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A two-sided q-analogue of the Coxeter complex

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Abstract

We define a complex of bimodules over the Iwahori–Hecke algebra associated to a finite Coxeter group, calculate its cohomology and show that it induces a derived equivalence over its module category extending the Morita equivalence given by a certain algebra automorphism. We show further that when tensored with the index representation this complex becomes isomorphic to the one-sided q-analogue of the Coxeter complex previously defined by V. Deodhar [On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials, J. Algebra 111 (2) (1987) 483–506] and A. Mathas [A q-analogue of the Coxeter complex, J. Algebra 164 (3) (1994) 831–848].

Let (W, S) be a finite Coxeter system, let R be a unitary commutative ring and let $q \in R^\times$. Let $\mathcal{H}$ be the *Iwahori–Hecke algebra* of (W, S) over R with parameter q; that is, $\mathcal{H}$ is the free R-module with basis $\{T_w \mid w \in W\}$ and multiplication given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q-1)T_w & \text{if } \ell(sw) < \ell(w) \end{cases}$$

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for $s \in S$, $w \in W$ and where ℓ is the usual length function on W. We fix a total order $S = \{s_i \mid 1 \leqslant i \leqslant |S|\}$ on the set S. For every subset $I \subset S$ let W_I be the subgroup of W generated by I (with the convention $W_\emptyset = 1$) and let $\mathcal{H}_I$ be the parabolic subalgebra of $\mathcal{H}$ having as R-basis the set $\{T_w \mid w \in W_I\}$; this algebra is isomorphic in the obvious way to the Iwahori–Hecke algebra of (W_I, I) over R with parameter q. The $\mathcal{H}$-module R endowed with the action $T_w.\lambda = q^{\ell(w)}\lambda$ for $w \in W$ and $\lambda \in R$ is called the *index representation* of $\mathcal{H}$.

Given an R-algebra A and an algebra automorphism α of A, we write ${}_\alpha A$ for the A-A-bimodule with the regular right action given by multiplication in A and the left action given by $a.a' = \alpha(a)a'$, for $a, a' \in A$. We denote by A-mod the category of finitely generated left A-modules and by $\mathcal{D}^\flat(A\text{-mod})$ the bounded derived category of A-mod.

Definition. Let D be the cochain complex of $\mathcal{H}$-$\mathcal{H}$-bimodules concentrated in degrees 0 to $|S|$ defined by setting

$$D^n = \bigoplus_{\substack{I \subset S \\ |I| = n}} \mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$$

for $0 \leqslant n \leqslant |S|$, with differential $d^n : D^n \to D^{n+1}$ given, for any integer n such that $0 \leqslant n < |S|$, by

$$d^n = \sum_{\substack{I \subset S \\ |I| = n}} \sum_{s_i \in S \setminus I} (-1)^i \pi_{I, I \cup \{s_i\}}$$

where $\pi_{I,J} : \mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{H}_J} \mathcal{H}$ is the canonical surjective homomorphism of $\mathcal{H}$-$\mathcal{H}$-bimodules, for any subsets I, J of S such that $I \subset J$.

Equivalently, D is the cochain complex associated with the coefficient system on the simplicial complex of subsets of S sending a subset I of S to $\mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$ and the inclusion $I \subset J$ to the canonical map $\pi_{I,J}$. One checks easily that $d^{n+1} \circ d^n = 0$. Note that $D^0 = \mathcal{H} \otimes_R \mathcal{H}$ and $D^{|S|} = \mathcal{H}$.

Theorem 1. *Let α be the algebra automorphism of $\mathcal{H}$ sending T_w to $(-q)^{\ell(w)}(T_{w^{-1}})^{-1}$ for all $w \in W$. The complex D has cohomology concentrated in degree zero, isomorphic to the $\mathcal{H}$-$\mathcal{H}$-bimodule ${}_\alpha\mathcal{H}$. In particular, the functor $D \otimes_{\mathcal{H}} - : \mathcal{D}^\flat(\mathcal{H}\text{-mod}) \to \mathcal{D}^\flat(\mathcal{H}\text{-mod})$ is an equivalence of categories which extends the Morita equivalence on $\mathcal{H}$-mod given by the automorphism α.*

Corollary. *For any $\mathcal{H}$-module M the complex $D \otimes_{\mathcal{H}} M$ has cohomology concentrated in degree 0 isomorphic to ${}_\alpha M$.*

An equivalent result to the above theorem has independently been obtained by Parshall and Scott [PS, 3.4]. Following [Iw, 5.4] the automorphism α of $\mathcal{H}$ has first been considered by O. Goldmann.

