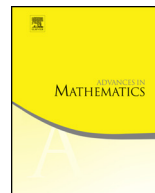




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Mean conservation of nodal volume and connectivity measures for Gaussian ensembles

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ABSTRACT

We study in depth the nesting graph and volume distribution of the nodal domains of a Gaussian field, which have been shown in previous works to exhibit asymptotic laws. A striking link is established between the asymptotic mean connectivity of a nodal domain (i.e. the vertex degree in its nesting graph) and the positivity of the percolation probability of the field, along with a direct dependence of the average nodal volume on the percolation probability. Our results support the prevailing ansatz that the mean connectivity and volume of a nodal domain is conserved for generic random fields in dimension $d = 2$ but not in $d \geq 3$, and are applied to a number of concrete motivating examples.

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1. Introduction

1.1. The connectivity measure for nodal domains of Euclidean Gaussian fields

Let $d \geq 2$ and $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a centred stationary C^3 -smooth Gaussian field. We are interested in the topological structure of the nodal set $\mathcal{A}(F) := F^{-1}(0)$, of high importance in various disciplines including oceanography [24], engineering [31,39] and cosmology (see for example [11,30] and the references therein). A *nodal component* of F is a connected component of $\mathcal{A}(F)$, and a *nodal domain* is a connected component of the complement $\mathbb{R}^d \setminus \mathcal{A}(F)$; as, under very mild conditions on the law of F to be prescribed below, F and ∇F have a.s. no common zeros by Bulinskaya's Lemma, its nodal set has no self-intersections. We may further encode the topological structure of $\mathcal{A}(F)$ as follows: let Ω_F be the (a.s. locally finite) collection of nodal domains of F , let \mathcal{C}_F be the collection of nodal components of F , and define the *nesting graph* $G = G(F) = (V, E)$ with vertex set $V = V(F) = \Omega_F$ and edge set $E = E(F) = \mathcal{C}_F$ so that two domains $v_1, v_2 \in V$ are adjacent in G via $e \in E$ if the corresponding domains share e as a common boundary component. Fig. 1 exhibits a fragment of the nesting graph G for some sample function F_ω .

Sarnak and Wigman [36] studied the nesting graph G for a generic stationary Gaussian field F . Observe that, by Jordan's Theorem, G is a.s. an (infinite) tree, and so the structure of G is largely encapsulated by the degrees of the vertexes V ('the connectivity of the nodal domains'). Under very mild extra assumptions (to be given in §1.3), Sarnak–Wigman established a law of large numbers for the connectivity in the following sense. Let $B(R) \subseteq \mathbb{R}^d$ denote the ball of radius R , and let $G(R) = (V(R), E(R))$ be the restriction of G to $B(R)$, i.e. the graph induced by restricting G to the vertices $V(R) \subseteq V$ that correspond to domains that are *fully contained within* $B(R)$; $G(R)$ might fail to be a tree but is necessarily a collection of disjoint trees (a forest). Letting

$$d(v) = d_R(v) \in \mathbb{Z}_{\geq 0}$$

denote the degree of $v \in V(R)$ (w.r.t. $G(R)$), define the *empirical connectivity measure*

$$\mu_{\Gamma(F);R} = \frac{1}{|V(R)|} \sum_{v \in V(R)} \delta_{d(v)} \quad (1.1)$$

on $\mathbb{Z}_{\geq 0}$. Sarnak–Wigman showed [36, Theorem 3.3] that, for a wide class of stationary Gaussian fields F , as $R \rightarrow \infty$ the (random) probability measure $\mu_{\Gamma(F);R}$ tends to a *deterministic* probability measure $\mu_{\Gamma(F)}$ on $\mathbb{Z}_{\geq 0}$ ('the limit connectivity measure') that depends on the law of F . More precisely, they proved that

$$\mathcal{D}(\mu_{\Gamma(F);R}, \mu_{\Gamma(F)}) \rightarrow 0$$

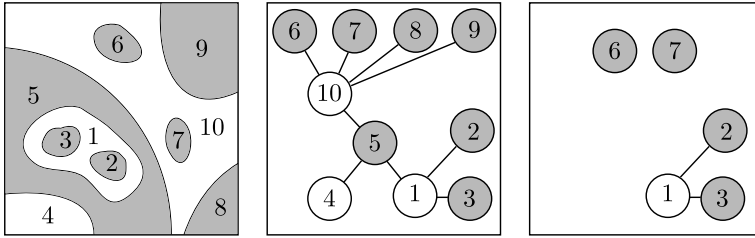


Fig. 1. Left: A sketch of the nodal domains of a function (grey and white are positive and negative domains respectively). Middle: The connectivity graph of the nodal domains; note that some of the domains might connect outside of the box, but this graph takes into account only the domains of the restricted function. Right: the corresponding graph $G(R)$, whose vertices correspond to nodal domains that are completely inside the box; typically, this graph is not a tree but a forest.

in probability as $R \rightarrow \infty$, with $\mathcal{D}(\cdot, \cdot)$ the total variation distance on probability measures on $\mathbb{Z}_{\geq 0}$ (see (1.7) below).

The properties of the limit connectivity measure $\mu_{\Gamma(F)}$ are of fundamental importance, and Sarnak–Wigman raised [36, p. 13] the question of the *mean* connectivity of the limit measure $\mu_{\Gamma(F)}$. Since G (and hence $G(R)$) contains no cycles, the mean of the empirical connectivity measures $\mu_{\Gamma(F);R}$ satisfy

$$\begin{aligned} \sum_{k=0}^{\infty} k \cdot \mu_{\Gamma(F);R}(k) &= \frac{1}{|V(R)|} \sum_{v \in V(R)} d(v) = \frac{1}{|V(R)|} \cdot 2|E(R)| \\ &\leq \frac{1}{|V(R)|} \cdot 2(|V(R)| - 1) = 2 - \frac{2}{|V(R)|} \leq 2. \end{aligned}$$

One can deduce via Fatou’s lemma [36, p. 31] that

$$\sum_{k=0}^{\infty} k \cdot \mu_{\Gamma(F)}(k) \leq 2;$$

i.e. the mean connectivity of the limit measure $\mu_{\Gamma(F)}$ is bounded by 2. It is then crucial to determine whether the equality

$$\sum_{k=0}^{\infty} k \cdot \mu_{\Gamma(F)}(k) = 2 \tag{1.2}$$

holds, for if it does *not*, then this indicates a non-local ‘escape of topology’ when passing to the limit.

Numerical experiments of Barnett–Jin (presented within [36]) seem to indicate that (1.2) *fails* for the monochromatic random wave and some band-limited Gaussian fields on \mathbb{R}^2 , the motivational examples of [36] (see §2.1 below for definitions). On the other hand, this may be a numerical artefact due to the slow convergence of the series on the l.h.s. of (1.2), reflecting the slow conjectured decay of $\mu_{\Gamma(F)}(k)$. Indeed, borrowed from

percolation theory (as inspired by [9]), it is plausible [36,18] that $\mu_{\Gamma(F)}(k)$ decays only as

$$\mu_{\Gamma(F)}(k) \approx \frac{1}{k^\alpha}, \quad \alpha = \frac{187}{91} = 2.0549\dots, \quad (1.3)$$

where α is the ‘Fisher exponent’ that describes the *area* distribution of percolation clusters in Bernoulli percolation on \mathbb{Z}^2 (suggesting [18] that the connectivity of a typical domain is proportional to its area); the numerical investigations of Barnett–Jin for band-limited Gaussian fields showed consistency with (1.3), although the results were not conclusive.

In this manuscript we address the question of whether (1.2) holds for a wide class of smooth Gaussian fields. We believe that, contrary to Barnett–Jin’s numerics, our results serve as striking evidence that (1.2) *does* hold for generic fields on \mathbb{R}^2 , including all the examples considered by Sarnak–Wigman. More precisely, our main result (Theorem 1.3 below) shows that (1.2) is essentially equivalent to the nodal domains of F *failing to percolate*, in a sense to be made rigorous. Since, in light of [1] (implying, under milder conditions, no percolation for random fields possessing positive correlations), and [3,5,9], the nodal domains of a generic F do not percolate if $d = 2$, and, in line with [12,15,40], do percolate in higher dimensions (see the discussion in §1.2 below), we believe that the equality (1.2) holds for generic random fields on \mathbb{R}^d *if and only if* $d = 2$. To support our statement we establish this claim rigorously for a particular class of Gaussian fields on \mathbb{R}^2 , including the important special case of the Bargmann–Fock field (see §2.1.1 for details).

1.2. Percolation probabilities for random fields

The study of the percolation of excursion sets of random fields was initiated by Molchanov–Stepanov [25]. For a random field

$$F : \mathbb{R}^d \rightarrow \mathbb{R}$$

and a number $u \in (-\infty, +\infty)$ one is interested whether the excursion set² $F^{-1}(u, \infty)$ *percolates*, i.e. contains an *unbounded component*. They found [25] that, much like in lattice percolation, there exists a *critical level* $u^* = u^*(F) \in [-\infty, +\infty]$, finite or infinite, so that for $u > u^*$, $F^{-1}(u, +\infty)$ does not percolate a.s., whereas for $u < u^*$, $F^{-1}(u, +\infty)$ does percolate a.s. (with little information at $u = u^*$, although it is expected that there is no percolation at the critical level). Various criteria for when the critical level u^* is finite were also addressed [25,26], along with other related questions.

In our case one is only interested in the *nodal set*, being the boundary of $F^{-1}(0, +\infty)$, and so the question whether $u^* > 0$ or $u^* \leq 0$ is crucial; indeed if $u^* > 0$ then a.s. there

² In the original treatment $F^{-1}(-\infty, u)$ is studied.

exist giant percolating nodal domains (likely unique up to sign) that cover a positive proportion of the entire space (see, e.g., [10, Theorem 16 on p. 76]). Although this question has been resolved rigorously in only a few special cases, the picture that has emerged from the physics literature (see, e.g., [9]) is that $u^* = 0$ for generic centred random fields on \mathbb{R}^2 , with no percolation of the nodal domains. Early work of Alexander [1] proved that the level lines $\{F(x) = u\}$ of a stationary-ergodic planar positive-correlated Gaussian field are a.s. bounded, which by the symmetry of centred Gaussian fields implies immediately that $u^* \leq 0$. Moreover, Bogomolny–Schmidt [9] gave a heuristic argument demonstrating that $u^* = 0$ for the monochromatic random wave on the plane, essentially by comparing the random wave model to critical Bernoulli percolation on the square lattice \mathbb{Z}^2 (i.e. where every edge is included independently with probability $p = 1/2$). Very recent results [3,33,27] have confirmed that $u^* = 0$ for a family of planar Gaussian fields with positive and rapidly-decaying correlations, and also verified the absence of percolation of the nodal domains; an important example to which these results apply is the Bargmann–Fock field (see §2.1.1 below).

On the other hand, numerical experiments recently conducted by Barnett–Jin (presented within [36]) indicate that, somewhat surprisingly, a generic centred random field F on \mathbb{R}^d , $d \geq 3$, *does* possess giant nodal component consuming a huge proportion of the space. Sarnak [35] observed that this distinction could be attributed to the fact that, for $d \geq 3$, the critical level u^* is likely to be strictly positive, i.e. the nodal domains correspond to the supercritical regime for $d \geq 3$ (whereas for $d = 2$ they correspond to the critical regime). This is consistent with recent results of Drewitz–Prévost–Rodriguez [15] who proved that $u^* > 0$ for a family of *strongly correlated* Gaussian fields on \mathbb{Z}^3 , including the important case of the massless harmonic crystal system with correlations decaying as

$$r_F(x, y) \approx \frac{1}{|x - y|}, \quad x, y \in \mathbb{Z}^3,$$

(the fact that $0 \leq u^* < \infty$ had been previously established in [12]).

