Embedding of Selmer group associated to relative Frobenius morphism of an abelian variety into the group of homomorphism of vector bundles

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Abstract

In this thesis we show that the Selmer group of the relative Frobenius morphism of an abelian variety defined on the function field of a projective and smooth curve C over a finite field k_0 of characteristic $p > 0$ can be embedded into the group of homomorphisms between two natural vector bundles over C.

Warning: Colored text and colored diagrams are contained, readers are suggested to print this thesis with a color printer.

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Contents

1. Introduction

Let C be a smooth projective curve over a finite field of characteristic p. We denote the function field of C by K. Let $\mathcal{A}$ be a semi-abelian scheme over C with the generic fiber A which is an abelian variety over K.[1] Naturally, we have the relative Frobenius morphism $F_{A/K} : A \to A^{(p)}$. It induces a map on rational points $F_{A/K} : A(K) \to A^{(p)}(K)$. To study the properties of the group $A^{(p)}(K)/F_{A/K}(A(K))$, we construct the corresponding Selmer group of A/K associated to $F_{A/K}$; this naturally contains $A^{(p)}(K)/F_{A/K}(A(K))$ as a subgroup. In this thesis, we will show that the Selmer group can be embedded in the group of homomorphisms of some vector bundles over C, which is a finite group.

Classically, one constructs the Selmer group of an abelian variety A/K associated to an isogeny f, where K is a number field and f is the multiplication map by m on $A(K)$, and shows that this group is finite to imply that $A(K)/f(A(K))$ is finite. This is *the weak Mordell-Weil Theorem*, see [2, C.4].

In the following subsections, we will define smooth projective curves, sites and sheaves, semi-abelian schemes, the Frobenius morphisms and the Selmer groups, and give some related lemmas and remarks. We will also gradually introduce our specific situation.

Let us give a quick introduction to our main theorem. First, let A be an abelian variety over a global field K of characteristic p which can be viewed as the function field of a smooth projective curve C. Assume there exists a semi-abelian scheme $\mathcal{A}$ over the curve C such that its generic fiber is A.

We can view the semi-abelian scheme $\mathcal{A}$ as a family of objects (fibers) over the curve, where the objects are closely related to abelian varieties. At the points (of the curve) where the fibers are indeed abelian varieties, they behave "well". We can show that the locus of nice points (i.e. subset of the curve such that the fibers of $\mathcal{A}$ are abelian varieties) is open dense. Furthermore, the locus of bad points, say E, is closed and finite. This set must be taken seriously, since it is the main obstacle of our expected result. We can view this set as the set of singularities which we need to handle.

One way to handle these singularities is by compactification, which is a global method; this is done in D. Rössler's paper [3, Corollaire 1.5]. He proved a stronger result [3, Corollaire 1.4] which generalizes our statement to higher dimension, with the fallback of requiring an extra condition: $A[m](K) \cong (\mathbb{Z}/m\mathbb{Z})^{2g}$ for some m s.t. $(p, m) = 1$.[2] As he mentioned in [3, Remark 1.6], the existence of the hypothesis is there forced by the nature of the proof, which is based on the existence of the moduli stack $A_{g,m}$ of abelian varieties of dimension g over k_0 with level-m structure that contains A, see [4, I, Def. 4.4]. In this thesis, we handle these singularities locally. If this method can be generalized to higher dimension, we will not require the existence of the above moduli stack, and hence we can get rid of the restriction $A[m](K) \cong (\mathbb{Z}/m\mathbb{Z})^{2g}$ for some m s.t. $(p, m) = 1$ mentioned above.

There is a property of this set of singularities, saying that those "singularities" are nice singularities in the sense that locally one may compensate by a rational function with only poles of order one. This property is the central reason that the embedding of the Selmer group.

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[1] According to [1, theorem 4.4.], it is very possible that $\mathcal{A} \cong N(A)^0$ where $N(A)^0$ is the identity component of the Néron model of A, given that semi-abelian scheme $\mathcal{A}$ exists.

[2] Note that the condition $A[m](K) \cong (\mathbb{Z}/m\mathbb{Z})^{2g}$ for some m s.t. $(p, m) = 1$ on [3, Corollaire 1.4] does not apply to [3, Corollaire 1.5], the proof of [3, Corollaire 1.5] uses a variant of the proof for [3, Thm 1.3] which does not require the condition $A[m](K) \cong (\mathbb{Z}/m\mathbb{Z})^{2g}$ when $\dim V = 1$.

In characteristic p, our ultimate goal is to study the rational points $A^{(p)}(K)$. But this is difficult to do directly. We thus consider the map induced by the relative Frobenius morphism $F_{A/K} : A(K) \to A^{(p)}(K)$. There is a useful tool to study the cokernel, $\frac{A^{(p)}(K)}{F_{A/K}(A(K))}$, called the Selmer group. There are a lot of technical details going on in the Selmer group. I will give an analogous example to show the central idea.

Consider the set of integer solutions of the equation $x^3 + y^3 + z^3 = 33$. The integer solutions are also (or descend to) rational solutions and solutions in $\mathbb{Z}_p$ for any prime p. Hence the set of integral solutions can be embedded into the "intersection" of rational solutions and $\mathbb{Z}_p$-solutions. Similarly, we can descend the relative Frobenius map to different completions of K, and for each of them we obtain a group in which $\frac{A^{(p)}(K)}{F_{A/K}(A(K))}$ always embeds. We call the intersection, in which $\frac{A^{(p)}(K)}{F_{A/K}(A(K))}$ naturally embeds, the *Selmer group*.

We state our main theorem 3.1 here:

Theorem. *There is a natural injective homomorphism*

$$\phi : \mathrm{Sel}^{(F_{A/K})}(A/K) \hookrightarrow \mathrm{Hom}_C(F_C^* \omega_{\ker F_{\mathcal{A}/C}/C}, \Omega^1_{C/k_0}(E)).$$

And this map ϕ is functorial for morphisms of semi-abelian schemes over C.

Notation: $F_{\mathcal{A}/C} : \mathcal{A} \to \mathcal{A}^{(p)}$ is the relative Frobenius morphism of $\mathcal{A}/C$, and similar for $F_{A/K}$. Then $\ker F_{\mathcal{A}/C}$ is a group scheme over C. For any group scheme G/S, we denote $e = e_{G/S} : S \to G$ as the unit section and define $\omega_{G/S} := e^*\Omega^1_{G/S}$ where $\Omega^1_{G/S}$ is the sheaf of relative differentials of degree 1 of G over S.

Note that $F_C^* \omega_{\ker F_{\mathcal{A}/C}/C}$ and $\Omega^1_{C/k_0}(E)$ are both vector bundles over C, where $\Omega^1_{C/k_0}(E) := \Omega^1_{C/k_0} \otimes_{\mathcal{O}_C} \mathcal{O}(E)$ acts as the analogy of "locally one may compensate by a rational function with only poles of order one".

Since $\mathrm{Hom}_C(F_C^* \omega_{\ker F_{\mathcal{A}/C}/C}, \Omega^1_{C/k_0}(E)) = \Gamma(C, F_C^* \omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}(E))$ where $F_C^* \omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}(E)$ is a finite locally free sheaf over C and C is proper over k_0, by [5, tag 02O5], this global section is a finite k_0-module, hence a finite abelian group since k_0 is finite. We also state the corollary 3.8 here.

Corollary. *The group $\frac{A^{(p)}(K)}{F_{A/K}(A(K))}$ is finite.*

For other applications and context of this embedding, see D. Rössler's work [6, p.5].

1.1. Sketch of the proof

The following subsection is a sketch of the proof.

1.1.1. Reduce to local case

Denote $\tilde{\mathcal{F}}$ by the fppf sheaf on C generated by the $\mathcal{O}_C$-module $\mathcal{F} := F_C^* \omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}$, and it is finite locally free.

Pick any closed point $v \in C$, apply the second exact sequence (see section 2) to the commutative square

$$\begin{tikzcd}
C & \mathrm{Spec}\, K \arrow[l, "\eta"'] \\
\mathrm{Spec}\, \hat{\mathcal{O}}_v \arrow[u] & \mathrm{Spec}\, K_v \arrow[l] \arrow[u]
\end{tikzcd}$$

we have the following commutative diagram, where the cohomology groups are for *fppf*-sheaves.

$$\begin{tikzcd}
 & \mathrm{H}^1(K, \ker F_{A/K}) \arrow[rr, hook, red, "\gamma_1"] \arrow[dd, hook, blue, "\lambda_4"] & & \mathcal{F}_\eta \arrow[dd, hook, "\mu_4"] \\
\mathrm{H}^1(C, \ker F_{\mathcal{A}/C}) \arrow[ur, hook, blue, "\lambda_1"] \arrow[rr, hook, red, "\gamma_2"] \arrow[dr, hook, blue, "\lambda_2"] \arrow[dd, hook, blue, "\lambda_5"] & & \mathcal{F}(C) \arrow[ur, hook, "\mu_1"] \arrow[dr, hook, "\mu_2"] \arrow[dd, hook, "\mu_5"] & \\
 & \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[rr, hook, red, "\gamma_3"] & & \tilde{\mathcal{F}}(K_v) \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, blue, "\lambda_3"] \arrow[rr, hook, red, "\gamma_4"] & & \tilde{\mathcal{F}}(\hat{\mathcal{O}}_v) \arrow[ur, hook, "\mu_3"] &
\end{tikzcd} \tag{1.1}$$

Our desired map ϕ embeds in γ_1. To prove our theorem it suffices to show that:

for all $T \in \mathrm{Sel}^{(F_{A/K})}(A/K)$, for all v closed in C, denote t the uniformiser of $\hat{\mathcal{O}}_v$, we have

$$\begin{cases} \gamma_3(\lambda_4(T)) \in \tilde{\mathcal{F}}(\hat{\mathcal{O}}_v) = \mathrm{Im}\, \mu_3 & \forall v \in C \backslash \{E \cup \eta\} \\ t \cdot \gamma_3(\lambda_4(T)) \in \tilde{\mathcal{F}}(\hat{\mathcal{O}}_v) = \mathrm{Im}\, \mu_3 & \forall v \in E \end{cases}$$

We will later see that this assertion is trivially true for the set of good points $C \backslash (E \cup \{\eta\})$.

1.1.2. Construct a short exact sequence of group schemes over $\hat{\mathcal{O}}_v$

The Raynaud construction (see [4, II, 1]) provides an exact sequence of formal schemes over $\hat{\mathcal{O}}_v$, say

$$0 \longrightarrow \hat{G} \xrightarrow{\text{closed immersion}} \hat{\mathcal{A}}_{\hat{\mathcal{O}}_v} \xrightarrow{\text{faithfully flat}} \hat{\mathcal{B}} \longrightarrow 0 \tag{1.2}$$

where $\hat{\ }$ denotes formalization. Here $\mathcal{B}/\hat{\mathcal{O}}_v$ is an abelian scheme and G is a torus over $\hat{\mathcal{O}}_v$. By definition, G splits (i.e. isomorphic to a product of finitely many copies of multiplicative group scheme $\mathbb{G}_m$) fpqc-locally. We can find a suitable base change so that we can assume G splits.

If $v \notin E$, the assertion $\gamma_3(\lambda_4(T)) \in \mathrm{Im}\, \mu_3$ follows trivially from the valuative criterion of properness and the fact that $\mathcal{A}_{\hat{\mathcal{O}}_v} \cong \mathcal{B}$ is proper. So we can assume $v \in E$.

From the exact sequence of formal schemes over $\hat{\mathcal{O}}_v$, we can construct a short exact sequence of finite commutative group schemes over $\hat{\mathcal{O}}_v$:

$$0 \to \mu_{p,\hat{\mathcal{O}}_v}^{\oplus l} \to \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \xrightarrow{q} \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v} \to 0.$$

Apply the second exact sequence to the above SES, we obtain the following simplified commutative diagram whose middle plane is the bottom plane of diagram 1.1:

$$\begin{tikzcd}
& \mathrm{H}^1(K_v, \mu_{p,K_v}^{\oplus l}) \arrow[rr, hook, "e_{1,K_v}"] \arrow[dd, "\phi_{K_v}"] & & K_v^j \arrow[dd, "\phi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_{p,\hat{\mathcal{O}}_v}^{\oplus l}) \arrow[ur, hook, "a_1"] \arrow[rr, hook, "e_1"] \arrow[dd, "\phi"] & & \hat{\mathcal{O}}_v^j \arrow[ur, hook, "b_1"] \arrow[dd, "\phi^*"] & \\
& \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[rr, hook, "e_{2,K_v}=\gamma_3"] \arrow[dd, "\psi_{K_v}"] & & K_v^n \arrow[dd, "\psi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_2=\lambda_3"] \arrow[rr, hook, "e_2=\gamma_4"] \arrow[dd, "\psi"] & & \hat{\mathcal{O}}_v^n \arrow[ur, hook, "b_2=\mu_3"] \arrow[dd, "\psi^*"] & \\
& \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v}) \arrow[rr, hook, "e_{3,K_v}"] & & K_v^m \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_3"] \arrow[rr, hook, "e_3"] & & \hat{\mathcal{O}}_v^m \arrow[ur, hook, "b_3"] &
\end{tikzcd} \tag{1.3}$$

1.1.3. Three lemmas and diagram chase

With the following three results/lemmas, the assertion $t \cdot \gamma_3(\lambda_4(T)) \in \operatorname{Im} \mu_3$ is just a simple diagram chase:

1. $t \cdot \operatorname{Im} e_{1,K_v} \subset \operatorname{Im} b_1$;
2. ψ is surjective;
3. $\psi_{K_v}(\lambda_4(T)) \subset \operatorname{Im} a_3$.

The first result relies on the understanding of the second exact sequence. The second result reduces to showing $\mathrm{H}^2(\hat{\mathcal{O}}_v, \mu_p) = 0$ which is not hard. The third one is the essential ingredient of our paper. To prove it we need to deeply investigate the Raynaud exact sequence in the setting of rigid analytic space.

1.1.4. Show that $\psi_{K_v}(\lambda_4(T)) \subset \operatorname{Im} a_3$

By definition $\lambda_4(T)$ comes from $A_{K_v}^{(p)}(K_v)$. It suffices to show $\psi_{K_v}(\lambda_4(T))$ comes from $\mathcal{B}_{K_v}^{(p)}(K_v) = \mathcal{B}^{(p)}(\hat{\mathcal{O}}_v)$ so it lies in $\operatorname{Im} a_3$.

$$\begin{tikzcd}
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[r, "a_2=\lambda_3"] \arrow[d, "\psi"] & \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[d, "\psi_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{B}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[r, "a_3"] & \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})
\end{tikzcd}$$

$$\begin{tikzcd}
A^{(p)}_{K_v}(K_v) \arrow[d, "f_A"] & \mathcal{B}^{(p)}_{K_v}(K_v) \arrow[d, "f_{\mathcal{B}}"] \\
\mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[r, "\psi_{K_v}"] & \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})
\end{tikzcd}$$

So the problem is reduced to showing that $\mathrm{Im}(\psi_{K_v} \circ f_A) \subset \mathrm{Im}\, f_B$, or finding a map of sets $A^{(p)}_{K_v}(K_v) \to \mathcal{B}^{(p)}_{K_v}(K_v)$ that is compatible with $\mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \to \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})$. This is where we introduce the method of rigid-analytic space. A brief introduction and related results are introduced in Appendix B.

Transform the above diagram into rigid-analytic setting, we have

$$\begin{tikzcd}
(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[d, "\Psi_A"] & (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[d, "\Psi_{\mathcal{B}}"] \\
\mathrm{PHS}(\ker F_{A^{\mathrm{an}}_{K_v}/K_v}) \arrow[r, "s"] & \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v})
\end{tikzcd}$$

and the problem is reduced to showing that $\mathrm{Im}(s \circ \Psi_A) \subset \mathrm{Im}\, \Psi_{\mathcal{B}}$, or finding a map of sets $(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \to (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v)$ making the diagram commutes.

By the uniformization theorem [7, 1.1 & 1.2], there exists an extension

$$0 \to \mathbb{G}^{\oplus l}_{m,K_v} \to E \xrightarrow{\rho} \mathcal{B}_{K_v} \to 0$$

and a commutative diagram of abelian rigid K_v-groups where each arrow is a group homomorphism

$$\begin{tikzcd}
E^{\mathrm{an}} \arrow[r, "\rho^{\mathrm{an}}"] \arrow[d, "\sigma"] & \mathcal{B}^{\mathrm{an}}_{K_v} = \hat{\mathcal{B}}^{\mathrm{rig}} \\
A^{\mathrm{an}}_{K_v} & (\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}} \arrow[l, hook', "i_{\mathcal{A}}"] \arrow[ul, hook', "j"] \arrow[u, "\hat{\varphi}^{\mathrm{rig}}"']
\end{tikzcd}$$

Just like the scheme case, we can construct the following commutative diagram from the above one:

$$\begin{tikzcd}
& (E^{\mathrm{an}})^{(p)}(K_v) \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[dd] \\
(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[dd] & & ((\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}})^{(p)}(K_v) \arrow[ll] \arrow[ul] \arrow[ur] \arrow[dd] & \\
& \mathrm{PHS}(\ker F_{E^{\mathrm{an}}/K_v}) \arrow[rr] \arrow[dl, "\cong"'] & & \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}) \\
\mathrm{PHS}(\ker F_{A^{\mathrm{an}}_{K_v}/K_v}) & & \mathrm{PHS}(\ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}) \arrow[ll, "\cong"] \arrow[ul, "\cong"] \arrow[ur] &
\end{tikzcd}$$

The proof of the three isomorphisms in the bottom plane is not trivial. We can also show that $(E^{\mathrm{an}})^{(p)}(K_v) \to (A^{\mathrm{an}}_{K_v})^{(p)}(K_v)$ is surjective. So any element in $(A^{\mathrm{an}}_{K_v})^{(p)}(K_v)$ can be lifted to an element in $(E^{\mathrm{an}})^{(p)}(K_v)$, and we can map it into $(\mathcal{B}_{K_v}^{\mathrm{an}})^{(p)}(K_v)$. This is the map of sets $(A_{K_v}^{\mathrm{an}})^{(p)}(K_v) \to (\mathcal{B}_{K_v}^{\mathrm{an}})^{(p)}(K_v)$ we want to construct. With a simple diagram chase, we can see that it is compatible with $\mathrm{PHS}(\ker F_{A_{K_v}^{\mathrm{an}}/K_v}) \to \mathrm{PHS}(\ker F_{\mathcal{B}_{K_v}^{\mathrm{an}}/K_v})$. The result follows.

This completes the sketch of the proof of our main theorem.

1.2. Smooth projective curve C

To formally introduce the situation, we start with the concrete definition of the curve.

Situation 1.1. Let C be a smooth, projective, geometrically integral scheme of dimension 1 over a finite field k_0 of characteristic p. Define $K = K(C)$, and η as the generic point.

We now state and prove some properties which we will later use in our proof.

Lemma 1.2. *Let C be as above, then we have*

i. C is projective, smooth, separated, universally closed, proper, flat, quasi-compact, of finite type over k_0.

ii. C is Noetherian, regular, normal, geometrically integral and of dimension 1.

iii. Every non-empty open affine subset of C is the spectrum of a Dedekind domain, i.e. a Noetherian normal domain of dimension 1.

iv. Every non-generic point v is closed, the stalk $\mathcal{O}_{C,v}$ is a discrete valuation ring, and we may denote it as $\mathcal{O}_v$, whose fraction field is K. Also the residue field $\kappa(v)$ is a finite extension of k_0. Hence C is also a factorial scheme.

v. Every non-empty open subset contains the generic point η and hence must be dense. Every closed subset is either C or a finite set of closed points.

vi. Every valuation ring of K contains k_0. There is a one-to-one correspondence between the set of non-generic points and the set of non-trivial valuations on K.

vii. K is transcendental of degree 1 over k_0.

viii. Every non-generic point $\{v\}$ is an effective Cartier divisor on C, i.e. a closed subscheme whose ideal sheaf $\mathcal{I}_{\{v\}} \subset \mathcal{O}_C$ is an invertible $\mathcal{O}_C$-module, see [5, tag 01WR].

ix. Every non-generic point $\{v\}$ is a prime Weil divisor on C, i.e. an integral closed subscheme of codimension 1 (see [5, tag 0BE2]), and vice versa. And the associated sheaf $\mathcal{O}(\{v\})$ defined as

$$\Gamma(U, \mathcal{O}(\{v\})) := \{t \in K(C)^* : \mathrm{Div}\,|_U t + \{v\}|_U \geq 0\} \cup \{0\}$$

is isomorphic to $\mathcal{I}_{\{v\}}^{\vee}$, where $\mathrm{Div}\,|_U s := \sum_Y \mathrm{val}_Y(s)[Y]$ is the divisor of zeros and poles of the rational section s, over the open subscheme U, and $\cdot|_U : \mathrm{Weil}\,X \to \mathrm{Weil}\,U$, $\sum_Y n_Y[Y] \mapsto \sum_{Y \cap U \neq \emptyset} n_Y[Y \cap U]$ denotes restriction map from X to U (cf. [8, Chap. 14.2]) .

Proof.

i. We know that projectivity implies proper which is equivalent to separated, universally closed and of finite type. And smooth implies flatness. Quasi-compactness follows from being of finite type.

ii. Since C is of finite type over k_0, any field is Noetherian, and any scheme which is of finite type over a Noetherian scheme is Noetherian, we have C is Noetherian. Since smoothness over a field implies regularity [9, 6.26], and regular schemes are normal, we know that C is a regular and normal scheme. And C is geometrically integral scheme of dimension 1 by definition.

iii. Let U be a non-empty open affine subset, we want to show that $\mathcal{O}_C(U)$ is a Noetherian normal domain of dimension 1. By [5, tag 0A21] we have that $\dim U = \dim C = 1$. Since C is an integral scheme, $\mathcal{O}_C(U)$ is an integral domain. By (*ii*) we know C is normal and integral, so $\mathcal{O}_C(U)$ is a Noetherian normal domain of dimension 1 [10, 4.1, 1.5].

iv. Let v be a non-generic point of C, let U be an arbitrary open affine subset containing v, say $U = \operatorname{Spec} R$, where R is a Dedekind domain by (*iii*). Then v corresponds to a non-zero prime ideal $\mathfrak{p}$ of R. In Dedekind domain, every non-zero prime ideal is maximal, and the localization at each maximal ideal is a discrete valuation ring (abbr. DVR), thus v is a closed point of U and $\mathcal{O}_{C,v} = \mathcal{O}_{U,v}$ is a DVR. Let $\{U_i\}$ be an open affine cover of C, then $\{v\} \cap U_i$ must be closed in each U_i, so v is closed in C.

Note the residue field $\kappa(v) = \mathrm{Frac}\frac{R}{\mathfrak{p}} = \frac{R}{\mathfrak{p}}$. Since C is of finite type over k_0, R is a finitely generated k_0-algebra, so is $\frac{R}{\mathfrak{p}}$. By Nullstellensatz, any finitely generated k_0 algebra which is also a field must be finite over k_0. Hence $\kappa(v)$ is a finite extension of k_0.

Next we show that the fraction field of $\mathcal{O}_v = \mathcal{O}_{C,v}$ is K. Since C is an integral scheme, we have a sequence of injection of integral domains $\mathcal{O}_C(U) \hookrightarrow \mathcal{O}_v \hookrightarrow \mathcal{O}_{C,\eta} = K$. Also K is the fraction field of $\mathcal{O}_C(U)$, it must also be the fraction field of $\mathcal{O}_v$. Now every stalk is either a DVR or a field, they are all unique factorization domains, so C is factorial.

v. Let U be a non-empty open subset, if $\eta \notin U$, then U^c is a proper closed subset containing the closure of η, which is C, contradiction. So we must have $\eta \in U$. Let Z be a closed subset of C, so Z is Noetherian, hence has a finite number of irreducible components. If $\eta \in Z$, then $Z = C$. Otherwise $\eta \notin Z$ and every point in Z is closed by (*iii*), and each point is an irreducible component, thus Z must be a finite set of closed points.

vi. Giving a valuation ring of K is the same as giving a valuation v on K. Since k_0 is finite, every non-zero element of it is of finite order, say $x^m = 1, \text{where}\, x \in k_0^*$. Hence $v(x^{m+1}) = (m+1) \cdot v(x) = v(x)$, then we have $v(x) = 0$ and x lies in the valuation ring. Given a non-trivial valuation ring $\mathcal{O}$ on K, we have a sequence of injection $k_0 \hookrightarrow \mathcal{O} \hookrightarrow K$. Hence we have the following commutative diagram.

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "g"] \arrow[d, "i_2"'] & C \arrow[d, "h"] \\
\operatorname{Spec} \mathcal{O} \arrow[r, "i_1"] & \operatorname{Spec} k_0
\end{tikzcd}$$

where g is the inclusion of generic point and h is the structure morphism of C and it is proper. By valuative criterion for properness, see [10, 3.3.2, 3.26], there exists a unique morphism

$j : \operatorname{Spec}\mathcal{O} \to C$ making the following diagram commute

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "g"] \arrow[d, "i_2"'] & C \arrow[d, "h"] \\
\operatorname{Spec}\mathcal{O} \arrow[r, "i_1"] \arrow[ur, dotted, "j"] & \operatorname{Spec} k_0
\end{tikzcd}$$

Let γ be the closed point of $\operatorname{Spec}\mathcal{O}$, let $v = j(\gamma)$. Then j induces a map of local rings $\mathcal{O}_v \to \mathcal{O}_\gamma = \mathcal{O}$, and j must factor through $\operatorname{Spec}\mathcal{O}_v$:

$$\begin{tikzcd}
\operatorname{Spec}\mathcal{O}_\gamma \arrow[r, "j_2"] \arrow[d, equal] & \operatorname{Spec}\mathcal{O}_v \arrow[d, "l"] \\
\operatorname{Spec}\mathcal{O} \arrow[r, "j"] & C
\end{tikzcd}$$

If $v = \eta$ then $l = g : \operatorname{Spec} K \to C$. And we have $g = j \circ i_2 = l \circ j_2 \circ i_2 = g \circ j_2 \circ i_2$. Since $\operatorname{Spec}\mathcal{O}_\eta = \operatorname{Spec} K \to C$ is a monomorphism in the category of schemes by [5, tag 01L9], we have $j_2 \circ i_2 = \mathrm{id}_{\operatorname{Spec} K}$, i.e. we have a map $K \to \mathcal{O} \hookrightarrow K$ which is identity, it implies $\mathcal{O} \hookrightarrow K$ is surjective which is a contradiction.

Thus $v = j(\gamma)$ is a closed point, and it is not hard to see that both $j \circ i_2$ and g factors through $\operatorname{Spec}\mathcal{O}_v \to C$. Again since $\operatorname{Spec}\mathcal{O}_v \to C$ is a monomorphism in the category of schemes by [5, tag 01L9], we have a commutative diagram

$$\begin{tikzcd}
K & \mathcal{O}_v \arrow[l, hook', "a'"] \arrow[dl, "c'"] \\
\mathcal{O} \arrow[u, hook, "b'"] &
\end{tikzcd}$$

To show c' is an injection it suffices to consider that if $c'(t) = 0$, then $b' \circ c'(t) = a'(t) = 0$, so $t = 0$. Hence c' is an injection. Also c' is induced as a local ring map by $j(\gamma) = v$, so $\mathcal{O}$ dominates $\mathcal{O}_v$ in K. Since valuation rings are maximal among local sub-rings of K by domination, we must have $\mathcal{O} = \mathcal{O}_v$, and this point is unique since the morphism j is unique. So we know that every non-trivial valuation on K gives rise to a unique non-generic point of C, and for every non-generic point v of C, $\mathcal{O}_v$ determines a non-trivial valuation on K and the process is clearly mutual.

vii. Since C is of finite type over k_0 and it is integral of dimension 1, the function field K is transcendental of degree 1 over k_0 by [10, 2.5.3, 5.19].

viii. Every non-generic point is closed, and we equip it with the reduced induced closed subscheme structure. By (*ii*) and (*iv*), every non-generic point $\{x\}$ as an irreducible closed subscheme is of codimension 1 and we have that $\mathcal{O}_{C,x}$ is a DVR so a unique factorization domain (abbr. UFD). By [5, tag 0AGA], every non-generic point can be viewed as an effective Cartier divisor on C.

ix. Every non-generic point is an irreducible closed subset of codimension 1 with the unique reduced induced closed subscheme structure, so it is integral and it is a prime Weil divisor on C. Reversely every prime Weil divisor on C as a set is an irreducible codimension 1 closed subset of C, hence it is not C and contains at least one non-generic point, which is closed and irreducible and codimension 1. Hence that non-generic point must be the whole divisor. They have the same closed subscheme structure since they are both reduced.

