + Ax_1 + \cdots + A^{n-1}x_{n-1}$. Then there is a generator B of a C_0-semigroup on a Banach space $Y \supset X$ such that $D(A) = D(B) \cap X$, $Bx = Ax$ for all $x \in D(A)$ and B has a bounded inverse which satisfies $\|B^{-n}\| \le c_n$ for all $n \ge 1$. Moreover, there is an isometric algebra homomorphism

$$\{R(\lambda, A) \colon \lambda \in \rho(A)\}' \to \mathcal{B}(Y)$$

which maps $R(\lambda, A)$ to $R(\lambda, B)$ whenever $\lambda \in \rho(A)$.

Proof. The proof again uses a quotient of $\ell_1(X)$. This time we use a weighted ℓ_1-norm. Let $w \colon \mathbb{N} \to \mathbb{R}$ be the weight defined by

$$w(n) = \begin{cases} 1 & n = 0 \\ c_n & n \ge 1. \end{cases}$$

The norms of the powers of the right shift R on $\ell_1(X, w)$ are

$$\|R^n\| = \sup_{k \ge 0} \frac{w(k+n)}{w(k)} \le c_n \quad (n \ge 1). \tag{3.11}$$

Let T be the semigroup generated by A. Similarly to Proposition 2.6.4, the operators

$$\tilde{T}(t)(x_n) := (T(t)x_n) \quad ((x_n) \in \ell_1(X, w), t \ge 0)$$

form a semigroup on $\ell_1(X, w)$ which has a generator C. The resolvent of C is given by

$$R(\lambda, C)(x_n) = (R(\lambda, A)x_n) \quad ((x_n) \in \ell_1(X, w), \lambda \in \rho(A)),$$

and the domain of C is

$$D(C) = \{(x_n) \in \ell_1(X, w) \colon x_n \in D(A) \text{ for all } n \ge 0, (Ax_n) \in \ell_1(X, w)\}.$$

Let J be the closure of the subspace

$$\{\, f - CRf \colon f \in D(C)\,\}$$

in $\ell_1(X, w)$. The subspace J is invariant under the semigroup $\tilde{T}$ and under the right shift R. Let $Y = \ell_1(X, w) \,/\, J$.

Define the embedding $\rho\colon X \to Y$ as $\rho(x) = x\,\mathbf{e}_0 + J$. We want to show that ρ is isometric. We get an inequality in one direction by $\|\rho(x)\| \le \|x\mathbf{e}_0\| = \|x\|$. For the reverse inequality, let $(y_k)_{k=0}^{n-1}$ be a finite sequence in $\mathcal{D}$. We can write

$$A^n y_0 = (Ay_{n-1}) + A(Ay_{n-2} - y_{n-1}) + \cdots + A^{n-1}(Ay_0 - y_1)$$

which by assumption gives $\|y_0\| \le \sum_{k=1}^n c_k \|Ay_{k-1} - y_k\|$ if we set $y_n = 0$. If we combine this with the triangle inequality $\|x\| \le \|x - y_0\| + \|y_0\|$ we get

$$\|x\| \le \|x - y_0\| + \sum_{k=1}^{n} c_k \|Ay_{k-1} - y_k\|$$

for all finite sequences $(y_k)_{k=0}^{n-1}$ in $\mathcal{D}$. In other words

$$\|x\| \le \|\rho(x) + f - CRf\|, \tag{3.12}$$

where $f \in \ell_1(X, w)$ is a finitely supported sequence with values in $\mathcal{D}$. Finite sequences with values in $\mathcal{D}$ are dense in $D(C)$ in the graph norm of C (see the proof of Theorem 3.3.1). So (3.12) holds for all $f \in D(C)$. This shows that $\|x\| \le \|\rho(x)\|$. Thus ρ is an isometric embedding.

The remainder of the proof is as in the proof of Theorem 3.3.1, the only difference being the estimate $\|R^n\| \le c_n$ $(n \ge 1)$ in (3.11). □

In this proof, we again used a quotient construction as in Theorem 3.1.1. This allows us to define the unbounded extension in a natural way, since the C_0-semigroup generated by A enables us to apply standard constructions for bounded operators. As the proof shows, the property that is used for obtaining an isometric embedding is that, for a finite sequence of elements $y_i \in D(A)$, we have

$$\|y_0\| \le \sum_{k=1}^{n} c_k \|Ay_{k-1} - y_k\|.$$

This was the condition used in [BM05, Theorem 2.1], in a formulation for Banach algebras.

Finally, we give a result for exponentially bounded k-times integrated semigroups, which is a consequence of Proposition 3.4.1.

Corollary 3.4.2. *Let A be a densely defined generator of a k-times integrated semigroup S on a Banach space X for some $k \ge 1$. Assume that $\|S(t)\| \le Me^{\omega t}$ $(t \ge 0)$ for some $M \ge 1, \omega \ge 0$. Let (c_n) be a positive submultiplicative sequence. Assume that*

$$\|x\| \le c_n\|x_0\| + c_{n-1}\|x_1\| + \cdots + c_1\|x_{n-1}\|$$

whenever $A^n x = x_0 + Ax_1 + \cdots + A^{n-1}x_{n-1}$ for $x, x_i \in D(A^\infty)$. Then X embeds continuously into a Banach space Y on which A has an extending C_0-semigroup generator B which has an inverse B^{-1} with $\|B^{-n}\| \le c_n$ $(n \ge 1)$.

Proof. Let $\|\cdot\|_Y$ be the norm from the Sandwich Theorem (see Proposition 2.5.11), that is

$$\|x\|_Y := \sup_{t\ge 0} \|e^{-bt}T(t)R(\mu_0, A)^k x\|_X \quad (x \in X)$$

for some $\operatorname{Re}\mu_0 > b > \omega$ so that $\mu_0 \in \rho(A)$, and

$$T(t)x = S(t)A^k x + \frac{t^{k-1}}{(k-1)!}A^{k-1}x + \cdots + tAx + x \quad (x \in X, t \ge 0).$$

Recall that on the completion Y of X under this norm, the resolvent of A extends to the resolvent of a C_0-semigroup generator C. We will show that C satisfies the assumptions of Proposition 3.4.1.

First, let us show that $D(A^\infty)$ is a core of $D(C)$ and contained in $D(C^\infty)$. Since $R(\lambda, C)$ extends $R(\lambda, A)$ for $\lambda \in \rho(A)$, it is clear that $D(A^\infty) \subset D(C)$. Since $D(A^\infty)$ is dense in X, X is dense in Y, and X is continuously embedded in Y, we get that $D(A^\infty)$ is dense in Y. Let $z \in D(C)$ and let (x_n) be a sequence in $D(A^\infty)$ such that $\|x_n - (\lambda - C)z\|_Y \to 0$ as $n \to \infty$. Then

$$\begin{aligned} \|R(\lambda, C)x_n - z\|_Y &\to 0, \text{ and} \\ \|(\lambda - C)R(\lambda, C)x_n - (\lambda - C)z\|_Y &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

So $R(\lambda, C)x_n$ converges to z in the graph norm of C, and $R(\lambda, C)x_n = R(\lambda, A)x_n \in D(A^\infty)$ for all $n \ge 0$. This shows that $D(A^\infty)$ is a core of $D(C)$.

Let $y, y_i \in D(A^\infty)$ with $C^n y = \sum_{i=0}^{n-1} C^i y_i$. Then $e^{-bt}T(t)R(\mu_0, A)^k z \in D(A^\infty)$ for $z = y, y_i$, since $D(A^\infty)$ is invariant under $R(\mu_0, A)$ and $S(t)$ (by [ABHN01, Lemma 3.2.2.b)], for example) and thus also under $T(t)$ $(t \ge 0)$. So

$$C^n e^{-bt}T(t)R(\mu_0, A)^k y = \sum_{i=0}^{n-1} C^i e^{-bt}T(t)R(\mu_0, A)^k y_i$$

for every $t \ge 0$. Since $C_X = A$ by the Sandwich Theorem, we get by assumption

$$\|e^{-bt}T(t)R(\mu_0, A)^k y\| \le \sum_{i=0}^{n-1} c_{n-i}\|e^{-bt}T(t)R(\mu_0, A)^k y_i\|$$

and thus

$$\|y\|_Y \le \sum_{i=0}^{n-1} c_{n-i}\|y_i\|_Y.$$

We can now apply Proposition 3.4.1 and get the required space Y and operator B. $\square$

3.5 Left inverses and dissipative extensions on Hilbert space

So far we have been discussing extensions of operators on larger Banach (or Hilbert) spaces. Often, one sees extensions of (typically unbounded) operators which are defined on the same space. This is, for example, the case for dissipative operators on Hilbert space (see [Dav07, Section 10.4]). A densely defined dissipative operator A on a Hilbert space X always has an extension to a maximal dissipative operator B on the same Hilbert space X. Here, we say that B extends A if $D(A) \subset D(B)$ and $Ax = Bx$ for all $x \in D(A)$. The extension B generates a C_0-semigroup on X ([Dav07, Theorem 10.4.2]). We can obtain a maximal dissipative extension B of a dissipative operator A on a Hilbert space using the Cayley transform. If

$$\tilde{C} = (I + A)(I - A)^{-1}, \qquad D(\tilde{C}) = \operatorname{ran}(I - A)$$

then $\tilde{C}$ is a contractive operator. Let P be the orthogonal projection of X onto the closure of $\operatorname{ran}(I - A)$. Then the product $C = \tilde{C}P$ is a contractive operator and C is the Cayley transform of a maximal dissipative operator B. We demonstrate this in the following example.

Example 3.5.1. Let $\tilde{C}$ be the operator

$$(0, x_1, x_2, \dots) \mapsto (x_1, x_2, \dots).$$

on ℓ_2. Clearly, $\tilde{C}$ is contractive and its domain is not dense. Let us show that $\operatorname{ran}(I + \tilde{C})$ is dense. Let $z = (z_k) \in \operatorname{ran}(I + \tilde{C})^{\perp}$ and take the standard sequence $\mathbf{e}_n$ for $n \geq 1$. Then

$$0 = \langle z, (I + \tilde{C})\mathbf{e}_n \rangle = z_{n-1} + z_n \quad (n \geq 1).$$

So $z_k = (-1)^n z_0$ for $n \geq 1$, and

$$\|z\|^2 = \sum_{n=0}^{\infty} |z_0|^2$$

and thus $z_n = 0$ for all $n \geq 0$. Hence $\operatorname{ran}(I + \tilde{C})^{\perp} = \{0\}$ and $\operatorname{ran}(I + \tilde{C})$ is dense.

The operator $\tilde{C}$ is the Cayley transform of the operator $A = (\tilde{C} - I)(I + \tilde{C})^{-1}$ with domain $D(A) = \operatorname{ran}(I + \tilde{C})$ on ℓ_2. An explicit formula for A is

$$(x_0, x_1, x_2, \dots) \mapsto (z_0, z_1, z_2, \dots)$$

where

$$z_k = x_k - 2\sum_{n=0}^{k-1}(-1)^{k-n}x_n \quad (k \geq 0). \tag{3.13}$$

A bounded extension C of $\tilde{C}$, defined on all of ℓ_2, is the operator

$$(x_0, x_1, x_2, \dots) \mapsto (x_1, x_2, x_3, \dots).$$

This operator is clearly contractive, and $I + C$ has dense range since it is an extension of $I + \tilde{C}$. Moreover, C is the Cayley transform of the operator B which satisfies

$$(x_0, x_1, x_2, \dots) \mapsto (z_0, z_1, z_2, \dots)$$

where

$$z_k = -x_k + 2\sum_{n=1}^{\infty}(-1)^{n-1}x_{k+n} \quad (k \geq 0). \tag{3.14}$$

For $(x_n) \in D(A) = \operatorname{ran}(I + \tilde{C})$, (3.13) and (3.14) give the same value of z_k. So B is an extension of A on ℓ_2.

We can also find a dissipative extension on a larger Hilbert space. We use Theorem 3.1.5 for this. We first need the following lemma.

Lemma 3.5.2. *Let $C \in \mathcal{B}(X)$ be an expansive operator on a Hilbert space. Let D be the (minimal) invertible extension of C obtained from Theorem 3.1.5. Then* $\operatorname{ran}(I + D)$ *is dense in Y if* $\operatorname{ran}(I + C)$ *is dense in X and* $\ker(I + D) = \{0\}$ *if* $\ker(I + C) = \{0\}$.

Proof. As in Theorem 3.1.5, let $C = UP$ be the polar decomposition of C into an isometry U and a positive, invertible operator P. The minimal extension D of C, constructed after the proof of Theorem 3.1.5, was defined by the matrix

$$D = \begin{pmatrix} C & I - UU^* \\ 0 & U^* \end{pmatrix}$$

on a subspace Y of $X \oplus X$ and satisfies $\|D^{-1}\| \leq 1$. Here, Y is the closed linear span of

$$X \times \{U^k(I - UU^*)x \colon x \in X, k \in \mathbb{N}\}.$$

The images of the operators $U^k(I - UU^*)$ are pairwise orthogonal in X for $k \geq 0$. We show first that $I + D$ is injective if $I + C$ is injective. Let $(x, z) \in \ker(I + D)$. Then

$$0 = ((I + C)x + (I - UU^*)z, (I + U^*)z) \tag{3.15}$$

which implies that $z = -U^*z$. Since z is in the closed linear span of $\operatorname{ran}(U^k(I - UU^*))$, and U is an isometry which is completely non-unitary, we find that $z = 0$. To see this, let $y \in \overline{\operatorname{span}}\{U^k(I - UU^*)x \colon x \in X\}$. There exist $x_k \in X$ such that

$$y = \sum_{k=0}^{\infty} U^k(I - UU^*)x_k.$$

The operator $U^k(I - UU^*)(U^*)^k$ is a projection onto $\operatorname{ran}(U^k(I - UU^*))$. We see that $U^k(I - UU^*)(U^*)^k y = U^k(I - UU^*)x_k$ for all $k \geq 0$. Hence,

$$y = \sum_{k=0}^{\infty} U^k(I - UU^*)(U^*)^k y. \tag{3.16}$$

We apply this to z with $z = -U^*z$. Let $z_0 = (I - UU^*)z$. Then

$$z_0 = (-1)^k(I - UU^*)(U^*)^k z \quad (k \geq 0)$$

and thus

$$z = \sum_{k=0}^{\infty} U^k(-1)^k z_0.$$

Since $\operatorname{ran}(I - UU^*) \perp \operatorname{ran} U$ and since U preserves orthogonality, we have

$$\|z\|^2 = \sum_{k=0}^{\infty} \|z_0\|^2$$

so that we must have $z = 0$. Now, Equation (3.15) reduces to $(I+C)x = 0$ and if $I+C$ is injective we get $(x, z) = 0$. This shows that $\ker(I + D) = \{0\}$ if $\ker(I + C) = \{0\}$.

Assume that $\operatorname{ran}(I + C)$ is dense in X. Let $(x, y) \in Y$ with $x \in X$ and $y \in \overline{\operatorname{span}}\{U^k(I - UU^*)x \colon x \in X\}$. We want to find $(u, v) \in Y$ such that

$$\|(x, y) - (I + D)(u, v)\| < \epsilon$$

for a given $\epsilon > 0$. First, let

$$v_n = \sum_{0 \leq k \leq i \leq n} (-1)^{i-k} U^k(I - UU^*)(U^*)^i y.$$

for $n \in \mathbb{N}$. Using that $U^*U = I$ and $U^*(I - UU^*) = 0$ we get

$$\begin{aligned}
(I + U^*)v_n &= \sum_{0 \le k \le i \le n} (-1)^{i-k} U^k (I - UU^*)(U^*)^i y \\
&\quad + \sum_{0 \le k \le i \le n} (-1)^{i-k} U^* U^k (I - UU^*)(U^*)^i y \\
&= \sum_{0 \le k \le i \le n} (-1)^{i-k} U^k (I - UU^*)(U^*)^i y \\
&\quad + \sum_{0 \le k < i \le n} (-1)^{i-k-1} U^k (I - UU^*)(U^*)^i y \\
&= \sum_{k=0}^{n} U^k (I - UU^*)(U^*)^k y.
\end{aligned}$$

Since (3.16) holds for y, we see that $\|(I + U^*)v_n - y\| \to 0$ as $n \to \infty$. So we can assume that $\|y - (I + U^*)v_n\| < \epsilon/2$ by choosing n large enough. By density of $\operatorname{ran}(I + C)$ in X, there is some $u \in X$ such that $\|x - (I - UU^*)v_n - (I + C)u\| < \epsilon/2$. So

$$\begin{aligned}
\|(x, y) - (I + D)(u, v_n)\|^2 &= \|(x - (I + C)u - (I - UU^*)v_n, y - (I + U^*)v_n)\|^2 \\
&= \|x - (I + C)u - (I - UU^*)v_n\|^2 + \|y - (I + U^*)v_n\|^2 \\
&< \epsilon^2.
\end{aligned}$$

□

Theorem 3.5.3. *Assume that A is a densely defined, dissipative operator with non-empty resolvent set on a Hilbert space X. Then there exist a Hilbert space $Y \supset X$ and a generator B of a C_0-semigroup of contractions such that $D(A) = D(B) \cap X$ and $Bx = Ax$ for all $x \in D(A)$.*

Proof. Since A is dissipative, it satisfies

$$\operatorname{Re}\langle Ax, x\rangle \le 0 \quad (x \in D(A)).$$

Substitute A by $aA + ib$ for some $a > 0, b \in \mathbb{R}$, if necessary, to assume that $1 \in \rho(A)$ or $-1 \in \rho(A)$. If $1 \in \rho(A)$ then A generates a semigroup of contractions by the Lumer-Phillips theorem (Theorem 2.3.3) and we can choose $Y = X$, so in the following we will assume that $-1 \in \rho(A)$. Define

$$C := (I - A)(I + A)^{-1}, \quad D(C) = \operatorname{ran}(I + A) = X.$$

Then C is a bounded operator such that $I + C = 2(I + A)^{-1}$ is injective and has dense range $D(A)$. Moreover, we can recover A from C as

$$A = (I - C)(I + C)^{-1}.$$

To see that C is expansive take $y = (I + A)^{-1}x$ in

$$\|(I - A)y\|^2 - \|(I + A)y\|^2 = -4\,\mathrm{Re}\,\langle Ay, y\rangle \geq 0 \quad (y \in D(A)).$$

We can extend C to a bounded invertible operator D with $\|D^{-1}\| \leq 1$ on some Hilbert space $Y \supset X$ using Theorem 3.1.5, and by Lemma 3.5.2 we see that the operator

$$B = (I - D)(I + D)^{-1}, \quad D(B) = \mathrm{ran}\,(I + D)$$

is well-defined and densely defined. We have

$$(I + B) = 2I(I + D)^{-1}, \quad (I - B) = 2D(I + D)^{-1}$$

so that $-1, 1 \in \rho(B)$ and

$$(I - B)(I + B)^{-1} = D.$$

Furthermore, B is dissipative since its Cayley transform D^{-1} is contractive (see Proposition 2.3.4). So B generates a C_0-semigroup of contractions by the Lumer-Phillips theorem (Theorem 2.3.3).

It remains to show that B restricts to A on X. Let $(x, 0) \in D(B)$. By definition, there is $(y, z) \in Y$ such that

$$(x, 0) = (I + D)(y, z) = ((I + C)y + (I - UU^*)z, (I + U^*)z).$$

So $z = -U^*z$ which implies $z = 0$, as in the proof of Lemma 3.5.2. Thus $x = (I + C)y \in D(A)$ and

$$B(x, 0) = (I - D)(y, 0) = ((I - C)(I + C)^{-1}x, 0) = (Ax, 0).$$

On the other hand, let $x \in D(A)$, so $x = (I + C)y$ for some $y \in X$. Then $(x, 0) = (I + D)(y, 0) \in D(B)$ and again $B(x, 0) = (Ax, 0)$. □

Remark 3.5.4. Let A be a dissipative operator such that $-A$ is the generator of a C_0-semigroup T on a Hilbert space X. Then A has an extension B on a Hilbert space $Y \supset X$ which generates a C_0-semigroup of contractions, by Theorem 3.5.3. We could ask whether $-B$ is again the generator of a C_0-semigroup S. Since the theorem states that B generates a C_0-semigroup, S would then be a C_0-group on a Hilbert space $Y \supset X$ such that $S(-t)$ is contractive and $S(t)x = T(t)x$ for all $x \in X$ and $t \geq 0$. This would give us the bound which we failed to obtain in Theorem 3.2.14.

Since we already have that B generates a semigroup on Y, we can try to apply Theorem 2.4.3 to show that B generates a group on Y. This means, we only have to find out whether the resolvent of B exists on a left half plane and has less than exponential growth in the real part. We will find the resolvent of B. Using the definitions from the proof of Theorem 3.5.3, we have

$$\lambda - B = (\lambda(I+D) - (D-I))(I+D)^{-1} = (1-\lambda)(\tfrac{1+\lambda}{1-\lambda} - D)(I+D)^{-1} \quad (\lambda \in \mathbb{C}).$$

So $\lambda \in \rho(B) \setminus \{1\}$ if and only if $\frac{1+\lambda}{1-\lambda} \in \rho(D)$, and then

$$R(\lambda, B) = \frac{1}{1-\lambda}(I+D)R(\tfrac{\lambda+1}{1-\lambda}, D).$$

Similarly, we get

$$R(\lambda, A) = \frac{1}{1-\lambda}(I+C)R(\tfrac{\lambda+1}{1-\lambda}, C)$$

if $\lambda \in \rho(A) \setminus \{1\}$. Using the formula from the proof of Theorem 3.1.5 for the resolvent of D, that is

$$R(\mu, D) = \begin{pmatrix} R(\mu, C) & R(\mu, C)(I - UU^*)R(\mu, U^*) \\ 0 & R(\mu, U^*) \end{pmatrix}$$

for $\mu \in \rho(C)$, we get

$$R(\lambda, B) = \begin{pmatrix} R(\lambda, A) & (R(\lambda, A) + \frac{1}{1-\lambda})(I - UU^*)R(\frac{1+\lambda}{1-\lambda}, U^*) \\ 0 & \frac{1}{1-\lambda}(I + U^*)R(\frac{1+\lambda}{1-\lambda}, U^*) \end{pmatrix}.$$

This holds for all $\lambda \in \rho(A)$ with $\lambda \neq 1$. If $-A$ generates a C_0-semigroup, this holds in particular for $\operatorname{Re}\lambda < -1$ after rescaling A suitably. In order to apply Theorem 2.4.3, we would have to show that for each $\alpha > 0$ there is some $r < 0$ such that $\sup_{\operatorname{Re}\lambda < r} e^{\alpha \operatorname{Re}\lambda} \|R(\lambda, B)\|$ is finite. It is not clear if this holds in general.

Let T be a C_0-semigroup on a Hilbert space X. We say that T is left invertible if there are $L(t) \in \mathcal{B}(X)$ such that $L(t)T(t) = I$ $(t \geq 0)$. If $T(t)$ is left invertible for some $t > 0$ then $T(t)$ is left invertible for all $t \geq 0$. Zwart showed that the left inverses can be chosen to form a C_0-semigroup. The following result is from [Zwa13].

Theorem 3.5.5 ([Zwa13, Theorem 2]). *Let A be the generator of a C_0-semigroup T on a Hilbert space X. Then T is left invertible if and only if $-A$ can be extended to the generator of a C_0-semigroup on X.*

We will extend the operator $-A$ in Proposition 3.5.7 below in the case where $-A$ is dissipative.

First, let us see how left inverses are related to families of projections onto $\operatorname{ran} T(t)$ $(t \geq 0)$. Assume that L and T are C_0-semigroups such that $L(t)T(t) = I$ for all $t \geq 0$. Then

$$P(t) := T(t)L(t) \quad (t \geq 0)$$

is strongly continuous, $P(t)$ is a projection onto $\operatorname{ran} T(t)$ for each $t \geq 0$, and

$$\|P(t)\| \leq \|T(t)\| \|L(t)\| \quad (t \geq 0).$$

Moreover, the product of P with itself is given by

$$P(s)P(t) = P(\max\{s,t\}) \quad (s,t \geq 0)$$

and the product of T and P satisfies

$$P(t-s)T(t) = T(t) \qquad (t \geq s) \tag{3.17}$$

and

$$T(t)P(s) = P(t+s)T(t) \quad (t,s \geq 0). \tag{3.18}$$

Assume now that T is a left invertible C_0-semigroup and that P is a strongly continuous family of projections $P(t)$ onto $\operatorname{ran} T(t)$ $(t \geq 0)$ satisfying (3.17) and (3.18). Let

$$L(t) := T(t)^{-1} P(t) \quad (t \geq 0).$$

Then L satisfies the semigroup property and it is strongly continuous. So L is a C_0-semigroup, and we have $L(t)T(t) = I$ for all $t \geq 0$. If we assume that

$$\|T(t)x\| \geq c\|x\| \quad (x \in X, t \geq 0) \tag{3.19}$$

for some $c > 0$, we get the norm estimate

$$\|L(t)\| \leq c^{-1} \|P(t)\| \quad (t \geq 0).$$

It would be ideal if $P(t)$ were just the orthogonal projection onto $\operatorname{ran} T(t)$ as we would then have $\|L(t)\| \leq c^{-1}$ $(t > 0)$. But this is not always the case (see Example 3.2.12).

Assume that T is a C_0-semigroup which satisfies (3.19), and L is a left inverse C_0-semigroup for T. We can first define the projections P and then get the C_0-semigroup $L' = L$ back from T and P, via $L'(t) = T(t)^{-1}P(t)$ $(t \geq 0)$. This leaves us with the rather bad estimate

$$\|L'(t)\| \leq c^{-1} \|T(t)\| \|L(t)\| \quad (t \geq 0). \tag{3.20}$$

We will estimate the norm of a particular left inverse that was found by Zwart. The estimate we will find shares with the estimate (3.20) that the left inverse is of at least the same exponential type as the semigroup T. The left inverse we will look at is that of Theorem 3.5.6 for a C_0-semigroup satisfying (3.19). The extension is a special case of [Zwa13, Theorem 3].

Theorem 3.5.6. *Let A be the generator of a C_0-semigroup T on a Hilbert space X. The following are equivalent.*

1. *$T(t)$ is left invertible for some/all $t > 0$.*

2. *There exists a C_0-semigroup L on X such that $L(t)T(t) = I$ for all $t \geq 0$.*

Assume that T is a C_0-semigroup on a Hilbert space X such that

$$\|T(t)x\| \geq c\|x\| \quad (x \in X, t \geq 0)$$

for some $1 > c > 0$. We want to esimate $\|L(t)\|$ $(t \geq 0)$, where L is the left inverse C_0-semigroup from Theorem 3.5.6. Zwart constructs the C_0-semigroup L as follows. Let

$$\omega_2 > \omega_1 > \omega_0(T), \quad \omega_3 > \omega_0(T),$$

so $\|T(t)\| \leq M_{\omega_i} e^{\omega_i t}$ $(t \geq 0, i = 1, 2, 3)$. Define an inner product and an operator Q by

$$\langle x_1, x_2 \rangle' = \langle x_1, Qx_2 \rangle = \int_0^\infty e^{-2\omega_2 t} \langle T(t)x_1, T(t)x_2 \rangle \,\mathrm{d}t \quad (x_1, x_2 \in X).$$

Then Q is positive and invertible. We get bounds on Q and Q^{-1} via

$$\|x\|'^2 \geq \int_0^\infty e^{-2\omega_2 t} c^2 \|x\|^2 = \frac{c^2}{2\omega_2} \|x\|^2 \quad (x \in X)$$

and

$$\|x\|'^2 \leq \int_0^\infty e^{-2\omega_2 t} M_{\omega_1}^2 e^{2\omega_1 t} \|x\|^2 = \frac{M_{\omega_1}^2}{2(\omega_2 - \omega_1)} \|x\|^2 \quad (x \in X).$$

The operator $Q^{-1}A^*Q - 2\omega_2 I + Q^{-1}$ generates the C_0-semigroup L which is a left inverse of T. Now,

$$\begin{aligned} \|L(t)\| &\leq \|Q\| \|Q^{-1}\| \|T(t)\| e^{-2\omega_2 t} e^{\|Q^{-1} t\|} \\ &\leq \frac{M_{\omega_1}^2}{2(\omega_2 - \omega_1)} \frac{2\omega_2}{c^2} M_{\omega_3} e^{\omega_3 t} e^{-2\omega_2 t} e^{\frac{2\omega_2}{c^2} t} \qquad (t \geq 0). \\ &= \frac{1}{c^2} \frac{\omega_2 M_{\omega_1}^2 M_{\omega_3}}{(\omega_2 - \omega_1)} e^{(\omega_3 + (\frac{2}{c^2} - 2)\omega_2)t} \end{aligned}$$

This seems to be a very bad estimate, since the exponential type in this estimate is

$$\omega_3 + (\frac{2}{c^2} - 2)\omega_2 > \omega_0(T)$$

as $c < 1$ and $\omega_2 > \omega_0(T) \geq 0$.

