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KEEPING UP WITH THE JONESES AND UNEMPLOYMENT RISK

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Keeping Up With the Joneses and Unemployment Risk

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Abstract

This paper characterises the dynamic behaviour of a growing economy where individuals 'keep up with the Joneses' and face uninsurable labour income risk. Idiosyncratic uncertainty about future labour income reduces the marginal propensity to consume out of financial wealth and raises the effective rate of discount in the aggregate consumption Euler equation. The higher the average rate of income growth, the higher the saving rate. If individuals have uncertain lifetimes, a higher mortality rate reduces the marginal propensity to consume out of wealth, and raises the ratio of marginal utilities between employment and unemployment.

Keywords : precautionary saving; comparison utility; consumption; growth.

JEL Classification : B40; D81; E21.

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1 Introduction

This paper characterises the dynamic behaviour of a growing economy, where individuals have preferences defined in terms of how consumption compares to average consumption and face uninsurable labour income risk. While there is no aggregate risk, individuals are unable to hedge against the idiosyncratic risk by trading contingent claims. Individual labour income follows a continuous time Markov process. A natural interpretation is that individuals face unemployment risk and attempt to 'keep up with the Joneses'[1]. A restriction on the form of preferences makes the problem tractable

Ours is a tractable framework to analyse the impact of risk aversion in a growing economy. Consumer preferences are defined in terms of own consumption relative to average consumption, where the utility function is exponential. This framework is convenient and plausible in terms of its positive description of behaviour. For instance, the coefficient of absolute risk aversion is decreasing in average consumption (DARA), and the mean coefficient of relative risk aversion is constant (CRRA) on a balanced growth path[2]. Also, consistent with empirical evidence, an exogenous increase in the growth rate causes an increase in the saving rate. The analysis contrasts a partial equilibrium approach, where factor prices are exogenous, and a general equilibrium approach, where factor prices are determined by capital accumulation[3]. A simple aggregate consumption Euler equation is derived, which may be

[1]The framework developed in this paper could be applied to issues of mobility up and down income classes, and to issues of habit persistence, at the expense of increasing the number of relevant variables.

[2]If utility depends on absolute consumption only, the exponential utility function exhibits constant absolute risk aversion (CARA). This is no longer necessarily true if utility depends on relative consumption, because the coefficient of absolute risk aversion is a function of the reference index.

[3]Using standard terminology, the economy in partial equilibrium is referred to as a small, open economy, while the economy in general equilibrium is referred to as a closed economy. Alternatively, the partial equilibrium analysis can be interpreted as the equilibrium of a closed economy, where externalities result in a linear production function, and endogenous growth.

interpreted intuitively. Idiosyncratic labour income uncertainty raises the effective rate of discount in the aggregate consumption Euler equation. The accumulation of financial wealth reduces consumption growth and raises the marginal propensity to consume out of financial wealth (mpc).

The hypothesis that individuals compare their current level of consumption to some reference index has a long tradition in economics. The class of utility functions associated with this hypothesis is labelled 'comparison utility'. The hypothesis has several variants. In Ryder and Heal (1973), current preferences depend on some 'customary' level of consumption as well as current consumption, where the customary level of consumption is an exponentially-weighted average of past *consumption* levels. In Becker and Murphy (1988), consumption is a form of investment in 'learning' about one's own tastes, summarised by the stock of 'consumption capital', where current preferences depend on both current consumption and consumption capital. In De la Croix and Michel (1999), tastes are inherited, inasmuch as children become accustomed to the standard of living afforded by their parents so that adults have aspirations related to the standard of living of their youth. In Gali (1994), individuals compare their own consumption to the average level in the economy, 'keeping up with the Joneses'. This suggests the existence of a social dimension to consumption, perhaps due to a form of social competition. Society's average consumption affects individual preferences but is taken as given by the individual ; there is therefore a consumption externality. Following Gali (1994), this paper assumes that preferences are characterised by 'keeping up with the Joneses'.

It has been known at least since Merton (1971) that a tractable consumption function can be computed if utility is exponential and the economy is stationary. This result has been used in a number of contexts, for instance in models of search unemployment, e.g. Flemming (1978), Acemoglu and Shimer (1999), Hassler, Rodriguez Mora, Storesletten, and Zilibotti (1999); and in models of precautionary saving, e.g. Kimball and Mankiw (1989), Caballero (1990), Caballero (1991), Weil

(1993). However, what seems to have been missing is a way of applying this result to a growing environment. Recent studies of search unemployment and growth, for instance, have either resorted to numerical solutions or assumed risk-neutral consumers. The following studies are but a few examples where consumers are assumed to be risk-neutral or to enjoy free access to perfect insurance markets : Aghion and Howitt (1994), Merz (1995), Andolfatto (1996), Shi and Wen (1997), Mortensen and Pissarides (1998). Our framework is a relatively parsimonious way of relaxing the (unrealistic) assumption of risk-neutrality[4].

In general, if individual labour income follows a Markov process, the optimal consumption function has a closed-form solution only in a few special cases. For instance, if the utility function is quadratic, the optimal consumption function is proportional to the sum of financial wealth and human wealth, where human wealth is defined as the present discounted value of expected future labour income. Thus, despite the presence of uncertainty, optimal consumption satisfies a version of the permanent income hypothesis. However, the quadratic utility function implies increasing absolute risk aversion, an implausible description of behaviour. A more realistic description of behaviour demands that an increase in uncertainty raise expected marginal utility, and induce individuals to defer consumption. The constant elasticity of intertemporal substitution utility function has this desirable property, but is intractable. The exponential utility function also has this property, and is tractable.

If utility is defined in terms of a single consumption good, the exponential utility function exhibits constant absolute risk aversion and constant absolute prudence. An unrealistic implication is that negative levels of consumption are not necessarily ruled out. Moreover, if individual labour income follows a continuous-time Markov process, the characterisation of consumption does not apply to a growing economy.

[4]This is to our knowledge the only study to report a closed-form expression for the growing economy when labour income follows a Markov process.

By contrast, our framework is applicable to growing and stationary economies alike. In addition, our analysis shows that the model is consistent with empirical evidence on risk aversion and on the effects of growth on saving.

Section 2 introduces the model. Section 3 examines the case where there is a good state and a bad state, e.g. work and unemployment, and where households seek to 'keep up with the Joneses'. Both the open and closed economy are analysed. The model is also extended to account for finite lives. Section 4 reports the solution when individuals care about both absolute and relative consumption. Section 5 concludes.

2 The Model

Consider an infinitely lived individual who maximises :

$$E_0 \int_0^\infty e^{-\rho s} u(c_s, z_s) ds,$$

$$u(c, z) = \frac{-1}{\eta} \exp[-\eta \; v(c, z)], \tag{1}$$

$$v(c, z) = c/z^\nu,$$

where, at time t, c_t is the flow of consumption and z_t is the reference index to which consumption is compared. The sub-utility function $v(c_t, z_t)$ captures the relationship between own consumption and the reference index. Here the sub-utility function is defined in terms of a ratio. In (1) the parameter $\nu \geq 0$ is an index of the importance of the reference index in the consumer's utility. If $\nu = 0$, only the absolute level of consumption matters, while if $\nu = 1$, only relative consumption matters. For any value of ν between 0 and 1, both absolute and relative consumption matter.

For a complete description of preferences, the reference index, z_t, must be specified. The appendix reports the solution of the optimal consumption problem in the case where, at any time t, the individual takes z_t as given. In the core of the paper, the reference index is taken to be average consumption, $z_t = C_t$. Individual labour income follows a continuous time Markov process. Let $\mathbf{\Lambda} = [\lambda_{ij}]$ be the Markov transition matrix among the J states, where λ_{ij}, $i \neq j$, denotes the instantaneous probability of moving from state i to state j and where $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$. The transition probabilities are constant.

Consider the problem of an individual in state i at time t. Let $V(k_t, z_t, i, t)$ denote the expected present discounted value of the utility of consumption when the initial state at time t is state i, for a given initial level of financial wealth k_t, and for a given level of the reference index z_t. The individual maximises (1) subject to :

$$\dot{k}_{t+s} = r_{t+s} k_{t+s} + w^i_{t+s} - c_{t+s}, \tag{2}$$

$$\dot{w}^i_{t+s} = \gamma \, w^i_{t+s}, \quad \text{for all i}, \tag{3}$$

$$k_t \;\text{ given, and}$$

$$\lim_{s \to \infty} e^{-rs} k_{t+s} = 0 \;\text{ with probability one.} \tag{4}$$

where ρ is the rate of pure time preference, $r_{t+s} > 0$ is the rate of interest at time $t + s$, w^i_{t+s} is the flow of labour income in state i at time $t + s$, and $\gamma \geq 0$ is the growth rate of labour income in each and every state, conditional upon being in that state (income grows at the same rate in all states). The stationary economy obtains as a special case, $\gamma = 0$.

