Utility Maximisation: Non-Concave Utility and Non-Linear Expectation

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Abstract

Since the birth of mathematical finance, portfolio selection has been one of the topics which have attracted a lot of interest, with models formulated in discrete and continuous time and developed in complete and incomplete markets. In conventional or neoclassical finance, many models are based off the assumption that agents make decisions by maximising their expected utility. Deviations between models and market observations have generated a recent field of study, behavioural finance, which incorporates psychology, sociology and finance together to resolve observed phenomenon like bubbles which conventional finance cannot explain.

In this thesis, we will be restricting ourselves to the complete continuous market and look at a new formulation of expected utility maximisation with behavioural finance elements incorporated into it, namely S-shaped utilities and probability distortions. We consider the three general cases of expected utility maximisation: utility from terminal wealth, utility from consumption and utility from terminal wealth and consumption. We shall review the neoclassical problems and then explore the cases with behavioural elements installed.

Key Words Portfolio Selection, continuous time, martingale approach, S-shaped function, probability distortion, cumulative prospect theory

Contents

Chapter 1

Background

1.1 Introduction

Finance is essentially the science of funds management [1] though specifically we will be dealing with mathematical finance which, to this date, can be portrayed by two distinct divisions, namely, neoclassical finance and behavioural finance [2]. The former of these two disciplines revolves around the assumption that all agents are rational wealth maximisers, spawning topics such as mean-variance portfolio selection and expected utility maximisation [3]. The latter field of study addresses the inconsistency between the financial theoretical framework and practice. This branch incorporates psychology and sociology with finance [4] to explain the deviation of human judgements and the process of decision making in the real world.

The financial sciences have undergone revolutionary changes during the past few decades due to the proliferation of high speed computers and the synergy of stochastic models with modern financial theory. However, despite the meteoric rise in research interests, behavioural finance has not been maintaining a parallel pace in its progression compared to its counterpart: indeed, some may describe it as being at an infant stage of development only gaining recognition in its own right since the 1980s [5].

Studies of neoclassical mathematical finance began in approximately the early 1950s, and since then, there have been many major breakthroughs within this field such as the Black Scholes Option Pricing model [2]. On the other hand, behavioural finance, as we have mentioned, is still a relatively new field arising from the need to explain the observations of regularly occurring anomalies such as market bubbles and crashes which conventional economic theory cannot account for [6]. Consequently, it is no surprise that research has been largely limited to a qualitative and empirical nature, differing from the neoclassical case, which basks in the range of mathematical and quantitative techniques available.

The limited mathematical treatment of behavioural finance is not due to the lack of interest, but rather, extensions of existing theories cannot be translated from the neoclassical to the behavioural sense because the known mathematical techniques break down, thus addressing the need for new unconventional approaches and the formulation of new models [3].

We shall delve into the notion that people have to make decisions under uncertain conditions. In this dissertation, we will be focusing exclusively on expect utility maximisation (EUM), incorporating various features of behavioural finance. Hence, we will present some general background information about expected utility theory and its inadequacies and thusly, outline the necessity for integrating psychological factors into the model.

1.2 Utility and Expected Utility Theory (EUT)

Utility

Within neoclassical finance, EUM is by far, one of the predominant investment decision rules in financial portfolio selection [7]. The concept of utility, in economical terms, can be described as an agents measure of relative satisfaction and is often modelled to be affected by the consumption and possession of wealth [8]. Furthermore, utility can be characterized by cardinal and ordinal utility which are approximate analogues to the notion of absolute and relative quantities. Cardinal utility captures the magnitude of utility differences whilst ordinal utility describes a way to define an order and not the strength of preferences, thus it essentially gives the ranking [9].

First Use of Expected Utility Model

Expected utility theory was developed to capture the idea of decision making under risk and uncertainty. The expected utility model was first proposed by Nicholas Bernoulli in 1713 [10] but was only formally elaborated in 1738 by Daniel Bernoulli in order to solve the famous problem, the St. Petersburg paradox [11]. Bernoulli resolved this problem by arguing that when agents are faced with decisions under uncertainty, assuming that they displayed risk aversion, tend to maximise the value of a logarithmic cardinal utility function rather than maximising their expected monetary payoff [12].

EUT Assumptions

In 1944, von Neumann and Morgenstern formally developed EUT based on an axiomatic system in their formulation of game theory [13] and are contingent on the assumption that all agents are rational. There are also several other key assumptions [7]:

1. They evaluate wealth according to final asset positions;
2. They are uniformly risk averse when faced with decisions under uncertainty;
3. They are able to evaluate probabilities objectively.

1.2.1 The von Neumann-Morgenstern Axioms

Before we present the axioms, we must firstly define some terms and notations. Resuming back to the idea of utility as briefly touched upon earlier, we see that it makes much more sense to rank utilities rather than adding them together. For example, one may think that A is preferable to B, however, one cannot quantify it exactly as, say A being fifty times preferable to B. This observation is what economists call the "law of diminishing returns" and hence, it is difficult to compare the utility of the A with fifty times the utility of the B. The way to get around this is to consider the probabilities. A lottery, prospect or gamble, is a finite set of outcomes each assigned with a probability of it occuring [14]. If the agent can choose between these various lotteries, then it is possible to additively compare A and B. In our case, we can now compare B with probability 1, to A with probability p or nothing with probability $1 - p$. By adjusting p, the point at which the B becomes preferable defines the ratio of the utilities of the two options.

Formally, a lottery L is defined as follows[15]: if options W and Z have probabilities p and $1 - p$ respectively in the L, then we can express it as a linear combination:

$$L = pW + (1 - p)Z$$

In a more general case where a lottery L has n possible options Z_i each with probability p_i of occuring, we can write:

$$L = \sum_{i=1}^{n} p_i Z_i \tag{1.1}$$

The notation $A \preceq B$ means B is preferred to A and $A \sim B$ means that the agent is indifferent between A and B. If agents can choose between lotteries, then she will have what is known as a utility function, which can be added and multiplied by real numbers. Hence the utility of an arbitrary lottery can be calculated as a linear combination of the utility of its constituents. We shall discuss the utility function $u(.)$ in the next subsection.

$$u_1(L) = \sum_{i=1}^{n} p_i u(Z_i) \tag{1.2}$$

In EUT, a utility function exists for an agent, who is deemed a rational decision maker, if she satisfies the following four axioms [13]:

1. **Completeness**
   For any two simple lotteries L and M, either $L \preceq M$ or $M \preceq L$ or $L \sim M$;

2. **Transitivity**
   For any three lotteries L, M, N, if $L \preceq M$ and $M \preceq N$, then $L \preceq N$;

3. **Convexity/Continuity**
   If $L \preceq M \preceq N$, then $\exists p \in [0, 1]$ such that the lottery $pL + (1 - p)N$ is equally preferable to M;

4. **Independence**
   For any three lotteries L, M, N, $L \preceq M$ if and only if $pL + (1-p)N \preceq pM + (1 - p)N$

1.2.2 Utility Function

In the von Neumann and Morgenstern expected utility formulation [13], the concept of utility can be mathematically captured in the form of a utility function. Let $x_i \in \Re$ be an outcome and $\mathbf{x} = \Re$ be a set of possible outcomes. An outcome x_i, is defined as the result of an event, A_i. The probability event A_i occuring, with an outcome x_i, is given by p_i. $u : \mathbf{x} \to [-\infty, +\infty)$ denotes a utility function such that the value of $u(x)$ is a measure of the agents preferences derived from the outcome x. Now,

$$x \succeq y \iff u(x) \geq u(y) \tag{1.3}$$

This represents an important property of a utility function; order preservation. Furthermore, we note that the utility function essentially derives a ranking, the actual magnitude is meaningless. In EUT, agents are modelled by the value of utility functions defined on final asset positions. If x and y represent wealth, then (1.3) clearly shows the monotonicity of the utility function, illustrating increasing utility which agents can gain from increasing wealth. Most utility functions used in theory are generally well-behaved. There exists preferences which do not possess the well-behaved properties rendering difficulty in analysis. Lexicographic preferences, for example, cannot be represented by a continuous utility function [17], but we shall only consider the case where utility functions exist, are continuous and increasing since agents are assumed to prefer more wealth to less. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the set of outcomes of a prospect or lottery and $\mathbf{p} = (p_1, \ldots, p_n)$ be the corresponding probabilities p_i associated with each outcome x_i, and furthermore, we condense this lottery by $(p_1, x_1; \ldots; p_n, x_n)$ or $(\mathbf{x}, \mathbf{p})$,

$$\begin{aligned} E[u(\mathbf{x})] = V(\mathbf{x}, \mathbf{p}) &= p_1 u(x_1) + p_2 u(x_2) + \ldots + p_n u(x_n) \\ &= \sum_{i=1}^{n} p_i u(x_i) \end{aligned} \tag{1.4}$$

Concavity Property

One of the key assumptions of EUT is that agents are uniformly risk averse. The effect of this assumption implicates the concavity property for the utility function. Expressing this mathematically, $\forall 0 \leq \alpha \leq 1, x, y \in \mathbf{x}$, the inequality,

$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$$

is satisfied.

People naturally differ in their degree of risk adversity. The shape of the utility function captures this aspect as well as other interesting features. Due to the curves concave nature, we get what economists coin as the “law of diminishing marginal utility”. Assuming the derivative of the utility function exists and is defined, concavity implies a decreasing derivative function, hence when the value of x is small, we get a large gradient and when x is relatively large, we have a small gradient [13].

Let us illustrate this with an example. Consider an agent who currently has \$0. The satisfaction attributed to the additional marginal utility gained by an increase of \$1 would obviously have a greater effect than a scenario when she had \$1,000,000 to begin with.

Examples of utility functions include:

Exponential Utility

$$u(x) = 1 - \exp(-\gamma x)$$

with $x \in \Re$ denoting the agents wealth and $\gamma > 0$ a constant

Power Utility

$$u(x) = \begin{cases} \frac{1}{\gamma}x^\gamma & x \geq 0 \\ -\infty & x < 0 \end{cases}$$

with $\gamma < 1$, $\gamma \neq 0$

Logarithmic Utility

$$u(x) = \begin{cases} \log(x) & x \geq 0 \\ -\infty & x < 0 \end{cases}$$

1.3 Behavioural Elements

1.3.1 Violations of Expected Utility Theory

The axioms proposed in von Neumann and Morgensterns expect utility theory has long been criticised to be inconsistent with the way people do decision making in the real world. Repeated observations and a substantial amount of empirical evidence have suggested a systematic violation of the assumptions which EUT is premised upon. This topic has been elaborated in great detail in a range of academic literatures and published works, for example, "The Journal of Economic Methodology - Discovered preferences and the experimental evidence of violations of expected utility theory"[18].

Paradoxes

Consider the following observation which clearly demonstrates the invalidity of the EUT independence axiom. This experiment is known as the Allais Paradox, which was formulated by the French economist Maurice Allais to demonstrate the inconsistency of EUT and the real world [19].

In this test, we compare the participants choices in two different experiments, each of which consists of a choice between two gambles which we shall call A and B.

Experiment 1: Choose between
Gamble 1A: winning \$1 million with a probability of 100%
Gamble 1B: winning \$1 million with a probability of 89%, \$0 with a probability of 1% and \$5 million with a probability of 10%

Experiment 2: Choose between
Gamble 2A: winning \$0 with a probability of 89%, \$1 million with a probability of 11%
Gamble 2B: winning \$0 with a probability of 90%, \$5 million with a probability of 10%

It has been observed that most of the agents outcomes of these two experiments are choices 1A and 2B for experiments 1 and 2 respectively [20]. If the same agent undergoes the two experiments and chooses 1A and 2B (which is empirically demonstrated to be highly likely), then this observation serves as a counterexample to the expected utility model, which asserts that she should choose either 1A and 2A or 1B and 2B.