We can use the maximal dissipative extension to obtain an equivalence similar to that of Theorem 3.5.6. The next Proposition discusses the special case where the left inverse C_0-semigroup L is contractive. The equivalence between conditions 1 and 2 in Proposition 3.5.7 was first established by Goldberg and Smith ([GS78]) and it also holds on Banach space (see Theorem 2.4.4). We present a new proof.

Proposition 3.5.7. *Let A be the generator of a C_0-semigroup T on a Hilbert space X. The following are equivalent.*

1. *T is expansive: $\|T(t)x\| \geq \|x\|$ $(x \in X, t \geq 0)$.*
2. *A is accretive: $\|(\lambda - A)x\| \geq (-\lambda)\|x\|$ $(x \in D(A), \lambda < 0)$.*
3. *There exists a C_0-semigroup L of contractions on X such that $L(t)T(t) = I$ $(t \geq 0)$.*

Proof. Assume that T is expansive. By Theorem 3.2.2, there is an extending C_0-group S of T on a Banach space $Y \supset X$ such that $\|S(-t)\| \leq 1$ for all $t \geq 0$. Let B be the generator of S. Since $S(-t)T(t)x = x$ for all $x \in X$, we have

$$\begin{aligned} 0 &= (S(-t)T(t)x)' \\ &= S(-t)AT(t)x - BS(-t)T(t)x \quad (x \in D(A), t \geq 0). \\ &= S(-t)(A - B)T(t)x \end{aligned}$$

This shows $Bx = Ax$ for $x \in D(A)$. Since $\|S(-t)\| \leq 1$ for $t \geq 0$, we have that $-B$ is dissipative. Hence $-A$ is dissipative, or in other words, A is accretive.

Assume that A is accretive. Then $-A$ is dissipative. Let B be a maximal dissipative extension of $-A$. Then B generates a C_0-semigroup L of contractions on X. Moreover,

$$\begin{aligned} (L(t)T(t)x)' &= L(t)AT(t)x + BL(t)T(t)x \\ &= L(t)AT(t)x + L(t)BT(t)x = 0 \end{aligned} \quad (t \geq 0) \tag{3.21}$$

for $x \in D(A)$, since B extends $-A$. This shows that $L(t)T(t)x = x$ for all $t \geq 0, x \in X$.

Let L be a C_0-semigroup of contractions such that $L(t)T(t) = I$ for all $t \geq 0$. Then

$$\|x\| = \|L(t)T(t)x\| \leq \|T(t)x\| \quad (x \in X, t \geq 0).$$

So T is expansive. □

The result by Goldberg and Smith, that is the equivalence between conditions 1 and 2 in Proposition 3.5.7, holds on Banach space and not only on Hilbert space. The proof in [GS78] works differently. The extension for a dissipative operator generating a C_0-semigroup (from Theorem 3.5.3) cannot be obtained for Banach space in the same way as for Hilbert space. This is because on Banach space, contractivity of the Cayley transform of A neither implies nor is implied by dissipativity of A. The Cayley transforms of generators of C_0-semigroups on Banach space are characterised as follows. In [EZ08], the Cayley transform V of an operator A with $1 \in \rho(A)$ was defined as

$$V := (A+I)(A-I)^{-1}.$$

They give the following result for operators on Banach space.

Theorem 3.5.8 ([EZ08, Theorem 2.2]). *Let $V \in \mathcal{B}(X)$ and $M \geq 1$. The following are equivalent.*

1. *V is the Cayley transform of the generator of a C_0-semigroup of contractions.*

2. *$V - I$ is injective and has dense range, $(1, \infty) \subset \rho(A)$ and*

$$\|(I-V)R(\mu, V)\| \leq \frac{2M}{(1+\mu)} \quad (\mu > 1).$$

In the proof of Theorem 3.1.5, which is employed in Theorem 3.5.3, we have an operator C on a Hilbert space which is expansive and which can be extended to an operator D with contractive inverse, and D^{-1} is the Cayley transform of some a C_0-semigroup generator. If we want to follow this construction on Banach space, we would seek a bounded extension C with inverse $V = C^{-1}$ which satisfies the assumptions of Theorem 3.5.8. The extension results we have available do not give us an estimate on $\|(I-V)R(\mu, V)\|$ for $\mu > 1$.

Let us go back to Hilbert spaces. There is not yet a generalisation of Proposition 3.5.7 which discusses the following cases. We would like to characterise the following three properties:

(i) T is uniformly bounded below: $\|T(t)x\| \geq c\|x\|$ $(x \in X, t \geq 0)$ for some $c > 0$.

(ii) A satisfies $\|(\lambda - A)^n x\| \geq c(-\lambda)^n \|x\|$ $(n \geq 1, x \in D(A^n), \lambda < 0)$ for some $c > 0$.

(iii) There exists a bounded C_0-semigroup L with $\|L(t)\| \leq M$ and $L(t)T(t) = I$ for all $t \geq 0$.

It is easy to see that (iii) implies (i), with $c = \frac{1}{M}$. We will see that (iii) also implies (ii) (this is shown in last paragraph of this section). A possible guess is that (i) and (iii) are related as in the case where $M = c = 1$. The next example shows that (i) does not imply (iii) when $c < 1$.

Example 3.5.9. Let $M > 1 > c > 0$. For $n > c^{-1}$, let T_n be the C_0-semigroup from Example 3.2.12 on a Hilbert space X_n. So

$$\|T(t)x\| \geq c\|x\|, \quad \|T(t)\| \leq \sqrt{2}(KM)^t \quad (x \in X, t \geq 0)$$

for some $K > 1$, and if S_n is any extending C_0-group of T_n on an extending Banach space then $\sup_{t\geq 0} \|S_n(-t)\| \geq n$. Define operators

$$T(t) = \bigoplus_{n>c^{-1}} T_n(t) \quad (t \geq 0)$$

on the infinite ℓ_2-direct sum $X = \bigoplus_{n>c^{-1}} X_n$. Then $\|T(t)\| \leq \sqrt{2}(KM)^t$ and $\|T(t)x\| \geq c\|x\|$ for all $x \in X$ and $t \geq 0$. If S is a C_0-group extending T on a Banach space $Y \supset X$ then $\sup_{t\geq 0} \|S(-t)\| = \infty$. We show that T does not have a bounded left inverse C_0-semigroup. Let $T(t)x = T(t_1)x_1 + \cdots + T(t_n)x_n$ for some $x_i \in X$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$ and let L be a left inverse C_0-semigroup. Then

$$\|x\| = \|L(t)T(t)x\| \leq \|L(t - t_1)\|\|x_1\| + \cdots + \|L(t - t_n)\|\|x_n\|.$$

If L were bounded, say by $C > 0$, this would verify the assumptions of Theorem 3.2.4 for the constant submultiplicative function $c(\cdot) = C$, so an extending group S with $\|S(-t)\| \leq C$ $(t \geq 0)$ would exist on some Banach space $Y \supset X$. This is not the case, so T does not have a bounded left inverse C_0-semigroup.

A special case of (iii) is when the C_0-semigroup L is similar to a C_0-semigroup of contractions. We know by Proposition 3.5.7 that if T is expansive then there is a contractive left inverse C_0-semigroup L. It is not difficult to see that if T is similar to an expansive C_0-semigroup, then there is a left inverse C_0-semigroup L which is similar to a contractive C_0-semigroup (and thus L is bounded). There are bounded C_0-semigroups which are not equivalent to a contractive C_0-semigroup. Chernoff found a bounded C_0-semigroup S which is not similar to a C_0-semigroup of contractions and which has a bounded generator A ([Che76]). If we let $T(t) := e^{-tA}$ $(t \geq 0)$, we see that T has S as its bounded left inverse C_0-semigroup. But T is not similar to an expansive C_0-semigroup.

We now show that (iii) implies (ii), with $c = \frac{1}{M}$. Assume that there is a left inverse C_0-semigroup L for T, and that $\|L(t)\| \leq M$ $(t \geq 0)$. The generator of L extends $-A$, and by Theorem 2.4.2 we have that A satisfies (ii). It is not known if the lower bound (ii) implies the existence of a left inverse C_0-semigroup. But it is more likely that (ii) and (i) are equivalent. This would generalise the equivalence given in Theorem 2.4.4, which also holds on Banach space. We formulate this as an open question.

Open Question 3.5.10. Let T be a C_0-semigroup with generator A on a Banach space X. Let $c > 0$ and consider the following inequalities.

$$\|T(t)x\| \geq c\|x\| \qquad (x \in X, t \geq 0),$$
$$\|\lambda^{-n}(\lambda - A)^n x\| \geq c\|x\| \qquad (n \geq 1, x \in D(A^n), \lambda < 0).$$

Are these bounds equivalent, at least on Hilbert space?

3.6 Reducing the spectrum

Let us go back to inverses of bounded operators and take a different view on them. Recall that for an operator $T \in \mathcal{B}(X)$, the approximate point spectrum is defined as the set

$$\sigma_{\mathrm{ap}}(T) = \{\lambda \in \mathbb{C} \colon (\lambda - T) \text{ is not bounded below}\}.$$

So if $\lambda \in \mathbb{C} \setminus \sigma_{\mathrm{ap}}(T)$, the operator $\lambda - T$ is either invertible and $\lambda \in \rho(T)$ or $\lambda - T$ has an invertible extension U by Theorem 3.1.1. In that case, we can define an extension $S = \lambda - U$ for T and find that $\lambda \in \rho(S)$, $\rho(T) \subset \rho(S)$ and $\|R(\lambda, S)\| \leq c^{-1}$ where $c > 0$ is such that $\|(\lambda - T)x\| \geq c\|x\|$ $(x \in X)$. Of course, one can try to find an extension S of an operator T such that $\sigma_{\mathrm{ap}}(T) = \sigma(S)$. This is possible as was shown by Read in [Rea88] (see [Rea84] for an earlier result on Banach algebras, and [Mül88] for how to use the Banach algebra result to infer the operator case).

Theorem 3.6.1 ([Rea88])**.** *Let X be a Banach space and $T \in \mathcal{B}(X)$. Then there is a Banach space $Y \supset X$ and a unital isometric homomorphism $\varphi \colon \{T\}' \to \mathcal{B}(Y)$ such that $\varphi(V)x = Vx$ for all $x \in X$ and $V \in \{T\}'$ and for $S = \varphi(T)$ we have $\sigma(S) = \sigma_{\mathrm{ap}}(T)$.*

Read's proof is not based on shift operators but it uses multiplication operators. A good presentation of the proof can be found in [Mül03, Section I.4]. We will give the proof for a partial result in Section 3.6.3. Here, we only sketch the idea to illustrate the connection to the proof by Arens.

In Arens' proof, we looked at vector-valued sequences and used the right shift R on $\ell_1(X)$. Let $f \in \ell_1(X)$ be such a sequence. We can define a function $\hat{f} \colon \mathbb{D} \to X$ as the Fourier transform of this sequence, namely

$$\hat{f}(z) = \sum_{n \in \mathbb{N}} f(n) z^n \quad (z \in \mathbb{D}).$$

This function $\hat{f}$ is bounded and holomorphic on the open unit disc $\mathbb{D}$ and it extends continuously to the closure $\overline{\mathbb{D}}$. The Fourier transform intertwines the shift operator R on $\ell_1(X)$ with the multiplication operator M_z, that is

$$\widehat{Rf}(z) = z\hat{f}(z) = (M_z\hat{f})(z) \quad (z \in \mathbb{D}).$$

For $T \in \mathcal{B}(X)$ we see that $\widehat{Tf}(z) = T\hat{f}(z)$. In Theorem 3.1.1, we were interested in a subspace $\{(I - TR)f\}$ which identifies with $\{(I - TM_z)\hat{f}\}$ on the Fourier transforms. Here, the operator T is such that $\|Tx\| \geq \|x\|$ for all $x \in X$, which implies that $\mathbb{D} \cap \sigma_{\text{ap}}(T) = \emptyset$. An idea of Read was to use instead functions on an (almost) complimentary domain $U \subset \mathbb{C}$. Rather than using some U with $U \cap \sigma_{\text{ap}}(T) \neq \emptyset$, we work with open sets U such that $\sigma_{\text{ap}}(T) \subset U$, such as

$$U_\epsilon = \{z \in \mathbb{C}\colon \operatorname{dist}(z, \sigma_{\text{ap}}(T)) < \epsilon\}.$$

On the space $H^\infty(U, X)$ of bounded holomorphic functions $f\colon U \to X$, we can use a quotient construction using the set $\{(M_z - T)f\}$, so T will be identified with M_z. The multiplication operators for the functions $(\lambda - z)^{-1}$ are bounded on $H^\infty(U, X)$ if $\lambda \notin U$ and they serve as the resolvent for T on the quotient. Finally, a limit construction (for a sequence U_ϵ letting $\epsilon \to 0$) allows an inverse for every $\lambda \notin \sigma_{\text{ap}}(T)$.

Consider unbounded operators. Unlike in Arens' Theorem, the proof of Read's Theorem does not seem to allow a change that would make it work for unbounded operators, not even for C_0-semigroup generators. We will explore this in Section 3.6.3 for a preliminary result of Read's theorem. However, it is possible to obtain Read's theorem for generators of C_0-semigroups. We will use the isometric homomorphism φ from Theoren 3.6.1 for this.

Theorem 3.6.2. *Let A be the generator of a C_0-semigroup T on a Banach space X. Then there is a generator B of a C_0-semigroup on a Banach space $Y \supseteq X$ such that $\sigma(B) = \sigma_{\text{ap}}(A)$, $D(A) = D(B) \cap X$ and $Ax = Bx$ for all $x \in D(A)$. Moreover, there exists a unital isometric homomorphism*

$$\varphi\colon \{R(\lambda, A)\colon \lambda \in \rho(A)\}' \to \mathcal{B}(Y)$$

with $\varphi(R(\lambda, A)) = R(\lambda, B)$ for all $\lambda \in \rho(A)$.

Proof. Let us note first that the spectrum of A and the spectrum of $R(\mu, A)$ for $\mu \in \rho(A)$ are related by in Lemma 2.1.2. Let T be the C_0-semigroup generated by A. Fix $\mu > \omega_0(T)$ and apply Theorem 3.6.1 to the operator $R(\mu, A)$. We get a

Banach space $Z \supset X$, a unital isometric homomorphism $\varphi\colon \{R(\mu, A)\}' \to \mathcal{B}(Z)$ and $\sigma(R_\mu) = \sigma_{\text{ap}}(R(\mu, A))$ where $R_\mu = \varphi(R(\mu, A))$. We want to show that $R_\mu = R(\mu, B)$ for a suitable B.

The operators $R_\lambda = \varphi(R(\lambda, A))$ ($\lambda \in \rho(A)$) still satisfy the resolvent identity. So they form a pseudo-resolvent and we have to show that they are injective. For this, we will use a C_0-semigroup on a subspace of Z. Let Y be the closure of $\operatorname{ran} R_\mu$. Note that if U commutes with $R(\mu, A)$, then Y is invariant under $\varphi(U)$. In particular, we can define $S(t) = \varphi(T(t))\restriction_Y$ for $t \geq 0$. We show that S is strongly continuous on Y. Since φ is isometric, we have that $\|S(t)\| = \|T(t)\|$ ($t \geq 0$) and

$$\|(S(t) - I)R_\mu\| = \|(T(t) - I)R(\mu, A)\| \to 0 \quad \text{as } t \to 0$$

(see Lemma 2.2.2). This implies that S is a C_0-semigroup on Y. We can now show that $R_\mu\restriction_Y = R(\mu, B)$, where B is the generator of S on Y. By Lemma 2.2.2 we have

$$\begin{aligned} &\Big\|(\mu - \frac{1}{t}(S(t) - I))R_\mu^2 - R_\mu\Big\| \\ &= \Big\|(\mu - \frac{1}{t}(T(t) - I))R(\mu, A)^2 - R(\mu, A)\Big\| \to 0 \quad \text{as } t \to 0. \end{aligned}$$

This shows that $R_\mu = (\mu - B)R_\mu^2$ on Z. Hence $R_\mu\restriction_Y = R(\mu, B)$. Since the operators $\{R_\lambda\restriction_Y\colon \lambda \in \rho(A)\}$ form a pseudo-resolvent, we also have $R_\lambda\restriction_Y = R(\lambda, B)$ for all $\lambda \in \rho(A)$.

It remains to show that $\sigma(R_\mu) = \sigma(R(\mu, B))$, since all assertions other than $\sigma(B) = \sigma_{\text{ap}}(A)$ clearly hold. Assume that $\lambda \in \rho(R_\mu)$ and that $\lambda - R_\mu$ has a bounded inverse W on Z. Then $WR_\mu = R_\mu W$ and hence the subspace Y is invariant under W. This implies that $\lambda \in \rho(R(\mu, B))$, and $\sigma(R(\mu, B)) \subset \sigma(R_\mu)$. Equality then follows from

$$\sigma_{\text{ap}}(R(\mu, A)) \subset \sigma(R(\mu, B)) \subset \sigma(R_\mu) = \sigma_{\text{ap}}(R(\mu, A)).$$

By Lemma 2.1.2 we get $\sigma_{\text{ap}}(A) = \sigma(B)$. □

The problem with Theorems 3.6.1 and 3.6.2 is that they do not give any estimate on the resolvent for the extension. In Theorem 3.6.1, for example, we do not have estimates on $\|R(\lambda, S)\|$ for values $\lambda \in \rho(S) \setminus \rho(T)$. We can show that bounds of one form or another do exist, using a standard argument. We shall be interested in the case of a bounded operator T on a Banach space X such that

$$\|(\lambda - T)x\| \geq c\|x\| \quad (x \in X, \lambda \in \mathbb{T}) \tag{3.22}$$

for some $c > 0$. We want to find an extension S of T with $\mathbb{T} \subset \rho(S)$ and give a bound $\sup_{\lambda\in\mathbb{T}} \|R(\lambda, S)\| \le K$. We show now that there is a bound K which depends only on $\|T\|$ and c.

For contradiction, let $M > 1$ and $c > 0$. Assume that for each $n \in \mathbb{N}$, there is an operator T_n on a Banach space X_n such that $\|T_n\| \le M$ and (3.22) holds, and if S_n is any extension of T_n then $\sup_{\lambda\in\mathbb{T}} \|R(\lambda, S_n)\| \ge n$. Define $T = \bigoplus_{n\ge1} T_n$ on the ℓ_∞-direct sum $X = \bigoplus_{n\ge1} X_n$. Then $\|T\| \le M$ and

$$\begin{aligned}\|(\lambda - T)(x_n)\| &= \sup_{n\ge1} \|(\lambda - T_n)x_n\| \\ &\ge \sup_{n\ge1} c\|x_n\| = c\|(x_n)\| \quad ((x_n) \in X, \lambda \in \mathbb{T}).\end{aligned}$$

By Theorem 3.6.1 there is an extension S of T with $\mathbb{T} \subset \rho(S)$. Since $\mathbb{T}$ is compact this implies that $\{\|R(\lambda, S)\| \colon \lambda \in \mathbb{T}\}$ is bounded, say by K. Using the (isometric) inclusion of X_n in X, we see that the extension S of T is also an extension of T_n. Hence $n \le \|R(\lambda, S)\| \le K$ for every $n \ge 1$. This is a contradiction. So for each T which satisfies (3.22) there is an extension S with an upper bound $K > 0$ for $\{\|R(\lambda, S)\| \colon \lambda \in \mathbb{T}\}$, and K depends only on c and $\|T\|$.

Open Question 3.6.3. Find $K = K(c, \|T\|)$ such that the following holds. If T is a bounded operator which satisfies (3.22), there is an extension S of T with $\|S\| = \|T\|$, $\mathbb{T} \subset \rho(S)$ and

$$\sup_{\lambda\in\mathbb{T}} \|R(\lambda, S)\| \le K(c, \|T\|).$$

We will consider weighted shift operators T_w on $\ell_1(\mathbb{Z})$ which satisfy (3.22) in Section 3.6.1 and obtain an extension S and a norm bound for $\|(\lambda - S)^{-1}\|$ $(\lambda \in \mathbb{T})$.

For generators of C_0-semigroups, it would be interesting to have a positive answer to the following open question.

Open Question 3.6.4. Let A be the generator of a C_0-semigroup. Assume that

$$\|(A - is)x\| \ge c\|x\| \qquad (x \in D(A), s \in \mathbb{R}). \tag{3.23}$$

for some $c > 0$. Does A have an extension B which is the generator of a C_0-semigroup and such that $i\mathbb{R} \subset \rho(B)$ and

$$\sup_{s\in\mathbb{R}} \|R(is, B)\| < \infty?$$

Assume that A generates a C_0-semigroup T on a Hilbert space. Then (3.23) is equivalent to the C_0-semigroup T being quasi-hyperbolic (see Section 4.7.1). This was shown in [BT10]. There is a Hilbert space version of Read's result ([Rea87]) which can be applied to the C_0-semigroup T to show that, in this case, there is such an extension B on a Hilbert space. Let us investigate in Sections 3.6.2 and 3.6.3 the difficulties to be overcome in order to construct such an extension using the method from Read's extension.

3.6.1 An extension of a weighted shift

We consider extensions of a weighted shift

$$T_w \mathbf{e}_n = w_n \mathbf{e}_{n+1} \quad (n \in \mathbb{Z})$$

on $X = \ell_1(\mathbb{Z})$. We are interested in the case when $\sigma_{\mathrm{ap}}(T_w) \neq \sigma(T_w)$. Ridge's characterisation of the spectrum of a shift operator (Proposition 2.6.5) gives that $\sigma_{\mathrm{ap}}(T_w)$ consists of two (possibly degenerate) annuli, and we are particularly interested in cases where the unit circle lies between the two annuli. By Proposition 2.6.5, this is the case when $0 \leq r^-(T_w) < 1 < i^+(T_w)$ where $r^-(T_w)$ and $i^+(T_w)$ are defined by (2.11) in Section 2.6. If that is the case, T_w satisfies

$$\|T_w x - \lambda x\| \geq c\|x\| \quad (x \in X, \lambda \in \mathbb{T})$$

for some $c > 0$. It is sufficient to look at the value $\lambda = 1$ for this lower bound, since we have that $\|T_w x - \lambda x\| = \|T_w y - y\|$ where $y_n = \lambda^{-n} x_n$ ($n \in \mathbb{Z}$), and $\|y\| = \|x\|$. We make a further restriction on the weight w and assume that

$$\begin{aligned} &w_n \geq 1 \qquad (n \geq 0), \\ &w_n \leq 1 \qquad (n < 0), \\ &0 \leq r^-(T_w) < 1 < i^+(T_w). \end{aligned}$$

We will also assume that T_w is bounded so $w_n \leq \|T_w\|$ ($n \in \mathbb{Z}$). We will proceed in the following order.

1. Find an isomorphic embedding $j\colon X \to Y$ where $jT_w = Sj$ and S is a direct sum of two shifts with simple annuli as their spectra.
2. Find a bound on $\|(S - I)^{-1}\|$ by considering both summands of S separately.

A model for the extension Ridge's characterisation of the spectrum for shift operators shows that the approximate point spectrum consists of two annuli. The outer radius $r^-(T_w)$ of the inner annulus is determined by an expression involving the numbers w_n for negative n, while the inner radius $i^+(T_w)$ is determined by w_n for positive n. We shall seek an extension S of T_w which is of the form $S = T_\alpha \oplus T_\beta$ on a space $\ell_1(\mathbb{Z}) \oplus \ell_1(\mathbb{Z})$ for some weights α, β. Here, T_α will have the spectral bound $\max\{r^-(T_w), \frac{1}{2}\}$, and $\sigma(T_\beta)$ will have the inner radius $\min\{i^+(T_w), 2\}$.

Consider

$$Y = \ell_1(\mathbb{Z} \times \{0,1\}) \simeq \ell_1(\mathbb{Z}) \oplus \ell_1(\mathbb{Z})$$

and take the standard double basis $(\mathbf{f}_n^0), (\mathbf{f}_n^1)$. Define

$$S\mathbf{f}_n^0 := \beta_n \mathbf{f}_{n+1}^0, \quad S\mathbf{f}_n^1 := \alpha_n \mathbf{f}_{n+1}^1 \quad (n \in \mathbb{Z}).$$

for some weights $(\alpha_n), (\beta_n)$. So S is the sum of two weighted shifts, and $\sigma_{\mathrm{ap}}(S), \sigma(S)$ are the union of the two corresponding spectra. Let

$$g_n = \begin{cases} \mathbf{f}_n^0 + \dfrac{\alpha_0 \cdots \alpha_{n-1}}{\beta_0 \cdots \beta_{n-1}} \mathbf{f}_n^1 & (n \geq 0) \\ \dfrac{\alpha_n \cdots \alpha_{-1}}{\beta_n \cdots \beta_{-1}} \mathbf{f}_n^0 + \mathbf{f}_n^1 & (n \leq 0). \end{cases}$$

Note $g_0 = \mathbf{f}_0^0 + \mathbf{f}_0^1$ in both cases. Then

$$S g_n = \begin{cases} \beta_n g_{n+1} & (n \geq 0) \\ \alpha_n g_{n+1} & (n < 0). \end{cases}$$

Given a weight w, we choose the weights α and β in the following way. They shall satisfy

$$w_n = \begin{cases} \beta_n & (n \geq 0), \\ \alpha_n & (n < 0), \end{cases}$$
$$\alpha_n = \frac{1}{2} \quad (n \geq 0),$$
$$\beta_n = 2 \quad (n < 0).$$

The map

$$j \colon \ell_1(\mathbb{Z}) \to Y, \quad \mathbf{e}_n \mapsto g_n$$

intertwines T_w with $S = (T_\alpha \oplus T_\beta)$, or $Sj = jT_w$. We find that j is an isomorphism, but j is not isometric. We have

$$1 \leq \|g_n\| \leq 1 + \frac{1}{2^{|n|}} \quad (n \in \mathbb{Z})$$

so that $\|(x_n)\| \leq \|j(x_n)\| \leq 2\|(x_n)\|$ for all $(x_n) \in X$. Also, we have that $\|g_n\| \to 1$ as $n \to \infty$ and as $n \to -\infty$.

Recall that T_w satisfies the inequality

$$\|(\lambda - T_w)x\| \geq c\|x\| \quad (x \in X, \lambda \in \mathbb{T}) \tag{3.24}$$

for some $c > 0$. By choice of the weights α and β, we clearly have that

$$\sigma(T_\alpha) \subset \mathbb{D}, \quad \sigma(T_\beta) \subset \mathbb{C} \setminus \overline{\mathbb{D}}.$$

Hence, T_α and T_β also satisfy an inequality of the form (3.24), for possibly different values of c. We shall relate c, the lower bound of T_w, and the lower bounds of T_α and T_β.