Kimball and Mankiw (1989) have characterised optimal consumption when individual labour income follows a Markov process. If $\nu = 0$ and $\gamma = 0$, we are back to the analysis of Kimball and Mankiw. The conditions under which the present model has a closed-form solution are derived in the appendix. It is shown that $\gamma > 0$ is incompatible with $\nu = 0$, so that the characterisation of optimal consumption given by Kimball and Mankiw does not apply to an economy in balanced growth. By contrast, if $\nu > 0$, optimal consumption can be characterised when $\gamma > 0$. This is the case in particular if individuals care about relative consumption only, $\nu = 1$, a situation referred to as 'keeping up with the Joneses'. The appendix derives the solution of the problem of optimal consumption with comparison utility, for any ν

$\in [0, 1]$. The core of the paper focuses on $\nu = 1$. Section 4 comments on the general case $\nu \in [0, 1]$.

The solution to the optimal consumption problem involves a linear consumption function, with time-varying slope and intercept. The differential equation that governs the dynamics of the slope (the marginal propensity to consume out of financial wealth) is a Bernouilli equation. It may therefore be solved in closed-form. The differential equation that governs the dynamics of the intercept, however, does not have a known closed-form solution, but may be easily analysed numerically, graphically, or by linear approximation. The model yields a tractable Euler consumption equation that accounts for uncertainty.

3 Keeping Up With the Joneses

This section characterises aggregate consumption in an economy where individuals face constant transition probabilities between two states, e.g. work and unemployment, and where they care only about relative consumption, 'keeping up with the Joneses'[5]. Individuals maximise :

$$E_0 \left[\frac{-1}{\eta} \int_0^\infty e^{-\rho s} \; e^{-\eta \; (c_s/C_s)} \; ds \right]$$

subject to a budget constraint, (2), given that labour income w_t^i alternates randomly between two states and follows a deterministic trend γ, (3), given an initial level of financial wealth k_0, and given a transversality condition that prevents consumers from borrowing indefinitely, (4). Individuals take average consumption C_t as given, neglecting the impact their actions have on average consumption. Population is normalised to one, so that C_t denotes both average and aggregate consumption.

The appendix shows that consumer optimisation yields a simple state-contingent

[5] See e.g. Gali (1994).

consumption function of the form

$$c_t^i = m_t\ k_t + a_t^i\ C_t,\ \ i = 1, 2,$$

where m_t is the slope of the consumption function (the marginal propensity to consume out of financial wealth), and where $a_t^i\ C_t$ is the intercept. The fact that the marginal propensity to consume out of financial wealth is independent of the state the consumer happens to be in makes it easy to aggregate every individual consumption functions into an aggregate consumption function.

Assuming an arbitrarily large number of individuals facing independent and identical transition probabilities, the distribution of consumers across states converges to the stationary distribution associated with the Markov chain. Let $\mathbf{\Pi}$ denote the stationary distribution associated with $\mathbf{\Lambda}$. Let C_t, K_t, and W_t denote per capita aggregates. The transition matrix is

$$\mathbf{\Lambda} = \begin{bmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{bmatrix}.$$

The associated stationary distribution between the two states is

$$\mathbf{\Pi} = \begin{bmatrix} \dfrac{\lambda_{21}}{\lambda_{12} + \lambda_{21}} & \dfrac{\lambda_{12}}{\lambda_{12} + \lambda_{21}} \end{bmatrix}.$$

Let the ratio of the marginal utilities between the two states be defined as

$$x_t \equiv \frac{u'(c_t^2/C_t)}{u'(c_t^1/C_t)} = e^{-\eta\ \Delta a_t}.$$

Let the per capita cross-sectional dispersion in labour income be written

$$\Delta \bar{w}_t \equiv \frac{w_t^2 - w_t^1}{W_t}.$$

Define the following short-hand notation,

$$\phi(x_t) \equiv \frac{\lambda_{12}\ \lambda_{21}}{\lambda_{12} + \lambda_{21}}\ \left(x_t + \frac{1}{x_t} - 2\right)$$

$$\beta(x_t) \equiv -\lambda_{12}(1 - x_t) + \lambda_{21}(1 - 1/x_t),$$

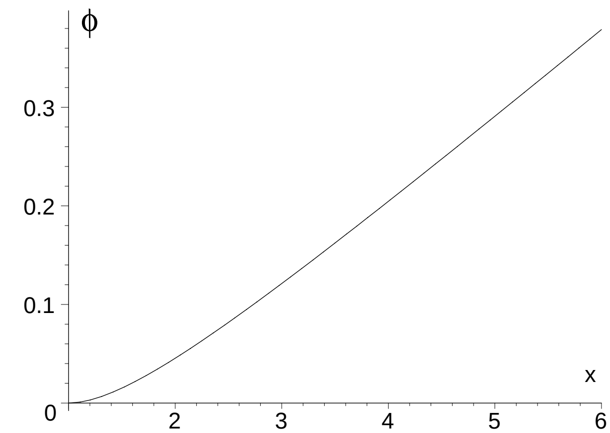

Fig. 1: **The 'Risk Premium'** $\phi(x)$

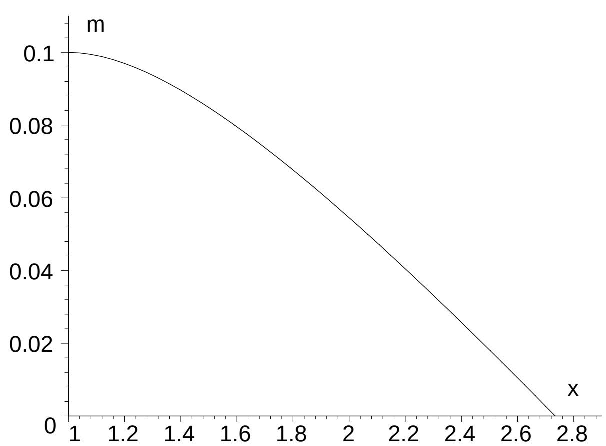

Fig. 2: **The mpc** $m(x)$

where $x_t \neq 0$. Note that, as the difference between the two states vanishes, $x_t \to 1$, $\phi_t \to 0$ and $\beta_t \to 0$.

Assuming that agents are infinitely lived, the dynamics of aggregate consumption may be characterised in terms of aggregate consumption, C_t, of the ratio of marginal utilities in the two states, x_t, of the marginal propensity to consume out of financial wealth, m_t, and of the aggregate level of financial wealth K_t :

$$\dot{C}_t/C_t = r_t + \phi(x_t) - \rho \tag{5a}$$

$$\dot{x}_t/x_t = m_t \log(x_t) + \beta(x_t) + \eta\ \Delta\bar{w}_t\ m_t\ (W_t/C_t) \tag{5b}$$

$$\dot{m}_t/m_t = m_t + \phi(x_t) - \rho. \tag{5c}$$

$$\dot{K}_t = r_t K_t + W_t - C_t, \tag{5d}$$

Equations (5a)-(5d) characterise the dynamics of aggregate consumption under uncertainty. In addition, individuals satisfy the transversality condition (4).

In partial equilibrium, for a given rate of interest r_t and a given level of aggregate labour income W_t, the dynamic characterisation of aggregate consumption is independent of financial wealth. This special property is a consequence of the exponential utility function assumed here, and of the assumption that individuals have infinite lives. If individuals have finite lives, the dynamics of aggregate consumption depend on financial wealth. In general equilibrium, the rate of interest and aggregate income are determined by capital accumulation. In this case, the dynamics of aggregate consumption depend on financial wealth, even if individuals have infinite lives. Both the partial equilibrium and general equilibrium cases are instructive. The partial equilibrium case may be interpreted as the equilibrium of a small open economy, while the general equilibrium case may be interpreted as the equilibrium of a closed economy.

If the cross-sectional dispersion in labour income is zero, $\bar{w}_t = 0$, for all $t \geq 0$, the two income states are equivalent and there is therefore no uncertainty. It follows that $x_t = 1$ and $\phi(x_t) = 0$. In this case, the system reduces to $\dot{C}_t = (r_t - \rho)C_t$. Thus, if there is no uncertainty, the Euler consumption equation of an economy where individuals have exponential preferences defined in terms of relative consumption happens to be the same as the Euler consumption equation of an economy where individuals have logarithmic preferences defined in terms of absolute consumption. The term $\phi(x_t)$ may be interpreted as a 'risk premium', the extra growth over $r_t - \rho$ reflecting extra precautionary saving. It is instructive to graph the curve $\phi(x_t)$. Figure 1 on Page 9 depicts the 'risk premium' (for the case $\Delta w_t \leq 0 \Rightarrow x_t \geq 1$). Other things equal, the higher the transitions probabilities, the higher the 'risk premium'[6]. The presence of uninsurable idiosyncratic risk raises the expected growth rate of aggregate consumption.