We use (1.4) to express these agents choices in mathematical terms, yielding,

Experiment 1 Choosing 1A over 1B implies:
$1.00u(\$1million) > 0.89u(\$1million) + 0.01u(\$0million) + 0.1u(\$5million)$
$\implies 0.11u(\$1million) > 0.01u(\$0million) + 0.1u(\$5million)$

Experiment 2 Choosing 2B over 2A implies:
$0.89u(\$0million) + 0.11u(\$1million) < 0.9u(\$0million) + 0.1u(\$5million)$
$\implies 0.11u(\$1million) < 0.01u(\$0million) + 0.1u(\$5million)$

Hence, under EUT principles, we arrive at a contradiction and this observation is known as the "common consequence effect" [21].

This is just one of many paradoxes which arise by assuming the axioms of EUT [3]. Other famous ones include the Ellsberg paradox [22], Friedman and Savage puzzle [23] and the Equity Premium puzzle [24].

Observations

The paradoxes serve as counterexamples to EUT as it illustrates how people in the real world do not generally make decisions the way the EUT framework assumed. These inconsistencies appear because of the following behavioural traits which agents generally display in real life.

1. **Risk Adverse and Risk Loving Attitudes**
It was assumed that agents are uniformly risk averse when faced with decisions under uncertainty. Empirical evidence suggests that agents have a different attitude towards risk depending on whether a gain or a loss is involved [25], so this axiom needs to be addressed.

The fourfold pattern of risk attitudes is a term which summarises the rules of decision making [26]: Agents display risk-averse behaviour in gains involving moderate probabilities and of small probability losses and display risk-seeking behaviour in losses involving moderate probabilities and of small probability gains. This observation explains the success of societys insurance and lottery industries.

Attempts have been made to incorporate this fourfold pattern feature in terms of a utility function which contains concave and convex regions, representing areas of risk adversity and risk loving respectively [27]. In Tversky and Wakkers paper on Risk attitudes and Decision Weights, a proposition was made for a non linear transformation of the probability scale [28].

2. **Non-Linear Preferences**
Observe the difference which one may perceive when knowing that men have a 1% chance of contracting disease X, whilst women have a 2% chance. We may perceive the risk for men as twice the risk for women. However, this 1% difference has less of an impact if the chances of contracting disease X for men is, say, 80% and for women 81%. Hence it is with this that we note that preferences between risky prospects are not linear in probabilities.

This effect again strengthens the need for a probability distortion as suggested by Tversky and Wakker. Kahneman and Tversky have elaborated on this observation as "losses loom larger than gains"[29] to illustrate the tendency of agents to overweigh small probabilities and underweigh moderate and large probabilities [30].

3. **Asymmetric Attitude towards loss and gains**
Since we have now established the various risk loving and risk adverse attitudes from [25], we can now elaborate on the impact which gains and losses induce in the agents psychologically. "Losses loom larger than gains" suggests that losses pack a much bigger impact than what is perceived by an agent than gains. We may see this effect in action in what is coined as the "disposition effect". This refers to the pattern that agents tend to avoid realizing paper losses and seek to realize paper gains. For example, if someone

buys a stock at $50 then it drops to $40 before rising to $45, most people do not want to sell the stock until it exceeds above $50 [6]. Empirical evidence [31] is also available, backing up this third observation.

EUT Assumptions Revisited

Further evidence [32] proposes that contrary to EUT, the final state of wealth if not always significant but what is important is the notion of a "reference point" which is derived from the selection provided and the agents personal expectations. This reference point is critical in that it separates the gains from the losses which, as we have seen, determines the convexity or concavity of the utility function [20]. This idea plays a main part in the formulation of Prospect Theory which we shall discuss in the next section.

Let us conclude by comparing the original key assumptions outlined in the previous section. Agents are assumed to be rational, evidently this is not always the case as some will give into their emotions. Furthermore, the key assumptions may be corrected to be [7]:

1. They evaluate assets on gains and losses which are defined with respect to a reference point, not on final wealth positions;

2. They are not uniformly risk averse: they are risk averse on gains and risk loving on losses, and significantly more sensitive to losses than gains;

3. They overweigh small probabilities and under weigh large probabilities.

With all these observations and paradoxes in mind, it is clear that if a more accurate representation is required, then a new model, which incorporates these psychological components, is needed.

1.3.2 Prospect Theory & Cumulative Prospect Theory

Prospect Theory

The discrepancies of EUT ultimately called for extensions which should attempt to accommodate all the psychological characteristics which were not previously accounted for. Experimental data and general intuition was the main input used by many in an attempt to formulate more adequate models [34]. PT was one of the major breakthroughs and was developed by Daniel Kahneman and Amos Tversky in 1979 to provide a more psychologically realistic alternative. The model described in the original paper provided a descriptive and normative way of capturing peoples behaviours under uncertainty [33].

In the paper, the choices presented to agents where decisions have to be carried out are referred to as prospects represented mathematically as $(\mathbf{x}, \mathbf{p})$, where the outcome x_i is associated with the probability p_i and $\sum p_i = 1$.

PT characterises decision making as a two stage process; editing and evaluation. In the editing stage all the possible outcomes of the decision are ordered following some heuristic. In the evaluation stage, we introduce a concept similar to the utility function, known as the "value function". In this stage, people "act" as though they were computing the value function, based on the potential outcomes and their respective probabilities and then select the alternative having a higher utility.

Define the prospect $(\mathbf{x}, \mathbf{p})$ as in the previous section. The value function is given by the following formula:

$$V(\mathbf{x}, \mathbf{p}) = w(p_1)v(x_1) + w(p_2)v(x_2) + \ldots + w(p_n)v(x_n) = \sum_{i=1}^{n} w(p_i)v(x_i) \tag{1.5}$$

where $x_1, x_2, \ldots$ are the potential outcomes and $p_1, p_2, \ldots$ their associated probabilities.

We notice a striking resemblance between PT and EUT. The key differences are:

1. The value function described in PT (1.5) is analogous to the utility function used in EUT (1.4) but whilst the utility function is defined on final asset positions, the value function considers how much the asset position deviates from the reference point. This phenomenon is known as the framing effect.

2. The utility function is concave throughout but the value function has concave and convex regions, separated by the reference point to take into account different attitudes to risk. The shape of the value curve has consequently earned the description "S-shaped function", a term which we may refer to throughout this dissertation.

3. The idea of probability distortion is introduced here via the use of probability weighting functions $w(p)$. These are employed in the formula for the value function in (1.5) in place of normal probabilities p

as used in the utility function in (1.4). The probability distortion is a monotonic transformation and captures the notion of the distinction between subjective and objective probabilities.

A reference point must be defined in PT, but it has been discussed in the paper "Behavioural Portfolio Selection in Continuous Time" [7] that there is a natural outcome or benchmark, assumed to be zero which serves are a base point to distinguish between gains from losses. Hence, for simplicity, zero shall be our reference point, implying that positive values of x are considered gains and negative values are considered losses. Like the utility function, we shall restrict ourselves to continuous and strictly increasing value functions in this dissertation [20].

The main success of PT is its ability to explain the real life observations which EUT cannot; however, there are some theoretical issues with this theory [35]:

1. It gave rise to violations of first-order stochastic dominance
2. It is not compatible with prospects with a large number of outcomes
3. Source preference - the source of uncertainty is not distinguished

The stochastic dominance property in point (1) is a form of stochastic ordering. A prospect A is said to have first-order stochastic dominance over another prospect B if for any outcome x, A gives at least as high a probability of receiving at least x as does B, and for some value x, A gives a higher probability of receiving at least x [36].

This can be expressed in mathematical notion: if $P(A \geq x) \geq P(B \geq x) \forall x$ and for some x, $P(A \geq x) > P(B \geq x)$. In terms of cumulative distribution functions of the two gambles, A dominating B means that $F_A(x) \leq F_B(x) \forall x$ with strict inequality at some x.

Since PT does not always satisfy the stochastic dominance then it could be the case that prospect A might be preferred to prospect B even though the probability of receiving x or greater is at least as high under prospect B as it is under prospect A $\forall x$, and is greater for some value of x.

Point (3) refers to the notion of source preference. Observations demonstrate that choices between prospects depend not only on the degree of uncertainty but also on the source of uncertainty. Source preference is demonstrated when an agent prefers to bet on a proposition drawn from one source than a proposition drawn from another source[37], like they may prefer to make decisions based on their own judgement.

Cumulative Prospect Theory (CPT)

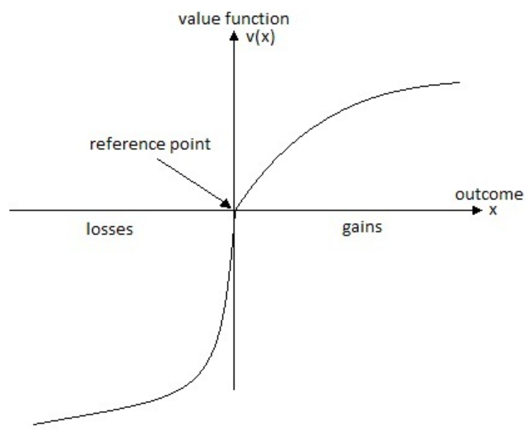
PT has made a good attempt at addressing some of the issues present in EUT but as we have just seen, it is not without flaws. Consequently, many others have proposed more generalised models for choice under uncertainty, for example, the Rank-Dependent Expected Utility Model (RDEUM), originally called Anticipated Utility Model is a generalised expected utility model for decision making under uncertainty [38]. It was formulated by Quiggin for the purpose of explaining the behaviour observed in the Allais paradox mentioned previously, as well as for the observation that people buy lottery and insurance.

The crucial idea of RDEUM was to apply probability distortions to the cumulative probabilities as opposed to the individual probabilities in order to include non linear preferences. This solved the problem of the violation of stochastic dominance as encountered in PT [39]. The authors of PT, Tversky and Kahneman took onboard this idea and developed it further to CPT.

CPT addresses all the issues faced in PT: it satisfies stochastic dominance, be applied to any uncertain prospects with an unlimited number of outcomes and accommodates the notion of source dependence. A more comprehensive discussion of these properties can be found in [25].

1.3.3 Value Function & Probability Distortion

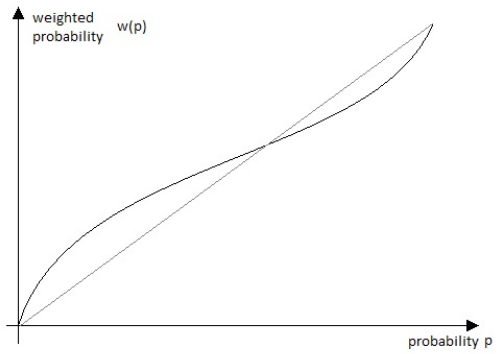
S-Shaped Value Function



One of the key features of CPT is the value function which takes into account the agents different attitudes towards risk. The value function is concave for gains and convex for losses illustrating the agents risk averse and risk loving behaviour for gains and losses respectively.

The notion of loss aversion is also incorporated into the value function. Notice the asymmetry of the shape: mathematically speaking the absolute value for $v(x)$ for $x \geq 0$ is less than the absolute value $v(x)$ for $x < 0$. As with the utility function, the value function also exhibits the law of diminishing marginal returns. Also notice the reference point in wealth that defines gains and losses.

Non-Linear Probability Distortions



Another one of the key elements of Kahneman and Tverskys CPT is a probability distortion that is a non linear transformation of the probability scale, which enlarges a small probability and diminishes a large probability.

The probability distortion or decision weighting function is denoted by $w(.)$ enabling the overweighing and underweighing of probabilities and maps [0,1] onto [0,1]. Notice that $w(0) = 0$ and $w(1) = 1$ which is intuitively consistent because it does not make sense to distort events which are certain to occur or won't occur.