To state our results we make use of the following notion of *percolation probability*:

Definition 1.1 (*Percolation probability associated to a Gaussian field*). Let $d \geq 2$ and $F : \mathbb{R}^d \rightarrow \mathbb{R}$ a C^3 -smooth stationary Gaussian field.

(1) For two closed sets $A, B \subseteq \mathbb{R}^d$, we define the event

$$\left\{ A \overset{F}{\longleftrightarrow} B \right\}$$

that there exists a nodal domain of F whose closure intersects both A and B (' A and B are connected by a nodal domain of F '). If A is the boundary of a nodal domain, then this means that there is a nodal domain *adjacent* to A whose closure intersects B .

- (2) The percolation probability associated to F is the probability

$$\mathcal{P}^F := \mathcal{P}r \left(\left\{ 0 \overset{F}{\longleftrightarrow} \infty \right\} \right)$$

of the event that the origin is contained in an unbounded nodal domain of F (note the slight abuse of notation, where we replace a point with the corresponding singleton). Equivalently,

$$\mathcal{P}^F := \lim_{R \rightarrow \infty} \mathcal{P}r \left(\left\{ 0 \overset{F}{\longleftrightarrow} \partial[-R, R]^d \right\} \right)$$

is the limit probability of the event that the origin is contained in a nodal domain of F intersecting the boundary of a large cube $[-R, R]^d \subseteq \mathbb{R}^d$.

- (3) We say that F *percolates* if the associated percolation probability $\mathcal{P} = \mathcal{P}^F$ is strictly positive.

It is evident that, for a continuous stationary random field, $\mathcal{P} = 0$ if $u^* < 0$, and $\mathcal{P} > 0$ if $u^* > 0$; it is moreover strongly believed, and in some cases rigorously known, that $\mathcal{P} = 0$ in the case $u^* = 0$. In light of the above discussion, it is natural to expect that, for a generic centred random field on \mathbb{R}^d , $\mathcal{P} > 0$ if and only if $d \geq 3$.

1.3. Mean (non-)conservation for connectivity measures, Euclidean case

A centred continuous Gaussian field $F : \mathbb{R}^d \rightarrow \mathbb{R}$ is uniquely determined, via Kolmogorov's Theorem, by its covariance function

$$r_F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad r_F(x, y) := \mathbb{E}[F(x) \cdot F(y)].$$

If F is stationary then, with the usual abuse of notation,

$$r_F(x, y) = r_F(0, x - y) =: r_F(x - y),$$

where now $r_F : \mathbb{R}^d \rightarrow \mathbb{R}$. Equivalently, F is determined by the *spectral measure* $\rho = \rho_F$ of F , which is the Fourier transform of r_F ; ρ is a positive measure on \mathbb{R}^d by Bochner's Theorem, and without loss of generality we may assume that ρ is a *probability* measure (this corresponds to fixing $r_F(0) = 1$). As in Nazarov–Sodin [38,29] and Sarnak–Wigman [36], we make the following basic assumptions on ρ :

Definition 1.2 (*Axioms on the spectral measure*).

- ($\rho 1$) The measure ρ has no atoms.
 ($\rho 2$) For some $p > 6$,

$$\int_{\mathbb{R}^d} \|\lambda\|^p d\rho(\lambda) < \infty.$$

($\rho 3$) The support of ρ does not lie in a hyperplane in \mathbb{R}^d .

These assumptions imply, respectively, that f is ergodic, has C^3 -smooth sample paths a.s., and is non-degenerate.

Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a centred stationary Gaussian field whose spectral measure ρ_F satisfies the axioms ($\rho 2$)–($\rho 3$). Nazarov–Sodin considered the total number $\mathcal{N}(F; R)$ of nodal domains of F lying entirely within a large ball $B(R)$, and proved [38,29] that there exists a number $c_{NS}(\rho) \geq 0$ (“the Nazarov–Sodin constant of F ”) such that, as $R \rightarrow \infty$, we have

$$\mathbb{E}[\mathcal{N}(F; R)] = c_{NS}(\rho) \cdot \text{Vol } B(R) + o_{R \rightarrow \infty}(R^d).$$

Under the additional assumption ($\rho 1$), they moreover established the convergence in mean

$$\mathbb{E} \left[\left| \frac{\mathcal{N}(F; R)}{\text{Vol } B(R)} - c_{NS}(\rho) \right| \right] \rightarrow 0, \quad \text{as } R \rightarrow \infty, \quad (1.4)$$

in particular implying a version of the law of large numbers

$$\mathcal{P}r \left(\left| \frac{\mathcal{N}(F; R)}{\text{Vol } B(R)} - c_{NS}(\rho) \right| > \epsilon \right) \rightarrow 0, \quad \text{as } R \rightarrow \infty, \quad (1.5)$$

for every $\epsilon > 0$. We will also need the assumption

($\rho 4$) $c_{NS}(\rho) > 0$,

satisfied in most natural examples, which endows $\mathcal{N}(F; R)$ with the proper asymptotic scaling in (1.4) (in fact, if ($\rho 4$) fails then $\mathcal{N}(F; R) = 0$ a.s.).

Recall the construction of the (random) nesting graph $G = G(F) = (V, E)$ corresponding to F in §1.1, and recall also the restriction $G(R)$ to the ball $B(R)$ and the empirical connectivity measure $\mu_{\Gamma(F); R}$ defined in (1.1). Under the assumptions ($\rho 1$)–($\rho 4$), Sarnak–Wigman [36] established the existence of a (deterministic) probability measure $\mu_{\Gamma(F)}$ on $\mathbb{Z}_{\geq 0}$, such that for every $\epsilon > 0$,

$$\lim_{R \rightarrow \infty} \mathcal{P}r \left(\mathcal{D}(\mu_{\Gamma(F); R}, \mu_{\Gamma(F)}) > \epsilon \right) = 0, \quad (1.6)$$

where the distance function $\mathcal{D}(\cdot, \cdot)$ is defined as

$$\mathcal{D}(\mu_1, \mu_2) := \sup_{A \subseteq \mathbb{Z}_{\geq 0}} |\mu_1(A) - \mu_2(A)|. \quad (1.7)$$

Moreover, under a mild further condition on ρ_F , Sarnak–Wigman [36] showed that $\mu_{\Gamma(F)}$ charges the whole of $\mathbb{Z}_{\geq 1}$. Our first principal result asserts that, under the above assump-

tions on F , the mean connectivity of the limit distribution $\mu_{\Gamma(F)}$ is equal to 2 if and only if the percolation probability \mathcal{P} is zero:

Theorem 1.3. *Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous stationary Gaussian field whose spectral measure satisfies $(\rho 1)$ – $(\rho 4)$. Let $\mu_{\Gamma(F)}$ be the limit connectivity measure defined in (1.6). Denote $\mathcal{P} = \mathcal{P}^F$ to be the percolation probability associated to F as in Definition 1.1. Then the equality*

$$\sum_{k=0}^{\infty} k \cdot \mu_{\Gamma(F)}(k) = 2$$

holds if and only if $\mathcal{P} = 0$.

1.4. Mean (non-)conservation for volume distribution

Other than the connectivity, one is also interested in the *empirical volume distribution* of the nodal domains. Recall that $\mathcal{N}(F; R)$ denotes the total number of nodal domains of F entirely lying inside a large ball $B(R)$. For $t > 0$, let $\mathcal{N}(F, t; R)$ denote the number of such nodal domains entirely lying in $B(R)$ of volume $< t$. Refining the work of Nazarov–Sodin, Beliaev–Wigman [6, Theorem 3.1] established that, under the same assumptions $(\rho 1)$ – $(\rho 4)$ on ρ , the empirical volume distribution $\mathcal{N}(F, t; R)$ obeys a law of large numbers. More precisely, there exists a (deterministic) cumulative distribution function $\Psi_F : \mathbb{R}_{>0} \rightarrow [0, 1]$ such that, for all continuity points t of $\Psi_F(\cdot)$,

$$\mathbb{E} \left[\left| \frac{\mathcal{N}(F, t; R)}{c_{NS}(\rho) \cdot \text{Vol } B(R)} - \Psi_F(t) \right| \right] \rightarrow 0 \quad (1.8)$$

as $R \rightarrow \infty$, or equivalently (in light of (1.4)),

$$\mathbb{E} \left[\left| \frac{\mathcal{N}(F, t; R)}{\mathcal{N}(F; R)} \cdot \mathbb{1}_{\mathcal{N}(F; R) > 0} - \Psi_F(t) \right| \right] \rightarrow 0;$$

(the indicator controls the negligible event that $B(R)$ contains no nodal domains).

Similarly to the question as to whether (1.2) holds for the limit connectivity distribution, one is also interested in the mean volume of the limit distribution Ψ_F :

$$\int_0^{\infty} (1 - \Psi_F(t)) dt. \quad (1.9)$$

Bearing in mind that, in light of Nazarov–Sodin’s (1.4), the ‘empirical mean volume’ should be about

$$\frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} \approx \frac{\text{Vol } B(R)}{c_{NS}(\rho) \cdot \text{Vol } B(R)} = \frac{1}{c_{NS}(\rho)},$$

one might expect the mean (1.9) to be equal to $\frac{1}{c_{NS}(\rho)}$. As such, one wishes to verify whether the triple equality

$$\int_0^\infty (1 - \Psi_F(t)) dt = \frac{1}{c_{NS}(\rho)} = \lim_{R \rightarrow \infty} \frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} \cdot \mathbb{1}_{\mathcal{N}(F; R) > 0} \quad (1.10)$$

holds, where the convergence of the r.h.s. of (1.10) is understood *in mean*; for if (1.10) fails, then this indicates an ‘escape of mass’ in the limit. Our second main result (Theorem 1.5 below) again verifies a connection between the ‘escape of mass’ and the percolation probability \mathcal{P} . Indeed, compared to Theorem 1.3, we are able to explicitly quantify the failure of (1.10) as a function of the percolation probability (see (1.12)). To state our result in full, we need to introduce one further assumption on the Gaussian field F :

Definition 1.4 (*Nodal lower concentration*). Suppose that the number of nodal domains of a stationary Gaussian field $F : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies the law of large numbers (1.5). Then we say that F satisfies the *nodal lower concentration property* if, for every $\epsilon > 0$,

$$\mathcal{P}r \left(\frac{\mathcal{N}(F; R)}{\text{Vol } B(R)} < c_{NS}(\rho) - \epsilon \right) = o_{R \rightarrow \infty} \left(\frac{1}{R^d} \right). \quad (1.11)$$

Compared to the law of large numbers (1.5), the nodal lower concentration property (1.11) *quantifies* the decay of the lower tail of $\mathcal{N}(F; R)/\text{Vol } B(R)$. Rivera–Vanneuville [32, Theorem 1.4] and Beliaev–Muirhead–Rivera [4] recently proved that F satisfies the nodal lower concentration property provided that the covariance function of F decays sufficiently quickly. In particular, it is sufficient that

$$r_F(x) \leq |x|^{-3d-\delta}$$

for some $\delta > 0$ and x sufficiently large.