The 'risk premium' $\phi(x_t)$ determines the evolution of consumption, C_t, in equation (5a). It also determines the evolution of the marginal propensity to consume out of wealth, m_t, in equation (5c). In a steady state, the marginal propensity to consume out of wealth m is related to the ratio of marginal utilities x according to $m = \rho - \phi(x)$. Figure 2 on Page 9 depicts the steady-state relation. The marginal propensity attains a maximum when $x = 1$. The higher the degree of uncertainty, the lower m. The steady-state marginal propensity to consume out of wealth is thus negatively related to the degree of uncertainty.

'Keeping up with the Joneses' drives a wedge between the coefficient of relative risk aversion RRA_t and the inverse of the elasticity of intertemporal substitution σ_t. Following Constandinides (1990), the coefficient of relative risk aversion is defined in terms of the value function, $RRA \equiv \frac{-k\ V_{kk}}{V_k}$. This is the appropriate definition for an intertemporal economy. For a single individual,

$$RRA_t^i = \frac{\eta\ m_t\ k_t^i}{C_t},$$

for $i = 1, 2$. Taking the mathematical expectation over the distribution of financial

[6]The calibration is $\lambda_{12} = 0.1$, $\lambda_{21} = 1$, $\rho = 0.1$, $r = 0.1$, $\gamma = 0.02$, $\Delta\bar{w} = -0.33$. The parameter η is chosen such that $\overline{RRA}$ in equation (6) is about 3 ($\overline{RRA}$ is an endogenous variable).

wealth holdings k_t^i yields the *mean* coefficient of relative risk aversion,

$$\overline{RRA}_t = \frac{\eta \; m_t}{C_t/K_t} \tag{6}$$

Thus, the mean coefficient of relative risk aversion is constant on a balanced growth path. Likewise, let $ARA_t \equiv \frac{-V_{kk}}{V_k}$ be the coefficient of absolute risk aversion. For a single individual,

$$ARA_t = \frac{\eta \; m_t}{C_t}.$$

The coefficient of absolute risk aversion is thus independent of the level of financial wealth, and *decreasing* in consumption, DARA. The degree of risk aversion also pins down the degree of prudence. In particular, the coefficient of absolute prudence is decreasing in average consumption[7].

The parameter η in the utility function is not *per se* a measure of absolute risk aversion. Nevertheless, η contributes to the measures of risk aversion defined above. To gauge the implied order of magnitude of η, consider an economy in balanced growth. The mean coefficient of relative risk aversion is equal to $\eta(r - \gamma)$ divided by C_t/K_t. With a ratio of consumption to financial wealth equal to 20%, and a rate of interest net of growth equal to 5%, the implied mean coefficient of relative risk aversion is $\eta/4$. Thus, a mean coefficient of relative risk aversion equal to 3 requires $\eta = 12$.

The elasticity of intertemporal substitution in consumption is defined as

$$\sigma_t \equiv E_t \left[\frac{-u'(c_t^i/C_t)}{c^i u''(c_t^i/C_t)} \right]$$

With the exponential utility function, the elasticity of intertemporal substitution in consumption is constant, $\sigma = 1/\eta$. It follows that, in balanced growth, the wedge

[7] Remark. If, inappropriately, the coefficient is defined in terms of the direct utility function, the coefficient of relative risk aversion is constant and equal to η, both in and out of a balanced growth path. Likewise, the coefficient of absolute risk aversion, inappropriately defined, is equal to η/C_t.

between the coefficient of relative risk aversion and the inverse of the elasticity of intertemporal substitution is equal to $r - \gamma$ divided by C_t/K_t. The back of the envelope calculations conducted above suggests that this wedge could be about 1/4, or 25%.

'Keeping up with the Joneses' drives a wedge between the coefficient of relative risk aversion and the inverse of the elasticity of intertemporal substitution. The path of aggregate consumption growth is more smooth if preferences depend on relative consumption than if they depend on absolute consumption only.

3.1 The Open Economy

In the open economy, the dynamic characterisation of aggregate consumption is independent of financial wealth. The initial level of financial wealth determines the initial level of consumption, via the budget constraint, and has no other impact on the subsequent dynamics. The dynamics of aggregate consumption are given by equations (5a)-(5c). To economise on notation, let c_t denote, in this section, the aggregate consumption-labour income ratio, $c_t \equiv C_t/W_t$. In balanced growth, $\dot{C}_t/C_t = \gamma$, which implies a constant consumption ratio, c. The parameter restriction $r > \gamma$ is imposed for sensible results. From equations (5a), (5b), and (5c), it follows that a balanced growth path is characterised by

$$\phi(x) = \rho + \gamma - r \tag{7a}$$

$$m = r - \gamma \tag{7b}$$

$$c = \frac{-\eta(r-\gamma)\Delta\bar{w}}{\beta(x) + (r-\gamma)\log(x)}. \tag{7c}$$

The system formed by equations (7a)-(7c) characterises the long-run impact of the structural parameters on the 'risk premium' $\phi(x)$, on the marginal propensity to consume out of financial wealth m, and on the consumption ratio c. The higher the rate of pure time preference ρ, the higher the 'risk premium', the higher the marginal propensity to consume, and the lower the consumption ratio. The higher the rate of interest r, the lower the 'risk premium', the higher the marginal propensity to consume, and the higher the consumption ratio. The higher the steady growth of productivity γ, the higher the 'risk premium', and the lower the marginal propensity to consume. The higher the degree of uncertainty in the economy, as measured by $\phi'(x)$, the lower the consumption ratio.

The dynamics of consumption can be analysed graphically with the help of a (partial) phase diagram in the (x_t, c_t) plane. Equation (7a) is a quadratic equation in x. It is straightforward to check that the equation has two distinct real roots provided $\rho > r - \gamma$, that is provided ρ is sufficiently high. Moreover the two roots are on either side of 1. Note that the denominator of equation (7c) is positive or negative as x is greater or less than 1. It follows that $\Delta\bar{w} > 0$ implies $x < 1$; while $\Delta\bar{w} < 0$ implies $x > 1$; and obviously $\Delta\bar{w} = 0$ implies $x = 1$. Equations (7a) and (7c) may be plotted to construct the phase diagram. Figure 3 on Page 15 shows the phase diagram in the case $\Delta\bar{w} < 0$. The thick line is the *projection* on the (x_t, c_t) plane of the one-dimensional stable manifold. For any initial level of financial wealth consistent with balanced growth, there is a unique triplet (c_0, x_0, m_0) such that the economy is on the stable manifold and converges to the steady state.

If the initial stock of financial wealth is below its steady-state level, aggregate consumption is initially below its steady-state level. It follows that expected consumption growth is initially higher, and the marginal propensity to consume out of financial wealth initially lower, than in steady state. Individuals accumulate financial wealth and gradually increase their consumption. During the transition, expected consumption growth falls, the marginal propensity to consume out of financial wealth rises, and the saving rate falls.

It is instructive to compare the characterisation of aggregate consumption, in this economy populated by infinitely-lived individuals subject to uninsurable labour income risk, with an economy populated by individuals who have uncertain lifetimes – the model of Blanchard (1985). In the model of Blanchard, with perfect annuity