To illustrate how this is utilised, we first consider expected utility for a random variable $X \geq 0$, which by definition of the conventional expected value is given by,

$$E(u(X)) = \int_0^{+\infty} u(x)\,\mathrm{d}F_X(x) \tag{1.6}$$

where X is a random payoff with $F_X(.)$ as the cumulative distribution function (CDF) and $u(.)$ is a utility function. We adopt a difference in notation as we are considering the continuous case. This expression shows that $u(.)$ can be regarded as a non linear distortion on payment when evaluating the mean of X. The following equality for the value function has been proposed in [43] when $X \geq 0$:

$$\alpha(X) = \int_0^{+\infty} w(P(X > x))\, \mathrm{d}x$$

where $w(.)$ is the probability distortion. This equation involves the use of Choquet integrals with respect to the capacity $w \circ P$. [42] Using Fubini's Theorem, we may rewrite this equation as:

$$\alpha(X) = \int_0^{+\infty} x\, \mathrm{d}(-w(1 - F_X(x)))$$

This demonstrates that $\alpha(X)$ involves a distortion on the CDF. Furthermore,

$$\alpha(X) = \int_0^{+\infty} xw'(1 - F_X(x))\, \mathrm{d}F_X(x) \tag{1.7}$$

If we compare this formula to the equation of expected utility in (1.6), we see that the term $w'(1 - F_X(x))$ puts a weight on the payment x. This equation captures the agents subjective behaviour as well because if $w(.)$ is convex, the value of $w'(p)$ is greater around 1 than around $p = 0$ and vice versa, thus encapsulating the idea of the risk attitudes observed in real life [3].

1.3.4 Continuous Time Behavioural Portfolio Selection

The CPT portfolio choice model incorporates psychological features at the expense of introducing the following technical components: reference points, value functions and probability distortions. Since then, behavioural portfolio choice has attracted a great deal of research interests, however, so far a plethora of studies have been largely confined to the single period setting such as the works of [40] and [41] which place greater emphasis on qualitative properties [7]. Despite the significant rise in this field of study, there have been limited advances on the continuous time behavioural portfolio choice problem which is the setting that we shall we examining.

In order to progress further, let us backtrack a bit and consider the case of EUM. We shall only concern ourselves in the continuous time portfolio choice case in which essentially there are two main approaches:

1. Stochastic Control and Dynamic Programming approach
2. Martingale or Duality approach

In EUT it was assumed that agents are uniformly risk adverse implicating global concavity of the utility function. We have argued for an S-shaped function which contains convex and concave regions however, global convexity/concavity is a necessary condition for traditional optimisation techniques hence we cannot simply employ them here.

Another problem arises from the non-linear probability distortions as proposed in CPT. The nice properties associated with the normal additive probability and linear expectation are no longer applicable. Let us illustrate this by considering a non-linear expectation operator $\tilde{E}(.)$ and two random variables X and Y, then:

$$\tilde{E}(X+Y) \neq \tilde{E}(X) + \tilde{E}(Y)$$

Furthermore, the dynamic consistency of the conditional expectation with respect to a filtration, which is the foundation of the dynamic programming principle, is absent due to the distorted probability [3]. Combining these difficulties together makes the problem even more difficult to tackle.

Jin and Zhou's paper on "Behavioural Portfolio Selection in Continuous Time" [7] established an innovative approach in tackling the continuous time CPT model, taking into account the non-globality property of the value function and the probability distortions. Their approach can be broken down into the following steps:

1. The S-shaped value function is addressed by decomposing the problem, by parameterising some key variables, into a gain part problem and a loss part problem.
2. The gain part problem is a Choquet maximisation problem involving a concave utility function and a probability distortion. The difficulty arising from the distortion is overcome by a technique known as quantile formulation which changes the decision variable from the random variable X to its quantile function $G(.)$.
3. The loss part problem is solved by noting that the existence of corner-point solutions, which are step functions in a function space.

Chapter 2

The Model

2.1 Neoclassical Elements

We will be considering the model as outlined in Karatzas and Shreve [44] where $n + 1$ financial assets are being traded in the market and the time horizon T is assumed to be finite. The asset price processes will be assumed to be It$\hat{o}$ processes on a filtered probability space $(\Omega, F, P, \{F_t\}_{t\geq 0})$ which is defined on a standard F_t-adapted n-dimensional Brownian motion $W(t) = (W^1(t), W^2(t), \ldots, W^n(t))'$ with $W^i(t) = (W^i(t))_{0\leq t\leq T}$ for $i = 1, \ldots, d$ and $W(0) = 0$. The prime in the $W(t)$ equation denotes a transposition.

We will also be working in the continuous, complete market setting so $n + 1$ assets are being traded continuously. One of the $n+1$ assets, is called a bank account, which we shall denote $S_0(t)$. It is riskless and its price process is as follows:

$$dS_0(t) = r(t)S_0(t)dt, t \in [0, T]; S_0(0) = s_0 > 0 \tag{2.1}$$

Here, $r(.)$ is an F_t-progressively measurable, scalar valued stochastic process and is known as the interest rate. It satisfies the condition $\int_0^T |r(t)|\, \mathrm{d}t < \infty$ a.s.. The other n assets are called stocks, which we shall denote $S_i(t)$. They are risky and their price processes satisfy the following equation:

$$dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dW(t)\right), i = 1, \ldots, n \tag{2.2}$$

We can further simplify this equation further using vectors and condensing it into the following form,

$$dS(t) = diag(S(t))[\mu(t)dt + \sigma(t)dW(t)] \tag{2.3}$$

where diag(S) denotes an nxn diagonal matrix with $S_1, S_2, \ldots, S_n$ along the diagonal. $\mu_i(.)$ and $\sigma_{ij}(.)$ are known as the appreciation and dispersion (or volatility) rates respectively. They are F_t-progressively measurable stochastic processes which satisfy:

$$\int_0^T ||\mu(t)|| \, \mathrm{d}t < \infty, \sum_{i=1}^n \sum_{j=1}^n \int_0^T (\sigma_{ij}(t))^2 \, \mathrm{d}t < \infty, a.s. \tag{2.4}$$

where $||.||$ denotes the Euclidean norm.

The agent is assumed to be a "small investor", which means that his actions do not influence the market prices. She begins with an initial endowment x and can consume while investing in the above market settings in the horizon $[0, T]$, so she may choose a consumption $c(t)$ and portfolio process $\pi(t)$.

A consumption process $c(t)$ is a non-negative F_t-adapted process which satisfies the condition $\int_0^T c(t) \, \mathrm{d}t < \infty$ a.s.. We shall denote $X(t) = (X)_{0 \le t \le T}(t)$ as the agents wealth process and also define processes for the number of riskless and risky assets she buys as $H_0(t)$ and $H_i(t)$ respectively,

$$H_0(t) = \frac{1}{S_0(t)} \left(X(t) - \sum_{i=1}^n \pi_i(t) \right)$$

$$H_i(t) = \frac{\pi_i(t)}{S_i(t)}, i = 1, \ldots, n, 0 \le t \le T \tag{2.5}$$

The wealth process $X(t)$, is deduced to be in the following stochastic differential equation form:

$$\begin{aligned} dX(t) &= H_0(t) dS_0(t) + \sum_{i=1}^n H_i(t) dS_i(t) - c(t) dt \\ &= r(t) \left(X(t) - \sum_{i=1}^n \pi_i(t) \right) dt + \sum_{i=1}^n \pi_i(t) \left(\mu_i(t) dt + \sum_{j=1}^n \sigma_{ij}(t) dW_j(t) \right) - c(t) dt \\ &= (r(t) X(t) - c(t)) dt + \pi(t)' [(\mu(t) - r(t) \mathbf{1}_n) dt + \sigma(t) dW(t)], \quad (2.6) \end{aligned}$$

where $X(0) = x$ and $\mathbf{1}_n$ denotes an n-dimensional vector of 1s. We made use of (2.1) and (2.2) to get from the first equality to the second equality and simple rearranging gives (2.6).

The discount factor is defined to be

$$\zeta(t) = \frac{1}{S_0(t)} = \exp(-\int_0^t r(s) \, \mathrm{d}s), 0 \le t \le T$$

Therefore, using Itô's lemma,

$$d\zeta(t) = -r(t)\zeta(t)dt \tag{2.7}$$

Using the stochastic differential equations (2.6) and (2.7), we can write $d(\zeta(t)X(t))$ as,

$$\begin{aligned} d(\zeta(t)X(t)) &= \zeta(t)dX(t) + X(t)d\zeta(t) + d < \zeta, Z >_t \\ &= \zeta(t)\pi'(t)[(\mu(t) - r(t)\mathbf{1}_n)dt + \sigma(t)dW(t)] - \zeta(t)c(t)dt \end{aligned}$$

Hence, we may now write,

$$\zeta(t)X(t) = x + \int_0^t \zeta(s)\pi'(s)[(\mu(s) - r(s))\mathbf{1}_n ds + \sigma(s)dW(s)] - \int_0^t \zeta(s)c(s)\,\mathrm{d}s, 0 \le t \le T \tag{2.8}$$

For a wealth process $X(t)$, and a given intial endowment x, portfolio process π and consumption process c, for notation simplicity, we can write $X(t) = X^{x,\pi,c}(t)$.

Definition 2.1

1. The portfolio process $\pi = (\pi_1, \dots, \pi_n)'$ is also said to be admissible if it is an $\Re^n$-valued, F_T-progressively measurable process that satisfies the condition,

$$\int_0^T (||\pi(t)'\sigma(t)||^2 + |\pi'(\mu(t) - r(t)\mathbf{1}_n)|)\,\mathrm{d}t < \infty, a.s. \tag{2.9}$$

2. An admissible process $\pi(.)$ is said to be tame if the discounted wealth process $X(t)/S_0(t)$ is almost surely bounded from below.

We denote $A(x)$ to be the set containing admissible consumption-portfolio pairs. Consider $\varepsilon(.)$, the stochastic exponential, where for any process Y, we have the following properties,

$$\begin{aligned} \varepsilon(Y) &= \exp(-Y - \frac{1}{2} < Y >) \\ (\lambda' \cdot W)(t) &= \int_0^t \lambda'(s)\,\mathrm{d}W(s), 0 \le t \le T \end{aligned}$$

Therefore, using the definitions, we have,

$$\varepsilon(-\lambda' \cdot W)(t) = \exp\left(-\int_0^t \lambda'(s)\,\mathrm{d}W(s) - \frac{1}{2}\int_0^t ||\lambda(s)||^2\,\mathrm{d}s\right), 0 \le t \le T \tag{2.10}$$

Theorem 2.1_(Karatzas & Shreve 1998 [44] Th. 1.4.2)_
1. If a continuous Itô process market (2.1) and (2.2) is arbitrage-free, then there exists an $\Re^n$-valued progressively measurable process λ, the market price of risk process, such that the following equations relating λ to the risk premium $\mu - r\mathbf{1}_n$ admit at least one solution:

$$\mu(t) - r(t)\mathbf{1}_n = \sigma(t)\lambda(t), 0 \leq t \leq T, a.s.$$

2. Conversely, if such a process λ exists, that satisfies the above requirements as well as,

$$\int_0^T ||\lambda(t)||^2 \, \mathrm{d}t < \infty, a.s., E[\varepsilon(-\lambda')_T \cdot W] = 1$$

then the market is arbitrage free.

The conditions in the second part of this theorem are satisfied if:

$$E\left[\exp\left(\frac{1}{2}\int_0^T ||\lambda||^2 \, \mathrm{d}t\right)\right] < \infty$$

from Novikov's theorem [45].

In light of the above theorem, we will make the following assumptions:

Assumption 2.1

1. There exists $c \in \Re$ such that $\int_0^T r(s) \, \mathrm{d}s \geq c$, a.s.
2. Rank$(\sigma(t)) = n$, a.e. $t \in [0, T]$, a.s.
3. There exists an $\Re^n$-valued, uniformly bounded, F_t-progressively measurable process $\lambda(.)$ such that $\sigma(t)\lambda(t) = \mu(t) - r(t)$, a.e. $t \in [0, T]$, a.s.

Under these assumptions, there exists a unique risk neutral probability measure Q defined by

$$Z(t) = \frac{dQ}{dP}|_{F_t} = \varepsilon(-\lambda' \cdot W)(t), 0 \leq t \leq T \tag{2.11}$$

Now we define the pricing kernel or the state price density process by,

$$\begin{aligned} \rho(t) &= \zeta(t)Z(t), 0 \leq t \leq T \\ &= \exp\left\{-\int_0^t [r(s) + \frac{1}{2}||\lambda(s)||^2] \, \mathrm{d}s - \int_0^t \lambda'(s) \, \mathrm{d}W(s)\right\} \end{aligned} \tag{2.12}$$

where the second equation uses (2.10). Simplifying notation, we let

$$\begin{aligned}\rho &= \rho(T), X = X(T)\\ \bar{\rho} &= esssup\rho = sup\{a \in \Re : P\{\rho > a\} > 0\}\\ \underline{\rho} &= essinf\rho = inf\{a \in \Re : P\{\rho < a\} > 0\}\end{aligned}$$

It also follows from (2.12) that $0 < \rho < +\infty$ and $0 < E[\rho] < +\infty$. We will enforce an assumption on ρ to avoid undue technicality.