Theorem 1.5. *Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous stationary Gaussian field whose spectral measure satisfies $(\rho 1) - (\rho 4)$. Let $\Psi = \Psi_F$ be the limit volume distribution defined in (1.8). Denote $\mathcal{P} = \mathcal{P}^F$ to be the percolation probability associated to F as in Definition 1.1. Then*

(a) *The mean of the limit volume distribution Ψ_F is*

$$\int_0^\infty (1 - \Psi_F(t)) dt = (1 - \mathcal{P}) \cdot \frac{1}{c_{NS}(\rho)}. \quad (1.12)$$

(b) If F moreover satisfies the nodal lower concentration property, then the empirical volume mean converges to $\frac{1}{c_{NS}(\rho)}$ in mean, i.e.

$$\lim_{R \rightarrow \infty} \mathbb{E} \left[\left| \frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} \cdot \mathbb{1}_{\mathcal{N}(F; R) > 0} - \frac{1}{c_{NS}(\rho)} \right| \right] = 0. \quad (1.13)$$

Theorem 1.5 shows that the first equality of (1.10) holds if and only if $\mathcal{P} = 0$, much like (the less explicit) Theorem 1.3 regarding the connectivity measure. On the other hand, the second equality of (1.10) holds for all fields satisfying the nodal lower concentration property regardless of whether $\mathcal{P} = 0$. We leave open the question of whether in fact the empirical volume mean converges to $1/c_{NS}(\rho)$ in full generality. Indeed it is plausible that all Gaussian fields satisfying $(\rho 1)$ – $(\rho 4)$ satisfy the nodal lower concentration property.

Finally, we would like to point out that it is conceivable, that an elegant transport principle argument, of the type used within [7] in a discrete setup, of a similar flavour to our settings, would also yield Theorem 1.5(a). However, we were not able to see how to apply it without dealing with the same subtleties that required our attention within our own, ad hoc, proof.

1.5. Ensembles of Gaussian fields on a manifold

In applications, rather than dealing with a single random field on \mathbb{R}^d , one is often given an *ensemble* (or sequence) of Gaussian fields, all defined on some fixed Riemannian manifold, that converge to a local limit. In this setting the ‘escape of mass’ for the volume distribution has a slightly different meaning than (1.10) (see Theorem 1.9 below).

Let us first make precise the setting in which we work. Let \mathcal{M} be a smooth compact Riemannian d -dimensional manifold, and let $\{\Phi_L\}_{L \in \mathcal{L}}$ be an ensemble of Gaussian fields $\Phi_L : \mathcal{M} \rightarrow \mathbb{R}$, with $\mathcal{L} \subseteq \mathbb{R}$ some discrete index set. Given a point $x \in \mathcal{M}$ and a sufficiently small neighbourhood U of x , we may identify U with a Euclidean sub-domain via the exponential map. More precisely, the exponential map

$$\exp_x : T_x(\mathcal{M}) \rightarrow \mathcal{M}$$

is a local isometry between a sufficiently small neighbourhood $U \subseteq T_x(\mathcal{M}) \equiv \mathbb{R}^d$ and

$$\exp_x(U) \subseteq \mathcal{M},$$

and by the compactness of \mathcal{M} we can choose U independent of x (under the identification $T_x(\mathcal{M}) \equiv \mathbb{R}^d$, where we identify $0 \in U$ with $x \in \mathcal{M}$). Hence, for every x we may induce a Gaussian field on a domain in \mathbb{R}^d and scale it using the linear structure of \mathbb{R}^d . That is, for $U \subseteq \mathbb{R}^d$ so that $\exp_x : U \rightarrow \exp_x(U)$ is bijective, we define the *scaled* Gaussian fields $\Phi_{x;L} : L \cdot U \rightarrow \mathbb{R}$ on the increasing domains

$$L \cdot U = \{Lu : u \in U\}$$

to be

$$\Phi_{x;L}(u) := \Phi_L(\exp_x(u/L)). \quad (1.14)$$

The covariance function of $\Phi_{x;L}$ is the function

$$r_{x;L}(u, v) := \mathbb{E}[\Phi_L(\exp_x(u/L)) \cdot \Phi_L(\exp_x(v/L))] = r_{\Phi_L}(\exp_x(u/L), \exp_x(v/L)), \quad (1.15)$$

defined for $u, v \in L \cdot U$, where r_{Φ_L} is the covariance function of Φ_L . Following Nazarov–Sodin [38,29], we consider only the situation in which the ensemble $\{\Phi_L\}_{L \in \mathcal{L}}$ possesses a ‘translation invariant local limit’:

Definition 1.6 (*Scaling limits for Gaussian ensembles.*). Let $\{\Phi_L\}_{L \in \mathcal{L}}$ be a Gaussian ensemble on \mathcal{M} , and let $r_{x;L}$ be given by (1.15). We say that $\{\Phi_L\}_{L \in \mathcal{L}}$ possesses a *translation invariant local limit* as $L \rightarrow \infty$, if, for almost all $x \in \mathcal{M}$, there exists a continuous covariance kernel $K_x : \mathbb{R}^d \rightarrow \mathbb{R}$ of a stationary Gaussian field on \mathbb{R}^d , so that for all $R > 0$,

$$\lim_{L \rightarrow \infty} \sup_{|u|, |v| < R} |r_{x;L}(u, v) - K_x(u - v)| = 0. \quad (1.16)$$

Definition 1.6 is applicable to a number of motivational examples (e.g. Kostlan’s ensemble, or band-limited fields, see §2.1 below). Moreover, in these examples K_x is independent of x , and so we can associate to the ensemble a single limiting Gaussian field $F : \mathbb{R}^d \rightarrow \mathbb{R}$ with covariance $K = K_x$. We will also need the following notions of smoothness and non-degeneracy of $\{\Phi_L\}_{L \in \mathcal{L}}$, cf. [38, Definitions 2-3], holding in all the interesting examples.

Definition 1.7 (*Uniform smoothness and non-degeneracy, [38, Definitions 2-3]*).

(1) We say that $\{\Phi_L\}_{L \in \mathcal{L}}$ is C^{3-} -smooth, if for every $R > 0$,

$$\limsup_{L \rightarrow \infty} \sup \{ |\partial_u^i \partial_v^j r_{x;L}(u, v)| : |i|, |j| \leq 3, x \in \mathcal{M}, \|u\|, \|v\| \leq R \} < \infty.$$

(2) We say that $\{\Phi_L\}_{L \in \mathcal{L}}$ is non-degenerate, if for every $R > 0$,

$$\liminf_{L \rightarrow \infty} \inf \{ \mathbb{E} [|\partial_\xi \Phi_{x;L}(u)|^2] : \xi \in \mathcal{S}^{d-1}, x \in \mathcal{M}, |u| \leq R \} > 0.$$

Assume now that the above holds (i.e. $\{\Phi_L\}_{L \in \mathcal{L}}$ is C^{3-} -smooth, non-degenerate, possessing a translation invariant local limit independent of x). Let $\mathcal{N}(\Phi_L)$ denote be the total number of nodal domains of Φ_L , and for $t > 0$, let $\mathcal{N}(\Phi_L, t)$ denote be the

number of those of (Riemannian) volume $< t$. In this setting Nazarov–Sodin [38,29] proved that

$$\mathbb{E} \left[\left| \frac{\mathcal{N}(\Phi_L)}{L^d \text{Vol}(\mathcal{M})} - c_{NS}(\rho) \right| \right] \rightarrow 0, \quad (1.17)$$

and Beliaev–Wigman [6, Theorem 1.5] proved that,³ if $\Psi(\cdot) = \Psi_F(\cdot)$ is the cumulative distribution function for F (i.e. (1.8) is satisfied), then for every continuity point t of $\Psi(\cdot)$, one has

$$\mathbb{E} \left[\left| \frac{\mathcal{N}(\Phi_L, t/L^d)}{\mathcal{N}(\Phi_L)} - \Psi(t) \right| \right] \rightarrow 0,$$

i.e., after the natural scaling, the volume distribution law tends to Ψ in mean.

Since, by the virtue of (1.12) of Theorem 1.5 applied on F , we readily know that

$$\frac{1}{c_{NS}(\rho)} = \frac{1}{1 - \mathcal{P}} \cdot \int_0^\infty (1 - \Psi(t)) dt, \quad (1.18)$$

(i.e. the first equality in (1.10) holds for the limit law), the question is whether we can relate it to the empirical volume mean

$$\frac{\text{Vol}(\mathcal{M})}{\mathcal{N}(\Phi_L)/L^d} = \frac{L^d \cdot \text{Vol}(\mathcal{M})}{\mathcal{N}(\Phi_L)}, \quad (1.19)$$

as asserted in Theorem 1.9 below, for a wide class of ensembles. Note that the corresponding question for mean connectivity trivialises, since we can use Euler’s identity on the total nesting graph on \mathcal{M} to verify that the mean connectivity of the nodal domain on \mathcal{M} converges to two in all cases. Similarly to Theorem 1.5, to state our result we need to introduce an analogous ‘nodal lower concentration property’ (cf. Definition 1.4):

Definition 1.8 (*Nodal lower concentration for ensembles, cf. Definition 1.4*). Let $\{\Phi_L\}_{L \in \mathcal{L}}$ be an ensemble of Gaussian fields possessing a translation invariant local limit as $L \rightarrow \infty$ that is independent of x . Assume further that the spectral measure ρ of the Gaussian field F corresponding to the limit covariance K satisfies $(\rho 1) - (\rho 4)$. We say that $\{\Phi_L\}_{L \in \mathcal{L}}$ satisfies the nodal lower concentration property if, for every $\epsilon > 0$,

$$\mathcal{P}r \left(\frac{\mathcal{N}(F; R)}{L^d \text{Vol}(\mathcal{M})} < c_{NS}(\rho) - \epsilon \right) = o_{L \rightarrow \infty} \left(\frac{1}{L^d} \right).$$

As an example, it is known [28, Theorem 1.1] that the random spherical harmonics (see §2.1.2 below) satisfy the nodal lower concentration property (in fact they satisfy the

³ Though stated only for band-limited functions, it is valid in the aforementioned setting.

vastly stronger exponential concentration property). Moreover it was recently shown [4] that the Kostlan ensemble of random homogeneous polynomials (see §2.1.1 below) also satisfies the nodal lower concentration property. On the other hand, imposing the lower concentration property merely on the limit random fields of a Gaussian ensemble (Definition 1.4) is unlikely to yield the lower concentration property for the ensemble, as the former does not control correlations on macroscopic scales. We are now in a position to state our theorem, asserting the asymptotic equality of (1.18) and (1.19) under suitable conditions:

Theorem 1.9. *Let $\{\Phi_L\}_{L \in \mathcal{L}}$ be a Gaussian ensemble on \mathcal{M} , C^{3-} -smooth and non-degenerate, possessing a translation invariant local limit K as $L \rightarrow \infty$ that is independent of x . Assume further that the spectral measure ρ of the Gaussian field F corresponding to the limit covariance K satisfies $(\rho 1) - (\rho 4)$, and also that $\{\Phi_L\}_{L \in \mathcal{L}}$ satisfies the nodal lower concentration property in Definition 1.8. Then*

$$\frac{L^d \text{Vol}(\mathcal{M})}{\mathcal{N}(\Phi_L)} \rightarrow \frac{1}{c_{NS}(\rho)} \quad \text{in mean.} \quad (1.20)$$

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2. Outline of the paper

In this section we discuss some applications of our main results, and also present an outline of their proofs.

2.1. Applications

Our results apply to several important ensembles of Gaussian fields on manifolds such as the sphere and torus, as well as to their scaling limits. Some of the applications are rigorous consequences of our main theorems, while others are conjectural.

2.1.1. Kostlan's ensemble and the Bargmann–Fock limit field

The Kostlan ensemble of degree n homogeneous polynomials is a sequence of Gaussian fields $g_n : \mathbb{RP}^d \rightarrow \mathbb{R}$ defined on the real projective space as

$$g_n(x) = \sum_{|J|=n} a_J \cdot \binom{n}{J} x^J, \quad (2.1)$$

where $J = (j_0, j_1, \dots, j_d)$ is a multi-index, $|J| = j_0 + \dots + j_d$, $x = [x_0 : x_1 : \dots : x_d]$, $x^J = x_0^{j_0} \cdot \dots \cdot x_d^{j_d}$, and $\{a_J\}$ are i.i.d. standard Gaussians. In the case $d = 1$, the study of the zeros of g_n is a classical problem in probability theory going back to Shub and Smale [37], and for $d > 1$, the study of the nodal structures of g_n in the complex algebro-geometric context was initiated by Gayet–Welschinger [16].