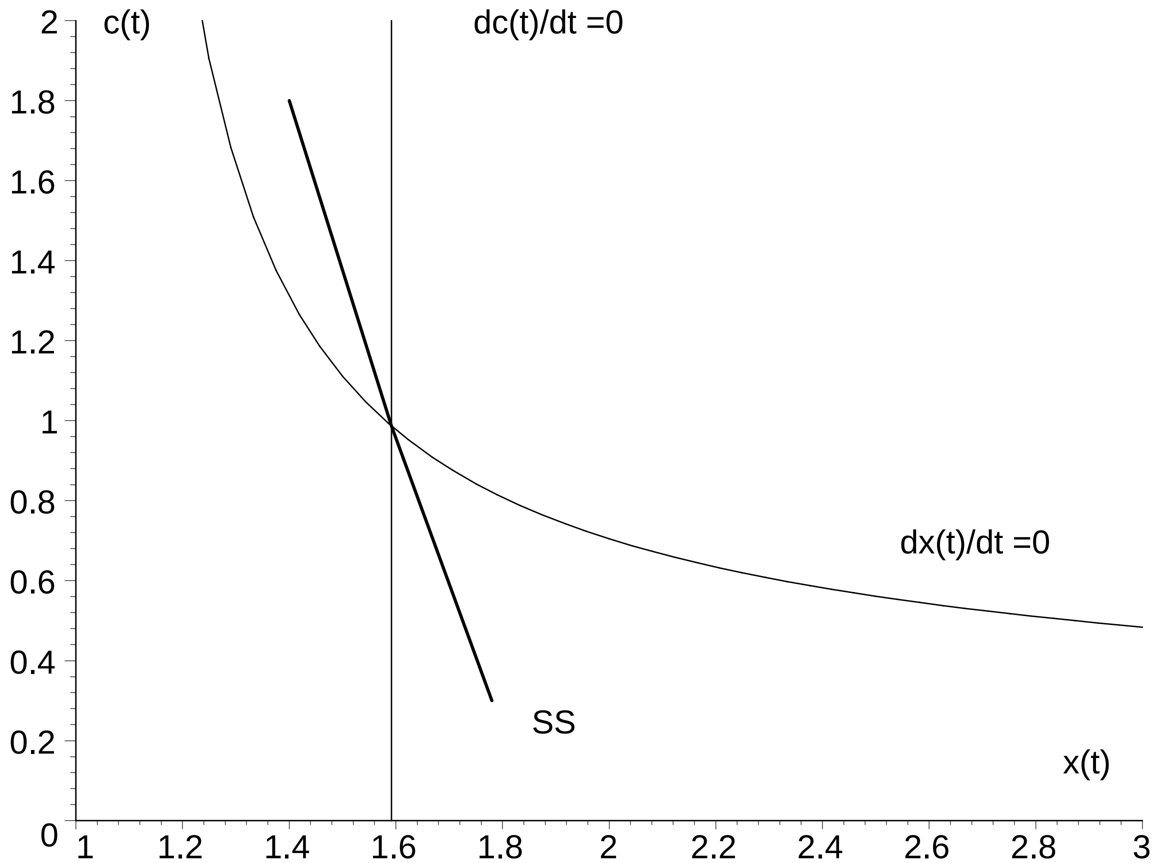

Fig. 3: **The Open Economy**

markets, uncertainty about the length of the horizon affects consumption in two ways. The provision of annuities ensures that individuals do not leave unintended bequests, and reduces saving. The provision of annuities also raises the rate of return on financial wealth, and encourages saving. The overall effect, if the intertemporal rate of substitution σ is less than or equal to unity, is to raise the marginal propensity to consume out of financial wealth. Thus, uncertainty about the length of life typically induces individuals to consume their wealth at a higher rate. On the other hand, idiosyncratic uncertainty about future labour income, when individuals cannot insure themselves, induces individuals, in aggregate, to consume their wealth at a lower rate. To see this, consider the model of Blanchard with logarithmic preferences and perfect annuity markets. The marginal propensity to consume out of wealth is equal to $\rho + p$, where p denotes the probability of death. By contrast, consider the present model in steady state. The marginal propensity to consume out of wealth is equal to $\rho - \phi$, where ϕ denotes a 'risk premium'.

The model exhibits a property which holds empirically and contradicts the standard permanent income model. An exogenous increase in the steady-state growth rate implies a reduction in the steady-state consumption ratio, i.e. an increase in steady-state saving. This may be seen by computing and evaluating $\partial c/\partial \gamma$ at the steady state,

$$\frac{\partial c}{\partial \gamma} = \frac{\eta \, \Delta \bar{w} \;\; [(r-\gamma)[r-\gamma+x\phi'(x)] + x\beta(x)\phi'(x)]}{x\phi'(x) \;\; [\beta(x) + (r-\gamma)\log(x)]^2} \;\; \leq 0.$$

As consumers have a desired ratio of consumption to income, an increase in the growth rate of income requires an increase in expected consumption growth. This is in accord with the empirical evidence gathered by Carroll and Weil (1994). 'Keeping up with the Joneses' can explain why income growth raises the saving rate.

To sum up, in the open economy, uncertainty about future labour income lowers the marginal propensity to consume out of financial wealth. As individuals accumulate financial wealth, the 'risk premium' falls and the marginal propensity to consume out of financial wealth rises.

3.2 The Closed Economy

In general equilibrium, the rate of interest r_t and labour income W_t are determined by capital accumulation. There are two factors of production, capital K_t and labour L_t. The number of efficient units of labour grows exogenously at rate γ. Let k_t denote capital per unit of effective labour, $k_t \equiv K_t/L_t$. Let $F(K_t, L_t)$ be the constant returns to scale production function and let δ denote the depreciation rate. Define $f(k_t) \equiv$

$F(k_t, 1) - (\delta + \gamma)k_t$. Factors are paid their marginal products[8]. Let $\Delta\bar{w}_t = \Delta\bar{w}$, for all $t \geq 0$. To economise on notation, let c_t denote, in this section, consumption per unit of effective labour, $c_t \equiv C_t/L_t$. In balanced growth, the closed economy is characterised by the following system

$$f'(k) + \phi(x) = \rho \tag{8a}$$

$$c = \frac{-\eta\ \Delta\bar{w}\ f'(k)\ [f(k) - kf'(k)]}{\beta(x) + f'(k)\ \log(x)} \tag{8b}$$

$$m = \rho - \phi(x) \tag{8c}$$

$$c = f(k) \tag{8d}$$

The system formed by equations (8a) -(8d) characterises the steady-state effects of the structural parameters on c, x, m, and k. Equation (8a) may be interpreted as follows. A higher degree of uncertainty reduces the effective rate of time preference, which encourages capital accumulation. On the other hand, a higher rate of growth of productivity reduces the effective rate of return on capital, which discourages capital accumulation. Equation (8c) shows that capital accumulation affects the consumption ratio in two ways. On the one hand, a higher level of capital reduces the marginal productivity of capital. This, in turn, reduces the marginal propensity to consume out of capital, and thus raises the consumption ratio. On the other hand, a higher level of capital raises the marginal productivity of labour. A higher marginal productivity of capital is associated with a higher wage, which raises the consumption ratio.

Manipulating equations (8a) -(8d) yields

$$\frac{\beta(x)}{\rho - \phi(x)} + \log(x) = -\eta\ \Delta\bar{w}\ [1 - \alpha(k)], \tag{9}$$

where $\alpha(k)$ denotes the capital income share, $\alpha(k) \equiv \frac{kf'(k)}{f(k)}$. The left-hand side of equation (9) is increasing in x. If the production technology is Cobb-Douglas,

[8]The subsequent qualitative analysis still holds if labour is paid a fraction of its marginal product, as in certain models of unemployment.

the capital income share α is independent of the capital-labour ratio. In this case, equation (9) gives x implicitly in terms of the structural parameters. The greater $\Delta\bar{w}$, or η, or ρ, the larger the 'risk premium' $\phi(x)$. The greater α, the smaller the 'risk premium' $\phi(x)$.

In general equilibrium, capital accumulation lowers the rate of interest, which discourages saving, but increases the wage, which encourages precautionary saving. If the elasticity of substitution between capital and labour is not too low, capital accumulation is associated with a fall in the 'risk premium'. If the elasticity is not too low, the system has one negative real root and is saddle-point stable[9]. There exists a one-dimensional stable manifold on which the economy converges to the steady-state equilibrium characterised by equations (8a)-(8d).

The dynamics of consumption may be analysed graphically by drawing the (partial) phase diagram in the (c_t, k_t) plane, corresponding to the following reduced-form system,

$$c = \frac{-\eta\ \Delta\bar{w}\ f'(k)\ [f(k) - kf'(k)]}{\beta[X(k)] + f'(k)\ \log[X(k)]} \tag{10a}$$

$$c = f(k) \tag{10b}$$

where $X(k)$ is defined as the inverse function of $\phi(x)$ evaluated at[10] $\rho - f'(k)$. Figure 4 on Page 19 shows the phase diagram associated with equations (10a)-(10b). The thick line is the *projection* on the (k_t, c_t) plane of the one-dimensional stable manifold. For any initial level of financial wealth k_0, there is a unique triplet $(c_0,\ x_0,\ m_0)$ such that the economy is on the stable manifold, and converges to the steady state.

If the elasticity of substitution is sufficiently low, there can be two negative eigenvalues, instead of one, and the outcome can be an indeterminacy of the balanced growth path. Indeterminacy is ruled out if production is Cobb-Douglas. More gener-

[9]The stability of the system is studied in the appendix.

[10]The function $\phi(x)$ is a quadratic form with two real roots on either side of 1. Given the sign of $\Delta\bar{w}$, only one of the roots is admissible. The inverse of $\phi(x)$ is therefore unique.