Assumption 2.2
$\rho(t)$ admits no atom, meaning that $P\{\rho(t) = a\} = 0 \; \forall a \in \Re^+$ and $\forall 0 \leq t \leq T$

Returning to $\rho(t) = \zeta(t)Z(t)$ then using Itō's lemma yields:

$$\begin{aligned}\rho(t)X(t) &= Z(t)\cdot\zeta(t)X(t)\\ d(\rho(t)X(t)) &= d(Z(t)\cdot\zeta(t)X(t))\\ &= Z(t)d(\zeta(t)X(t)) + \zeta(t)X(t)dZ(t) + d < Z, \zeta X >_t\\ &= \rho(t)(\pi'(t)\sigma(t) - X(t)\lambda'(t))dW(t) - \rho(t)c(t)dt \qquad (2.13)\end{aligned}$$

Integrating (2.13) gives,

$$\rho(t)X(t) + \int_0^t \rho(s)c(s)\,\mathrm{d}s = x + \int_0^t \rho(s)(\pi'(s)\sigma(s) - X(s)\lambda'(s))\,\mathrm{d}W(s), 0 \leq t \leq T \qquad (2.14)$$

Now, when $(c, \pi) \in A(x)$,

$$\rho(t)X(t) + \int_0^t \rho(s)c(s)\,\mathrm{d}s > 0 \qquad (2.15)$$

hence the process is a supermartingale, which means that,

$$E\left[\rho X + \int_0^T \rho(t)c(t)\,\mathrm{d}t\right] \leq x \qquad (2.16)$$

Inequality (2.16) forms our budget constraint for our optimisation problem with terminal wealth and consumption. Now, the next theorem is important for our expected utility maximisation problems which we will employ later, so now we simply state as follows:

Theorem 2.2 *(Karatzas & Shreve 1998 [44] Th. 3.3.5)*
In a complete market, where the risky assets satisfy the stochastic differential equation

$$dS(t) = diag(S(t))[\mu(t)dt + \sigma(t)dW(t)]$$

and the agent's wealth satisfies

$$\rho(t)X(t) + \int_0^t \rho(s)c(s)\,\mathrm{d}s = x + \int_0^t \rho(s)(\pi'(s)\sigma(s) - X(s)\lambda'(s))\,\mathrm{d}W(s)$$

and the above supermartingale condition Consider a contingent claim ξ, F_T-measurable random variable almost surely bounded from below such that,

$$E\left[\rho\xi + \int_0^T \rho(t)c(t)\,\mathrm{d}t\right] = x \geq 0$$

Then there exists a tame portfolio process $\pi(.)$ such that $(\pi, c) \in A(x)$ and $X = X^{x,\pi,c} = \xi$, a.s. and furthermore,

$$X(t) = \frac{1}{\rho(t)}E\left[\rho\xi + \int_t^T \rho(s)c(s)\,\mathrm{d}s|F_t\right], 0 \leq t \leq T$$

2.1.1 Utility and Dual Functions

We present a formal definition of a utility function as given in [44]:

A utility function is a concave, non-decreasing, upper semi continuous function $u : \Re \to [-\infty, \infty)$ satisfying:

1. the half-line $dom(u) = \{x \in \Re; u(x) > -\infty\}$ is a non-empty subset of $[0, \infty)$. If we define $\bar{x} = \inf\{x \in \Re : u(x) > -\infty\}$, hence $\bar{x} \in [0, \infty)$ and either $dom(u) = [\bar{x}, \infty)$ or $dom(u) = (\bar{x}, \infty)$, but we shall only consider the utility functions for which $\bar{x} = 0$.

2. u' is continuous, positive, and strictly decreasing on the interior of $dom(u)$, $u'(\infty) = \lim_{x\to\infty} u'(x) = 0$ and $u'(0) = \lim_{x\to-\infty} u'(x) = \infty$

We now introduce the convex dual or convex conjugate which we shall denote $\bar{u}(.)$, and for notational simplicity, we shall define $I(.)$ as the inverse of the marginal utility u', hence,

$$u'(I(y)) = I(u'(y)) = y, 0 < y < \infty \tag{2.17}$$

$u'(.)$ and $I(.)$ are continuous and strictly decreasing functions. The convex dual of u is the function,

$$\bar{u}(y) = \sup_{x \in dom(u)} \{u(x) - xy\}, 0 < y < \infty \tag{2.18}$$

$\bar{u}(.)$ is a convex, decreasing function and is continuously differentiable, from the above definition, it follows that,

$$\bar{u}(y) \geq u(x) - xy \tag{2.19}$$

The maximum of $u(x) - xy$ occurs ($u''(x) < 0$ due to its concave nature, implicating a maximum) when $u'(x) = y$ or equivalently $x = I(y)$, hence (2.19) achieves equality when,

$$\bar{u}(y) = u[I(y)] - yI(y) \tag{2.20}$$

Differentiating (2.20) with respect to y yields $\bar{u}'(y) = -I(y)$ for $0 < y < \infty$. Now the bidual relation,

$$u(x) = \inf_{y \in \Re^+} \{\bar{u}(y) + xy\}, x \in dom(u) \tag{2.21}$$

The minimum of $\bar{u}(y) + xy$ occurs when $y = u'(x)$. We also verify that the quantity obtained is indeed a minimum; since $u'(I(y)) = y$, it follows that $u''(I(y))I'(y) = 1$. Since $u''(.) < 0$, $I'(y) < 0$, and so $\bar{u}''(y) = -I'(y) > 0$. Therefore, we have,

$$u(x) = \bar{u}(u'(x)) + xu'(x) \tag{2.22}$$

2.1.2 Formulating the Expected Utility

Having introduced the notion of utility functions and convex duals, we now present the optimization problems which, given a utility function $u(.)$, all involve finding the optimal consumption portfolio pair $(c, \pi) \in A(x)$ where $A(x)$ is the admissible set, which yields maximum expected utility bounded by some constraint. Take note of the notation here, $u_2(.)$ denotes a utility function which takes only one argument, whilst $u_1(t, .)$ denotes another utility function which takes an additional time argument. For $u_1(t, .)$ we can define the subsistence consumption

$$\bar{c}(t) = \inf\{c \in \Re : u_1(t, .) > -\infty\}$$

and for $u_2(.)$, we remind you the subsistence terminal wealth

$$\bar{x} = \inf\{x \in \Re : u_2(x) > -\infty\}$$

Definition 2.2

1. $\Psi_1(y) = E\left[\int_0^T \rho(t) I_1(t, y\rho(t))\, \mathrm{d}t\right]$
2. $\Psi_2(y) = E\left[\rho I_2(y\rho)\right]$
3. $\Psi_3(y) = E\left[\int_0^T \rho(t) I_1(t, y\rho(t))\, \mathrm{d}t + \rho I_2(y\rho)\right], 0 < y < \infty$

Hence,

1. $\Psi_1(\infty) = E\left[\int_0^T \rho(t)\bar{c}\, \mathrm{d}t\right]$
2. $\Psi_2(\infty) = E\left[\rho\bar{x}\right]$
3. $\Psi_3(\infty) = E\left[\int_0^T \rho(t)\bar{c}\, \mathrm{d}t + \rho\bar{x}\right]$

From the properties of the pricing kernel, the above three quantities are finite. These quantities will be considered during optimisation. There are three models for the expected utility, namely:

1. **Expected Utility from Consumption Only**

$$J_1(c(.)) = E\left[\int_0^T u_1(t, c(t))\, \mathrm{d}t\right]$$

subject to the constraint

$$E\left[\int_0^T \rho(t) c(t)\, \mathrm{d}t\right] \leq x$$

2. **Expected Utilty from Terminal Wealth Only**

$$J_2(X) = E\left[u_2(X)\right]$$

where subject to the constraint

$$E\left[\rho X\right] \leq x$$

3. **Expected Utility from Consumption and Terminal Wealth**

$$J_3(c(.), X) = E\left[\int_0^T u_1(t, c(t))\, \mathrm{d}t + u_2(X)\right]$$

subject to the constraint

$$E\left[\rho X + \int_0^T \rho(t) c(t)\, \mathrm{d}t\right] \leq x$$

2.2 Behavioural Elements

2.2.1 S-Shaped Value Function and Probability Distortions

Having formulated the problems for the neoclassical cases, we now move on to the behavioural aspect. As discussed, in order to incorporate the psychological factors into the neoclassical finance model, to account for different attitudes towards risk, non linear preferences etc, we need to introduce our S-shaped value function and non-linear probability distortions.

Instead of a globally convex utility function as used in the conventional EUM model, we utilise a function, encapsulating the risk preferences dictated by CPT, which has a reference point B at terminal time T separating regions where terminal wealth is interpreted as gains (excess over B) and losses (shortfall from B). The reference point B is an F_T-measurable random variable with $E(\rho B) < +\infty$. The functions $u_+ : \Re^+ \to \Re^+$ and $u_- : \Re^+ \to \Re^+$ are utility functions which measure the gains and losses respectively. They are both strictly increasing, concave and $u_+(0) = u_-(0) = 0$. $u_+(.)$ is strictly concave and twice differentiable, with $u'_+(0+) = +\infty$ and $u'_+(+\infty) = 0$.

Hence the agent value function

$$u(x) = u_+(x^+)\mathbf{1}_{x\geq 0}(x) - u_-(x^-)\mathbf{1}_{x<0}(x) \tag{2.23}$$

where a^+ denotes the positive part and a^- denotes the negative part of a real number a. Combined in this form, we see that this captures the risk seeking and risk averse attitude towards gains and losses respectively, via the S-shaped function, when x is negative, the function in this region is convex, and when x is positive, the function in this region is concave.

Our non-linear probability distortions for gains and losses will be represented by the two functions $w_+ : [0,1] \to [0,1]$ and $w_- : [0,1] \to [0,1]$ respectively. Both of them are differentiable and strictly increasing, with $w_+(0) = w_-(0) = 0$ and $w_+(1) = w_-(1) = 1$.

2.2.2 Formulating the Behavioural Criterion

Now, for the value of a contingent claim, CPT treated only the discrete case, so now we present the continuous case as used in [7]; given a contingent claim X, an F_T-measurable random variable, we denote $V(X)$ to be its value, and it is defined as:

$$V(X-B) = \tilde{E}[u(X-B)] = V_+((X-B)^+) - V_-((X-B)^-) \qquad (2.24)$$

where, for any random variable $Y \geq 0$, $V_+(.)$ and $V_-(.)$ are defined as:

$$V_+(Y) = \int_0^\infty w_+(P\{u_+(Y) > y\})\,\mathrm{d}y, V_-(Y) = \int_0^\infty w_-(P\{u_-(Y) > y\})\,\mathrm{d}y \qquad (2.25)$$

We have mentioned in chapter 1 that B can be assumed to be 0, so our problem is:

$$V(X) = \tilde{E}[u(X)] = V_+(X^+) - V_-(X^-) \qquad (2.26)$$

Chapter 3

Utility from Terminal Wealth

In this chapter, we will aim to solve the problem of expected utility maximisation from terminal wealth only subject to constraints which we will describe later. We will first review the neoclassical case with no probability distortions and a global concave utility function, and solve it before looking at the behavioural case where probability distortions are present and we utility the S-shaped value function.