Tensoring D by the index representation yields the q-analogue C_q of the Coxeter complex defined by Deodhar in [Deo] and by Mathas in [Ma]:

Theorem 2. *There is an isomorphism of complexes of left $\mathcal{H}$-modules $D \otimes_{\mathcal{H}} R \cong C_q$.*

There are several reasons to think that the derived equivalence in Theorem 1 should in fact be a homotopy equivalence. For $q = 1$ the complex C_q is isomorphic to the Coxeter complex C of RW-modules, and the complex D is isomorphic to the "diagonally induced" Coxeter complex of RW-RW-bimodules $\mathrm{Ind}_{\Delta W}^{W \times W}(C)$ (see, e.g., [LS]), of which it is known that it induces a homotopy equivalence (consequence of [Ri1, §8]). Furthermore, the complex D and the complex X inducing the Alvis–Curtis duality defined for example by M.Cabanes and J.Rickard in [CaRi] are shown in [Sch, 4.5.2] to be related via the Schur functor. Since one expects X to induce a homotopy equivalence [CaRi, 5.2] one should certainly also expect D to induce a homotopy equivalence.

Conjecture. *The functor $D \otimes_{\mathcal{H}} - : \mathcal{K}^\flat(\mathcal{H}\text{-mod}) \to \mathcal{K}^\flat(\mathcal{H}\text{-mod})$ is an equivalence of categories.*

For J a subset of S, we denote by ${}^J W$ the full set of left coset representatives of W/W_J given by all w in W such that for all s in J, $\ell(ws) > \ell(w)$. In particular, the elements of ${}^J W$ are the unique elements of minimal length in their coset. Therefore whenever we have for $w \in W$ that $w = xv$ with $x \in {}^J W$ and $v \in W_J$ then $\ell(w) = \ell(x) + \ell(v)$. Let w_0 be the longest element in W. In order to prove Theorem 1 we need the following lemma.

Lemma. *For any $w \in W$ we have $T_{w_0} T_w T_{w_0}^{-1} = T_{w_0 w w_0^{-1}}$; in particular the R-linear map on $\mathcal{H}_W$ sending T_w to $T_{w_0 w w_0^{-1}}$ is an inner algebra automorphism of $\mathcal{H}_W$.*

Proof. We will show that $T_{w_0} T_w = T_{w_0 w w_0^{-1}} T_{w_0}$ by induction over $\ell(w)$. If $s \in S$ then $T_{w_0} T_s = q T_{w_0 s} + (q-1) T_{w_0} = q T_{(w_0 s w_0^{-1}) w_0} + (q-1) T_{w_0} = T_{w_0 s w_0^{-1}} T_{w_0}$. If $w = w's$ with $s \in S$ and $\ell(w') < \ell(w)$ then by the above calculation and by induction we get $T_{w_0} T_w = T_{w_0} T_{w'} T_s = T_{w_0 w' w_0^{-1}} T_{w_0} T_s = T_{w_0 w' w_0^{-1}} T_{w_0 s w_0^{-1}} T_{w_0} = T_{w_0 w w_0^{-1}} T_{w_0}$. The last equality holds since $\ell(w_0 w w_0^{-1}) = \ell(w_0 w' w_0^{-1}) + \ell(w_0 s w_0^{-1})$. The result follows. □

Proof of Theorem 1. By shifting and renumbering the terms we view the augmented Coxeter complex C associated to the finite Coxeter group (W, S) as a cochain complex concentrated in degrees 0 to $|S|$. Therefore the degree n component C^n of C is equal to the direct sum of the permutation modules $R[W/W_I]$, where I runs over the set of subsets of S such that $|I| = n$ (see, for example, [Hu, 1.15]). In particular, the degree 0 component of C is the regular RW-module RW. It is well known that C has cohomology concentrated in degree 0, isomorphic to the sign representation of W.

We denote by $\tilde{C}$ the subcomplex of C, which in all positive degrees coincides with C, and whose degree 0 term is the R-submodule of RW with basis $W - \{w_0\}$. Then $\tilde{C}$ is an acyclic complex of R-modules.

The complex $\tilde{C}$ is known to have a filtration by acyclic subcomplexes $\tilde{C}_{(p)}$ defined for any positive integer p as follows: the degree n component of $\tilde{C}_{(p)}$ is the direct sum, taken over all subsets $I \subset S$ such that $|I| = n$, of the R-submodule of $R[W/W_I]$ generated by the cosets wW_I such that $\ell(w) \leqslant p$.