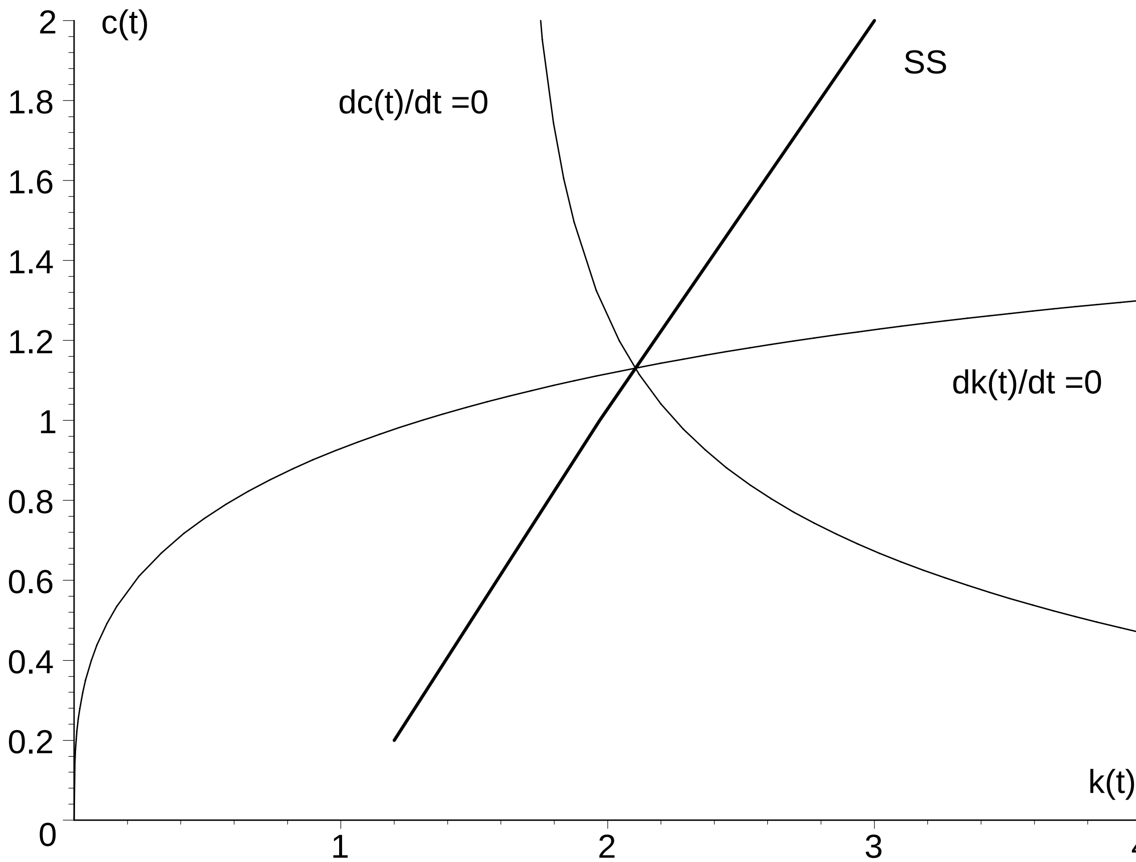

Fig. 4: **The Closed Economy**

ally, the balanced growth path is locally determinate if the elasticity of substitution is not too low.

Suppose the elasticity of substitution between capital and labour is not too low. If the initial level of capital is below its steady-state level, the aggregate consumption ratio is initially below its steady-state level. Consumption growth is initially higher, and the marginal propensity to consume initially lower, than in steady state. The economy accumulates capital and gradually increases consumption. During the transition, the 'risk premium' and consumption growth fall, and the marginal propensity to consume out of financial wealth rises. With Cobb-Douglas production, the saving rate falls during the transition[11].

3.3 Finite Lives

This section introduces finite lives in the model. An economy where, abstracting from risk, the marginal propensity to consume out of financial wealth is high requires a higher 'risk premium'. This follows from equation (7a) in the case of an open economy, and from equation (9) in the case of a closed economy. An economy where individuals have finite lives is characterised by a higher marginal propensity to consume out of financial wealth than an economy where individuals have infinite lives. Thus, a given degree of uncertainty about future labour income has a greater impact on aggregate consumption in an economy where individuals have finite lives.

Following Blanchard (1985), individuals face uncertain lifetimes and have access to perfect annuity markets. If individuals have exponential preferences defined in terms of relative consumption, and face labour income risk, aggregate consumption c_t (in effective units) and the marginal propensity to consume out of financial wealth m_t satisfy

$$\dot{c}_t = (r_t + \phi_t - \rho)\; c_t - \eta\; p\; m_t\; k_t \tag{11}$$

$$\dot{m}_t / m_t = m_t + \phi_t - \rho - p, \tag{12}$$

where $\phi_t = \phi(x_t)$ is determined as before by equation (5b), and where aggregate financial wealth k_t (in effective units) satisfies

$$\dot{k}_t = r_t k_t + w_t - c_t \tag{13}$$

In the model studied by Blanchard (1985), individuals have isoelastic preferences defined in terms of absolute consumption, and there is no labour income uncertainty.

[11] See the appendix.

The equations of the Blanchard model are[12]

$$\dot{c}_t = \sigma(r_t - \rho)\ c_t - p\ m_t\ k_t \tag{14}$$

$$\dot{m}_t/m_t = m_t - [(1-\sigma)(r_t + p) + \sigma(\rho + p)], \tag{15}$$

where σ denotes the elasticity of intertemporal substitution, and where aggregate financial wealth k_t satisfies equation (13). The coefficient of relative risk aversion in the model of Blanchard may be computed by applying the above definition (the following also holds if individuals have infinite lives),

$$RRA_t = \frac{1}{\sigma}\frac{m_t}{K_t/C_t}. \tag{16}$$

The effects of the parameters η and σ^{-1} on the coefficient of relative risk aversion may be contrasted by comparing equations (6) and (16). Consider an economy in steady state. The marginal propensity to consume out of financial wealth is equal to $\rho + p - \phi$ in our model, and $r + p - \sigma(r - \rho)$ in the Blanchard model. In both models, a higher probability of death p raises the steady-state marginal propensity to consume out of wealth m. In the Blanchard model, if $r > \rho$, then the higher σ^{-1}, the higher the marginal propensity to consume out of wealth m. If $r < \rho$, the opposite is true. In our model, the effect of η is implicit in the 'risk premium' $\phi(x)$. The higher the 'risk premium' $\phi(x)$, the lower the marginal propensity to consume out of wealth m. Thus, other things equal, the higher η the higher $\phi(x)$, and the lower m. For a given level of the marginal propensity to consume m, a higher mortality rate p implies a higher 'risk premium' $\phi(x)$.

4 Absolute and Relative Consumption

For completeness, this section characterises optimal consumption when individuals care about both absolute and relative consumption. In this case, individual prefer-

[12] These equations correspond to the equations on page 234 of Blanchard (1985), where the marginal propensity to consume out of wealth, m_t, is denoted Δ_t^{-1} by Blanchard, and where the elasticity of intertemporal substitution σ is denoted σ^{-1} by Blanchard.

ences are given by :

$$E_0 \left[-\frac{1}{\eta} \int_0^{\infty} e^{-\rho s} e^{-\eta \ c_s^i / C_s^\nu} \right], \tag{17}$$

where $0 \leq \nu \leq 1$. Aggregate consumption is characterised by the following system

$$\dot{C}_t = \frac{[r_t + \phi(x_t) - \rho] \ C_t^\nu}{(1-\nu) \ \eta + C_t^{\nu-1}} \tag{18a}$$

$$\dot{m}_t / m_t = m_t - r_t + \frac{\nu \ [r_t + \phi(x_t) - \rho] \ C_t^{\nu-1}}{(1-\nu) \ \eta + C_t^{\nu-1}} \tag{18b}$$

$$\dot{x}_t / x_t = m_t \log(x_t) + \beta(x_t) + \frac{\eta \ m_t \ \Delta w_t}{C_t^\nu}. \tag{18c}$$

Except in the case $\nu = 1$, the system of equations (18a)-(18c) does not have a steady state with (strictly) positive growth. In optimal consumption problems, balanced growth usually requires utility functions to be homogeneous with respect to their arguments. The utility function defined in (17) is not homogeneous unless $\eta = 1$. If $\eta = 1$, the utility function is homogeneous of degree zero with respect to c_t^1 and c_t^2.

In a stationary state where $\nu \neq 0$, and $x \neq 1$, aggregate consumption is equal to

$$C = \left[\frac{-\eta \ r \ \Delta w}{r \log(x) + \beta(x)} \right]^{\frac{1}{\nu}}.$$

The aggregate level of consumption in stationary state, C, is lower the higher ν is, that is, the more important relative consumption is in the utility function.