3.1 Neoclassical Problem

Problem Statement

$$\textbf{Maximise}: V_2(x) = \sup_{(c,\pi)\in A_2(x)} J_2(X) = \sup_{(c,\pi)\in A_2(x)} E\left[u_2(X)\right] \tag{3.1}$$

$$\textbf{s.t.}: \begin{cases} E\left[\rho X\right] & = x \\ A_2(x) & = \{(c,\pi) \in A(x) : E\min[0, u_2(X)] > -\infty\} \end{cases}$$

where $X = X^{x,\pi,c}$ is the terminal wealth with a given initial endowment $x \geq 0$ with associated portfolio process $\pi(.)$ and consumption process $c(.)$, $u_2(.)$ is the utility function, $A_2(x) = \{(c,\pi) \in A(x) : E\min[0, u_2(X)] > -\infty\}$ and $A(x)$ is the set of admissible consumption-portfolio pairs.

Solution

We will proceed to solve the EUM for terminal wealth for the neoclassical case first, recall definition 2.1,

$$\Psi_2(y) = E\left[\rho I_2(y\rho)\right], \Psi_2(\infty) = E\left[\rho \bar{x}\right] \tag{3.2}$$

where $I_2(.)$ is the inverse of the marginal utility u_2' and the quantity $\Psi_2(y) < \infty, \forall y \in (0,\infty)$. Hence, $\Psi_2(y)$ is non-increasing and continuous on $(0,\infty)$, and strictly decreasing on $(0,r_2)$ where r_2 is defined as,

$$r_2 = \sup\{y > 0 : \Psi_2(y) > \Psi_2(\infty)\} > 0 \tag{3.3}$$

If we restrict the argument of the function Ψ_2 to $(0,r_2)$ then it will have a strictly decreasing inverse function Φ_2, hence, by definition,

$$\Psi_2(\Phi_2(x)) = x, \forall x \in (\Psi_2(\infty),\infty) \tag{3.4}$$

In view of theorem 2.2, the problem (3.1) can be essentially restated as:

$$\textbf{Maximise} : E\left[u_2(\xi)\right] \tag{3.5}$$

$$\textbf{s.t.} : \begin{cases} E\left[\rho\xi\right] & = x \\ \xi \text{ is an} & \text{a.s. lower bounded } F_T\text{-measurable random variable} \end{cases}$$

This problem may be tackled using the method of Lagrangian multipliers yielding,

$$\begin{aligned} &= E\left[u_2(\xi)\right] - y\left(E\left[\rho\xi\right] - x\right) \\ &= xy + E[u_2(\xi) - y\rho\xi] \\ &\leq xy + E\left[\bar{u}_2(y\rho)\right] \end{aligned} \tag{3.6}$$

where the inequality arose from the convex dual, whose definition is given as (2.19) $\bar{u}(y) \geq u(x) - xy$. By definition of the respective convex duals, equality holds, if and only if,

$$\xi^* = (u_2')^{-1}(y\rho) = I_2(y\rho) \tag{3.7}$$

We may find the value of the Lagrange multiplier y by substituting these optimal values into the contraint equation $E[\rho\xi^*] = x$, which, recalling (3.2) and (3.4), gives,

$$\begin{aligned} \Psi_2(y) &= E[\rho I_2(y\rho)] = x \\ y &= \Phi_2(x) \end{aligned}$$

Therefore, the optimal quantities are,

$$\xi^*(x) = I_2(\Phi_2(x)\rho)$$

Hence, when $\xi^*(x) = X^{x,c,\pi*}(x) = X^*(x)$

$$V_2(x) = E\left[u_2(X^*(x))\right] \tag{3.8}$$

3.2 Behavioural Problem

Having looked at the neoclassical case, we now introduce the S-shaped value function and probability distortions. However, first we need to look at the issue of ill-posedness before we proceed.

3.2.1 Ill-Posedness

An EUM problem is said to be ill-posed if the optimal value for the problem is not finite. In EUT, the utility function is globally concave, enabling the problem to be well-posed, i.e. have a finite optimal value, for most cases. However, since we wish to use a function which contains convex and concave regions, coupled with non-linear probability distortions, ill-posedness is now an issue and conditions must be enforced in order achieve well-posedness.

The following theorems are established in [7], and serve as a good basis for well-posedness conditions.

Theorem 3.1*(Jin & Zhou 2008 [7] Th. 3.1 and 3.2)*
The above maximisation problem is ill-posed under either of the following two conditions:

1. There exists a non-negative F_T-measurable random variable X such that $E(\rho X) < +\infty$ and $V_+(X) = +\infty$
2. $u_+(+\infty) = +\infty$, $\bar{\rho} = +\infty$ and $w_-(x) = x$

The first point is intuitive because it essentially states that the model is ill-posed if the non-negative contingent claim has a finite price yet it has an infinite prospective value, which means that the agent may purchase the claim which has a finite price, but yet be able to attain an infinite value at the end.

The second point shows that the utility on gains can be infinitely large, then a probability distortion on losses is necessary in order to ensure well-posedness. Now we move onto the case where we utilise probability distortions and S-shaped functions as defined at the end of the previous chapter assuming the conditions outlined in theorem 3.1. The problem with purely terminal wealth has been addressed in literature [7] so we will merely review the main steps. Take note that previously, we have used the term x to represent the initial endowment, but now, to avoid unnecessary cumbersome notation, the initial endowment for this single behavioural problem will be denoted x_0. In essence, we wish to solve the following maximisation problem:

Problem Statement

$$\textbf{Maximise} : V(X) = \int_0^\infty w_+(P\{u_+(X^+) > y\})\,\mathrm{d}y + \int_0^\infty w_-(P\{u_-(X^-) > y\})\,\mathrm{d}y \tag{3.11}$$

$$\textbf{s.t.} : \begin{cases} E[\rho X] & = x_0 \\ X \text{ is an} & \text{a.s. lower bounded, } F_T\text{-random variable} \end{cases}$$

where u_+ and u_- are functions which map $\Re^+ \rightarrow \Re^+$ forming our S-shaped value function given in (2.23) and w_+ and w_- are the probability distortions mapping $[0,1] \rightarrow [0,1]$. In this optimisational problem, X is the decision variable. Once optimal X^* is obtained, the optimal portfolio is then the one replicating X^*. The quantity we wish to maximise contains probability distortions reflecting on the subjective attitudes and the constraint doesn't, because constraints are objective issues.

Solution

The key idea to tackling problem (3.11) is by breaking it down into three sub-problems. For any feasible solution X in (3.11), X^+ corresponds to the gain in the event $A = \{X \geq 0\}$ with replication cost $x_+ = E[\rho X^+]$ and X^- corresponds to the loss in the event $A^c = X < 0$ with $E[\rho X^-] = x_+ - x_0$. We wish to maximise $V(X) = V_+(X^+) - V_-(X^-)$ so the idea is to split the problem into components; we firstly consider $V_+(X)$ which is a quantity relating to gains so we wish to maximise $V_+(X)$ subject to constraints, then we consider $V_-(X)$, a quantity relating to the measurement of losses so we wish to minimise $V_-(X)$. If $v_+(A, x_+)$ and $v_-(A, x_+)$ are the maximal and minimal values for $V_+(X)$ and $V_-(X)$ respectively, then the next sub-problem is to maximise $v_+(A, x_+) - v - (A, x_+)$. Now we look at the sub-problems:

Sub-Problem 1

$$\textbf{Maximise} : V_+(X) = \int_0^\infty w_+(P\{u_+(X) > y\})\,\mathrm{d}y \tag{3.12}$$

$$\textbf{s.t.} : \begin{cases} E(\rho X) & = x_+ \\ X & \geq 0 \text{ a.s., } x = 0 \text{ a.s. on } A^c \end{cases}$$

This first sub-problem focuses on the $V_+(X)$ quantity in $V(X) = V_+(X^+) - V_-(X^-)$ and the parameters are $A \in F_T$ and $x_+ \geq x_0^+ \geq 0$. It addresses the event where our terminal wealth X is a gain so we aim to maximise this and our decision variable is X. The optimal value of this problem is denoted $v_+(A, x_+)$ and is defined as follows:

1. $P(A) > 0 \implies$ the feasible region for (3.12) is non-empty and $v_+(A, x_+)$ is the supremum of $V_+(X)$
2. $P(A) = 0$ and $x_+ = 0 \implies X = 0$ and $v_+(A, x_+) = 0$
3. $P(A) = 0$ and $x_+ > 0 \implies$ there is no feasible solution for (3.12) $v_+(A, x) = -\infty$

Sub-Problem 2

$$\textbf{Minimise} : V_-(X) = \int_0^\infty w_-(P\{u_-(X) > y\})\,\mathrm{d}y \tag{3.13}$$

$$\textbf{s.t.} : \begin{cases} E(\rho X) & = x_+ - x_0,\ x \geq 0 \text{ a.s.},\ X = 0 \text{ a.s. on A} \\ X \text{ is} & \text{upper bounded a.s.} \end{cases}$$

The second sub-problem focuses on the $V_-(X)$ quantity in $V(X) = V_+(X^+) - V_-(X^-)$ and the parameters are $A \in F_T$ amd $x_+ \geq x_0^+ \geq 0$ with X upper bounded. It addresses the event where our terminal wealth X is a loss so we aim to minimise this and our decision variable here is again X. The optimal value of this problem is denoted $v_-(A, x_+)$ and is defined as follows:

1. $P(A) < 1 \implies$ the feasible region of (3.13) is non-empty and $v_-(A, x_+)$ is the infimum of $V_-(X)$
2. $P(A) = 1$ and $x_+ = x_0 \implies X = 0$ and $v_-(A, x_+) = 0$
3. $P(A) = 1$ and $x_+ \neq x_0 \implies v_-(A, x) = +\infty$

Sub-Problem 3

$$\textbf{Maximise} : v_+(A, x_+) - v_-(A, x_+) \tag{3.14}$$

$$\textbf{s.t.} : \begin{cases} A \in F_T & x_+ \geq x_0^+ \\ x_+ = 0 & \text{when } P(A) = 0,\ x_+ = x_0 \text{ when } P(A) = 1 \end{cases}$$

In this third sub-problem, we wish to find the optimal split between the quantities which relate to gains and losses. The decision variables in this maximisation problem here are x_+, a real number and A, a random event. The random event A causes difficulty in approaching this problem so in [7], Jin and Zhou made use of the following theorem:

Theorem 3.2*(Jin & Zhou 2008 [7] Th. 5.1)*
For any feasible pair (A, x_+) of sub-problem (3.14), there exists $c \in [\underline{\rho}, \bar{\rho}]$ such that $\bar{A} := \{\omega : \rho \leq c\}$ satisfies,

$$v_+(\bar{A}, x_+) - v_-(\bar{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+)$$

Moreover, if sub-problem (3.12) admots an optimal solution with parameters (A, x_+), then the above inequality is strict unless $P(A \cap \bar{A}^c) + P(A^c \cup \bar{A}) = 0$.

The consequence of this theorem is that instead of considering random events A, we only need to consider the events of the type $A = \{\rho \leq c\}$ where $c \in [\underline{\rho}, \bar{\rho}]$. Hence, (3.14) may be rewritten:

Sub-Problem 3A

$$\textbf{Maximise} : v_+(c, x_+) - v_-(c, x_+) \tag{3.15}$$

$$\textbf{s.t.} : \begin{cases} \underline{\rho} \leq c \leq \bar{\rho} & x_+ \geq x_0^+ \\ x_+ = 0 & \text{when } c = \underline{\rho},\ x_+ = x_0 \text{ when } c = \bar{\rho} \end{cases}$$

where for notational simplicity, $v_+(c, x_+) = v_+(\{\omega : \rho \leq c\}, x_+)$ and $v_-(c, x_+) = v_-(\{\omega : \rho \leq c\}, x_+)$. In anticipation of the solutions, we make the following assumptions in order that the quantities $v_+(c, x_+)$ and X_+^* have an explicit form.