Next we show that $\mathcal{I}_{\{v\}}^{\vee} \cong \mathcal{O}(\{v\})$. Since C is integral, any restriction map is injective and we have

$$\mathcal{O}_C(U) = \bigcap_{u \in U} \mathcal{O}_{C,u} = \{t \in K(C)^* | \forall u \in U - \eta, \operatorname{val}_u t \geq 0\} \cup \{0\}$$

Similar statement holds for any quasi-coherent ideal sheaf $\mathcal{I}$, i.e. any restriction map is injective and $\mathcal{I}(U) = \cap_{u \in U} \mathcal{I}_u$. Hence

$$\mathcal{I}_{\{v\}}(U) = \bigcap_{u \in U} \mathcal{I}_{\{v\},u} = \begin{cases} \bigcap_{u \in U} \mathcal{O}_{C,u} = \mathcal{O}_C(U) & \text{if } v \notin U \\ \mathfrak{m}_v \cap \bigcap_{u \in U - v} \mathcal{O}_{C,u} & \text{if } v \in U \end{cases}$$

Now we can determine $\mathcal{I}_{\{v\}}^{\vee}$. We know $\mathcal{I}_{\{v\}}^{\vee}(U) = \operatorname{Hom}_{\mathcal{O}_C|_U}(\mathcal{I}_{\{v\}}|_U, \mathcal{O}_C|_U)$. For any $\phi \in \operatorname{Hom}_{\mathcal{O}_C|_U}(\mathcal{I}_{\{v\}}|_U, \mathcal{O}_C|_U)$, denote $V = U - v$, we claim that ϕ is determined by $\phi(V)(1)$. If $v \notin U$, then $V = U$ and $\mathcal{I}_{\{v\}}|_U = \mathcal{O}_C|_U$, clearly ϕ is determined by $\phi(U)(1)$. If $v \in U$, we know ϕ is uniquely determined by its restrictions to stalks, for points x other than v, ϕ_x is determined by $\phi(V)(1)$; and for $\phi_v : \mathcal{I}_{\{v\},v} \to \mathcal{O}_{C,v}$, consider $s \in \mathcal{I}_{\{v\},v} = \mathfrak{m}_v$, there exists some open $W \subset U$ s.t. $s \in \mathcal{O}_C(W)$. Pick any $w \in W - v$, we have $\phi_v(s) = \phi(W)(s) = \phi_w(s) = s \cdot \phi_w(1)$. Hence $\phi_v(s)$ is also determined by $\phi(V)(1)$, so is ϕ.

Clearly the zero element can be $\phi(V)(1)$. For an element $t \in K(C)^*$ to be $\phi(V)(1)$, we need $\forall u \in U, \forall s \in \mathcal{I}_{\{v\},u}, st \in \mathcal{O}_{C,u}$. Equivalently we need

$$\forall u \in U - \eta, \forall s \in \mathcal{I}_{\{v\},u}, \operatorname{val}_u(st) = \operatorname{val}_u(t) + \operatorname{val}_u(s) \geq 0$$

Equivalently

$$\forall u \in U - \eta, \operatorname{val}_u(t) + \min_{s \in \mathcal{I}_{\{v\},u}} \operatorname{val}_u(s) \geq 0$$

Equivalently

$$\operatorname{val}_u(t) \geq \begin{cases} 0 & \text{if } u \in U - \eta - v \\ -1 & \text{if } u = v \end{cases}$$

Equivalently (we use the fact that the set of prime Weil divisors is the same as the set of non-generic points here)

$$\operatorname{Div}|_U t + \{v\}|_U \geq 0$$

Hence we have $\mathcal{I}_{\{v\}}^{\vee} \cong \mathcal{O}(\{v\})$ where the identification of ring structures and restriction maps follow trivially.

□

1.3. Site and sheaves

The following subsection is very general and but we need to introduce the definition of *fppf*-sheaves on the big *fppf* site of a scheme S. Readers can skip this subsection.

Definition 1.3. Let $\mathcal{C}$ be a category, a family of morphisms with fixed target in $\mathcal{C}$ is given by an object $U \in \operatorname{Ob}(\mathcal{C})$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i \in I}$ to indicate this.

Definition 1.4. A site is a pair $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ where $\mathcal{C}$ is a category and $\mathrm{Cov}(\mathcal{C})$, the coverings of $\mathcal{C}$, is a set whose elements are of the form $\{U_i \to U\}_{i\in I}$, i.e. a family of morphisms with fixed target, satisfying the following axioms:

1. If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.
2. If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and for each i we have $\{V_{ij} \to U_i\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
3. If $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Next we define presheaves/sheaves on a site.

Definition 1.5. Let $\mathcal{C}$ be a category, a presheaf of abelian groups on $\mathcal{C}$ is a contravariant functor $\mathcal{F} : \mathcal{C} \to \mathrm{Ab}$.

Definition 1.6. Let $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$ be a site, let $\mathcal{F}$ be a presheaf of abelian groups, then $\mathcal{F}$ is a sheaf if for all $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$, the following diagram is an equalizer:

$$\begin{tikzcd}\mathcal{F}(U) \arrow[r] & \prod_{i\in I}\mathcal{F}(U_i) \arrow[r, shift left, "\mathrm{pr}_1^*"] \arrow[r, shift right, "\mathrm{pr}_2^*"'] & \prod_{(i_1,i_2)\in I\times I}\mathcal{F}(U_{i_1}\times_U U_{i_2})\end{tikzcd}$$

Next we can define the big *fppf* site of S and an *fppf*-sheaf on it.

Definition 1.7. Let S be a scheme, the big *fppf* site of S, denoted by $(Sch/S)_{fppf}$, is the pair $(Sch/S, \mathrm{Cov}(Sch/S))$, where Sch/S is the category of S-schemes, and $\{f_i : U_i \to U\}_{i\in I} \in \mathrm{Cov}(Sch/S)$ if each f_i is a morphism of S-schemes, and it is flat, locally of finite presentation and $U = \bigcup_i f(U_i)$.

An ***fppf*-sheaf** (of abelian groups on the big *fppf* site of a scheme S) is just a sheaf of abelian groups on the big *fppf* site of a scheme S.

Remark 1.8. The category of sheaves of abelian groups on the big *fppf* site of a scheme S is an abelian category, see [11, Thm. 8.11].

This fact is quite useful since in an abelian category we can apply all sorts of homological lemmas, e.g. the snake lemma. The following proposition states that any commutative S-group scheme can be viewed as a sheaf of abelian groups.

Proposition 1.9. *Let G be a commutative S-group scheme, we define the presheaf $\mathcal{F}_G$ associated to G by $\mathcal{F}_G(T) := \mathrm{Hom}_S(T, G)$. Then $\mathcal{F}_G$ is an fppf sheaf on the big fppf site of S, $(Sch/S)_{\mathrm{fppf}}$.*

Proof. See [12, II, Cor. 1.7]. ☐

1.4. Semi-abelian scheme

Definition 1.10. We call A/K an **abelian variety** if A is a group scheme over K and it is proper and geometrically integral over K.

The concept of abelian variety is central to arithmetic geometry, and there are many equivalent definitions. The following remark states that there are $2 \times 4 \times 2 = 16$ equivalent definitions of abelian varieties.

Remark 1.11. Let {projective, proper}, {geometrically irreducible, irreducible, geometrically connected, connected}, {smooth, geometrically reduced} be three sets of properties, pick one from each of them, and let A/K be a group scheme with the chosen properties. Then it agrees with the definition of abelian variety.

Proof. Clearly projectivity implies properness; and geometrically irreducibility implies the other three which in turn implies connectivity. By [5, tag 056T] we know that smoothness of K-schemes implies geometrically reducedness. Thus it suffices to show that "abelian varieties are projective, geometrically irreducible and smooth", and "proper connected and geometrically reduced group schemes over K is an abelian variety".

By [5, tag 0BF9], abelian varieties are projective and smooth. Geometrically irreducibility comes from the definition. Next we show that proper connected and geometrically reduced group schemes over K are abelian varieties, i.e. they are proper and geometrically integral.

It suffices to show geometrically irreducibility. A group scheme over the field K must contain a K-rational point, the unit section. In this case, A is connected implies it is geometrically connected by [5, tag 04KV]. Hence after base change to any field extension K' of K, $A_{K'}$ is connected. Plus, since every connected group scheme over a field is irreducible by [5, tag 0B7Q], $A_{K'}$ is irreducible. So A is geometrically irreducible. The result follows. ☐

The definition of abelian varieties does not directly state it is abelian, but it is indeed true.

Remark 1.12. An abelian variety is a commutative group scheme.

Proof. See [5, tag 0BFD]. ☐

Cohomology arises everywhere in mathematics, so it is natural to define exact sequences in different cases. The definition of exact sequence of abelian groups can be easily defined, but not for commutative group schemes because cokernels of commutative group schemes do not behave as well as cokernels of abelian groups. We will define the exact sequence of commutative group schemes over a scheme S as follows, but this only behaves well under certain conditions.

Definition 1.13. Let S be a scheme, then a sequence of commutative group scheme over S

$$0 \to G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \to 0$$

is **exact** if f is an isomorphism onto $\ker g$ and g is faithfully flat.

We will only use the above definition under two "nice" conditions:

1. G_1/S is flat and locally of finite presentation and g is locally of finite presentation.

2. the base scheme is the spectrum of a field and all of G_1, G_2 and G_3 are of finite type over the base field.

Remark 1.14. The second condition implies the first one.

Proof. Let $S = \operatorname{Spec} k$ and G_1, G_2, G_3 be finite type commutative group schemes over k. Since $\operatorname{Spec} k$ is Noetherian, they are also of finite presentation. By [5, tag 02FV], the morphism between them must be locally of finite presentation. In particular, G_1/S and g are locally of finite presentation. Since S is the spectrum of a field, G_1/S is flat. ☐

And for the case of first condition we present a remark below showing that the above definition of exactness agrees with the one in the abelian category of *fppf*-sheaves on the big *fppf* site of S, $(Sch/S)_{fppf}$. The later definition of exactness is quite nice since it lives in an abelian category. As we mentioned before, we can thus apply homological lemmas like the snake lemma.

Remark 1.15. Fix a base scheme S, let $0 \to T \overset{\phi}{\to} G \overset{q}{\to} H \to 0$ be a sequence of commutative group schemes over S s.t. T is flat and locally of finite presentation over S. Then the following two definitions of exactness of the above sequence agree.

1. ϕ is an isomorphism onto $\ker q$ and q is faithfully flat and locally of finite presentation.

2. It is exact as *fppf*-sheaves in the abelian category of *fppf*-sheaves on the big *fppf* site of S, $(Sch/S)_{\text{fppf}}$.

Proof. (1)⇒(2): Since ϕ is an isomorphism onto $\ker q$, we can view the sequence as $0 \to \ker q \overset{\phi}{\to} G \overset{q}{\to} H \to 0$. So $(\ker q, \phi)$ represents the kernel sheaf of q and the corresponding injection of sheaves. It remains the show that $q : G \to H$ represents a surjective map of sheaves. The result follows from q is *fppf*, [5, tag 02WJ] and [5, tag 05VM].

(2)⇒(1): Since it's an exact sequence of sheaves, T represents the kernel sheaf of $q : G \to H$, which is also represented by the group scheme $\ker q$. The injection of kernel sheaf into G is represented by the immersion of $\ker q$ into G. By Yoneda lemma, we must have $\phi : T \to G$ is an isomorphism onto $\ker q$.

By exactness, H is the *fppf* quotient of $\phi : T \to G$ in the sense of the definition in [13, 4.28], then by [13, 4.35], it's also a geometric quotient, in the sense of [13, 4.12]. The construction of geometric quotient makes $q : G \to H$ surjective.

By [13, 4.32] we have that flatness and the property of being locally finite presentation are *fppf* local on the target, then by [13, 4.33], the fact T/S is flat and locally of finite presentation implies $q : G \to H$ is flat and locally of finite presentation. Since we showed it is surjective, we have that q is faithfully flat and locally of finite presentation. ☐

With the definition of abelian varieties and exact sequences, we can now define the semi-abelian scheme, cf. [4, Chap. I, Def. 2.3].

Definition 1.16. Let C be a scheme, we say T is a **torus** over C if T is a commutative group scheme which is *fppf*-locally isomorphic to a product of finitely many copies of the multiplicative group scheme $\mathbb{G}_{m,C}$ over C, i.e. for any point $s \in C$, there exists an open neighborhood U of s, an *fppf* morphism $C' \to U$ and a non-negative integer r s.t. $T \times_C C' \cong \mathbb{G}_{m,C}^{\oplus r}$. We say $\pi : \mathcal{A} \to C$ is a **semi-abelian scheme** if $\mathcal{A}$ is a smooth separated commutative group scheme over C via π with geometrically connected fibers, such that each fiber $\mathcal{A}_v$, where $v \in C$, is an extension of an abelian variety $\mathcal{B}_v$ by a torus T_v over the residue field $\kappa(v)$, i.e. a short exact sequence (abbr. SES) $0 \to T_v \to \mathcal{A}_v \to \mathcal{B}_v \to 0$.

Remark 1.17. Let $\pi : \mathcal{A} \to C$ be a semi-abelian scheme, pick a point $v \in C$, then the exact sequence of commutative group scheme over $\kappa(v)$, $0 \to T_v \to \mathcal{A}_v \xrightarrow{g} \mathcal{B}_v \to 0$, satisfies the condition 2 mentioned above, i.e. they are all of finite type over $\kappa(v)$.

Proof. Since $\mathcal{B}_v$ is an abelian variety, it is of finite type. $\mathcal{A}_v$ is a connected group scheme over a field so irreducible, by [5, tag 0B7Q]. Thus $\mathcal{A}_v$ is quasi-compact by [5, tag 0B7P]. Hence the structure morphism of $\mathcal{A}_v$ is quasi-compact since the pre-image of the open affine set $\operatorname{Spec} \kappa(v)$ is quasi-compact. Also the structure morphism is smooth as a base change of π, hence locally of finite presentation, thus locally of finite type. Combined with quasi-compactness, it is of finite type.

Since $\mathcal{B}_v$ is an abelian variety so separated, $\ker g = T_v$ is a closed subgroup scheme of $\mathcal{A}_v$ by [13, 3.13]. The structure morphism of T_v is hence the composition of the structure morphism of $\mathcal{A}_v$ and a closed immersion, and both of them are of finite type, hence it is of finite type. ☐

It is also worth noting that the structure morphism of a semi-abelian scheme is of finite presentation. This fact helps showing that the set of "bad points" E in our main theorem is closed. This is vital since if E is not a closed set, then we cannot define $\mathcal{O}(E)$ and hence $\Omega^1_{C/k_0}(E)$ does not make sense.

Proposition 1.18. *Let $\pi : \mathcal{A} \to C$ be a semi-abelian scheme, then π is of finite presentation.*

Proof. It follows from [5, tag 01UA] and [14, Exposé VIB, Corollaire 5.5]. ☐

Situation 1.19. Continuing situation 1.1, let A be an abelian variety over K, and $\pi : \mathcal{A} \to C$ be a semi-abelian scheme such that the generic fiber is A. Define $E \subset C$ to be the set of "bad points",

$$E := \{v \in C : \mathcal{A}_v \text{ is not an abelian variety over } \kappa(v)\}.$$

Lemma 1.20. *In above situation, we have*

1. *the set E is a finite closed subset.*

2. *the closed set E can be viewed as an effective Cartier divisor on C, and we have $E = \sum_{v \in E}\{v\}$ as a Cartier divisor. Denote $\mathcal{I}_E \subset \mathcal{O}_C$ by the ideal sheaf defining it.*

3. *the closed set E can be viewed as a Weil divisor as $\sum_{v\in E}\{v\}$, a finite sum of prime divisors. The associated sheaf $\mathcal{O}(E)$ is defined as*

$$\Gamma(U, \mathcal{O}(E)) := \{t \in K(C)^* : \operatorname{Div}|_U t + E|_U \geq 0\} \cup \{0\}$$

then we have an isomorphism $\mathcal{I}_E^\vee \cong \mathcal{O}(E)$ which sends the canonical inclusion $\mathcal{I}_E \hookrightarrow \mathcal{O}_C$ to 1.

4. *the global sections $\Gamma(C, \mathcal{O}(E))$ of $\mathcal{O}(E)$ equals the zero element union the set of rational functions $t \in K(C)^*$ s.t. for any non-generic point v, we have*

$$val_v(t) \geq \begin{cases} 0 & \text{if } v \notin E \\ -1 & \text{if } v \in E \end{cases}$$

Proof.

1. Since the definition of abelian variety can be proper, connected and smooth group schemes by remark 1.11, the set E can also be characterized by $\mathcal{A}_v$ is not proper over $\kappa(v)$. Since π is of finite presentation, by [9, Appx. E, p. 580, (6)], we know the complement set $C\backslash E = \{v \in V : \mathcal{A}_v \text{ is proper over } \kappa(v)\}$ is constructible (i.e. finite union of locally closed subset according to [9, Def. 10.12]) and contains the generic point η. By [5, tag 005K], this set contains a dense open subset of C (taking $Z = C$). So E is a subset of a proper closed subset of C, by lemma 1.2 (v), it is also finite.

2. Since C is reduced, every close subset has a unique closed subscheme structure. Thus we can treat E as a closed subscheme. By lemma 1.2 (viii), every non-generic point is an effective Cartier divisor already, then by [5, tag 0C4R], we have E is an effective Cartier divisor and it equals $\sum_{v\in E}\{v\}$.

3. The procedure is almost identical to the proof of lemma 1.2 (ix). Similarly we have

$$\mathcal{O}_C(U) = \bigcap_{u\in U} \mathcal{O}_{C,u} = \{t \in K(C)^* | \forall u \in U, \operatorname{val}_u t \geq 0\} \cup \{0\}$$

$$\mathcal{I}_E(U) = \bigcap_{u\in U} \mathcal{I}_{E,u} = \begin{cases} \bigcap_{u\in U} \mathcal{O}_{C,u} = \mathcal{O}_C(U) & \text{if } E\cap U = \emptyset \\ \bigcap_{v\in E\cap U} \mathfrak{m}_v \cap \bigcap_{u\in U-E} \mathcal{O}_{C,u} & \text{otherwise} \end{cases}$$

We know $\mathcal{I}_E^\vee(U) = \operatorname{Hom}_{\mathcal{O}_C|_U}(\mathcal{I}_E|_U, \mathcal{O}_C|_U)$, for any $\phi \in \operatorname{Hom}_{\mathcal{O}_C|_U}(\mathcal{I}_E|_U, \mathcal{O}_C|_U)$, denote $V = U - E$, we claim that ϕ is determined by $\phi(V)(1)$. We know ϕ is uniquely determined by its restrictions to stalks: for points $x \notin E$, clearly $\phi_x : \mathcal{O}_{C,x} \to \mathcal{O}_{C,x}$ is determined by $\phi(V)(1)$; for points $x \in E$, we have $\phi_x : \mathfrak{m}_x \to \mathcal{O}_{C,x}$, let $s \in \mathfrak{m}_x$, there exists some open $W \subset U$ s.t. $s \in \mathcal{O}_C(W)$. Shrink it further so that W doesn't contain any other point in E. Pick any $w \in W - x$, we have $\phi_x(s) = \phi(W)(s) = \phi_w(s) = s \cdot \phi_w(1) = s \cdot \phi(V)(1)$. Hence $\phi_x(s)$ is also determined by $\phi(V)(1)$, so is ϕ.

Clearly the zero element can be $\phi(V)(1)$. For an element $t \in K(C)^*$ to be $\phi(V)(1)$, we need $\forall u \in U, \forall s \in \mathcal{I}_{E,u}, st \in \mathcal{O}_{C,u}$. Equivalently we need

$$\forall u \in U - \eta, \forall s \in \mathcal{I}_{E,u}, \operatorname{val}_u(st) = \operatorname{val}_u(t) + \operatorname{val}_u(s) \geq 0$$

Equivalently

$$\forall u \in U - \eta, \operatorname{val}_u(t) + \min_{s\in\mathcal{I}_{E,u}} \operatorname{val}_u(s) \geq 0$$

Equivalently

$$\operatorname{val}_v(t) \geq \begin{cases} 0 & \text{if } v \in U - E - \eta \\ -1 & \text{if } v \in E \cap U \end{cases}$$

Equivalently (we use the fact that the set of prime Weil divisors is the same as the set of non-generic points here)

$$\operatorname{Div}|_U t + E|_U \geq 0$$

Hence we have $\mathcal{I}_E^\vee \cong \mathcal{O}(E)$ where the identification of ring structures and restriction maps follow trivially. And it is easy to see that the canonical inclusion $\mathcal{I}_E \hookrightarrow \mathcal{O}_C$ is induced by taking $\phi(V)(1) = 1$.

4. Follows trivially from above proof.

□

For any $\mathcal{O}_C$-module $\mathcal{L}$, we define the twist of $\mathcal{L}$ by E as $\mathcal{L}(E) := \mathcal{L} \otimes_{\mathcal{O}_C} \mathcal{O}(E)$. So we have $F_{C,*}\Omega^1_{C/k_0,\mathrm{cl}}(E) = F_{C,*}\Omega^1_{C/k_0,\mathrm{cl}} \otimes_{\mathcal{O}_C} \mathcal{O}(E)$.

1.5. Frobenius morphism

In characteristic p, the Frobenius map naturally arises. We can define the Frobenius map of rings, fields, algebras simply by sending t to t^p. Similarly we define the absolute Frobenius morphism of schemes over characteristic p.

Definition 1.21. Let S be a scheme of characteristic p, then we define the absolute Frobenius morphism $F_S : S \to S$ induced by $\mathcal{O}_S \to \mathcal{O}_S, s \mapsto s^p$.

We then define the *relative Frobenius morphism* which we will use to construct the Selmer group.

Definition 1.22. Let X be a scheme over S of characteristic p with structure morphism $\pi : X \to S$, then we define $X^{(p)}$ as the fiber product $X \times_S S$ along π and F_S. And we define the **relative Frobenius morphism** of X/S as $F_{X/S} : X \to X^{(p)}$ along the morphisms $F_X : X \to X$ and $\pi : X \to S$, see the following diagram.

$$\begin{tikzcd}
X \arrow[rd, dashed, "F_{X/S}"] \arrow[rrd, bend left, "F_X"] \arrow[rdd, bend right, "\pi"'] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

Remark 1.23. Both absolute Frobenius morphism and relative Frobenius morphism are functorial. Moreover, if X is a S-group scheme, then $X^{(p)}$ naturally has a structure of S-group, and $F_{X/S}$ is a S-group morphism, the proof is identical to the proof of lemma B.13.

Definition 1.24. A scheme S of characteristic p is said to be perfect if F_S is an isomorphism. A finite group scheme N over S is said to have height h if $F^h_{N/S} : N \to N^{(p^h)}$ is zero (i.e. $F^h_{N/S}$ factors through the unit section of $N^{(p^h)}$) and $F^{h-1}_{N/S}$ is not zero, cf. [15, III, The Frobenius morphism].

If A is an abelian variety over K of characteristic p, then the relative Frobenius morphism $F_{A/K} : A \to A^{(p)}$ naturally induces a map of rational points,

$$F_{A/K} : A(K) \to A^{(p)}(K)$$

which gives us a pathway to study the ultimate goal, the structure of $A^{(p)}(K)$.

The following lemma verifies one of the conditions for above relative Frobenius morphism to be an isogeny, which will be defined in the next subsection.

Lemma 1.25. *Let G be a smooth scheme over a locally Noetherian scheme S of characteristic p, then the relative Frobenius morphism $F_{G/S}$ is faithfully flat.*

Proof. Clearly relative Frobenius morphism is a homeomorphism hence surjective. Since faithfully flat is just surjective and flat, it suffices to show flatness. We would like to use this lemma [5, tag 039E]. There are four conditions, which we will show one by one. We only care about the red arrows in the following commutative diagram.

$$\begin{tikzcd}
G \arrow[rrd, bend left, "F_G"] \arrow[rd, "F_{G/S}"] \arrow[rdd, bend right, "\pi"'] & & \\
& G^{(p)} \arrow[r] \arrow[d, "\pi^{(p)}"] & G \arrow[d, "\pi"] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

1. G is locally of finite presentation over S: it follows from the smoothness of π.

2. G is flat over S: it follows from the smoothness of π.

3. For every $s \in S$, the morphism $F_{G/S,s} = F_{G_s/\kappa(s)} : G_s \to G_s^{(p)}$ is flat: Since G/S is smooth, $G_s/\kappa(s)$ is smooth. By [10, Chap. 4.3, Ex. 3.13], we have that the relative Frobenius morphism of $G_s/\kappa(s)$ is flat.

4. $G^{(p)}$ is locally of finite type over S: since $\pi^{(p)}$ is the base change of the smooth morphism π, it is also smooth hence locally of finite presentation thus locally of finite type.

Then the result follows from [5, tag 039E]. ☐

Corollary 1.26. *Let G be a smooth commutative group scheme over a locally Noetherian scheme S of characteristic p, then*

$$0 \to \ker F_{G/S} \to G \overset{F_{G/S}}{\to} G^{(p)} \to 0$$

is a short exact sequence in the sense of remark 1.15, i.e. $\ker F_{G/S}$ *is flat and locally of finite presentation over S, $F_{G/S}$ is faithfully flat and locally of finite presentation. In particular, this sequence is exact as fppf-sheaves.*

Proof. By above lemma we know $F_{G/S}$ is faithfully flat. Since G is smooth over S, G is locally of finite presentation over S. Thus $F_{G/S}$ is finite by [5, tag 0CCD], so locally of finite presentation. Pull back $F_{G/S}$ w.r.t. the unit section of $G^{(p)}$, we can see that $\ker F_{G/S}$ is flat and locally of finite presentation over S. The result follows. ☐

Corollary 1.27. *In the above corollary, we have a long exact sequence of abelian groups:*

$$0 \to \ker F_{G/S}(S) \to G(S) \to G^{(p)}(S) \to \mathrm{H}^1_{\mathrm{fppf}}(S, \ker F_{G/S}) \to \mathrm{H}^1_{\mathrm{fppf}}(S, G) \to \mathrm{H}^1_{\mathrm{fppf}}(S, G^{(p)})$$

where $\mathrm{H}^1_{\mathrm{fppf}}(S, \mathcal{F})$ denotes the first order fppf cohomology group over S when $\mathcal{F}$ is an fppf-sheaf on the big fppf site of S.

Proof. Just apply the *fppf* cohomology to the short exact sequence. ☐

The following lemma simply verifies one of the conditions of a cited result used in the proof of our main theorem.

Lemma 1.28. *Let X be a group scheme over S of characteristic p, let $N := \ker F_{X/S}$, then N is a group scheme over S and $F_{N/S} = 0$, i.e. for any S-scheme Y, the induced group homomorphism $N(Y) \to N^{(p)}(Y) : f \to F_{N/S} \circ f$ is trivial, more precisely, $F_{N/S} \circ f : Y \to N^{(p)}$ factors through the unit section of $N^{(p)}$.*

Proof. Since X/S is a group scheme, the relative Frobenius morphism $F_{X/S}$ is a homomorphism of S-group schemes, thus $N = \ker F_{X/S}$ is a subgroup scheme of X. Similarly $N^{(p)}$ is a subgroup scheme of $X^{(p)}$. By functoriality of relative Frobenius morphism, we have a commutative diagram of group schemes over S.

$$\begin{tikzcd}
N \arrow[r, "F_{N/S}"] \arrow[d, "i"] & N^{(p)} \arrow[d, "i^{(p)}"] \\
X \arrow[r, "F_{X/S}"] & X^{(p)}
\end{tikzcd}$$

For any S-scheme Y, it induces a commutative diagram of group homomorphisms:

$$\begin{tikzcd}
N(Y) \arrow[r, "\tilde{F}_{N/S}"] \arrow[d, "\tilde{i}"] & N^{(p)}(Y) \arrow[d, "\tilde{i}^{(p)}"] \\
X(Y) \arrow[r, "\tilde{F}_{X/S}"] & X^{(p)}(Y)
\end{tikzcd}$$

Since N is the kernel of $F_{X/S}$, $\tilde{F}_{X/S} \circ \tilde{i}(f) = \tilde{i}^{(p)} \circ \tilde{F}_{N/S}(f)$ is trivial. Since $N^{(p)}$ is a subgroup scheme of $X^{(p)}$, $\tilde{i}^{(p)}$ is an injection, thus $\tilde{F}_{N/S}(f)$ is trivial so we may write $F_{N/S} = 0$. ☐

1.6. Selmer group of an isogeny between abelian varieties over a field

To define the Selmer group (associated to an isogeny of varieties), we must first define what an isogeny is.

Definition 1.29. Let A, B be two abelian varieties over K, then a morphism of K-group schemes $f : A \to B$ is an **isogeny** if it is finite, flat and surjective.

Next we define the notation of separable isogeny, and later we will define the Selmer group associated to a separable isogeny and then generalize it to arbitrary isogenies.

Lemma 1.30. *Let $f : A \to B$ be an isogeny between abelian varieties over K. Then the following conditions are equivalent:*

1. *The function field $k(A)$ is a separable field extension of $k(B)$.*

2. *f is an étale morphism.*

3. $\ker f$ *is an étale group scheme.*

In this case we say f is a ***separable isogeny****.*

Proof. See [13, Prop. 5.6 (i)]. ☐

At the level of separably closed field, a separable isogeny induces a surjective map of rational points (to be proved in the next lemma). We need this surjectivity to form a short exact sequence of abelian groups by extending a kernel to the left, from which we can apply the Galois cohomology and obtain a long exact sequence, which will be used to construct the Selmer group.