5 Conclusion

This paper develops a simple framework to study a growing economy where individuals face idiosyncratic labour income risk. Labour income alternates between two states, e.g. work and unemployment. A restriction on the form of the utility function makes the problem tractable. Preferences are defined in terms of relative consumption. An increase in average consumption raises the marginal utility of an individual's consumption, because individuals attempt to 'keep up with the Joneses'. Tractability stems from the property that all individuals have the same marginal propensity to consume out of financial wealth, irrespective of their date of birth, and irrespective of their wealth holdings. The propensity of individuals to accumulate financial wealth as a buffer against risk depends only on relative consumption in the different states.

'Keeping up with the Joneses' drives a wedge between the coefficient of relative risk aversion and the inverse of the elasticity of intertemporal substitution. The path of aggregate consumption growth is more smooth if preferences depend on relative consumption than if they depend on absolute consumption only. Idiosyncratic uncertainty about future labour income reduces the marginal propensity to consume out of financial wealth, raises the effective rate of discount in the consumption Euler equation, and stimulates saving. An increase in expected income growth raises the saving rate. If individuals have uncertain lifetimes, a higher mortality rate reduces the marginal propensity to consume out of wealth, and raises the ratio of marginal utilities between the state of employment and unemployment.

6 Appendix

6.1 Appendix A. Optimal Consumption with Comparison Utility.

Let V_k and V_t denote the partial derivatives of the value function. In continuous time, the Hamilton-Jacobi-Bellman (HJB) equation may be written

$$\rho V(k_t, z_t, i, t) = \max_{c_t} \Big\{ u(c_t, z_t) + \sum_{j \neq i} \lambda_{ij} \left[V(k_t, z_t, j, t) - V(k_t, z_t, i, t) \right] + V_k(k_t, z_t, i, t)\, \dot{k}_t + V_t(k_t, z_t, i, t) \Big\}\ , \quad i = 1...J. \tag{19}$$

The problem is recursive. The assumption of an infinite horizon means that the HJB equation cannot be solved backwards starting from the last period. However, under certain conditions, given sufficient time the optimal value function will converge to a stationary form. Note that in a stationary state, the term V_t in the HJB equation vanishes.

Maximisation of the right-hand side of the HJB equation yields the first-order condition :

$$\frac{\partial u(c_t, z_t)}{\partial c_t} = V_k(k_t, z_t, i, t)\ , \quad i = 1...J. \tag{20}$$

The problem has a closed-form solution in the special case where the consumer has exponential utility, $u(c, z) = \frac{-1}{\eta} \exp[-\eta v(c, z)]$. A strategy for solving the problem consists in guessing a special form of the value function and verifying that it is the solution. In the special case where $\nu = 0$ and $\gamma = 0$ (the reference index vanishes), and where the rate of interest r is constant, the solution is known, from e.g. Flemming (1978) and Kimball and Mankiw (1989),

$$V(k_t, i, t) = \frac{-1}{\eta r}\ \exp[-\eta\ c(k_t, i, t)]\ , \tag{21}$$

$$c(k_t, i, t) = r k_t + a_t^i\ , \quad i = 1...J, \tag{22}$$

where a_t^i is the intercept of the consumption function in state i. This solution can be obtained by the 'guess and verify method' as follows.

Consider the first-order condition (20). Assuming exponential utility, letting $\nu = 0$, $\gamma = 0$, and guessing the solution,

$$\exp[-\eta c(k_t, i, t)] = \exp[-\eta(rk_t + a_t^i)]\,, \quad i = 1...J,$$

which implies $c(k_t, i, t) = rk_t + a_t^i$.

Combining equations (2), (21) and (22) into the HJB equation (19) shows that the guess is indeed correct, and that the intercept of each consumption functions, for each state, must solve the following system of differential equations,

$$\dot{a}_t^i = r(a_t^i - w_t^i) + \frac{1}{\eta}\sum_{j \neq i} \lambda_{ij}\, e^{\eta(a_t^i - a_t^j)} + \frac{1}{\eta}\,(r - \rho - \sum_{j \neq i} \lambda_{ij})\,,\ i = 1...J. \tag{23}$$

Given constant values of income w^i in each state i, it may be shown[13] that the system converges, in the long run, to a unique stationary state, with $\dot{a}_t^i = 0$ and $a_t^i = a^i$ for all t.

In the special case where there are only two states, e.g. employed and unemployed, the steady-state consumption functions are,

$$c^1 = rk + w^1 - \frac{r - \rho - \lambda_{12}\,(1 - x)}{\eta\, r} \tag{24a}$$

$$c^2 = rk + w^2 - \frac{r - \rho - \lambda_{21}\,(1 - 1/x)}{\eta\, r}\,, \tag{24b}$$

where x_t is the ratio of the marginal utilities between the two states,

$$x_t \equiv \frac{u'(c_t^2)}{u'(c_t^1)}. \tag{25}$$

The ratio x_t is the solution of the differential equation,

$$\dot{x}_t / x_t = r \log(x_t) - \lambda_{12}(1 - x_t) + \lambda_{21}(1 - 1/x_t) - r\ \eta(w_t^1 - w_t^2). \tag{26}$$

For a steady state to exist, x_t must either be constant or growing at a constant rate. Let $g_t \equiv \dot{x}_t / x_t$. In a steady state g must satisfy

$$\log(x_t) - (\lambda_{12}/r)\,(1 - x_t) + (\lambda_{21}/r)\,(1 - 1/x_t) - g/r = \eta(w_t^1 - w_t^2). \tag{27}$$

[13] See Kimball and Mankiw (1989).

Suppose that labour income in each state, w_t^i, is growing at the constant rate γ. It is immediately apparent that the system does not have a steady state if $\gamma > 0$. Suppose for instance that $w_t^1 > w_t^2$. This implies that over time the difference $w_t^1 - w_t^2$ tends to ∞ at rate γ. For a steady state to exist the left-hand side of equation (27) must grow at the same rate. The presence of the constant terms makes it impossible. In addition, the terms in x_t and the terms in $\log(x_t)$ cannot grow at the same rate. But even a constant value of x, i.e. $g = 0$, is incompatible with steady growth. Without imposing further constraints on the problem, such as explicitly restricting consumption to positive values or introducing borrowing constraints, it cannot be guaranteed that the solution given by equations (22) and (23) will always satisfy the transversality condition (4). Thus, if labour income grows at a constant rate, there is no simple solution to the problem. It will be shown that a simple solution can be found, when labour income is growing, if utility depends on the ratio of consumption over a reference index that grows at the same rate.

The 'guess and verify' method can be applied to find the solution when $0 < \nu \leq 1$, and when the rate of interest is a function of time, r_t. It is convenient to rewrite the problem in terms of the transformed variables, $\hat{k}_t \equiv k_t/(z_t)^\nu$, $\hat{c}_t \equiv c_t/(z_t)^\nu$, and $\hat{w}_t^i \equiv w_t^i/(z_t)^\nu$. Let $\dot{z}_t/z_t \equiv \mu_t$. Consumers choose the modified control variable $\hat{c}_t$ to maximise the objective function, subject to the law of motion for the modified state variable, $\hat{k}_t$,

$$\dot{\hat{k}}_t = (r_t - \nu\mu_t)\ \hat{k}_t + \hat{w}_t^i - \hat{c}_t. \tag{28}$$

and subject to the law of motion for the modified income process,

$$\dot{\hat{w}}_t^i = (\gamma - \nu\mu_t)\ \hat{w}_t^i. \tag{29}$$

The solution is found by guessing that there exist m_t and a_t^i such that

$$V(\hat{k}_t, z_t, i, t) = \frac{-1}{\eta m_t}\ \exp[-\eta\ \hat{c}(\hat{k}_t, z_t, i, t)] \tag{30}$$

$$\hat{c}(\hat{k}_t, z_t, i, t) = m_t\ \hat{k}_t + a_t^i\ , \quad i = 1...J, \tag{31}$$

where the presence of the reference index z_t is implicit in m_t and a_t^i, but also in the modified state variable $\hat{k}_t$. This guess yields consumption functions of the form

$$c(k_t, z_t, i, t) = m_t \ k_t + a_t^i (z_t)^\nu \ , \quad i = 1...J, \tag{32}$$

where the slope of the consumption function, or marginal propensity to consume out of wealth, is m_t, and where the intercept is $a_t^i(z_t)^\nu$. A novel feature is that the slope of the consumption function is not necessarily constant and generally state-dependent. The validity of this guess is now checked.