An estimate for T_α. Let us find an estimate for

$$\|(\lambda - T_\alpha)^{-1}\| \quad (\lambda \in \mathbb{T}).$$

For $k < 0$ and $n \geq 0$ with $k + n < 0$ let

$$x = \mathbf{e}_{k+1} + w_{k+1}\mathbf{e}_{k+2} + \cdots + w_{k+1} \cdots w_{k+n-1}\mathbf{e}_{k+n}.$$

Then

$$c \leq \frac{\|T_w x - x\|}{\|x\|} \leq \frac{1 + w_{k+1} \cdots w_{k+n}}{1 + w_{k+1} + \cdots + w_{k+1} \cdots w_{k+n-1}} \leq \frac{2}{n w_{k+1} \cdots w_{k+n-1}}.$$

This shows that

$$w_k \cdots w_{k+n-1} \leq \frac{2}{cn}$$

for all $n \geq 1$ and $k < 0$ with $k + n \leq 0$. If $k \geq 0$ and $n \geq 1$ then

$$\alpha_k \cdots \alpha_{k+n-1} \leq \frac{1}{2^n} \leq \frac{1}{2n}.$$

Finally, if $k \geq 1$ and $n \geq 1$ then

$$w_{-k} \cdots w_{-1}\alpha_0 \cdots \alpha_{n-1} \leq \frac{2}{ck}\frac{1}{2^n} \leq \frac{2}{c(n+k)}.$$

Since $0 < c < 1$, this shows that

$$\alpha_k \cdots \alpha_{k+n-1} \leq \frac{2}{cn} \quad (k \in \mathbb{Z}, n \geq 1).$$

This implies $\|T_\alpha^n\| \le \frac{2}{cn}$ for all $n \ge 1$. For $x \in X$,

$$\begin{aligned}(1 - \|T_\alpha^n\|)\|x\| &\le \|T_\alpha^n x - x\| \\ &= \|(I + T_\alpha + \cdots + T_\alpha^{n-1})(T_\alpha x - x)\| \\ &\le n\|T_\alpha x - x\|.\end{aligned}$$

So $\|T_\alpha x - x\| \ge \frac{1}{n}(1 - \frac{2}{cn})\|x\|$. Let $n = \lfloor \frac{4}{c} \rfloor$. Then $n \ge 4$ and $n > \frac{4}{c} - 1$ so that $\frac{4}{cn} < 1 + \frac{1}{n} \le \frac{5}{4}$ Hence

$$\|T_\alpha x - x\| \ge \frac{c}{4}(1 - \frac{5}{8}) \ge \frac{c}{12} \quad (x \in X). \tag{3.25}$$

An estimate for T_β. We want to estimate

$$\|(\lambda - T_\beta)^{-1}\| \quad (\lambda \in \mathbb{T}).$$

For $k \ge 0$ and $n \in \mathbb{N}$, let

$$x = \mathbf{e}_k + w_k \mathbf{e}_{k+1} + \cdots + (w_k \cdots w_{k+n-2})\mathbf{e}_{k+n-1}.$$

Since we have that $w_n \ge 1$ for $n \ge 0$, we get that

$$\begin{aligned} c &\le \frac{\|T_w x - x\|}{\|x\|} = \frac{1 + w_k \cdots w_{k+n-1}}{1 + w_k + \cdots + (w_k \cdots w_{k+n-2})} \\ &\le \frac{1 + w_k \cdots w_{k+n-1}}{n} \end{aligned} \quad (k \ge 0, n \in \mathbb{N}). \tag{3.26}$$

Let $\gamma_n^n := \inf\{\beta_k \cdots \beta_{k+n-1} \colon k \in \mathbb{Z}\}$ so that $\|T_\beta^n x\| \ge \gamma_n^n \|x\|$ for all $x \in \ell_1(\mathbb{Z})$. To get an estimate for $\frac{1+\gamma_n^n}{n}$, we complement (3.26) by

$$c \le 1 \le \frac{1 + 2^n}{n} \quad (n \ge 1),$$

and by

$$\begin{aligned}\beta_{-k} \cdots \beta_{-1}\beta_0 \cdots \beta_{n-1} + 1 &= 2^k w_0 \cdots w_{n-1} + 1 \\ &\ge (2^k - 1) + (w_0 \cdots w_{n-1} + 1) \quad (k, n \ge 1) \\ &\ge c(k + n).\end{aligned}$$

So we have

$$c \le \frac{1 + \gamma_n^n}{n} \quad (n \ge 1),$$

and we can combine this with the estimate

$$\begin{aligned}(\gamma_n^n - 1)\|x\| &\le \|T_\beta^n x - x\| \\ &\le \|(I + T_\beta + T_\beta^2 + \cdots + T_\beta^{n-1})(T_\beta x - x)\| \\ &\le \frac{\|T_\beta\|^n - 1}{\|T_\beta\| - 1}\|T_\beta x - x\|.\end{aligned}$$

Thus

$$\|T_\beta x - x\| \geq c_n \|x\| \quad \text{where } c_n = \frac{\|T_\beta\| - 1}{\|T_\beta\|^n}(\gamma_n^n - 1). \tag{3.27}$$

We optimize over n and choose $n = \lfloor \frac{1}{\log \|T_\beta\|} + \frac{2}{c} + 1 \rfloor$. Then

$$\frac{1}{\log \|T_\beta\|} + \frac{2}{c} + 1 \geq n \geq \frac{1}{\log \|T_\beta\|} + \frac{2}{c}$$

so that

$$(\gamma_n^n - 1) \geq n\frac{\gamma_n^n + 1}{n} - 2 \geq nc - 2 \geq \frac{1}{\log \|T_\beta\|},$$

$$\frac{1}{\|T_\beta\|^n} \geq \left(\frac{1}{\|T_\beta\|}\right)^{1/\log \|T_\beta\|} \|T_\beta\|^{-(1+2/c)} \geq \frac{1}{e}\|T_\beta\|^{-(1+2/c)}.$$

Thus, by (3.27) and $c \leq \|T_\beta\| - 1$, we find that

$$\|T_\beta x - x\| \geq \frac{c}{\log \|T_\beta\|} \frac{1}{e\|T_\beta\|^{2/c+1}} \|x\| \quad (x \in \ell_1(\mathbb{Z})). \tag{3.28}$$

Now, we can combine the inequalities (3.25) and (3.28) to obtain an estimate for $\|(I - S)^{-1}\|$. The dominant expression is the one in (3.28), so we get that

$$\|(I - S)^{-1}\| \leq e\frac{\log \|T_\beta\|}{c} \|T_\beta\|^{2/c+1}.$$

Finally, note that $\|T_\beta\| = \max\{2, \|T_w\|\}$, and we have a bound for $\|(I - S)^{-1}\|$ in terms of $\|T_w\|$ and the lower bound c.

Remark 3.6.5. We defined an isomorphic embedding $j\colon \ell_1(X) \to \ell_1(X) \oplus \ell_1(X)$ with $\|(x_n)\| \leq \|j(x_n)\| \leq 2\|(x_n)\|$ for all $(x_n) \in \ell_1(X)$. We can use j to define an isometric embedding $\tilde{j}$ by

$$\tilde{j}(\mathbf{e}_n) := \frac{g_n}{\|g_n\|} \quad (n \in \mathbb{Z}).$$

We would then find that

$$S\tilde{j}(\mathbf{e}_n) = \frac{\|g_{n+1}\|}{\|g_n\|}\tilde{j}(T_w \mathbf{e}_{n+1}).$$

Note that asymptotically, we have $\frac{\|g_{n+1}\|}{\|g_n\|} \to 1$ as $n \to \infty$ and as $n \to -\infty$.

3.6.2 Lower bounds of unbounded operators

In Theorem 3.6.2 we constructed an extension for a C_0-semigroup generator A by applying Theorem 3.6.1 to the resolvent of A, and then showing that the extension of the resolvent is again the resolvent of an operator. Let us see how lower bounds of the form (3.23) for A are related to lower bounds for its resolvent.

If $\mu \in \rho(A)$ then

$$\lambda - A = (\mu - \lambda)\big((\mu-\lambda)^{-1} - R(\mu, A)\big)(\mu - A) \quad (\lambda \in \mathbb{C}).$$

So (3.23) holds for A with $c > 0$ if and only if

$$|\mu - \lambda| \big\| \big((\mu-\lambda)^{-1} - R(\mu, A)\big)x \big\| \geq c \|R(\mu, A)x\| \qquad (x \in X).$$

As we mentioned in Section 3.6 and as we will see in Section 3.6.3, the extension in Theorem 3.6.1 uses a quotient of a space $H^\infty(U, X)$ of bounded holomorphic X-valued functions. The open set U contains the approximate point spectrum of the operator in question. We now see how this approach, applied to an unbounded operator, relates to the construction for one of its resolvent operators.

Let A be the generator of a C_0-semigroup T with $\omega_0(T) > 0$ and assume that A satisfies (3.23) for some $c > 0$. By Theorem 2.4.2, there is some $\eta > \omega_0(T)$ such that $\|R(\lambda, A)\| \leq \frac{2}{c}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \eta$. The open set

$$U = \{z \in \mathbb{C} \colon \operatorname{Re} z < -\frac{c}{2} \text{ or } \frac{c}{2} < \operatorname{Re} z < \omega_0(T) + \eta\}$$

contains the approximate point spectrum of A and $(\lambda - A)$ is uniformly bounded below for $\lambda \notin U$, that is

$$\|(\lambda - A)x\| \geq \frac{c}{2}\|x\| \quad (x \in D(A)).$$

Taking the transformation $f(z) := \frac{1}{\mu - z}$ $(z \in \mathbb{C})$ for some $\mu \in \rho(A)$, we get an open set $V = f(U)$ which includes $\sigma_{\mathrm{ap}}(R(\mu, A)) \setminus \{0\}$. There is a bijection $\pi \colon H^\infty(V, X) \to H^\infty(U, X)$ defined by

$$\pi(g)(z) := g(f(z)) \qquad (z \in U)$$

with inverse

$$\pi^{-1}(g)(w) = g(f^{-1}(w)) = g(\mu - \frac{1}{w}) \qquad (w \in f(U)).$$

If A is unbounded, we have that $0 \in \sigma_{\mathrm{ap}}(R(\mu, A))$. If we were to take an open neighbourhood $\tilde{V}$ for $\sigma_{\mathrm{ap}}(R(\mu, A))$ and map it to a set containing $\sigma_{\mathrm{ap}}(A)$ using f^{-1}, we would get a set $\tilde{U} = f^{-1}(\tilde{V})$ with bounded complement. We could instead try to work with the set U directly, perhaps by introducing further constraints on the functions $g \in H^\infty(U, X)$ that we use, such as continuity at ∞. This makes sense, since for these functions we have that $\pi(g)$ is continuous at 0 which is true for any function which is holomorphic on some set $\tilde{V}$ containing $\sigma_{\mathrm{ap}}(R(\mu, A)) \ni 0$. This seems to be a sensible constraint. Moreover, this defines a closed subspace of $H^\infty(U, X)$ as we work with the supremum norm.

We might also consider using another restriction on the functions $g \in H^\infty(U, X)$ that we want to work with. In the proof of Theorem 3.6.2 for a generator A of a C_0-semigroup T, we had to restrict to a subspace on which the C_0-semigroup induced by T is strongly continuous. So we could consider only those g such that $\|T(t)g - g\|_U \to 0$ as $t \to 0$. We will see which parts of the proof of Read's theorem we can adapt to C_0-semigroup generators in Section 3.6.3.

3.6.3 Adapting Read's proof

We can try to adapt the proof of Read's theorem to C_0-semigroup generators, as we successfully did for various other proofs including that of Arens' theorem (Theorem 3.1.1). With operators satisfying the inequality (3.23) in mind, we work towards an intermediate result which is a weaker statement than Theorem 3.6.1. The proposition for bounded operators is as follows.

Proposition 3.6.6 (see [Mül03])**.** *Let T be a bounded operator on a Banach space X and let U be an open neighbourhood of $\sigma_{\mathrm{ap}}(T)$. Then there exists an isometric embedding $X \subset Y$ and an isometric homomorphism $\varphi\colon \{T\}' \to \mathcal{B}(Y)$ such that $\|\varphi(V)\| = \|V\|$, $\varphi(V)\restriction_X = V$ for all $V \in \{T\}'$, and for the extension $S = \varphi(T)$ we have $\sigma(S) \subset \overline{U}$.*

We will see at the end of this section that we also get an estimate on the resolvent, of the form

$$\|R(\lambda, S)\| \leq K \operatorname{dist}(\lambda, U)^{-1} \quad (\lambda \in \sigma(T) \setminus \overline{U})$$

for some $K > 0$. In the context of an (unbounded) operator A satisfying (3.23) for some $c > 0$, we would want to choose an open set

$$U \subset \mathbb{C} \setminus \left\{ z \in \mathbb{C}\colon -\frac{c}{2} \leq \operatorname{Re} z \leq \frac{c}{2} \right\}.$$

We would hope to get an extension B of A such that the vertical strip is contained in the resolvent set of B, and $\|R(is, B)\| \leq \frac{2K}{c}$ $(s \in \mathbb{R})$ for some $K > 0$.

The idea of the proof of Proposition 3.6.6 was already outlined at the beginning of Section 3.6 after the statement of Theorem 3.6.1. What it essentially comes down to is taking a quotient space of $H^\infty(U, X)$ to identify M_z and the operator T, so that $M_{(\lambda-z)^{-1}}$ are the required resolvent operators for $\lambda \notin \overline{U}$. The problematic part is showing that the inclusion of the space X into the quotient is isomorphic (renorming then gives an isometric embedding). This proof was split into several steps in [Mül03, Chapter I, Section 4], which we take as an orientation. The result giving the isomorphic embedding is Lemma 3.6.10 below.

First, we have to introduce some notation. For an operator A, let

$$d(A) := \inf\{\|Ax\| \colon x \in D(A), \|x\| = 1\}$$

be its lower bound. For a fixed operator, the lower bound gives a function $d(\cdot - A)$ on $\mathbb{C}$ which is continuous by the inequality

$$|\|(z_1 - A)x\| - \|(z_2 - A)x\|| \leq |z_1 - z_2|\|x\| \quad (z_1, z_2 \in \mathbb{C}, x \in D(A)).$$

Note that the approximate point spectrum is the preimage of $\{0\}$ under this map.

Lemma 3.6.7. *Let A be an operator on a Banach space X with non-empty resolvent set $\rho(A)$. Let G be an open connected subset of $\mathbb{C} \setminus \sigma_{\mathrm{ap}}(A)$ and let U be a non-empty open subset of G. Suppose that $f \colon G \to X$ and $g \colon U \to D(A) \subset X$ are analytic functions which satisfy*

$$f(z) = (A - z)g(z) \quad (z \in U). \tag{3.29}$$

Then it is possible to extend g analytically to G and the extension satisfies (3.29) *for all $z \in G$.*

Proof. By G_0 we denote the largest open domain $U \subset G_0 \subset G$ to which g can be analytically extended and on which it satisfies the equation. We show that $G_0 = G$ by using the connectedness of G.

Let $\lambda \in G$ and take a continuous path $\varphi \colon [0, 1] \to G$ with $\varphi(0) \in U$ and $\varphi(1) = \lambda$. Let Z be the (closed) image of φ and choose some $r > 0$ such that $\operatorname{dist}(Z, \mathbb{C} \setminus G) > r$ and $\inf_{t \in [0,1]} d(A - \varphi(t)) \geq r$. Let

$$M = \max\{\|f(z)\| \colon \operatorname{dist}(z, Z) \leq r\}$$

and let $\mu \in G_0 \cap Z$. On some neighbourhood $U_1 \subset G$ of μ, f and g are defined and satisfy (3.29) for $z \in U_1$. Both f and g have a Taylor series

$$f(z) = \sum_{i=0}^{\infty} f_i (z - \mu)^i,$$
$$g(z) = \sum_{i=0}^{\infty} g_i (z - \mu)^i$$

for z near μ with coefficients $f_i \in X, g_i \in D(A)$ $(i \geq 0)$. Then by the Cauchy integral formula we have $\|f_i\| \leq \frac{M}{r^i}$ $(i \geq 0)$. Comparing coefficients in (3.29) gives

$$f_0 = (A - \mu) g_0, \qquad (A - \mu) g_i = g_{i-1} + f_i \quad (i \geq 1).$$

So we can estimate the norm of the g_i by $\|g_0\| \leq d(A - \mu)^{-1} \|f_0\| \leq r^{-1} M$ and $\|g_i\| \leq (i+1) M r^{-(i+1)}$. This shows the convergence of the Taylor series of g for $|z - \mu| < r$, hence these z belong to G_0. Since we chose r independently of the point $\mu \in Z \cap G_0$, we see that the intersection $Z \cap G_0$ is both open and closed in Z. By connectedness of Z it is all of Z. As the choice of λ is arbitrary, we see $G_0 = G$. □

Lemma 3.6.8. *Let A be an operator on a Banach space X with non-empty resolvent set. Let U be a neighbourhood of $\sigma_{\mathrm{ap}}(A)$ such that the operators $\lambda - A$ have a uniform lower bound $c > 0$ for $\lambda \in \mathbb{C} \setminus U$. Let $g \colon U \to D(A) \subset X$ be a bounded analytic function satisfying $x = (A - z) g(z)$ for $z \in U$. Then $x = 0$.*

Proof. For each connected component G of $\mathbb{C} \setminus \sigma_{\mathrm{ap}}(A)$ the intersection $U \cap G$ is non-empty. So by Lemma 3.6.7, we can extend g to an entire function satisfying

$$x = (A - z) g(z) \quad (z \in \mathbb{C}).$$

The function g is assumed to be bounded on U and for $\lambda \in \mathbb{C} \setminus U$ we calculate $c \cdot \|g(\lambda)\| \leq \|(\lambda - A) g(\lambda)\| = \|x\|$. So g is constant by Liouville's theorem. Use distinct values $z_1, z_2 \in \mathbb{C}$ to get $x = 0$. □

Remark 3.6.9. Let $T \in \mathcal{B}(X)$ and let $U \supset \sigma_{\mathrm{ap}}(T)$ be an open set. Since $\{\lambda \in \mathbb{C} \colon |\lambda| \leq 2\|T\|\} \setminus U$ is compact, the continuous function $d(\cdot - T)$ attains its lower bound on this set. For $\lambda \in \mathbb{C}$ with $|\lambda| > 2\|T\|$ we have $\|(\lambda - T)x\| \geq \|T\| \|x\|$ for all $x \in X$. So $(\lambda - T)$ has a uniform lower bound for $\lambda \in \mathbb{C} \setminus U$ for any such U.

Remember that $H^\infty(U, X)$ denotes the set of bounded holomorphic X-valued functions on U. Denote the supremum norm over U by $\|\cdot\|_U$. The following lemma will allow us to define a quotient norm on this vector space for which the embedding $X \to H^\infty(U, X)$ by constant functions is an isomorphism, and which identifies T and M_z.

Lemma 3.6.10. *Let T be a bounded operator on a Banach space X and U be an open neighbourhood of $\sigma_{\mathrm{ap}}(T)$. Then there exists a constant $K > 0$ such that $\|x\| \leq K\|g\|_U$ whenever $x \in X, f, g \in H^\infty(U, X)$ and $x = g(z) + (T - z)f(z)$ $(z \in U)$.*

Proof. By the inclusion $H^\infty(V, X) \subset H^\infty(U, X)$ for $U \subset V$ via restrictions of functions, which is continuous for the supremum norm, and as $\sigma_{\mathrm{ap}}(T)$ is bounded, we may assume that U is bounded. For proof by contradiction we assume there are $x_n \in X, f_n, g_n \in H^\infty(U, X)$ such that $x_n = g_n(z)+(T-z)f_n(z)$ for $z \in U$ and $\|x_n\| = 1$, and $\|g_n\|_U \to 0$ as $n \to \infty$.

First note that the operator T induces an operator to the quotient $Q(X) = \ell^\infty(X) / c_0(X)$ since $c_0(X)$ is invariant under T. Here, we use the pointwise application $T(u_n) := (Tu_n)$. Take the element $x = (x_n) + c_0(X) \in Q(X)$ with norm $\|x\| = \limsup_{n\to\infty}\|x_n\| = 1$ and the function $g(z) := (g_n(z))$ with values in $c_0(X)$. On the set

$$Z = \left\{z \in U\colon\ \mathrm{dist}(z, \sigma_{\mathrm{ap}}(T)) \geq \tfrac{1}{2}\,\mathrm{dist}(\sigma_{\mathrm{ap}}(T), \mathbb{C} \setminus U)\right\}$$

the function $d(\cdot - T)$ has a lower bound $r > 0$. By the maximum modulus principle we have

$$\begin{aligned}\sup_{n\geq 0}\|f_n(z)\|_U =& \sup_{n\geq 0}\|f_n(z)\|_Z \leq r^{-1}\sup_{n\geq 0}\|(T - z)f_n(z)\|_Z \\ =& r^{-1}\sup_{n\geq 0}\|x_n - g_n(z)\|_Z < \infty\end{aligned}$$

so there is a bounded function $f\colon U \to Q(X)$ given by $f(z) = (f_n(z)) + c_0(X)$. We show that f is holomorphic. Let $\lambda \in U, 0 < s < \mathrm{dist}(\lambda, \mathbb{C} \setminus U), M = \sup_{n\geq 0}\|f_n\|_U$ and let

$$f_n(z) = \sum_{i=0}^{\infty} f_{n,i}(z - \lambda)^i \quad (|z - \lambda| < \mathrm{dist}(\lambda, \mathbb{C} \setminus U))\,.$$

On the coefficients we have the norm estimates $\|f_{n,i}\| \leq \frac{M}{s^i}$ so for each i the coefficients $(f_{n,i})_n$ are in $\ell^\infty(X)$ and $f \in H^\infty(U, \ell^\infty(X))$. Let us pass to the quotient $Q(X)$. As the approximate point spectrum for T on $Q(X)$ is the same as on X, the equation

$$(x + c_0(X)) = (T - z)(f(z) + c_0(X)) \quad (z \in U)$$

and Lemma 3.6.8 imply $x \in c_0(X)$ which contradicts $\|x\| = 1$. □

Let us try to find a statement similar to Lemma 3.6.10 for an unbounded operator A in place of the bounded operator T. Since we cannot assume that U is bounded, we assume instead that $\lambda - A$ is uniformly bounded below for $\lambda \notin U$. Assume there exist $x_n \in X$,

$$f_n, g_n \in \{f \in H^\infty(U, X)\colon f \text{ is continuous at } \infty\}$$

such that $x_n = g_n(z) + (A - z)f_n(z)$ for $z \in U$, $\|x_n\| = 1$ and $\|g_n\|_U \to 0$ as $n \to \infty$. Then $f_n(\infty) = 0$ for $n \geq 0$. We want to show that $\sup_{n\geq 0} \|f_n\|_U < \infty$, using the uniform lower bound of $A - z$ near the boundary of U. Assume $d(\lambda - A) \geq 2r$ for $\lambda \in \mathbb{C} \setminus U$ and let

$$Z = \{z \in U\colon \operatorname{dist}(z, \partial U) < r\},$$

where ∂U denotes the boundary of U. For $\lambda \in Z$ we have $\|(\lambda - A)x\| \geq r\|x\|$ for all $x \in D(A)$. Since f is continuous at ∞, the maximum modulus principle shows that $\|f_n\|_U = \|f_n\|_Z$ for all $n \in \mathbb{N}$. So

$$\begin{aligned} \sup_{n\geq 0} \|f_n(z)\|_U &= \sup_{n\geq 0} \|f_n(z)\|_Z \leq r^{-1} \sup_{n\geq 0} \|(A - z)f_n(z)\|_Z \\ &= r^{-1} \sup_{n\geq 0} \|x_n - g_n(z)\|_Z < \infty. \end{aligned}$$

Hence $f(z) := (f_n(z))$ is bounded on U. The map, $f\colon U \to \ell_\infty(X)$ is holomorphic by the same reasoning as in the proof of Lemma 3.6.10. Moreover, the function $((A-z)f_n(z))+c_0(X)$ with values in the quotient $\ell_\infty(X)/c_0(X)$ is equal to $(x_n)+c_0(X)$. We want to conclude that $x = 0$ using Lemma 3.6.8. To that end, we would want to have an operator A on the quotient $\ell_\infty(X) / c_0(X)$ and be able to write $(x_n)+c_0(X) = (A - z)(f_n(z)) + c_0(X)$. However, we cannot write

$$(A - z)(f_n(z)) + c_0(X) = ((A - z)f_n(z)) + c_0(X)$$

since we do not have that the operator

$$A(x_n) := (Ax_n), \qquad D(A) = \{(x_n) \in \ell_\infty(X)\colon x_n \in D(A), (Ax_n) \in \ell_\infty(X)\}$$

is well defined on the quotient $\ell_\infty(X) / c_0(X)$. This means we cannot use this strategy for unbounded operators.

We now state a proof for Proposition 3.6.6.

Proof of Proposition 3.6.6. Take the Banach space $H^\infty(U, X)$ and the closure J of the subspace $\{(T - z)f(z)\colon f \in H^\infty(U, X)\}$. We can define an embedding $\rho\colon X \to Y$ into the quotient $Y = H^\infty(U, X) / J$ by using the constant functions. The embedding ρ satisfies $\|\rho(x)\|_Y \leq \|x\|_X$ $(x \in X)$. By Lemma 3.6.10 there is a $K > 0$ such that

$$\|\rho(x)\|_Y = \inf_{f\in H^\infty(U,X)} \|x - (T - z)f\|_U \geq K^{-1}\|x\| \quad (x \in X),$$

so ρ is an isomorphism.

Every operator V commuting with T extends to an operator Y, using pointwise multiplication on H^∞ and observing invariance of J. This extension of V is bounded by $\|V\|$. We denote the extension of T by S.

For $\lambda \in \mathbb{C}\backslash\overline{U}$ the function $(\lambda - z)^{-1}$ is in $H^\infty(U, \mathbb{C})$. So we define the multiplication operator $R\colon f(z) \mapsto (\lambda - z)^{-1} f(z) + J$ for $f \in H^\infty(U, X)$ and observe

$$R(\lambda - S)f(z) + J = (\lambda - z)^{-1}((\lambda - z) + (z - S))f(z) = f(z) + J.$$

Similarly $(\lambda - S)Rf(z) + J = f(z) + J$. Hence $\lambda - S$ is invertible on the quotient Y and $\lambda \notin \sigma(S)$.