Assumption 3.1

1. $F_\rho^{-1}(z)/w'_+(z)$ is a non-decreasing function in $z \in (0, 1]$ and $\liminf_{x \to +\infty} \left(\frac{-x u''_+(x)}{u'_+(x)} \right) < 0$
2. $E \left[u_+ \left((u'_+)^{-1} \left(\frac{\rho}{w'_+(F_\rho(\rho))} \right) \right) w'_+(F_\rho(\rho)) \right] < +\infty$

where we denoted the cumulative distribution function of ρ as $F(.)$. In [7], Jin and Zhou introduce another optimisational problem (3.16). This optimisational problem is related by not the same as (3.15):

Sub-Problem 3B

$$\textbf{Maximise} : v(c, x_+) = E \left[u_+ \left((u'_+)^{-1} \left(\frac{\lambda(c, x_+)\rho}{w'_+(F_\rho(\rho))} \right) \right) w'_+(F_\rho(\rho)) \mathbf{1}_{\rho \leq c} \right]$$

$$-u_-\left(\frac{x_+ - x_0}{E[\rho \mathbf{1}_{\rho>c}]}\right) w_-(1-F_\rho(c)) \tag{3.16}$$

$$\textbf{s.t.}: \begin{cases} \underline{\rho} \le c \le \bar{\rho} & x_+ \ge x_0^+ \\ x_+ = 0 & \text{when } c = \underline{\rho},\ x_+ = x_0 \text{ when } c = \bar{\rho} \end{cases}$$

where $\lambda(c, x_+)$ satisfies $E\left[(u'_+)^{-1}(\frac{\lambda(c,x_+)\rho}{w'_+(F_\rho(\rho))})\rho \mathbf{1}_{\rho\le c}\right] = x_+$ and using the convention that $u_-\left(\frac{x_+ - x_0}{E[\rho \mathbf{1}_{\rho>c}]}\right) w_-(1-F_\rho(c)) = 0$ when $c = \bar{\rho}$ and $x_+ = x_0$

Jin and Zhou [7] proved that (3.15) and (3.16) have the same supremum values hence if we solve (3.16) we also obtain the solution for (3.15). The problem of finding the optimal quantities for expected utility maximisation for terminal wealth in the behavioural context has been calculated in literature [7] and the results are presented in the proceeding theorem.

Theorem 3.1 *(Jin & Zhou 2008 [7] Th. 4.1)*
Under assumptions (3.1) and assuming that $u_-(.)$ is strictly concave at 0:

1. If X^* is optimal for (3.11), then $c^* = F_\rho^{-1}(P\{X^* \ge 0\})$, $x_+^* = E[\rho(X^*)^+]$, where $F_\rho(.)$ is the cumulative distribution function for ρ are optimal for (3.16).

2. If (c^*, x_+^*) is optimal for (3.16) problem and X_+^* is optimal for (3.12) with parameters $(\{\rho \le c^*\}, x_+^*)$, then

$$X^* = \left[(u'_+)^{-1}\left(\frac{\lambda(c^*, x_+^*)\rho}{w'_+(F_\rho(\rho))}\right)\right]\mathbf{1}_{\rho\le c^*} - \left[\frac{x_+^* - x_0}{E[\rho \mathbf{1}_{\rho>c^*}]}\right]\mathbf{1}_{\rho>c^*} \tag{3.17}$$

is optimal for (3.11), where $\lambda(c, x_+)$ satisfies $E\left[(u'_+)^{-1}(\frac{\lambda(c,x_+)\rho}{w'_+(F_\rho(\rho))})\rho \mathbf{1}_{\rho\le c}\right] = x_+$ and $F_\rho(.)$ is the cumulative density function for the terminal state pricing kernel.

This explicit form of terminal wealth (3.17) tells us that whether the agent perceives the terminal wealth as a gain or a loss is determined by c^* : if the terminal state pricing density is less than or equal to c^* then the agent perceives a gain and if the terminal state pricing density is bigger than the c^*, then a perceived loss is attained. So the probability of attaining a perceived gain is given by $P(\rho \le c^*) = F_\rho(c^*)$[3].

In order to replicate this optimal terminal value, at time 0, the agent buys a contingent claim with payoff $\left[(u'_+)^{-1}\left(\frac{\lambda(c^*,x_+^*)\rho}{w'_+(F_\rho(\rho))}\right)\right]\mathbf{1}_{\rho\le c^*}$ at cost x_+^*. However, since $x_+^* \geq x_0$ the agent will also need to sell a claim at cost $x_+^* - x_0$ with payoff $\left[\frac{x_+^* - x_0}{E[\rho \mathbf{1}_{\rho > c^*}]}\right] \mathbf{1}_{\rho > c^*}$ to fund her purchase.

The case considered here was when the reference point $B = 0$. The general form [3]when $B \neq 0$ is:

$$X^* = \left[(u_+')^{-1}\left(\frac{\lambda(c^*, x_+^*)\rho}{w_+'(F_\rho(\rho))}\right) + B\right] \mathbf{1}_{\rho \leq c^*} - \left[\frac{x_+^* - (x_0 - E[\rho B])}{E[\rho \mathbf{1}_{\rho > c^*}]} - B\right] \mathbf{1}_{\rho > c^*}$$

where $c^* := F_\rho^{-1}(P\{X^* \geq B\})$, $x_+^* = E[\rho(X^* - B)^+]$ and $\lambda(c, x_+)$ satisfies $E\left[(u_+')^{-1}(\frac{\lambda(c, x_+)\rho}{w_+'(F_\rho(\rho))})\rho \mathbf{1}_{\rho \leq c}\right] = x_+$ and $F_\rho(.)$.

Chapter 4

Utility from Consumption & Terminal Wealth

In this chapter, we will aim to solve the problem of expected utility maximisation from terminal wealth and consumption subject to constraints which we will describe later. We will first review the neoclassical case with no probability distortions and a global concave utility function, and solve it before looking at the behavioural case where probability distortions are present and the utility is an S-shaped value function.

4.1 Neoclassical Problem

Problem Statement

$$\textbf{Maximise}: V_3(x) = \sup_{(c,\pi)\in A_3(x)} E\left[\int_0^T u_1(t, c(t))\,\mathrm{d}t + u_2(X)\right] \tag{4.1}$$

$$\textbf{s.t.}: \begin{cases} E\left[\rho X + \int_0^T \rho(t)c(t)\,\mathrm{d}t\right] & = x \\ A_3 = A_1(x) \cap A_2(x) \end{cases}$$

where $X = X^{x,\pi,c}$ is the terminal wealth with a given initial endowment $x \geq 0$ with associated portfolio process $\pi(.)$ and consumption process $c(.)$, $u_1(.)$ and u_2 are utility functions, $A_1(x) = \{(c,\pi) \in A(x) : E\min[0, \int_0^T u_1(t, c(t))]\,\mathrm{d}t > -\infty\}$ $A_2(x) = \{(c,\pi) \in A(x) : E\min[0, u_2(X)] > -\infty\}$, $A_3 = A_1(x) \cap A_2(x)$ and $A(x)$ is the set of admissible consumption-portfolio pairs

Solution

Again, we will start from the conventional EUM for maximising expected utility with cusumption and terminal wealth. Recall definition 2.1,

$$\Psi_3(y) = E\left[\int_0^T \rho(t) I_1(t, y\rho(t))\,\mathrm{d}t + \rho I_2(y\rho)\right], \Psi_3(\infty) = E\left[\int_0^T \rho(t)\bar{c}\,\mathrm{d}t + \bar{x}\right] \tag{4.2}$$

where $I_1(.)$ and $I_2(.)$ are the inverses of the marginal utilities u_1' and u_2' respectively and the quantity $\Psi_3(y) < \infty, \forall y \in (0, \infty)$. Hence, $\Psi_3(y)$ is non-increasing and continuous on $(0, \infty)$, and strictly decreasing on $(0, r_3)$ where r_3 is defined as,

$$r_3 = \sup\{y > 0 : \Psi_3(y) > \Psi_3(\infty)\} > 0 \tag{4.3}$$

If we restrict the argument of the function Ψ_3 to $(0, r_3)$ then it will have a strictly decreasing inverse function Φ_3, hence, by definition,

$$\Psi_3(\Phi_3(x)) = x, \forall x \in (\Psi_3(\infty), \infty) \tag{4.4}$$

As usual, in view of theorem 2.2, the problem (4.1) can be essentially restated:

$$\textbf{Maximise}: E\left[\int_0^T u_1(t, c(t))\,\mathrm{d}t + u_2(\xi)\right] \tag{4.5}$$

$$\textbf{s.t.}: \begin{cases} E\left[\int_0^T \rho(t)c(t)\,\mathrm{d}t + \rho\xi\right] & = x \geq 0 \\ \xi \text{ is an a.s. bounded} & F_T \text{ measurable random variable} \end{cases}$$

where $c(.)$ is the consumption process, and $u_1(.)$ and $u_2(.)$ are utility functions. This problem may be tackled using the method of Lagrangian multipliers yielding,

$$\begin{aligned} &= E\left[\int_0^T u_1(t, c(t))\,\mathrm{d}t + u_2(\xi)\right] - y\left(E\left[\int_0^T \rho(t)c(t)\,\mathrm{d}t + \rho\xi\right] - x\right) \\ &= xy + E\int_0^T [u_1(t, c(t)) - y\rho(t)c(t)]\,\mathrm{d}t + E[u_2(\xi) - y\rho\xi] \\ &\leq xy + E\left[\int_0^T \bar{u}_1(t, y\rho(t))\,\mathrm{d}t + \bar{u}_2(y\rho)\right] \end{aligned} \tag{4.6}$$

where the inequality arises from the convex dual which, by definition of the respective convex duals, equality holds, if and only if,

$$c^*(t) = (u_1')^{-1}(t, y\rho(t)) = I_1(t, y\rho(t)), \xi^* = (u_2')^{-1}(y\rho) = I_2(y\rho) \tag{4.7}$$

for $0 \leq t \leq T$. We may find the value of the lagrange multiplier y by substituting these optimal values into the contraint equation $E\left[\int_0^T \rho(t)c^*(t)\,\mathrm{d}t + \rho\xi^*\right] = x$, which, using (4.2) and (4.4), gives,

$$\begin{aligned} \Psi_3(y) &= E\left[\int_0^T \rho(t)I_1(t, y\rho(t))\,\mathrm{d}t + \rho I_2(y\rho)\right] = x \\ y &= \Phi_3(x) \end{aligned} \tag{4.8}$$

Therefore, the optimal quantities are,

$$c_t^*(x) = I_1(t, \Phi_3(x)\rho(t)), \xi^*(x) = I_2(\Phi_3(x)\rho) \tag{4.9}$$

Hence, when $\xi^*(x) = X^{x,c,\pi*}(x) = X^*(x)$

$$V_3(x) = E\left[\int_0^T u_1(t, c_t^*(x))\,\mathrm{d}t + u_2(X^*(x))\right] \tag{4.10}$$

4.2 Behavioural Problem

Our conventional problem was $V_3(x) = \sup_{(c,\pi)\in A_3(x)} E\left[\int_0^T u_1(t, c(t))\,\mathrm{d}t + u_2(X)\right]$, however, due to the non-linearity of distorted expectation, when looking at EUM with terminal wealth and consumption from a behavioural point of view, we have three cases which we may consider:

Problem 1

$$\tilde{E}\left(\int_0^T u_1(c(t))\,\mathrm{d}t + u_2(X)\right) \tag{4.11}$$

Problem 2

$$\tilde{E}\left(\int_0^T u_1(c(t))\,\mathrm{d}t\right) + \tilde{E}\left[u_2(X)\right] \tag{4.12}$$

Problem 3

$$\int_0^T \tilde{E}\left(u_1(c(t))\right)\,\mathrm{d}t + \tilde{E}[u_2(X)] \tag{4.13}$$

Each of these problems fall under the terminal wealth and consumption category but the one which is taken onboard by the majority remains open for discussion. (4.11) resembles the case where an agent considers the terminal wealth and their consumption process as a whole, (4.12) resembles the case where an agent wishes to maximise their terminal wealth and consumption process separately but considers their overall consumption through the horizon $[0,T]$ and (4.13) is similar to problem 2, only the agent considers their consumption "incrementally" as opposed to overall.