Equation (22) above is replaced by equation (31). Substituting equations (2), (20), and (31) into the HJB equation (19) confirms that the guess is indeed correct, and that a_t^i solves the system of differential equations (33) [instead of (23)],

$$\begin{aligned} \dot{a}_t^i = m_t \ (a_t^i - \hat{w}_t^i) \ + \ \frac{1}{\eta} \sum_{j \neq i} \lambda_{ij} \ e^{\eta(a_t^i - a_t^j)} \ + \ \frac{1}{\eta} \ (m_t - \dot{m}_t / m_t - \rho - \sum_{j \neq i} \lambda_{ij}) \\ + \ m_t \ (m_t - r_t + \nu \mu_t - \dot{m}_t / m_t) \ \hat{k}_t \ , \quad i = 1...J. \end{aligned} \tag{33}$$

As it is written, equation (33) depends non-linearly on the modified state variable $\hat{k}_t$. This would be incompatible with the guess that the modified consumption function is linear in $\hat{k}_t$. If the guess is correct, therefore, the term in $\hat{k}_t$ must vanish. Setting the term in $\hat{k}_t$ equal to zero in equation (33) implies

$$\dot{a}_t^i = m_t \ (a_t^i - \hat{w}_t^i) + \frac{1}{\eta} \sum_{j \neq i} \lambda_{ij} \ e^{\eta(a_t^i - a_t^j)} \ + \frac{1}{\eta} \ (r_t - \nu \mu_t - \rho - \sum_{j \neq i} \lambda_{ij}) \tag{34a}$$

$$\dot{m}_t = m_t \ (m_t - r_t + \nu \mu_t) \ , \quad i = 1...J, \tag{34b}$$

where (34b) has been used to eliminate $\dot{m}_t / m_t$ in (34a).

Equation (34a) gives the law of motion of the intercept, and equation (34b) gives the law of motion of the slope. Neither the slope nor the intercept depends on the modified state variable $\hat{k}_t$, or on any other endogenous variable. They depend on labour income w_t^i, on the reference index z_t [recall that $\hat{w}_t^i = w_t^i / (z_t)^\nu$], on the law of motion of the reference index μ_t, and on the structural parameters, all of which are exogenous to the consumer's problem. This confirms that the guess is correct. To ensure that a steady state exists and that the transversality condition is satisfied, however, a restriction on the parameters of the model is needed.

Given constant values of modified income $\hat{w}^i$, in each state, and a constant rate of growth of the reference index μ, the system of differential equations (34a)-(34b) has a unique steady state. However, if income grows at constant rate $\gamma > 0$, as in equation (3), modified income $\hat{w}^i$ can be constant only if $\mu = \gamma/\nu$. In other words, for a steady state to exist when labour income is growing at rate γ, the reference index must be growing at rate γ/ν. This condition will not generally be satisfied for simple economic examples. Indeed it seems difficult to justify that in an economy growing at rate γ, the reference index should grow at a different rate. However, if $\nu = 1$, and if, in steady state, the reference index is growing at rate $\mu = \gamma$, then the condition $\mu = \gamma/\nu$ is satisfied and thus the system (34a)-(34b) has a unique stationary state. Thus if consumers care only about relative consumption, i.e. $\nu = 1$, it is possible to find a closed-form solution for precautionary saving in the growing economy. Rather than imposing the condition that $\mu = \gamma/\nu$, the stronger set of conditions is imposed :

$$\nu = 1 \ , \qquad \gamma = \mu > 0. \tag{35}$$

The restrictions in (35) imply that, in a steady state, the reference index, in each state, grows at the same rate as labour income. Outside a steady state, however, the variables may grow at different rates. With (35) imposed, the system can be re-written,

$$\dot{a}_t^i = m_t \ (a_t^i - w_t^i/z_t) + \frac{1}{\eta} \sum_{j \neq i} \lambda_{ij} \ e^{\eta(a_t^i - a_t^j)} \ + \frac{1}{\eta} \ (r_t - \mu_t - \rho - \sum_{j \neq i} \lambda_{ij}) \tag{36a}$$

$$\dot{m}_t = (m_t)^2 - (r_t - \mu_t) \ m_t \ , \quad i = 1 ... J. \tag{36b}$$

Equation (36a) is the analogue of equation (23) and it is also non-linear. Three differences are apparent. First, modified income, w_t/z_t, 'replaces' income w_t. Secondly, m_t, rather than r appears in the first term on the right-hand side. Thirdly, the growth correction term $-\mu_t$ appears on the right hand side. Equation (18b) implies that the slope of the consumption function is not necessarily constant, unlike when utility depends only on absolute consumption. A useful characteristic of the system is that it is block-recursive. The intercept depends on the slope, while the slope is determined independently. The system may therefore be solved in two steps, solving for the slope first, solving for the intercept next. Equation (18b) is non-linear, but can be recognised as a Bernouilli equation. A simple change of variable will transform it into a linear equation, which may then be solved explicitly, given a sequence for $\{\mu_t\}$, for all $t > 0$. Given constant values of μ, then $m = r - \mu$. Given the condition imposed in (35), $\mu = \gamma$, the stationary state value of the slope is, therefore,

$$m = r - \gamma. \tag{37}$$

Clearly, for meaningful results, the following restriction on the parameters is necessary, $r > \gamma$. The full solution to (18b) is given below, but first an example is discussed.

Assuming $\nu = 1$, the consumption function may be re-written [instead of (32)],

$$c(k_t, z_t, i, t) = m_t \ k_t + a_t^i z_t, \quad i = 1...J. \tag{38}$$

In the special case where there are only two states, the steady-state consumption functions are

$$c_t^1 = (r-\gamma)k_t + w_t^1 - \left(\frac{r-\gamma-\rho+\lambda_{12}\ (x-1)}{\eta\ (r-\gamma)}\right)\ z_t \tag{39a}$$

$$c_t^2 = (r-\gamma)k_t + w_t^2 - \left(\frac{r-\gamma-\rho+\lambda_{21}\ (1/x-1)}{\eta\ (r-\gamma)}\right)\ z_t, \tag{39b}$$

where x, defined earlier in (25), is the solution of :

$$\log(x) + \frac{\lambda_{12}\ (x-1)}{r-\gamma} - \frac{\lambda_{21}\ (1/x-1)}{r-\gamma} = \eta(\hat{w}^1 - \hat{w}^2). \tag{40}$$

Equations (39a) and (39b) can be compared with equations (24a) and (24b). It is immediately apparent that, given a constant value of x (the ratio of marginal utilities in the two states), consumption and wealth can grow at the same rate as labour income only if the reference index too is growing at the same rate, i.e. only if $\gamma = \mu$. Equation (40) can be compared with equation (26). Again, it is apparent that for x to be constant, $\hat{w}^1$ and $\hat{w}^2$ must be constant, which can happen only if labour income and the reference index are growing at the same rate, i.e. only if $\gamma = \mu$. *In a steady state*, each consumption function is linear in wealth, labour income and the reference index. This is no longer true outside a steady state since x_t is a non-linear (implicit) function of z_t.

The slope is the solution of a Bernouilli differential equation whose solution can be found by a simple change of variable. Let $\zeta_t \equiv -1/m_t$ and define $R_t \equiv r - \mu_t$. Substituting ζ_t into equation (18b),

$$\dot{\zeta}_t = R_t \, \zeta_t \, + 1.$$

This is a linear equation in ζ_t, with forward-looking solution,

$$\zeta_t = \int_0^\infty e^{-\int_0^s R_v dv} \, ds \, + B \, e^{\int_0^t R_v dv},$$

where B is a constant of integration. However, for the transversality condition (4) to be satisfied, the constant of integration B must be equal to zero,

$$\zeta_t = \int_0^\infty e^{-\int_0^s R_v dv} \, ds.$$

The value of the slope may be found from ζ_t by simply reversing the change of variable, $m_t = -1/\zeta_t$. It may be checked that, asymptotically, $\lim_{t\to\infty} \zeta_t = -1/\lim_{t\to\infty} R_t = -1/(r-\gamma)$, which is consistent with $\lim_{t\to\infty} m_t = r - \gamma$.