Let I be a subset of S, the set $\{xW_I \mid x \in {}^IW\}$ is an R-basis of $R[W/W_I]$, and recall that the set of tensors $\{T_x \otimes T_w \mid x \in {}^IW,\ w \in W\}$ is an R-basis of $\mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$. Thus there is a unique isomorphism of right $\mathcal{H}$-modules

$$R[W/W_I] \otimes_R \mathcal{H} \cong \mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$$

sending $xW_I \otimes T_w$ to $T_x \otimes T_w$, where $x \in {}^IW$ and $w \in W$. Taking the direct sum over the set of subsets of S of these isomorphisms yields an isomorphism of graded right $\mathcal{H}$-modules

$$\eta\colon\ C \otimes_R \mathcal{H} \cong D.$$

This graded map is not in general compatible with the differentials, so this is not an isomorphism of complexes, in general. But if we set $\tilde{D} = \eta(\tilde{C} \otimes_R \mathcal{H})$ and $\tilde{D}_{(p)} = \eta(\tilde{C}_{(p)} \otimes_R \mathcal{H})$, it is nevertheless obvious that $\tilde{D}$ and $\tilde{D}_{(p)}$ are subcomplexes of right $\mathcal{H}$-modules of D. Then η induces graded isomorphisms between subsequent quotients on both sides,

$$\big(\tilde{C}_{(p)} \otimes_R \mathcal{H}\big)/\big(\tilde{C}_{(p-1)} \otimes_R \mathcal{H}\big) \cong \tilde{D}_{(p)}/\tilde{D}_{(p-1)} \tag{1}$$

for any positive integer p. The crucial point here is to observe that this is actually now an isomorphism of complexes. To see the compatibility with the differentials of the quotient complexes, we apply exactly the argument Mathas used in formula (2.3) in [Ma] combined with Remark 2.8 in that paper: consider subsets $I \subset J \subset S$ such that $|I| + 1 = |J|$. Let $x \in {}^IW$ such that $\ell(x) = p$ and let $w \in W$. We distinguish then two cases: if $x \in {}^JW$ then the isomorphism in (1) commutes with the induced differentials because in this case the element $T_x \otimes T_w$ is also a basis element of $\mathcal{H} \otimes_{\mathcal{H}_J} \mathcal{H}$. If x is not in ${}^JW$ then $x = yv$ for some $y \in {}^JW$ and some $v \in W_J$, which implies that $\ell(y) < \ell(x) = p$. Thus $xW_I \otimes T_w$ belongs to the subcomplex $\tilde{C}_{(p-1)} \otimes_R \mathcal{H}$, and similarly, the image of $xW_I \otimes T_w$ in $\mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$ belongs to the subcomplex $\tilde{D}_{(p-1)}$.

This means that $\tilde{D}$ is filtered by subcomplexes such that subsequent quotients of this filtration are acyclic. By a standard long exact cohomology sequence argument it follows that the complex $\tilde{D}$ is acyclic.

The quotient complex $D/\tilde{D}$ is concentrated in degree 0, its degree 0 term is R-free and has as a basis the set $\{T_{w_0} \otimes T_w \mid w \in W\}$ and thus is isomorphic to $\mathcal{H}$ as a right $\mathcal{H}$-module. Therefore, the cohomology of D is concentrated in degree 0 and is isomorphic to $\mathcal{H}$ as a right $\mathcal{H}$-module.

In order to calculate the $\mathcal{H}$-$\mathcal{H}$-bimodule structure of $H^0(D)$, let $\varphi : \mathcal{H} \to H^0(D)$ be the isomorphism of right $\mathcal{H}$-modules such that

$$\varphi(T_w) = T_{w_0} \otimes T_w + \sum_{x \in W_1} T_x \otimes h_x(w)$$

for some uniquely determined elements $h_x(w) \in \mathcal{H}$, where W_i is the subset of W of all elements x such that $\ell(x) \leqslant \ell(w_0) - i$ for $0 \leqslant i \leqslant \ell(w_0)$. We can rewrite $\varphi(T_w)$ in terms of w_0, S and W_2:

$$\varphi(T_w) = T_{w_0} \otimes T_w + \sum_{s \in S} T_{w_0 s} \otimes h_{w_0 s}(w) + \sum_{x \in W_2} T_x \otimes h_x(w). \tag{2}$$