For any field K, we denote the **separable closure** of K in its **algebraic closure** as K^s. Note that if K is perfect (e.g. a number field), then $K^s = \overline{K}$.

Lemma 1.31. *Let $f : A \to B$ be a separable isogeny between abelian varieties over K, then the induced group homomorphism $f : A(K^s) \to B(K^s)$ is surjective.*

Proof. Let $b \in B$ be a K^s-valued point, i.e. $\kappa(b) \subset K^s$. Since f is surjective there exists $a \in A$ s.t. $f(a) = b$. To show that a is a K^s-valued point, it suffices to show that $\kappa(a)/\kappa(b)$ is a separable field extension. It follows from the fact that f is étale by definition (2) of separable isogeny. □

The following lemma simply states that $A^{(p)}/K$ is a well-defined abelian variety and $F_{A/K}$ is a well-defined isogeny, so that we can define the Selmer group associated to them.

Lemma 1.32. *Let A/K be an abelian variety of characteristic p, then $A^{(p)}/K$ is also an abelian variety and $F_{A/K} : A \to A^{(p)}$ is an isogeny between them.*

Proof. The definition of abelian variety ensures that any base change of fields is also an abelian variety, so $A^{(p)}/K$ is also an abelian variety. We know $F_{A/K}$ is a group homomorphism. Apply lemma 1.25, we have that $F_{A/K}$ is surjective and flat. By [5, tag 0CCD], it is also finite. Hence it is an isogeny. □

Just as mentioned above, with the SES (short exact sequence) extended by the surjection $f : A(K^s) \to B(K^s)$, we now apply Galois cohomology theory and it induces a long exact sequence as following.

Lemma 1.33. *Let $f : A \to B$ be a separable isogeny between abelian varieties over K. Denote $N := \ker f$. Then we have an exact sequence of abelian groups and $G_K :=$ Gal(K^s, K)-modules,*

$$0 \to N(K^s) \to A(K^s) \to B(K^s) \to 0$$

and it induces a long exact sequence of abelian groups:

$$0 \to N(K) \to A(K) \to B(K) \to \mathrm{H}^1(G_K, N(K^s)) \to \mathrm{H}^1(G_K, A(K^s)) \to \mathrm{H}^1(G_K, B(K^s))$$

where $\mathrm{H}^1(G_K, M)$ denotes the first order Galois cohomology group where M is an abelian G_K-module.

Proof. The first exact sequence follows from lemma 1.31 and the definition of kernel of group schemes. The second exact sequence follows from the Galois cohomology applied to the first exact sequence. □

We denote $\mathrm{H}^1(G_K, A(K^s))[f]$ as the kernel of $\mathrm{H}^1(G_K, A(K^s)) \to \mathrm{H}^1(G_K, B(K^s))$. So we have an exact sequence truncated from the long exact sequence

$$0 \to B(K)/f(A(K)) \to \mathrm{H}^1(G_K, N(K^s)) \to \mathrm{H}^1(G_K, A(K^s))[f] \to 0$$

For any non-trivial valuation v on K, we have a completion K_v. Then we have a morphism of pairs $(\mathrm{Gal}(K^s/K), A(K^s)) \to (\mathrm{Gal}(K_v^s/K_v), A_{K_v}(K_v^s))$ in the sense of [16, I, 2.4], it induces

$\mathrm{H}^r(\mathrm{Gal}(K^s/K), A(K^s)) \to \mathrm{H}^r(\mathrm{Gal}(K_v^s/K_v), A_{K_v}(K_v^s))$ which commutes with the long exact sequence. Hence we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B(K)/f(A(K)) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^1(G_K, N(K^s)) \arrow[r] \arrow[d, "\alpha_v"] & \mathrm{H}^1(G_K, A(K^s))[f] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_{K_v}(K_v)/f_{K_v}(A_{K_v}(K_v^s)) \arrow[r, "\delta_v"] & \mathrm{H}^1(G_{K_v}, N_{K_v}(K_v^s)) \arrow[r, "\beta_v"] & \mathrm{H}^1(G_{K_v}, A_{K_v}(K_v^s))[f_v] \arrow[r] & 0
\end{tikzcd}$$

Thus we can define the Selmer group of A/K associated to $f : A \to B$ as following:

$$\mathrm{Sel}^{(f)}(A/K) := \{t \in \mathrm{H}^1(G_K, N(K^s)) \mid (\beta_v \circ \alpha_v)(t) = 0 \text{ for all non-trivial valuation } v \text{ on } K\}$$

and note that $\mathrm{Im}\,\delta \subset \mathrm{Sel}^{(f)}(A/K)$. We can see that the understanding of Selmer group, tells us about the structure of $B(K)/f(A(K))$. Through this paper, we give an injection of the Selmer group of A/K associated to the relative Frobenius morphism into some other abelian group which is easier to work with. It is interesting to note that the relative Frobenius morphism as an isogeny is a **purely inseparable morphism/isogeny**, cf. [13, Chap. 5.2, Prop. 5.15].

We can identify the first order Galois cohomology groups $\mathrm{H}^1(G_K, A(K^s))$ with the group of principal homogeneous spaces of A/K (cf. [17]). The later group can be proved to be isomorphic to the first order *fppf* cohomology group $\mathrm{H}^1_{\mathrm{fppf}}(K, A)$ where A/K is considered as an *fppf*-sheaf (cf. [12, III, (2.10)&(4.7)]). Hence we can use *fppf* cohomology instead of Galois cohomology. Next we define the Selmer group associated to arbitrary isogeny (not necessarily separable like the number field case).

Lemma 1.34. *Let $f : A \to B$ be an isogeny between abelian varieties over K. Denote $N := \ker f$. Then we have an exact sequence of fppf-sheaves*

$$0 \to N \to A \to B \to 0$$

And it induces a long exact sequence of abelian groups:

$$0 \to N(K) \to A(K) \to B(K) \to \mathrm{H}^1_{\mathrm{fppf}}(K, N) \to \mathrm{H}^1_{\mathrm{fppf}}(K, A) \to \mathrm{H}^1_{\mathrm{fppf}}(K, B)$$

where $\mathrm{H}^1_{\mathrm{fppf}}(K, \mathcal{F})$ denotes the first order fppf cohomology group over K when $\mathcal{F}$ is an fppf-sheaf on the big fppf site of $\mathrm{Spec}\, K$.

Proof. The map f is an isogeny so it is faithfully flat and finite. As the structure morphism of $\ker f$ is just the pull-back of f w.r.t. the unit section of B, it is also faithfully flat and finite. So the short exact sequence satisfies the conditions in remark 1.15. Hence it is an exact sequence of *fppf*-sheaves. Then we apply the *fppf cohomology and get the long exact sequence of abelian groups.* ☐

Next the Selmer group is defined just as above but with every Galois cohomology group replaced by an *fppf* cohomology group. Now we are ready to define the Selmer group in our situation.

Situation 1.35. Continue situation 1.19. Recall that C is a projective smooth geometrically integral curve over a finite field k_0, with function field K. A is an abelian variety over K. Since $F_{A/K}$ is an isogeny we have the following commutative diagram for each closed point v on C:

$$\begin{tikzcd}
0 \arrow[r] & A^{(p)}(K)/F_{A/K}(A(K)) \arrow[r, "\delta"] \arrow[d] & \mathrm{H}^1_{\mathrm{fppf}}(K, \ker F_{A/K}) \arrow[r] \arrow[d, "\alpha_v"] & \mathrm{H}^1_{\mathrm{fppf}}(K, A)[F_{A/K}] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A^{(p)}_{K_v}(K_v)/F_{A_{K_v}/K_v}(A_{K_v}(K_v)) \arrow[r, "\delta_v"] & \mathrm{H}^1_{\mathrm{fppf}}(K_v, \ker F_{A_{K_v}/K_v}) \arrow[r, "\beta_v"] & \mathrm{H}^1_{\mathrm{fppf}}(K_v, A_{K_v})[F_{A_{K_v}/K_v}] \arrow[r] & 0
\end{tikzcd}$$

And

$$\mathrm{Sel}^{(F_{A/K})}(A/K) := \{t \in \mathrm{H}^1_{\mathrm{fppf}}(K, \ker F_{A/K}) \mid (\beta_v \circ \alpha_v)(t) = 0\ \forall v \text{ closed in } C\}$$

2. The Second Exact Sequence

In this section we state an injection of abelian groups which will be heavily used in our proof, and prove its functoriality under various circumstance. It is a corollary of “The Second Exact Sequence” in [15, III, 5.6], and the completed proof can be found in [18, 2].

In the original paper [18, 2] the explicit description of the map is only stated but not proved. Thus we will prove it ourselves, and use that to show functoriality and generalize the theorem.

Let $\pi : X \to S$ be a morphism of schemes of characteristic p where S is perfect and X is locally Noetherian. Write X' as X considered as S-scheme via $\pi' : X \overset{F_X}{\to} X \overset{\pi}{\to} S$. Write $\Omega^1_{X'/S,\mathrm{cl}} = \ker(d : \Omega^1_{X'/S} \to \Omega^2_{X'/S})$ for the sheaf of closed differential forms on X' relative to S. Let N be a finite flat commutative group scheme [3] of height one over X. In our situation everything is locally Noetherian so it won't matter.

Denote $e = e_{N/X} : X \to N$ the unit section of N and define $\omega_{N/X} = e^*\Omega^1_{N/X}$.

Theorem 2.1. *Let $f : X_{\mathrm{fppf}} \to X_{\text{ét}}$ be the morphism of sites defined by the identity map. If $\pi : X \to S$ is smooth, then we have an exact sequence of étale sheaves:*

$$0 \to R^1 f_* N \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}} \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \to 0 \tag{2.1}$$

Proof. See [18, 2]. ☐

We want to generalize the condition of smoothness of X/S. So we first identify their appearance in the proof.

The exact sequence 2.1 is deduced from an exact sequence of fppf sheaves over X, see [19, Thm.1.3] (with $(\varphi, \psi, X, S) := (F_X, \mathrm{id}_X, X', X)$):

$$0 \to \mathbb{G}_{m,X} \to F_{X,*}\mathbb{G}_{m,X'} \to F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} \to \Omega^1_{X'/X} \to 0 \tag{2.2}$$

which only requires that $F_X : X' \to X$ has a rank one p base Zariski locally, i.e. there is an open covering $\{U_\alpha\}$ of X s.t. $\forall \alpha$, $\exists f_1, ..., f_n \in \Gamma(U_\alpha, \mathcal{O}_{X'})$ s.t. $\{f_1^{e_1}...f_n^{e_n} | 0 \le e_i < p\}$ defines a basis of $F_{X,*}\mathcal{O}_{X'}$ as an $\mathcal{O}_X$-module. We can immediately see that $\Omega^1_{X'/X}$ is finite locally free with basis $\{df_i | 1 \le i \le n\}$ over U_α. When X is smooth over a perfect scheme S, $F_X = F_{X/S}$ satisfies the condition that F_X has a rank one p base Zariski locally, see [19, 1.1 & 1.2] and [5, tag 0CCD].

Another implicit usage of the smoothness of X/S is that: if X is smooth over a perfect scheme S, then X is reduced. But this result can also be deduced from the fact $F_X : X' \to X$ has a rank one p base Zariski locally: for each point $x \in X$ we know that $F_{\mathcal{O}_{X,x}} : \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}$ is finite free and in particular injective, so if $a^n = 0$ for some $a \in \mathcal{O}_{X,x}$ and $n > 0$, then $a^{p^e} = 0$ for some $p^e \ge n$ so $a = 0$ and $\mathcal{O}_{X,x}$ is reduced.

The perfectness of the base scheme S is not important throughout the proof, we just need the perfect condition to identify $\Omega^1_{X/S}$, $\Omega^1_{X/X^{(p)}}$, $\Omega^1_{X'/S}$, and $\Omega^1_{X'/X}$ as the same quasi-coherent $\mathcal{O}_X$-module if we identify X with X', so that we can plant S into the final sequence.

---

[3] In [18], there is no assumption of locally Noetherianness on X, the authors probably mean “finite locally free commutative group scheme” when they mention “finite flat commutative group scheme” because they freely used Cartier Duality which only works for finite locally free commutative group schemes. They are equivalent under locally Noetherian condition, see [5, tag 02KB].

That means, if we replace $\Omega^1_{X'/S,\mathrm{cl}}$ by $\Omega^1_{X'/X,\mathrm{cl}}$ and replace $\Omega^1_{X/S}$ by $\Omega^1_{X/X^{(p)}}$ in 2.1:

$$0 \to R^1 f_* N \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \Omega^1_{X/X^{(p)}} \to 0\ ,$$

then S does not even need to appear in the exact sequence. Here is a simple proof of their identification.

Lemma 2.2. *Let S be a perfect scheme of characteristic p, and $\pi : X \to S$ be a morphism of schemes, then we can identify $\Omega^1_{X/S}$, $\Omega^1_{X/X^{(p)}}$, $\Omega^1_{X'/S}$, and $\Omega^1_{X'/X}$ as the same quasi-coherent $\mathcal{O}_X$-module if we identify X with X'.*

Proof. By considering the composition $X' \overset{F_X}{\to} X \overset{\pi}{\to} S$, we have an exact sequences of quasi-coherent $\mathcal{O}_{X'}$-modules:

$$F_X^*\Omega^1_{X/S} \to \Omega^1_{X'/S} \to \Omega^1_{X'/X} \to 0$$

where the first map is zero as $dx^p = px^{p-1}dx = 0$. Hence $\Omega^1_{X'/S} \cong \Omega^1_{X'/X}$.

We have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\pi"] & X' \arrow[d, "\pi'"] \\ S \arrow[r, "F_S"] & S \end{tikzcd}$$

Thus $\Omega^1_{X/S} \cong \mathrm{id}_X^*\Omega^1_{X'/S}$. Also it's easy to see that $F_X = F_{X/S} : X' = X \to X^{(p)} \cong X$, thus $\Omega^1_{X/X^{(p)}} \cong \Omega^1_{X'/X}$.

Hence we can identify $\Omega^1_{X/S}$, $\Omega^1_{X/X^{(p)}}$, $\Omega^1_{X'/S}$, and $\Omega^1_{X'/X}$ as the same quasi-coherent $\mathcal{O}_X$-module if we identify X with X'. ☐

Also note that $\Omega^1_{X'/S,\mathrm{cl}}$ is in general not a quasi-coherent $\mathcal{O}_{X'}$-module but $F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}$ is a quasi-coherent $\mathcal{O}_X$-module since $F_{X,*}\Omega^1_{X'/S,\mathrm{cl}} = F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} = \ker(F_{X,*}d : F_{X,*}\Omega^1_{X'/X} \to F_{X,*}\Omega^2_{X'/X})$ and $F_{X,*}d$ is a map of quasi-coherent $\mathcal{O}_X$-modules: $r \cdot xdy = r^p xdy \mapsto dr^p x \wedge dy = r^p dx \wedge dy = r \cdot dx \wedge dy$.

So we have the same exact sequence if we replace the condition of smoothness of X/S by that F_X has a rank one p base Zariski locally. Hence we claim the following theorem which is more useful for us because we will deal with X which may not be smooth over S, e.g. function field of a curve that is not finitely presented over the base field, like $(X, S) = (\operatorname{Spec} K, \operatorname{Spec} k_0)$ in the diagram 3.1.

We can see that this is more like a result over X instead over S, so it is not surprising that the smoothness of X/S is not used in the proof except generating the initial sequence 2.2 and deducing the fact X is reduced.

Theorem 2.3. *Let $f : X_{\mathrm{fppf}} \to X_{\mathrm{\acute{e}t}}$ be the morphism of sites defined by the identity map. If $F_X : X' \to X$ has a rank one p base Zariski locally, then we have an exact sequence of étale sheaves:*[4]

$$0 \to R^1 f_* N \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}} \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \Omega^1_{X/S} \to 0$$

---

[4] If we use the sequence

$$0 \to R^1 f_* N \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \Omega^1_{X/X^{(p)}} \to 0$$

instead, then we don't need S to be perfect. This fact that the perfectness of S can be dropped in a way won't be used in the thesis because S is always assumed to be perfect.

Corollary 2.4. *We have an injection* $\Phi_N : \mathrm{H}^1_{\mathrm{fppf}}(X, N) \hookrightarrow \Gamma(X, \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$.

Proof. It suffices to show $R^1 f_* N(X) = \mathrm{H}^1_{\mathrm{fppf}}(X, N)$. We first show that the étale sheaf f_*N is trivial, i.e. for all schemes Y étale over X, every morphism $Y \to N$ factors through the unit section $e : X \to N$. We know that X is reduced from the fact $F_X : X' \to X$ has a rank one p base Zariski locally. By [5, tag 03PC] (8), Y is also reduced. By [5, tag 0356], $Y \to N$ must factors through $Y \to N_{\mathrm{red}} \to N$ so it suffices to show $X = N_{\mathrm{red}}$. N is finite over X so separated, then $e : X \to N$ is a closed immersion by [5, tag 047G]. Denote $\mathcal{I} \subset \mathcal{O}_N$ the ideal sheaf defining X. As N is of height one, the absolute Frobenius morphism $F_N : N \to N$ factors through $e : X \to N$, so $\mathcal{I}^p = 0$. Hence X and N share the same underlying space and $X = N_{\mathrm{red}}$. So $f_*N = 0$.

By [5, tag 0732], there exists a spectral sequence $E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(X, R^q f_* N)$ converging to $E^{p+q} = \mathrm{H}^{p+q}_{\mathrm{fppf}}(X, N)$. By [20, Chap. 0, 2.3.2], we have an exact sequence

$$0 \to E_2^{1,0} \to E^1 \to E_2^{0,1} \to E_2^{2,0} \to E^2$$

Since $f_*N = 0$, we have $E_2^{n,0} = \mathrm{H}^n_{\text{ét}}(X, f_*N) = 0$. So the sequence degenerates to

$$0 \to 0 \to E^1 \xrightarrow{\sim} E_2^{0,1} \to 0 \to E^2$$

So we have $\mathrm{H}^1_{\mathrm{fppf}}(X, N) \cong \mathrm{H}^0_{\text{ét}}(X, R^1 f_* N)$. □

Since N is of height one, $N = \ker F_{N/X}$. By [21, II, 2.1.2], N is Zariski-locally of the form $\frac{\mathcal{O}_X[y_1,\dots,y_n]}{(y_1^p,\dots,y_n^p)}$ for some n, so we have $\omega_{N/X} \cong \frac{(y_1,\dots,y_n)}{(y_1,\dots,y_n)^2}$ Zariski-locally, hence it is locally free of finite rank, so is its dual. Thus $\omega^\vee_{N/X}$ is a flat $\mathcal{O}_X$-module by [5, tag 05P2]. So the functor $\omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \bullet$ is exact and we have an injection of quasi-coherent modules (the injection as fppf and étale sheaves is proved latter):

$$\omega^\vee_{N/X} \otimes F_{X,*}\Omega^1_{X'/S,\mathrm{cl}} \hookrightarrow \omega^\vee_{N/X} \otimes F_{X,*}\Omega^1_{X'/S}.$$

By [10, 6, 4.21], we have isomorphisms of quasi-coherent $\mathcal{O}_X$-modules

$$\omega^\vee_{N/X} \otimes F_{X,*}\Omega^1_{X'/S,\mathrm{cl}} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$$

$$\omega^\vee_{N/X} \otimes F_{X,*}\Omega^1_{X'/S} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S})$$

Thus we have compatible identifications of abelian groups

$$\Gamma(X, \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) = \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$$

$$\Gamma(X, \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S}) = \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S})$$

Sometimes we work with $\omega^\vee_{N/X} \otimes F_{X,*}\Omega^1_{X'/S}$ instead because it is easier to deal with.

So we have an injection $\mathrm{H}^1_{\mathrm{fppf}}(X, N) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$ and in particular $\mathrm{H}^1_{\mathrm{fppf}}(X, N) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S})$ this is closed related to our main result, the injection $\mathrm{Sel}^{(F_{A/K})}(A/K) \hookrightarrow \mathrm{Hom}_C(\omega_{\ker F_{\mathcal{A}/C}/C}, F_{C,*}(\Omega^1_{C/k_0}(E)))$, because the later injection embeds in the former injection.

2.1. Some useful facts and their proofs

Before we state and prove the explicit description of the injection

$$\mathrm{H}^1_{\mathrm{fppf}}(X, N) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}),$$

we need to prove a few facts

1. We can identify $\mathrm{H}^1_{\mathrm{fppf}}(X, N)$ with isomorphism classes of representable N-torsors over X (cf. [12, III, (2.10)&(4.3)&(4.7)]). From now on we may just write $\mathrm{H}^1(X, N)$.
2. X is a regular scheme, and $F_X : X' \to X$ is an *fppf* covering.
3. N-torsors over X always pullback to the trivial torsor over X' via $F_X : X' \to X$.
4. For any N-torsor P over X, the trivialization $X' \to P$ is unique.
5. $F_{X,*}\Omega^1_{X'/X,\mathrm{cl}}$ is a finite locally free $\mathcal{O}_X$-module and for any morphism $f : Y \to X$, denote $g : X' \times_X Y \to Y$ the canonical projection, we have $f^*F_{X,*}\Omega^1_{X'/X} = g_*\Omega^1_{X'\times Y/Y}$ and $f^*F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} = g_*\Omega^1_{X'\times Y/Y,\mathrm{cl}}$.
6. Let $f : Y \to X$ a morphism of S-schemes s.t. $F_{Y/X} : Y' \to X' \times_X Y$ is an isomorphism, denote $g : X' \times_X Y \to Y$ the canonical projection, then $F_Y : Y' \to Y$ has a rank one p base Zariski locally, $f^*F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} = F_{Y,*}\Omega^1_{Y'/Y,\mathrm{cl}}$ and $f^*(F_X^*\omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) = F_Y^*\omega^\vee_{N_Y/Y} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$.
7. In Situation 1.35, every arrow of the following commutative diagram has isomorphic relative Frobenius morphism.

$$\begin{tikzcd}
\operatorname{Spec} K_v \arrow[r] \arrow[d] & \operatorname{Spec} \hat{\mathcal{O}}_v \arrow[d] \\
\operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}_v \arrow[d] \\
\operatorname{Spec} K \arrow[r] & C
\end{tikzcd}$$

8. The injection of quasi-coherent modules

$$\omega^\vee_{N/X} \otimes F_{X,*}\Omega^1_{X'/S,\mathrm{cl}} \hookrightarrow \omega^\vee_{N/X} \otimes F_{X,*}\Omega^1_{X'/S}$$

induces an injection of sheaves on the big τ-site where $\tau \in \{$Zariski, fppf, étale, smooth, syntomic$\}$ and the small étale site. Note that this is not true in general.

Fact 1 is already proved.

Fact 2: X is a regular scheme, and $F_X : X' \to X$ is an *fppf* covering.

Proof of fact 2. By assumption there is an open covering $\{U_\alpha\}$ of X s.t. $\forall \alpha, \exists f_1, ..., f_n \in \Gamma(U_\alpha, \mathcal{O}_{X'})$ s.t. $\{f_1^{e_1}...f_n^{e_n} | 0 \leq e_i < p\}$ defines a basis of $F_{X,*}\mathcal{O}_{X'}$ as an $\mathcal{O}_X$-module, in particular, for each $x \in X$, the map $F_{\mathcal{O}_{X,x}} : \mathcal{O}_{X,x} \to \mathcal{O}_{X',x}$ makes $\mathcal{O}_{X',x}$ a finite free $\mathcal{O}_{X,x}$-module, so it is a flat $\mathcal{O}_{X,x}$-module.

Kunz show that the Frobenius map of a Noetherian local ring is flat if and only if the local ring is regular, c.f. [22]. The original statement required that {the local ring is reduced and the Frobenius map is flat}, but flat local ring map is faithfully flat so injective, and an injective Frobenius map implies the ring is reduced, say $x^n = 0$ then $x^{p^e} = 0$ for some $p^e \geq n$, so $x = 0$. Thus we can remove the {reduced} condition.

Therefore $\mathcal{O}_{X,x}$ is a regular local ring for all $x \in X$. So X is a regular scheme. $F_X : X' \to X$ is faithfully flat, quasi-compact and surjective thus it is an *fppf* covering. ☐

Fact 3: N-torsors over X always pullback to the trivial torsor over X' via $F_X : X' \to X$.

It was proved in [18, 2.3] but the descent process is not rigorously verified. Thus I provide my own proof as follows.

Lemma 2.5. *Let X be a locally Noetherian scheme of characteristic p, and N be a finite flat commutative group scheme of height one on X with structure morphism $N \xrightarrow{g} X$. Then the pullback map via $F_X : X' \to X$:*

$$\mathrm{H}^1(X, N) \to \mathrm{H}^1(X', N)$$

is trivial.

Proof. Let $h : P \to X$ be an N-torsor over X. To show that P is trivial over X', it is equivalent to construct a morphism of X-schemes: $X' \to P$. As $N \to X$ is finite, flat and surjective (surjectivity follows from the surjectivity of $\mathrm{id}_X : X \xrightarrow{e} N \to X$), $P \to X$ is also finite, flat and surjective via *fppf* descent, see lemma A.6 in the appendix. In particular it is affine, so quasi-compact and quasi-separated. Hence $h_*\mathcal{O}_P$ and $g_*\mathcal{O}_N$ are quasi-coherent $\mathcal{O}_X$-algebras and finite locally free $\mathcal{O}_X$-modules. There is an anti-equivalence of categories between {schemes affine over X} and {quasi-coherent sheaves of $\mathcal{O}_X$-algebras}, see [5, tag 01SA]. So to construct a morphism of X-schemes from X' to P is the same to construct a morphism of quasi-coherent sheaves of $\mathcal{O}_X$-algebras from $h_*\mathcal{O}_P \to F_{X,*}\mathcal{O}_{X'}$.

Clearly the induced map $\mathcal{O}_X \to h_*\mathcal{O}_P$ (resp. $g_*\mathcal{O}_N$) is injective by checking on the stalks. As N is of height one, we can easily get the following commutative diagram:

$$\begin{tikzcd}
N \arrow[r, "F_N"] \arrow[d, "g"] & N \arrow[d, "g"] \\
X \arrow[r, "F_X"] \arrow[ur, "e \circ F_X"] & X
\end{tikzcd}$$

Note that $P \to X$ is faithfully flat and quasi-compact, and $P \times_X P \to P$ has a section $P \xrightarrow{\Delta} P \times_X P$ which is the diagonal map, we can see that $P \times_X P$ is an *fppf*-torsor of $N \times_X P$ over P with a section, hence there exists an isomorphism of P-schemes: $\alpha : N \times_X P \xrightarrow{\sim} P \times_X P$.

So we have the following commutative diagram, the front layer and the back layer are the same, and they are connected by absolute Frobenius morphisms.

$$\begin{tikzcd}
& N \times_X P \arrow[rr, shift left, "\alpha"] \arrow[dl, "F_{N\times_X P}"'] \arrow[dd] & & P \times_X P \arrow[ll, shift left, "\alpha^{-1}"] \arrow[dl, "F_{P\times_X P}"'] \arrow[ddll, bend left] \\
N \times_X P \arrow[rr, shift left, "\alpha"] \arrow[dd] & & P \times_X P \arrow[ll, shift left, "\alpha^{-1}"] \arrow[ddll, bend left] & \\
& X \times_X P \arrow[dl, "F_{X\times_X P}"'] & & \\
X \times_X P & & &
\end{tikzcd} \tag{2.3}$$

Combining the above diagram with another obvious commutative diagram we have:

$$\begin{tikzcd}
& N \times_X P \arrow[rr, shift left, "\alpha"] \arrow[dl, "F_{N\times_X P}"'] \arrow[dd] & & P \times_X P \arrow[ll, shift left, "\alpha^{-1}"] \arrow[dl, "F_{P\times_X P}"'] \arrow[rr] \arrow[ddll, bend left] & & P \times_X X \arrow[dl, "F_{P\times_X X}"] \arrow[dd] \\
N \times_X P \arrow[rr, shift left, "\alpha"] \arrow[dd] & & P \times_X P \arrow[ll, shift left, "\alpha^{-1}"] \arrow[rr] \arrow[ddll, bend left] & & P \times_X X \arrow[dd] & \\
& X \times_X P \arrow[dl, "F_{X\times_X P}"'] \arrow[rrrr] & & & & X \arrow[dl, "F_X"] \\
X \times_X P \arrow[rrrr] & & & & \color{red}{X} &
\end{tikzcd} \tag{2.4}$$

We can show every arrow of above diagram is affine by 4 basic facts:

1.$N \to X$ and $P \to X$ are affine.

2.Absolute Frobenius morphisms are affine.

3.Base changes of affine morphisms are affine.

4.If the composition $X \to Y \to Z$ is affine and $Y \to Z$ is separated, then $X \to Y$ is affine.