These three problems are all subject to the contraint $E(\int_0^T c(t)\rho(t)\,\mathrm{d}t+\rho X) = x_0$. Out of these three problems, (4.12) and (4.13) seem more realistic as people usually wish to plan their terminal wealth and consumption process separately although this requires empirical evidence to support this claim. Agents may also anticipate unexpected changes throughout their consumption process and so, may prefer to maximise their subjective expectation for consumption for all time, rather than throughout the entire time horizon $[0,T]$, hence for these reasons, we will explore (4.13). Like in the previous chapter in the behavioural problem section, we wish to denote our initial endowment by x_0. Hence our maximisation problem is:

Problem Statement

$$\textbf{Maximise}:\left[\int_0^T \tilde{E}[u_1(c(t))]\,\mathrm{d}t + \tilde{E}u_2(X)\right] \tag{4.14}$$

$$\textbf{s.t.}:\begin{cases} E\left(\int_0^T c(t)\rho(t)\,\mathrm{d}t + \rho X\right) & = x_0 \\ c(t)\geq 0, X \text{ is an a.s. lower bounded,} & F_T \text{ random variable} \end{cases}$$

where $u_1(.)$ and $u_2(.)$ are S-shaped value functions, $\tilde{E}(.)$ is the distorted expectation operator capturing subjective decision making and $E(.)$ is the normal expectation operator.

Solution

We shall tackle this problem by splitting the quantities up, so our problem, $\int_0^T \tilde{E}u(c(t))\,\mathrm{d}t + \tilde{E}u(X)$ subject to $\int_0^T E(c(t)\rho(t))\,\mathrm{d}t + E(\rho X) = x_0$, may be divided when we allocate our initial endowment x_0 into two funds; one to fund the quantity to achieve our desired consumption which we denote x_1 and the other one to fund the quantity to achieve our desired terminal wealth which we denote x_2, hence $x_1 + x_2 = x_0$. Consequently, we have two subproblems; problem (4.15) deals with the consumption aspect, whilst problem (4.16) deals with the terminal wealth aspect.

Sub-Problem 1:

$$\textbf{Maximise}: V_1(c(t)) = \max \left[\int_0^T \tilde{E} u_1(c(t))\, \mathrm{d}t \right] \tag{4.15}$$

$$\textbf{s.t.}: \int_0^T E(c(t)\rho(t))\, \mathrm{d}t = x_1 \geq 0, c(t) \geq 0$$

Sub-Problem 2:

$$\textbf{Maximise}: V_2(X) = \left[\tilde{E} u_2(X) \right] \tag{4.16}$$

$$\textbf{s.t.}: \begin{cases} E(\rho X) & = x_2 \geq 0 \\ X \text{ is an} & \text{a.s. lower bounded, } F_T \text{ random variable} \end{cases}$$

(4.16) has already been addressed in the previous chapter so the optimal value for (4.16) is:

$$X^* = \left[(u'_{2+})^{-1} \left(\frac{\lambda(c^*, x^*_+)\rho}{w'_+(F_\rho(\rho))} \right) \right] \mathbf{1}_{\rho \leq c^*} - \left[\frac{x^*_+ - x_2}{E[\rho \mathbf{1}_{\rho > c^*}]} \right] \mathbf{1}_{\rho > c^*}$$

where x^*_+ is the price of terminal gains and $\lambda(c, x_+)$ satisfies $E\left[(u'_{2+})^{-1} (\frac{\lambda(c, x_+)\rho}{w'_+(F_\rho(\rho))}) \rho \mathbf{1}_{\rho \leq c} \right] = x_+$. Hence we only need to consider (4.15). This problem can be solved by splitting it further into two sub-problems:

Sub-Problem 1A:

$$\textbf{Maximise}: \max_{c(t)} \tilde{E}[u_1(c(t))] \tag{4.17}$$

$$\textbf{s.t.}: \begin{cases} E(c(t)\rho(t)) & = b(t) \\ c(t) & \geq 0 \end{cases}$$

where $\rho(t)$ is a given strictly positive random variable, with no atom whose cumulative distribution function is given as $F_{\rho(t)}(.)$. We denote the optimal value for this problem as $v[b(t)]$. $c(t)$ is the consumption process and the decision variable here with parameter $b(t)$.

Sub-Problem 1B:

$$\textbf{Maximise}: \int_0^T v[b(t)]\,\mathrm{d}t \tag{4.18}$$

$$\textbf{s.t.}: \begin{cases} \int_0^T b(t)\,\mathrm{d}t & = x_1 \\ \qquad b(t) & \geq 0 \end{cases}$$

Here, $b(t)$ is the decision variable. We proceed by solving (4.17) then (4.18). In order to solve (4.17), we need to employ a technique known as quantile formulation in order to tame the distorted expectation into a more manipulatable form [46]. Now $c(t) \geq 0$ a.s., so we treat it as a non-negative random variable. Furthermore, recall (2.23) the definition of our S-shaped value function $u(x) = u_+(x^+)\mathbf{1}_{x\geq 0}(x) - u_-(x^-)\mathbf{1}_{x<0}(x)$, for $x \in \Re$. Now for $u(c(t))$,since $c(t)$ is positive, we need only concern ourselves with the positive part and so, by definition,

$$\tilde{E}u_1(c(t)) = \int_0^{+\infty} w\{P(u_1(c(t)) > y)\}\,\mathrm{d}y$$

where $w : [0,1] \to [0,1]$ is a strictly increasing, differentiable function with $w(0) = 0$, $w(1) = 1$ and $u(.)$ is a strictly concave, strictly increasing, twice differentiable function with $u(0) = 0$, $u'(0) = +\infty$, $u'(+\infty) = 0$. For (4.17) since $c(t)$ is considered for one time only, we condense the notation from $c(t)$ to c_t and $\rho(t)$ to ρ_t. Now, our maximisation problem may be rewritten as:

$$\textbf{Maximise}: \alpha_1(c_t) = \int_0^{+\infty} w\{P(u_1(c_t) > y)\}\,\mathrm{d}y \tag{4.19}$$

$$\textbf{s.t.}: E[\rho_t c_t] = b_t \geq 0, c(t) \geq 0$$

The probability distortion $w(.)$ in $\alpha_1(c_t)$ means that it is not globally concave or convex in c_t. Quantile formulation, the technique which we will employ, changes the decision variable from c_t to it's quantile function which we denote $G(.)$, the effect of this transformation is that concavity is recovered.

Now if we have two random variables X and Y, and $X \sim Y$ then $\alpha(X) = \alpha(Y)$ by the law-invariance property of $\alpha(.)$ [3]. Let $Z := 1 - F_{\rho_t}(\rho_t)$, implying that $Z \sim U(0,1)$ and hence, because $\rho(t)$ is assumed to be atomless, we may rearrange the equation, we have, $\rho_t = F_{\rho_t}^{-1}(1 - Z)$ a.s.[46]. Let G be the quantile function of a non-negative random variable such that G is non-decreasing with $G(0-) = 0$ and $G(+\infty) = 1$. It is proved in [7] that if (4.19) admits an optimal solution $c(t)^*$ whose quantile function is $G(.)$, then $c(t)^* = G^{-1}(1 - F_\rho(\rho)) = G^{-1}(Z)$. So we may transform the above optimisation problem as:

$$\textbf{Maximise}: \beta_1(G) = \int_0^{+\infty} w(P\{u_1(G^{-1}(Z)) > y)\}\,\mathrm{d}y \tag{4.20}$$

$$\textbf{s.t.} \begin{cases} E[G^{-1}(Z)F_{\rho_t}^{-1}(1-Z)] & = b(t) \\ G \text{ is the distribution function} & \text{of a non-negative random variable} \end{cases}$$

We now rewrite $\beta_1(G)$ into a more manipulatable form, denote $\bar{w}(x) = w(1-x)$, $x \in [0,1]$, and $\bar{u_1} = \sup_{x\in\Re^+} u_1(x)$

$$\begin{aligned} \beta_1(G) &= \int_0^{\bar{u_1}} \bar{w}(P\{u_1(G^{-1}(Z)) \geq y)\}\,\mathrm{d}y \\ &= \int_0^{\bar{u_1}} \bar{w}(P\{Z \leq G(u_1^{-1}(y))\})\,\mathrm{d}y \\ &= \int_0^{\bar{u_1}} \bar{w}(G^{-1}(u_1^{-1}(y)))\,\mathrm{d}y \\ &= \int_0^1 u_1(G^{-1}(\bar{w}^{-1}(t)))\,\mathrm{d}t \\ &= -\int_0^1 u_1(G^{-1}(s))\bar{w}'(s)\,\mathrm{d}s \\ &= \int_0^1 u_1(G^{-1}(s))w'(1-s)\,\mathrm{d}s \\ &= E[u_1(G^{-1}(Z))w'(1-Z)] \end{aligned}$$

If we let $\Sigma = \{g : [0,1) \to \Re^+, g(0) = 0\}$ where g is non-decreasing and left continuous and $g = G^{-1}$, we can rewrite our problem as

$$\textbf{Maximise}: \tilde{\beta_1}(g) = E[u_1(g(Z))w'(1-Z)] \tag{4.21}$$

$$\textbf{s.t.}: E[g(Z)F_{\rho_t}^{-1}(1-Z)] = b(t), g \in \Sigma$$

When our optimisation problem is written in this form, we may employ the technique of Lagrangian multipliers to remove the constraint.

$$\textbf{Maximise}: \tilde{\beta_1^\lambda}(g) = E[u_1(g(Z))w'(1-Z) - \lambda g(Z)F_{\rho_t}^{-1}(1-Z)] \tag{4.22}$$

$$\textbf{s.t.}: g \in \Sigma$$

where λ may be determined by the original linear constraint. Differentiating this with respect to $g(z) \in \Re^+$, yields $u_1'(g(Z))w'(1-Z) - \lambda F_\rho^{-1}(1-Z)$, equating this to 0 and rearranging gives us:

$$g(z) = (u_1')^{-1}\left(\frac{\lambda F_{\rho_t}^{-1}(1-z)}{w'(1-z)}\right) \tag{4.23}$$

It may be verified that this quantity is indeed a maximum because if we differentiate again, we get $u_1''(g(Z))w'(1-Z)$, since $u_1(.)$ is concave, $u_1''(.) < 0$ and also $w'(.) > 0$, hence the second derivative is a negative quantity indicating the function $g(z)$ yields a maximum. As mentioned above $g = G^{-1}$ must be non-decreasing, so if $F_{\rho_t}^{-1}(z)/w'(z)$ is non-decreasing in $z \in (0, 1]$, then $g(z)$ is non-decreasing in $z \in [0, 1)$ which satisfies the conditions required to be a solution for the above maximisation problem otherwise, it doesn't. In the previous chapter, it was an assumption that $F_{\rho_t}^{-1}(z)/w'(z)$ is non-decreasing in $z \in (0, 1]$ hence, with this assumption, optimal g is guaranteed. Rewriting and summarising the calculations, we have:

$$\textbf{Maximise} : \max_{c(t)} \tilde{E} u_1(c(t))$$

$$\textbf{s.t.} : \begin{cases} E(c(t)\rho(t)) & = b(t) \\ c(t) & \geq 0 \end{cases}$$

and since $Z = 1 - F_\rho(\rho)$,

$$c_t^*(\lambda) = (u_1')^{-1}\left(\frac{\lambda \rho_t}{w'(F_{\rho_t}(\rho_t))}\right), \lambda > 0 \tag{4.24}$$

where λ satisfies $E(\rho_t c_t^*(\lambda)) = b_t$, hence $\lambda = f[b(.)]$. Having attained the optimal solution for (4.17) we now we wish to solve problem (4.18):

$$\textbf{Maximise} : \int_0^T v[b(t)] \, \mathrm{d}t$$

$$\textbf{s.t.} : \begin{cases} \int_0^T b(t) \, \mathrm{d}t & = x_1 \\ b(t) & \geq 0 \end{cases}$$

Again, we apply the Lagrangian method to get rid of the contraint, by introducing the Lagrangian multiplier ϕ

$$\textbf{Maximise} : \int_0^T v[b(t)] \, \mathrm{d}t - \phi\{\int_0^T b(t) \, \mathrm{d}t - x_1\} \tag{4.25}$$

Equivalently,

$$\textbf{Maximise} : \int_0^T v[b(t)] - \phi b(t) \, \mathrm{d}t \tag{4.26}$$

Differentiating the integrand with respect to $b(t)$ and equating it to 0 yields $v'[b^*(t)] - \phi = 0$ and therefore

$$b^*(t) = (v')^{-1}(\phi) \tag{4.27}$$

where ϕ satisfies $\int_0^T (v')^{-1}(\phi)\,\mathrm{d}t = x_1$, hence $\phi = \psi(x_1)$. Inserting this into (4.27) gives:

$$b^*(t) = (v')^{-1}(\phi(x_1)) \tag{4.28}$$

Now for continuous random variable $X \geq 0$,

$$\begin{aligned}
v'(X) = \tilde{E}'u_1(X) &= \int_0^\infty w'(1 - F_X(y))\,\mathrm{d}y \\
&= -\int_0^\infty w'(1 - F_X(y)) f_X(y)\,\mathrm{d}y \\
v''(X) &= \tilde{E}''u_1(X) \\
&= \int_0^\infty w''(1 - F_X(y)) f_X^2(y)\,\mathrm{d}y - \int_0^\infty w'(1 - F_X(y)) \frac{df_X(y)}{dx}\,\mathrm{d}y
\end{aligned}$$

Since $w'(.) > 0$, $v''(b(t)) < 0$ implicating that the quantity $b^*(t)$ is optimal for the problem.