Since the complex D is concentrated in degrees 0 to $|S|$ and the cohomology of D is concentrated in degree 0, $\varphi(T_w)$ is in the kernel of the differential d^0. For $s \in S$, denote by d_s the component of d^0 whose image is in $\mathcal{H} \otimes_{\mathcal{H}_{\{s\}}} \mathcal{H}$. We observe next that for all $w \in W$, we have

$$h_{w_0 s}(w) = -T_s T_w. \tag{3}$$

Indeed, if $x \in W$ and $h \in \mathcal{H}$, we write the image of $T_x \otimes h$ in the direct sum decomposition

$$\mathcal{H} \otimes_{\mathcal{H}_s} \mathcal{H} = \bigoplus_{y \in {}^{\{s\}}W} T_y \otimes \mathcal{H}$$

as $T_x \otimes h$ if $x \in {}^{\{s\}}W$ and as $T_{xs} \otimes T_s h$ otherwise. Then

$$\begin{aligned} d_s\big(\varphi(T_w)\big) &= T_{w_0 s} \otimes T_s T_w + T_{w_0 s} \otimes h_{w_0 s}(w) + \sum_{t \in S,\ t \neq s} T_{w_0 ts} \otimes T_s h_{w_0 t}(w) \\ &\quad + \sum_{x \in W_2 \cap {}^{\{s\}}W} T_x \otimes h_x(w) + \sum_{x \in W_2 \setminus {}^{\{s\}}W} T_{xs} \otimes T_s h_x(w) \end{aligned}$$

since $w_0 ts \in {}^{\{s\}}W$ when $t \neq s$ in S. Equation (3) is then a consequence of the direct sum decomposition above and the fact that for x in $W_2 \setminus {}^{\{s\}}W$ the element xs is in W_3.

In order to calculate the left action of $\mathcal{H}$ on $H^0(D)$ we make further use of the isomorphism φ. Let $t \in S$ and $w \in W$. We denote the left action of an element T_t on an element $\varphi(T_w)$ of $H^0(D)$ by $T_t.\varphi(T_w)$. More precisely, since we would like to calculate the left action of $\mathcal{H}$ on $H^0(D)$ we have to consider $\varphi^{-1}(T_t.\varphi(T_w))$. Thus if $T_t.\varphi(T_w) = \sum_{w \in W} T_w \otimes h_w$ we only consider the value of h_{w_0} corresponding to T_{w_0}. Using the decomposition in (2) we must calculate the different terms of

$$T_t.\varphi(T_w) = T_t.T_{w_0} \otimes T_w + \sum_{s \in S} T_t.T_{w_0 s} \otimes h_{w_0 s}(w) + \sum_{x \in W_2} T_t.T_x \otimes h_x(w). \tag{4}$$

By definition of the multiplication in $\mathcal{H}$ we have

$$T_t.T_{w_0} \otimes T_w = q T_{t w_0} \otimes T_w + (q-1) T_{w_0} \otimes T_w$$

and only the term $(q-1)T_{w_0} \otimes T_w$ is going to contribute. For the middle term of Eq. (4), we obtain for $s \in S$

$$T_t.T_{w_0s} \otimes h_{w_0s}(w) = \begin{cases} T_{tw_0s} \otimes h_{w_0s}(w) & \text{if } \ell(tw_0s) > \ell(w_0s), \\ (qT_{tw_0s} + (q-1)T_{w_0s}) \otimes h_{w_0s}(w) & \text{if } \ell(tw_0s) < \ell(w_0s), \end{cases}$$

but $\ell(tw_0s) > \ell(w_0s)$ implies that $tw_0s = w_0$ and therefore $T_{tw_0s} = T_{w_0}$. Thus the only term we need to consider is the term $T_t.T_{w_0t'} \otimes h_{w_0t'}(w)$ where t' is the unique element in S such that $w_0^{-1}tw_0 = t'$. Therefore t' is the image of t by the automorphism of W given by conjugation with w_0. It now follows by the preceding paragraph that

$$T_t.T_{w_0s} \otimes h_{w_0s}(w) = T_{w_0} \otimes h_{w_0(w_0tw_0^{-1})}(w) = -T_{w_0} \otimes T_{w_0tw_0^{-1}}T_w.$$

The elements of the form $T_t.T_x \otimes h_x(w)$ with $x \in W_2$ of the third term in Eq. (4) are not going to contribute to the $\mathcal{H}$-$\mathcal{H}$-bimodule structure, since for all $t \in S$, $w_0 \notin tW_2$.