We can view the whole diagram as a diagram over red $\color{red}{X}$. Because every arrow is affine, and there is an anti-equivalence of categories between {schemes affine over X} and {quasi-coherent sheaves of $\mathcal{O}_X$-algebras}, we can convert the diagram into a diagram of quasi-coherent $\mathcal{O}_X$-algebras as following, which is ultimately simplified.

$$\begin{tikzcd}
& \bullet \arrow[rr, shift left, green, "\color{green}{\delta}"] & & \mathcal{G} \arrow[ll, shift left, "\delta^{-1}"] & & \bullet \arrow[ll, hook, blue] \\
\bullet \arrow[ur, green] \arrow[rr, shift left, "\delta"] & & \bullet \arrow[ll, shift left, green, "\color{green}{\delta^{-1}}"] \arrow[ur] & & \mathcal{F}_1 \arrow[ll, green] \arrow[ur] & \\
& \bullet \arrow[uu] \arrow[uurr, bend right] & & & & \mathcal{F}_2 \arrow[llll] \arrow[uu, hook, blue] \\
\bullet \arrow[uu] \arrow[ur] \arrow[uurr, bend right] & & & & \mathcal{O}_X \arrow[llll] \arrow[uu] \arrow[ur] &
\end{tikzcd} \tag{2.5}$$

We define $\phi_1 : \mathcal{F}_1 \to \mathcal{G}$ to be the composition of green arrows and $\phi_2 : \mathcal{F}_2 \hookrightarrow \mathcal{G}$ to be the composition of blue arrows. And in the end we want to construct a map of quasi-coherent algebras from $\mathcal{F}_1$ to $\mathcal{F}_2$

which proves the lemma. Note that the category of quasi-coherent modules on a scheme X is abelian, see [5, tag 077P].

For now let's assume the composition $\mathcal{F}_1 \overset{\phi_1}{\to} \mathcal{G} \to \operatorname{Coker} \phi_2$ is zero. Then we have

1. $\mathcal{F}_1 \overset{\phi_1}{\to} \mathcal{G}$ factors through $\mathcal{F}_2 = \ker(\mathcal{G} \to \operatorname{Coker} \phi_2)$.

2. $\exists! \theta : \mathcal{F}_1 \to \mathcal{F}_2$ which is a morphism of $\mathcal{O}_X$-modules s.t. the following diagram commutes

$$\begin{tikzcd}
\mathcal{F}_1 \arrow[r, "\phi_1"] \arrow[dr, "\theta"'] & \mathcal{G} \arrow[r] & \operatorname{Coker} \phi_2 \\
& \mathcal{F}_2 \arrow[u, hook, "\phi_2"'] \arrow[ur, "0"'] &
\end{tikzcd}$$

3. Since $\mathcal{F}_2 \overset{\phi_2}{\to} \mathcal{G}$ is injective, it is a monomorphism in the category of quasi-coherent $\mathcal{O}_X$-modules. So we can cancel ϕ_2 in the following equality:

$$(\mathcal{O}_X \to \mathcal{F}_1 \overset{\theta}{\to} \mathcal{F}_2 \overset{\phi_2}{\hookrightarrow} \mathcal{G}) = (\mathcal{O}_X \to \mathcal{F}_1 \overset{\phi_1}{\to} \mathcal{G}) = (\mathcal{O}_X \to \mathcal{F}_2 \overset{\phi_2}{\hookrightarrow} \mathcal{G})$$

   so we have

$$(\mathcal{O}_X \to \mathcal{F}_1 \overset{\theta}{\to} \mathcal{F}_2) = (\mathcal{O}_X \to \mathcal{F}_2)$$

4. Now we just need to prove θ is a morphism of quasi-coherent $\mathcal{O}_X$-algebras. We claim for any open subset $U \subset X$, $\theta(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ is a ring map, note $\phi_1(U)$ and $\phi_2(U)$ are ring maps.

5. Write θ, ϕ_1, ϕ_2 for $\theta(U), \phi_1(U), \phi_2(U)$. Then $\phi_2 \circ \theta(1) = \phi_1(1) = 1$, by injectivity of ϕ_2, $\theta(1) = 1$. Clearly $\theta(a + b) = \theta(a) + \theta(b)$ and $\theta(0) = 0$, this comes from the fact that θ is a morphism between quasi-coherent $\mathcal{O}_X$-modules.

6. Let $a, b \in \mathcal{F}_1(U)$,

$$\begin{aligned}
\phi_2 \circ \theta(ab) &= \phi_1(ab) \\
&= \phi_1(a)\phi_1(b) \\
&= \phi_2(\theta(a))\phi_2(\theta(b)) \\
&= \phi_2(\theta(a)\theta(b))
\end{aligned}$$

   by injectivity of ϕ_2 we have $\theta(ab) = \theta(a)\theta(b)$.

7. Thus θ is a morphism of quasi-coherent $\mathcal{O}_X$-algebras. Hence it induces a morphism of X-schemes $X' \to P$ which proves the lemma.

We now prove the composition $\mathcal{F}_1 \overset{\phi_1}{\to} \mathcal{G} \to \operatorname{Coker} \phi_2$ is zero. It suffices to show it is zero restricting to stalks. Pick arbitrary $x \in X$, say $R := \mathcal{O}_{X,x}$, $S := (g_* \mathcal{O}_N)_x$ and $T := (h_* \mathcal{O}_P)_x$. Then S and T are finite free R-modules. So the following left diagram induces the right diagram.

$$\begin{tikzcd}
N \arrow[r, "F_N"] \arrow[d, "g"] & N \arrow[d, "g"] \\
X \arrow[r, "F_X"] \arrow[ur, "e \circ F_X"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
S & S \arrow[l, "s^p \mapsfrom s"', "F_S"] \arrow[dl, "\varphi"] \\
R \arrow[u, hook] & R \arrow[l, "r^p \mapsfrom r"', "F_R"] \arrow[u, hook]
\end{tikzcd}$$

Hence $\varphi(s) = s^p \in R \subset S$ where we view R as a subring of S. The diagram 2.5 restricting to x is equivalent to the following:

$$\begin{tikzcd}
 & S \otimes_R T \arrow[rr, green, shift left, "\delta"] & & T \otimes_R T \arrow[ll, shift left, "\delta^{-1}"] & & T \otimes_R R \arrow[ll, blue, hook'] \\
S \otimes_R T \arrow[ur, green, "F_{S\otimes_R T}"] \arrow[rr, shift left, "\delta"] & & T \otimes_R T \arrow[ll, green, shift left, "\delta^{-1}"] \arrow[ur, "F_{T\otimes_R T}"] & & T \otimes_R R \arrow[ll, green, hook'] \arrow[ur, "F_{T\otimes_R R}"] & \\
 & R \otimes_R T \arrow[uu, hook] \arrow[uurr, hook, bend right] & & & & R \arrow[llll, hook'] \arrow[uu, blue, hook] \\
R \otimes_R T \arrow[uu, hook] \arrow[ur, "F_{R\otimes_R T}"] \arrow[uurr, hook, bend right] & & & & R \arrow[llll, hook'] \arrow[uu] \arrow[ur, "F_R"] &
\end{tikzcd} \tag{2.6}$$

It suffices to show that the image of the composition of green arrows lies in R. Note that it is also the image of $T \otimes_R R \hookrightarrow T \otimes_R T \overset{Frob}{\to} T \otimes_R T$, $t \otimes 1 \mapsto t \otimes 1 \mapsto t^p \otimes 1$. Fix $t \otimes 1 \in T \otimes_R R$, assume that $\delta^{-1}(t \otimes 1) = \sum_i s_i \otimes t_i$ for some $s_i, t_i \in T$.

$$\begin{aligned}
\delta \circ F_{S\otimes_R T} \circ \delta^{-1}(t \otimes 1) &= \delta \circ F_{S\otimes_R T}(\sum_i s_i \otimes t_i) \\
&= \delta(\sum_i s_i^p \otimes t_i^p) \\
&\text{since } s_i^p = \varphi(s_i) \in R \\
&= \delta(1 \otimes (\sum_i s_i^p \cdot t_i^p)) \\
&= 1 \otimes (\sum_i s_i^p \cdot t_i^p)
\end{aligned}$$

So we have $t^p \otimes 1 = 1 \otimes (\sum_i s_i^p \cdot t_i^p)$.

Claim: Let A, B be R-algebras which are also finite free R-modules where R is a local ring, if $a \otimes 1 = 1 \otimes b \in A \otimes_R B$, then $a = b \in R$.

Proof of the claim. It suffices to show that for any finite free R-modules which is an R-algebra, it has a basis containing 1. Then we just compare the coefficients of the R-basis and the result follows. Let $\{a_1, ..., a_n\}$ be an R-basis of A. Then $\exists r_i \in R$ s.t. $1 = \sum_{i=1}^n r_i \cdot a_i$. Note that for any faithfully flat ring map $R \to A$ and any ideal $I \subset R$, we have $(I \otimes_R A) \cap R = I$. Thus $\langle r_1, ..., r_n \rangle = (\langle r_1, ..., r_n \rangle \otimes_R A) \cap R = A \cap R = R$. In particular, there exists $c_i \in R$ s.t. $1 = \sum_{i=1}^n r_i c_i$. We will later expand the vector $[r_1, ..., r_n]$ to a basis of R^n, i.e. expand it to an invertible $n \times n$ matrix over R.

This is standard, write $\mathbf{r} := [r_1, ..., r_n] \in R^n$ and $\mathbf{c} := [c_1, ..., c_n] \in R^n$ and consider $f : R^n \to R, \mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{c}$. Then $f(\mathbf{r}) = 1$ and $R^n = R \cdot \mathbf{r} \oplus \ker f$ via $\mathbf{v} = f(\mathbf{v}) \cdot \mathbf{r} + (\mathbf{v} - f(\mathbf{v}) \cdot \mathbf{r})$.

So $\ker f$ is a direct summand of R^n hence projective over the local ring R. Thus it is free. Hence R^n has a basis containing $[r_1, ..., r_n]$ and they form an invertible matrix over R, say $\mathbf{M}$.

Apply $\mathbf{M}$ to the basis of A, $[a_1, ..., a_n]$, we get another basis starting from $1 = \sum_{i=1}^n r_i \cdot a_i$. The result follows. ☐

By the above claim we know that $t^p \in R$ which is exactly what we want to show. Then the lemma follows from the above argument. ☐

Fact 4: For any N-torsor P over X, the trivialization $X' \to P$ is unique.

We need the help of the following lemma:

Lemma 2.6. *Let Y be a reduced scheme and $\pi : X \to Y$ be an affine morphism of schemes with one surjective section $e : Y \to X$. Then the section is unique.*

Proof. It's easy to see that π and e are homeomorphisms inverse to each other, so any other sections must be topologically the same as e. Assume the result is true for affine scheme Y. Then for general scheme Y, let $g : Y \to X$ be another section, and U be an affine open subset of Y. Then U is a reduced scheme and $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is an affine morphism with one surjective section $e|_U : U \to \pi^{-1}(U)$. Then $g|_U : U \to \pi^{-1}(U)$ must agree with $e|_U$ by assumption. Thus g agrees with e on an open affine cover so $g = e$. Thus we can assume Y is affine, so X is affine. So the problem reduces to: Let $f : R \to A$ be a ring map where R is a reduced ring, and $e : A \to R$ be a retraction s.t. $\operatorname{Spec} R \to \operatorname{Spec} A$ is surjective. We want to show e is the only retraction map.

Let $I := \ker e$ so $R \cong \frac{A}{I}$. We know $\operatorname{Spec} R$ is homeomorphic to $\operatorname{Spec} A$. So A and $\frac{A}{I}$ have the same set of prime ideals. Thus I is a sub-ideal of the nilradical of A, i.e. $\forall a \in I, \exists n \geq 0$ s.t. $a^n = 0$.

Also $R \cong \frac{A}{I}$ is reduced thus $0 = \mathcal{N}(R) \supset \frac{\mathcal{N}(A)+I}{I}$ where $\mathcal{N}(\bullet)$ denotes the nilradical ideal. Hence $\mathcal{N}(A) \subset I \subset \mathcal{N}(A)$ and then $I = \mathcal{N}(A)$. Let $g : A \to R$ be another retraction map and $a \in \mathcal{N}(A)$, then $\exists n \geq 0$ s.t. $a^n = 0$. Then $0 = g(a^n) = g(a)^n$ so $g(a) = 0$. Thus g factors through $A \xrightarrow{e} \frac{A}{I} \cong R$. There is only one R-algebra map $R \to R$ and it is the identity map. Hence $g = e$ and the result follows. ☐

Proof of fact 4. We know $P(X') = P_{X'}(X') = N_{X'}(X')$. We only need to show $N_{X'} \to X'$ has only one section. We know $X' = X$ is reduced. $N_{X'} \to X'$ is affine by base change. Also $N_{X'} \to X'$ is still a finite flat group scheme of height one by base change. In particular, the absolute Frobenius morphism on $N_{X'}$ factors through the unit section $X' \to N_{X'}$. Hence the unit section is surjective, then we apply the above lemma and know that $N_{X'} \to X'$ has only one section, the unit section. It follows that $P(X')$ is singleton. ☐

Fact 5: $F_{X,*}\Omega^1_{X'/X,\mathrm{cl}}$ is a finite locally free $\mathcal{O}_X$-module and for any morphism $f : Y \to X$, denote $g : X' \times_X Y \to Y$ the canonical projection, we have $f^* F_{X,*}\Omega^1_{X'/X} = g_*\Omega^1_{X'\times Y/Y}$ and $f^*F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} = g_*\Omega^1_{X'\times Y/Y,\mathrm{cl}}$.

Proof of fact 5. The statement is affine-locally so we can assume $X = \operatorname{Spec} R$ is affine and R' is a free over R with a basis $\{x_1^{e_1}...x_n^{e_n} | 0 \leq e_i < p\}$, hence $\Omega^1_{R'/R} = \bigoplus R' \cdot dx_i = \bigoplus_{e_k,i} R x_1^{e_1} \cdots x_n^{e_n} dx_i$. So $\Omega^2_{R'/R} = \bigoplus_{e_k,i<j} R x_1^{e_1} \cdots x_n^{e_n} dx_i \wedge dx_j$. Let M be the $\mathbb{F}_p$-vector space with basis $\{x_1^{e_1} \cdots x_n^{e_n} dx_i\}$ and M^2 be the vector space with basis $\{x_1^{e_1} \cdots x_n^{e_n} dx_i \wedge dx_j\}$ and define $M \xrightarrow{d_M} M^2$, $d(udv) = du \wedge dv$ as usual. Then we know

$$\Omega^1_{R'/R} = R \otimes_{\mathbb{F}_p} M,\ \Omega^2_{R'/R} = R \otimes_{\mathbb{F}_p} M^2$$

as free R-modules and $d = \mathrm{id}_R \otimes d_M : \Omega^1_{R'/R} \to \Omega^2_{R'/R}$ as map of R-modules. As R is flat over $\mathbb{F}_p$, $\Omega^1_{R'/R,\mathrm{cl}} = \ker d = R \otimes_{\mathbb{F}_p} \ker d_M$ is a finite free R-module.

If S is a R-algebra, it is easy to see that $\Omega^1_{S\otimes R'/S} \cong S \otimes_R \Omega^1_{R'/R} \cong S \otimes_{\mathbb{F}_p} M$ and $\Omega^2_{S\otimes R'/S} = S \otimes_{\mathbb{F}_p} M^2$, and $\Omega^1_{S\otimes R'/S,\mathrm{cl}} = \ker d_S = S \otimes_{\mathbb{F}_p} \ker d_M = S \otimes_R R \otimes_{\mathbb{F}_p} \ker d_M = S \otimes_R \Omega^1_{R'/R,\mathrm{cl}}$. The result follows. ☐

Fact 6: Let $f : Y \to X$ a morphism of S-schemes s.t. $F_{Y/X} : Y' \to X' \times_X Y$ is an isomorphism, denote $g : X' \times_X Y \to Y$ the canonical projection, then $F_Y : Y' \to Y$ has a rank one p base Zariski locally, $f^* F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} = F_{Y,*}\Omega^1_{Y'/Y,\mathrm{cl}}$ and $f^*(F_X^*\omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) = F_Y^*\omega^\vee_{N_Y/Y} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$.

Proof. Since S is perfect, $F_Y = F_{Y/S}$ and it suffices to show that $F_{Y/S}$ is a rank one purely inseparable Galois covering (or piG for short), see [19, 1, Def 1 & Prop 1.1]. We know $F_{X/S} = F_X$ is a piG. Consider the following commutative diagram

$$\begin{tikzcd}
Y \arrow[d, "F_{Y/X}"] \arrow[dd, bend right, "F_{Y/S}"'] & & \\
Y^{(p/X)} \arrow[r] \arrow[d, "h"] & X \arrow[d, "F_{X/S}"] & \\
Y^{(p/S)} \arrow[r] \arrow[d] & X^{(p/S)} \arrow[r] \arrow[d] & S \arrow[d, "F_S"] \\
Y \arrow[r] & X \arrow[r] & S
\end{tikzcd}$$

It's easy to see that all the squares are Cartesian. Since piG morphisms are stable under base change, h is a piG as a base change of $F_{X/S}$. By assumption $F_{Y/X}$ is an isomorphism so $F_{Y/S} = h \circ F_{Y/X}$ is a piG. Thus F_Y has a rank one p base Zariski locally.

By fact 5, $f^* F_{X,*}\Omega^1_{X'/X,\mathrm{cl}} = g_*\Omega^1_{X'\times Y/Y,\mathrm{cl}} \cong F_{Y,*}\Omega^1_{Y'/Y,\mathrm{cl}}$. Since the Frobenius morphisms of both X and Y have a rank one p base Zariski locally and S is perfect, we can identify $\Omega^1_{X/S}$ with $\Omega^1_{X'/X}$ when we identify X and X', similarly we can identify $\Omega^1_{Y/S}$ with $\Omega^1_{Y'/Y}$ when we identify Y and Y'. So to prove $f^*(F_X^*\omega^\vee_{N/X} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}) = F_Y^*\omega^\vee_{N_Y/Y} \otimes_{\mathcal{O}_Y} \Omega^1_{Y/S}$ is the same as to prove $f^*(F_X^*\omega^\vee_{N/X} \otimes_{\mathcal{O}_{X'}} \Omega^1_{X'/X}) = F_Y^*\omega^\vee_{N_Y/Y} \otimes_{\mathcal{O}_{Y'}} \Omega^1_{Y'/Y}$. Consider the following commutative diagram

$$\begin{tikzcd}
Y' \arrow[r, "F_Y"] \arrow[d, "f"] & Y \arrow[r, "e_{N_Y}"] \arrow[d, "f"] & N_Y \arrow[d] \\
X' \arrow[r, "F_X"] & X \arrow[r, "e_N"] & N
\end{tikzcd}$$

since $F_{Y/X}$ is an isomorphism, the left square (so both are) is Cartesian.

We have

$$\begin{aligned}
& f^*(F_X^*\omega^\vee_{N/X} \otimes_{\mathcal{O}_{X'}} \Omega^1_{X'/X}) \\
=& f^* F_X^*\omega^\vee_{N/X} \otimes_{\mathcal{O}_{Y'}} f^*\Omega^1_{X'/X} \\
=& F_Y^* f^*\omega^\vee_{N/X} \otimes_{\mathcal{O}_{Y'}} \Omega^1_{Y'/Y} \\
=& F_Y^*(f^*\omega_{N/X})^\vee \otimes_{\mathcal{O}_{Y'}} \Omega^1_{Y'/Y} \\
=& F_Y^*\omega^\vee_{N_Y/Y} \otimes_{\mathcal{O}_{Y'}} \Omega^1_{Y'/Y}
\end{aligned}$$

The result follows. ☐

Fact 7: In Situation 1.35, every arrow of the following commutative diagram has isomorphic relative

Frobenius morphism.

$$\begin{tikzcd}
\operatorname{Spec} K_v \arrow[r] \arrow[d] & \operatorname{Spec} \hat{\mathcal{O}}_v \arrow[d] \\
\operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} \mathcal{O}_v \arrow[d] \\
\operatorname{Spec} K \arrow[r] & C
\end{tikzcd}$$

Proof of fact 7. We first claim that if $f : Y \to S$ is a morphism with isomorphic relative Frobenius morphism and $g : X \to Y$ is a morphism of scheme, then $g : X \to Y$ has isomorphic relative Frobenius morphism if and only if $f \circ g : X \to S$ has isomorphic relative Frobenius morphism.

Proof of the claim. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[d, "F_{X/Y}"] \arrow[dd, bend right, "F_{X/S}"'] & & \\
X^{(p/Y)} \arrow[r] \arrow[d, "h"] & Y \arrow[d, "F_{Y/S}"] & \\
X^{(p/S)} \arrow[r] \arrow[d] & Y^{(p/S)} \arrow[r] \arrow[d] & S \arrow[d, "F_S"] \\
X \arrow[r, "g"] & Y \arrow[r, "f"] & S
\end{tikzcd}$$

It's easy to see that every square above is Cartesian. So if $F_{Y/S}$ is an isomorphism, then so is h. So $F_{X/S}$ is an isomorphism if and only if $F_{X/Y}$ is an isomorphism. ☐

So it suffices to show that $\operatorname{Spec} K \to C$, $\operatorname{Spec} \mathcal{O}_v \to C$, $\operatorname{Spec} \hat{\mathcal{O}}_v \to \operatorname{Spec} \mathcal{O}_v$ and $\operatorname{Spec} K_v \to \operatorname{Spec} \hat{\mathcal{O}}_v$ have isomorphic relative Frobenius morphisms. Note that every flat monomorphism is weakly étale by [5, tag 094X], hence has isomorphic relative Frobenius morphism by [5, tag 0F6W].

Claim: Let X be a scheme and $x \in X$, then $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ is a flat monomorphism.

Proof of the claim. Take an open affine neighborhood U of x, as $U \hookrightarrow X$ is a flat monomorphism, we can assume X is affine, say $X = \operatorname{Spec} A$. By [5, tag 01L3], it suffices to show that $A \to A_{\mathfrak{p}}$ is flat and $\Delta : A_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}} \to A_{\mathfrak{p}}$ is an isomorphism for any prime ideal $\mathfrak{p}$ of A.

Any localization map $A \to S^{-1}A$ is flat by [5, tag 00HT]. Take $S := A - \mathfrak{p}$, then we have $A_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}} = S^{-1}A \otimes_A A_{\mathfrak{p}} \cong S^{-1}A_{\mathfrak{p}} = A_{\mathfrak{p}}$. Denote the isomorphism as $\phi : A_{\mathfrak{p}} \xrightarrow{\sim} A_{\mathfrak{p}} \otimes_A A_{\mathfrak{p}}, a \mapsto 1 \otimes a$. We can see that $\Delta \circ \phi = \mathrm{id}$ so Δ must be an isomorphism (the same proof works for any multiplicative set S). ☐

Morphisms $\operatorname{Spec} K \to C$, $\operatorname{Spec} \mathcal{O}_v \to C$ and $\operatorname{Spec} K_v \to \operatorname{Spec} \hat{\mathcal{O}}_v$ are of the form $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ (x can be the generic point). So they are flat monomorphisms so have isomorphic relative Frobenius morphism.

Now we consider $\phi : \mathcal{O}_v \to \hat{\mathcal{O}}_v$. Denote $(R, I, \kappa) = \mathcal{O}_v$ which is a DVR. Here $I = t_v R$ for an uniformiser $t_v \in R$. We need to show $F_{\hat{R}/R} : R' \otimes_R \hat{R} \to \hat{R}, a \otimes b \mapsto ab^p$ is an isomorphism. By [5, tag 00MA] and the fact R' is a finite free R-module, the canonical map $R' \otimes_R \hat{R} \to \widehat{R'}, a \otimes b \mapsto ab^p$ is an isomorphism, where $\widehat{R'}$ is the completion of R' with respect to $F_R(I)R' = t_v^p R'$. And $\widehat{R'}$ is set theoretically the same as the completion of R with respect to I^p, so the same as the completion of R with respect to I by [5, tag 0319]. So the canonical isomorphism identifies with $F_{\hat{R}/R}$. □

Fact 8: The injection of quasi-coherent modules

$$\omega_{N/X}^\vee \otimes F_{X,*}\Omega^1_{X'/S,\mathrm{cl}} \hookrightarrow \omega_{N/X}^\vee \otimes F_{X,*}\Omega^1_{X'/S}$$

induces an injection of sheaves on the big τ-site where $\tau \in$ {Zariski, fppf, étale, smooth, syntomic} and the small étale site.

Proof of fact 8. In general the functor taking quasi-coherent modules to sheaves on the big τ-site where $\tau \in$ {Zariski, fppf, étale, smooth, syntomic} is not left exact, but in this case we do have an injection of τ-sheaves. We check this by directly verifying the injection on every Y-valued point: if Y is an X-scheme with structure morphism $f : Y \to X$, denote $g : Y \times_X X' \to Y$, we have

$$\begin{aligned}\Gamma(Y, \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) &= \Gamma(Y, \mathit{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}))\\ &= \mathrm{Hom}_{\mathcal{O}_Y}(f^*\omega_{N/X}, f^*F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})\\ &\text{by fact 5}\\ &= \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, g_*\Omega^1_{Y\times_X X'/Y,\mathrm{cl}})\end{aligned}$$

and similarly

$$\Gamma(Y, \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S}) = \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, g_*\Omega^1_{Y\times_X X'/Y})$$

and we have a canonical injection $\mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, g_*\Omega^1_{Y\times_X X'/Y,\mathrm{cl}}) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, g_*\Omega^1_{Y\times_X X'/Y})$. So we have an injection of sheaves. □

If we have an injection of τ-sheaves $\mathcal{F} \hookrightarrow \mathcal{G}$ and a singleton τ-covering $Y \to X$ (e.g. the fppf covering $X' \to X$ mentioned in fact 2), so we have injections $\mathcal{F}(X) \hookrightarrow \mathcal{F}(Y)$ and $\mathcal{G}(X) \hookrightarrow \mathcal{G}(Y)$, then $\mathcal{F}(Y) \cap \mathcal{G}(X) = \mathcal{F}(X)$ by basic sheaf properties.

In particular, to verify if an element of $\Gamma(X, \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S})$ lies in $\Gamma(X, \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$, it suffices to check if its restriction in $\Gamma(X', \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S})$ lies in $\Gamma(X', \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$.

2.2. Three explicit descriptions of Φ_N

Now that we have finished proving the useful facts, we are ready to give three explicit descriptions of the injection $\Phi_N : \mathrm{H}^1(X, N) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$, and later we will prove they coincide.

The first description comes directly from the proof of [18, 2.4]: Consider the nerve of the covering $X' \to X$:

$$X''' \substack{\rightarrow \\ \rightarrow \\ \rightarrow} X'' = X' \times_X X' \rightrightarrows X' \to X$$

Let $N' := F_* N_{X'}$ etc. ..., we know that (see [18, 2.5])

$$\begin{aligned}
\mathrm{H}^1(X, N) &= R^1 f_* N(X) \\
&= \underline{\check{H}}^1_{\text{ét}}(X'/X, f_* N)(X) \\
&= \frac{\ker(f_* N'' \to f_* N''')}{\mathrm{Im}(f_* N' \to f_* N'')}(X) \\
&= \ker(f_* N'' \to f_* N''')(X) \text{ (as } f_* N' = 0) \\
&= f_* \ker(N'' \to N''')(X) \\
&\subset N''(X) \\
&= N(X'')
\end{aligned}$$

The correspondence between $\mathrm{H}^1(X, N)$ and $\ker(N'' \to N''')(X)$ is as in the proof of [12, III, 4.6]: there is a one-to-one correspondence between isomorphism classes of torsors that become trivial on X' (in this case all of them) and 1-cocycles for $\{X' \to X\}$ with values in N. Now we can start to describe the injection. Let P be a torsor, then $P(X')$ has a unique element ψ, and there exists a unique element $\alpha \in N(X'') = N(X' \times_X X')$ s.t.

$$\alpha \cdot (X'' \overset{\mathrm{pr}_1}{\to} X' \overset{\psi}{\to} P) = (X'' \overset{\mathrm{pr}_2}{\to} X' \overset{\psi}{\to} P).$$

It's easy to see that $\alpha|_{12} + \alpha|_{23} = \alpha|_{13}$ under $\mathrm{pr}_{12}, \mathrm{pr}_{13}, \mathrm{pr}_{23} : X''' \to X''$. Now we arrive at $\alpha \in \ker(N'' \to N''')(X)$. We have

$$\mathrm{H}^1(X, N) = \ker(N'' \to N''')(X) = \mathrm{Coker}(N \to N')(X)$$

Recall from [18, 2.2] the map $\mathrm{Coker}(N \to N') \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_* \Omega^1_{X'/S,\mathrm{cl}}$ is induced from $N' \to \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_* \Omega^1_{X'/S,\mathrm{cl}}$ defined as

$$\mathit{Hom}_{\mathrm{gr}}(N^\vee, F_* \mathbb{G}_{m,X'}) \overset{\mathit{Hom}_{\mathrm{gr}}(N^\vee, F_* d \ln)}{\longrightarrow} \mathit{Hom}_{\mathrm{gr}}(N^\vee, F_* \Omega^1_{X'/S,\mathrm{cl}}).$$

Though we have a surjective map of fppf abelian sheaves $N' \to \mathrm{Coker}(N \to N') = \ker(N'' \to N''') \subset N''$, we cannot in general lift $\alpha \in \mathrm{Coker}(N \to N')(X)$ to an element of $N'(X)$. We need to pull back to some fppf covering to do that.