Let Δa_t denote $a_t^2 - a_t^1$, and let Δw_t denote $w_t^2 - w_t^1$. Equation (36a yields

$$\Delta \dot{a}_t = m_t \left(\Delta a_t - \frac{\Delta w_t}{C_t} \right) + \frac{\lambda_{21}}{\eta} \, e^{\eta \, \Delta a_t} - \frac{\lambda_{12}}{\eta} \, e^{-\eta \, \Delta a_t} \, + \frac{\lambda_{21} - \lambda_{12}}{\eta}$$

Assuming that agents are infinitely lived, the dynamics of aggregate consumption may be characterised in terms of C_t, Δa_t, and m_t :

$$\dot{C}_t = [r_t + \varphi(\Delta a_t) - \rho]\ C_t$$
$$\Delta \dot{a}_t = m_t\ \Delta a_t + \varkappa(\Delta a_t) - \frac{m_t\ \Delta w_t}{C_t}$$
$$\dot{m}_t = (m_t)^2 + [\varphi(\Delta a_t) - \rho]\ m_t,$$

where

$$\varkappa(\Delta a_t) \equiv \frac{\lambda_{21} - \lambda_{12}}{\eta}\ [\cosh(\eta \Delta a_t) - 1] + \frac{\lambda_{12} + \lambda_{21}}{\eta}\ \sinh(\eta \Delta a_t)$$
$$\varphi(\Delta a_t) \equiv \frac{2\ \lambda_{12}\ \lambda_{21}}{\lambda_{12} + \lambda_{21}}\ [\cosh(\eta \Delta a_t) - 1].$$

An alternative characterisation may be obtained in terms of x_t, rather than Δa_t, by using

$$x_t \equiv \frac{u'(c_t^2)}{u'(c_t^1)} = e^{-\eta\ \Delta a_t}.$$

6.2 Appendix B. The Open Economy.

The stability of the system may be checked by computing the determinant and trace of the Jacobian matrix. The determinant is equal to

$$\frac{\eta (r - \gamma)^2\ x\ \Delta \bar{w}\ \phi'(x)}{c},$$

where

$$\phi'(x) = \frac{\lambda_{12}\ \lambda_{21}}{\lambda_{12} + \lambda_{21}} \left(1 - \frac{1}{x^2} \right).$$

The sign of the determinant depends on the sign of the product $\phi'(x)\ \Delta \bar{w}$. Since $\Delta \bar{w} < 0$ implies $x > 1$, and $\Delta \bar{w} > 0$ implies $x < 1$, it follows that $\phi'(x)$ and $\Delta \bar{w}$ are of opposite sign. The determinant is therefore always negative, implying that the product of the three characteristic roots is negative. The trace is positive,

$$2(r - \gamma) + \lambda_{12}\ x^2 + \frac{\lambda_{21}}{x} \quad > 0.$$

The trace is positive, so the system has at least one positive root. But since the product of the roots is negative, there must be two positive real roots and one negative real root. The system is therefore saddle-point stable.

6.3 Appendix C. The Closed Economy.

The stability of the system may be analysed by applying the Routh-Hurwitz stability theorem. Let the characteristic polynomial be written

$$\lambda^4 - \mathrm{Tr}\ \lambda^3 + \mathrm{M}_2\lambda^2 - \mathrm{M}_1\lambda + \mathrm{Det}.$$

The Routh-Hurwitz theorem states that the number of positive roots of the characteristic polynomial is equal to the number of variations of sign in the scheme

$$1 \quad -\mathrm{Tr} \quad +\mathrm{M}_2 \quad -\mathrm{M}_1 \quad +\mathrm{Det},$$

The following is sufficient for stability : Tr > 0, Det < 0, and $\mathrm{M}_2 \geq 0$. The determinant Det is equal to

$$xcf'f''\phi' \left\{ \underbrace{\frac{f' + x\phi'}{x\phi'}}_{+} \quad - \underbrace{\frac{\beta + f' \log(x)}{w}}_{+} \quad \underbrace{\left(k + \frac{f'w}{ff''}\right)}_{+\,-} \quad \underbrace{+\ \frac{\beta}{f'}}_{+} \right\}.$$

The sign of the determinant is generally ambiguous. It is negative if the elasticity of substitution between capital and labour is not too low. The signs of the expressions marked by the braces are established in the case $x \geq 1$. The case $x < 1$ is obtained by symmetry. From $x \geq 1$, it follows that $\phi'(x) > 0$ and $\log(x) > 0$, which establishes the sign of the first two terms marked by braces. Lastly, the sign of the third term marked by braces depends on the sign of $1 - \epsilon_{KL}$, where ϵ_{KL} denotes the elasticity of substitution between capital and labour, $\epsilon_{KL} \equiv \frac{F_K(K,L)F_L(K,L)}{F(K,L)F_{12}(K,L)} = \frac{-f'w}{fkf''}$. The trace Tr is positive,

$$3f' + x\beta' > 0.$$

The principal minor M_2 is equal to

$$\underbrace{3(f')^2 + 2xf'\beta' + \beta x\phi'}_{+} \quad + \underbrace{ff''}_{-}$$

The principal minor M_2 is positive provided f'' is low enough. For instance, $\epsilon_{KL} \geq \frac{f-kf'}{3f'k}$ is sufficient for $M_2 > 0$. If production is Cobb-Douglas, the sufficient condition is $\epsilon_{KL} \geq \frac{1-\alpha}{3\alpha}$.

If the elasticity of substitution between capital and labour ϵ_{KL} is not too close to zero, the system has three positive roots and one negative root. The system is therefore saddle-point stable. The sufficient condition is not necessary. Saddle-point stability can arise even if the sufficient condition is not satisfied.

6.4 Appendix D. The Saving Rate.

This section shows that, if production is Cobb-Douglas, the saving rate falls as capital is accumulated. The saving rate is defined as

$$s_t \equiv 1 - \frac{c_t}{f(k_t)}$$

Let $\chi_t \equiv \frac{c_t}{f(k_t)}$. In steady-state, the saving rate is equal to $s = \frac{\alpha(\delta+\gamma)}{f'(k)}$. After a few manipulations, the rate of growth of χ_t may be written

$$\frac{\dot{\chi}_t}{\chi_t} = \underbrace{\alpha(1-s_t)}_{+} + \underbrace{\phi(x_t) - \phi(x)}_{+} + \underbrace{\alpha(\delta+\gamma)\left(1-\frac{1}{s}\right)}_{-}.$$

The saving rate must fall during the transition, until $\dot{\chi}_t = 0$.

References

Acemoglu, Daron and Robert Shimer (1999) "Efficient Unemployment Insurance" *Journal of Political Economy 107* :893-928.

Aghion, Philippe and Peter Howitt (1994) "Growth and Unemployment" *Review of Economic Studies 61* : 477-494.

Andolfatto, David (1996) "Business Cycles and Labor-Market Search" *American Economic Review 86* : 112-132.

Becker, Gary S. and Kevin M. Murphy (1988) "A Theory of Rational Addiction" *Journal of Political Economy 96* : 675-700.

Blanchard, Olivier J. (1985) "Debt, Deficits, and Finite Horizons" *Journal of Political Economy 93* : 223-247.

Caballero, Ricardo J. (1991) "Earnings Uncertainty and Aggregate Wealth Accumulation" *American Economic Review 81 (4)* 859-871.

Caballero, Ricardo J. (1990) "Consumption Puzzles and Precautionary Saving" *Journal of Monetary Economics 25* 113-136.

Campbell and Cochrane (1999) "By Force of Habit : A Consumption-Based Explanation of Aggregate Index Market Behaviour" *Journal of Political Economy 107.*

Carroll, Christopher D. and David N. Weil (1994) "Saving and Growth: a Reinterpretation" *Carnegie-Rochester Conference Series on Public Policy 40* : 133-192.

Constandinides, George M. (1990) "Habit Formation: A Resolution of the Equity Premium Puzzle" *Journal of Political Economy 98* :519-543.

De la Croix, David and Philippe Michel (1999) "Optimal Growth when Tastes Are Inherited" *Journal of Economic Dynamics and Control 23* :519-537.

Flemming, John S. (1978) "Aspects of Optimal Unemployment Insurance" Journal of Public Economics 10 : 403-425.

Gali, Jordi (1994) "Keeping Up With the Joneses : Consumption Externalities, Portfolio Choice, and Asset Prices" *Journal of Money, Credit, and Banking 26* : 1-8.

Hassler, Rodriguez Mora, Storesletten, and Zilibotti (1999) "Equilibrium Unemployment Insurance" IIES Seminar Paper No. 667.

Kimball, Miles S. and N. Gregory Mankiw (1989) "Precautionary Saving and the Timing of Taxes" *Journal Of Political Economy 97 (4)* : 863-879.

Merton, Robert C. (1971) "Optimum Consumption and Portfolio Rules in a Continuous-Time Model" *Journal of Economic Theory 3* : 373-413.

Merz, Monika (1995) "Search in the Labour Market and the Real Business Cycle" *Journal of Monetary Economics 36* : 269-300.

Mortensen, Dale T. and Christopher A. Pissarides (1998) "Technological Progress, Job Creation and Job Destruction" *Review of Economic Dynamics 1* : 733-753.

Ryder, Harl E. Jr. and Geoffrey M. Heal (1973) "Optimal Growth and Intertemporally Dependent Preferences" *The Review of Economic Studies 40* 1-31.

Shi, Shouyong and Quan Wen (1997) "Labor Market Search and Capital Accumulation : Some Analytical Results" *Journal of Economic Dynamics and Control 21* : 1747-1776.

Weil, Philippe (1993) "Precautionary Savings and the Permanent Income Hypothesis" *Review of Economic Studies 60* : 367-383.