Alternatively to (2.1), one may restrict g_n on the unit sphere $\mathcal{S}^d \hookrightarrow \mathbb{RP}^d$, and consider $g_n : \mathcal{S}^d \rightarrow \mathbb{R}$; with this identification, g_n is the centred Gaussian field with covariance

$$r_{g_n}(x, y) := \mathbb{E}[g(x) \cdot g(y)] = \langle x, y \rangle^n = \cos^n(\theta(x, y)),$$

where $\theta(\cdot, \cdot)$ is the angle (or spherical distance) between two spherical points. The upshot is that, with this representation, g_n is rotation invariant, with uniformly rapidly decaying correlations, and rapid convergence towards the scaling limit Bargmann–Fock random field $F_{BF} : \mathbb{R}^d \rightarrow \mathbb{R}$, with covariance $r_{BF}(x, y) = e^{-\|x-y\|^2/2}$. In particular, the ensemble $\{g_n\}$ possesses F_{BF} as its translation invariant scaling limit around every point $x \in \mathcal{S}^d$ (scaling by \sqrt{n}). As is evident from its covariance, F_{BF} is stationary and isotropic, with rapid, super-exponential decay of correlations.

In the case $d = 2$, it is known [3] that the percolation probability $\mathcal{P}^{F_{BF}} = 0$ of the Bargmann–Fock field vanishes, and, moreover [33], the critical level u^* is equal to zero. Hence by Theorem 1.3 the mean of the limit connectivity measure of F_{BF} and, what is the same, the limit connectivity measure of g_n , are both equal to exactly 2. In higher dimensions the positivity of $\mathcal{P}^{F_{BF}}$ is not known, however, in accordance with Sarnak’s insight (explained at the end of §1.2) we believe that $\mathcal{P}^{F_{BF}} > 0$, so that (1.2) *should not* hold. The uniform rapid decay of correlations of both F_{BF} and $\{g_n\}$ imply (see the comments after Definitions 1.4 and 1.8) the nodal lower concentration property, so that Theorem 1.5(b) applies to F_{BF} , and Theorem 1.9 applies to $\{g_n\}$.

2.1.2. Spherical harmonics, arithmetic random waves, and their scaling limits

For $\ell \geq 1$ the degree- ℓ spherical harmonics are the harmonic polynomials on \mathbb{R}^{d+1} of degree ℓ restricted to the unit sphere \mathcal{S}^d ; they constitute a linear space of dimension

$$M_{d;\ell} = \frac{2\ell + d - 1}{\ell + d - 1} \binom{\ell + d - 1}{d - 1}$$

satisfying the Schrödinger equation

$$\Delta_{\mathcal{S}^d} T_\ell + \lambda_{d;\ell} T_\ell = 0,$$

with (spherical) Laplace eigenvalues $\lambda_{d;\ell} = \ell(\ell + d - 2)$. For a L^2 -orthonormal basis $\mathcal{E} = \{\eta_{\ell;1}, \dots, \eta_{\ell;M_{d;\ell}}\}$ we define the random fields on \mathcal{S}^d

$$T_\ell(x) = \frac{1}{\sqrt{M_{d;\ell}}} \sum_{j=1}^{M_{d;\ell}} a_j \eta_{\ell;j}(x),$$

with a_j standard i.i.d. Gaussians; the law of T_ℓ is independent of the choice of \mathcal{E} . Equivalently, T_ℓ is the (uniquely defined) centred Gaussian field on \mathcal{S}^d , with covariance function $\mathbb{E}[T_\ell(x) \cdot T_\ell(y)] = P_{d;\ell}(\cos(\theta(x, y)))$, where $\theta(x, y)$ is again the spherical distance, and $P_{d;\ell}$ is the degree- ℓ Gegenbauer polynomial (so, in particular, for $d = 2$ these are the Legendre polynomials).

The Gaussian ensemble $\{T_\ell\}_{\ell \geq 1}$ is important in mathematical physics, cosmology, natural sciences and other disciplines; the fields T_ℓ appear in the Fourier expansion of any isotropic L^2 -summable Gaussian field on \mathcal{S}^d , hence its importance in the study of the Cosmic Microwave Background (CMB), where T_ℓ , $\ell \rightarrow \infty$, corresponds to high precision experimental measurements. By the standard asymptotics for the Gegenbauer polynomials, $\{T_\ell\}$ possesses a translation invariant local limit, namely, the stationary Gaussian field F with the spectral measure being the hypersurface measure on $\mathcal{S}^{d-1} \subseteq \mathbb{R}^d$. For example, for $d = 2$ these are the planar isotropic monochromatic waves ('Berry's Random Wave Model'), believed [8] to represent generic (deterministic) Laplace eigenfunctions on two-dimensional manifolds. For the ensemble $\{T_\ell\}$ the exponential nodal concentration was established [28], stronger than the mere nodal lower concentration property required for the application of Theorem 1.9. As for the percolation probability, we believe that $\mathcal{P} > 0$ if and only if $d \geq 3$ (see §2.1.3).

Another manifold where the solutions for the Schrödinger equation can be written explicitly is the d -dimensional torus $\mathbb{T}^d = \mathbb{R}^d/[0, 1]^d$. We may write a general solution to Schrödinger equation as

$$f_n(x) = \frac{1}{\sqrt{r_d(n)}} \sum_{\|\vec{\lambda}\|^2 = n} a_{\vec{\lambda}} e(\langle \vec{\lambda}, x \rangle),$$

where $n \geq 1$ and the summation is over all lattice points $\vec{\lambda} = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$ satisfying $\|\vec{\lambda}\|^2 = \lambda_1^2 + \dots + \lambda_d^2 = n$ (i.e. $\vec{\lambda}$ is on a radius \sqrt{n} centred $(d-1)$ -hypersphere), $x = (x_1, \dots, x_d) \in \mathbb{T}^d$, $a_{\vec{\lambda}} \in \mathbb{C}$ are some complex-valued coefficients satisfying

$$a_{-\vec{\lambda}} = \overline{a_{\vec{\lambda}}}. \quad (2.2)$$

One can endow $\{f_n\}$ with a Gaussian probability measure by taking $\{a_{\vec{\lambda}}\}$ to be standard Gaussian i.i.d. (save for (2.2)); the resulting ensemble is referred to as 'Arithmetic Random Waves'.

For $d \geq 3$, the ensemble $\{f_n(x)\}$ possesses the same translation invariant local limit as $\{T_\ell\}$, whereas for $d = 2$ this limit arises for generic index sequences, with other scaling limits for exceptional thin index sequences [14,20]; it is known that the nodal structures of f_n are related to the number theoretic properties of these exceptional numbers [19,21]. For f_n the exponential concentration was established by Rozenstein [34] for $d \geq 3$, and $d = 2$ with n generic, stronger than needed for an application of Theorem 1.9.

2.1.3. Band-limited functions

The examples in §2.1.2 are particular cases of *band-limited* random Gaussian functions for a generic smooth compact d -manifold \mathcal{M} (where no spectral degeneracy is expected), put forward by Sarnak–Wigman [36]. Let Δ be the Laplace–Beltrami operator on \mathcal{M} , $\{\varphi_j\}_{j \geq 1}$ the (discrete) orthonormal basis of $L^2(\mathcal{M})$ consisting of eigenfunctions satisfying

$$\Delta \varphi_j + \lambda_j \varphi_j = 0,$$

with corresponding sequence of eigenvalues $\lambda_j \geq 0$ nondecreasing, $\lambda_j \rightarrow \infty$. Fix a number $\alpha \in [0, 1]$ (the ‘band’), and, given a spectral parameter $T \rightarrow \infty$, we define the α -band limited random function to be

$$\Phi_T(x) = \sum_{\alpha \cdot T \leq \sqrt{\lambda_j} \leq T} a_j \varphi_j(x), \quad (2.3)$$

with a_j standard Gaussian i.i.d., where for $\alpha = 1$ it is understood that the summation in (2.3) is in the range

$$T - \eta(T) \leq \sqrt{\lambda_j} \leq \sqrt{\lambda_j},$$

and $\eta(\lambda) = o(T)$ with $\sqrt{T}\eta(T) \rightarrow \infty$. It is known [23,17,22] that $\{\Phi_T\}$ possesses a translation invariant local limit, with the limit kernel K being the Fourier transform of the characteristic function of the annulus

$$\{y \in \mathbb{R}^d : \alpha \leq |y| \leq 1\},$$

independent of x (for $\alpha = 1$, the unit sphere $\mathcal{S}^{d-1} \subseteq \mathbb{R}^d$). Equivalently, the scaling limit random field F of Φ_T at every point is stationary and rotation invariant (isotropic), and its spectral measure is the characteristic function of the above annulus.

For this fundamental ensemble Sarnak–Wigman [36] established a limit connectivity measure $\mu_{\Gamma,d,\alpha}$ on $\mathbb{Z}_{\geq 0}$ that charges all of $\mathbb{Z}_{\geq 0}$, with some extra care required [13] for the case $\alpha = 1$ in which the support of the corresponding spectral measure does not contain an interior point; Beliaev–Wigman [6] proved the analogous results for the limit volume distribution for nodal domains. For the limit random field F , it is not known whether the percolation probability \mathscr{P} is positive, nor, in light of the fact that the covariance function r_F of F decays too slowly, whether the nodal lower concentration property (1.11) holds. We believe that the nodal lower concentration property *should hold* for both F and $\{\Phi_T\}_T$ for all $\alpha \in [0, 1]$, in all dimensions $d \geq 2$, and, in accordance with Sarnak’s insight (explained at the end of §1.2), we believe that $\mathscr{P} > 0$ if and only if $d \geq 3$, $\alpha \in [0, 1]$ arbitrary. If our intuition is correct, the upshot is that (1.2) holds if and only if $d = 2$, whereas (1.13) and Theorem 1.9 hold for all $d \geq 2$.

2.2. Outline of the proofs of the main results

The proof of Theorem 1.3 and Theorem 1.5(a) are based on an analysis of the contribution from *boundary components* to, respectively, the total connectivity and volume of the nodal domains. Rather than work with a radius R ball $B(R)$, it will be convenient to redefine $B(R)$ to be the cube $[-R, R]^d$. None of the conclusions in [29,36,6] are affected by this change; notably each of (1.4), (1.6), and (1.8) remain valid.

The proofs of Theorem 1.3 and 1.5 (a) are divided into three steps each:

Step 1. First we define an appropriate *quantification* of the contribution from the boundary components to the total connectivity and volume; let us focus first on the connectivity. Recall that $G(R) = (V(R), E(R))$ denotes the nesting graph of the nodal domains that are *fully contained* in $B(R)$. We can similarly define the nesting graph $\overline{G}(R) = (\overline{V}(R), \overline{E}(R))$ of *all* nodal domains of the field $F|_{B(R)}$; this is the graph with vertices $\overline{V}(R)$ the connected components of $B(R) \setminus \mathcal{A}(F)$ and edges $\overline{E}(R)$ which are the connected components of $B(R) \cap \mathcal{A}(F)$ that record adjacency among the nodal domains of $F|_{B(R)}$. One advantage of $\overline{G}(R)$ over $G(R)$ is that it is a.s. a tree (by Jordan's Theorem); hence we have by Euler's formula that

$$\sum_{v \in \overline{V}(R)} \overline{d}(v) = 2(\overline{\mathcal{N}}(F; R) - 1), \quad (2.4)$$

where $\overline{\mathcal{N}}(F; R) = |\overline{V}(R)|$ denotes the number of nodal domains of $F|_{B(R)}$.