In this case, we can simply use the fppf covering $\{F_X : X' \to X\}$: α is not necessarily in the image of the map $N'(X) \to N''(X)$, but the image of α restricting to $N''(X')$, which is $\alpha|_{23}$ if we identify the restriction map as $N(X'') = N''(X) \to N''(X') = N(X''')$, lies in the image of $N'(X') \to N''(X')$ because when we identify the map with $N(X'') = N'(X') \to N''(X') = N(X''')$, the image of α is $\alpha|_{13} - \alpha|_{12} = \alpha|_{23}$. See the following commutative diagram:

$$\begin{tikzcd}
N'(X) \arrow[r, equal] & N(X') \arrow[r, "\phi \mapsto \phi|_2 - \phi|_1"] \arrow[d, "\phi \mapsto \phi|_2"] & N(X'') \arrow[r, equal] \arrow[d, "\beta \mapsto \beta|_{23}"] & N''(X) \\
N'(X') \arrow[r, equal] & N(X'') \arrow[r, "\beta \mapsto \beta|_{13} - \beta|_{12}"] & N(X''') \arrow[r, equal] & N''(X')
\end{tikzcd}$$

So the image $\Phi_N(P)$ of α under the map $\mathrm{Coker}(N \to N')(X) \to \Gamma(X, \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S,\mathrm{cl}})$ is the unique element s.t. after restricting to X', is the image of α under the map $N'(X') \to \Gamma(X', \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S,\mathrm{cl}})$, defined as in the following commutative diagram:

$$\begin{tikzcd}
N'(X') \arrow[r, "\partial"] \arrow[d, phantom, sloped, "\cong"] & \Gamma(X', \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S,\mathrm{cl}}) \arrow[d, phantom, sloped, "\cong"] \\
\mathit{Hom}_{\mathrm{gr}}(N^\vee, F_*\mathbb{G}_{m,X'})(X') \arrow[r, "\mathit{Hom}_{\mathrm{gr}}(N^\vee{,}F_*d\ln)"] & \mathit{Hom}_{\mathrm{gr}}(N^\vee, F_*\Omega^1_{X'/S,\mathrm{cl}})(X')
\end{tikzcd}$$

i.e. $\Phi_N(P) \in \Gamma(X, \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S,\mathrm{cl}})$ is the unique element s.t. $\Phi_N(P)|_{X'} = \partial\alpha$.

The second description was mentioned in [18, Rmk. 2.7] but without proof of equivalence: Let P be an N-torsor over X, there is a unique trivialization $\psi : X' \to P$ over X, then there exists a unique element $\alpha \in N(X'') = N(X' \times_X X')$ s.t.

$$\alpha \cdot (X'' \overset{\mathrm{pr}_1}{\to} X' \overset{\psi}{\to} P) = (X'' \overset{\mathrm{pr}_2}{\to} X' \overset{\psi}{\to} P).$$

After appending the diagonal morphism $X' \overset{\triangle}{\to} X''$ in the front of the above relation, we have

$$(X' \overset{\triangle}{\to} X'' \overset{\alpha}{\to} N) \cdot (X' \overset{\psi}{\to} P) = (X' \overset{\psi}{\to} P)$$

so we must have $(X' \overset{\triangle}{\to} X'' \overset{\alpha}{\to} N) = (X' \to X \overset{e}{\to} N)$ is the unit element in $N(X')$. We can decompose the diagonal morphism $\Delta : X' \to X''$ into $\Delta : X' \overset{i}{\to} \underline{\mathrm{Spec}}_{\mathcal{O}_X} F_*(\mathcal{O}_{X'}[\Omega^1_{X'/X}]) \overset{j}{\to} X''$, where j corresponds to $R' \otimes_R R' \to R'[\Omega^1_{R'/R}], (a, b) \mapsto (ab, adb)$ and i corresponds to $R'[\Omega^1_{R'/R}] \to R', (a, bdc) \mapsto a$.

All concerned schemes are affine over X so we can look at this affine-locally. Assume $X = \mathrm{Spec}\, R$, $X' = \mathrm{Spec}\, R'$, $N = \mathrm{Spec}\, A$ with augmentation ideal I. So we have

$$(A \overset{\alpha^\#}{\to} R' \otimes_R R' \to R'[\Omega^1_{R'/R}] \to R') = (A \to \frac{A}{I} \to R')$$

Thus I must map to $\Omega^1_{R'/R} \subset R'[\Omega^1_{R'/R}]$ which is a square zero ideal. Hence α induces a map $\omega_{A/R} \cong \frac{I}{I^2} \to \Omega^1_{R'/R}$. So globally α induces a map $\omega_{N/X} \to F_{X,*}\Omega^1_{X'/S}$ which is an element in $\mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S})$. Later we will show it lies in $\mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) = \Gamma(X, \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S,\mathrm{cl}})$, which is equivalent to show its restriction in $\Gamma(X', \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S})$ lies in $\Gamma(X', \omega_{N/X}^\vee \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$.

Our third and final description was mentioned [3, 1.1]: Let P be an N-torsor over X with the unique trivialization $\psi : X' \to P$, we have an isomorphism of torsors $\xi : X' \times_X P \to X' \times_X N$ over X' s.t. the following diagram commutes:

$$\begin{tikzcd}
X' \arrow[r, equal] \arrow[d, "(\mathrm{id}_{X'}{,}\psi)"] & X' \arrow[d, "e_{X'}"] \\
X' \times_X P \arrow[r, "\xi"] & X' \times_X N
\end{tikzcd}$$

After expansion we have

$$\begin{tikzcd}
X' \arrow[r, equal] \arrow[d, "\psi"] & X' \arrow[r, equal] \arrow[d, "(\mathrm{id}_{X'},\psi)=\psi_{X'}"] & X' \arrow[r, "F_X"'] \arrow[d, "e_{X'}"] & X \arrow[d, "e"] \\
P \arrow[d] & X' \times_X P \arrow[l] \arrow[r, "\xi"] \arrow[d] & X' \times_X N \arrow[r, "\mathrm{pr}_N"] \arrow[d] & N \arrow[d] \\
X & X' \arrow[l, "F_X"'] \arrow[r, equal] & X' \arrow[r, "F_X"] & X
\end{tikzcd}$$

From the diagram we can easily see that

$$\begin{aligned}
F_X^*\omega_{N/X} &= F_X^* e^* \Omega^1_{N/X} \\
&= (e \circ F_X)^* \Omega^1_{N/X} \\
&= (\mathrm{pr}_N \circ e_{X'})^* \Omega^1_{N/X} \\
&= e_{X'}^* \mathrm{pr}_N^* \Omega^1_{N/X} \\
&\cong e_{X'}^* \Omega^1_{X' \times_X N/X'} \\
&\cong \psi_{X'}^* \Omega^1_{X' \times_X P/X'} \\
&\cong \psi^* \Omega^1_{P/X} \\
&\to \Omega^1_{X'/X}
\end{aligned}$$

which identifies with an element in $\mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S})$. We now show that the third and second descriptions agree.

We know that the group action of $N(X'')$ on $P(X'')$ is the same as $N_{X''}(X'')$ on $P_{X''}(X'')$. Through $X'' \overset{\mathrm{pr}_1}{\to} X'$, we pullback $\xi : X' \times_X P \to X' \times_X N$ to $\xi'' : X'' \times_X P \to X'' \times_X N$ s.t. the morphism

$$X'' \overset{(\mathrm{id}_{X''},\psi\circ\mathrm{pr}_1)}{\to} X'' \times_X P \overset{\xi''}{\to} X'' \times_X N \overset{\mathrm{pr}_N}{\to} N$$

is the unit element in the group $N(X'')$.

Denote $g_1 = (\mathrm{id}_{X''}, \psi \circ \mathrm{pr}_1) : X'' \to X'' \times_X P$ and $g_2 = (\mathrm{id}_{X''}, \psi \circ \mathrm{pr}_2)$. By definition we have $\alpha \cdot g_1 = g_2$. Since ξ'' is an isomorphism of torsors, we know that

$$\alpha \cdot (\mathrm{pr}_N \circ \xi'' \circ g_1) = (\mathrm{pr}_N \circ \xi'' \circ g_2).$$

As $\mathrm{pr}_N \circ \xi'' \circ g_1$ is the unit element, we have

$$\alpha = \mathrm{pr}_N \circ \xi'' \circ g_2 : X'' \overset{g_2}{\to} X'' \times_X P \to X'' \times_X N \to N.$$

By a simple diagram chase in the following diagram:

$$\begin{tikzcd}
X'' \arrow[r, equal] \arrow[d, "g_2"] & X'' \arrow[d, "(\mathrm{pr}_1,\psi\circ\mathrm{pr}_2)"] & \\
X'' \times_X P \arrow[r, "(\mathrm{pr}_1,\mathrm{id}_P)"] \arrow[d, "\xi''"] & X' \times_X P \arrow[d, "\xi"] & \\
X'' \times_X N \arrow[r, "(\mathrm{pr}_1,\mathrm{id}_N)"] & X' \times_X N \arrow[r] & N
\end{tikzcd}$$

we know that $\alpha = (X'' \overset{(\mathrm{pr}_1,\psi\circ\mathrm{pr}_2)}{\to} X' \times_X P \xrightarrow{\xi} X' \times_X N \to N)$, now we can consider the following commutative diagram:

$$\begin{tikzcd}
X' \times_X X' \arrow[r, equal] \arrow[d, "(\mathrm{pr}_1,\psi\circ\mathrm{pr}_2)"] & X' \times_X X' \arrow[d, "((\mathrm{pr}_1,\psi\circ\mathrm{pr}_1),(\mathrm{pr}_1,\psi\circ\mathrm{pr}_2))"] & \underline{\mathrm{Spec}}_{\mathcal{O}_X} F_*(\mathcal{O}_{X'}[\Omega^1_{X'/X}]) \arrow[l] \arrow[d] \\
X' \times_X P \arrow[d, "\xi"] & (X' \times_X P) \times_{X'} (X' \times_X P) \arrow[l, "\mathrm{pr}_2"] \arrow[d, "\xi\times\xi"] & \underline{\mathrm{Spec}}_{\mathcal{O}_{X'}} h_{X',*}(\mathcal{O}_{X'\times_X P}[\Omega^1_{X'\times_X P/X'}]) \arrow[l] \arrow[d] \\
X' \times_X N \arrow[d] & (X' \times_X N) \times_{X'} (X' \times_X N) \arrow[l, "\mathrm{pr}_2"] \arrow[d] & \underline{\mathrm{Spec}}_{\mathcal{O}_{X'}} g_{X',*}(\mathcal{O}_{X'\times_X N}[\Omega^1_{X'\times_X N/X'}]) \arrow[l] \arrow[d] \\
N & N \times_X N \arrow[l, "\mathrm{pr}_2"] & \underline{\mathrm{Spec}}_{\mathcal{O}_X} g_*(\mathcal{O}_N[\Omega^1_{N/X}]) \arrow[l]
\end{tikzcd}$$

The verification of commutativity is straightforward. The only non-obvious square is the top right square. We may assume everything is affine, then the top right square corresponds to

$$\begin{tikzcd}
R' \otimes_R R' \arrow[r, "x_1\otimes x_2\mapsto(x_1x_2,x_1dx_2)"] & R'[\Omega^1_{R'/R}] \\
(R' \otimes_R S) \otimes_{R'} (R' \otimes_R S) \arrow[u, "a_1\otimes b_1\otimes a_2\otimes b_2"', "\mapsto a_1a_2\psi^{\#}(b_1)\otimes\psi^{\#}(b_2)"] \arrow[r, "a_1\otimes b_1\otimes a_2\otimes b_2\mapsto(a_1a_2\otimes b_1b_2,(a_1a_2\otimes b_1)d(1\otimes b_2))"'] & R' \otimes_R S[\Omega^1_{R'\otimes_R S/R'}] \arrow[u, "(x_1\otimes y_1,x_2\otimes y_2d(1\otimes y_3))"', "\mapsto(x_1\psi^{\#}(y_1),x_2\psi^{\#}(y_2)d\psi^{\#}(y_3))"]
\end{tikzcd}$$

So the commutativity can be checked directly. Clearly the red arrows correspond to the second description and the blue ones correspond to the third description, and they identify with the same element in $\mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S})$.

Next we show the first and second descriptions agree. Recall that in the first description we need $\Phi_N(P) \in \Gamma(X, \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S,\mathrm{cl}})$ to be the unique element s.t. $\Phi_N(P)|_{X'} = \partial\alpha$. So it suffices to check their agreement restricting to $\underline{X'}$ (we add underline to distinguish it from other X's).

We will compare them in $\mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'} \times_X N)^\vee, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'},\mathrm{cl}})$: the map induced by the first description will be called δ_α, and the one induced by the second description will be called ξ_α. We will show that for all V over $\underline{X'}$, and all $\phi \in \mathrm{Hom}_{V-\mathrm{gr}}(V \times_X N, \mathbb{G}_{m,V})$, we have

$$\delta_\alpha(V)(\phi) = \xi_\alpha(V)(\phi)$$

which completes the proof.

δ_α from the first description:

From the first description we know ∂ is determined as following:

$$\begin{tikzcd}
F_*N_{X'}(\underline{X'}) \arrow[r, dotted, "\partial"] \arrow[d, phantom, sloped, "\cong"] & \Gamma(\underline{X'}, \omega^\vee_{N/X} \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S,\mathrm{cl}}) \arrow[d, phantom, sloped, "\cong"] \\
\mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'} \times_X N)^\vee, F'_*\mathbb{G}_{m,\underline{X'}\times_X X'}) \arrow[r, "\mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'}\times_X N)^\vee, d\ln)"] & \mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'} \times_X N)^\vee, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'},\mathrm{cl}})
\end{tikzcd}$$

The isomorphism on the left column is

$$\begin{aligned}F_*N_{X'}(\underline{X'}) &\cong N_{\underline{X'}\times_X X'}(\underline{X'}\times_X X')\\ &\cong N^{\vee\vee}_{\underline{X'}\times_X X'}(\underline{X'}\times_X X')\\ &= \mathrm{Hom}_{\underline{X'}\times_X X'-\mathrm{gr}}((\underline{X'}\times_X X'\times_X N)^{\vee}, \mathbb{G}_{m,\underline{X'}\times_X X'})\\ &\cong \mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'}\times_X N)^{\vee}, F'_*\mathbb{G}_{m,\underline{X'}\times_X X'})\end{aligned}$$

and the isomorphism on the right column is proved in [23, Chap. 1, 1.4].

So in particular $\alpha \in N(\underline{X'}\times_X X')$ induces an element $\mathrm{ev}_\alpha \in N^{\vee\vee}_{\underline{X'}\times_X X'}(\underline{X'}\times_X X')$ which further induces an element $\epsilon_\alpha \in \mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'}\times_X N)^{\vee}, F'_*\mathbb{G}_{m,\underline{X'}\times_X X'})$. The complete description of ϵ_α is as following:

Given $s : V \to \underline{X'}$, denote $q = s\times\mathrm{id}_{X'} : V\times_X X' \to \underline{X'}\times_X X'$ and $q' = (\mathrm{pr}_V, \alpha\circ q) : V\times_X X' \to V\times_X N$, we have

$$\begin{array}{rcl}\epsilon_\alpha(V) : (\underline{X'}\times_X N)^{\vee}(V) & \to & F'_*\mathbb{G}_{m,\underline{X'}\times_X X'}(V)\\ \mathrm{Hom}_{V-\mathrm{gr}}(V\times_X N, \mathbb{G}_{m,V}) & \to & \mathbb{G}_{m,\underline{X'}\times_X X'}(V\times_X X')\\ \mathrm{Hom}_{V-\mathrm{gr}}(V\times_X N, \mathbb{G}_{m,V}) & \to & \mathcal{O}_{V\times_X X'}(V\times_X X')^{\times}\\ \phi & \mapsto & q'^{\#}\circ\phi'^{\#}(t)\end{array}$$

where ϕ induces $\phi' \in \mathrm{Hom}(V\times_X N, \mathbb{G}_m)$ and $\phi'^{\#} : \mathbb{Z}[t,t^{-1}] \to \mathcal{O}_{V\times_X N}(V\times_X N)$.

In other words, for given V, we have that α induces a map $q' : V\times_X X' \to V\times_X N$. Then given a group homomorphism $\phi : V\times_X N \to \mathbb{G}_{m,V}$, we use the following map to get an element in $\mathcal{O}_{V\times_X X'}(V\times_X X')^{\times}$ from $t \in \mathbb{Z}[t,t^{-1}]$.

$$\begin{tikzcd}V\times_X X' \arrow[r, "q'"] & V\times_X N \arrow[r, "\phi"] \arrow[rr, bend right, "\phi'"'] & \mathbb{G}_{m,V} \arrow[r] & \mathbb{G}_m\end{tikzcd}$$

After appending the $d\ln$ map, we have $\delta_\alpha \in \mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'}\times_X N)^{\vee}, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'},\mathrm{cl}})$ whose description is as following:

$$\begin{array}{rcl}\delta_\alpha(V) : (\underline{X'}\times_X N)^{\vee}(V) & \to & F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'},\mathrm{cl}}(V)\\ \mathrm{Hom}_{V-\mathrm{gr}}(V\times_X N, \mathbb{G}_{m,V}) & \to & \Gamma(V, F'_{V,*}\Omega^1_{V\times_X X'/V,\mathrm{cl}})\\ \phi & \mapsto & (q'^{\#}\circ\phi'^{\#}(t))^{-1}d(q'^{\#}\circ\phi'^{\#}(t))\end{array}$$

where $F'_V = \mathrm{id}_V\times F_X : V\times_X X' \to V$. Clearly we can replace $\mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'}\times_X N)^{\vee}, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'},\mathrm{cl}})$ by $\mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'}\times_X N)^{\vee}, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'}})$.

So we have

$$\delta_\alpha(V)(\phi) = (q'^{\#}\circ\phi'^{\#}(t))^{-1}d(q'^{\#}\circ\phi'^{\#}(t)).$$

ξ_α from the second description:

In the second description we know the following map (the red arrow)

$$\underline{\mathrm{Spec}}_{\mathcal{O}_X} F_*(\mathcal{O}_{X'}[\Omega^1_{X'/X}]) \to X'\times_X X' \overset{\alpha}{\to} N$$

corresponds to an element in $\Gamma(X, \omega^{\vee}_{N/X} \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S}) = \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S})$. After pulling back via $\underline{X'} \overset{F_X}{\rightarrow} X$, denote $F' : \underline{X'} \times_X X' \to \underline{X'}$ the projection of the left factor, the following map

$$\underline{\mathrm{Spec}}_{\mathcal{O}_{\underline{X'}}} F'_*(\mathcal{O}_{\underline{X'}\times_X X'}[\Omega^1_{\underline{X'}\times_X X'/\underline{X'}}]) \to (\underline{X'}\times_X X')\times_{\underline{X'}}(\underline{X'}\times_X X') \cong \underline{X'}\times_X(X'\times_X X') \overset{\mathrm{id}_{\underline{X'}}\times\alpha}{\rightarrow} \underline{X'}\times_X N$$

corresponds to an element in $\Gamma(\underline{X'}, \omega^{\vee}_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S}) = \mathrm{Hom}_{\mathcal{O}_{\underline{X'}}}(\omega_{N_{\underline{X'}}/\underline{X'}}, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'}})$. And we claim this process is compatible with the restriction map

$$\Gamma(X, \omega^{\vee}_{N/X} \otimes_{\mathcal{O}_X} F_*\Omega^1_{X'/S}) \to \Gamma(\underline{X'}, \omega^{\vee}_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S})$$

and we omit the proof because it's a straightforward calculation.

By [23, Chap. 1, 1.4], we have an isomorphism

$$\mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'} \times_X N)^{\vee}, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'}}) \cong \mathrm{Hom}_{\mathcal{O}_{\underline{X'}}}(\omega_{\underline{X'}\times_X N}, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'}}).$$

So the above map induces an element $\xi_\alpha \in \mathrm{Hom}_{\underline{X'}-\mathrm{gr}}((\underline{X'} \times_X N)^{\vee}, F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'}})$ in this way:

Given $s : V \to \underline{X'}$, and $\phi \in (\underline{X'} \times_X N)^{\vee}(V) = \mathrm{Hom}_{V-\mathrm{gr}}(V \times_X N, \mathbb{G}_{m,V})$, it induces a map

$$\begin{tikzcd}
\underline{\mathrm{Spec}}_{\mathcal{O}_V} F'_{V,*}(\mathcal{O}_{V\times_X X'}[\Omega^1_{V\times_X X'/V}]) \arrow[rr] \arrow[d, "\tau"] & & (V \times_X X') \times_V (V \times_X X') \arrow[dll, "\sim"'] & \\
V \times_X (X' \times_X X') \arrow[r, "\mathrm{id}_V\times\alpha"] & V \times_X N \arrow[r, "\phi"] \arrow[rr, bend right, "\phi'"'] & \mathbb{G}_{m,V} \arrow[r] & \mathbb{G}_m
\end{tikzcd}$$

and we get an element $\xi_\alpha(V)(\phi) \in \Gamma(V, F'_{V,*}\Omega^1_{V\times_X X'/V})$ from $t \in \mathbb{Z}[t, t^{-1}]$, i.e.

$$\begin{array}{rcl}
\xi_\alpha(V) : (\underline{X'} \times_X N)^{\vee}(V) & \to & F'_*\Omega^1_{\underline{X'}\times_X X'/\underline{X'}}(V) \\
\mathrm{Hom}_{V-\mathrm{gr}}(V \times_X N, \mathbb{G}_{m,V}) & \to & \Gamma(V, F'_{V,*}\Omega^1_{V\times_X X'/V}) \\
\phi & \mapsto & \tau^{\#}(\phi'^{\#}(t))
\end{array}$$

So we have

$$\xi_\alpha(V)(\phi) = \tau^{\#}(\phi'^{\#}(t)).$$

Compare $\delta_\alpha(V)(\phi)$ and $\xi_\alpha(V)(\phi)$:

And now we just need to check that $(q'^{\#} \circ \phi'^{\#}(t))^{-1}d(q'^{\#} \circ \phi'^{\#}(t)) = \tau^{\#}(\phi'^{\#}(t))$ using the fact $\alpha|_{13} - \alpha|_{12} = \alpha|_{23}$.

Consider the following two different paths:

$$\begin{tikzcd}
 & \underline{\mathrm{Spec}}_{\mathcal{O}_V} F'_{V,*}(\mathcal{O}_{V\times_X X'}[\Omega^1_{V\times_X X'/V}]) \arrow[d] & \\
 & (V\times_X X')\times_V (V\times_X X') \arrow[dl, red, "q'^{-1}\times q'"'] \arrow[dr, blue, "\sim", "\omega"'] & \\
(V\times_X N)\times_V (V\times_X N) \arrow[dr, red, "m_{V\times_X N}"] & & V\times_X (X'\times_X X') \arrow[dl, blue, "\mathrm{id}_V\times\alpha"] \\
 & V\times_X N \arrow[d, "\phi'"] & \\
 & \mathbb{G}_m &
\end{tikzcd}$$

Clearly the path with the blue arrows induces $\tau^{\#}(\phi'^{\#}(t))$. We claim that the element $(q'^{\#}\circ\phi'^{\#}(t))^{-1}d(q'^{\#}\circ\phi'^{\#}(t))$ is induced by the path with the red arrows, and later we will show that $m_{V\times_X N}\circ(q'^{-1}\times q') = (\mathrm{id}_V\times\alpha)\circ\omega$ and then the result follows.

Assume everything is affine, $\phi\in(\underline{X'}\times_X N)^{\vee}(V) = (V\times_X N)^{\vee}(V)$ identifies with a group-like element in $\mathcal{O}_{V\times_X N}(V\times_X N)$ (i.e. a group-like element $x\in\mathcal{O}_{V\times_X N}(V\times_X N)$ is a unit in $\mathcal{O}_{V\times_X N}(V\times_X N)$ such that $m^{\#}_{V\times_X N}(x) = x\otimes x$), hence $m^{\#}_{V\times_X N}(\phi'^{\#}(t)) = \phi'^{\#}(t)\otimes\phi'^{\#}(t)$. Also we have $(q'^{-1})^{\#}\circ\phi'^{\#}(t) = (q'^{\#}\circ\phi'^{\#}(t))^{-1}$. Hence the above path with the red arrows actually induces $\phi\mapsto(q'^{\#}\circ\phi'^{\#}(t))^{-1}d(q'^{\#}\circ\phi'^{\#}(t))$.

If X is not affine, pick an open affine covering $\{X_i\}_{i\in I}$ of X. For each $i\in I$, pick an open affine covering $\{V_{ij}\}_{j\in J_i}$ for $s^{-1}(X_i)\subset V$. Let $N_i\subset N$ be the preimage of X_i in N, N_i is affine as N is affine over X. We can see that $V\times_X X'$ is covered by open affines $\{V_{ij}\times_{X_i} X'_i\}_{i\in I, j\in J_j}$. Restricting to each $V_{ij}\times_{X_i} X'_i$, we know the above path with the red arrows induces $\phi\mapsto(q'^{\#}\circ\phi'^{\#}(t))^{-1}d(q'^{\#}\circ\phi'^{\#}(t))$. The first claim then follows from the identity axiom of sheaves.

The final statement $m_{V\times_X N}\circ(q'^{-1}\times q') = (\mathrm{id}_V\times\alpha)\circ\omega$ follows from the following commutative diagram where top row represents $m_{V\times_X N}\circ(q'^{-1}\times q')$ and the bottom row represents $(\mathrm{id}_V\times\alpha)\circ\omega$:

$$\begin{tikzcd}
(V\times_X X')\times_V (V\times_X X') \arrow[r, "q'^{-1}\times q'"] \arrow[d] & (V\times_X N)\times_V (V\times_X N) \arrow[r, "m_{V\times_X N}"] \arrow[d] & V\times_X N \arrow[d] \\
(\underline{X'}\times_X X')\times_{\underline{X'}} (\underline{X'}\times_X X') \arrow[r, "(\mathrm{pr}_1{,}\alpha^{-1})\times(\mathrm{pr}_1{,}\alpha)"] \arrow[d, equal] & (\underline{X'}\times_X N)\times_{\underline{X'}} (\underline{X'}\times_X N) \arrow[r, "m_{\underline{X'}\times_X N}"] & \underline{X'}\times_X N \arrow[d, equal] \\
(\underline{X'}\times_X X')\times_{\underline{X'}} (\underline{X'}\times_X X') \arrow[r] & \underline{X'}\times_X (X'\times_X X') \arrow[r, "\mathrm{id}_{\underline{X'}}\times\alpha"] & \underline{X'}\times_X N \\
(V\times_X X')\times_V (V\times_X X') \arrow[r, "\omega"] \arrow[u] & V\times_X (X'\times_X X') \arrow[r, "\mathrm{id}_V\times\alpha"] \arrow[u] & V\times_X N \arrow[u]
\end{tikzcd}$$

the middle commutative square represents the fact $-\alpha|_{12}+\alpha|_{13} = \alpha|_{23}$, and the first row (resp. the fourth row) is the base change of the second row (resp. the third row) w.r.t. $V\to\underline{X'}$. So we have $m_{V\times_X N}\circ(q'\times q'^{-1}) = (\mathrm{id}_V\times\alpha)\circ\omega$ and hence $\xi_\alpha = \delta_\alpha$. So the first and second descriptions agree.

Now we can conclude that all the three descriptions of Φ_N are valid.

Example 2.7. If $N = \mu_p : (\ker(\mathbb{G}_m \overset{p\text{th power}}{\to} \mathbb{G}_m))$, and X is the spectrum of a UFD (unique factorization domain), so $\mathrm{H}^1(X, \mathbb{G}_m) = \mathrm{Pic}(X) = 0$ and we have $\mathrm{H}^1(X, \mu_p) = \frac{\mathcal{O}_X(X)^\times}{(\mathcal{O}_X(X)^\times)^p}$. Also $\omega_{\mu_p/X} \cong \mathcal{O}_X$, and Φ_N is

$$\begin{aligned} \frac{\mathcal{O}_X(X)^\times}{(\mathcal{O}_X(X)^\times)^p} &\to \Omega^1_{X/S,\mathrm{cl}} \\ \overline{a} &\mapsto \frac{da}{a} \end{aligned}$$

Proof. This example is also in [15, III, 5.9]. A complete proof of this statement is consisted of lemma 3.3, lemma 3.4 and lemma 3.5. ☐

2.3. Functoriality lemmas

After we stated the three descriptions of the injection

$$\Phi_N : \mathrm{H}^1(X, N) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$$

and proved their equivalences, we can prove some functoriality lemmas.