We have to change the norm on Y to get an isometric embedding. We can choose the norm

$$\|y\| = \inf\left\{\|x\|_X + K\|y - \rho(x)\|_Y \colon x \in X\right\}.$$

This norm is equivalent to $\|\cdot\|_Y$ with $\min\{1, K\}\|y\|_Y \leq \|y\| \leq K\|y\|_Y$ for all $y \in Y$. For $x \in X$, we have

$$\|x\|_X \leq \|y\|_X + \|x - y\|_X \leq \|y\|_X + k\|\rho(x - y)\|_Y \quad (y \in X)$$

which implies $\|x\|_X \leq \|\rho(x)\|$, and we have

$$\begin{aligned}\|\rho(x)\| &= \inf\{\|y\|_X + k\|\rho(x - y)\| \colon y \in X\} \\ &\leq \|x\|_X + k\|0\| \leq \|x\|_X\end{aligned}$$

by choosing $y = x$. So X is isometrically embedded in Y. For $y \in Y$, we can bound $\|Sy\|$ by

$$\begin{aligned}\|Sy\| &= \inf\left\{\|x\|_X + k\|Sy - \rho(x)\|_Y \colon x \in X\right\} \\ &\leq \inf\left\{\|Tx\|_X + k\|Sy - S\rho(x)\|_Y \colon x \in X\right\} \\ &\leq \|T\| \inf\left\{\|x\|_X + k\|y - \rho(x)\|_Y \colon x \in X\right\} \\ &= \|T\|\|y\|.\end{aligned}$$

The isometric embedding gives the last required inequality, $\|S\| \geq \|T\|$. □

Remark 3.6.11. In the proof of Proposition 3.6.6, we can bound the resolvent operators for the extension S in the following way. Let $y \in Y$ and remember that the resolvent $R(\lambda, S)$ for $\lambda \notin U$ was defined by $M_{(\lambda - z)^{-1}}$. Using the two norms $\|\cdot\|$ and $\|\cdot\|_Y$ on Y, we find

$$\|(\lambda - z)^{-1}y\| \leq K\|(\lambda - z)^{-1}y\|_Y \leq K \operatorname{dist}(\lambda, U)^{-1}\|y\|_Y \leq K \operatorname{dist}(\lambda, U)^{-1}\|y\| \quad (y \in Y).$$

Chapter 4

Maximal Parts

In this chapter we introduce the concept of *maximal parts*. The idea of maximal parts is to use seminorms to describe continuous embeddings of Banach spaces, with respect to some properties of an operator or a family of operators. The properties we will consider are contractivity, expansiveness, dissipativity and accretivity. In the definition of the seminorms, we will find the same formulations and ideas that are used for invertible extensions. Moreover, we will find that we need invertible extensions to prove maximality of some seminorms, such as the expansive seminorm in Section 4.1. The results in this section are all new, apart from Theorems 4.7.2 and 4.7.3. The main results of this chapter are Theorem 4.2.6 on a dissipative seminorm, Proposition 4.2.16 on the universal property of this seminorm, and the two applications of this result. The first application to quasi-hyperbolic semigroup leads to Theorem 4.7.1 which is a characterisation for the existence of a continuous embedding into a quasi-hyperbolic semigroup. This result relates to the work in [BT10]. The second application in Theorem 4.7.4 is a generation theorem which stands in relation to the generation theorem by Goldberg and Smith ([GS78]) as the Feller-Miyadera-Phillips theorem stands in relation to the Hille-Yosida theorem (see Theorems 2.4.1 and 2.4.2).

4.1 Contractive and expansive parts of a bounded operator

To understand what a certain maximal part of an operator is, we look at the simplest case first. We will define the maximal contractive part of a bounded operator. This is very prototypical and the arguments in the following sections will follow the same line of reasoning, although in Section 4.2 some properties such as maximality, for example, will be less obvious than they are in the contractive case.

Let T be a bounded operator on a Banach space X with norm $\|\cdot\|$. There are seminorms $\|\cdot\|'$ on X such that

1. $\|x\|' \leq \|x\|$ for all $x \in X$,
2. $\|Tx\|' \leq \|x\|'$ for all $x \in X$.

If a seminorm $\|\cdot\|'$ satisfies 1 and 2, we say that it is a contractive seminorm for T. There is clearly a largest seminorm amongst those $\|\cdot\|'$, which is given by

$$\sup\left\{\|x\|' \colon \|\cdot\|' \text{ satisfies 1 and 2}\right\} \quad (x \in X).$$

It turns out that the largest of these seminorms is given by

$$\|x\|_T = \inf\left\{\sum_{i=0}^{n}\|x_i\| \colon n \geq 0, x = \sum_{i=0}^{n} T^i x_i\right\} \quad (x \in X). \tag{4.1}$$

It is fairly easy to check that $\|\cdot\|_T$ is a seminorm, and since $x = T^0 x$ we see that $\|x\|_T \leq \|x\|$ for all $x \in X$. To show that T is contractive in $\|\cdot\|_T$, let $x \in X$ and assume that

$$x = \sum_{i=0}^{n} T^i x_i. \tag{4.2}$$

Multiply (4.2) by T to see that $Tx = \sum_{i=1}^{n+1} T^i x_{i-1}$ holds. So $\|Tx\|_T \leq \|x\|_T$ for all $x \in X$. Now, we want to show that indeed every other contractive seminorm $\|\cdot\|'$ for T is dominated by $\|\cdot\|_T$. So let $\|\cdot\|'$ be a seminorm which satisfies 1 and 2, let $x \in X$ and assume that (4.2) holds. Then

$$\|x\|' \leq \sum_{i=0}^{n}\|T^i x_i\|' \leq \sum_{i=0}^{n}\|x_i\|' \leq \sum_{i=0}^{n}\|x_i\|.$$

So we have that $\|x\|' \leq \|x\|_T$.

Note that any operator U commuting with T is bounded in $\|\cdot\|_T$. More precisely, we have

$$\|Ux\|_T \leq \|U\|\|x\|_T \quad (x \in X), \tag{4.3}$$

where $\|U\|$ is the usual operator norm of U in $\mathcal{B}(X)$. This can be seen from the definition of $\|\cdot\|_T$. In particular, (4.3) holds for the resolvent of T.

We now use $\|\cdot\|_T$ to find a contractive map $\pi \colon X \to X_T$ to a Banach space X_T such that π intertwines T and a contractive operator $\mathcal{T}$. We will find that $\rho(T) \subset \rho(\mathcal{T})$. Let

$$N = \{x \in X \colon \|x\|_T = 0\}$$

be the (norm closed) null space of $\|\cdot\|_T$ and let X_T be the completion of the quotient space $X \,/\, N$ in the norm induced by $\|\cdot\|_T$ (for which we will use the same notation). Since N is invariant under T, the operator

$$\hat{T}(x+N) := Tx + N \quad (x \in X)$$

is well-defined on $X \,/\, N$, and it is contractive since $\|Tx\|_T \leq \|x\|_T$. So $\hat{T}$ extends continuously to a contractive operator $\mathcal{T} \in \mathcal{B}(X_T)$. Similarly, for any operator $U \in \mathcal{B}(X)$ commuting with T we get an operator $\mathcal{U} \in \mathcal{B}(X_T)$ with $\|\mathcal{U}\| \leq \|U\|$ by (4.3), and the map $U \mapsto \mathcal{U}$ is a contractive algebra homomorphism. Let $\pi\colon X \to X_T$ be the natural map which is induced by the quotient map from X to $X \,/\, N$. It is clear that π is contractive and has dense range. Moreover, it satisfies $\pi U = \mathcal{U}\pi$ for the operators U and their associated operators $\mathcal{U}$, and in particular $\pi T = \mathcal{T}\pi$. This also implies that $\sigma(\mathcal{T}) \subset \sigma(T)$ and $\pi R(\lambda, T) = R(\lambda, \mathcal{T})\pi$ for $\lambda \in \rho(T)$.

Definition 4.1.1. The triple $(X_T, \mathcal{T}, \pi)$ is called the *contractive part* of T.

We say that a triple (Y, S, ρ) is an intertwining triple if Y is a Banach space, $\rho\colon X \to Y$ is a contractive linear map, $S \in \mathcal{B}(Y)$ is contractive and $S\rho = \rho T$. For an intertwining triple (Y, S, ρ) we can define a seminorm on X by

$$\|x\|' = \|\rho(x)\| \quad (x \in X).$$

We have that $\|x\|' \leq \|\rho\|\|x\| \leq \|x\|$ and

$$\|Tx\|' = \|\rho(Tx)\| = \|S\rho(x)\| \leq \|\rho(x)\| = \|x\|'$$

for all $x \in X$. So $\|\cdot\|'$ is a contractive seminorm for T and thus $\|x\|' \leq \|x\|_T$ $(x \in X)$. This inequality corresponds to a universal property of the triple $(X_T, \mathcal{T}, \pi)$, which is stated in the next proposition.

Proposition 4.1.2. *Let $(X_T, \mathcal{T}, \pi)$ be the contractive part of an operator $T \in \mathcal{B}(X)$. Let (Y, S, ρ) be an intertwining triple. Then there is a unique contractive map $\sigma\colon X_T \to Y$ such that $\rho = \sigma\pi$ and $\sigma\mathcal{T} = S\sigma$.*
Moreover, if some intertwining triple (Y, S, ρ) has this universal property, there is a unique isometry $\sigma\colon X_T \to Y$ intertwining $\mathcal{T}$ and S.

Proof. Since $\|\rho(\cdot)\|$ defines a contractive seminorm for T, we see that $\|\rho(x)\| \leq \|x\|_T$ for $x \in X$ by maximality of $\|x\|_T$. Hence if we define $\sigma\colon \pi(x) \mapsto \rho(x)$ for $x \in X$, then σ is well-defined and contractive on a dense subset of X_T. We want to extend σ

to X_T. Since σ is contractive, it extends uniquely to X_T. Also, we have $\sigma\pi = \rho$ by construction. Moreover,

$$\sigma\mathcal{T}\pi = \sigma\pi T = \rho T = S\rho = S\sigma\pi$$

which proves that $\sigma\mathcal{T} = S\sigma$ on the dense subset $X \,/\, N \subset X_T$. Continuity of $\sigma\mathcal{T}$ and $S\sigma$ implies that the equality holds on X_T. This proves the universal property of the contractive part of T.

Assume that (Y, S, ρ) is another triple with this universal property. Then there are contractive maps $\sigma\colon X_T \to Y$ and $\sigma'\colon Y \to X_T$ which intertwine $\mathcal{T}$ and S. Since $\sigma'\sigma\pi = \sigma'\rho = \pi$, we see that $\sigma'\sigma = I_{X_T}$ by density of $\operatorname{ran}\pi$ in X_T. But σ and σ' are contractive, so σ must be isometric. Hence X_T is isometrically embedded in Y. If Y is minimal in the sense that for no non-trivial closed subspace Z with $\rho(X) \subset Z \subset Y$ the triple $(Z, S{\restriction_Z}, \sigma)$ has the universal property, then we even get that $Y = \sigma X_T$. $\square$

Consider the dual map $\pi'\colon {X_T}' \to X'$. Since π has dense range, we get that π' is injective. This means, we can identify ${X_T}'$ with a subspace of X'. Let us show that this subspace is given by

$$W := \left\{\varphi \in X'\colon \sup_{n\geq 0} \|(T')^n\varphi\| < \infty\right\}.$$

Let $\xi \in {X_T}'$ and $n \geq 0$. Then

$$|(T')^n\pi'\xi(x)| = |\xi(\pi T^n x)| \leq \|\xi\|\|T^n x\|_T \leq \|\xi\|\|x\| \quad (x \in X).$$

This shows that $\pi'\xi \in W$ and $\sup_{n\geq 0}\|(T')^n\pi'\xi\| \leq \|\xi\|$. On the other hand, if $\varphi \in W$, $x \in X$ and $x = \sum_{i=0}^n T^i x_i$, then

$$|\varphi(x)| \leq \sum_{i=0}^{n} |\varphi(T^i x_i)| \leq \sum_{i=0}^{n} \|(T')^i\varphi\|\|x_i\|.$$

This shows that $|\varphi(x)| \leq (\sup_{n\geq 0}\|(T')^n\varphi\|)\|x\|_T$ $(x \in X)$. This means that ξ defined by $\xi(\pi x) := \varphi(x)$ $(x \in X)$ is a well-defined functional on $X \,/\, N$ which extends continuously to X_T. In particular, we have $\varphi = \pi'\xi$ and $\|\xi\| \leq \sup_{n\geq 0}\|(T')^n\varphi\|$. This shows that $W = \operatorname{ran}(\pi')$ and

$$\|\xi\| = \sup_{n\geq 0}\|(T')^n\pi'\xi\| \quad (\xi \in {X_T}').$$

Using the Hahn-Banach theorem, we get the following proposition.

Proposition 4.1.3. *Let $T \in \mathcal{B}(X)$. The seminorm $\|\cdot\|_T$ is given by*

$$\|x\|_T = \sup\left\{|\varphi(x)|\colon \varphi \in X',\ \sup_{n\geq 0}\|(T')^n\pi'\varphi\| = 1\right\} \quad (x \in X). \tag{4.4}$$

To see some effects of this renorming, let us look at two basic examples. The first one is a multiplication operator $T = M_h$ which multiplies functions from the space $X = L_p(\Omega, \mu)$ $(1 \leq p \leq \infty)$ by a bounded measurable function $h\colon \Omega \to \mathbb{C}$. If $f \in X$ is supported on $\{|h| \geq c\}$ for some $c > 1$ then $\frac{f}{h^n} \in X$, $f = T^n \frac{f}{h^n}$ and thus $\|f\|_T \leq \frac{\|f\|}{c^n}$ for all $n \geq 1$. So for general $f \in X$ and $c > 1$ we get

$$\|f\|_T \leq \|f\chi_{\{|h|<c\}}\|_T + \|f\chi_{\{|h|\geq c\}}\|_T \leq \|f\chi_{\{|h|<c\}}\|.$$

Thus,

$$\|f\|_T \leq \lim_{c\to 1^+}\|f\chi_{\{|h|<c\}}\| \quad (f \in X). \tag{4.5}$$

Since T is contractive in the seminorm $\lim_{c\to 1^+}\|f\chi_{\{|h|<c\}}\|$ $(f \in X)$, and since this seminorm is dominated by $\|\cdot\|$, we have equality in (4.5). For $1 \leq p < \infty$, we can apply the dominated convergence theorem to simplify this to

$$\|f\|_T = \|f \cdot \chi_{\{|h|\leq 1\}}\| \quad (f \in X).$$

But on $L_\infty(1, 2)$ with $h(x) = x$, for example, we get $\|f\|_T \neq 0 = \|f\chi_{\{|h|\leq 1\}}\|$ if we take, say, $f(x) = 1$ for all $x \in (1, 2)$.

The other standard example is shift operators. Let T be the right shift on the weighted space $\ell_1(w)$ with norm $\|\cdot\|_{1,w}$ (see Section 2.6). The contractive norm $\|\cdot\|_T$ is again a weighted ℓ_1-norm for the weight

$$v(n) = \min\{w(k)\colon k \leq n\} \quad (n \in \mathbb{N}).$$

Indeed, we can use the maximality property of $\|\cdot\|_T$ to see this as follows. A simple calculation shows that $\|\cdot\|_T$ attains the values $\|\mathbf{e}_n\|_T = v(n)$ on the basic sequences $\mathbf{e}_n$ for $n \in \mathbb{N}$. Since $\|\cdot\|_T$ has to satisfy the triangle inequality we have

$$\|(x_n)\|_T \leq \sum_{n\in\mathbb{N}}|x_n|\|\mathbf{e}_n\|_T = \|(x_n)\|_{1,v} \quad ((x_n) \in \ell_1(w))$$

and hence $\|\cdot\|_T$ is dominated by the norm $\|\cdot\|_{1,v}$. On the other hand, the weight v is decreasing so the shift T satisfies $\|Tx\|_{1,v} \leq \|x\|_{1,v}$ $(x \in X)$ and $\|\cdot\|_{1,v}$ is dominated by the norm $\|\cdot\|_{1,w}$. Since the norm $\|\cdot\|_T$ is maximal with these two properties, it must be equal to $\|\cdot\|_{1,v}$.

For shifts on weighted ℓ_p spaces for $1 < p \leq \infty$, however, the norm $\|\cdot\|_T$ is no longer of such a simple form. We will now look at shifts on finite dimensional ℓ_∞-spaces.

Example 4.1.4 (Two-dimensional shift)**.** Let X be the vector space $\mathbb{C}^2$ equipped with the norm

$$\|(\alpha,\beta)\| = \max\{|\alpha|, w|\beta|\} \quad ((\alpha,\beta)\in X)$$

for some $w > 1$. Let T be the right shift

$$T(\alpha,\beta) = (0,\alpha) \quad ((\alpha,\beta)\in X).$$

Let $x = (\alpha,\beta)\in X$. To calculate $\|x\|_T$, we want to minimise

$$\|x_1\| + \|x_2\| \quad (x_1, x_2 \in X,\ x = x_1 + Tx_2).$$

So we can take $x_1 = (\alpha, t)$ and $x_2 = (\beta - t, 0)$ for some $t \in \mathbb{C}$. We now seek to minimise the function

$$f(t) := \max(|\alpha|, w|t|) + |\beta - t| \quad (t \in \mathbb{C}).$$

Clearly, we can assume that $\alpha, \beta \geq 0$ and $0 \leq t \leq \beta$. The derivative in $t > 0$ is then given by

$$f'(t) = \begin{cases} -1 & wt < \alpha \\ w - 1 > 0 & wt > \alpha. \end{cases}$$

So for $\alpha \geq w\beta$ we minimise $f(t)$ by choosing $t = \beta$. Otherwise, $f(t)$ is smallest when $wt = \alpha$. We thus see that

$$\begin{aligned} \|x\|_T &= \begin{cases} |\alpha|\dfrac{w-1}{w} + |\beta| & \text{if } |\alpha| \leq |\beta|w \\ |\alpha| & \text{if } |\alpha| \geq |\beta|w \end{cases} \\ &= \max\Big\{\,|\alpha|,\ |\alpha|\frac{w-1}{w} + |\beta|\,\Big\}. \end{aligned}$$

In the following example, we show that for a shift T on an n-dimensional weighted ℓ_∞-space, the contractive norm $\|x\|_T$ is again given as a maximum over a finite number of functionals evaluated at $|x|$. Here, $|x|$ is the (possibly finite) sequence $(|x_k|)$ if $x = (x_k)$. The following argument is of a geometric nature. We show that for a shift operator, the unit ball of the contractive norm on the first k coordinates is determined by the original norm on the first k coordinates only, and can hence be calculated recursively.

Example 4.1.5 (n-dimensional shift)**.** Let X be the vector space $\mathbb{C}^n$ equipped with the norm

$$\|x\| = \max\{w_i|x_i| \colon i = 1,\ldots,n\} \quad (x \in \mathbb{C}^n)$$

for some $w_i > 0$ $(i = 1, \ldots, n)$. Let T be the shift operator

$$T(x_1, x_2, \ldots, x_n) = (0, x_1, \ldots, x_{n-1}).$$

Then the contractive norm $\|\cdot\|_T$ of the shift T can be calculated recursively over elements of the form

$$x = (x_1, \ldots, x_k, 0, \ldots, 0)$$

since for such an x, $\|x\|_T$ only depends on the first k values of w. We show this now by using the definition (4.1) of $\|\cdot\|_T$. Note that we can fix the variable n in (4.1) and assume it is equal the dimension of the space, since $T^i = 0$ for $i \geq n$.

If $k = 1$ then the definition shows that

$$\|(x_1, 0, \ldots, 0)\|_T = \|(x_1, 0, \ldots, 0)\| = w_1|x_1|.$$

For $2 \leq k \leq n$, the norm $\|x\|_T$ is given by the minimum of

$$\sum_{i=0}^{n} \|y_i\| \quad (y_i \in X, \ \sum_{i=0}^{n} T^i y_i = x).$$

Among the minimising tuples (y_i), there is one for which the j-th coordinate of y_i is 0 when $j > k - i$ and thus $y_i = 0$ if $i > k$. Indeed, for any minimising tuple $(\tilde{y}_i)$, we can obtain another minimising tuple (y_i) by setting these coordinates equal to 0. Now, let (y_i) be such a minimising tuple and let

$$y = \sum_{i=1}^{k} T^{i-1} y_i.$$

We then have $x = y_0 + Ty$. Since at most the first $k - 1$ coordinates of y are non-zero, we can assume that we already know $\|y\|_T$ for which we know $\|y\|_T \leq \sum_{i=1}^{k} \|y_i\|$. This shows that $\|x\|_T$ is given by

$$\|(x_1, \ldots, x_k, 0, \ldots, 0)\|_T = \min \left\{ \|x - Ty\| + \|y\|_T \colon y = (y_1, \ldots, y_{k-1}, 0, \ldots, 0) \right\}$$

which depends on $w_1, \ldots, w_k$ only, as $\|y\|_T$ can be calculated from $w_1, \ldots, w_{k-1}$ only.

Let us find a geometric description of the unit ball of the norm $\|\cdot\|_T$ using that it is recursive in the dimension n. Let B be the closed unit ball of $\|\cdot\|$,

$$B_{n-1} = \left\{ y \in \mathbb{C}^n \colon y = (y_1, \ldots, y_{n-1}, 0), \|y\|_T \leq 1 \right\} \tag{4.6}$$

and define C as the convex hull of $B \cup B_{n-1} \cup TB_{n-1}$. We will show that C is the closed unit ball of $\|\cdot\|_T$. For this, we first show that C is the unit ball of a norm on $\mathbb{C}^n$.

For C to be the unit ball of a norm, it has to be convex, balanced ($\lambda C \subset C$ for $|\lambda| \leq 1$) and it has to satisfy $\bigcup_{k\geq 1} kC = \mathbb{C}^n$. It is clear that C is convex. To show that C is balanced, it suffices to consider $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Let $|\lambda| = 1$, let $u \in C$ and write u as the convex combination of elements $x \in B_{n-1}, y \in TB_{n-1}$ and $z \in B$. So there are numbers $\alpha, \beta, \gamma \geq 0$ with

$$u = \alpha x + \beta y + \gamma z \quad (\alpha + \beta + \gamma = 1). \tag{4.7}$$

Since B and B_{n-1} are balanced sets, $\lambda u \in C$. Finally, $B \subset C$ we have that

$$\bigcup_{k\geq 1} kC \supseteq \bigcup_{k\geq 1} kB = \mathbb{C}^n.$$

So C is indeed the unit ball of a norm $\|\cdot\|$.

Let us show next that $\|Tx\|_C \leq \|x\|_C$ for $x \in X$, that is $u \in C$ implies $Tu \in C$. Let $u \in C$ and write u as a convex combination as in (4.7). It is trivial that $Tx \in TB_{n-1}$, since $x \in B_{n-1}$. To deal with Ty for $y \in TB_{n-1}$, write $y = Ty'$ where $y' = (y_1, \ldots, y_{n-1}, 0)$ and $\|y'\|_T \leq 1$. We want to show that $Ty = T^2y' \in TB_{n-1}$. Let P_i be the projection onto the first i coordinates ($i \geq 1$). Clearly, $\|P_i\| \leq 1$ and also $P_iT^j = T^jP_{i-j}$. So

$$T^2y' = P_nT^2y' = T^2P_{n-2}y'$$

and it is sufficient to show that $TP_{n-2}y' \in B_{n-1}$. But if $y' = \sum_{i=0}^n T^iv_i$ for some $v_i \in X$ then

$$TP_{n-2}y' = \sum_{i=0}^{n} P_{n-1}T^{i+1}v_i$$

and thus $\|TP_{n-2}y'\|_T \leq \|y'\|_T \leq 1$. Hence $TP_{n-2}y' \in B_{n-1}$ and $Ty = T^2P_{n-2}y' \in TB_{n-1}$. To handle $z \in B$, let $z' = (0, \ldots, 0, z_n)$ so that $Tz' = 0$. Since

$$z - z' = (z_1, \ldots, z_{n-1}, 0)$$

and by definition of the norm $\|z-z'\|_T \leq \|z-z'\| \leq \|z\| \leq 1$ we have that $z-z' \in B_{n-1}$. So again, $Tz \in TB_{n-1}$. Thus, Tu is a convex combination of elements in TB_{n-1} and by convexity of TB_{n-1}, $Tu \in TB_{n-1} \subset C$. This is what we wanted to show. Hence $\|Tx\|_C \leq \|x\|_C$ for all $x \in X$.

We can now show that C is the unit ball of $\|\cdot\|_T$. Note first that we have $\|\cdot\|_C \leq \|\cdot\|_T$ by maximality of $\|\cdot\|_T$. So we only have to show that the unit ball of $\|\cdot\|_T$ contains C. The unit ball of $\|\cdot\|_T$ must contain B since $\|\cdot\|_T \leq \|\cdot\|$, and it certainly contains B_{n-1} and TB_{n-1}. So C is contained in the unit ball of $\|\cdot\|_T$, and we get that $\|\cdot\|_C = \|\cdot\|_T$ and that C is the closed unit ball of $\|\cdot\|_T$.

We can use the description of C to show that for all dimensions $n \in \mathbb{N}$ the norm $\|\cdot\|_T$ is given by

$$\|x\|_T = \max\{\, L(|x|) \colon L \in F \,\} \quad (x \in X) \tag{4.8}$$

where F is a finite set of linear functionals $L \colon X \to \mathbb{C}$. Note that this is equivalent to showing that $C = \{\, x \in \mathbb{C}^n \colon L(|x|) \leq 1 \text{ for all } L \in F \,\}$. Since $x \in C$ if and only if $|x| \in C$, what we are really interested in is real convex sets. Following [Grü67, pp. 31–32], we say that K is a convex polytope if $K \subset \mathbb{R}^n$ is a compact convex set with a finite number of extreme points. It was shown in [Grü67] that this is equivalent to K being the intersection of a finite family of halfspaces $\{\, x \in \mathbb{R}^n \colon L(x) \leq c \,\}$. We will apply this to the set

$$C_+ = \{\, |x| \colon x \in C \,\} \subset \mathbb{R}^n.$$

First of all, note that the set B_{n-1} in (4.6) is equal to the embedded unit ball of the contractive norm of the shift on $\mathbb{C}^{n-1}$ with weights $w_1, \ldots, w_{n-1}$. Thus, we will use induction on the dimension n. For $n = 1$, the shift is the zero operator and thus trivially contractive. So $\|\cdot\|_T = \|\cdot\|$ and our claim holds. Assume that the contractive norm for the shift on weighted $n-1$ dimensional space is given by (4.8). Then B_{n-1} and its image TB_{n-1} have only a finite number of extreme points by the induction hypothesis, and so does B. Since C, the unit ball of $\|\cdot\|_T$, is the convex hull of the union of these sets, its extreme points are contained among those of B_{n-1}, TB_{n-1} and B. In particular, the set

$$C_+ = \{\, x \in C \colon x \geq 0 \,\} = \{\, |x| \colon x \in C \,\}$$

of positive elements of C, has a finite number of extreme points. So there is a finite set F' of pairs (L, c) where L is a functional and $c \in \mathbb{R}$ such that

$$C_+ = \bigcap_{(L,c) \in F'} \{\, x \in \mathbb{R}^n \colon L(x) \leq c \,\}.$$

After reducing F' to a minimal set, let

$$F = \{\, c^{-1} L \colon (L, c) \in F', c \neq 0 \,\}.$$

We will show that

$$C_+ = \{\, x \geq 0 \colon L(x) \leq 1 \text{ for all } L \in F \,\}. \tag{4.9}$$

To see that C_+ is included in the set on the right hand side of (4.9), let $x \in C_+$ and $(L, c) \in F'$. Since $0 \in C_+$ we get that $0 = L(0) \leq c$. Thus, if $c > 0$ we have that

$c^{-1}L(x) \leq 1$. Now let x be in the set on the right hand side of (4.9), that is $x \geq 0$ and $L(x) \leq 1$ for all $L \in F$. Let $(L, c) \in F'$. If $c \neq 0$ then $c^{-1}L \in F$ so that $L(x) \leq c$. If $c = 0$, note that for some small $t > 0$ we have $tx \in C_+$. This follows since $x = |x|$ and since C contains an ε-ball around 0. Thus, $tL(x) = L(tx) \leq 0$ and thus $L(x) \leq 0$. So either way, $L(x) \leq c$ and thus $x \in C_+$. Hence (4.9) holds. Since $x \in C$ if and only if $|x| \in C_+$ we find that indeed

$$\|x\|_T = \max\{\, L(|x|) \colon L \in F \,\} \quad (x \in X).$$

Remark 4.1.6. Let T be the shift operator on $\mathbb{C}^n$ and $\|\cdot\|$ some norm on $\mathbb{C}^n$. The geometric description for the unit ball of $\|\cdot\|_T$ is still given by the convex hull of

$$B \cup B_{n-1} \cup TB_{n-1}$$

where B is the closed unit ball of $\|\cdot\|$ and

$$B_{n-1} = \big\{ y \in \mathbb{C}^n \colon y = (y_1, \dots, y_{n-1}, 0), \|y\|_T \leq 1 \big\}.$$

The arguments in Example 4.1.5 are still valid since they only depend on T being a shift operator.