Definition 2.1 (*Boundary connectivity*). The boundary connectivity is defined to be

$$\mathcal{C}(R) := \sum_{v \in \overline{V}(R)} \overline{d}(v) - \sum_{v \in V(R)} d(v),$$

where $\overline{d}(v)$ denotes the degree of the vertex v in $\overline{G}(R)$ (recall that $d(v)$ denotes the degree of the vertex v in $G(R)$).

Observe that, similarly to (2.4), by Euler's formula

$$\sum_{v \in V(R)} d(v) = 2(\mathcal{N}(F; R) - T(R)),$$

where $T(R)$ denotes the number of connected components of the union of the closure of all the $D \in V(R)$ (since, in generally, $G(R)$ is a *union* of trees). Hence, combining with (2.4),

$$\mathcal{C}(R) = 2(T(R) - 1) - 2(\overline{\mathcal{N}}(F; R) - \mathcal{N}(F; R)). \quad (2.5)$$

Notice also that $\overline{\mathcal{N}}(F; R) - \mathcal{N}(F; R)$ equals the number of nodal domains of $F|_{B(R)}$ that intersect $\partial B(R)$. It is simple to deduce that this has negligible expectation in the limit:

Proposition 2.2. *Let F be a continuous stationary Gaussian field with spectral measure ρ satisfying $(\rho 2) - (\rho 3)$. Then as $R \rightarrow \infty$,*

$$\frac{\mathbb{E}[\overline{\mathcal{N}}(F; R) - \mathcal{N}(F; R)]}{\text{Vol } B(R)} \rightarrow 0.$$

Together, these observations show that

$$\begin{aligned} \liminf_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{C}(R)]}{\text{Vol } B(R)} &= \liminf_{R \rightarrow \infty} \frac{2\mathbb{E}[T(R)]}{\text{Vol } B(R)} \quad \text{and} \\ \limsup_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{C}(R)]}{\text{Vol } B(R)} &= \limsup_{R \rightarrow \infty} \frac{2\mathbb{E}[T(R)]}{\text{Vol } B(R)} \end{aligned} \quad (2.6)$$

i.e. $\mathbb{E}[\mathcal{C}(R)]$ and $2\mathbb{E}[T(R)]$ are asymptotically equivalent in the large R limit. This fact that will greatly assist the asymptotic analysis of $\mathcal{C}(R)$ that we undertake in Section 3 in order to prove Theorem 1.5.

The notion of ‘boundary volume’, analogous to boundary connectivity applied in course of proving Theorem 1.3, is defined in a significantly simpler manner:

Definition 2.3 (*Boundary volume*). The boundary volume $\mathcal{V}(R)$ is the total volume of the connected components of $B(R) \setminus \mathcal{A}(F)$ that intersect the boundary $\partial B(R)$.

Since the nodal set $\mathcal{A}(F)$ is a set of zero volume, the boundary volume $\mathcal{V}(R)$ can also be expressed as

$$\mathcal{V}(R) = \text{Vol}(\{x \in B(R) : x \overset{F}{\leftarrow} \partial B(R)\}).$$

The definitions of boundary connectivity and boundary volume can both be extended to the setting of a Gaussian field $\Phi : \mathcal{M} \rightarrow \mathbb{R}$ on a compact Riemannian manifold M , although we formalise this only in the case of the volume:

Definition 2.4 (*Boundary volume on a manifold*). Fix $x_0 \in \mathcal{M}$, and $r > 0$ sufficiently small so that $\exp_{x_0}(\cdot)$ is a bijection on the radius- r ball inside $T_{x_0}(\mathcal{M})$. Then we define $\mathcal{V}_{\Phi; x_0}(r)$ to be the total volume of the nodal domains of Φ , restricted to the radius- r geodesic ball centred at x_0 , that intersect the boundary of this ball.

Step 2. The next step is to link the quantities $\mathcal{C}(R)$ and $\mathcal{V}(R)$ to the percolation probability \mathcal{P} . In the case of the connectivity, the following proposition roughly asserts that the contribution to the connectivity from the boundary is negligible, as a fraction of the total volume of $B(R)$, if and only if $\mathcal{P} = 0$:

Proposition 2.5. *Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous stationary Gaussian field with spectral measure ρ satisfying $(\rho 2) - (\rho 3)$, and let $\mathcal{P} = \mathcal{P}^F$ be the associated percolation probability. Then*

(a) If $\mathcal{P} = 0$, then

$$\lim_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{C}(R)]}{\text{Vol } B(R)} = 0.$$

(b) Conversely, if $\mathcal{P} > 0$, and if in addition the spectral measure ρ satisfies $(\rho 4)$, then

$$\liminf_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{C}(R)]}{\text{Vol } B(R)} > 0.$$

One interpretation of Proposition 2.5 is that if $\mathcal{P} > 0$ then the postulated percolating giant nodal domains (see §1.2) make a non-negligible contribution to the total connectivity; we believe it to be of independent interest.

For the volume, we identify a more direct relationship between the contribution from the boundary components and the percolation probability:

Proposition 2.6.

(a) Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous stationary Gaussian field with spectral measure ρ satisfying $(\rho 2)$ – $(\rho 3)$ with associated percolation probability $\mathcal{P} = \mathcal{P}^F$. Then

$$\lim_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{V}(R)]}{\text{Vol } B(R)} = \mathcal{P}. \quad (2.7)$$

(b) Let $\{\Phi_L\}_{L \in \mathcal{L}}$ be a Gaussian ensemble on \mathcal{M} possessing a translation invariant local limit K as $L \rightarrow \infty$ that is independent of x . Suppose the spectral measure $\rho = \rho_F$ corresponding to the limit field F satisfies $(\rho 2)$ – $(\rho 3)$ and has associated percolation probability $\mathcal{P} = \mathcal{P}^F$. Then for every $x_0 \in \mathcal{M}$,

$$\lim_{R \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{\mathbb{E}[\mathcal{V}_{\Phi_L; x_0}(R/L)]}{\text{Vol } B(R/L)} = \mathcal{P}. \quad (2.8)$$

Although Proposition 2.6(b) is not used in the proof of our main theorems, we believe it to be of independent interest in its own right. Proposition 2.6 implies that in the case $\mathcal{P} = 0$ the total volume of the nodal components inside $B(R)$ that touch the boundary is negligible. On the other hand, for deterministic reasons there are boundary components of diameter $O(R)$. As illustrated by Fig. 2, which shows the boundary components for the Bargmann-Fock field, the typical structure of the boundary components is to have many holes, accounting for their negligible total volume even though their diameter might be large. In particular, the rate of the convergence of the expression on the l.h.s. of (2.7) (and (2.8)) to the limit is expected to be slow.

Step 3. The final step is to express the mean connectivity and mean volume of the limit measures in terms of the asymptotics formulae for $\mathcal{C}(R)$ and $\mathcal{V}(R)$ that appear in Propositions 2.5 and 2.6 above:

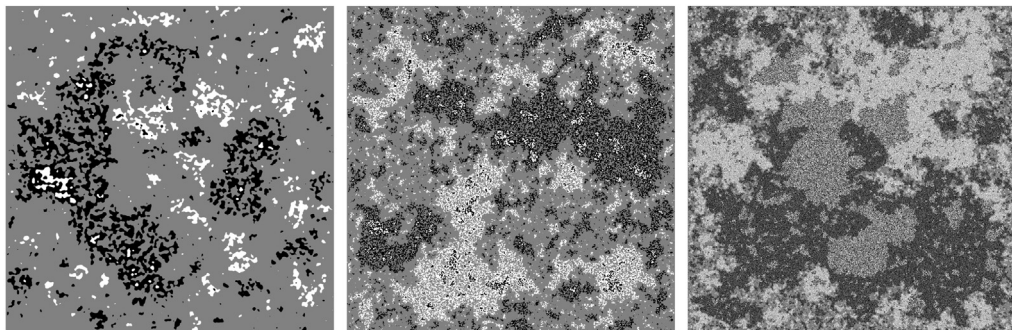


Fig. 2. Interior nodal domains of the Bargmann-Fock field. The black and white are positive and negative nodal domains respectively, that are entirely contained inside $B(R)$, and the grey are all nodal domains that are connected to the boundary. Note that the grey components are interlaced with most of the black and white domains, including far away from the boundary. Left: $R = 200$, middle: $R = 800$, right: $R = 3200$.

Proposition 2.7. *Let F be a continuous stationary Gaussian field with spectral measure ρ satisfying $(\rho 1) - (\rho 4)$, and let $\mathcal{P} = \mathcal{P}^F$ be the associated percolation probability. Then*

$$\begin{aligned} \frac{1}{c_{NS}(\rho)} \left(1 - \limsup_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{V}(R)]}{\text{Vol } B(R)} \right) &\leq \int_0^\infty (1 - \Psi_F(t)) dt \\ &\leq \frac{1}{c_{NS}(\rho)} \left(1 - \liminf_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{V}(R)]}{\text{Vol } B(R)} \right) \end{aligned} \quad (2.9)$$

and

$$2 - \frac{1}{c_{NS}(\rho)} \limsup_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{C}(R)]}{\text{Vol } B(R)} \leq \sum_{k=0}^{\infty} k \cdot \mu_{\Gamma(F)}(k) \leq 2 - \frac{1}{c_{NS}(\rho)} \liminf_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{C}(R)]}{\text{Vol } B(R)}. \quad (2.10)$$

The proof of Propositions 2.2, 2.5 and 2.6 will be given in Section 3, whereas the proof of Proposition 2.7 will be given in Section 4; by combining these propositions we deduce the proofs of Theorem 1.3 and Theorem 1.5(a). The proof of Theorem 1.5(b) and Theorem 1.9 are more straightforward, and are completed in Section 5.

3. Analysis of the contribution of boundary components

In this section we undertake an analysis of the boundary connectivity $\mathcal{C}(R)$ and volume $\mathcal{V}(R)$, linking their asymptotics to the percolation probability \mathcal{P} , and, in particular, prove Propositions 2.2, 2.5 and 2.6.

3.1. Proof of Proposition 2.2

The proof of Proposition 2.2 is standard [38]. The boundary $\partial B(R)$ can be decomposed as the disjoint union of $3^d - 1$ boundary cubes C_i of intermediate dimensions $0 \leq i \leq d-1$; by standard Morse theory arguments, the number of boundary components

$$\overline{\mathcal{N}}(F; R) - \mathcal{N}(F; R)$$

is bounded above by the sum, over the boundary cubes C_i , of the number of critical points of F restricted to C_i . By stationarity and the Kac-Rice formula [2, Theorem 6.3] (applicable by $(\rho 2)$ – $(\rho 3)$), the expected number of critical points of F restricted to a cube C of dimension $i \geq 1$ is equal to

$$\text{Vol}(C) \cdot \varphi_{\nabla_C f(0)}(0) \cdot \mathbb{E}[|\det \nabla_C^2 f(0)| \mid \nabla_C f(0) = 0],$$

where $\text{Vol}(C)$ is the i -dimensional volume of C , ∇_C and ∇_C^2 are respectively the gradient and Hessian of F restricted to C , and $\varphi_{\nabla_C f(0)}(0)$ denotes the Gaussian density of $\nabla_C f(0)$ at the value 0. Since the quantity

$$s = s(C) = \varphi_{\nabla f(0)}(0) \cdot \mathbb{E}[|\det \nabla^2 f(0)| \mid \nabla f(0) = 0]$$

depends only on the i directions that span the cube C , and since $s(C) > 0$ by $(\rho 2)$ – $(\rho 3)$, the expected number of critical points on each boundary cube C_i is proportional to its volume with a constant depending only on the spanning axis directions. Hence in particular

$$\mathbb{E}[\overline{\mathcal{N}}(F; R) - \mathcal{N}(F; R)] = O(R^{d-1}).$$

3.2. Proof of Proposition 2.6

The proof of Proposition 2.6 rests on a simple deterministic lemma. For each $R > 0$ and $x \in B(R)$, let $d_R^-(x)$ denote the distance between x and $\partial B(R)$ (i.e. the distance between x and the closest point on $\partial B(R)$ to x), and let

$$d_R^+(x) := 2R - d_R^-(x)$$

be the distance between x and the farthest point of $\partial B(R)$, also lying on the axis connecting x with its closest point on $\partial B(R)$.