Lemma 2.8. *The map Φ_N is functorial over N, i.e. if $f : N \to M$ is a morphism of finite flat commutative group schemes of height 1 over X, then there exists a commutative diagram as following:*

$$\begin{tikzcd}
\mathrm{H}^1(X, N) \arrow[r, hook, "\Phi_N"] \arrow[d, "f'"] & \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) \arrow[d, "i"] \\
\mathrm{H}^1(X, M) \arrow[r, hook, "\Phi_M"] & \mathrm{Hom}_{\mathcal{O}_X}(\omega_{M/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})
\end{tikzcd}$$

Proof. Here $f' : \mathrm{H}^1(X, N) \to \mathrm{H}^1(X, M)$ is the canonical map induced by $f : N \to M$. We know an element of $\mathrm{H}^1(X, N) = \check{\mathrm{H}}^1(X'/X, N)$ is a 1-cocycle for $\{X' \to X\}$ of the form $\alpha \in N(X'')$ s.t. $\alpha|_{12} + \alpha|_{23} = \alpha|_{13}$ under $\mathrm{pr}_{12}, \mathrm{pr}_{13}, \mathrm{pr}_{23} : X''' \to X''$. Then $f \circ \alpha \in M(X'')$ satisfies similar relation as f is a morphism of group schemes and hence preserves group actions, so $f \circ \alpha = f'(\alpha) \in \mathrm{H}^1(X, M) = \check{\mathrm{H}}^1(X'/X, M)$.

Let $\mathcal{I}$ and $\mathcal{J}$ be the augmentation ideal of N and M respectively, then $i : \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) \to \mathrm{Hom}_{\mathcal{O}_X}(\omega_{M/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$ is induced by the map $\omega_{M/X} \cong \frac{\mathcal{J}}{\mathcal{J}^2} \to \frac{\mathcal{I}}{\mathcal{I}^2} \cong \omega_{N/X}$.

Let $\alpha \in \mathrm{H}^1(X, N)$, $\Phi_N(\alpha)$ is induced by $\mathcal{O}_N \overset{\alpha^\#}{\to} \mathcal{O}_{X''} \to \mathcal{O}_{X'}[\Omega^1_{X'/X}]$, and $i \circ \Phi_N(\alpha)$ is induced by $\mathcal{O}_M \overset{f^\#}{\to} \mathcal{O}_N \overset{\alpha^\#}{\to} \mathcal{O}_{X''} \to \mathcal{O}_{X'}[\Omega^1_{X'/X}]$ which also induces the map $\Phi_M(f \circ \alpha)$ as $\alpha^\# \circ f^\# = (f \circ \alpha)^\#$. Thus $i \circ \Phi_N = \Phi_M \circ f'$. ☐

Lemma 2.9. *Φ_N is functorial over X, i.e. if Y is another scheme over S s.t. $F_Y : Y' \to Y$ has a rank one p base Zariski locally, and $f : Y \to X$ is a morphism of schemes over S, then there exists a commutative diagram as following:*

$$\begin{tikzcd}
\mathrm{H}^1(X, N) \arrow[r, hook, "\Phi_N"] \arrow[d, "f'"] & \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) \arrow[d, "i"] \\
\mathrm{H}^1(Y, N_Y) \arrow[r, hook, "\Phi_{N_Y}"] & \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, F_{Y,*}\Omega^1_{Y'/S,\mathrm{cl}})
\end{tikzcd}$$

Proof. Let $\alpha \in \mathrm{H}^1(X, N) = \check{\mathrm{H}}^1(X'/X, N)$, then it induces the map $(X'' \times_X Y \overset{\alpha \times \mathrm{id}_Y}{\to} N \times_X Y) \in \check{\mathrm{H}}^1(Y \times_X X'/Y, N_Y)$. Clearly the covering $\{Y' \overset{F_Y}{\to} Y\}$ is a refinement of $\{Y \times_X X' \to Y\}$, then it induces the map $(Y'' \to Y \times_X X'' \overset{\mathrm{id}_Y \times \alpha}{\to} Y \times_X N) \in \check{\mathrm{H}}^1(Y'/Y, N) = \mathrm{H}^1(Y, N)$, which is exactly $f'(\alpha)$.

Denote $F' : Y \times_X X' \to Y$ the projection map on the first factor. The map i is the following composition:

$$\mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) \to \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, F'_*\Omega^1_{X' \times_X Y/Y,\mathrm{cl}}) \to \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, F_{Y,*}\Omega^1_{Y'/S,\mathrm{cl}})$$

where the first map is just a pull-back and the second map is induced by $F'_*\Omega^1_{X'\times_X Y/Y,\mathrm{cl}} \to F_{Y,*}\Omega^1_{Y'/Y,\mathrm{cl}}$. The above composition of maps can embed into the following composition

$$\mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S}) \xrightarrow{j} \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, F'_*\Omega^1_{X'\times_X Y/Y}) \xrightarrow{h} \mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, F_{Y,*}\Omega^1_{Y'/S})$$

Using the second description of $\Phi_N(\alpha)$, $\Phi_N(\alpha)$ is induced by the map

$$\underline{\mathrm{Spec}}_{\mathcal{O}_X}\, F_*(\mathcal{O}_{X'}[\Omega^1_{X'/X}]) \to X' \times_X X' \xrightarrow{\alpha} N$$

and $j \circ \Phi_N(\alpha)$ is induced by the map

$$\underline{\mathrm{Spec}}_{\mathcal{O}_Y}\, F'_*(\mathcal{O}_{Y\times_X X'}[\Omega^1_{Y\times_X X'/Y}]) \to (Y \times_X X') \times_Y (Y \times_X X') \cong Y \times_X (X' \times_X X') \xrightarrow{\mathrm{id}_Y \times \alpha} Y \times_X N$$

Then $h \circ j \circ \Phi_N(\alpha)$ is induced by the map

$$\underline{\mathrm{Spec}}_{\mathcal{O}_Y}\, F_{Y,*}(\mathcal{O}_{Y'}[\Omega^1_{Y'/Y}]) \to \underline{\mathrm{Spec}}_{\mathcal{O}_Y}\, F'_*(\mathcal{O}_{Y\times_X X'}[\Omega^1_{Y\times_X X'/Y}]) \to Y \times_X X'' \xrightarrow{\mathrm{id}_Y \times \alpha} Y \times_X N$$

From the other direction, $\Phi_{N_Y} \circ f'(\alpha)$ is induced by the map

$$\underline{\mathrm{Spec}}_{\mathcal{O}_Y}\, F_{Y,*}(\mathcal{O}_{Y'}[\Omega^1_{Y'/Y}]) \to Y'' \to Y \times_X X'' \xrightarrow{\mathrm{id}_Y \times \alpha} Y \times_X N$$

So the result follows from the following trivial commutative diagram:

$$\begin{tikzcd}
\underline{\mathrm{Spec}}_{\mathcal{O}_Y}\, F_{Y,*}(\mathcal{O}_{Y'}[\Omega^1_{Y'/Y}]) \arrow[r] \arrow[d] & Y'' \arrow[d] \\
\underline{\mathrm{Spec}}_{\mathcal{O}_Y}\, F'_*(\mathcal{O}_{Y\times_X X'}[\Omega^1_{Y\times_X X'/Y}]) \arrow[r] & Y \times_X X''
\end{tikzcd}$$

□

If moreover $f : Y \to X$ has isomorphism relative Frobenius morphism, the group $\mathrm{Hom}_{\mathcal{O}_Y}(\omega_{N_Y/Y}, F_{Y,*}\Omega^1_{Y'/S,\mathrm{cl}})$ can be written as $\mathrm{Hom}_{\mathcal{O}_Y}(f^*\omega_{N/X}, f^*F_{X,*}\Omega^1_{X'/S,\mathrm{cl}})$, see fact 6.

Corollary 2.10. *If $f : Y \to X$ is a morphism of schemes over S with isomorphism relative Frobenius morphism. Denote $\mathcal{G}$ by the fppf sheaf on X generated by the $\mathcal{O}_X$-module $\omega_{N/X} \otimes_{\mathcal{O}_X} F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}$. Due to fact 6, the above commutative diagram can be written as:*

$$\begin{tikzcd}
\mathrm{H}^1(X, N) \arrow[r, hook, "\Phi_N"] \arrow[d, "f'"] & \Gamma(X, \mathcal{G}) \arrow[d, "i"] \\
\mathrm{H}^1(Y, N_Y) \arrow[r, hook, "\Phi_{N_Y}"] & \Gamma(Y, \mathcal{G})
\end{tikzcd}$$

2.4. A variant form of the second exact sequence

Note that we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S,\mathrm{cl}}) \hookrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\omega_{N/X}, F_{X,*}\Omega^1_{X'/S}) \cong \mathrm{Hom}_{\mathcal{O}_{X'}}(F_X^*\omega_{N/X}, \Omega^1_{X'/S}).$$

If we identify X' with X, then $\Omega^1_{X'/S} = \Omega^1_{X/S}$ so the last group can be written as $\mathrm{Hom}_{\mathcal{O}_X}(F_X^*\omega_{N/X}, \Omega^1_{X/S})$.

Moreover, we have

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_X}(F_X^*\omega_{N/X}, \Omega^1_{X/S})\\
=&\Gamma(X, \mathit{Hom}_{\mathcal{O}_X}(F_X^*\omega_{N/X}, \Omega^1_{X/S}))\\
=&\Gamma(X, (F_X^*\omega_{N/X})^\vee \otimes_{\mathcal{O}_X} \Omega^1_{X/S})\\
&\text{since } \omega_{N/X} \text{ is finite locally free}\\
=&\Gamma(X, F_X^*\omega_{N/X}^\vee \otimes_{\mathcal{O}_X} \Omega^1_{X/S})
\end{aligned}$$

Since we are going to deal with $\Omega^1_{X/S}(E)$, we better transform the second exact sequence and their corresponding functoriality lemma into the form s.t. the target group is of the form $\mathrm{Hom}_{\mathcal{O}_X}(F_X^*\omega_{N/X}, \Omega^1_{X/S}) = \Gamma(X, F_X^*\omega_{N/X}^\vee \otimes_{\mathcal{O}_X} \Omega^1_{X/S})$.

Corollary 2.11. *Let $\pi : X \to S$ be a morphism of schemes of characteristic p where S is perfect and X is locally Noetherian. Let N be a finite flat commutative group scheme over X of height 1. Suppose $F_X : X' \to X$ has a rank one p base Zariski locally. Denote $\tilde{\mathcal{F}}_N$ by the fppf sheaf on X generated by the $\mathcal{O}_X$-module $\mathcal{F}_N := F_X^*\omega_{N/X}^\vee \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$, then we have an injection of abelian groups*

$$\Psi_N : \mathrm{H}^1(X, N) \to \Gamma(X, \tilde{\mathcal{F}}_N).$$

Proof. It follows from Corollary 2.4 and the above discussion. ☐

Corollary 2.12. *Ψ_N is functorial over N, i.e. if $f : N \to M$ is a morphism of finite flat commutative group schemes of height 1 over X, then there exists a commutative diagram as following:*

$$\begin{tikzcd}
\mathrm{H}^1(X, N) \arrow[r, hook, "\Psi_N"] \arrow[d, "f'"] & \Gamma(X, \tilde{\mathcal{F}}_N) \arrow[d, "i"] \\
\mathrm{H}^1(X, M) \arrow[r, hook, "\Psi_M"] & \Gamma(X, \tilde{\mathcal{F}}_M)
\end{tikzcd}$$

Proof. The proof is identical to that of lemma 2.8 with $\Omega^1_{X'/S,\mathrm{cl}}$ replaced by $\Omega^1_{X'/S}$. ☐

The above corollary will be used in the diagram 3.7.

Corollary 2.13. *Ψ_N is functorial over X in the sense that if $f : Y \to X$ is a morphism of schemes over S with isomorphism relative Frobenius morphism, then $F_Y : Y' \to Y$ has a rank one p base Zariski locally and we have the following commutative diagram*

$$\begin{tikzcd}
\mathrm{H}^1(X, N) \arrow[r, hook, "\Psi_N"] \arrow[d, "f'"] & \Gamma(X, \tilde{\mathcal{F}}_N) \arrow[d, "i"] \\
\mathrm{H}^1(Y, N_Y) \arrow[r, hook, "\Psi_{N_Y}"] & \Gamma(Y, \tilde{\mathcal{F}}_N)
\end{tikzcd}$$

Proof. We can use fact 6 to identify $\Gamma(Y, \tilde{\mathcal{F}}_N)$ with $\mathrm{Hom}_{\mathcal{O}_Y}(F_Y^*\omega_{N_Y/Y}, \Omega^1_{Y/S})$. Then it is a direct result of corollary 2.10. ☐

The above corollary will be used in diagram 3.1 and diagram 3.7.

3. The main theorem

For any S-group scheme G, denote the unit section as $e_{G/S} : G \to S$ and let $\omega_{G/S} := e^*\Omega^1_{G/S}$.

From now on, all cohomology groups are of *fppf* sheaves unless explicitly specified. Now we are ready to state our main theorem.

Theorem 3.1. *There is a natural injective homomorphism*

$$\phi : \mathrm{Sel}^{(F_{A/K})}(A/K) \hookrightarrow \mathrm{Hom}_C(F_C^*\omega_{\ker F_{\mathcal{A}/C}/C}, \Omega^1_{C/k_0}(E)).$$

And this map ϕ is functorial for morphisms of semi-abelian schemes over C.

This theorem is also proved in Corollaire 1.5 in D. Rössler's paper [3], and this thesis provides an alternative proof.

Proof. As $\mathcal{A}$ is smooth over C, in particular locally of finite type, we know $F_{\mathcal{A}/C}$ is finite by [5, tag 0CCD] so $\ker F_{\mathcal{A}/C}$ is finite over C as the base change of $F_{\mathcal{A}/C}$ by the unit section. By lemma 1.25, we know $F_{\mathcal{A}/C}$ is flat so $\ker F_{\mathcal{A}/C}$ is flat over C. Together with lemma 1.28, we know $\ker F_{\mathcal{A}/C}$ is a finite flat commutative group scheme of height one over C. Since C is smooth over the perfect field k_0, $F_C = F_{C/k_0}$ has a rank one p base Zariski locally, see [19, 1.1 & 1.2] and [5, tag 0CCD]. So we can apply the second exact sequence and utilize its functoriality.

Denote $\tilde{\mathcal{F}}$ by the fppf sheaf on C generated by the finite locally free module $\mathcal{F} := F_C^*\omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}$ (the finite locally freeness of $\omega^\vee_{\ker F_{\mathcal{A}/C}/C}$ is in Section 2 and the finite locally freeness of Ω^1_{C/k_0} comes from the fact that C is smooth over k_0).

Pick any closed point $v \in C$, apply fact 7 and Corollary 2.13 to the commutative square

$$\begin{tikzcd}
C & \mathrm{Spec}\, K \arrow[l, "\eta"'] \\
\mathrm{Spec}\, \hat{\mathcal{O}}_v \arrow[u] & \mathrm{Spec}\, K_v \arrow[l] \arrow[u]
\end{tikzcd}$$

we have the following commutative diagram, where the cohomology groups are for *fppf*-sheaves, the rows are injective,

$$\begin{tikzcd}
 & \mathrm{H}^1(K, \ker F_{A/K}) \arrow[rr, hook, "\gamma_1"] \arrow[dd, hook, "\lambda_4"] & & \mathcal{F}_\eta \arrow[dd, hook, "\mu_4"] \\
\mathrm{H}^1(C, \ker F_{\mathcal{A}/C}) \arrow[ur, hook, "\lambda_1"] \arrow[rr, hook, "\gamma_2"] \arrow[dr, hook, "\lambda_2"] \arrow[dd, hook, "\lambda_5"] & & \mathcal{F}(C) \arrow[ur, hook, "\mu_1"] \arrow[dr, hook, "\mu_2"] \arrow[dd, hook, "\mu_5"] & \\
 & \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[rr, hook, "\gamma_3"] & & \tilde{\mathcal{F}}(K_v) \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "\lambda_3"] \arrow[rr, hook, "\gamma_4"] & & \tilde{\mathcal{F}}(\hat{\mathcal{O}}_v) \arrow[ur, hook, "\mu_3"] &
\end{tikzcd}$$

(3.1)

Next we show the arrows on the right plane are all injective, the above right plane is just:

$$\begin{tikzcd}
\mathcal{F}(C) \arrow[r, "\mu_1"] \arrow[d, "\mu_5"] & \mathcal{F}_\eta \arrow[d, "\mu_4"] \\
\tilde{\mathcal{F}}(\hat{\mathcal{O}}_v) \arrow[r, "\mu_3"] & \tilde{\mathcal{F}}(K_v)
\end{tikzcd}$$

As $\mathcal{F}$ is locally free and C is an integral scheme, all restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ and $\mathcal{F}(U) \to \mathcal{F}_x$ are injective. So μ_1 is injective. μ_5 is a composition of $\mathcal{F}(C) \to \mathcal{F}_v$ and $\mathcal{F}_v \to \mathcal{F}_v \otimes_{\mathcal{O}_v} \hat{\mathcal{O}}_v$, the former is injective, and later is the base change of the injective ring map $\mathcal{O}_v \to \hat{\mathcal{O}}_v$ w.r.t. the finite free module $\mathcal{F}_v$ so also injective, hence μ_5 is injective. Similarly μ_3 (resp. μ_4) is the base change of the injective map $\hat{\mathcal{O}}_v \to K_v$ (resp. $K \to K_v$) w.r.t. the finite free module $\mathcal{F}_v$ (resp. $\mathcal{F}_\eta$), so they are also injective. Now the left plane embeds into the right plane, by a simple diagram chase we can show every arrow in the left plane is injective.

We have

$$\begin{aligned}
\operatorname{Hom}_C(\omega_{\ker F_{\mathcal{A}/C}/C}, F_{C,*}(\Omega^1_{C/k_0}(E))) &= \Gamma(C, \mathcal{H}om_{\mathcal{O}_C}(\omega_{\ker F_{\mathcal{A}/C}/C}, F_{C,*}(\Omega^1_{C/k_0}(E))) \\
&= \Gamma(C, \mathcal{H}om_{\mathcal{O}_C}(F_C^*\omega_{\ker F_{\mathcal{A}/C}/C}, \Omega^1_{C/k_0}(E)) \\
&= \Gamma(C, (F_C^*\omega_{\ker F_{\mathcal{A}/C}/C})^\vee \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}(E)) \\
&\text{since } \omega_{\ker F_{\mathcal{A}/C}/C} \text{ is finite locally free} \\
&= \Gamma(C, F_C^*\omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}(E)) \\
&= \Gamma(C, \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}(E)) \\
&\mathcal{O}(E) \text{ is an invertible module so finite locally free} \\
&\subset (\mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}(E))_\eta \\
&= \mathcal{F}_\eta
\end{aligned}$$

and our target is to show that the image of $\mathrm{Sel}^{(F_{A/K})}(A/K)$ under γ_1 lies in $\Gamma(C, \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}(E)) = \bigcap_{v \in C} \mathcal{F}_v \otimes_{\mathcal{O}_v} \mathcal{O}(E)_v \subset \mathcal{F}_\eta$.

Let $T \in \mathrm{Sel}^{(F_{A/K})}(A/K)$, equivalently, we want to show

$$\forall v \in C, \gamma_1(T) \in \mathcal{F}_v \otimes_{\mathcal{O}_v} \mathcal{O}(E)_v.$$

When $v \notin E$, $\mathcal{O}(E)_v = \mathcal{O}_v$. When $v \in E$, $\mathcal{O}(E)_v = t_v^{-1}\mathcal{O}_v$ where $t_v \in \mathcal{O}_v$ is a uniformiser. So it suffices to show that

$$\begin{cases} \gamma_1(T) \in \mathcal{F}_v & \forall v \in C \backslash \{E \cup \eta\} \\ t_v \cdot \gamma_1(T) \in \mathcal{F}_v & \forall v \in E \end{cases}$$

As $\mathcal{F}_\eta$ is a finite free K-module, it suffices to check the order of each component s_i of $\gamma_1(T)$ in $\mathcal{F}_\eta \cong K^n$ with respect to $\mathcal{O}_v$, which can also be checked in K_v. Thus it suffices to show that

$$\begin{cases} \gamma_3(\lambda_4(T)) \in \tilde{\mathcal{F}}(\hat{\mathcal{O}}_v) = \operatorname{Im} \mu_3 & \forall v \in C \backslash \{E \cup \eta\} \\ t_v \cdot \gamma_3(\lambda_4(T)) \in \tilde{\mathcal{F}}(\hat{\mathcal{O}}_v) = \operatorname{Im} \mu_3 & \forall v \in E \end{cases} \tag{3.2}$$

And functoriality will be proved in the end.

Fix a closed point v. To simplify notation write $t = t_v$ and $s = \gamma_3(\lambda_4(T))$, so we want to show $s \in \operatorname{Im} \mu_3$ if $v \notin E$ and $t \cdot s \in \operatorname{Im} \mu_3$ if $v \in E$. The equation eventually happens in the bottom plane of above diagram 3.1, so we discard the top plane, and investigate the bottom plane (constructed by applying the second exact sequences to $\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}$) by inserting it into a sequence of planes (constructed by applying the second exact sequences to the sequence $\mu_{p,\hat{\mathcal{O}}_v}^{\oplus l} \to \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \to \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$), see diagram 3.7. Then we prove our theorem by diagram chasing.

To start, the Raynaud construction (see [4, II, 1]) provides an exact sequence of formal schemes over $\hat{\mathcal{O}}_v$, say

$$0 \longrightarrow \hat{G} \xrightarrow{\text{closed immersion}} \hat{\mathcal{A}}_{\hat{\mathcal{O}}_v} \xrightarrow{\text{faithfully flat}} \hat{\mathcal{B}} \longrightarrow 0 \tag{3.3}$$

where $\hat{\cdot}$ denotes formalization. Here $\mathcal{B}/\hat{\mathcal{O}}_v$ is an abelian scheme and G is a torus over $\hat{\mathcal{O}}_v$.

To explain this construction in short: Denote $S = \operatorname{Spec} \hat{\mathcal{O}}_v$, $S_n = \operatorname{Spec} \frac{\hat{\mathcal{O}}_v}{t^{n+1}\hat{\mathcal{O}}_v}$ and $\mathcal{A}_n = \mathcal{A}_{\hat{\mathcal{O}}_v} \times_S S_n$ for $n \geq 1$. By definition of semi-abelian schemes, there exists a short exact sequence $0 \to G_0 \to \mathcal{A}_0 \to \mathcal{B}_0 \to 0$ over $S_0 = \operatorname{Spec} \kappa(v)$ (note $\mathcal{O}_v$ and $\hat{\mathcal{O}}_v$ have the same residue field by [5, tag 05GG]), where G_0 is a torus and $\mathcal{B}_0$ is an abelian variety.

By [4, I, Thm. 2.2], we can lift the closed immersion $G_0 \to \mathcal{A}_0$ to $G_1 \to \mathcal{A}_1$ over S_1, i.e. there exists a torus G_1 embedding in $\mathcal{A}_1$ as a closed immersion and the embedding is a group homomorphism, and it descends to $G_0 \to \mathcal{A}_0$ after the base change $S_0 \to S_1$. The existence of the quotient $\mathcal{B}_1$ of $\mathcal{A}_1$ by G_1 is guaranteed by [13, 4.39], and we must have $\mathcal{B}_1 \times_{S_1} S_0 \cong \mathcal{B}_0$, see [13, 4.30].

Inductively we have an exact sequence of fppf-sheaves $0 \to G_n \to \mathcal{A}_n \to \mathcal{B}_n \to 0$ for all $n \geq 0$, which is also an exact sequence of commutative group schemes in the sense of definition 1.13 by remark 1.15. Results in [14, XII] ensure G_n is a maximal torus of $\mathcal{A}_n$ and $\mathcal{B}_n$ are abelian schemes. The existence of G and $\mathcal{B}$ over S is proved in [4, II, 1] using the fact $\hat{\mathcal{O}}_v$ is normal.

By definition, G splits (i.e. isomorphic to a product of finitely many copies of multiplicative group scheme $\mathbb{G}_m$) fpqc-locally. As $\hat{\mathcal{O}}_v$ is a Noetherian local ring, G splits after a finite étale base change $\hat{\mathcal{O}}_v \to R$ which is surjective on the spectrum, see [14, Expose X, Cor. 3.3]. Now we want to find a suitable unramified base change so we can assume G splits.

Reduction to the case G splits:

As $\operatorname{Spec} R \to \operatorname{Spec} \hat{\mathcal{O}}_v$ is surjective, there exists a prime $\mathfrak{q}$ of R lying over (0). By the going up property of integral ring map, there exists a prime $\mathfrak{p} \supset \mathfrak{q}$ lying over $t\hat{\mathcal{O}}_v$. And $R_{\mathfrak{p}}$ is our potential candidate for new $\hat{\mathcal{O}}_v$ and $R_{\mathfrak{q}}$ is our potential candidate for new K_v.

We have a flat local ring map $\hat{\mathcal{O}}_v \to R_{\mathfrak{p}}$ (resp. $K_v \to R_{\mathfrak{q}}$) which must be faithfully flat and hence injective, see [5, tag 00HR]. By [5, tag 02GU], we have $tR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ (resp. $(0) = \mathfrak{q}R_{\mathfrak{q}}$) and the induced residue field extension is finite separable. In particular, $R_{\mathfrak{q}} = R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}}$ is a field which is finite separable over K_v.

Clearly, R is a Noetherian ring since R is a finite $\hat{\mathcal{O}}_v$-module. We know R is a regular ring by [5, tag 07NF]. So $R_{\mathfrak{p}}$ is a regular local ring so a Noetherian normal domain, see [5, tag 00NP] and [5, tag 0567]. We also have $\dim(R_{\mathfrak{p}}) = 1$ by the fact $tR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ and [5, tag 00ON]. So $R_{\mathfrak{p}}$ is a discrete valuation ring and it's minimal prime ideal (0) must be $\mathfrak{q}R_{\mathfrak{p}}$. Hence the fraction field of $R_{\mathfrak{p}}$ is indeed $R_{\mathfrak{q}}$.

Since $tR_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ is the maximal ideal of the DVR $R_{\mathfrak{p}}$, t is also an uniformiser of $R_{\mathfrak{p}}$. And the discrete valuation on $R_{\mathfrak{q}}$ (determined by its valuation ring $R_{\mathfrak{p}}$) extends the discrete valuation on K_v (determined by its valuation ring $\hat{\mathcal{O}}_v$). By [24, 7.38], we know that the integral closure of $\hat{\mathcal{O}}_v$ in $R_{\mathfrak{q}}$ defines the same extended valuation on $R_{\mathfrak{q}}$ and $R_{\mathfrak{q}}$ is complete with this valuation. So $R_{\mathfrak{p}}$ must be the integral closure of $\hat{\mathcal{O}}_v$ in $R_{\mathfrak{q}}$. Hence $R_{\mathfrak{p}}$ is a complete discrete valuation ring and it is finite over $\hat{\mathcal{O}}_v$ by [5, tag 032L].

$$\begin{tikzcd}
\hat{\mathcal{O}}_v \arrow[r] \arrow[d] & R_{\mathfrak{p}} \arrow[d] \\
K_v \arrow[r] & R_{\mathfrak{q}}
\end{tikzcd}$$

If $x \in K_v \cap R_{\mathfrak{p}}$, then x is integral over $\hat{\mathcal{O}}_v$. Plus the fact $\hat{\mathcal{O}}_v$ is integrally closed in K_v, we have $x \in \hat{\mathcal{O}}_v$. So we have $K_v \cap R_{\mathfrak{p}} = \hat{\mathcal{O}}_v$. Now we are ready to check if we can safely assume $\hat{\mathcal{O}}_v = R_{\mathfrak{p}}$, i.e. if it suffices to check the result on $R_{\mathfrak{p}}$.

Clearly G splits over $R_{\mathfrak{p}}$. The ring map $\hat{\mathcal{O}}_v \to R_{\mathfrak{p}}$ is finite, flat, and formally unramified by [5, tag 02GU], thus unramified by [5, tag 02G5], thus weakly étale by [5, tag 094X], thus has isomorphic relative Frobenius morphism by [5, tag 0F6W]. The finite separable field extension $K_v \to R_{\mathfrak{q}}$ is étale so has isomorphic relative Frobenius morphism by [5, tag 0EBS]. By the proof of fact 7, we know both $\hat{\mathcal{O}}_v \to K_v$ and $R_{\mathfrak{p}} \to R_{\mathfrak{q}}$ have isomorphic relative Frobenius morphism. Thus by fact 5, every arrow of the following commutative diagram is a pull-back:

$$\begin{tikzcd}
F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}} \otimes \Omega^1_{\hat{\mathcal{O}}_v/k_0} \arrow[r] \arrow[d] & F^*_{K_v}\omega^\vee_{\ker F_{\mathcal{A}_{K_v}/K_v}} \otimes \Omega^1_{K_v/k_0} \arrow[d] \\
F^*_{R_{\mathfrak{p}}}\omega^\vee_{\ker F_{\mathcal{A}_{R_{\mathfrak{p}}}/R_{\mathfrak{p}}}} \otimes \Omega^1_{R_{\mathfrak{p}}/k_0} \arrow[r] & F^*_{R_{\mathfrak{q}}}\omega^\vee_{\ker F_{\mathcal{A}_{R_{\mathfrak{q}}}/R_{\mathfrak{q}}}} \otimes \Omega^1_{R_{\mathfrak{q}}/k_0}
\end{tikzcd}$$

We know $F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}} \otimes \Omega^1_{\hat{\mathcal{O}}_v/k_0} = \tilde{\mathcal{F}}(\hat{\mathcal{O}}_v)$ is a finite free module (mentioned above), so we can assume it's isomorphic to $\hat{\mathcal{O}}_v^{\oplus n}$ for some $n \in \mathbb{N}$. Then the above diagram is just:

$$\begin{tikzcd}
\hat{\mathcal{O}}_v^{\oplus n} \arrow[r] \arrow[d] & K_v^{\oplus n} \arrow[d] \\
R_{\mathfrak{p}}^{\oplus n} \arrow[r] & R_{\mathfrak{q}}^{\oplus n}
\end{tikzcd}$$

Let $s \in K_v^{\oplus n}$. To check if s or $t \cdot s$ are in $\hat{\mathcal{O}}_v^{\oplus n} = R_{\mathfrak{p}}^{\oplus n} \cap K_v^{\oplus n}$, it suffices to check in $R_{\mathfrak{p}}^{\oplus n}$.