Example 4.1.7 (n-dimensional operator)**.** We can use the formula in Proposition 4.1.3 to calculate the contractive norm for general linear transformations on n-dimensional space. Let X be a finite dimensional normed space and T a linear map on X. Let T' be the dual map. By (4.4), the contractive norm of T is given by

$$\|x\|_T = \sup\left\{ |\varphi(x)| \colon \varphi \in X',\ \sup_{k \geq 0} \|(T')^k \varphi\| = 1 \right\} \quad (x \in X).$$

Choose a basis of X' to write T' in Jordan normal form with Jordan blocks J_{λ_i} for some $\lambda_i \in \mathbb{C}$. The power $(T')^k$ is a block diagonal matrix with entries $J_{\lambda_i}^k$. Let J_λ be one such Jordan block matrix of size $m \geq 1$, and $v_1, \dots, v_m$ be the corresponding vectors for which the block takes the Jordan form. We have

$$(T')^k v_i = \binom{k}{i-1} \lambda^{k-i+1} v_1 + \cdots + \binom{k}{0} \lambda^k v_i \quad (i = 1, \dots, k).$$

Since the v_i are linearly independent, we have for $i = 1, \dots, m$ that $\{\|(T')^k v_i\| \colon k \geq 1\}$ is bounded if and only if the coefficients $\big\{ \binom{k}{j-1} |\lambda^{k-j+1}| \colon k \geq 1 \big\}$ are bounded. This implies that $|\lambda| < 1$ or that $|\lambda| = 1$ and $i = 1$. This describes the subspace of functionals which we have to use to determine $\|\cdot\|_T$.

We now turn to expansive operators. An operator $T \in \mathcal{B}(X)$ is expansive if

$$\|Tx\| \geq \|x\| \quad (x \in X).$$

If T is invertible then T is expansive if and only if T^{-1} is contractive. But in general, an expansive operator is not surjective. We will define the expansive part of an operator in a similar way as we did at the beginning of this section for the contractive part. The first step is to find the largest seminorm on X which is dominated by $\|\cdot\|$ and in which a fixed operator is expansive. For $T \in \mathcal{B}(X)$, this seminorm is given by

$$\|x\|_T^- = \inf\left\{\sum_{i=0}^{n}\|x_i\| : n \geq 0, T^n x = \sum_{i=0}^{n} T^i x_i\right\} \quad (x \in X). \tag{4.10}$$

Again, it is easy to verify that $\|\cdot\|_T^-$ is a seminorm which is dominated by $\|\cdot\|$ and that T is expansive in this seminorm. To see that $\|\cdot\|_T^-$ is the largest such seminorm, we need the following lemma which is similar to [BY01, Proposition 2.2]. Recall that $T \in \mathcal{B}(X)$ is super-expansive if whenever $T^n x = \sum_{i=0}^{n} T^i x_i$ for some $x, x_i \in X$ then

$$\|x\| \leq \sum_{i=0}^{n}\|x_i\|.$$

Lemma 4.1.8. *Let $T \in \mathcal{B}(X)$ and $\|\cdot\|$ be a seminorm on X. Then*

$$\|Tx\| \geq \|x\| \quad (x \in X)$$

if and only if T is super-expansive in $\|\cdot\|$.

Proof. Assume that T is expansive in $\|\cdot\|$ and let $x \in X$. If $T^n x = \sum_{i=0}^{n} T^i x_i$ then

$$\begin{aligned}
\|x_0\| &= \left\|T^n(x - x_n) - \sum_{i=1}^{n-1} T^i x_i\right\| \\
&\geq \left\|T^{n-1}(x - x_n) - \sum_{i=1}^{n-1} T^{i-1} x_i\right\| \\
&\geq \left\|T^{n-1}(x - x_n) - \sum_{i=2}^{n-1} T^{i-1} x_i\right\| - \|x_1\| \\
&\geq \cdots \\
&\geq \|T(x - x_n)\| - \|x_{n-1}\| - \cdots - \|x_1\| \\
&\geq \|x - x_n\| - \|x_{n-1}\| - \cdots - \|x_1\|,
\end{aligned}$$

so we see that

$$\|x\| \leq \|x - x_n\| + \|x_n\| \leq \|x_0\| + \|x_1\| + \cdots + \|x_n\|.$$

Hence T is super-expansive in $\|\cdot\|$.

On the other hand, if T is super-expansive in $\|\cdot\|$ let $x \in X$. Then $T(x) = T(0) + (Tx)$ implies $\|x\| \leq \|Tx\|$ and T is expansive in $\|\cdot\|$. $\square$

Let us use this lemma to show that $\|\cdot\|_T^-$ is the maximal expansive seminorm subject to being dominated by the norm $\|\cdot\|$ on X. If $\|\cdot\|'$ is another such seminorm and if $T^n x = \sum_{i=0}^n T^i x_i$ for some $x, x_i \in X$ then by Lemma 4.1.8 we get

$$\|x\|' \leq \|x_0\|' + \cdots + \|x_n\|' \leq \|x_0\| + \cdots + \|x_n\|.$$

Hence $\|x\|' \leq \|x\|_T^-$ for all $x \in X$.

Next, we construct a triple $(X_T^-, \mathcal{T}, \pi)$ which will serve as the expansive part. We follow the construction of the contractive part. Let

$$N = \{x \in X \colon \|x\|_T^- = 0\}$$

and let X_T^- be the completion of $X \,/\, N$ in the norm $\|\cdot\|_T^-$. Let $\pi \colon X \to X_T^-$ be the map induced by the quotient map. Again, π is contractive, has dense range and π intertwines T and an expansive operator $\mathcal{T} \in \mathcal{B}({X_T}^-)$ with $\|\mathcal{T}\| \leq \|T\|$.

We can also use the triple $(X_T^-, \mathcal{T}, \pi)$ to establish maximality of the seminorm $\|\cdot\|_T^-$. If $\|\cdot\|'$ is another seminorm dominated by $\|\cdot\|$ and such that $\|Tx\|' \geq \|x\|'$ for all $x \in X$, we can repeat the quotient construction for the seminorm $\|\cdot\|'$ and get a triple $(Y, \tilde{S}, \rho)$ consisting of a Banach space Y, an expansive operator $\tilde{S} \in \mathcal{B}(Y)$ and a contractive map $\rho \colon X \to Y$ which intertwines T and S. Moreover, we have $\|\cdot\|' = \|\rho(\cdot)\|$. By Theorem 3.1.1, $\tilde{S}$ has an invertible extension S with contractive inverse S^{-1} on a Banach space containing Y. Assume that $T^n x = \sum_{i=0}^n T^i x_i$ holds on X. Then

$$\|x\|' = \|S^{-n}\rho T^n x\| \leq \sum_{i=0}^{n} \|S^{i-n}\rho x_i\| \leq \sum_{i=0}^{n} \|x_i\|.$$

This shows again that $\|x\|' \leq \|x\|_T^-$ for all $x \in X$.

Definition 4.1.9. The triple $(X_T^-, \mathcal{T}, \pi)$ is called the expansive part of T.

The universal property of the maximal expansive part can be shown in the same way as for the maximal contractive part, so we omit the proof. Here, we say that (Y, S, ρ) is an intertwining triple if S is expansive, $\rho \colon X \to Y$ is contractive and intertwines T and S.

Proposition 4.1.10. *Let $(X_T^-, \mathcal{T}, \pi)$ be the expansive part of an operator $T \in \mathcal{B}(X)$. Let (Y, S, ρ) be an intertwining triple. The there is a unique contractive map $\sigma\colon X_T^- \to Y$ such that $\rho = \sigma\pi$ and $\sigma\mathcal{T} = S\sigma$.*
Moreover, if some intertwining triple (Y, S, ρ) has this universal property, there is a unique isometry $\sigma\colon X_T^- \to Y$ intertwining $\mathcal{T}$ and S.

We now identify the image of the dual map of $\pi\colon X \to {X_T}^-$. Let $T \in \mathcal{B}(X)$ and let

$$W := \{\varphi \in X' \colon \mathfrak{n}(\varphi) < \infty\}$$

where

$$\mathfrak{n}(\varphi) = \inf\Big\{\sup_{n\geq 0}\|\varphi_n\| \colon (\varphi_n)_{n\geq 0} \subset X', \varphi = \varphi_0, \varphi_n = T'\varphi_{n+1} \text{ for all } n \geq 0\Big\}$$

if the set of (φ_n) is non-empty, and $\mathfrak{n}(\varphi) = \infty$ otherwise. Let us show that $W = \operatorname{ran}\pi'$ and that $\|x\| = \mathfrak{n}(\pi x)$ for all $x \in X_T^-$. Let $\xi \in ({X_T}^-)'$. Define ξ_1 on $\mathcal{T}{X_T}^-$ by

$$\xi_1(\mathcal{T}x) := \xi(x) \qquad (x \in X_T^-).$$

Then $|\xi_1(\mathcal{T}x)| \leq \|\xi\|\|x\| \leq \|\xi\|\|\mathcal{T}x\|$ for all $x \in X_T^-$. Extend ξ_1 using the Hahn-Banach theorem to $\xi_1 \in ({X_T}^-)'$, so that $\xi = \mathcal{T}'\xi_1$ and $\|\xi_1\| \leq \|\xi\|$. Iterating the construction of functionals in this way, we get a sequence (ξ_n) with $\xi_n = \mathcal{T}'\xi_{n+1}$ and $\|\xi_n\| \leq \|\xi\|$ for all $n \geq 1$. Hence $\pi'(\xi) \in W$ and $\mathfrak{n}(\pi'\xi) \leq \|\xi\|$.

Now let $\varphi \in W$ with a corresponding sequence (φ_n). We want to find $\xi \in ({X_T}^-)'$ such that $\varphi = \pi'(\xi)$. In other words, we have to find an $M > 0$ such that $|\varphi(x)| \leq M\|x\|_T^-$ for all $x \in X$. The required ξ is then the unique continuous extension of $\xi(\pi x) := \varphi(x)$ $(x \in X)$. Let $x \in X$ and assume that $T^n x = \sum_{i=0}^n T^i x_i$. Then

$$|\varphi(x)| = |\varphi_n(T^n x)| \leq \sum_{i=0}^n \|\varphi_{n-i}\|\|x_i\| \leq \big(\sup_{k\geq 0}\|\varphi_k\|\big)\sum_{i=0}^n \|x_i\|$$

which shows that $|\varphi(x)| \leq \mathfrak{n}(\varphi)\|x\|_T^-$ for all $x \in X$. Hence the functional ξ defined by $\xi(\pi x) = \varphi(x)$ $(x \in X)$ lies in $(X_T^-)'$, $\varphi = \pi'\xi$ and $\|\xi\| \leq \mathfrak{n}(\varphi)$. In summary, we have $W = \operatorname{ran}\pi'$ and $\|\xi\| = \mathfrak{n}(\pi'\xi)$ for all $\xi \in ({X_T}^-)'$. Consequently, we have the following proposition.

Proposition 4.1.11. *Let $T \in \mathcal{B}(X)$. The seminorm $\|\cdot\|_T^-$ is given by*

$$\|x\|_T^- = \sup\{|\varphi(x)| \colon \varphi \in X', \mathfrak{n}(\varphi) = 1\} \quad (x \in X).$$

4.2 The dissipative part of an operator

In Section 4.1 we defined the contractive part and the expansive part of an operator. We will proceed here in a similar manner, to introduce the dissipative part of an operator. To start with, we again consider seminorms. Let A be a closed operator with domain $D(A) \subset X$. Recall that A is dissipative if

$$\|(\lambda - A)x\| \geq \lambda \|x\|$$

for all $x \in D(A)$ and $\lambda > 0$. Assume that $\|\cdot\|'$ is a seminorm on X which is dominated by $\|\cdot\|$ and such that

$$\|(\lambda - A)x\|' \geq \lambda \|x\|'$$

for all $x \in D(A)$ and $\lambda > 0$. We say that A is $\|\cdot\|'$-*dissipative*. Let

$$N = \{\, x \in X \colon \|x\|' = 0 \,\}$$

be the null space of the seminorm $\|\cdot\|'$ and let Y be the $\|\cdot\|'$-completion of the quotient space $X \,/\, N$. If we assume that the operator

$$B(x + N) := Ax + N, \quad D(B) = \{x + N \colon x \in D(A)\}$$

is well-defined and closable on Y then the contractive quotient map $\rho \colon X \to Y$ has dense range and intertwines A and B. Moreover, B is dissipative on Y. It is an open problem, for which operators A the operator B is well-defined.

If we are instead given a contractive map $\rho \colon X \to Y$ and a dissipative operator B on Y, we can define a seminorm $\|x\|' = \|\rho(x)\|$ on X. If ρ intertwines A and B we get that A is $\|\cdot\|'$-dissipative. Among these spaces Y we want to find the largest one in an appropriate sense, which we expect to correspond to the maximal dissipative seminorm on X. Again, it is easy to see that there is a maximal seminorm $|||\cdot|||$. Define $|||\cdot|||$ by

$$|||x||| := \sup \left\{ \|x\|' \right\} \quad (x \in X)$$

where the supremum is taken over all seminorms $\|\cdot\|'$ such that

1. $\|x\|' \leq \|x\|$ for all $x \in X$,

2. $\|(\lambda - A)x\|' \geq \lambda \|x\|'$ for all $x \in D(A)$ and $\lambda > 0$.

This clearly defines a seminorm and it is indeed maximal with respect to these properties. We can rephrase dissipativity of A by saying that the operators $\lambda^{-1}(\lambda - A)$ are expansive for all $\lambda > 0$. For $\mu > 0$, let $\|\cdot\|_\mu$ be the maximal seminorm which is dominated by $\|\cdot\|$ and for which $\mu^{-1}(\mu - A)$ is expansive. Then

$$\begin{aligned}\|\lambda^{-1}(\lambda - A)^{-1}x\|_\mu &= \|\lambda^{-1}(\lambda - \mu + \mu - A)x\|_\mu \\ &\geq \frac{\mu}{\lambda}\|x\|_\mu - \frac{|\lambda - \mu|}{\lambda}\|x\|_\mu = \|x\|_\mu\end{aligned} \tag{4.11}$$

for $0 < \lambda \leq \mu$ and $x \in D(A)$. So the limit

$$\lim_{\mu\to\infty} \|x\|_\mu \quad (x \in X) \tag{4.12}$$

defines a seminorm in which A is dissipative. Moreover, (4.12) is the maximal seminorm in which A is dissipative.

Assume that A generates a C_0-semigroup T. It is not clear whether the C_0-semigroup is bounded in the maximal norm $|||\cdot|||$. If we had

$$|||T(t)x||| \leq M_t|||x||| \quad (x \in X)$$

for some $t > 0, M_t > 0$, then the null space N for the seminorm $|||\cdot|||$ would be invariant under $T(t)$. However, it is not clear whether this always holds.

In Section 4.2.1, we define a seminorm $|||\cdot|||_A$ for an operator A with non-empty resolvent set. The seminorm is dominated by the original norm $\|\cdot\|$ of X, and A is dissipative for $|||\cdot|||_A$. In addition, any operator U which commutes with the resolvent of A satisfies

$$|||Ux|||_A \leq \|U\| |||x|||_A \quad (x \in X),$$

where $\|U\|$ is the norm of U in $\mathcal{B}(X)$. This will be shown in Theorem 4.2.6. We will verify additional properties of $|||\cdot|||_A$ in Section 4.2.2, which include certain maximality results (Proposition 4.2.14 and 4.2.16) in the case when A is the generator of a semigroup.

4.2.1 A dissipative seminorm

Let A be an operator on a Banach space X with non-empty resolvent set $\rho(A)$, and let $x, x_i \in X$, $n \geq 0$ and $\mu \in \mathbb{C}$. We say that $P(x, x_i, \mu, n)$ holds if there is $\alpha \in \rho(A)$ such that

$$\mu^{-n}(\mu - A)^n(\alpha - A)^{-n}x = \sum_{i=0}^{n} \mu^{-i}(\mu - A)^i(\alpha - A)^{-n}x_i. \tag{4.13}$$

It is clear that $P(x, x_i, \mu, n)$ holds for all $\mu \in \mathbb{C}, n \geq 0$, when $x_n = x$ and $x_i = 0$ for $i < n$.

Lemma 4.2.1. *Equation* (4.13) *is independent of* $\alpha \in \rho(A)$.

Proof. Given $\alpha, \beta \in \rho(A)$, apply the bounded operator $(\alpha - A)^n(\beta - A)^{-n}$ to (4.13). □

Definition 4.2.2. Let A be an operator on a Banach space X with non-empty resolvent set $\rho(A)$.

1. For $\mu > 0$, define $|||\cdot|||_{A,\mu} \colon X \to \mathbb{R}$ by

$$|||x|||_{A,\mu} = \inf\left\{ \sum_{i=0}^{n} \|x_i\| \colon n \geq 0, x_i \in X \text{ and } P(x, x_i, \mu, n) \text{ holds} \right\} \quad (x \in X).$$

2. Define a map $|||\cdot|||_A \colon X \to \mathbb{R}$ by

$$|||x|||_A = \inf\left\{ |||x|||_{A,\mu} \colon \mu > 0 \right\} \quad (x \in X).$$

We will first present some properties of the maps $|||\cdot|||_{A,\mu}$ and use these to establish some properties of $|||\cdot|||_A$.

Lemma 4.2.3. *Let* $\mu \in \mathbb{C}$. *The map* $|||\cdot|||_{A,\mu}$ *is a seminorm and it is dominated by* $\|\cdot\|$.

Proof. To show that $|||\cdot|||_A$ is dominated by $\|\cdot\|$, let $x \in X$. Take $n \in \mathbb{N}$, $x_n = x$ and $x_i = 0$ for $i < n$ where $\alpha \in \rho(A)$. Then $P(x, x_i, \mu, n)$ holds and $|||x|||_{A,\mu} \leq \|x\|$.

To show that $|||\cdot|||_A$ is a seminorm, we have to verify the triangle inequality. Let $x, y \in X$ and $x_i, y_i \in X$ such that $P(x, x_i, \mu, n)$ and $P(y, y_i, \mu, m)$ hold for some $m, n \in \mathbb{N}$. Multiplying (4.13) by $\mu^{-k}(\mu - A)^k(\alpha - A)^{-k}$ for $k = m$ gives that $P(x, x'_i, \mu, n + m)$ holds for

$$x'_i = \begin{cases} x_{i-m} & i \geq m \\ 0 & \text{otherwise} \end{cases}$$

and similarly $P(y, y'_i, \mu, n + m)$ holds for

$$y'_i = \begin{cases} y_{i-n} & i \geq n \\ 0 & \text{otherwise.} \end{cases}$$

Then simply adding the resulting equations shows that $P(x + y, x'_i + y'_i, \mu, n + m)$ holds. Now,

$$\sum_{i=0}^{n+m} \|x'_i + y'_i\| \leq \sum_{i=0}^{n} \|x_i\| + \sum_{i=0}^{m} \|y_i\|$$

and thus $|||x+y|||_{A,\mu} \leq |||x|||_{A,\mu} + |||y|||_{A,\mu}$. To see that $|||\cdot|||_{A,\mu}$ is absolutely homogeneous, multiply both sides in (4.13) by some $\lambda \in \mathbb{C}$. Then $P(\lambda x, \lambda x_i, \mu, n)$ holds and thus $|||\lambda x|||_{A,\mu} = |\lambda| |||x|||_{A,\mu}$. □

Lemma 4.2.4. *Let $0 < \lambda \leq \mu$. Then*

$$|||\lambda^{-1}(\lambda - A)x|||_{A,\mu} \geq |||x|||_{A,\mu} \tag{4.14}$$

for all $x \in D(A)$.

Proof. Consider the case when $\lambda = \mu$ first. If $P(\mu^{-1}(\mu - A)x, x_i, \mu, n)$ holds for some $x \in D(A)$, multiply the corresponding equation of the form (4.13) by $(\alpha - A)^{-1}$. Then $P(x, x_i, \mu, n+1)$ holds with $x_{n+1} = 0$. So by definition of $|||\cdot|||_{A,\mu}$, (4.14) holds for $\lambda = \mu$. Now, let $0 < \lambda < \mu$. Then by the reverse triangle inequality we have for $x \in D(A)$ as in (4.11) that

$$|||\lambda^{-1}(\lambda - A)x|||_{A,\mu} = |||x|||_{A,\mu}.$$

□

Lemma 4.2.5. *Let $0 < \lambda < \mu$. Then*

$$|||x|||_{A,\mu} \leq |||x|||_{A,\lambda}$$

for all $x \in X$.

Proof. Let $x \in X$ and assume that $P(x, x_i, \lambda, n)$ holds. We will show that

$$|||x|||_{A,\mu} \leq \sum_{i=0}^{n} \|x_i\| \tag{4.15}$$

using induction on $n \geq 0$. Note that for the base case $n = 0$. So (4.13) reduces to $x = x_0$ and Lemma 4.2.3 gives (4.15).

For the inductive step let $k \in \mathbb{N}$ and assume that (4.15) holds for $n = k$ whenever $P(x, x_i, \lambda, k)$ holds. Let x, x_i be such that $P(x, x_i, \lambda, k+1)$ holds. First, note that $x - x_{k+1} \in D(A)$. In fact, rearranging (4.13) and using that $(\lambda - A)(\alpha - A)^{-1} = (\lambda - \alpha)(\alpha - A)^{-1} + I$ commutes with $(\alpha - A)^{-1}$ shows that

$$\lambda^{-(k+1)}(\lambda - A)^{(k+1)}(\alpha - A)^{-(k+1)}(x - x_{k+1}) = (\alpha - A)^{-1}y \tag{4.16}$$

where $y = \sum_{i=0}^{k} \lambda^{-i}(\lambda - A)^i(\alpha - A)^{-k}x_i$. This shows that

$$\left((\lambda - \alpha)(\alpha - A)^{-1} + 1\right)^{k+1}(x - x_{k+1}) \in D(A).$$

Now, if $((\lambda - \alpha)(\alpha - A)^{-1} + 1)u = (\alpha - A)^{-1}v$ for some $u, v \in X$ then

$$u = (\alpha - A)^{-1}\left(v - (\lambda - \alpha)u\right) \in D(A).$$

This shows recursively that $((\lambda - A)(\alpha - A))^i(x - x_{k+1}) \in D(A)$ for $i = k, k-1, \ldots, 0$. Thus (4.16) becomes

$$\lambda^{-k}(\lambda - A)^k(\alpha - A)^{-k}(\lambda^{-1}(\lambda - A))(x - x_{k+1}) = y.$$

This shows that $P(\lambda^{-1}(\lambda - A)(x - x_{k+1}), x_i, \lambda, k)$ holds. By the induction hypothesis we get that

$$|||\lambda^{-1}(\lambda - A)(x - x_{k+1})|||_{A,\mu} \leq \sum_{i=0}^{k} \|x_i\|.$$

Combining this with the estimate (4.14) from Lemma 4.2.4 and the fact that $|||\cdot|||_{A,\mu}$ is dominated by $\|\cdot\|$ shows that

$$|||x|||_{A,\mu} \leq |||x - x_{k+1}|||_{A,\mu} + |||x_{k+1}|||_{A,\mu} \leq \sum_{i=0}^{k+1} \|x_i\|,$$

completing the proof. □

Theorem 4.2.6. *Let A be an operator on a Banach space X with non-empty resolvent set $\rho(A)$. Then $|||\cdot|||_A$ is a seminorm dominated by $\|\cdot\|$ such that*

$$|||(\lambda - A)x|||_A \geq \lambda\, |||x|||_A$$

for all $\lambda > 0$ and $x \in D(A)$. Furthermore, any bounded operator U that commutes with the resolvent of A satisfies $|||Ux|||_A \leq \|U\|\, |||x|||_A$ for all $x \in X$.

Proof. It is clear that $|||x|||_A \leq \|x\|$ for all $x \in X$. By Lemma 4.2.5 we can write $|||x|||_A = \lim_{\mu\to\infty} |||x|||_{A,\mu}$. As the limit of seminorms, $|||\cdot|||_A$ is itself a seminorm. Let $\lambda > 0$ and take limits in (4.14) as $\mu \to \infty, \mu > \lambda$. This shows that A is dissipative for $|||\cdot|||_A$.

Let U be an operator which commutes with the resolvent of A, and assume that $P(x, x_i, \mu, n)$ holds. Multiply (4.13) by U to see that $P(Ux, Ux_i, \mu, n)$ holds. Thus, $|||Ux|||_A \leq \|U\| \sum_{i=0}^{n} \|x_i\|$ and we see that indeed $|||Ux|||_A \leq \|U\| |||x|||_A$ for all $x \in X$. □

Example 4.2.7. Let (Ω, μ) be a measure space and let $h\colon \Omega \to \mathbb{C}$ be a measurable function. Let $A = M_h$ be the multiplication operator on $L_p(\Omega, \mu)$ for some $1 \leq p < \infty$. The spectrum of A is the closure of the essential range of h and $D(A)$ is dense if (Ω, μ) is σ-finite. Assume that $\rho(A)$ is non-empty. Then

$$|||f|||_A = \|f \cdot \chi_D\|_p \tag{4.17}$$

for $f \in L_p(\Omega, \mu)$ where $D = \{\, x \in \Omega\colon\ \operatorname{Re} h(x) \le 0 \,\}$. To show this, let

$$N(\lambda, q) = \{\, x \in \Omega\colon\ |h(x) - \lambda| < q\lambda \,\}$$

for $0 < q < 1$ and $\lambda > 0$. Fix $\lambda > 0$ and $0 < q < 1$ and let $f \in L_p(\Omega, \mu)$ vanish outside $N(\lambda, q)$. Then $\lambda^{-n}(\lambda - h)^n f \in L_p(\Omega, \mu)$ which implies $f \in D(A^n)$ for all $n \in \mathbb{N}$. Pick some $\alpha \in \rho(A)$ and write

$$\lambda^{-n}(\lambda - A)^n(\alpha - A)^{-n} f = (\alpha - A)^{-n}\lambda^{-n}(\lambda - A)^n f.$$

By definition of $|||\cdot|||_A$ we get

$$\begin{aligned} |||f|||_A &\le \|\lambda^{-n}(\lambda - A)^n f\| \\ &\le \|\lambda^{-n}(\lambda - h)^n \chi_{N(\lambda,q)}\|_\infty \|f\| \\ &\le q^n \|f\| \end{aligned}$$

Since this holds for all $n \in \mathbb{N}$ we get that $|||f|||_A = 0$ if f vanishes outside $N(\lambda, q)$.

Now consider arbitrary $f \in L_p(\Omega, \mu)$. Then $|||f\chi_{N(\lambda,q)}|||_A = 0$ by the above. If $x \in \Omega \setminus D$ then $\operatorname{Re} h(x) > 0$ and $h(x)$ is contained in circles $\{z \in \mathbb{C}\colon |z - \lambda| < \lambda\}$ for all large $\lambda > 0$. Also, $h(x)$ is contained in $\{z \in \mathbb{C}\colon |z - \lambda| < \lambda - \frac{1}{\lambda}\}$ for possibly larger $\lambda > 0$. So $x \in N(\lambda, \lambda - \lambda^{-1})$ for some $\lambda > 1$ and the complement $\Omega \setminus D$ is equal to the union $\bigcup_{\lambda > 1} N(\lambda, \lambda - \lambda^{-1})$. So

$$|||f\chi_{\Omega\setminus D}|||_A = \lim_{\lambda\to\infty} |||f\chi_{N(\lambda,\lambda-\lambda^{-1})}|||_A = 0$$

by the dominated convergence theorem.