Lemma 3.1. *Let $g : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be a non-increasing function and define $g_\infty = \lim_{s \rightarrow \infty} g(s)$. Then, for every $r \in \mathbb{R}$, as $R \rightarrow \infty$ we have the limits*

$$\frac{\int_{x \in B(R)} g((d_R^-(x) - r)^+) dx}{\text{Vol } B(R)} \rightarrow g_\infty \quad (3.1)$$

and

$$\frac{\int_{x \in B(R)} g((d_R^+(x) - r)^+) dx}{\text{Vol } B(R)} \rightarrow g_\infty$$

The proof of Lemma (3.2) will be given immediately after the proof of Proposition 2.6.

Proof of Proposition 2.6 assuming Lemma 3.1. Let us begin with part (a). Observe first that, for each $R > 0$,

$$\mathcal{V}(R) = \text{Vol}(\{x \in B(R) : x \overset{F}{\longleftrightarrow} \partial B(R)\}) = \int_{x \in B(R)} \mathbb{1}\{x \overset{F}{\longleftrightarrow} \partial B(R)\} dx,$$

and so

$$\mathbb{E}[\mathcal{V}(R)] = \int_{x \in B(R)} \mathcal{P}r[x \overset{F}{\longleftrightarrow} \partial B(R)] dx. \quad (3.2)$$

Notice also that, by the stationarity of F , and in light of the fact that for every $x \in B(R)$,

$$B_x(d_R^-(x)) \subseteq B(R) \subseteq B_x(d_R^+(x)),$$

we have for every $x \in B(R)$,

$$\mathcal{P}r[0 \overset{F}{\longleftrightarrow} \partial B_{d_R^+(x)}] \leq \mathcal{P}r[x \overset{F}{\longleftrightarrow} \partial B(R)] \leq \mathcal{P}r[0 \overset{F}{\longleftrightarrow} \partial B_{d_R^-(x)}]. \quad (3.3)$$

Hence, by substituting (3.3) into (3.2), we obtain the inequality

$$\frac{\int_{x \in B(R)} \mathcal{P}r[0 \overset{F}{\longleftrightarrow} \partial B_{d_R^+(x)}] dx}{\text{Vol } B(R)} \leq \frac{\mathbb{E}[\mathcal{V}(R)]}{\text{Vol } B(R)} \leq \frac{\int_{x \in B(R)} \mathcal{P}r[0 \overset{F}{\longleftrightarrow} \partial B_{d_R^-(x)}] dx}{\text{Vol } B(R)}. \quad (3.4)$$

Applying Lemma 3.1 to the non-increasing function $g(s) = \mathcal{P}r[0 \overset{F}{\longleftrightarrow} \partial B(s)]$, yields that both the l.h.s. and the r.h.s. of (3.4) converge, as $R \rightarrow \infty$, to the limit

$$\mathcal{P} := \lim_{s \rightarrow \infty} \mathcal{P}r[0 \overset{F}{\longleftrightarrow} \partial B(s)],$$

and therefore so does $\mathbb{E}[\mathcal{V}(R)]/\text{Vol } B(R)$, which is the statement of Proposition 2.6 (a).

We now turn to part (b). Recall the definition of the scaled random fields $\Phi_{x;L}$ in (1.14), with covariance $r_{x;L}$ given by (1.15). The assumed locally-uniform convergence (1.16) of the covariance kernels $r_{x;L}$ to K ensure that, on any compact domain, the random field $\Phi_{x;L}$ converges in law, in the C^0 topology, to the translation invariant local limit field F . Next we observe that the function h that maps a C^3 -smooth function on a piece-wise smooth compact domain $D \subseteq \mathbb{R}^d$ to the total volume of the nodal domains that intersect ∂D is continuous in the C^0 topology up to a null set of F . This is since the set of discontinuities of h is contained in the set of functions such that there is a critical point of $F|_D$ or $F|_{\partial D}$ with height zero, which is indeed a null set for F by Bulinskaya's Lemma, valid by (ρ_2) – (ρ_3) (see e.g. [2, Proposition 6.12]).

Hence, by the Continuous Mapping Theorem, we have the convergence in law

$$\frac{\mathcal{V}_{\Phi_L;x_0}(R/L)}{\text{Vol } B(R/L)} \xrightarrow{\text{Law}} \frac{\mathcal{V}(R)}{\text{Vol } B(R)}. \quad (3.5)$$

Since the random variables on both r.h.s. and l.h.s. of (3.5) are clearly bounded, their means also converge, i.e.

$$\lim_{L \rightarrow \infty} \frac{\mathbb{E}[\mathcal{V}_{\Phi_L;x_0}(R/L)]}{\text{Vol } B(R/L)} = \frac{\mathbb{E}[\mathcal{V}(R)]}{\text{Vol } B(R)}.$$

In light of part (a) of Proposition 2.6, we have the result upon taking the limit $R \rightarrow \infty$. \square

Proof of Lemma 3.1. Since g is non-increasing, we have the inequality

$$\frac{\int_{x \in B(R)} g((d_R^\pm(x) - r)^+) dx}{\text{Vol } B(R)} \geq g_\infty.$$

Hence in light of the trivial inequality $d^+(\cdot) \geq d^-(\cdot)$, to prove both statements of Lemma 3.1, it is sufficient to prove (3.1) for $r > 0$ only. Moreover, without loss of generality we may assume that $g(0) = 1$ and $g_\infty = 0$ (as otherwise we may pass to $(g(\cdot) - g_\infty)/(g(0) - g_\infty)$). Integrating over $d - 1$ -dimensional cubic shells (technically justified by dividing the cube into $2d$ identical right-pyramids and applying the smooth co-area formula),

$$\begin{aligned} \int_{x \in B(R)} g((d_R^-(x) - r)^+) dx &= 2^d d \int_{s=0}^{R-r} s^{d-1} g((R-r-s) ds + 2^d d \int_{s=R-r}^R s^{d-1} ds \\ &= 2^d d \int_{s=0}^{R-r} s^{d-1} g(R-r-s) ds + 2^d (R^d - (R-r)^d), \end{aligned}$$

and so it remains to show that for every $\varepsilon, r > 0$ there exists an $R > 0$ sufficiently large so that

$$\int_{s=0}^{R-r} s^{d-1} g(R-r-s) ds < \varepsilon R^d.$$

Let $t > 0$ be such $g(t) \leq (d/2)\varepsilon$, and recall that g is bounded by 1. Then

$$\int_{s=0}^{R-r} s^{d-1} g(R-r-s) ds = \int_{s=0}^{R-r-t} s^{d-1} g(R-r-s) ds + \int_{s=R-r-t}^{R-r} s^{d-1} g(R-r-s) ds$$

$$\begin{aligned}
&\leq (d/2)\varepsilon \int_{s=0}^{R-r-t} s^{d-1} ds + \int_{s=R-r-t}^{R-r} s^{d-1} ds \\
&\leq (d/2)\varepsilon \int_{s=0}^R s^{d-1} ds + \int_{s=R-r-t}^{R-r} R^{d-1} ds \\
&\leq (\varepsilon/2)R^d + tR^{d-1},
\end{aligned}$$

which is less than εR^d for sufficiently large R . \square

3.3. Proof of Proposition 2.5

As a preparation towards proving Proposition 2.5 we will need three auxiliary lemmas. The first is a simple deterministic bound on the connected components of a set $S \subseteq B(R)$:

Lemma 3.2. *There exists an absolute constant $c > 0$ with the following property. For every $R > 1$ and $S \subseteq B(R)$ closed subset, and $\varepsilon > 0$, the number of connected components of $B(R) \setminus S$ whose volume is at least ε is bounded above by*

$$K(S) \cdot (\varepsilon^{-1} + 1) + cR^{d-1},$$

where

$$K(S) := \#\{k \in \mathbb{Z}^d : S \text{ intersects the cube } [0, 1]^d + k\}. \quad (3.6)$$

Our next lemma, borrowed from [38,36], shows that, under the usual assumptions on F , the limit volume distribution Ψ_F exhibits at most power behaviour at the neighbourhood of the origin, i.e. yields an upper bound for the (asymptotic) number of *small nodal domains*:

Lemma 3.3. *Let F be a continuous stationary Gaussian field with spectral measure ρ satisfying $(\rho 2)$ – $(\rho 3)$. Then there exist constants $c_1, c_2, t_0 > 0$ such that, for all $t_0 > t > 0$,*

$$\Psi_F(t) < c_1 t^{c_2}.$$

Finally we state a simple consequence of $\mathcal{P} > 0$, namely that it guarantees the existence, with positive probability, of a nodal domain that (i) lies fully inside a small ball $B(r)$ and (ii) is connected to the boundary $\partial B(R)$ by another nodal domain:

Lemma 3.4. *Assume that $\mathcal{P} > 0$, and assume also that the spectral measure ρ satisfies $(\rho 2)$ – $(\rho 4)$. Recall that $V(r)$ denotes the set of nodal domains that are fully contained within $B(r)$. Then there exists a number $r > 0$ such that*

$$\liminf_{R \rightarrow \infty} \mathcal{P}r[\exists D \in V(r) : \partial D \xleftrightarrow{F} \partial B(R)] > 0.$$

We are now ready to prove Proposition 2.5. Recall that $V(R)$ is the set of nodal domains that are fully contained within $B(R)$, and $\overline{V}(R)$ is the set of nodal domains of the field F restricted to $B(R)$; hence $\overline{V}(R) \setminus V(R)$ is the set of nodal domain of $F|_{B(R)}$ that intersect $\partial B(R)$. Recall also that $\mathcal{N}(F; R) = |V(R)|$ and $\overline{\mathcal{N}}(F; R) = |\overline{V}(R)|$.

Proof of Proposition 2.5 assuming Lemmas 3.2–3.4. We begin with part (a). For each $R > 0$, define

$$S_R := \cup_{D_i \in \overline{V}(R) \setminus V(R)} D_i = \{x \in B(R) : x \xleftrightarrow{F} \partial B(R)\}$$

to be the union of the nodal domains of F restricted to $B(R)$ that intersect the boundary. Let $T(R)$ be the number of connected components of $B(R) \setminus S_R$, and observe that this agrees with the definition given immediately after Definition 2.1. Equation (2.5) states that

$$\mathcal{C}(R) = 2(T(R) - 1) - 2(\overline{\mathcal{N}}(F; R) - \mathcal{N}(F; R))$$

and hence, in view of Proposition 2.2, to establish the result it is sufficient to show that

$$\lim_{R \rightarrow \infty} \frac{\mathbb{E}[T(R)]}{\text{Vol } B(R)} = 0. \quad (3.7)$$

Applying Lemma 3.2, for every $\varepsilon > 0$ we have that

$$T(R) \leq \mathcal{N}(F, \varepsilon; R) + K(S_R)(\varepsilon^{-1} + 1) + cR^{d-1}, \quad (3.8)$$

where $K(S_R)$ was introduced in (3.6), and $c > 0$ is an absolute constant. Suppose that $\varepsilon > 0$ is a continuity point of the limit volume distribution Ψ_F . By (1.8), as $R \rightarrow \infty$,

$$\frac{\mathbb{E}[\mathcal{N}(F, \varepsilon; R)]}{\text{Vol } B(R)} \rightarrow c_{NS}(\rho) \cdot \Psi_F(\varepsilon).$$

Since Lemma 3.3 implies that, as $\varepsilon \rightarrow 0$,

$$\Psi_F(\varepsilon) \rightarrow 0,$$

we deduce that

$$\lim_{\varepsilon \rightarrow 0} \lim_{R \rightarrow \infty} \frac{\mathbb{E}[\mathcal{N}(F, \varepsilon; R)] + cR^{d-1}}{\text{Vol } B(R)} = 0, \quad (3.9)$$

where the limit as $\varepsilon \rightarrow 0$ is understood as being taken on a subsequence of continuity points of Ψ_F .