Note that both $R_{\mathfrak{p}}$ and $\hat{\mathcal{O}}_v$ are DVRs, they share the same uniformiser, and both of their residue fields are finite. In the remaining proof, $R_{\mathfrak{p}}$ (resp. $R_{\mathfrak{q}}$) is a perfect replacement of $\hat{\mathcal{O}}_v$ (resp. K_v) because they share all the properties needed to complete the proof.

For example, using the proof of fact 7, we can see that every arrow in the following diagram has isomorphic relative Frobenius morphism.

$$\begin{tikzcd}
\operatorname{Spec} R_{\mathfrak{q}} \arrow[r] \arrow[d] & \operatorname{Spec} R_{\mathfrak{p}} \arrow[d] \\
\operatorname{Spec} K \arrow[r] & C
\end{tikzcd}$$

So we can replace $\hat{\mathcal{O}}_v$ with $R_{\mathfrak{p}}$ and K_v with $R_{\mathfrak{q}}$ in diagram 3.1 with similar proof.

From now on, we mean $R_{\mathfrak{p}}$ when we write $\hat{\mathcal{O}}_v$ and we mean $R_{\mathfrak{q}}$ when we write K_v. I will leave a note whenever a property of $\hat{\mathcal{O}}_v$ (resp. K_v) non-trivially passes to $R_{\mathfrak{p}}$ (resp. $R_{\mathfrak{q}}$).

Now we can safely assume G splits over $\hat{\mathcal{O}}_v$, i.e.

$$G = \mathbb{G}_m^{\oplus l}$$

for some $l \in \mathbb{N}$.

If $v \notin E$ then $\mathcal{A}_{\hat{\mathcal{O}}_v} \cong \mathcal{B}$ is already proper and $l = 0$, and by the valuation criterion of properness we have

$$\begin{aligned} \mathcal{A}_{\hat{\mathcal{O}}_v}(\hat{\mathcal{O}}_v) &= A_{K_v}(K_v) \\ \mathcal{A}^{(p)}_{\hat{\mathcal{O}}_v}(\hat{\mathcal{O}}_v) &= A^{(p)}_{K_v}(K_v) \\ \mathcal{A}^{(p)}_{\hat{\mathcal{O}}_v}(\hat{\mathcal{O}}_v)/F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}(\mathcal{A}_{\hat{\mathcal{O}}_v}(\hat{\mathcal{O}}_v)) &= A^{(p)}_{K_v}(K_v)/F_{A_{K_v}/K_v}(A_{K_v}(K_v)) \end{aligned}$$

By corollary 1.27, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{A}^{(p)}_{\hat{\mathcal{O}}_v}(\hat{\mathcal{O}}_v)/F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}(\mathcal{A}_{\hat{\mathcal{O}}_v}(\hat{\mathcal{O}}_v)) \arrow[r] \arrow[d, equal] & \mathrm{H}^1_{\mathrm{fppf}}(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[d, "\lambda_3"] \\ A^{(p)}_{K_v}(K_v)/F_{A_{K_v}/K_v}(A_{K_v}(K_v)) \arrow[r, "\delta_v"] & \mathrm{H}^1_{\mathrm{fppf}}(K_v, \ker F_{A_{K_v}/K_v}) \end{tikzcd}$$

so in this case $\operatorname{Im}\delta_v \subset \operatorname{Im}\lambda_3$. In particular, $\lambda_4(T) \in \operatorname{Im}\lambda_3$ so $s = \gamma_3(\lambda_4(T)) \in \operatorname{Im}\mu_3$. Thus, we only need to deal with the bad points $v \in E$ and we can assume $l \geq 1$.

The following lemma shows that the short exact sequence of formal schemes induces an exact sequence of finite commutative group schemes over $\hat{\mathcal{O}}_v$ whose middle term is $\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}$, so we can investigate the bottom plane with it. (to be continued) □

Lemma 3.2. *There exists a short exact sequence of finite commutative group schemes:*

$$0 \to \mu^{\oplus l}_{p,\hat{\mathcal{O}}_v} \to \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \overset{q}{\to} \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v} \to 0 \tag{3.4}$$

Proof. We first show the construction process of this sequence. Recall the construction of Raynaud sequence from the fact that $\mathcal{A}_{\hat{\mathcal{O}}_v}$ is a semi-abelian scheme over $\operatorname{Spec}\hat{\mathcal{O}}_v$. Let $W := \operatorname{Spec}(\hat{\mathcal{O}}_v/t^m\hat{\mathcal{O}}_v)$ where $m \geq 1$. The above exact sequence of formal schemes over $\hat{\mathcal{O}}_v$ is equivalent to an inverse system of exact sequences in both senses of remark 1.15:

$$0 \to G_W \to \mathcal{A}_W \overset{q_m}{\to} \mathcal{B}_W \to 0$$

So $G_W = \ker q_m$ and q_m is faithfully flat and locally of finite presentation.

Now, we pull back the sequence w.r.t. $W \overset{F_W}{\to} W$, we have an exact sequence

$$0 \to G_W^{(p)} \to \mathcal{A}_W^{(p)} \overset{q_m^{(p)}}{\to} \mathcal{B}_W^{(p)} \to 0$$

Next, we connect the two exact sequences by relative Frobenius morphisms which are universal homeomorphisms and integral by [5, tag 0CCB]. Hence we have the following commutative diagram.

$$\begin{tikzcd} 0 \arrow[r] & G_W \arrow[r] \arrow[d, "F_{G_W/W}"] & \mathcal{A}_W \arrow[r, "q_m"] \arrow[d, "F_{\mathcal{A}_W/W}"] & \mathcal{B}_W \arrow[r] \arrow[d, "F_{\mathcal{B}_W/W}"] & 0 \\ 0 \arrow[r] & G_W^{(p)} \arrow[r] & \mathcal{A}_W^{(p)} \arrow[r, "q_m^{(p)}"] & \mathcal{B}_W^{(p)} \arrow[r] & 0 \end{tikzcd}$$

Naturally, we consider the kernel of the relative Frobenius morphisms. In this case, the kernels of homomorphisms between group schemes and the kernels of *fppf* sheaves agrees (the group kernel clearly represents the sheaf kernel). So we have

$$\begin{tikzcd}
 & \ker F_{G_W/W} \arrow[r] \arrow[d] & \ker F_{\mathcal{A}_W/W} \arrow[r, "\tilde{q}_m"] \arrow[d] & \ker F_{\mathcal{B}_W/W} \arrow[d] & \\
0 \arrow[r] & G_W \arrow[r] \arrow[d, "F_{G_W/W}"] & \mathcal{A}_W \arrow[r, "q_m"] \arrow[d, "F_{\mathcal{A}_W/W}"] & \mathcal{B}_W \arrow[r] \arrow[d, "F_{\mathcal{B}_W/W}"] & 0 \\
0 \arrow[r] & G_W^{(p)} \arrow[r] & \mathcal{A}_W^{(p)} \arrow[r] & \mathcal{B}_W^{(p)} \arrow[r] & 0
\end{tikzcd}$$

By applying the snake lemma to the abelian category of *fppf* sheaves, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker F_{G_W/W} \arrow[r] \arrow[d] & \ker F_{\mathcal{A}_W/W} \arrow[r, "\tilde{q}_m"] \arrow[d] & \ker F_{\mathcal{B}_W/W} \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
0 \arrow[r] & G_W \arrow[r] \arrow[d, "F_{G_W/W}"] & \mathcal{A}_W \arrow[r, "q_m"] \arrow[d, "F_{\mathcal{A}_W/W}"] & \mathcal{B}_W \arrow[r] \arrow[d, "F_{\mathcal{B}_W/W}"] & 0 \\
0 \arrow[r] & G_W^{(p)} \arrow[r] \arrow[d] & \mathcal{A}_W^{(p)} \arrow[r] & \mathcal{B}_W^{(p)} \arrow[r] & 0 \\
 & \operatorname{Coker} F_{G_W/W} = 0 & & &
\end{tikzcd}$$

As an *fppf* sheaf, $\operatorname{Coker} F_{G_W/W} = 0$, see [12, II, 2.18 (b)]. We have an exact sequence of *fppf* sheaves:

$$0 \to \ker F_{G_W/W} \to \ker F_{\mathcal{A}_W/W} \xrightarrow{\tilde{q}_m} \ker F_{\mathcal{B}_W/W} \to 0 \tag{3.5}$$

Because $\ker F_{G_W/W} = \mu_{p,W}^{\oplus l}$ is flat and locally of finite presented over W by corollary 1.26 ($\mathbb{G}_m$ is smooth), this is an exact sequence in both senses of exactness by remark 1.15, in particular $\tilde{q}_m$ is faithfully flat.

As $G_W, \mathcal{A}_W, \mathcal{B}_W$ are all smooth over W, thereby locally of finite type over W, then $F_{G_W/W}$, $F_{\mathcal{A}_W/W}$ and $F_{\mathcal{B}_W/W}$ are finite by [5, tag 0CCD] and faithfully flat by lemma 1.25. Then by base change via unit section, (3.5) is an exact sequence of finite flat commutative group schemes over W (as W is Noetherian, they are also finite locally free, see [5, tag 02KB]). With different m, they form an inverse system of sequences (i.e. a sequence of formal schemes over $\hat{\mathcal{O}}_v$). By [25, 8.4.6] applied with $X = Y = \hat{\mathcal{O}}_v$, we have an equivalence from the category of finite $\hat{\mathcal{O}}_v$-schemes to the category of finite $\hat{\mathcal{O}}_v$-formal schemes. Thus from (3.5) we obtain a sequence of finite $\hat{\mathcal{O}}_v$-schemes which must be

$$0 \to \ker F_{G/\hat{\mathcal{O}}_v} = \mu_{p,\hat{\mathcal{O}}_v}^{\oplus l} \to \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \xrightarrow{q} \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v} \to 0 \tag{3.6}$$

Now we want to prove this is an exact sequence of commutative group schemes in both senses of exactness. Recall the first one is $\mu_{p,\hat{\mathcal{O}}_v}^{\oplus l} = \ker q$ and q is faithfully flat and locally of finite presented, and the second one is being exact as *fppf* sheaves. Since $\mu_{p,\hat{\mathcal{O}}_v}^{\oplus l}$ is already flat and locally of finite presented, those two definitions already agree.

First of all, from the equivalence of categories, the sub-categories of group objects are equivalent, and their kernel agree. Hence $\mu_{p,\hat{\mathcal{O}}_v}^{\oplus l} = \ker q$ and the corresponding morphism is an immersion of kernel. By lemma 1.28, $\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$ are of height 1. So the absolute Frobenius morphism of $\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$ factors through the unit section $\operatorname{Spec}\hat{\mathcal{O}}_v \to \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$ of $\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$, hence the unit section of $\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$ is surjective. As q is a group homomorphism, the unit section of $\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$ factors through q, so q is surjective. Now we are left to show that q is flat.

The plan is to use the flatness criterion in [5, tag 0523]. To use the lemma's notation, denote $R = S = \mathcal{O}(\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v})$, $M = \mathcal{O}(\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v})$, and $I = tR$. So if we can show that R is Noetherian, S is a local ring, $IS = tS$ is contained in its maximal ideal and M is finite over S, then $\frac{M}{I^n M}$ is flat over $\frac{R}{I^n}$ for all $n \geq 1$ implies that M is flat over R.

It's easy to see R is Noetherian as R is finite over $\hat{\mathcal{O}}_v$. We know the unit section $\operatorname{Spec}\hat{\mathcal{O}}_v \to \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$ is both injective and surjective, hence bijective. Hence $R = S$ has only two prime ideals. The finite ring map $\hat{\mathcal{O}}_v \to S$ is integral, so by the going up property of integral ring maps (see [5, tag 00GU]), we know one of the prime ideals of S contains the other one. So S is a local ring and its unique maximal ideal lies over $t\hat{\mathcal{O}}_v$, in particular tS is contained in S's maximal ideal. The fact M is finite over S follows from [5, tag 035D]. Hence we have M is flat over R.

We have shown that $\mu_{p,\hat{\mathcal{O}}_v}^{\oplus l} \to \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}$ is an isomorphism onto $\ker q$ and q is faithfully flat and locally of finite presentation. Thus it is an exact sequence of finite commutative group schemes over $\hat{\mathcal{O}}_v$. □

continued proof of theorem 3.1. By remark 1.15, we know

$$0 \to \mu_{p,\hat{\mathcal{O}}_v}^{\oplus l} \to \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \xrightarrow{q} \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v} \to 0$$

is also an exact sequence of *fppf*-sheaves, so it induces a sequence of cohomology groups. Combined with the second exact sequence functorially (see lemma 2.8), we have a commutative diagram whose middle plane is the bottom plane of diagram 3.1.

$$\begin{tikzcd}
& \mathrm{H}^1(K_v, \mu_{p,K_v}^{\oplus l}) \arrow[rr, hook, "e_{1,K_v}"] \arrow[dd, "\phi_{K_v}"] & & F^*_{K_v}\omega^\vee_{\mu^{\oplus l}_{p,K_v}/K_v} \otimes_{K_v} \Omega^1_{K_v/k_0} \arrow[dd, "\phi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_{p,\hat{\mathcal{O}}_v}^{\oplus l}) \arrow[ur, hook, "a_1"] \arrow[rr, hook, "e_1"] \arrow[dd, "\phi"] & & F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\mu^{\oplus l}_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0} \arrow[ur, hook, "b_1"] \arrow[dd, "\phi^*"] & \\
& \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[rr, hook, "e_{2,K_v}=\gamma_3"] \arrow[dd, "\psi_{K_v}"] & & F^*_{K_v}\omega^\vee_{\ker F_{A_{K_v}/K_v}/K_v} \otimes_{K_v} \Omega^1_{K_v/k_0} \arrow[dd, "\psi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_2=\lambda_3"] \arrow[rr, hook, "e_2=\gamma_4"] \arrow[dd, "\psi"] & & F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0} \arrow[ur, hook, "b_2=\mu_3"] \arrow[dd, "\psi^*"] & \\
& \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v}) \arrow[rr, hook, "e_{3,K_v}"] & & F^*_{K_v}\omega^\vee_{\ker F_{\mathcal{B}_{K_v}/K_v}/K_v} \otimes_{K_v} \Omega^1_{K_v/k_0} \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_3"] \arrow[rr, hook, "e_3"] & & F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0} \arrow[ur, hook, "b_3"] &
\end{tikzcd} \tag{3.7}$$

The right plane looks scary but it's just a bunch of $\hat{\mathcal{O}}_v^m$'s and K_v^n's. We will soon simplify it. Let's assume there exists $j, n, m \in \mathbb{N}$ s.t.

$$\begin{aligned} F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\mu^{\oplus l}_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}\otimes_{\hat{\mathcal{O}}_v}\Omega^1_{\hat{\mathcal{O}}_v/k_0} &\cong \hat{\mathcal{O}}_v^j \\ F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}\otimes_{\hat{\mathcal{O}}_v}\Omega^1_{\hat{\mathcal{O}}_v/k_0} &\cong \hat{\mathcal{O}}_v^n \\ F^*_{\hat{\mathcal{O}}_v}\omega^\vee_{\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}\otimes_{\hat{\mathcal{O}}_v}\Omega^1_{\hat{\mathcal{O}}_v/k_0} &\cong \hat{\mathcal{O}}_v^m \end{aligned}$$

Hence the above diagram simplifies to the following diagram.

$$\begin{tikzcd}
 & \mathrm{H}^1(K_v,\mu^{\oplus l}_{p,K_v}) \arrow[rr, hook, "e_{1,K_v}"] \arrow[dd, "\phi_{K_v}"] & & K_v^j \arrow[dd, "\phi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v,\mu^{\oplus l}_{p,\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_1"] \arrow[rr, hook, "e_1"] \arrow[dd, "\phi"] & & \hat{\mathcal{O}}_v^j \arrow[ur, hook, "b_1"] \arrow[dd, "\phi^*"] & \\
 & \mathrm{H}^1(K_v,\ker F_{A_{K_v}/K_v}) \arrow[rr, hook, "e_{2,K_v}=\gamma_3"] \arrow[dd, "\psi_{K_v}"] & & K_v^n \arrow[dd, "\psi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v,\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_2=\lambda_3"] \arrow[rr, hook, "e_2=\gamma_4"] \arrow[dd, "\psi"] & & \hat{\mathcal{O}}_v^n \arrow[ur, hook, "b_2=\mu_3"] \arrow[dd, "\psi^*"] & \\
 & \mathrm{H}^1(K_v,\ker F_{\mathcal{B}_{K_v}/K_v}) \arrow[rr, hook, "e_{3,K_v}"] & & K_v^m \\
\mathrm{H}^1(\hat{\mathcal{O}}_v,\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_3"] \arrow[rr, hook, "e_3"] & & \hat{\mathcal{O}}_v^m \arrow[ur, hook, "b_3"] &
\end{tikzcd} \tag{3.8}$$

Recall our goal is to show that $t\cdot\gamma_3(\lambda_4(T))\in\operatorname{Im}\mu_3$, or with the new name, $t\cdot e_{2,K_v}(\lambda_4(T))\in\operatorname{Im}b_2$.

At this step, a few groups can already be calculated explicitly. Note $\mathcal{O}_v$ and $\hat{\mathcal{O}}_v$ have the same residue field by [5, tag 05GG]. And this is a finite extension of k_0 since v is a closed point (note that we are actually using $R_{\mathfrak{p}}$ whose residue field is a finite extension of the residue field of $\mathcal{O}_v$). So $\kappa(v)$ is finite thus perfect. By [26, Chap. 2, §4, Thm. 2], if we write $L=\kappa(v)$, then we can identify $\hat{\mathcal{O}}_v\cong L[[t]]$ and $K_v\cong L((t))$. $\square$

Lemma 3.3. *We have that*

$$\mathrm{H}^1(K_v,\mu_p)\cong\frac{L((t))^\times}{(L((t))^\times)^p},\ \mathrm{H}^1(\hat{\mathcal{O}}_v,\mu_p)\cong\frac{L[[t]]^\times}{(L[[t]]^\times)^p}$$

and

$$\mathrm{H}^1(K_v,\mu_p^{\oplus l})\cong\left(\frac{L((t))^\times}{(L((t))^\times)^p}\right)^{\oplus l},\ \mathrm{H}^1(\hat{\mathcal{O}}_v,\mu_p^{\oplus l})\cong\left(\frac{L[[t]]^\times}{(L[[t]]^\times)^p}\right)^{\oplus l}.$$

Proof. Consider the long exact sequence of *fppf* cohomology associated to the exact sequence of *fppf* sheaves

$$0\to\mu_p\to\mathbb{G}_m\xrightarrow{p}\mathbb{G}_m\to 0.$$

To evaluate $\mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p)$ and $\mathrm{H}^1(K_v, \mu_p)$, we need to know $\mathrm{H}^1(\hat{\mathcal{O}}_v, \mathbb{G}_m) \cong \mathrm{Pic}(\hat{\mathcal{O}}_v)$ and $\mathrm{H}^1(K_v, \mathbb{G}_m) \cong \mathrm{Pic}(K_v)$ first. We know $\hat{\mathcal{O}}_v$ is a DVR (discrete valuation ring) and so a UFD (unique factorization domain), and K_v is a field so it is a UFD too. Thus both of their Picard groups are zero by [5, tag 0BCH].

So we have a commutative diagram of abelian groups where the rows are exact.

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{H}^0(K_v, \mu_p) \arrow[r, hook] & \mathrm{H}^0(K_v, \mathbb{G}_m) \arrow[r, "t\mapsto t^p"] & \mathrm{H}^0(K_v, \mathbb{G}_m) \arrow[r] & \mathrm{H}^1(K_v, \mu_p) \arrow[r] & 1 \\
1 \arrow[r] \arrow[u] & \mathrm{H}^0(\hat{\mathcal{O}}_v, \mu_p) \arrow[r, hook] \arrow[u, hook] & \mathrm{H}^0(\hat{\mathcal{O}}_v, \mathbb{G}_m) \arrow[r, "t\mapsto t^p"] \arrow[u, hook] & \mathrm{H}^0(\hat{\mathcal{O}}_v, \mathbb{G}_m) \arrow[r] \arrow[u, hook] & \mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p) \arrow[r] \arrow[u, hook] & 1 \arrow[u]
\end{tikzcd}$$

Hence the above diagram is just:

$$\begin{tikzcd}
1 \arrow[r] & \ker F_{K_v^\times} \arrow[r, hook] & L((t))^\times \arrow[r, "F_{K_v^\times}"] & L((t))^\times \arrow[r] & \frac{L((t))^\times}{(L((t))^\times)^p} \arrow[r] & 1 \\
1 \arrow[r] \arrow[u] & \ker F_{\hat{\mathcal{O}}_v^\times} \arrow[r, hook] \arrow[u, hook] & L[[t]]^\times \arrow[r, "F_{\hat{\mathcal{O}}_v^\times}"] \arrow[u, hook] & L[[t]]^\times \arrow[r] \arrow[u, hook] & \frac{L[[t]]^\times}{(L[[t]]^\times)^p} \arrow[r] \arrow[u, hook] & 1 \arrow[u]
\end{tikzcd}$$

In particular, we have

$$\mathrm{H}^1(K_v, \mu_p) \cong \frac{L((t))^\times}{(L((t))^\times)^p}, \ \mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p) \cong \frac{L[[t]]^\times}{(L[[t]]^\times)^p}$$

The rest of the statement follows from a similar argument. $\square$

Lemma 3.4. *Let $X = \mathrm{Spec}\, A$ be an affine scheme of characteristic p. Then*

$$\omega_{\mu_{p,X}/X} \cong \mathcal{O}_X$$

as an $\mathcal{O}_X$-module, hence $\omega^\vee_{\mu_{p,X}/X} \cong \mathcal{O}_X$. Also $\omega^\vee_{\mu_{p,X}^{\oplus l}/X} \cong \mathcal{O}_X^{\oplus l}$.

Proof. Clearly we have $\mu_{p,X} = \mathrm{Spec}\, \frac{A[x]}{(x^p-1)}$. Let $e : X \to \mu_{p,X}$ be the unit section, which is represented by $\frac{A[x]}{(x^p-1)} \overset{x\mapsto 1}{\to} A$. By definition, $\omega_{\mu_{p,X}/X} = e^*\Omega^1_{\mu_{p,X}/X}$. Because everything is affine and every concerned module is quasi-coherent, we may view it as a module over a ring, instead of an $\mathcal{O}_X$-module.

First, $\Omega^1_{\mu_{p,X}/X} = \Omega^1_{\frac{A[x]}{(x^p-1)}/A} = \frac{A[x]}{(x^p-1)}dx \cong \frac{A[x]}{(x^p-1)}$ as an $\frac{A[x]}{(x^p-1)}$-module (basically because of characteristic p). Then $e^*\Omega^1_{\mu_{p,X}/X}$ is just $\frac{A[x]}{(x^p-1)}dx \otimes_{\frac{A[x]}{(x^p-1)}} A$ where A is an $\frac{A[x]}{(x^p-1)}$-module via $x \mapsto 1$, which is just A. Hence $\omega_{\mu_{p,X}/X} \cong \mathcal{O}_X$ as an $\mathcal{O}_X$-module.

If M and N are two commutative group schemes over X, we have

$$\begin{aligned}
\omega_{M\times N/X} &= e^*_{M\times N}\Omega^1_{M\times N/X} \\
&= e^*_{M\times N}(\mathrm{pr}^*_M\Omega^1_{M/X} \oplus \mathrm{pr}^*_N\Omega^1_{N/X}) \\
&= e^*_M\Omega^1_{M/X} \oplus e^*_N\Omega^1_{N/X} \\
&= \omega_{M/X} \oplus \omega_{N/X}
\end{aligned}$$

Then $\omega^\vee_{M\times N/X} = \mathcal{H}om_{\mathcal{O}_X}(\omega_{M\times N/X}, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(\omega_{M/X} \oplus \omega_{N/X}, \mathcal{O}_X)$. Use [5, tag 03EN] and the fact that finite products agree with finite coproducts in the abelian category of $\mathcal{O}_X$-modules, we have $\mathcal{H}om_{\mathcal{O}_X}(\omega_{M/X} \oplus \omega_{N/X}, \mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\omega_{M/X}, \mathcal{O}_X) \oplus \mathcal{H}om_{\mathcal{O}_X}(\omega_{N/X}, \mathcal{O}_X) \cong \omega^\vee_{M/X} \oplus \omega^\vee_{N/X}$. It follows that $\omega^\vee_{\mu_{p,X}^{\oplus l}/X} \cong (\omega^{\oplus l}_{\mu_{p,X}/X})^\vee \cong \mathcal{O}_X^{\oplus l}$. $\square$

Thus we have

$$\omega^{\vee}_{\mu_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \cong \mathcal{O}_{\hat{\mathcal{O}}_v},\ \omega^{\vee}_{\mu_{p,K_v}/K_v} \cong \mathcal{O}_{K_v}.$$

The following lemma states that the fact which we want to show in the middle plane, is already true in the top plane.

Lemma 3.5. *We have that* $t \cdot \operatorname{Im} e_{1,K_v} \subset \operatorname{Im} b_1$.

Proof. We focus on the top plane of diagram 3.8. Since e_1 and e_{1,K_v} are functorial, both of them decompose into a direct sum of l pieces of maps. For example, $e_1 : \mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p^{\oplus l}) \to F^*_{\hat{\mathcal{O}}_v} \omega^{\vee}_{\mu^{\oplus l}_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0} \cong (F^*_{\hat{\mathcal{O}}_v} \omega^{\vee}_{\mu_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0})^{\oplus l}$ equals the direct sum of l copies of

$$e_1 : \mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p) \to F^*_{\hat{\mathcal{O}}_v} \omega^{\vee}_{\mu_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0}.$$

Explicitly, we apply the second exact sequence functorially with $\mu_p^{\oplus l} \overset{\mathrm{pr}_i}{\to} \mu_p$, so we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p^{\oplus l}) \arrow[r] \arrow[d, "\wr"] & F^*_{\hat{\mathcal{O}}_v} \omega^{\vee}_{\mu^{\oplus l}_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0} \arrow[d, "\wr"] \\
\bigoplus_{i=1}^{l} \mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p) \arrow[r] & \bigoplus_{i=1}^{l} F^*_{\hat{\mathcal{O}}_v} \omega^{\vee}_{\mu_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0}
\end{tikzcd}$$

where the left vertical arrow is an isomorphism because cohomology commutes with limits and colimits. It suffices to show $t \cdot \operatorname{Im} e_{1,K_v} \subset \operatorname{Im} b_1$ on each component so we can safely assume $l = 1$.

Note that we already show that $\mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p) \cong \frac{L[[t]]^{\times}}{(L[[t]]^{\times})^p}$ and $F^*_{\hat{\mathcal{O}}_v} \omega^{\vee}_{\mu_{p,\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0} \cong \mathcal{O}_{\hat{\mathcal{O}}_v} \otimes_{\hat{\mathcal{O}}_v} \Omega^1_{\hat{\mathcal{O}}_v/k_0} \cong \Omega^1_{\hat{\mathcal{O}}_v/k_0} \cong \Omega^1_{L[[t]]/k_0}$.

We claim that $e_1 : \frac{L[[t]]^{\times}}{(L[[t]]^{\times})^p} \to \Omega^1_{L[[t]]/k_0}$ can be represented by the map $\overline{a} \overset{d\ln}{\mapsto} \frac{da}{a}$. This is clearly well defined since $\frac{dab^p}{ab^p} = \frac{b^p da}{ab^p} = \frac{da}{a}$.

We follow the second description of the second exact sequence. An element $\overline{a}$ in $\mathrm{H}^1(\hat{\mathcal{O}}_v, \mu_p)$ can be represented by the μ_p-torsor $P_a = \operatorname{Spec} \frac{L[[t]][x]}{(x^p - a)}$ over $X := \operatorname{Spec} \hat{\mathcal{O}}_v$. The action of μ_p on P_a is obvious. Define $X' = X \overset{F_X}{\to} X$. So $F_X^* P_a$ is a torsor over X' which must be uniquely trivialized, by $X' \overset{\varphi_a}{\to} P_a$, $\frac{L[[t]][x]}{(x^p - a)} \to L[[t]]$, $cx \mapsto c^p a$. It gives rise to two maps $X' \times_X X' = X'' \to P_a$, namely $\varphi_a \circ p_1, x \mapsto a \otimes 1$ and $\varphi_a \circ p_2, x \mapsto 1 \otimes a$. As $P_a(X'')$ is a $\mu_p(X'')$-torsor, there exists a unique map $h_a : X'' \to \mu_{p,X}$ s.t. $h_a \cdot (\varphi_a \circ p_1) = \varphi_a \circ p_2$, we can see that it's determined by $x \mapsto a^{-1} \otimes a$.