For $f \in L_p(\Omega, \mu)$, assume that $P(f\chi_D, g_i, \lambda, n)$ holds with $\lambda > 0$. Let $f_i = g_i\chi_D$. Then $P(f\chi_D, f_i, \lambda, n)$ holds and $\|f_i\| \le \|g_i\|$ for $i = 0, 1, \ldots, n$. The function $\lambda^{-1}(\lambda - h)$ has the inverse $\lambda(\lambda - h)^{-1}$ on the set $D \subset \Omega$, and $|\lambda(\lambda - h)^{-1}| \le 1$ on D. Equation (4.13) shows that

$$\|f\chi_D\| \le \sum_{i=0}^{n} \|\lambda^{n-i}(\lambda - h)^{i-n} f_i\| \le \sum_{i=0}^{n} \|f_i\|$$

which implies that $|||f\chi_D|||_A = \|f\chi_D\|$. Finally,

$$|||f\chi_D|||_A - |||f\chi_{\Omega\setminus D}|||_A \le |||f|||_A \le |||f\chi_D|||_A + |||f\chi_{\Omega\setminus D}|||_A$$

holds for $f \in L_p(\Omega, \mu)$ which proves (4.17).

4.2.2 The maximal dissipative part

Let A be an operator on a Banach space X with non-empty resolvent set. We saw in Theorem 4.2.6 that the map $|||\cdot|||_A$ from Definition 4.2.2 is a seminorm with respect to which A is dissipative. The beginning of Section 4.2 explained how a seminorm can be used to get a new Banach space. Here, we will consider the new Banach space obtained from the seminorm $|||\cdot|||_A$ and show that the quotient map intertwines A with a dissipative operator. Let

$$N := \{ x \in X \colon |||x|||_A = 0 \}.$$

Define

$$X_A := \overline{X \,/\, N}$$

as the completion of the quotient space $X \,/\, N$ in the norm induced by $|||\cdot|||_A$. We denote the norm on X_A also by $|||\cdot|||_A$. On the quotient space $X \,/\, N$, define

$$R_\lambda(x + N) := R(\lambda, A)x + N \quad (x \in X, \lambda \in \rho(A)). \tag{4.18}$$

These operators are well-defined and bounded in $|||\cdot|||_A$ by Theorem 4.2.6. So we can extend them continuously to the completion X_A. We want the family

$$\{ R_\lambda \colon \lambda \in \rho(A) \}$$

to be the resolvent of an operator. It is certainly a pseudo-resolvent. We need to show that R_λ is injective (see Section 2.1). Motivated by a property of sectorial operators, we restrict our attention to *quasi sectorial* operators which we define as follows.

Definition 4.2.8. An operator A on a Banach space X is quasi sectorial if A is densely defined and there exist $M > 0$, $\delta \in \mathbb{T}$ and $a > 0$ such that $\delta(a, \infty) \subset \rho(A)$ and

$$\|\lambda R(\lambda, A)\| \leq M$$

for all $\lambda \in \delta(a, \infty)$.

Remark 4.2.9. Clearly, any sectorial operator A is quasi sectorial, as can be seen by choosing $\delta = 1$ and $a = 0$. Assume B is quasi sectorial for some numbers $\delta \in \mathbb{T}$ and $a > 0$. Then $A = a - \delta^{-1}B$ is sectorial.

If A is quasi sectorial for some values δ, a, M as in Definition 4.2.8, we can approximate the identity operator using the resolvent of A, as follows. For $x \in D(A)$ and $\lambda \in \rho(A)$ we have

$$\|\lambda R(\lambda, A)x - x\| \leq \|R(\lambda, A)x\| \|Ax\| \leq \frac{M}{|\lambda|} \|Ax\|$$

which tends to 0 as $\lambda \to \infty$ for $\lambda \in \delta(a, \infty)$. Since $D(A)$ is dense in X and the operators $\lambda R(\lambda, A)$ are uniformly bounded, this proves strong convergence $\lambda R(\lambda, A)x \to x$ as $\lambda \to \infty$ for $x \in X$. As an example for such operators take δA where A generates a C_0-semigroup and $\delta \in \mathbb{C}$.

What is most important for us is that A turns into a dissipative operator on X_A, as shown in the next Proposition.

Proposition 4.2.10. *Let A be a quasi sectorial operator on a Banach space X and let $\pi\colon X \to X_A$ be the natural contractive map. There is a dissipative operator $\mathcal{A}$ on X_A such that $\pi A \subset \mathcal{A}\pi$. Moreover, there is a unital contractive algebra homomorphism*

$$\varphi\colon \{R(\lambda, A)\colon \lambda \in \rho(A)\}' \to \mathcal{B}(X_A)$$

such that $\varphi(U)\pi = \pi U$ for all U commuting with the resolvent of A. Moreover, each $\varphi(U)$ is uniquely determined by this equation and we have that $\varphi(R(\lambda, A))$ is the resolvent of $\mathcal{A}$.

Proof. For $\lambda \in \rho(A)$, let $R_\lambda \in \mathcal{B}(X_A)$ be the operator defined in (4.18). We show first that $R_\lambda\colon \rho(A) \to \mathcal{B}(X_A)$ is the resolvent of an operator. We have already seen that it is a pseudo-resolvent. So it is sufficient to show that $\ker R_\lambda = \{0\}$ for some $\lambda \in \rho(A)$. Since A is quasi sectorial, there are $\delta \in \mathbb{T}, a > 0$ such that

$$|||\lambda R_\lambda \pi(x) - \pi(x)|||_A \le \|\lambda R(\lambda, A)x - x\| \to 0 \qquad (x \in X)$$

as $\lambda \to \infty, \lambda \in \delta(a, \infty)$. The operators R_λ are uniformly bounded in $|||\cdot|||_A$ by quasi-sectoriality of A and by Theorem 4.2.6. Uniform boundedness of $\{\lambda R_\lambda\colon \lambda \in \delta(a, \infty)\}$ and density of $\pi(X)$ in X_A imply that

$$|||\lambda R_\lambda y - y|||_A \to 0 \qquad (y \in X_A). \tag{4.19}$$

Now let $y \in \ker R_\lambda$ for some $\lambda \in \rho(A)$. Since the kernel of R_λ is independent of λ (see Section 2.1) we get from (4.19) that $y = 0$. So $R_\lambda = R(\lambda, \mathcal{A})$ for some operator $\mathcal{A}$ on X_A for all $\lambda \in \rho(A)$ and $\mathcal{A}$ has dense domain $D(\mathcal{A}) = \operatorname{ran} R_\lambda$. To show that $\pi A \subset \mathcal{A}\pi$, let $x \in D(A)$, $\lambda \in \rho(A)$ and $y = (\lambda - A)x$. Then

$$\pi(x) = x + N = R(\lambda, A)y + N = R_\lambda(y + N) \in D(\mathcal{A})$$

and $\mathcal{A}\pi(x) = \pi(Ax)$. To see that $\mathcal{A}$ is dissipative, notice that

$$|||(\lambda - \mathcal{A})\pi(x)|||_A = |||(\lambda - A)x|||_A \ge \lambda |||\pi(x)|||_A \quad (\lambda > 0) \tag{4.20}$$

for $x \in D(A)$. We can use that $\pi(D(A))$ is a core of $\mathcal{A}$ to get (4.20) for all $x \in D(\mathcal{A})$. Indeed, for $x = R_\lambda y \in D(\mathcal{A}) = \operatorname{ran} R_\lambda$, use density of $\pi(X)$ in X_A to find $y_n \in X$ with $|||\pi(y_n) - y|||_A \to 0$ as $n \to \infty$. Then $x_n := \pi(R(\lambda, A)y_n) \in \pi D(A)$ and

$$|||x_n - x|||_A \to 0 \text{ and } |||\mathcal{A}x_n - \mathcal{A}x|||_A \to 0$$

as $n \to \infty$. Hence $\mathcal{A}$ is dissipative.

Assume that U commutes with the resolvent of A. By Theorem 4.2.6, $|||Ux|||_A \leq \|U\| |||x|||_A$ for all $x \in X$. So the subspace N is invariant under U and the operator $\varphi(U)(x+N) = Ux+N$ is well-defined on $X\,/\,N$ with norm $\|\varphi(U)\| \leq \|U\|$. It extends continuously to X_A.

It is clear that $\varphi(U)$ is unique with $\varphi(U)\pi = \pi U$ since $\pi(X)$ is dense in X_A. Uniqueness shows that $\varphi(U_1)\varphi(U_2)\pi = \pi U_1 U_2 = \varphi(U_1 U_2)\pi$ if $U_1, U_2 \in \mathcal{B}(X)$ commute with the resolvent of A. Moreover, it is clear that $\varphi(R(\lambda, A)) = R(\lambda, \mathcal{A})$ for $\lambda \in \rho(A)$ by construction of $\mathcal{A}$. □

Definition 4.2.11. Let A be quasi sectorial. Define the *dissipative part* of A as the triple $(X_A, \mathcal{A}, \pi)$.

Proposition 4.2.12. *Let A be a quasi sectorial operator such that $\omega \in \rho(A)$ for some $\omega > 0$. Then $\mathcal{A}$ generates a C_0-semigroup T_A of contractions on X_A.*

Proof. By definition of $\mathcal{A}$, we have that $\omega \in \rho(A) \subset \rho(\mathcal{A})$. Moreover, $\mathcal{A}$ is densely defined and dissipative on X_A by Proposition 4.2.10. By the Lumer-Phillips theorem (Theorem 2.3.3), $\mathcal{A}$ generates a C_0-semigroup of contractions. □

Proposition 4.2.13. *Let $\delta \in \mathbb{T}$ and A be an operator such that δA generates a C_0-semigroup T. Let φ be the homomorphism from Proposition 4.2.10. Then $\delta\mathcal{A}$ generates the C_0-semigroup S defined by*

$$S(t) := \varphi(T(t)) \quad (t \geq 0).$$

Proof. Since δA generates a C_0-semigroup, A is quasi sectorial and Proposition 4.2.10 applies. Since $T(t)$ commutes with the resolvent of A, $S(t)$ is defined. To see that S is strongly continuous, let $x \in X$. We have

$$|||S(t)\pi(x) - S(s)\pi(x)|||_A \leq \|T(t)x - T(s)x\| \quad (s, t \geq 0)$$

which gives strong continuity of S on $X\,/\,N$, a dense subset of X_A. The operators satisfy $\|S(t)\| \leq \|T(t)\|$ $(t \geq 0)$. So S is strongly continuous on X_A and it is easy to see that $\delta\mathcal{A}$ is the generator of S, for example by using the Laplace transform. □

Proposition 4.2.14. *Let A be a quasi sectorial operator on a Banach space X and let $M \geq 1$. Assume there is a bounded linear map $\rho\colon X \to Y$ into a Banach space Y and a C_0-semigroup S on Y with $\|S(t)\| \leq M$ for all $y \in Y, t \geq 0$ with a generator B such that there is $\alpha \in \rho(A) \cap \rho(B)$ with $\rho R(\alpha, A) = R(\alpha, B)\rho$.*
Then $\|\rho(x)\| \leq M\|\rho\| |||x|||_A$ for all $x \in X$.

Proof. Let $x \in X, \mu > 0$, $n \geq 0$ and $x_i \in X$ such that $P(x, x_i, \mu, n)$ holds. Let $\alpha \in \rho(A) \cap \rho(B)$. Since ρ intertwines $R(\alpha, A)$ and $R(\alpha, B)$, applying ρ to (4.13) yields

$$\mu^{-n}(\mu - B)^n(\alpha - B)^{-n}\rho(x) = \sum_{i=0}^{n} \mu^{-i}(\mu - B)^i(\alpha - B)^{-n}\rho(x_i).$$

Since B is the generator of the bounded semigroup S on Y, the operators $\mu^k(\mu - B)^{-k}$ exist and satisfy $\|\mu^k(\mu - B)^{-k}\| \leq M$ for $\mu > 0$ and $k \geq 1$ by the Feller-Miyadera-Phillips theorem. We thus get

$$\begin{aligned}
\|\rho(\alpha - A)^{-n}x\| &= \|(\alpha - B)^{-n}\rho(x)\| \\
&\leq \sum_{i=0}^{n} \|\mu^{n-i}(\mu - B)^{-n+i}(\alpha - B)^{-n}\rho(x_i)\| \\
&\leq M \sum_{i=0}^{n} \|(\alpha - B)^{-n}\rho(x_i)\| \\
&\leq M\|\rho\| \sum_{i=0}^{n} \|(\alpha - A)^{-n}x_i\|.
\end{aligned}$$

We can replace x and x_i by the elements $(\alpha - A)^n(\beta - A)^{-n}x$ and $(\alpha - A)^n(\beta - A)^{-n}x_i$ for $\beta \in \rho(A)$. After multiplying by β^n, this shows that

$$\|\rho(\beta^n(\beta - A)^{-n}x)\| \leq M\|\rho\| \sum_{i=0}^{n} \|\beta^n(\beta - A)^{-n}x_i\|$$

for all $\beta \in \rho(A)$. Since A is quasi sectorial, we can take a limit as $\beta \to \infty$ in some direction to obtain

$$\|\rho(x)\| \leq M\|\rho\| \sum_{i=0}^{n} \|x_i\|.$$

Thus, we have that $\|\rho(x)\| \leq M\|\rho\| \, |||x|||_A$ for all $x \in X$. ☐

Remark 4.2.15. In Proposition 4.2.14 we considered triples (Y, S, ρ) where Y is a Banach space and $\rho\colon X \to Y$ a continuous map which intertwines T and a C_0-semigroup S on Y. We can restrict the semigroup S to the closed invariant subspace $\overline{\rho(X)}$ of Y. This subspace is isometrically isomorphic to the completion of the quotient space X / N' where $N' = \{x \in X : \|\rho(x)\| = 0\}$ with the induced norm $\|\rho(\cdot)\|$. The semigroup T induces a C_0-semigroup S on the quotient. This shows that we effectively look at a special class of seminorms in Proposition 4.2.14. We look at those seminorms $\|\cdot\|'$ on X such that

1. There exists $K > 0$ such that $\|x\|' \leq K\|x\|$ for all $x \in X$,
2. There exists $M > 0$ such that $\|T(t)\|' \leq M\|x\|'$ for all $t \geq 0$ and $x \in X$.

We can use Proposition 4.2.14 to find a universal property of the triple (X_A, T_A, π) which we will state in Proposition 4.2.16.

Let A be the generator of a C_0-semigroup T. We have a triple (X_A, T_A, π) where $\pi : X \to X_A$ is the quotient map with dense image (see Proposition 4.2.12). We can compare it with other *intertwining triples* (Y, S, ρ) where Y is a Banach space, $\rho : X \to Y$ is a continuous linear map, S is a C_0-semigroup of contractions on Y and $\rho T(t) = S(t)\rho$ for all $t \geq 0$. So (X_A, T_A, π) is an intertwining triple which moreover is contractive ($\|\pi\| \leq 1$) and minimal ($X_A = \overline{\pi(X)}$). Given any intertwining triple (Y, S, ρ), one can obtain a contractive minimal intertwining triple by replacing ρ by $\|\rho\|^{-1}\rho$ and Y by $\overline{\rho(X)}$.

Proposition 4.2.16. *Let A be the generator of a C_0-semigroup T on a Banach space X. The triple (T_A, X_A, π) has the following universal property.*

If (Y, S, ρ) is an intertwining triple for T on X where S is a C_0-semigroup of contractions, then there is a unique bounded linear map $\sigma : X_A \to Y$ such that $\sigma\pi = \rho$. Moreover, $S(t)\sigma = \sigma T_A(t)$ for all $t \geq 0$ and $\|\sigma\| = \|\rho\|$.

Proof. Define $\sigma : X / N \to Y$ by $\sigma(\pi x) = \rho(x)$ for $x \in X$. We have to show that σ is well-defined and bounded with respect to the norm $|||\cdot|||_A$ on X / N. By Proposition 4.2.14, the kernel N of π is contained in $\ker \rho$. Thus, σ is well-defined. Moreover,

$$\|\sigma(\pi x)\| = \|\rho x\| \leq \|\rho\| |||x|||_A = \|\rho\| \|\pi x\| \quad (x \in X)$$

so that $\|\sigma\| \leq \|\rho\|$. We can extend σ to the closure of X / N which is X_A. Uniqueness of σ is clear.

Moreover, we see $\|\rho\| = \|\sigma\pi\| \leq \|\sigma\|\|\pi\| = \|\sigma\|$ by contractivity of π, giving $\|\sigma\| = \|\rho\|$. Also,

$$\sigma T_A(t)\pi = \sigma\pi T(t) = \rho T(t) = S(t)\rho = S(t)\sigma\pi \qquad (t \geq 0)$$

which implies $\sigma T_A(t) = S(t)\sigma$ for all $t \geq 0$ by density of $\operatorname{ran} \pi$ in X_A. □

The universal property in Proposition 4.2.16 implies uniqueness of (X_A, T_A, π) in the following sense.

Proposition 4.2.17. *Let (Y, S, ρ) be a contractive intertwining triple with the universal property of Proposition 4.2.16. Then there is an isometric embedding $\sigma\colon X_A \to Y$ such that $\rho = \sigma\pi$ and $S(t)\sigma = \sigma T_A(t)$ for $t \geq 0$. If (Y, S, ρ) is minimal, then σ is surjective.*

Proof. By the universal property from Proposition 4.2.16, there are contractive maps $\sigma\colon X_A \to Y$ and $\tau\colon Y \to X_A$ such that $\rho = \sigma\pi$ and $\pi = \tau\rho$. This implies that $\rho = (\sigma\tau)\rho$ and $\pi = (\tau\sigma)\pi$. By density of $\operatorname{ran}\pi$ in X_A, we have that $\tau\sigma = I$.

To see that the embedding $\sigma\colon X_A \to Y$ is isometric, it suffices to show that $\|\sigma(\pi x)\| = \|\pi x\|$ for all $x \in X$. This follows from the estimates

$$\|\sigma(\pi x)\| \leq \|\sigma\|\|\pi x\| \quad (x \in X)$$

and

$$\|\pi(x)\| = \|\tau\sigma(\pi x)\| \leq \|\tau\|\|\sigma(\pi x)\| \quad (x \in X).$$

Finally, Y is minimal if and only if it is equal to the closed subspace σX_A. If this is the case, the isometry σ is surjective. □

Remark 4.2.18. From the proof of Proposition 4.2.17, it is easy to see that a triple (Y, S, ρ) with the universal property is minimal if and only if the map $\sigma\colon X_A \to Y$ is an isometric isomorphism.

Let A be an operator with non-empty resolvent set. Let $|||\cdot|||$ be the maximal dissipative seminorm for A, which was defined as a supremum over seminorms at the beginning of Section 4.2. It is not clear whether the resolvent operators of A are bounded in $|||\cdot|||$. For the seminorm $|||\cdot|||_A$, we know that this is the case by Theorem 4.2.6. To see how the seminorms compare, note that certainly $|||\cdot|||_A \leq |||\cdot|||$. Let N be the null space of $|||\cdot|||$ and N_A be the null space of $|||\cdot|||_A$. Then $N \subset N_A$. There are two ways in which $|||\cdot|||$ and $|||\cdot|||_A$ could possibly differ. We could have $N \neq N_A$, and we could have that the seminorms induce different norms on $X \,/\, N_A$.

Open Question 4.2.19. Assume that A generates a C_0-semigroup, and $|||\cdot|||$ is a seminorm which is dominated by $\|\cdot\|$ and such that $|||(\lambda - A)x||| \geq \lambda|||x|||$ for all $x \in D(A), \lambda > 0$. Do we have that $|||x||| \leq |||x|||_A$ for all $x \in X$?

Example 4.2.20. Recall from Example 4.2.7 that the seminorm $|||\cdot|||_A$ for a multiplication operator $A = M_h$ on the space $L_p(\Omega, \mu)$ for $1 \le p < \infty$ with non-empty resolvent set is given by

$$|||f|||_A = \|f\chi_{\{\operatorname{Re} h \le 0\}}\| \quad (f \in L_p(\Omega, \mu)).$$

So the quotient $X \,/\, N$ is complete under $|||\cdot|||_A$ and isometrically isomorphic to $L^p(\Omega', \mu{\restriction_{\Omega'}})$ where $\Omega' = \{x \in \Omega \colon \operatorname{Re} h(x) \le 0\}$ and $\mathcal{A} = M_{h\restriction_{\Omega'}}$. Note that A is possibly not the generator of a C_0-semigroup here, but $\mathcal{A}$ does generate a C_0-semigroup.

Example 4.2.21. Let A be the generator of the right shift semigroup T on $X = L_p(\mathbb{R}_+, w)$ with the norm

$$\|f\| = \left(\int_0^\infty |f(t)|^p w(t)\,\mathrm{d}t\right)^{1/p}$$

where $1 \le p < \infty$ and $w \colon \mathbb{R}_+ \to (0, \infty)$ is a continuous weight such that T is a C_0-semigroup. Define the new weight

$$v(t) := \inf\{\, w(s) \colon 0 \le s \le t \,\} \quad (t \ge 0).$$

and let $\|\cdot\|'$ be the norm defined by

$$\|f\|' = \left(\int_0^\infty |f(t)|^p v(t)^p\,\mathrm{d}t\right)^{1/p}.$$

Then clearly $\|\cdot\|' \le \|\cdot\|$ since $v \le w$, and v is decreasing so the right shift C_0-semigroup on $L_p(\mathbb{R}_+, v)$ is contractive. Proposition 4.2.14 gives $|||f|||_A \ge \|f\|'$ for all $f \in X$.

Let $p = 1$. We will show that $|||\cdot|||_A = \|\cdot\|'$. Let χ_I be the indicator function of a bounded interval $I \subset \mathbb{R}_+$. The function w is uniformly continuous on the compact set $[0, \sup I]$ so for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$|s - t| < \delta \text{ and } 0 \le s, t \le \sup I \Longrightarrow |w(t) - w(s)| < \varepsilon.$$

Split the interval I into n disjoint subintervals I_i, each of length $l < \delta$. We can write $\chi_I = \sum_{i=1}^n \chi_{I_i}$ and have

$$|||\chi_I|||_A \le \sum_{i=1}^n |||\chi_{I_i}|||_A.$$

Since T_A is a contractive C_0-semigroup, we have that $|||T(t)g|||_A \le |||g|||_A \le \|g\|$ for all $t \ge 0$ and $g \in X$. For $g = \chi_{I_i - t}$ where $I_i - t = \{\, x - t \colon x \in I_i \,\}$, this shows that

$$|||\chi_{I_i}|||_A \le \|\chi_{I_i - t}\|.$$

Let $i \in \{1, 2, \ldots, n\}$, choose $s \in [0, \inf I_i]$ such that $w(s) = v(\inf I_i)$, and let $t = \inf I_i - s$. Since I_i has length less than δ, we get $|||\chi_{I_i}|||_A \leq \|\chi_{I_i}\|' + 2\varepsilon \cdot l$. So

$$|||\chi_I|||_A \leq \sum_{i=0}^{n} \|\chi_{I_i}\|' + 2\varepsilon ln \leq \|\chi_I\|' + 2 \sup I \cdot \varepsilon,$$

where we use that $\|f + g\|' = \|f\|' + \|g\|'$ if f, g have disjoint support, and $ln \leq \sup I$. Since $\varepsilon > 0$ can be chosen arbitrarily, we see that $|||\chi_I|||_A = \|\chi_I\|'$ holds. Let f be a simple function, so $f = \sum_{i=0}^{n} \alpha_i \chi_{I_i}$ for some disjoint intervals I_i. Then

$$|||f|||_A \leq \sum_{i=0}^{n} |\alpha_i| |||\chi_{I_i}|||_A = \sum_{i=0}^{n} |\alpha_i| \|\chi_{I_i}\|' = \|f\|'.$$

This inequality requires that $p = 1$. By density of simple functions in X, this shows that $|||f|||_A = \|f\|'$ for all $f \in X$. A similar proof works for the right shift on $C_0(\mathbb{R}_+, w) \subset L_\infty(\mathbb{R}_+, w)$ which has dissipative part $C_0(\mathbb{R}_+, v)$.

It may happen that the dissipative part of a C_0-semigroup generator A is isomorphic to the space X on which A is defined. This is precisely the case when A generates a bounded C_0-semigroup. In fact, we can use the seminorm $\|\cdot\|_A$ to infer the Feller-Miyadera-Phillips theorem (Theorem 2.4.2) from the Hille-Yosida theorem (Theorem 2.4.1). This will be shown in Section 4.7.

4.2.3 A dual formulation of the dissipative seminorm

Let A be the generator of a C_0-semigroup and let $|||\cdot|||_A$ be its dissipative seminorm which gives the dissipative part $(X_A, \mathcal{A}, \pi)$. We will use the dual map of π to find a formula for $|||\cdot|||_A$ which is similar to those in Propositions 4.1.3 and 4.1.11. Since $\mathcal{A}$ is dissipative on X_A, the dual operators $(\lambda - \mathcal{A}')$ are surjective for $\lambda > 0$. Moreover, for any $\xi \in X_A'$ there is $\xi' \in X_A'$ with $\|\xi'\| \leq \|\xi\|$ and $\lambda^{-1}(\lambda - \mathcal{A}')\xi' = \xi$. We prove this below and use it to identify $\pi'(X_A') \subset X'$. For $\varphi \in X'$, let

$$\mathfrak{n}(\varphi) = \inf \Big\{ \sup_{n \geq 0} \|\varphi_n\| \colon (\varphi_n) \subset X' \text{ with } \varphi_0 = \varphi, \varphi_n \in D(A'^n) \text{ and } \exists \lambda > 0 \text{ s.t. } \varphi_n = \lambda^{-1}(\lambda - A')\varphi_{n+1} \ (n \geq 0) \Big\},$$

$\mathfrak{n}(\varphi) = \infty$ if the set is empty, and let

$$W = \Big\{ \varphi \in \bigcap_{n \geq 0, \lambda > 0} \operatorname{ran} (\lambda - A')^n \colon \mathfrak{n}(\varphi) \text{ is finite} \Big\}.$$

We want to show that $W = \pi'(X'_A)$. Let $\varphi \in W$ and $M = \mathfrak{n}(\varphi)$. If $x \in X$ and $P(x, x_i, \mu, n)$ holds for some $\mu > 0$ then there is $(\varphi_n) \subset X'$ such that

$$\varphi((\alpha - A)^{-n}x) = \varphi_n(\mu^{-n}(\mu - A)^n(\alpha - A)^{-n}x) = \sum_{i=0}^{n} \varphi_{n-i}((\alpha - A)^{-n}x_i)$$

and thus

$$|\varphi(\alpha^n(\alpha - A)^{-n}x)| \leq M \sum_{i=0}^{n} \|\alpha^n(\alpha - A)^{-n}x_i\| \quad (\alpha \in \rho(A)).$$

Since A is the generator of a C_0-semigroup, we can take the limit as $\alpha \to \infty, \alpha > \omega_0(T)$ and get

$$|\varphi(x)| \leq M|\!|\!|x|\!|\!|_A \qquad (x \in X).$$

This means that we can define ξ on X_A by $\xi(\pi x) := \varphi(x)$ $(x \in X)$ and have $\xi \in X'_A$ with $\|\xi\| \leq \mathfrak{n}(\varphi)$. For this ξ, we have $\pi'(\xi) = \varphi$.