Turning to bounding $K(S_R)$, we first claim that if $\mathscr{P} = 0$ then

$$\lim_{R \rightarrow \infty} \Pr[D \xleftrightarrow{F} \partial B(R)] = 0$$

for an arbitrary compact domain D . To this end, we observe that since 0 does not lie on a nodal component a.s., the nodal domain containing 0 covers a small cube $B(\varepsilon)$ with probability tending to 1 as $\varepsilon \rightarrow 0$. Hence if $\mathscr{P} = 0$ then it cannot be the case that

$$\liminf_{R \rightarrow \infty} \Pr[B(\varepsilon) \xleftrightarrow{F} \partial B(R)] > 0,$$

for arbitrary small $\varepsilon > 0$, since then

$$\liminf_{R \rightarrow \infty} \Pr[0 \xleftrightarrow{F} \partial B(R)] > 0,$$

which is in contradiction with $\mathscr{P} = 0$. Thus we have that

$$\lim_{R \rightarrow \infty} \Pr[B(\varepsilon) \xleftrightarrow{F} \partial B(R)] = 0$$

for sufficiently small $\varepsilon > 0$, and we deduce the claim by covering D with a finite number C_i of translations of $B(\varepsilon)$.

Next, applying Lemma 3.1 to the function $g(s) = \Pr[B(1) \xleftrightarrow{F} \partial B(s)]$ and the constant $r = 1$, and arguing as in the proof of Proposition 2.6, we deduce that

$$\lim_{R \rightarrow \infty} \frac{\mathbb{E}[K(S_R)]}{\text{Vol } B(R)} = 0. \quad (3.10)$$

Combining (3.9) and (3.10) and substituting these into (3.8), while sending first $R \rightarrow \infty$ and then $\varepsilon \rightarrow 0$, we arrive at (3.7).

Let us now establish part (b). As in part (a), it is sufficient to prove that

$$\liminf_{R \rightarrow \infty} \frac{\mathbb{E}[T(R)]}{\text{Vol } B(R)} > 0. \quad (3.11)$$

Fix $r > 0$ as in Lemma 3.4 and consider tiling $B(R)$ with $O(R^d)$ disjoint translations E_i of $B(r)$ (ignoring the leftover untiled space). Observe that, since E_i are disjoint,

$$\mathbb{E}[T(R)] \geq \sum_{E_i} \Pr[\exists D \subseteq V(E_i) : \partial D \xleftrightarrow{F} \partial B(R)].$$

Hence applying Lemma 3.1 to the function

$$g(s) = \Pr[\exists D \in V(E_i) : \partial D \xleftrightarrow{F} \partial B(s)]$$

and the constant r , and arguing as in the proof of Proposition 2.6, we deduce that

$$\liminf_{R \rightarrow \infty} \frac{\mathbb{E}[T(R)]}{\text{Vol } B(R)} \geq \lim_{s \rightarrow \infty} g(s). \quad (3.12)$$

Since $\lim_{s \rightarrow \infty} g(s)$ is strictly positive by the virtue of Lemma 3.4, so is the l.h.s. of (3.12), yielding (3.11). \square

We now prove the auxiliary Lemmas 3.2-3.4:

Proof of Lemma 3.2. Let $\mathcal{K}(S)$ denote the union of all cubes $[0, 1]^d + k$, $k \in \mathbb{Z}^d$, that intersect S ; by the definition (3.6) of $K(\cdot)$ we have

$$K(S) = \text{Vol}(\mathcal{K}(S)). \quad (3.13)$$

A connected component of $B(R) \setminus S$ is either contained within $\mathcal{K}(S)$ or is not. The number of components contained within $\mathcal{K}(S)$ with volume at least ε is bound above by

$$\text{Vol}(\mathcal{K}(S)) \cdot \varepsilon^{-1} = K(S) \cdot \varepsilon^{-1}, \quad (3.14)$$

by (3.13).

On the other hand, we may bound the number of those components not lying inside $\mathcal{K}(S)$ by invoking the geometric Lemma 3.5 below (with S taking the role of A and $\mathcal{K}(S)$ taking the role of B) to be at most $K(S) + cR^{d-1}$. Together with the bound (3.14) for those components lying inside $\mathcal{K}(S)$, this yields the statement of Lemma 3.2. \square

Lemma 3.5. *Let $B \subseteq B(R)$ be a finite union*

$$B = \bigcup_{k \in I} (k + [0, 1]^d)$$

of cubes of the form $k + [0, 1]^d$, $I \subseteq \mathbb{Z}^d$, and $A \subseteq B$ a closed set (Fig. 3). Then the number of connected components of $B(R) \setminus A$ not contained within B is at most

$$\text{Vol}(B) + c \cdot R^{d-1}. \quad (3.15)$$

Proof. First, every connected component of $\tilde{A} := B(R) \setminus A$ that is not contained within B intersects $\tilde{B} := B(R) \setminus B$ (as otherwise it would be fully contained in B), and hence contains a distinct connected component of $B(R) \setminus B$. This induces an injection between the connected components of \tilde{A} and the connected components of \tilde{B} , so the number of the former is bounded from above by the latter. Now we claim that the bound (3.15) is applicable for the number of connected components of \tilde{B} .

To this end we associate to each connected component C of \tilde{B} a distinct cube that is either in B or adjacent to $\partial B(R)$ in the following manner. Given a point $x = (x_1, x_2, \dots, x_d) \in C$ we increase x_1 until we escape C , i.e. find the smallest $x'_1 > x_1$ so that $x' = (x'_1, x_2, \dots, x_d) \notin C$; since C is open (being a complement of a closed set),

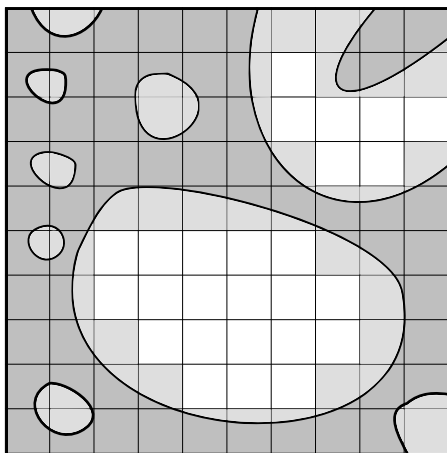


Fig. 3. An illustration of Lemma 3.5, depicting the set B (in grey) and the set $A \subseteq B$ (in dark grey). The number of connected components of $B(R) \setminus A$ that are not contained within B can be bounded by the number of connected components of $B(R) \setminus B$ (in white), which in turn can be bounded (up to boundary effects) by the volume of B ; in this figure there are two such components.

x' is in the interior of one of the faces of the cubes $[0, 1]^d + k \subseteq B$, or one of the cubes $[0, 1]^d + k$ intersecting $\partial B(R)$. These cubes are clearly distinct for different components A , so their number is bounded by (3.15), as claimed. \square

Proof of Lemma 3.3. This is a restatement of [38, Lemma 9] (the full proof given in [36, Lemma 4.12]). Although the result in [36] is stated only for certain special cases of F , its proof holds unimpaired for all F satisfying the axioms $(\rho 2)$ – $(\rho 3)$. It also yields the universality of the exponent c_2 , depending only on the dimension d (and the threshold t_0), although the constant c_1 also depends on the field F . \square

Proof of Lemma 3.4. Let \mathcal{E} denote the event that the positive excursion set

$$\{x \in \mathbb{R}^d : F(x) > 0\}$$

has an unbounded connected component. Since the percolation probability \mathcal{P} is positive, and by the symmetry of F and $-F$, the event \mathcal{E} has positive probability. Since moreover the random field F is ergodic (equivalent to $(\rho 1)$), and the event \mathcal{E} is translation invariant, we deduce that \mathcal{E} occurs a.s.

Now, let S denote the union of all the unbounded connected components of the positive excursion set $\{x \in \mathbb{R}^d : F(x) > 0\}$. Since we assumed $(\rho 4)$ that $c_{NS}(\rho) > 0$, there exists a positive density of bounded nodal domains, and so $0 \notin S$ with positive probability. Hence, for sufficiently large $r > 0$, and letting W denote the component of S^c that contains 0, the event

$$\mathcal{F} = \{0 \in S^c \text{ and } W \subset B(r)\}$$

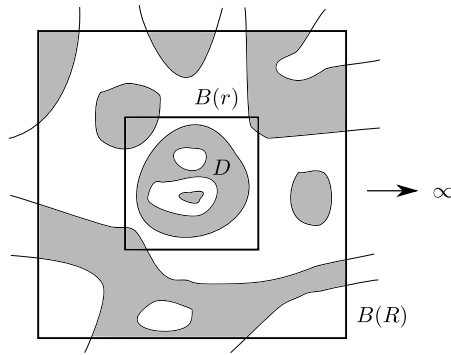


Fig. 4. With positive probability there is a nodal domain D lying entirely inside $B(r)$, and which is connected by an infinite component to the boundary of an arbitrary large square $B(R)$.

holds with positive probability. The set W is the union of the nodal domain D with all nodal domains that are inside of D , see Fig. 4.

Finally, assume the event $\mathcal{E} \cap \mathcal{F}$, and notice that W contains a nodal domain in $V(r)$ that has ∂W as a boundary component. Since W is a component of S^c , it must be the case that $\partial W \xrightarrow{F} \infty$, and so

$$\mathcal{E} \cap \mathcal{F} \subseteq \cap_{R>r} \{\exists D \in V(r) : \partial D \xrightarrow{F} \partial B(R)\}.$$

Since $\mathcal{P}r(\mathcal{E} \cap \mathcal{F}) = \mathcal{P}r(\mathcal{F}) > 0$, we deduce the result. \square

4. The mean connectivity and volume of the limit distribution

In this section we show how to express the mean of the limit connectivity and volume measures $\mu_{\Gamma(F)}$ and Ψ_F in terms of the asymptotic formulae for $\mathcal{C}(R)$ and $\mathcal{V}(R)$ (that appear in Propositions 2.5 and 2.6); in particular, we prove Proposition 2.7. Recall that, by (1.6), for every $k \geq 0$,

$$\mu_{\Gamma(F);R}(k) = \frac{1}{|V(R)|} \cdot \#\{v \in V(R) : d(v) = k\} \rightarrow \mu_{\Gamma(F)}(k)$$

in probability, as $R \rightarrow \infty$. Since we know from (1.4) that

$$\frac{|V(R)|}{\text{Vol } B(R)} = \frac{\mathcal{N}(F; R)}{\text{Vol } B(R)} \rightarrow c_{NS}(\rho) \quad \text{in mean,} \quad (4.1)$$

by the triangle inequality we can deduce, similarly to (1.8), that

$$\frac{\mathbb{E}[\#\{v \in V(R) : d(v) = k\}]}{c_{NS}(\rho) \cdot \text{Vol } B(R)} \rightarrow \mu_{\Gamma(F)}(k) \quad (4.2)$$

We restate (1.8) for convenience in the form

$$\frac{\mathbb{E}[\#\{v \in V(R) : \text{Vol}(v) < t\}]}{c_{NS}(\rho) \cdot \text{Vol } B(R)} \rightarrow \Psi_F(t),$$

valid at all continuity points of Ψ_F ; equivalently, in light of (4.1)

$$\begin{aligned} \frac{\mathbb{E}[\#\{v \in V(R) : \text{Vol}(v) \geq t\}]}{c_{NS}(\rho) \cdot \text{Vol } B(R)} &= \frac{\mathbb{E}[\mathcal{N}(F; R)] - \mathbb{E}[\#\{v \in V(R) : \text{Vol}(v) < t\}]}{c_{NS}(\rho) \cdot \text{Vol } B(R)} \\ &\rightarrow 1 - \Psi_F(t). \end{aligned} \quad (4.3)$$

We remark that passing to the complement (4.3) (i.e. working with domains of volume $\geq t$ and not $< t$) is an important technical step since it will eventually allow us to invoke the Monotone Convergence Theorem when working with the convergent integral $\int_0^\infty (1 - \Psi_F(t))dt$ in the proof of Proposition 2.7 below (see (4.6)).