Combining the optimal solutions for the two subproblems (4.24) and (4.28), we find that the optimal value for problem 1 is:

$$c_t^*(x_1) = (u_1')^{-1}\left(\frac{f\{(v')^{-1}(\psi(x_1))\}\rho_t}{w'(F_{\rho_t}(\rho_t))}\right) \tag{4.29}$$

We conclude this consumption part with the following theorem which reassures us that our splitting process indeed solves our optimisational problem (4.15). We will then resume with our original problem of this subsection in this chapter, optimising the expected utility with consumption and terminal wealth in the behavioural context.

Theorem 4.1

1. If c^* is optimal for (4.15) then $b^*(t) := E(c^*(t)\rho(t))$ optimal for (4.18) and $c^*(t)$ is optimal for (4.17) with $b(.) = b^*(.)$

2. If $b^*(.)$ is optimal for (4.18) and $c^*(.)$ is optimal for (4.17) with parameters $b(.) = b^*(.)$ then c^* is optimal for (4.15)

Proof

1. (i) We wish to show that $\int_0^T v[b(t)]\,\mathrm{d}t \le \int_0^T v[b^*(t)]\,\mathrm{d}t$, for all feasible $b(t)$ and $b^*(t) := E(c^*(t)\rho(t))$. For any $b(.)$ there exists a $c(.)$ such that $\tilde{E}u_1(c(t)) = v(b(t))\ \forall t \in [0,T]$ and $b(t) = E[c(t)\rho(t)]$. Hence,

$$\begin{aligned}
\int_0^T v[b(t)]\,\mathrm{d}t &= \int_0^T \tilde{E}u_1(c(t))\,\mathrm{d}t \\
&\le \max \int_0^T \tilde{E}u_1(c(t))\,\mathrm{d}t \\
&= \int_0^T \tilde{E}u_1(c^*(t))\,\mathrm{d}t \\
&= \int_0^T v[b^*(t)]\,\mathrm{d}t
\end{aligned}$$

(ii) Now we need to show that if $c^*(t)$ is optimal for (4.15), then $c^*(t)$ is optimal for (4.17) with $b(.) = b^*(.)$. $c^*(t)$ is optimal for (4.15), hence,

$$\int_0^T \tilde{E}(u_1(c(t)))\,\mathrm{d}t \le \int_0^T \tilde{E}(u_1(c^*(t)))\,\mathrm{d}t$$

for all $c(t)$ feasible for (4.15). Also, $c^*(t)$ with $b(.) = b^*(.)$ is feasible for (4.17)

$$\begin{aligned}
\tilde{E}(u_1(c(t))) &\le \tilde{E}(u_1(c^*(t))) \\
\tilde{E}u_1(c_t) &\le v[c^*(t)]
\end{aligned}$$

2. We need to show that if $b^*(.)$ is optimal for (4.18) and $c^*(t)$ is optimal for (4.17) with $b(.) = b^*(.)$ then c^* is optimal for (4.15). For all $c(.)$, define $b(.) = E[c(.)\rho(.)]$, then,

$$\begin{aligned}
\int_0^T \tilde{E}u_1(c(t))\,\mathrm{d}t &\le \int_0^T v[b(t)]\,\mathrm{d}t \\
&\le \int_0^T v[b^*(t)]\,\mathrm{d}t \\
&= \int_0^T \tilde{E}u_1(c^*(t))\,\mathrm{d}x
\end{aligned}$$

as required.

Summarising, from (4.29) $c_t^*(x_1) = (u_1')^{-1}\left(\frac{f\{(v')^{-1}(\psi(x_1))\}\rho_t}{w'(F_{\rho_t}(\rho_t))}\right)$ is the optimal consumption, and $X^*(x_2) = \left[(u_{2+}')^{-1}\left(\frac{\lambda(c^*,x_+^*)\rho}{w_+'(F_\rho(\rho))}\right)\right]\mathbf{1}_{\rho\le c^*} - \left[\frac{x_+^*-x_2}{E[\rho\mathbf{1}_{\rho>c^*}]}\right]\mathbf{1}_{\rho>c^*}$ is the optimal terminal wealth where where $\lambda(c,x_+)$ satisfies $E\left[(u_{2+}')^{-1}(\frac{\lambda(c,x_+)\rho}{w_+'(F_\rho(\rho))})\rho\mathbf{1}_{\rho\le c}\right] = x_+$ and $x_0 = x_1 + x_2$. Our original problem was to solve:

$$\textbf{Maximise}: \left[\int_0^T \tilde{E}[u_1(c(t))]\,\mathrm{d}t + \tilde{E}u_2(X)\right]$$

$$\textbf{s.t.}: \begin{cases} E\left(\int_0^T c(t)\rho(t)\,\mathrm{d}t + \rho X\right) & = x_0 = x_1 + x_2 \\ c(t) \ge 0, X \text{ is an a.s. lower bounded}, & F_T \text{ random variable} \end{cases}$$

The ratio of x_1 to x_2 may be determined as follows, for a given x_0, the optimal quantities $c_t^*(x_1)$ and $X^*(x_0 - x_1)$ are inserted into the above maximisation problem, so that for a given x_1,

$$\gamma(x_1) = \max \int_0^T \tilde{E}\left(u_1(c_t^*(x_1))\right)\,\mathrm{d}t + \tilde{E}[u_2(X^*(x_0 - x_1)) \tag{4.30}$$

By recalling notation from (4.15) and (4.16) we write $\int_0^T \tilde{E}\left(u_1(c_t^*(x_1))\right)\,\mathrm{d}t = V_1(x_1)$ and $\tilde{E}[u_2(X^*(x_0-x_1)) = V_2(x_0-x_1)$ hence we can neatly express the integrand as $V_1(x_1) + V_2(x_0 - x_1)$. Finally, to solve the terminal value and consumption maximisation problem, we work out:

$$\gamma(x_1) = \max_{0\le x_1\le x_0} V_1(x_1) + V_2(x_0 - x_1) \tag{4.31}$$

enabling us to work out the optimal splitting.

4.2.1 Algorithm

The main steps to solving (4.14) are outlined here,

1. Solve (4.15) with any $x_1 \ge 0$ to get optimal value $V_1(x_1)$ and optimal consumption $c_t^*(x_1)$
2. Solve (4.16) with any $x_2 \in \Re$ to get optimal value $V_2(x_2)$ and optimal terminal wealth $X^*(x_2)$
3. Solve (4.31) with $V_1(x_1)$ and $V_2(x_2)$ obtained in the previous steps

So now we have the following algorithm to solve problem (4.14).

Behavioural Expected Utility Maximisation with Consumption and Terminal Wealth Algorithm

1. Solve (4.17) for any $b(t) \geq 0$ to get its optimal value function $v[b(t)]$ and optimal solution $c^*(b(t))$
2. Plug $v(.)$ to problem (4.18) and solve it to get the optimal solution $b^*(t)$.
3. $c^*(t) = c^*(b^*(t))$ gives an optimal solution to (4.15). Obtain $V_1(c^*_t(x_1))$
4. Solve problem (3.12) with $(\{\omega : \rho \leq c\}, x_+)$, where $\underline{\rho} \leq c \leq \bar{\rho}$ and $x_+ \geq x_2^+$ are given, to obtain $v_+(c, x_+)$ and the optimal solution $X^*_+(c, x_+)$
5. Solve problem (3.16) to get (c^*, x^*_+)
6. If $(c^*, x^*_+) = (\bar{\rho}, x_2)$ then $X^*_+(\bar{\rho}, x_2)$ solves (4.16) otherwise $X^*(x_2) = \left[(u'_{2+})^{-1}\left(\frac{\lambda(c^*, x^*_+)\rho}{w'_+(F_\rho(\rho))}\right)\right]\mathbf{1}_{\rho \leq c^*} - \left[\frac{x^*_+ - x_2}{E[\rho \mathbf{1}_{\rho > c^*}]}\right]\mathbf{1}_{\rho > c^*}$ solves (4.16). Obtain $V_2(X^*(x_2))$
7. With $x_0 = x_1 + x_2$, solve problem (4.31) to find the optimal split

Chapter 5

Conclusions

Expected Utility Maximisation was regarded as one of the crowning achievements in optimal portfolio selection, however, the repeated violations in observation evidently suggests that this is not an adequate model of decision making under risk and uncertainty. Instead, EUT can be regarded more as a cornerstone for our quest towards exploring the optimal portfolio problem, as more empirical and theoretical research is conducted to close the gap between how agents really make decision and what is dictated by the mathematics.

CPT is generally accepted as the descriptive model for risky decision making, and work is currently being done to "mathematicalised" this behavioural finance concept. The psychological aspects are incorporated adequately; the once globally concave utility function used in EUT, is replaced with an S-shaped value function, whose regions of convexity and concavity are distinguished by a reference point, capturing the agents general risk attitude towards gains and losses. Introducing the probability distortions also takes into the notion that agents tend to overweigh small probabilities and underweigh large probabilities.

The results of chapter 4 are:

$$c_t^*(x_1) = (u_1')^{-1}\left(\frac{f\{(v')^{-1}(\psi(x_1))\}\rho_t}{w'(F_{\rho_t}(\rho_t))}\right)$$

$$X^*(x_2) = \left[(u_{2+}')^{-1}\left(\frac{\lambda\rho}{w_{2+}'(F_\rho(\rho))}\right)\right]\mathbf{1}_{\rho\le c^*} - \left[\frac{x_{2+}^* - x_2}{E[\rho\mathbf{1}_{\rho>c^*}]}\right]\mathbf{1}_{\rho>c^*}$$

for $x_0 = x_1 + x_2$ where x_0 is the initial endowment. We also formulated an algorithm to solve this problem.

5.1 Possible Extensions

As we have outlined before in the previous chapter on maximising expected utility with terminal wealth and consumption, if we introduce the behavioural criterion, this maximising problem may be interepreted in three different ways. The whole point of developing theories beyond EUT to behavioural finance was due to the fact that theory was not observed in practice. The psychological extensions were required to address these deviations, and hence, further research could be employed to see which of the three different intepretations, which all yield different values, most closely match real life.

Intuitively, out of the three problem intepretations, thought experiments might advocate the non existence of a general prevalent problem mainly due to the fact that for any single agent, she might employ different subjective methods of considering her subjective expected utility to maximise so context is very important. A simple example demonstrating this would be comparing the scenario when the agent is drawing up a monthly spending budget, so she wishes to plan her consumption over the time horizon and also wish to have a certain amount to pay off, say the rent and another scenario where say the agent makes decisions Taoistically and "go with the flow". (4.12) and (4.13) would probably be more suited to analysing the former and the latter scenarios respectively.

Since context is important, one may wish to restrict the scenario and investigate this problem in financial investment situations. As for mathematical extensions, solving each of the three problems would stand as a challenging problem itself as each would require different methods of deducing the optimal quantities due to the nature of the non-linear expectations.

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