Now, let $\xi \in X'_A$ and let $\lambda > 0$. We have to find $\xi_1 \in X'_A$ such that $\xi = \lambda^{-1}(\lambda - \mathcal{A}')\xi_1$. Define $\xi_1(\lambda^{-1}(\lambda - \mathcal{A})x) := \xi(x)$ for $x \in D(\mathcal{A})$. Then

$$|\xi_1(\lambda^{-1}(\lambda - \mathcal{A})x)| = |\xi(x)| \leq \|\xi\||\!|\!|x|\!|\!|_A \leq \|\xi\||\!|\!|\lambda^{-1}(\lambda - \mathcal{A})x|\!|\!|_A$$

so that ξ_1 has an extension in X'_A with $\|\xi_1\| \leq \|\xi\|$ by the Hahn-Banach theorem (we denote the extension by ξ_1, too). By definition of the adjoint operator, $\xi_1 \in D(\mathcal{A}')$ and $\lambda^{-1}(\lambda - \mathcal{A}')\xi_1 = \xi$. We can continue in this way to find ξ_n with $\|\xi_n\| \leq \|\xi\|$ and $\xi_n = \lambda^{-1}(\lambda - \mathcal{A}')\xi_{n+1}$ for all $n \in \mathbb{N}$. Since $\lambda > 0$ was arbitrary, $\pi'\xi \in W$ and $\mathfrak{n}(\pi'\xi) \leq \|\xi\|$. So we see that $W = \pi'(X'_A)$ and $\|\xi\| = \mathfrak{n}(\pi'\xi)$ for all $\xi \in X'_A$. We thus get the following proposition.

Proposition 4.2.22. *Let A be the generator of a C_0-semigroup on a Banach space X. Then*

$$|\!|\!|x|\!|\!|_A = \sup\Big\{|\varphi(x)| \colon \varphi \in X', \mathfrak{n}(\varphi) = 1\Big\} \quad (x \in X).$$

4.3 The accretive part of an operator

Recall that an operator A is *accretive* if $-A$ is dissipative, that is if

$$\|(\lambda - A)x\| \geq (-\lambda)\|x\| \tag{4.21}$$

for all $x \in D(A)$ and $\lambda < 0$. We thus consider seminorms $\|\cdot\|'$ such that the inequality (4.21) holds with $\|\cdot\|'$ in place of $\|\cdot\|$. Clearly, this holds if and only if $-A$ is $\|\cdot\|'$-dissipative. Hence if A has non-empty resolvent set $\rho(A)$, we can use the seminorm $|||\cdot|||_{-A}$ defined for $-A$ in Definition 4.2.2. Since $-A$ is dissipative for $|||\cdot|||_{-A}$, A is accretive for $|||\cdot|||_{-A}$. As mentioned in Open Question 4.2.19, it is not clear if the seminorm $|||\cdot|||_{-A}$ is maximal among the seminorms $\|\cdot\|'$ for which

1. $\|x\|' \leq \|x\|$ for all $x \in X$,
2. $\|(\lambda - A)x\|' \geq (-\lambda)\|x\|'$ for all $x \in D(A)$ and $\lambda < 0$.

Remark 4.3.1. In the introduction to this chapter (Section 1.2) we mentioned hyperbolic semigroups. A generator A of such a C_0-semigroup splits into the generator of the stable part on X_{s} and the generator of the unstable part on X_{u}, where $X = X_{\mathrm{s}} \oplus X_{\mathrm{u}}$. On these parts, the maximal dissipative and maximal accretive norms, respectively, are equivalent to the original norm restricted to the subspaces.

We can write the splitting of X as a (splitting) exact sequence

$$0 \to X_{\mathrm{u}} \to X \to X_{\mathrm{s}} \to 0$$

of contractive maps. If we let X_A be the dissipative part of A, we have a contractive map $X \to X_A$ which has a dense image but which might not be surjective. On the other hand, if we let X_{-A} be the accretive part of A which we will define below, we do not have a continuous map $X_{-A} \to X$. If we were trying to imitate or generalise this sequence, we would have to somehow find an outer accretive part. But even then, we would not expect exactness of the sequence at X. A closed subspace on which A generates an isometric C_0-semigroup should be contained in both the dissipative and the accretive part of A since A restricted to this subspace is both accretive and dissipative.

Let A be an operator with non-empty resolvent set. Let $|||\cdot|||_{-A}$ be the dissipative seminorm for $-A$ from Definition 4.2.2. Then A is accretive in $|||\cdot|||_{-A}$. By Theorem 4.2.6, we know that $|||\cdot|||_{-A}$ is dominated by $\|\cdot\|$ and that any operator U commuting with $R(\lambda, A)$ for some $\lambda \in \rho(A)$ satisfies $|||Ux|||_{-A} \leq \|U\| |||x|||_{-A}$ for all $x \in X$.

Let A be a quasi sectorial operator. Then $-A$ is quasi sectorial and we can define X_{-A} as in Section 4.2.2. We can apply Proposition 4.2.10 and find that there is an accretive operator $\mathcal{A}$ on X_{-A}, and that the map $\pi\colon X \to X_{-A}$ satisfies $\pi A \subset \mathcal{A}\pi$, and there is a homomorphism $\varphi\colon \{R(\lambda, A)\colon \lambda \in \rho(A)\}' \to \mathcal{B}(X_{-A})$.

In Proposition 4.2.12 we found a C_0-semigroup T_{-A} under the assumption that $\omega \in \rho(-A)$, or equivalently that $-\omega \in \rho(A)$, for some $\omega > 0$. If we assume instead that A generates a C_0-semigroup, Proposition 4.2.13 shows that $\mathcal{A}$ generates a C_0-semigroup T_A^- on $X_A^- := X_{-A}$. Moreover, since $\mathcal{A}$ is accretive on X_A^-, we know that T_A^- is expansive by Theorem 2.4.4.

Definition 4.3.2. Let A be a quasi sectorial operator. Define the *accretive part* of A as the triple $(X_A^-, \mathcal{A}, \pi)$.

We now want to find a universal property of (X_A^-, T_A^-, π) similar to that in Proposition 4.2.16. Let us start by finding a result analoguous to Proposition 4.2.14.

Proposition 4.3.3. *Let A be a quasi sectorial operator on a Banach space X. Assume that there is a bounded map $\rho\colon X \to Y$, a C_0-semigroup S on Y with $\|S(t)x\| \geq \|x\|$ for all $x \in X$ and $t \geq 0$, and that the generator B of S is such that there exists $\lambda \in \rho(A) \cap \rho(B)$ with $\rho R(\lambda, A) = R(\lambda, B)\rho$. Then $\|\rho(x)\| \leq \|\rho\| \|x\|_{-A}$ for all $x \in X$.*

Proof. Since the C_0-semigroup S is expansive, there is an extending C_0-group U on some Banach space $Z \supset Y$ with $\|U(-t)\| \leq 1$ for all $t \geq 0$, by Theorem 3.2.2. Let C be the generator of U. Since the operators commuting with U extend to Z by a homomorphism, $R(\lambda, A)$ extends to $R(\lambda, C)$ and we find that $\lambda \in \rho(C)$. Let us verify the assumptions of Proposition 4.2.14. We have that $R(-\lambda, -C)\rho = R(-\lambda, -B) = \rho R(-\lambda, -A)$ and $-\lambda \in \rho(-C) \cap \rho(-A)$. Moreover, $-C$ generates the C_0-semigroup $t \mapsto U(-t)$ which is a C_0-semigroup of contractions. So Proposition 4.2.14 yields $\|\rho(x)\| \leq \|\rho\| \|x\|_{-A}$ for all $x \in X$. □

Assume that A generates a C_0-semigroup T. In this section, we call a triple (Y, S, ρ) *intertwining*, if the C_0-semigroup S is expansive and $\rho\colon X \to Y$ intertwines T and S.

Proposition 4.3.4. *Let A be the generator of a C_0-semigroup T on a Banach space X. The triple (T_A^-, X_A^-, π) has the following universal property.*

Let (Y, S, ρ) be an intertwining triple. Then there is a unique bounded linear map $\sigma\colon X_A^- \to Y$ such that $\sigma\pi = \rho$. Moreover, $S(t)\sigma = \sigma T_A^-(t)$ for all $t \geq 0$ and $\|\sigma\| = \|\rho\|$.

Proof. Define σ on $\pi(X)$ by $\sigma(\pi x) = \rho(x)$ for $x \in X$ where $\pi\colon X \to X_{-A}$ is induced by the quotient map. We want to extend σ to X_A^-. By Proposition 4.3.3 we have

$$\|\sigma\pi(x)\| = \|\rho(x)\| \leq \|\rho\| \|x\|_{-A} = \|\rho\| \|\pi(x)\| \quad (x \in X).$$

Hence σ is well-defined and we can extend σ uniquely to X_A^- with $\|\sigma\| \leq \|\rho\|$. By construction, we have that $\sigma\pi = \rho$.

Moreover, we have $\|\rho\| = \|\sigma\pi\| \le \|\sigma\|$ since $\|\pi\| = 1$. Hence $\|\sigma\| = \|\rho\|$. To see that σ intertwines T_A^- and S, note that for $x \in X$ and $t \ge 0$

$$\sigma T_A^-(t)\pi(x) = \sigma\pi(T(t)x) = \rho(T(t)x) = S(t)\rho(x) = S(t)\sigma\pi(x).$$

By density of $\pi(X)$ in X_A^-, this gives $\sigma T_A^-(t) = S(t)\sigma$. □

We also have that (T_{-A}, X_{-A}) is unique with respect to the universal property from the last proposition (see also Proposition 4.2.17).

Proposition 4.3.5. *Let A be the generator of a C_0-semigroup T on a Banach space X. Let (Y, S, ρ) be an intertwining triple which has the universal property of (T_A^-, X_A^-, ρ) in Proposition 4.3.4. Then there is an isometric embedding $\sigma\colon X_A^- \to Y$ such that $\rho = \sigma\pi$.*

Proof. By the universal property for (T_A^-, X_A^-, π) there is a map $\sigma\colon X_A^- \to Y$ and by the universal property for (Y, S, ρ) there is a map $\tau\colon Y \to X_A^-$, and these maps satisfy $\rho = \sigma\pi$ and $\pi = \tau\rho$. So we get that $\rho = (\sigma\tau)\rho$ and $\pi = (\tau\sigma)\pi$. Since π has dense range, this shows that $\tau\sigma = I$ and we also get

$$\|x\| = \|\tau\sigma(x)\| \le \|\sigma(x)\| \quad (x \in X_A^-)$$

which together with $\|\sigma(x)\| \le \|x\|$ shows that σ is isometric. □

Example 4.3.6. Similarly to Example 4.2.20 we can find the accretive part of a multiplication operator $A = M_h$ on $X = L_p(\Omega, \mu)$. For $1 \le p < \infty$ it is given by the restriction to $\Omega' = \{\operatorname{Re} h \ge 0\}$. That is, $X_A^- = L_p(\Omega', \mu\restriction_{\Omega'})$ and $\mathcal{A} = M_{h\restriction_{\Omega'}}$. Note that for such an operator A, the subspace of functions supported on $\{\operatorname{Re} h = 0\}$ is isometrically contained in both the dissipative and the accretive part of A.

Example 4.3.7. In Example 4.2.21 we calculated the dissipative seminorm of the generator of the shift semigroup on a weighted $L_p(\mathbb{R}_+, w)$ space. For a continuous weight w, we can show that the accretive seminorm of the generator A of the shift semigroup on $L_p(\mathbb{R}_+, w)$ is given by the generator of the right shift semigroup on $L_p(\mathbb{R}_+, v)$ where

$$v(t) = \inf\{w(s)\colon s \ge t\} \quad (t \ge 0).$$

We can do the same for the generator of the shift on $L_p(\mathbb{R}, w)$. This show that we can have $|||\cdot|||_A = |||\cdot|||_{-A} = 0$, for example by taking a weight w which satisfies $w(t) \to 0$ as $t \to -\infty$ and $w(t) \to 0$ as $t \to \infty$, such as

$$w(t) = \begin{cases} e^{-t} & t \ge 0 \\ e^{t} & t \le 0. \end{cases}$$

4.4 The contractive part of a C_0-semigroup

In a similar way to how we defined the maximal contractive seminorm for an operator in Section 4.1, we can define a (maximal) seminorm $|||\cdot|||$ for a C_0-semigroup T in such a way that T becomes contractive. That is, $|||\cdot|||$ is maximal among the seminorms $\|\cdot\|'$ with

1. $\|x\|' \leq \|x\|$ for all $x \in X$,
2. $\|T(t)x\|' \leq \|x\|'$ for all $x \in X$ and $t \geq 0$.

We can also define a seminorm $\|\cdot\|_T$ in terms of T only. This definition, given below, resembles the definition of the contractive norm for an operator in (4.1) in Section 4.1. This seminorm is maximal and thus equal to $|||\cdot|||$.

Let A be the generator of the C_0-semigroup T. We know that T is contractive if and only if A is dissipative, for example by the Lumer-Phillips theorem (see Theorem 2.3.3). This time, dissipativity and contractivity are used in their traditional sense and thus refer to the norm $\|\cdot\|$ of the Banach space on which T is defined. We will use this to infer that $|||\cdot|||_A = \|\cdot\|_T$ when A is the generator of T.

Definition 4.4.1. Let $T\colon \mathbb{R}_+ \to \mathcal{B}(X)$ be a C_0-semigroup. Define

$$\|x\|_T = \inf\left\{ \sum_{i=0}^{n} \|x_i\| \colon n \geq 0, x_i \in X \text{ and } Q(x, x_i, n) \text{ holds} \right\} \quad (x \in X).$$

where we say that $Q(x, x_i, n)$ holds for T if there are $t_n \geq \cdots \geq t_1 \geq t_0 = 0$ such that $x = \sum_{i=0}^{n} T(t_i)x_i$.

The following proposition states properties of $\|\cdot\|_T$ which are easy to prove. The proof is omitted.

Proposition 4.4.2. *Let T be a C_0-semigroup. Then $\|\cdot\|_T$ is a seminorm which is dominated by $\|\cdot\|$ and*

$$\|T(t)x\|_T \leq \|x\|_T \quad (x \in X, t \geq 0).$$

Let

$$N = \{\, x \in X \colon \|x\|_T = 0 \,\}$$

and let X_T be the completion of the quotient space $X \,/\, N$ under the norm induced by $\|\cdot\|_T$. The map $\rho\colon X \to X_T$ induced by the quotient map is contractive, since

$\|x\|_T \le \|x\|$ for all $x \in X$. Furthermore, the subspace N is invariant under $T(t)$ for each $t \ge 0$. So

$$S(t)(x + N) := T(t)x + N \quad (x \in X, t \ge 0) \tag{4.22}$$

is well-defined on a dense subset of X_T. Moreover, the operators $S(t)$ are contractive and strongly continuous in t on $X \,/\, N$. Since the operators $S(t)$ are contractive, they extend continuously to X_T and S is strongly continuous on X_T. Moreover, they satisfy $S(t)\rho = \rho T(t)$ for all $t \ge 0$ by definition. We will show that $X_A = X_T$ and $S = T_A$, for which we need the following lemma.

Lemma 4.4.3. *Let T be a C_0-semigroup on a Banach space X and let $|||\cdot|||$ be a seminorm such that $|||x||| \le \|x\|$ and $|||T(t)x||| \le |||x|||$ for all $x \in X, t \ge 0$. Then $|||x||| \le \|x\|_T$ for all $x \in X$.*

Proof. Let $|||\cdot|||$ be a seminorm as stated in the Lemma. Let $x \in X$ and assume that $t_i \ge 0, x_i \in X$ satisfy $x = \sum_{i=0}^{n} T(t_i)x_i$. From the properties of $|||\cdot|||$ we see that

$$|||x||| \le \sum_{i=0}^{n} |||T(t_i)x_i||| \le \sum_{i=0}^{n} \|x_i\|. \qquad \square$$

Theorem 4.4.4. *Let T be a C_0-semigroup with generator A. Then*

$$|||x|||_A = \|x\|_T \quad (x \in X).$$

Proof. Apply Proposition 4.2.16 to A with $Y = X_T$ and the semigroup S from (4.22) on Y. The semigroup S is contractive and the map $\rho\colon X \to Y$ is contractive and intertwines T and S, so the assumptions of Proposition 4.2.16 are satisfied. We get a contractive map $\sigma\colon X_A \to X_T$ which intertwines S and T_A. This shows that

$$\|x\|_T = \|\rho(x)\|_T \le \|\sigma\| |||\pi(x)|||_A = |||x|||_A \quad (x \in X).$$

For the reverse inequality we use Lemma 4.4.3. The seminorm $|||\cdot|||_A$ satisfies the requirements. Indeed, the generator of T_A on X_A is dissipative so that T_A is contractive, and $\pi\colon X \to X_A$ intertwines T and T_A. This means that

$$|||T(t)x|||_A = |||T_A(t)\pi(x)|||_A \le |||\pi(x)|||_A = |||x|||_A \quad (x \in X, t \ge 0).$$

Hence $|||\cdot|||_A \le \|\cdot\|_T$ and the two seminorms are equal. $\square$

4.5 The expansive part of a C_0-semigroup

For the expansive part of a C_0-semigroup T we consider seminorms $\|\cdot\|'$ such that

1. $\|x\|' \leq \|x\|$ for all $x \in X$,
2. $\|T(t)x\|' \geq \|x\|'$ for all $x \in X$ and $t \geq 0$.

Assume that T is a C_0-group. Let $T^-(t) = T(-t)$. Then T^- is a C_0-semigroup, and $\|\cdot\|_{T^-}$ satisfies $\|\cdot\|_{T^-} \leq \|\cdot\|$ and

$$\|x\|_{T^-} = \|T^-(t)T(t)x\|_{T^-} \leq \|T(t)x\|_{T^-} \quad (t \geq 0).$$

Moreover, the generator A of T is such that $-A$ generates T^- and by Theorem 4.4.4 we have that $|||\cdot|||_{-A} = \|\cdot\|_{T^-}$. Without assuming that T is a C_0-group, we will define a seminorm $\|\cdot\|_T^-$ which satisfies 1 and 2 above, and we will show that $|||\cdot|||_{-A} = \|\cdot\|_T^-$.

Definition 4.5.1. Let $T\colon \mathbb{R}_+ \to \mathcal{B}(X)$ be a C_0-semigroup. We say that $Q^-(x, x_i, n)$ holds for T if there are $t_n \geq \cdots \geq t_1 \geq t_0 = 0$ such that

$$T(t_n)x = \sum_{i=0}^{n} T(t_i)x_i.$$

Define

$$\|x\|_T^- = \inf\left\{ \sum_{i=0}^{n} \|x_i\| \colon n \geq 0, x_i \in X \text{ and } Q^-(x, x_i, n) \text{ holds} \right\} \quad (x \in X).$$

The map $\|\cdot\|_T$ has the following properties, which can be easily proved. We omit the proof.

Proposition 4.5.2. *Let T be a C_0-semigroup on a Banach space X. Then $\|\cdot\|_T^-$ is a seminorm which is dominated by $\|\cdot\|$ and such that*

$$\|T(t)x\|_T^- \geq \|x\|_T^- \quad (x \in X, t \geq 0).$$

We can again repeat the quotient construction with respect to the null space N of the seminorm $\|\cdot\|_T^-$. This gives us a Banach space $X_T{}^-$ and an expansive C_0-semigroup S. The next lemma shows that $\|\cdot\|_T^-$ is maximal.

Lemma 4.5.3. *Let T be a C_0-semigroup on a Banach space X and let $|||\cdot|||$ be a seminorm such that $|||x||| \leq \|x\|$ and $|||T(t)x||| \geq |||x|||$ for all $x \in X, t \geq 0$. Then $|||x||| \leq \|x\|_T^-$ for all $x \in X$.*

Proof. Let $|||\cdot|||$ be a seminorm as described in the Proposition. Let $x \in X$ and assume that $t_i \geq 0, x_i \in X$ satisfy $t_i \leq t_n$ and $T(t_n)x = \sum_{i=0}^{n} T(t_i)x_i$. We need to show that

$$|||x||| \leq \sum_{i=0}^{n} \|x_i\|. \tag{4.23}$$

As in Lemma 4.1.8 we have that T is super-expansive in $|||\cdot|||$, so (4.23) follows from the fact that $|||\cdot||| \leq \|\cdot\|$. □

Theorem 4.5.4. *Let T be a C_0-semigroup with generator A. Then $|||x|||_{-A} = \|x\|_T^-$ for all $x \in X$.*

Proof. Apply Proposition 4.3.4 to A with $Y = X_T^-$ and the C_0-semigroup S on Y. The C_0-semigroup S on $X_T{}^-$ is expansive and the map $\rho\colon X \to Y$ is contractive and intertwines T and S, so the assumptions of the Proposition are satisfied. We get a contractive map $\sigma\colon X_A^- \to X_T$ which intertwines S and T_A^-. This shows that

$$\|x\|_T^- = \|\rho(x)\|_T^- \leq \|\sigma\| |||\pi(x)|||_{-A} = |||x|||_{-A} \quad (x \in X).$$

For the reverse inequality, use Lemma 4.5.3. The seminorm $|||\cdot|||_{-A}$ satisfies the assumptions. Indeed, the generator of T_A^- is accretive and T_A^- is expansive by Theorem 2.4.4, and $\pi\colon X \to X_A^-$ intertwines T and T_A^-. This implies that

$$|||T(t)x|||_{-A} = |||T_A^-(t)\pi(x)|||_{-A} \geq |||\pi(x)|||_{-A} = |||x|||_{-A}$$

for all $x \in X, t \geq 0$. Hence $|||\cdot|||_{-A} \leq \|\cdot\|_T^-$ and the two seminorms are equal. □

4.6 Generalisations

So far we have considered contractive and expansive parts only for a single operator and for C_0-semigroups. As in Section 3.2, we can work with representations of abelian semigroups. Here, talking about contractivity still makes sense, and expansiveness can be replaced by super-expansiveness. The methods are the same as for C_0-semigroups.

Going back to C_0-semigroups, we can replace contractivity by a different growth characterisation, namely boundedness and the corresponding notion for their generators. Since bounded C_0-semigroups are similar to contractive C_0-semigroups (on Banach space), the seminorms which arise are equivalent to the contractive seminorms.

4.6.1 Contractive and super-expansive parts of semigroup representations

Sections 4.4 and 4.5 discuss the contractive and expansive parts only in the context of C_0-semigroups. Contractivity is a notion which also makes sense for general semigroup representations, and expansiveness takes the more general form of super-expansiveness which complements contractivity as seen in Section 3.2. Here, we consider again representations of subsemigroups $\mathfrak{S}$ of a locally compact abelian group $\mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{S} - \mathfrak{S}$, the semigroup representations that also have been studied in [BY01].

As a reminder, we say that a representation T of a semigroup $\mathfrak{S}$ is super-expansive if $T(t_n)x = \sum_{i=0}^{n-1} T(t_i)x_i$ implies $\|x\| \leq \sum_{i=0}^{n-1}\|x_i\|$ where $t_i \in \mathfrak{S} \cup \{0\}$ and $t_n - t_i \in \mathfrak{S} \cup \{0\}$.

Definition 4.6.1. Let T be a representation of a semigroup $\mathfrak{S}$. We say that $Q(x, x_i, n)$ holds if there are $t_0, t_1, \ldots, t_{n-1}$ such that $t_i \in \mathfrak{S}\cup\{0\}$ and $x = \sum_{i=0}^{n-1} T(t_i)x_i$. Similarly, we say that $Q^-(x, x_i, n)$ holds if instead $T(t_n)x = \sum_{i=0}^{n-1} T(t_i)x_i$ for some $t_i \in \mathfrak{S} \cup \{0\}$ with $t_n - t_i \in \mathfrak{S} \cup \{0\}$. Define

$$\|x\|_T = \inf \left\{ \sum_{i=0}^{n-1}\|x_i\| \colon n \geq 0, x_i \in X \text{ and } Q(x, x_i, n) \text{ holds} \right\} \quad (x \in X)$$

and

$$\|x\|_T^- = \inf \left\{ \sum_{i=0}^{n-1}\|x_i\| \colon n \geq 0, x_i \in X \text{ and } Q^-(x, x_i, n) \text{ holds} \right\} \quad (x \in X).$$

Remark 4.6.2. For C_0-semigroups, this definition agrees with Definitions 4.4.1 and 4.5.1. For an operator $T \in \mathcal{B}(X)$, the definitions (4.1) of $\|\cdot\|_T$ and (4.10) of $\|\cdot\|_T^-$ in Section 4.1 agree with the ones in Defintion 4.6.1 for the representation of $\mathbb{N}$ given by $n \mapsto T^n$.

Proposition 4.6.3. *Let T be a representation of a semigroup $\mathfrak{S}$ on a Banach space X. The maps in Definition 4.6.1 are seminorms which are dominated by $\|\cdot\|$. Moreover, we have that*

1. $\|T(t)x\|_T \leq \|x\|_T$ *for all* $x \in X, t \in \mathfrak{S}$,
2. T *is super-expansive for* $\|\cdot\|_T^-$.

Proof. Let us start by showing that the maps are seminorms. If $Q(x, x_i, n)$ and $Q(y, y_i, m)$ hold then there exist $t_i \in \mathfrak{S} \cup \{0\}$ and $s_j \in \mathfrak{S} \cup \{0\}$ such that $x =$

$\sum_{i=0}^{n-1} T(t_i)x_i$ and $y = \sum_{j=0}^{m-1} T(s_j)y_j$. This implies that $Q(x+y, z_i, m+n)$ holds where

$$z_i = \begin{cases} x_i & i < n \\ y_{i-n} & n \leq i < m+n \end{cases}$$

and

$$u_i = \begin{cases} t_i & i < n \\ s_{i-n} & n \leq i < m+n. \end{cases}$$

If we assume instead that $Q^-(x, x_i, n)$ and $Q^-(y, y_j, m)$ hold then there exist $t_i \in \mathfrak{S} \cup \{0\}$ and $s_j \in \mathfrak{S} \cup \{0\}$ with $t_n - t_i \in \mathfrak{S} \cup \{0\}$ and $s_m - s_j \in \mathfrak{S} \cup \{0\}$ such that

$$T(t_n)x = \sum_{i=0}^{n-1} T(t_i)x_i, \quad T(s_n)y = \sum_{j=0}^{m-1} T(s_j)y_j.$$

This implies that $Q^-(x+y, z_i, m+n)$ holds where

$$z_i = \begin{cases} x_i & i < n \\ y_{i-n} & n \leq i < m+n \end{cases}$$

if we choose $u_i \in \mathfrak{S} \cup \{0\}$ by

$$u_i = \begin{cases} t_i + s_m & i < n \\ t_n + s_{i-n} & n \leq i, \end{cases}$$

as $(t_n + s_m) - u_i \in \mathfrak{S} \cup \{0\}$ for $0 \leq i < m+n$. This implies that the triangle inequality holds for $\|\cdot\|_T$ and for $\|\cdot\|_T^-$. Positive homogeneity is easy to check. The seminorms are dominated by $\|\cdot\|$ since we have that $x = T(0)x$ and $T(t)x = T(t)x$ $(t \in \mathfrak{S} \cup \{0\})$ hold for all $x \in X$, which directly gives

$$\|x\|_T \leq \|x\|, \qquad \|x\|_T^- \leq \|x\| \quad (x \in X).$$

Next, we show contractivity of T with respect to $\|\cdot\|_T$. Let $x \in X$ and $t \in \mathfrak{S}$. Assume that $Q(x, x_i, n)$ holds for some $t_i \in \mathfrak{S} \cup \{0\}$. Then $Q(T(t)x, x_i, n)$ holds with $t_i' = t + t_i \in \mathfrak{S} \cup \{0\}$. This proves that

$$\|T(t)x\|_T \leq \sum_{i=0}^{n-1} \|x_i\|$$

and $\|T(t)x\|_T \leq \|x\|_T$.