Finally we have obtained

$$\varphi^{-1}\big(T_t.\varphi(T_w)\big) = \big((q-1)T_1 - T_{w_0tw_0^{-1}}\big)T_w$$

and since $\alpha(T_t) = -q(T_t)^{-1} = (q-1)T_1 - T_t$ it follows from the lemma that

$$\varphi^{-1}\big(T_t.\varphi(T_w)\big) = T_{w_0}\alpha(T_t)(T_{w_0})^{-1}T_w.$$

That is, $H^0(D) \cong {}_\beta\mathcal{H}$ as $\mathcal{H}$-$\mathcal{H}$-bimodule, where β is the algebra automorphism obtained by composing α with the inner automorphism given by conjugation by T_{w_0}. Thus ${}_\beta\mathcal{H} \cong {}_\alpha\mathcal{H}$ and finally $H^0(D) \cong {}_\alpha\mathcal{H}$.

We are now able to prove the last part of the theorem: since $\mathcal{H}$ is a symmetric algebra and all the terms of D are finitely generated and projective as left and right $\mathcal{H}$-modules, by [Ri2] all we need to show is that

$$D \otimes_{\mathcal{H}} D^\vee \cong \mathcal{H} \quad \text{and} \quad D^\vee \otimes_{\mathcal{H}} D \cong \mathcal{H}$$

in the derived category of $\mathcal{H}$-$\mathcal{H}$-bimodules, where $D^\vee$ denotes the R-linear dual of D. However, since the cohomology of D is concentrated in degree 0, where it is isomorphic to $\mathcal{H}$, D is exact in every positive degree. Since the differential $d^{|S|-1}$ is surjective onto $\mathcal{H}$, its image is projective as a right $\mathcal{H}$-module. Therefore the short exact sequences of right $\mathcal{H}$-modules

$$0 \to \mathrm{Ker}\big(d^k\big) \to D^k \to \mathrm{Im}\big(d^k\big) \to 0$$

are split for all $k > 0$. Thus the cycles $Z^*(D)$ and the boundaries $B^*(D)$ are projective as right $\mathcal{H}$-modules. Therefore by the Künneth formula $H^*(D \otimes_{\mathcal{H}} D^\vee) \cong H^*(D) \otimes_{\mathcal{H}} H^*(D^\vee)$. Since $H^0(D) \cong {}_\alpha\mathcal{H}$ and $H^*(D^\vee) = H^0(D^\vee) \cong ({}_\alpha\mathcal{H})^\vee \cong \mathcal{H}_\alpha$ we obtain $H^*(D) \otimes_{\mathcal{H}} H^*(D^\vee) \cong {}_\alpha\mathcal{H} \otimes_{\mathcal{H}} \mathcal{H}_\alpha \cong {}_\alpha\mathcal{H}_\alpha$. This implies the first isomorphism since ${}_\alpha\mathcal{H}_\alpha \cong \mathcal{H}$ as $\mathcal{H}$-$\mathcal{H}$-bimodules. The second isomorphism is obtained in the same way by considering left instead of right $\mathcal{H}$-modules. □

Proof of Theorem 2. Let

$$G = - \otimes_{\mathcal{H}} R : (\mathcal{H} \otimes_R \mathcal{H}^\circ)\text{-mod} \to \mathcal{H}\text{-mod}$$

be the functor induced by tensoring with the index representation of $\mathcal{H}$. Since for $I \subset S$ the terms of D are direct sums of the $\mathcal{H}$-$\mathcal{H}$-bimodules $\mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$ and the terms of C_q are direct sums of $\mathcal{H}$-modules of the form $\mathcal{H} \otimes_{\mathcal{H}_I} R$, the terms of the complex $G(D) = D \otimes_{\mathcal{H}} R$ are isomorphic as left $\mathcal{H}$-modules to the terms of C_q.

In order to show that the differentials coincide we have to show that for $I \subset J \subset S$ the maps $G(\pi_{I,J})$ induce the differential of C_q. But if $T_x \otimes T_y \in \mathcal{H} \otimes_{\mathcal{H}_I} \mathcal{H}$ then its canonical image in $\mathcal{H} \otimes_{\mathcal{H}_I} R$ is equal to $T_x \otimes q^{\ell(y)}$ and similarly the canonical image of $\pi_{I,J}(T_x \otimes T_y) = T_x \otimes T_y \in \mathcal{H} \otimes_{\mathcal{H}_J} \mathcal{H}$ in $\mathcal{H} \otimes_{\mathcal{H}_J} R$ is equal to $G(\pi_{I,J})(T_x \otimes q^{\ell(y)}) = T_x \otimes q^{\ell(y)}$ and thus $G(\pi_{I,J})$ clearly induces the differential of C_q. □

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