Proof of Proposition 2.7. We first prove statement (2.9) of Proposition 2.7. To this end we let

$$R_i = 2^i,$$

and partition the cube

$$B(R_i) := [-R_i, R_i]^d$$

into 2^d disjoint cubes $C_{i-1;j}$, $j = 0, \dots, 2^d - 1$ of side length $2R_{i-1}$. We extend the notation of the nesting graph $G(R) = (V(R), E(R))$ to cover the cubes $C_{i-1;j}$, i.e. define

$$G(C_{i-1;j}) = (V(C_{i-1;j}), E(C_{i-1;j}))$$

analogously to $G(R) = (V(R), E(R))$. Notice that every nodal domain that is fully inside $B(R_i)$ is either fully inside one of the $C_{i-1;j}$ or intersects the boundary of at least one of the $C_{i-1;j}$. Hence, neglecting the latter domains, we have for every $t > 0$,

$$\#\{v \in V(R_i) : \text{Vol}(v) \geq t\} \geq \sum_{j=0}^{2^d-1} \#\{v \in V(C_{i-1;j}) : \text{Vol}(v) \geq t\}. \quad (4.4)$$

Taking expectations of both sides of (4.4), and upon exploiting the stationarity of F , this implies that

$$\mathbb{E}[\#\{v \in V(R_i) : \text{Vol}(v) \geq t\}] \geq 2^d \mathbb{E}[\#\{v \in V(R_{i-1}) : \text{Vol}(v) \geq t\}],$$

which in turn implies that the sequence

$$\varphi_F^i(t) := \frac{\mathbb{E}[\#\{v \in V(R_i) : \text{Vol}(v) \geq t\}]}{c_{NS}(\rho) \cdot \text{Vol}(B(R_i))}$$

is monotone increasing in $i \geq 1$. By (4.3), the sequence φ_F^i has the almost everywhere limit

$$\lim_{i \rightarrow \infty} \varphi_F^i(t) = 1 - \Psi_F(t), \quad (4.5)$$

and by applying the Monotone Convergence Theorem on (4.5), we obtain the equality

$$\lim_{i \rightarrow \infty} \int_0^\infty \varphi_F^i(t) dt = \int_0^\infty (1 - \Psi_F(t)) dt. \quad (4.6)$$

Observe that

$$\int_0^\infty \#\{v \in V(R_i) : \text{Vol}(v) \geq t\} dt = \sum_{v \in V(R_i)} \text{Vol}(v),$$

and so, interchanging expectation and integration,

$$\int_0^\infty \varphi_F^i(t) dt = \frac{\mathbb{E} \left[\sum_{v \in V(R_i)} \text{Vol}(v) \right]}{c_{NS}(\rho) \cdot \text{Vol}(B(R_i))}. \quad (4.7)$$

Combining (4.6) and (4.7), we conclude that

$$\int_0^\infty (1 - \Psi_F(t)) dt = \lim_{i \rightarrow \infty} \frac{\mathbb{E} \left[\sum_{v \in V(R_i)} \text{Vol}(v) \right]}{c_{NS}(\rho) \cdot \text{Vol}(B(R_i))}. \quad (4.8)$$

It remains to analyse the r.h.s. of equality (4.8). To this end we notice that, using the definition of the boundary volume $\mathcal{V}(R)$ in Definition 2.3, for every $R > 0$

$$\sum_{v \in V(R)} \text{Vol}(v) = \text{Vol}(B(R)) - \mathcal{V}(R). \quad (4.9)$$

Inserting (4.9) into (4.8) finally yields

$$\begin{aligned} \int_0^\infty (1 - \Psi_F(t)) dt &= \lim_{i \rightarrow \infty} \frac{\text{Vol}(B(R_i)) - \mathbb{E}[\mathcal{V}(R_i)]}{c_{NS}(\rho) \cdot \text{Vol}(B(R_i))} \\ &= \frac{1}{c_{NS}(\rho)} \left(1 - \lim_{i \rightarrow \infty} \frac{\mathbb{E}[\mathcal{V}(R_i)]}{\text{Vol}(B(R_i))} \right), \end{aligned}$$

completing the proof of (2.9).

We turn to statement (2.10), which is proved similarly. Arguing as for the first statement, and replacing integrals with sums whenever necessary, we arrive at the following analogue of the equality (4.8) for the connectivity:

$$\sum_{k=0}^{\infty} k \mu_{\Gamma(F)}(k) = \lim_{i \rightarrow \infty} \frac{\mathbb{E}[\sum_{v \in V(R_i)} d(v)]}{c_{NS}(\rho) \text{Vol}(B(R_i))}. \quad (4.10)$$

Recall (2.4), which states that

$$\sum_{v \in \bar{V}(R)} \bar{d}(v) = 2(\bar{\mathcal{N}}(F; R) - 1).$$

By the definition of the boundary connectivity $\mathcal{C}(R)$ in Definition 2.1, we therefore have

$$\begin{aligned} \sum_{v \in V(R)} d(v) &= 2(\bar{\mathcal{N}}(F; R) - 1) - \mathcal{C}(R) \\ &= 2(\mathcal{N}(F; R) - 1) - \mathcal{C}(R) + 2(\bar{\mathcal{N}}(F; R) - \mathcal{N}(F; R)). \end{aligned} \quad (4.11)$$

Inserting (4.11) into (4.10) yields

$$\sum_{k=0}^{\infty} k \mu_{\Gamma(F)}(k) = \lim_{i \rightarrow \infty} \frac{2\mathbb{E}[\mathcal{N}(F; R_i)] - \mathbb{E}[\mathcal{C}(R_i)] + 2\mathbb{E}[\bar{\mathcal{N}}(F; R) - \mathcal{N}(F; R)]}{c_{NS}(\rho) \text{Vol}(B(R_i))}.$$

Given Proposition 2.2 and the convergence in (1.4), this reduces to

$$\sum_{k=0}^{\infty} k \mu_{\Gamma(F)}(k) = 2 - \frac{1}{c_{NS}(\rho)} \cdot \lim_{i \rightarrow \infty} \frac{\mathbb{E}[\mathcal{C}(R_i)]}{\text{Vol}(B(R_i))},$$

completing the proof of (2.10). \square

5. The empirical mean volume

Recall that Nazarov–Sodin showed (1.4) that, under the assumptions $(\rho 1)$ – $(\rho 4)$,

$$\frac{\mathcal{N}(F; R)}{\text{Vol } B(R)} \rightarrow c_{NS}(\rho) \quad \text{in mean.}$$

In this section we verify that, under the additional nodal lower concentration property (1.11), the ‘reciprocal’ convergence

$$\frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} \mathbb{1}_{\mathcal{N}(F; R) > 0} \rightarrow \frac{1}{c_{NS}(\rho)} \quad \text{in mean}$$

holds, i.e. the ‘empirical volume mean’ converges to $1/c_{NS}(\rho)$ (see Theorem 1.5(b)). The proof of Theorem 1.5(b) only uses elementary properties of convergence in mean, and the proof of the related Theorem 1.9 is similar.

Proof of Theorem 1.5(b). For every fixed $\varepsilon \in (0, 1/c_{NS}(\rho))$ we may write

$$\mathbb{E} \left[\left| \frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} \mathbb{1}_{\mathcal{N}(F; R) > 0} - \frac{1}{c_{NS}(\rho)} \right| \right] = E_1 + E_2 + E_3 + E_4, \quad (5.1)$$

where

$$\begin{aligned} E_1 &= E_1(F; R) = \mathbb{E} \left[\left| \frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} - \frac{1}{c_{NS}(\rho)} \right| \mathbb{1}_{\text{Vol}(B(R))/\mathcal{N}(F; R) < 1/c_{NS}(\rho) - \varepsilon} \right], \\ E_2 &= E_2(F; R) = \mathbb{E} \left[\left| \frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} - \frac{1}{c_{NS}(\rho)} \right| \mathbb{1}_{\text{Vol}(B(R))/\mathcal{N}(F; R) \in [1/c_{NS}(\rho) - \varepsilon, 1/c_{NS}(\rho) + \varepsilon]} \right], \\ E_3 &= E_3(F; R) = \mathbb{E} \left[\left| \frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} - \frac{1}{c_{NS}(\rho)} \right| \mathbb{1}_{1/c_{NS}(\rho) - \varepsilon < \text{Vol}(B(R))/\mathcal{N}(F; R) < \infty} \right] \end{aligned}$$

and

$$E_4 = E_4(F; R) = \frac{1}{c_{NS}(\rho)} \mathcal{P}r(\mathcal{N}(F; R) = 0).$$

Next we bound each of the E_i , $i = 1, \dots, 4$ separately.

First,

$$E_1 \leq \frac{1}{c_{NS}(\rho)} \times \mathcal{P}r \left(\frac{\mathcal{N}(F; R)}{\text{Vol}(B(R))} > 1/(1/c_{NS}(\rho) - \varepsilon) \right) \rightarrow 0 \quad (5.2)$$

as $R \rightarrow \infty$, by (1.4), (see, e.g., the law of large numbers (1.5)). Second, trivially

$$E_2 < \varepsilon. \quad (5.3)$$

Next, since, being an integer, $\mathcal{N}(F; R) \geq 1$, we have

$$E_3 \leq \max\{\text{Vol } B(R), 1/c_{NS}\} \times \mathcal{P}r \left(\frac{\mathcal{N}(F; R)}{\text{Vol}(B(R))} < 1/(1/c_{NS}(\rho) - \varepsilon) \right) \rightarrow 0 \quad (5.4)$$

by the definition (1.11) of nodal lower concentration, and $\text{Vol } B(R) = O(R^d)$. Lastly,

$$E_4 \rightarrow 0, \quad (5.5)$$

since

$$\mathcal{P}r(\mathcal{N}(F; R) = 0) \rightarrow 0$$

by the law of large numbers (1.5). We finally collect (5.2), (5.3), (5.4) and (5.5), substitute these into (5.1), and take $\varepsilon \rightarrow 0$, to establish that

$$\mathbb{E} \left[\left| \frac{\text{Vol } B(R)}{\mathcal{N}(F; R)} \mathbb{1}_{\mathcal{N}(F; R) > 0} - \frac{1}{c_{NS}(\rho)} \right| \right] \rightarrow 0$$

as $R \rightarrow \infty$, which is the statement of Theorem 1.5(b). \square

The proof of Theorem 1.9 is almost identical to the above:

Proof of Theorem 1.9. The statement (1.20) of Theorem 1.9 follows from the same argument as presented within the proof of Theorem 1.5(b) above, where we replace (1.4) with its manifold version (1.17), and the nodal lower concentration property in Definition (1.4) with its manifold version in Definition 1.8. \square

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