For super-expansiveness of T in the seminorm $\|\cdot\|_T^-$, assume that

$$T(t)x = \sum_{i=0}^{n-1} T(t_i)x_i$$

where $t - t_i \in \mathfrak{S} \cup \{0\}$. Now, if $T(s_i)x_i = \sum_{k=0}^{n_i-1} T(t_{i,k})x_{i,k}$ holds for each i with some values s_i,$t_{i,k} \in \mathfrak{S}$ such that $s_i - t_{i,k} \in \mathfrak{S} \cup \{0\}$ then

$$T(t+s)x = \sum_{i=0}^{n-1} \sum_{k=0}^{n_i-1} T(t_i + s - s_i + t_{i,k})x_{i,k}$$

where $s = \sum_{i=0}^{n-1} s_i$ and we have $(s+t)-(t_i+s-s_i+t_{i,k}) = (t-t_i)+(s_i-t_{i,k}) \in \mathfrak{S}\cup\{0\}$. This implies

$$\|x\|_T^- \leq \sum_{i=0}^{n-1} \sum_{k=0}^{n_i-1} \|x_{i,k}\|.$$

Take the infimum over all such $x_{i,k}$ depending on the x_i to get $\|x\|_T^- \leq \sum_{i=0}^{n-1} \|x_i\|_T^-$. That is, T is super-expansive in $\|\cdot\|_T^-$. □

Proposition 4.6.4. *Let T be a representation of a semigroup $\mathfrak{S}$. Then $\|\cdot\|_T$ is the maximal contractive seminorm for T which is dominated by $\|\cdot\|$, and $\|\cdot\|_T^-$ is the maximal super-expansive seminorm for T which is dominated by $\|\cdot\|$.*

Proof. Assume there is another contractive seminorm $\|\cdot\|'$ for the semigroup representation T and assume that $\|x\|' \leq \|x\|$ for all $x \in X$. Let $x \in X$ and assume that $Q(x, x_i, n)$ holds. Then

$$\|x\|' = \left\|\sum_{i=0}^{n-1} T(t_i)x_i\right\|' \leq \sum_{i=0}^{n-1} \|x_i\|' \leq \sum_{i=0}^{n-1} \|x\|$$

so that $\|x\|' \leq \|x\|_T$. This shows that $\|\cdot\|_T$ is maximal.

To prove that $\|\cdot\|_T^-$ is maximal, let $\|\cdot\|'$ be a super-expansive seminorm for T such that $\|x\|' \leq \|x\|$ for all $x \in X$. Assume that $T(t_n)x = \sum_{i=0}^{n-1} T(t_i)x_i$ where $t_i \in \mathfrak{S}$, $t_n - t_i \in \mathfrak{S} \cup \{0\}$. Then

$$\|x\|' \leq \sum_{i=0}^{n-1} \|x_i\|' \leq \sum_{i=0}^{n-1} \|x_i\|.$$

So we see that $\|x\|' \leq \|x\|_T^-$ for all $x \in X$. □

4.6.2 Bounded parts for C_0-semigroups

Let T be a C_0-semigroup on a Banach space X. Assume that T is bounded, say by $M \geq 1$. Then T is contractive in the equivalent norm

$$\sup_{t \geq 0} \|T(t)x\| \quad (x \in X).$$

This implies that, up to a linear isomorphism, the bounded part and the contractive part of a semigroup are the same. It is easy to define a seminorm in which a semigroup is bounded by a given bound $M \geq 1$. For the C_0-semigroup T, let $\|\cdot\|_M$ be the supremum of all seminorms $\|\cdot\|'$ such that

1. $\|x\|' \leq \|x\|$ for all $x \in X$.
2. $\|T(t)x\|' \leq M\|x\|'$ for all $x \in X, t \geq 0$.

We can repeat the renorming construction above and get a seminorm $\sup_{t\geq 0}\|T(t)(\cdot)\|_M$ in which T is contractive and which satisfies

$$\sup_{t\geq 0}\|T(t)x\|_M \leq M\|x\| \quad (x \in X).$$

Hence $\|\cdot\|_M \leq M\|\cdot\|_T$ where $\|\cdot\|_T$ is the maximal contractive seminorm for T. On the other hand, we also have $\|\cdot\|_T \leq \|\cdot\|_M$ by choice of $\|\cdot\|_M$. So the two seminorms are equivalent. Let us turn to the formulation of the seminorm $\|\cdot\|_M$ in terms of T and $\|\cdot\|$. Define

$$\|x\|_{T,M} := \inf\left\{\|x_0\| + M\sum_{i=1}^{n}\|x_i\| \colon n \geq 0, x_i \in X, Q(x, x_i, n) \text{ holds}\right\}.$$

where Q is as in Definition 4.4.1. If $\|\cdot\|'$ is a seminorm dominated by $\|\cdot\|$ and such that $\|T(t)x\|' \leq M\|x\|'$ for all $x \in X, t \geq 0$ and $x = \sum_{i=0}^{n} T(t_i)x_i$ for some $t_i \geq 0$ and $t_0 = 0$ then

$$\|x\|' \leq \sum_{i=0}^{n}\|T(t_i)x_i\|' \leq \|x_0\| + M\sum_{i=1}^{n}\|x_i\|$$

so that $\|x\|' \leq \|x\|_{T,M}$. This shows that $\|\cdot\|_{T,M}$ is maximal with these properties.

We saw in Theorem 4.4.4 that the maximal contractive norm for T is equal to the dissipative norm $|||\cdot|||_A$ for the generator A of T. In the Feller-Miyadera-Phillips theorem, the property of A corresponding to T being bounded by M is

$$\|\lambda^{-n}(\lambda - A)^n x\| \geq \frac{1}{M}\|x\| \tag{4.24}$$

for all $\lambda > 0$, $n \geq 1$ and all $x \in D(A^n)$. So an immediate guess is that the maximal seminorm in which (4.24) holds is equal to $\|\cdot\|_{T,M}$ if A generates the semigroup T. We can show this in the same way as for Theorem 4.4.4, that is, by passing to quotient spaces.

For an explicit description of the seminorm for (4.24), assume that A has non-empty resolvent. For $\mu > 0$, let $|||\cdot|||_{A,\mu}$ be the seminorm from Definition 4.2.2 and let

$$\begin{aligned} |||x|||_{A,M,\mu} &:= \inf\left\{\|x_n\| + M\sum_{i=0}^{n-1}\|x_i\| \colon P(x, x_i, \mu, n)\right\} \\ &= \inf\left\{\|y\| + M|||x-y|||_{A,\mu} \colon y \in X\right\} \end{aligned} \qquad (x \in X)$$

for $\mu > 0$ where P is defined by (4.13) in Section 4.2.1. It is clear that $|||\cdot|||_{A,M,\mu}$ is a seminorm such that $|||x|||_{A,\mu} \leq |||x|||_{A,M,\mu} \leq M|||x|||_{A,\mu}$ for all $x \in X$. We can see that (4.24) holds for $0 < \lambda < \mu$ and all $n \geq 1$, since

$$|||\lambda^{-n}(\lambda - A)^n x|||_{A,M,\mu} \geq |||\lambda^{-n}(\lambda - A)^n x|||_{A,\mu} \geq |||x|||_{A,\mu} \geq \frac{1}{M}|||x|||_{A,M,\mu}$$

by Lemma 4.2.4. By Lemma 4.2.5, we get for $0 < \lambda < \mu$ that

$$\|y\| + |||x-y|||_{A,\mu} \leq \|y\| + |||x-y|||_{A,\lambda} \quad (x, y \in X)$$

and hence $|||x|||_{A,M,\mu} \leq |||x|||_{A,M,\lambda}$ for all $x \in X$. Therefore,

$$|||x|||_{A,M} := \lim_{\mu\to\infty}|||x|||_{A,M,\mu} \quad (x \in X)$$

is a seminorm which is dominated by $\|\cdot\|$ and satisfies (4.24). For the generator A of a C_0-semigroup, we also get maximality results as in Propositions 4.2.14 and 4.2.16. We omit them here. They leads to the following proposition.

Proposition 4.6.5. *Let A be the generator of a C_0-semigroup T and let $M \geq 1$. Then*

$$|||T(t)x|||_{A,M} \leq M|||x|||_{A,M}$$

for all $t \geq 0$ and $x \in X$.

Consequently, we have that $|||x|||_{A,M} = \|x\|_{T,M}$ for all $x \in X, M \geq 1$.

4.7 Applications

We will see how the maximal parts and the corresponding seminorms apply to two different topics. One of them is quasi-hyperbolic semigroups which were in fact a motivation in defining maximal parts. We will show in Theorem 4.7.1 that the seminorms can be used to characterise a continuous embedding of a C_0-semigroup into a quasi-hyperbolic semigroup. The other topic is generation theorems. We can use the dissipative seminorm to recover the Feller-Miyadera-Phillips Theorem from the Hille-Yosida theorem (see Theorem 4.7.2). A slightly greater effort is required to obtain a generalisation to a result by Goldberg and Smith. We will use the accretive seminorm to find the generalisation Theorem 4.7.4.

4.7.1 Quasi-hyperbolic semigroups

Recall that a quasi-hyperbolic semigroup is a restriction of a hyperbolic semigroup to a closed invariant subspace, and that a semigroup S is hyperbolic if the underlying space Y splits as a direct sum of a stable part Y_{s} and an unstable part Y_{u} such that there is $t > 0$ with

$$\|S(t)\restriction_{Y_{\mathrm{s}}}\| \leq \frac{1}{2}, \qquad \|(S(t)\restriction_{Y_{\mathrm{u}}})^{-1}\| \leq \frac{1}{2}.$$

Let P be the projection onto the stable part Y_{s} along the unstable part Y_{u} (that is $P(x+y) = x$ for $x \in Y_{\mathrm{s}}$ and $y \in Y_{\mathrm{u}}$). By the above, there exists some $M > 0$ such that the estimate $\|S(s)P\| \leq Me^{-s/t \ln 2}$ holds for all $s \geq 0$. Moreover for $\alpha = \ln 2/t > 0$ the map

$$\sup_{s\geq 0} \|e^{\alpha s} S(s)Px\| \quad (x \in X)$$

is a seminorm on Y which is equivalent to the norm $\|\cdot\|$ on Y_{s}. Furthermore, the semigroup $e^{\alpha s}S(s)P$ is contractive for this seminorm. Similarly, we get that

$$\sup_{s\geq 0} \|(e^{-\alpha s} S(s)\restriction_{Y_u})^{-1}(I - P)x\| \quad (x \in X)$$

is a seminorm on Y which is equivalent to $\|\cdot\|$ on Y_{u}, and $e^{-\alpha s}S(s)$ is expansive in this seminorm. Let T be a restriction of S to an invariant subspace X, so that T is a quasi-hyperbolic semigroup. Let $\|\cdot\|_\alpha$ be the maximal contractive seminorm for $e^{\alpha t}S(t)$ and let $\|\cdot\|_{-\alpha}$ be the maximal expansive seminorm for $e^{-\alpha t}S(t)$. Since

$$\|\cdot\|_\alpha + \|\cdot\|_{-\alpha}$$

is equivalent to $\|\cdot\|$ on Y, it is also equivalent to $\|\cdot\|$ on X. Moreover, we get that T has a uniform exponential decay and growth, respectively, in these seminorms, namely

$$\|T(t)x\|_\alpha \leq e^{-\alpha t}\|x\|_\alpha$$

and

$$\|T(t)x\|_{-\alpha} \geq e^{\alpha t}\|x\|_{-\alpha}$$

for all $x \in X$ and $t \geq 0$.

We can also reverse the arguments. Assume that T is a semigroup on X such that

$$\|\cdot\|_\alpha + \|\cdot\|_{-\alpha}$$

is equivalent to the norm $\|\cdot\|$ on X for some $\alpha > 0$. Here, the seminorms $\|\cdot\|_\alpha$ and $\|\cdot\|_{-\alpha}$ are the maximal contractive and maximal expansive seminorms for $e^{\alpha t}T(t)$ and

$e^{-\alpha t}T(t)$, respectively. Since our aim is to find properties of A which imply that $\|\cdot\|_\alpha + \|\cdot\|_{-\alpha}$ is a norm, we should note that

$$\|\cdot\|_\alpha = |||\cdot|||_{A+\alpha} \text{ and } \|\cdot\|_{-\alpha} = |||\cdot|||_{-A+\alpha}$$

by Theorems 4.4.4 and 4.5.4. Let X_α and $X_{-\alpha}$ be the completions of the quotients of X with respect to the null spaces of the respective seminorms $\|\cdot\|_\alpha, \|\cdot\|_{-\alpha}$ (that is, we pass to the maximal parts; we denote the C_0-semigroup on these spaces still by T). It is quite evident that the C_0-semigroup $(T(t), T(t))$ on the direct sum $X_\alpha \oplus X_{-\alpha}$ is hyperbolic. Using the diagonal embedding

$$\pi(x) = (\pi_\alpha(x), \pi_{-\alpha}(x))$$

where $\pi_{\pm\alpha} \colon X \to X_{\pm\alpha}$ are the quotient maps, we get an isomorphic embedding of X into $X_\alpha \oplus X_{-\alpha}$ and thus T is similar to a quasi-hyperbolic semigroup and thus T is itself quasi-hyperbolic.

A more direct approach is to use the following equivalent formulation (see [BT10]). A semigroup T is quasi-hyperbolic if and only if there exists $t > 0$ such that

$$\max\{\|T(2t)x\|, \|x\|\} \geq 2\|T(t)x\| \quad (x \in X).$$

By our assumption, there exists $c > 0$ such that

$$c\|x\| \leq \max\left\{\|x\|_\alpha, \|x\|_{-\alpha}\right\} \leq \|x\|$$

for all $x \in X$. Hence

$$\begin{aligned} 2\|T(t)x\| &\leq 2c^{-1} \max\left\{\|T(t)x\|_\alpha, \|T(t)x\|_{-\alpha}\right\} \\ &\leq 2c^{-1}e^{-\alpha t} \max\left\{\|x\|_\alpha, \|T(2t)x\|_{-\alpha}\right\} \\ &\leq 2c^{-1}e^{-\alpha t} \max\{\|x\|, \|T(2t)x\|\} \end{aligned}$$

and for $t \geq \frac{-\ln c + \ln 2}{\alpha}$ this shows that T is quasi-hyperbolic. Note that this discussion does not give a method of testing which values of α and c work for a given C_0-semigroup T.

Seminorms very similar to $\|\cdot\|_\alpha$ and $\|\cdot\|_{-\alpha}$ were already introduced in the context of quasi-hyperbolic operators (and not only C_0-semigroups) in [BT10, Section 2]. Given an operator $T \in \mathcal{B}(X)$, the motivation was to find a continuous embedding $X \to Y$ and a (quasi-) hyperbolic operator S on Y such that $S{\restriction_X} = T$. We can do the same thing for C_0-semigroups using $\|\cdot\|_\alpha$ and $\|\cdot\|_{-\alpha}$. We should note that the seminorms introduced in [BT10] are equal to the contractive and expansive seminorms for an operator T from Section 4.1 up to rescaling, assuming that T is a surjection. For $\alpha > 1$, they were defined in [BT10] as

$$p_{\alpha\pm}(x) = \inf\Big\{\sum_{r=1}^{n} q_{\alpha\pm}(x_r)\colon n \geq 1, x_r \in X, x = \sum_{r=1}^{n} x_r\Big\}$$

where

$$q_{\alpha+}(x) = \inf_{n\in\mathbb{N}} \frac{\|T^n x\|}{\alpha^n}, \qquad q_{\alpha-}(x) = \inf\Big\{\frac{\|y\|}{\alpha^n}: y \in X, n \in \mathbb{N}, T^n y = x\Big\}.$$

So we see that $p_{\alpha+}$ is the maximal seminorm for which $\alpha^{-1}T$ is expansive, and $p_{\alpha-}$ is the maximal seminorm for which αT is contractive (assuming T is surjective). We summarise this in a statement similar to [BT10, Theorem 2.7], but formulated here in a more general form and for C_0-semigroups.

Theorem 4.7.1. *Let T be a C_0-semigroup on a Banach space X. The following are equivalent.*

1. *There exists $\alpha > 0$ such that $\|\cdot\|_\alpha + \|\cdot\|_{-\alpha}$ is a norm on X.*
2. *There exist a Banach space Y and a quasi-hyperbolic semigroup S on Y such that X is continuously embedded in Y and $T(t) = S(t)\restriction_X$ for all $t \geq 0$.*
3. *There exist a Banach space Z and a hyperbolic semigroup U on Z such that X is continuously embedded in Z and $T(t) = U(t)\restriction_X$ for all $t \geq 0$.*

On Hilbert space, a C_0-semigroup generator A generates a quasi-hyperbolic semigroup if and only if there is $c > 0$ such that

$$\|(A - is)x\| \geq c\|x\| \qquad (s \in \mathbb{R}, x \in D(A)). \tag{4.25}$$

This was shown in [BT10, Corollary 3.10]. It is not true on Banach space (see [BT10, Example 3.3]). An open question which was formulated in [BT10, Section 4] is related to Theorem 4.7.1. Let A be the generator of a C_0-semigroup T. Assume there exists $c > 0$ such that (4.25) holds. Is there a Banach space Y, a quasi-hyperbolic semigroup S on Y and a continuous injection $\pi\colon X \to Y$ such that $\pi T(t) = S(t)\pi$ for all $t \geq 0$? The results in [BT10] show that $\|\cdot\|_\alpha + \|\cdot\|_{-\alpha}$ is a norm if $\alpha = 0$ and T is surjective, but we require this to be true for some positive α for the question to have a positive answer.

4.7.2 Generation theorems

It is well known that C_0-semigroups are uniquely determined by their generators. There are different theorems that describe generators of C_0-semigroups of a certain type. Often, information about the spectrum and resolvent of the generator is sufficient to infer properties and the behaviour of the C_0-semigroup. The most basic of these theorems are the Lumer-Phillips theorem and the Hille-Yosida theorem (Theorems 2.3.3 and 2.4.1). Both theorems discuss dissipative operators, and when they generate a C_0-semigroup of contractions. A generalisation of the Hille-Yosida theorem characterises generators of bounded C_0-semigroups. We will use the dissipative seminorm to infer the bounded case from the contractive case. The proof given for example in [EN00, Theorem II.3.8] very much motivated the construction of the dissipative seminorm. There, given a resolvent bound for the generator of a C_0-semigroup, an equivalent norm is constructed in which the generator becomes dissipative. Here is a different proof, following the same idea.

Theorem 4.7.2. *Let $M \geq 1$ and let A be a densely defined operator on a Banach space X. Then A generates a C_0-semigroup T with*

$$\|T(t)\| \leq M \quad (t \geq 0)$$

if and only if

$$\|\lambda^n R(\lambda, A)^n\| \leq M \quad (\lambda > 0, n \geq 1). \tag{4.26}$$

The new part in the following proof is assuming that the resolvent of A satisfies the bounds in the theorem, A generates a bounded C_0-semigroup. The proof for the other implication is included for completeness, and can be found in [EN00].

Proof. Let us assume that T is a C_0-semigroup with $\|T(t)\| \leq M$ for all $t \geq 0$. It has a generator A with a resolvent given by the Laplace transform

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t$$

for all $\lambda > 0$ and $x \in X$. Using the resolvent equation, we see that the powers of $R(\lambda, A)$ can be expressed in terms of its derivatives, and after differentiating under the integral sign we see that

$$R(\lambda, A)^n x = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t)x \, \mathrm{d}t \quad (x \in X, \lambda > 0, n \geq 1).$$

The integral can be estimated using the bound on T (see [EN00, Corollary II.1.11]).

Now assume that the resolvent of A satisfies the norm bound (4.26). Let $|||\cdot|||_A$ be the dissipative seminorm from Definition 4.2.2. We will show that it is equivalent to the norm $\|\cdot\|$. Assume that $P(x, x_i, \mu, n)$ holds for some $\mu > 0$. Since $\mu \in \rho(A)$ for $\mu > 0$ by assumption, it follows that $x = \sum_{i=0}^{n} \mu^{n-i}(\mu - A)^{i-n} x_i$ and

$$\begin{aligned} \|x\| &\leq \sum_{i=0}^{n} \|\mu^{n-i}(\mu - A)^{i-n}\| \|x_i\| \\ &\leq M \sum_{i=0}^{n} \|x_i\|. \end{aligned}$$

From this, we see that $\frac{1}{M}\|\cdot\| \leq |||\cdot|||_A \leq \|\cdot\|$. We have that $\|\lambda R(\lambda, A)\| \leq M$ for $\lambda > 0$, so A is quasi sectorial. Since $\lambda \in \rho(A)$ for $\lambda > 0$, we see that A generates a C_0-semigroup T of contractions on $(X, |||\cdot|||_A)$ by Proposition 4.2.12. We can estimate the bound of T in the norm $\|\cdot\|$ by

$$\|T(t)x\| \leq M|||T(t)x|||_A \leq M|||x|||_A \leq M\|x\| \quad (x \in X, t \geq 0).$$

Hence $\|T(t)\| \leq M$, as required. □

In this result we used the dissipative seminorm in a situation where it was equivalent to the norm on the Banach space that we started with. Since we also have an accretive seminorm, let us see what we know for C_0-semigroup generators that are accretive. The next result is due to Goldberg and Smith ([GS78], see Theorem 2.4.4). We already encountered this theorem in the Hilbert space setting as Proposition 3.5.7.

Theorem 4.7.3. *Let A be the generator of a C_0-semigroup T on a Banach space X. The following are equivalent.*

1. $\|T(t)x\| \geq \|x\|$ *for all* $x \in X, t \geq 0$.
2. $\|(\lambda - A)x\| \geq -\lambda\|x\|$ *for all* $x \in D(A), \lambda < 0$.

The second condition of this theorem is precisely accretivity of the generator A. So if we use the accretive seminorm for a generator A, we can look for a result where the second condition is replaced by one for which the accretive seminorm for A is equivalent to the original norm $\|\cdot\|$. We should note the following. Assume that T satisfies condition 1 from Theorem 4.7.3 with respect to a norm $\|\cdot\|'$ and let $\|\cdot\|$ be an equivalent norm. Then for some $c > 0$ and $m > 0$,

$$\|T(t)x\| \geq m\|T(t)x\|' \geq m\|x\|' \geq c\|x\| \quad (t \geq 0, x \in X).$$

However, if $\|T(t)x\| \geq c\|x\|$ $(x \in X, t \geq 0)$ for some $c > 0$ it is not always true that there is an equivalent norm $\|\cdot\|'$ in which $\|T(t)x\|' \geq \|x\|'$ for all $t \geq 0$ and $x \in X$ (see Example 3.2.11). So we cannot proceed as in Theorem 4.7.2 to obtain a version of Theorem 4.7.3 with expansiveness of T replaced by

$$\|T(t)x\| \geq c\|x\| \quad (t \geq 0, x \in X).$$

However expansiveness also appears in Theorem 3.2.2 which states that it is equivalent to the existence of an extending C_0-group S on a larger Banach space with $\|S(-t)\| \leq 1$ for all $t \geq 0$. The existence of an equivalent norm can be combined with this result and the more general version of Badea and Müller (Theorem 3.2.4) to get the following version of Theorem 4.7.3. Recall that $D(A^\infty) = \bigcap_{n \geq 0} D(A^n)$.

Theorem 4.7.4. *Let A be the generator of a C_0-semigroup T on a Banach space X, and $M \geq 1$. The following are equivalent.*

1. *There is an isometric embedding $X \subset Y$ and a C_0-group S on Y such that $S(t)x = T(t)x$ for all $t \geq 0$ and $x \in X$, $\|S(t)\| = \|T(t)\|$ for all $t \geq 0$ and $\|S(-t)\| \leq M$ for all $t \geq 0$.*

2. *We have that*

$$\|x\| \leq M \sum_{i=0}^{n-1} \|x_i\| \tag{4.27}$$

whenever

$$\mu^{-n}(\mu - A)^n x = \sum_{i=0}^{n-1} \mu^{-i}(\mu - A)^i x_i \tag{4.28}$$

for some $\mu < 0, n \geq 1$ and $x, x_i \in D(A^\infty)$.

Proof. Assume that T has an extension S as described in 1. Let B be the generator of S which contains $(-\infty, 0)$ in its resolvent set and satisfies $\|\lambda^n R(\lambda, B)^n\| \leq M$ for $\lambda < 0$. If there are elements $x, x_i \in D(A^\infty)$ and $\mu < 0$ such that (4.28) holds, we get

$$\|x\| = \|\mu^n R(\mu, B)^n \mu^{-n}(\mu - A)^n x\| \leq \sum_{i=0}^{n-1} \|\mu^{n-i} R(\mu, B)^{n-i}\| \|x_i\|$$

which shows that 2 holds.

Now assume that 2 holds. Let $|||\cdot|||_{-A}$ be the accretive seminorm for A from Definition 4.2.2. We will show first that (4.27) implies the equivalence of $|||\cdot|||_{-A}$ and the norm $\|\cdot\|$. In the definition of $|||x|||_{-A}$, we take an infimum over sums $\sum_{i=0}^{n} \|x_i\|$

for elements $x_i \in X$ which satisfy $P(x, x_i, \mu, n)$ from page 104 when formulated for $-A$. Note that this is the same as saying $P(x, x_i, -\mu, n)$ holds for A.

For $x \in X$, assume such x_i are given. The following can be found in [Dav80, pp. 28–29]. For $f \in C_c^\infty(0, \infty)$, define

$$T_f x = \int_0^\infty f(t)T(t)x \, \mathrm{d}t \quad (x \in X).$$

Then $T_f x \in D(A^\infty)$, T_f commutes with $R(\lambda, A)$ and A and there are f_k as above such that $T_{f_k} x \to x$ as $k \to \infty$ for all $x \in X$. If we take $y_k = T_{f_k} x$ and $y_k^i = T_{f_k} x_i$ we get by our assumption that

$$\|y_k\| \leq M \sum_{i=0}^n \|y_k^i\|.$$

Taking the limit for $k \to \infty$ in this inequality gives

$$\|x\| \leq M \sum_{i=0}^n \|x_i\|$$

which shows that $\|x\| \leq M |||x|||_{-A}$. This and Theorem 4.2.6 show that

$$|||x|||_{-A} \leq \|x\| \leq M |||x|||_{-A}$$

for all $x \in X$, and that A is $|||\cdot|||_{-A}$-accretive.

By equivalence of the norms, A also generates T on $(X, |||\cdot|||_{-A})$. From Theorem 4.7.3 we see that the C_0-semigroup T generated by A satisfies

$$|||T(t)x|||_{-A} \geq |||x|||_{-A} \quad (x \in X, t \geq 0).$$

So T satisfies the assumptions of Theorem 3.2.2, and we have an isometric embedding of $(X, |||\cdot|||_{-A})$ into some Banach space $\tilde{Y}$. On $\tilde{Y}$, there is a C_0-group $\tilde{S}$ with

$$\|\tilde{S}(-t)\| \leq 1 \quad (t \geq 0).$$

We will use this to show that T satisfies the assumptions of Theorem 3.2.4. Assume that $T(t_n)x = \sum_{i=1}^n T(t_i)x_i$ for $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$ and $x, x_i \in X$. Then

$$\begin{aligned}
\|x\| &\leq M |||x|||_{-A} \\
&= M \|\tilde{S}(-t_n)T(t_n)x\|_{\tilde{Y}} \\
&\leq M \sum_{i=1}^n \|\tilde{S}(t_i - t_n)\| |||x_i|||_{-A} \\
&\leq M \sum_{i=1}^n \|x_i\|
\end{aligned}$$

since $\tilde{S}(t)$ is contractive for negative t. We can apply Theorem 3.2.4 with the same constant M and get an extending C_0-group S on a Banach space Y which satisfies the required properties. $\square$

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