Using the anti-equivalence between schemes affine over X and quasi-coherent $\mathcal{O}_X$-algebras, h_a defines a map of quasi-coherent $\mathcal{O}_X$-algebras $\frac{\mathcal{O}_X[x]}{(x^p - 1)} \to F_{X,*}\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} F_{X,*}\mathcal{O}_{X'}$ which is determined by $x \mapsto a^{-1} \otimes a$. Let I be the augmentation ideal $(x - 1)$ of $\frac{\mathcal{O}_X[x]}{(x^p - 1)}$ and N be the nilradical ideal of $F_{X,*}\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} F_{X,*}\mathcal{O}_{X'}$. We have

$$\begin{tikzcd}
\frac{I}{I^2} \arrow[r] \arrow[d, equal] & \frac{N}{N^2} \arrow[d, equal, "y \otimes z \mapsto y dz"] \\
\omega_{\mu_{p,X}/X} \arrow[r] & F_{X,*}\Omega^1_{X'/k_0}
\end{tikzcd}$$

So h_a induces a map of quasi-coherent $\mathcal{O}_X$-modules $\omega_{\mu_p,X/X} \to F_{X,*}\Omega^1_{X'/k_0}$. And $\omega_{\mu_p,X/X}$ is isomorphic to $\mathcal{O}_X \cdot (x-1)$ so the map is determined by the image of $x-1$ which is $a^{-1}da = \frac{da}{a}$. Thus the corresponding map $F_X^*\omega_{\mu_p,X/X} \to \Omega^1_{X'/k_0}$ is also determined by $\frac{da}{a}$.

Similarly the map $e_{1,K_v} : \mathrm{H}^1(K_v, \mu_p) \to F^*_{K_v}\omega^\vee_{\mu_p,K_v/K_v} \otimes_{K_v} \Omega^1_{K_v/k_0}$ can be identified with $\frac{L((t))^\times}{(L((t))^\times)^p} \to \Omega^1_{L((t))/k_0}, \overline{a} \mapsto \frac{da}{a}$.

Thus the top plane of the diagram (3.8) is as following, assuming $l = 1$:

$$\begin{tikzcd}
\frac{L((t))^\times}{(L((t))^\times)^p} \arrow[r, "e_{1,K_v}=d\ln"] & \Omega^1_{L((t))/k_0} \\
\frac{L[[t]]^\times}{(L[[t]]^\times)^p} \arrow[u, "a_1"] \arrow[r, "e_1=d\ln"] & \Omega^1_{L[[t]]/k_0} \arrow[u, "b_1"']
\end{tikzcd}$$

Since every formal power series in $L[[t]]$ with a non-zero constant term is a unit, a general non-zero element in $L((t))$ is of the form $t^m f$ where m is an integer and $f \in L[[t]]^\times$. Hence a general element in $\frac{L((t))^\times}{(L((t))^\times)^p}$ is of the form $\overline{t^m f}$ where $0 \leq m \leq p-1$ is an integer and $f \in L[[t]]^\times$. So

$$d\ln(\overline{t^m f}) = \frac{d(t^m f)}{t^m f} = \frac{mt^{m-1}f + t^m df}{t^m f} = \frac{m}{t} + \frac{df}{f}$$

Hence

$$t \cdot d\ln(\overline{t^m f}) = m + t\frac{df}{f} = m + t \cdot d\ln\overline{f} \in \operatorname{Im} b_1$$

The result follows. □

Lemma 3.6. $\psi : \mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \to \mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v})$ *is surjective.*

Proof. We only need to show that $\mathrm{H}^2(\hat{\mathcal{O}}_v, \mu_p) = 0$. Consider the long exact sequence of *fppf* cohomology associated to the exact sequence of *fppf* sheaves

$$0 \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

It suffices to show that $\mathrm{H}^1(\hat{\mathcal{O}}_v, \mathbb{G}_m) = 0 = \mathrm{H}^2(\hat{\mathcal{O}}_v, \mathbb{G}_m)$. The first one is just $\mathrm{Pic}(\hat{\mathcal{O}}_v)$, which must be 0 since $\hat{\mathcal{O}}_v$ is a DVR so UFD, see [5, tag 0BCH]. Since $\mathbb{G}_{m,\hat{\mathcal{O}}_v}$ is a smooth quasi-projective commutative group scheme over $\hat{\mathcal{O}}_v$ (smoothness by direct calculation, affine morphism ⇒ quasi-affine morphism ⇒ quasi-projective morphism given finite type, see [5, tag 0B3H]), by [12, III, 3.9], we can identify $\mathrm{H}^2(\hat{\mathcal{O}}_v, \mathbb{G}_{m,\hat{\mathcal{O}}_v})$ as an étale cohomology group, hence it is the cohomological Brauer group over $\hat{\mathcal{O}}_v$, see [12, IV, §1, p. 136].

Since $\hat{\mathcal{O}}_v$ is a local ring of dimension 1, it is isomorphic to the Brauer group $\mathrm{Br}(\hat{\mathcal{O}}_v)$ by [12, IV, 2.17]. Since $\hat{\mathcal{O}}_v$ is a complete local ring thus Henselian, $\mathrm{Br}(\hat{\mathcal{O}}_v)$ is isomorphic to the Brauer group of the residue field $\mathrm{Br}(\kappa(v))$ by [12, IV, 2.13].

As the residue field $\kappa(v)$ is finite over k_0, $\kappa(v)$ is also finite as a set. Then, by [12, IV, 2.20 (a)], we have $\mathrm{Br}(\kappa(v)) = 0$[5], which implies the desired result. □

The next lemma is the essential ingredient of this paper. To prove it we need to deeply investigate the Raynaud exact sequence in the setting of rigid analytic space. With the following lemma the assertion 3.2 is just a simple diagram chase.

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[5] This is where we essentially use the fact k_0 is finite. If k_0 is just perfect, the group $\mathrm{Br}(\kappa(v))$ could be non-zero.

Lemma 3.7. $\psi_{K_v}(\lambda_4(T)) \subset \operatorname{Im} a_3$

Proof. By definition $\lambda_4(T)$ comes from $A^{(p)}_{K_v}(K_v)$. It suffices to show $\psi_{K_v}(\lambda_4(T))$ comes from $\mathcal{B}^{(p)}_{K_v}(K_v) = \mathcal{B}^{(p)}(\hat{\mathcal{O}}_v)$ so it lies in $\operatorname{Im} a_3$.

$$\begin{tikzcd}
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[r, "a_2=\lambda_3"] \arrow[d, "\psi"] & \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[d, "\psi_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{B}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[r, "a_3"] & \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})
\end{tikzcd}$$

Now it suffices find a map of sets $A^{(p)}_{K_v}(K_v) \to \mathcal{B}^{(p)}_{K_v}(K_v)$ that is compatible with $\mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \to \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})$. This is where we introduce the method of rigid-analytic space. A brief introduction and related results are introduced in Appendix B.

Before we apply the results in the Appendix, we need to verify some preliminary conditions: $\hat{\mathcal{O}}_v$ is a complete valuation ring of height 1 and $F_{\hat{\mathcal{O}}_v}$ is a finite map. We know that $\hat{\mathcal{O}}_v \cong L[[t]]$ where L is a finite field, is clearly a complete valuation ring of height 1. And $F_{\hat{\mathcal{O}}_v}$ is generated by $\{t^i : 0 \leq i \leq p-1\}$ so $F_{\hat{\mathcal{O}}_v}$ is finitely generated. It follows that K_v is a complete field with a non-trivial non-Archimedean absolute value and F_{K_v} is finite.

Our first objective is to convert the maps $A^{(p)}_{K_v}(K_v) \to \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v})$ and $\mathcal{B}^{(p)}_{K_v}(K_v) \to \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})$ to maps in the rigid analytic setting so we can apply rigid analytic methods.

By lemma B.2 and lemma B.14, we know

$$\begin{aligned}
A^{(p)}_{K_v}(K_v) &= (A^{(p)}_{K_v})^{\mathrm{an}}(K_v) = (A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \\
\mathcal{B}^{(p)}_{K_v}(K_v) &= (\mathcal{B}^{(p)}_{K_v})^{\mathrm{an}}(K_v) = (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v) \\
F_{A^{\mathrm{an}}_{K_v}/K_v} &= (F_{A_{K_v}/K_v})^{\mathrm{an}} \\
F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v} &= (F_{\mathcal{B}_{K_v}/K_v})^{\mathrm{an}}
\end{aligned}$$

As analytification preserves commutative group objects, $A^{\mathrm{an}}_{K_v}$, $(A^{\mathrm{an}}_{K_v})^{(p)}$, $\mathcal{B}^{\mathrm{an}}_{K_v}$, $(\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}$ are all commutative rigid K_v-groups. And $F_{A^{\mathrm{an}}_{K_v}/K}$ and $F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$ are group homomorphisms by lemma B.13. By [5, tag 0CCB] and [5, tag 0CCD], $F_{A_{K_v}/K_v}$ and $F_{\mathcal{B}_{K_v}/K_v}$ are surjective and finite, hence their analytifications are surjective and finite by [27, 5.2.1 (2)]. It follows that both $\ker F_{A_{K_v}/K_v}$ and $\ker F_{\mathcal{B}_{K_v}/K_v}$ are finite over K_v, and so are their analytifications.

We have

$$\begin{aligned}
(\ker F_{A_{K_v}/K_v})^{\mathrm{an}} &\cong \ker(F_{A_{K_v}/K_v})^{\mathrm{an}} \text{ as } (\cdot)^{\mathrm{an}} \text{ commutes with fiber products} \\
&\cong \ker F_{A^{\mathrm{an}}_{K_v}/K_v}
\end{aligned}$$

and similarly $(\ker F_{\mathcal{B}_{K_v}/K_v})^{\mathrm{an}} \cong \ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$. By lemma B.6, we have

$$\begin{aligned}
\mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) &\cong \mathrm{PHS}(\ker F_{A_{K_v}/K_v})^{\mathrm{an}} \cong \mathrm{PHS}(\ker F_{A^{\mathrm{an}}_{K_v}/K_v}) \\
\mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v}) &\cong \mathrm{PHS}(\ker F_{\mathcal{B}_{K_v}/K_v})^{\mathrm{an}} \cong \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v})
\end{aligned}$$

Now we apply proposition B.8: as $F_{A^{\mathrm{an}}_{K_v}/K_v}$ (resp. $F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$) is a group homomorphism which is surjective on the level of topological space, there exists a group homomorphism $(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \to$

PHS($\ker F_{A^{\mathrm{an}}_{K_v}/K_v}$) (resp. $(\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v) \to$ PHS($\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$)), which by construction is compatible with $A^{(p)}_{K_v}(K_v) \to \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v})$ (resp. $\mathcal{B}^{(p)}_{K_v}(K_v) \to \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})$). And the homomorphism $\mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \to \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})$ induces a homomorphism PHS($\ker F_{A^{\mathrm{an}}_{K_v}/K_v}$) $\to$ PHS($\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$) which is determined by $\ker F_{A^{\mathrm{an}}_{K_v}/K_v} \cong (\ker F_{A_{K_v}/K_v})^{\mathrm{an}} \overset{q^{\mathrm{an}}_{K_v}}{\to} (\ker F_{\mathcal{B}_{K_v}/K_v})^{\mathrm{an}} \cong \ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$ where $q : \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v} \to \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}$ is introduced in 3.4.

Now we have a diagram

$$\begin{tikzcd}
(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[d, "\Psi_A"] & (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[d, "\Psi_{\mathcal{B}}"] \\
\mathrm{PHS}(\ker F_{A^{\mathrm{an}}_{K_v}/K_v}) \arrow[r, "s"] & \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v})
\end{tikzcd}$$

and the problem is reduced to showing that $\mathrm{Im}(s \circ \Psi_A) \subset \mathrm{Im}\, \Psi_{\mathcal{B}}$, or finding a map of sets $(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \to (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v)$ making the diagram commutes.

By the uniformization theorem [7, 1.1 & 1.2], there exists an extension

$$0 \to \mathbb{G}^{\oplus l}_{m,K_v} \to E \overset{\rho}{\to} \mathcal{B}_{K_v} \to 0$$

in the category of commutative group schemes that are of finite type over K_v, and a commutative diagram of abelian rigid K_v-groups where each arrow is a group homomorphism

$$\begin{tikzcd}
E^{\mathrm{an}} \arrow[r, "\rho^{\mathrm{an}}"] \arrow[d, "\sigma"] & \mathcal{B}^{\mathrm{an}}_{K_v} = \hat{\mathcal{B}}^{\mathrm{rig}} \\
A^{\mathrm{an}}_{K_v} & (\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}} \arrow[l, hook', "i_{\mathcal{A}}"] \arrow[ul, hook', "j"] \arrow[u, "\hat{\varphi}^{\mathrm{rig}}"']
\end{tikzcd}$$

where M is a lattice of E^{an} s.t. $A^{\mathrm{an}}_{K_v} \cong \frac{E^{\mathrm{an}}}{M}$ via σ, σ is thus surjective, $i_{\mathcal{A}} : (\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}} \hookrightarrow A^{\mathrm{an}}_{K_v}$ is an open immersion constructed as in subsection B.5 (note that it is a group homomorphism by the same proof in lemma B.25), and j is also an open immersion and a group homomorphism. The above diagram induces the following commutative diagram, see lemma B.12 and lemma B.13:

$$\begin{tikzcd}
 & E^{\mathrm{an}} \arrow[rr] \arrow[dl] \arrow[dd, "F_{E^{\mathrm{an}}/K_v}"] & & \mathcal{B}^{\mathrm{an}}_{K_v} \arrow[dd, "F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}"] \\
A^{\mathrm{an}}_{K_v} \arrow[dd, "F_{A^{\mathrm{an}}_{K_v}/K_v}"] & & (\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}} \arrow[ll, hook'] \arrow[ul, hook'] \arrow[ur] \arrow[dd, "F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}"] & \\
 & (E^{\mathrm{an}})^{(p)} \arrow[rr] \arrow[dl] & & (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)} \\
(A^{\mathrm{an}}_{K_v})^{(p)} & & ((\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}})^{(p)} \arrow[ll, hook'] \arrow[ul, hook'] \arrow[ur] &
\end{tikzcd}$$

In order to apply corollary B.9 on above diagram, we need to show every vertical arrow is surjective with finite kernel. We have dealt with $F_{A^{\mathrm{an}}_{K_v}/K_v}$ and $F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$, the same argument can be applied to F_{E^{an}/K_v}. Using lemma B.25, we know the induced map $\ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v} \to \ker F_{A^{\mathrm{an}}_{K_v}/K_v}$ is an isomorphism so $\ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}$ is finite. So it's left to show $F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}$ is surjective (it is very likely that every relative Frobenius morphism of rigid K_v-space is a universally homeomorphism just like the scheme case). Recall that $F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}}$ is an identity map on the underlying topological space, so it is equivalent to show $((\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}})^{(p)} \to (\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}$ is injective. By embedding into $(A^{\mathrm{an}}_{K_v})^{(p)} \to A^{\mathrm{an}}_{K_v}$, it suffices to show $(A^{\mathrm{an}}_{K_v})^{(p)} \to A^{\mathrm{an}}_{K_v}$ is injective, which is equivalent to the surjection of $F_{A^{\mathrm{an}}_{K_v}/K_v}$ which is already proved. So we are safe to apply corollary B.9, and we have:

$$\begin{tikzcd}
 & (E^{\mathrm{an}})^{(p)}(K_v) \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[dd] \\
(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[dd] & & ((\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}})^{(p)}(K_v) \arrow[ll] \arrow[ul] \arrow[ur] \arrow[dd] & \\
 & \mathrm{PHS}(\ker F_{E^{\mathrm{an}}/K_v}) \arrow[rr] \arrow[dl] & & \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}) \\
\mathrm{PHS}(\ker F_{A^{\mathrm{an}}_{K_v}/K_v}) \arrow[rr, equal] & & \mathrm{PHS}(\ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Next we show that the open immersion of rigid groups $j : (\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}} \hookrightarrow E^{\mathrm{an}}$ leads to an isomorphism $\ker F_{E^{\mathrm{an}}/K_v} \cong \ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}$. This is completely general, let $j : U \hookrightarrow X$ be an open immersion of rigid K_v-groups (in the sense that j is also a group homomorphism). Considering the following commutative diagram,

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f"] \arrow[ddr, bend right] & & \\
 & U \arrow[r, "j"] \arrow[d, "F_U"] & X \arrow[d, "F_X"] \\
 & U \arrow[r, "j"] & X
\end{tikzcd}$$

we can see that $f(Z) \subset U$ so there exists a unique morphism $Z \to U$ making the diagram commute. Therefore the square is Cartesian, hence all the three squares below are Cartesian since the lower one is automatically Cartesian.

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[d, "F_{U/K_v}"] & X \arrow[d, "F_{X/K_v}"] \\
U^{(p)} \arrow[r, "j^{(p)}"] \arrow[d] & X^{(p)} \arrow[d] \\
U \arrow[r, "j"] & X
\end{tikzcd}$$

Now we consider the commutative cube:

$$\begin{tikzcd}
 & \ker F_{U/K_v} \arrow[rr] \arrow[dl] \arrow[dd] & & \ker F_{X/K_v} \arrow[dl] \arrow[dd] \\
\operatorname{Sp} K_v \arrow[rr, equal] \arrow[dd, "e_{U^{(p)}}"] & & \operatorname{Sp} K_v \arrow[dd, "e_{X^{(p)}}"] & \\
 & U \arrow[rr, "j"] \arrow[dl, "F_{U/K_v}"] & & X \arrow[dl, "F_{X/K_v}"] \\
U^{(p)} \arrow[rr, "j^{(p)}"] & & X^{(p)} &
\end{tikzcd}$$

We know that the bottom, left and right squares are Cartesian. The front square is also Cartesian because $j^{(p)} : U^{(p)} \to X^{(p)}$ is an open immersion (base change preserves open immersions, and for any open immersion $U \to X$ and any morphism $Z \to U$, we have $Z \times_X U \cong Z \times_U U \cong Z$). Therefore all the 6 faces of the cube are Cartesian, in particular, the top square is Cartesian and we have $\ker F_{U/K_v} \cong \ker F_{X/K_v}$.

We know $\ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v} \to \ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}$ is induced by $\hat{q} : (\ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v})\hat{} \to (\ker F_{\mathcal{B}/\hat{\mathcal{O}}_v})\hat{}$, so the map $\mathrm{PHS}(\ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}) \to \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v})$ is compatible with $\mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \to \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v})$. Using that $\ker F_{E^{\mathrm{an}}/K_v} \cong \ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v} \cong \ker F_{A^{\mathrm{an}}_{K_v}/K_v}$, we have the following commutative diagram:

$$\begin{tikzcd}
 & (E^{\mathrm{an}})^{(p)}(K_v) \arrow[rr] \arrow[dl] \arrow[dd] & & (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[dd] \\
(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \arrow[dd] & & ((\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}})^{(p)}(K_v) \arrow[ll] \arrow[ul] \arrow[ur] \arrow[dd] & \\
 & \mathrm{PHS}(\ker F_{E^{\mathrm{an}}/K_v}) \arrow[rr] \arrow[dl, "\cong"] & & \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v}) \\
\mathrm{PHS}(\ker F_{A^{\mathrm{an}}_{K_v}/K_v}) \arrow[rr, equal] & & \mathrm{PHS}(\ker F_{(\hat{\mathcal{A}}_{\hat{\mathcal{O}}_v})^{\mathrm{rig}}/K_v}) \arrow[ul, "\cong"] \arrow[ur] &
\end{tikzcd}$$

We know $\sigma : E^{\mathrm{an}} \to A^{\mathrm{an}}_{K_v}$ is a quotient map, so E^{an} and $A^{\mathrm{an}}_{K_v}$ are locally the same, hence the induced map of K_v-rational points is surjective. And base change doesn't change that property. Hence we have a surjection $(E^{\mathrm{an}})^{(p)}(K_v) \to (A^{\mathrm{an}}_{K_v})^{(p)}(K_v)$. And any element in $(A^{\mathrm{an}}_{K_v})^{(p)}(K_v)$ can be lifted to an element in $(E^{\mathrm{an}})^{(p)}(K_v)$, and we can map it into $(\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v)$. This is the map of sets $(A^{\mathrm{an}}_{K_v})^{(p)}(K_v) \to (\mathcal{B}^{\mathrm{an}}_{K_v})^{(p)}(K_v)$ we want to construct. With a simple diagram chase, we can see that it is compatible with $\mathrm{PHS}(\ker F_{A^{\mathrm{an}}_{K_v}/K_v}) \to \mathrm{PHS}(\ker F_{\mathcal{B}^{\mathrm{an}}_{K_v}/K_v})$. The result follows. $\square$

Proof of assertion (3.2). For convenience, we include diagram 3.8 again:

$$\begin{tikzcd}
 & \mathrm{H}^1(K_v, \mu^{\oplus l}_{p,K_v}) \arrow[rr, hook, "e_{1,K_v}"] \arrow[dd, "\phi_{K_v}"] & & K_v^j \arrow[dd, "\phi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \mu^{\oplus l}_{p,\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_1"] \arrow[rr, hook, "e_1"] \arrow[dd, "\phi"] & & \hat{\mathcal{O}}_v^j \arrow[ur, hook, "b_1"] \arrow[dd, "\phi^*"] & \\
 & \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \arrow[rr, hook, "e_{2,K_v}=\gamma_3"] \arrow[dd, "\psi_{K_v}"] & & K_v^n \arrow[dd, "\psi^*_{K_v}"] \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_2=\lambda_3"] \arrow[rr, hook, "e_2=\gamma_4"] \arrow[dd, "\psi"] & & \hat{\mathcal{O}}_v^n \arrow[ur, hook, "b_2=\mu_3"] \arrow[dd, "\psi^*"] & \\
 & \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v}) \arrow[rr, hook, "e_{3,K_v}"] & & K_v^m \\
\mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{B}/\hat{\mathcal{O}}_v}) \arrow[ur, hook, "a_3"] \arrow[rr, hook, "e_3"] & & \hat{\mathcal{O}}_v^m \arrow[ur, hook, "b_3"] &
\end{tikzcd} \tag{3.9}$$

Since ψ is surjective and $\psi_{K_v}(\lambda_4(T)) \in \operatorname{Im} a_3$, there exists $T' \in \mathrm{H}^1(\hat{\mathcal{O}}_v, \ker F_{\mathcal{A}_{\hat{\mathcal{O}}_v}/\hat{\mathcal{O}}_v})$ s.t.

$$a_3 \circ \psi(T') = \psi_{K_v}(a_2(T')) = \psi_{K_v}(\lambda_4(T))$$

Let $\varepsilon := a_2(T') - \lambda_4(T)$, then $\psi_{K_v}(\varepsilon) = 0$. By exactness of the column

$$\mathrm{H}^1(K_v, \mu^{\oplus l}_{p,K_v}) \xrightarrow{\phi_{K_v}} \mathrm{H}^1(K_v, \ker F_{A_{K_v}/K_v}) \xrightarrow{\psi_{K_v}} \mathrm{H}^1(K_v, \ker F_{\mathcal{B}_{K_v}/K_v}),$$

there exists $\varepsilon_0 \in \mathrm{H}^1(K_v, \mu^{\oplus l}_{p,K_v})$ s.t. $\phi_{K_v}(\varepsilon_0) = \varepsilon$. By lemma 3.5, there exists $\theta \in \hat{\mathcal{O}}_v^j$ s.t. $t \cdot e_{1,K_v}(\varepsilon_0) = b_1(\theta)$ so

$$\begin{aligned}
t \cdot \gamma_3(\lambda_4(T)) &= t \cdot e_{2,K_v}(\lambda_4(T)) \\
&= t \cdot e_{2,K_v}(a_2(T') - \varepsilon) \\
&= t \cdot e_{2,K_v}(a_2(T')) - t \cdot e_{2,K_v}(\varepsilon) \\
&= t \cdot b_2(e_2(T')) - t \cdot e_{2,K_v}(\phi_{K_v}(\varepsilon_0)) \\
&= t \cdot b_2(e_2(T')) - t \cdot \phi^*_{K_v}(e_{1,K_v}(\varepsilon_0)) \\
&= t \cdot b_2(e_2(T')) - \phi^*_{K_v}(t \cdot e_{1,K_v}(\varepsilon_0)) \\
&= t \cdot b_2(e_2(T')) - \phi^*_{K_v}(b_1(\theta)) \\
&= t \cdot b_2(e_2(T')) - b_2(\phi^*(\theta)) \\
&\in \operatorname{Im} b_2
\end{aligned}$$

This completes the proof of assertion (3.2) so we know ϕ is a well-defined injection. Now it remains to show the functoriality of ϕ with respect to morphisms of semi-abelian schemes over C, it follows from the functoriality of the second exact sequence (see corollary 2.12) and the fact ϕ embeds in the second exact sequence. Now we complete the proof of Prop 3.1. ☐

Corollary 3.8. *The group* $\frac{A^{(p)}(K)}{F_{A/K}(A(K))}$ *is finite.*

Proof. Let $\mathcal{F} = F_C^* \omega^{\vee}_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}(E)$, then $\mathrm{Hom}_C(\omega_{\ker F_{\mathcal{A}/C}/C}, F_{C,*}(\Omega^1_{C/k_0}(E))) = \mathcal{F}(C)$. We have shown that $\mathcal{F}$ is a finite locally free sheaf over C. Plus that C is proper over k_0, by [5, tag 02O5], $\mathcal{F}(C)$ is a finite k_0-module, hence a finite abelian group since k_0 is finite. With our theorem, it follows that $\mathrm{Sel}^{(F_{A/K})}(A/K)$ and its subgroup $\frac{A^{(p)}(K)}{F_{A/K}(A(K))}$ are both finite. □

4. Example

4.1. Construction of $\mathcal{A}/C$

Let k be a finite field of characteristic $p \neq 2, 3$. Let $\mathbb{P}^1_k = \operatorname{Proj} k[s,t]$ be the projective line. We want to construct an elliptic surface over $\mathbb{P}^1_k$ with semistable reduction at each fiber. The idea is to use glueing technique.

Let

$$A(T) = a_n T^n + \cdots + a_0$$
$$B(T) = b_m T^m + \cdots + b_0$$

be polynomials over k s.t. $-16(4A(T)^3 + 27B(T)^2) \neq 0$. Then

$$y^2 z = x^3 + A(\frac{t}{s})xz^2 + B(\frac{t}{s})z^3$$

defines a closed subset $\mathcal{E}_s$ of $\mathbb{P}^2_k \times_k D_+(s)$, where $D_+(s) \cong \operatorname{Spec} k[\frac{t}{s}]$

Let

$$\alpha(S) = a_n + \cdots + a_0 S^n = S^n A(\frac{1}{S})$$
$$\beta(S) = b_m + \cdots + b_0 S^m = S^m B(\frac{1}{S})$$

Suppose that $n = 4d, m = 6d$ for some integer $d \geq 1$.

Over $D_+(t)$, let $\mathcal{E}_t$ be the closed subset of $\mathbb{P}^2_k \times_k D_+(t)$ defined by

$$Y^2 Z = X^3 + \alpha(\frac{s}{t})XZ^2 + \beta(\frac{s}{t})Z^3.$$

By construction we have that $\mathcal{E}_s$ and $\mathcal{E}_t$ agree over $D_+(s) \cap D_+(t)$ (via $Y = (\frac{s}{t})^{3d}y, X = (\frac{s}{t})^{2d}x, Z = z$), so we can glue them to obtain $\mathcal{E}$ over $\mathbb{P}^1_k$.

Since $\mathbb{P}^1_k = D_+(s) \cup \{[0:1]\}$, the fibers of $\mathcal{E}$ are just the fibers over $D_+(s)$ and the fiber over $[0:1]$. Now we want to find suitable conditions for $A(T), B(T)$ s.t. all fibers are semistable reduction.

Let $p \in D_+(s)$, then $\mathcal{E}_p$ over $\kappa(p)$ is a good reduction if and only if $\Delta_{\kappa(p)} = -16(4A(\frac{t}{s})^3 + 27B(\frac{t}{s})^2) \neq 0$; it is a multiplicative reduction if and only if $\Delta_{\kappa(p)} = 0$ and $-2A(\frac{t}{s})B(\frac{t}{s}) \neq 0$ in $\kappa(p)$, in this case its singular point is $[-\frac{3B(\frac{t}{s})}{2A(\frac{t}{s})}, 0, 1] \in \mathbb{P}^2(\kappa(p))$ and if we remove this singular point, this fiber is isomorphic to a multiplicative group scheme $\mathbb{G}_m$ after we pass to a finite field extension of $\kappa(p)$, so this fiber is a torus, see [28, pg.57-60].

Claim 4.1. The condition $\Delta = -16(4A(\frac{t}{s})^3 + 27B(\frac{t}{s})^2) = 0$ defines a non-empty proper closed subset Z in $D_+(s) \cong \mathbb{A}^1_k = \operatorname{Spec} k[\frac{t}{s}]$.

Proof. Let $F(x) = 4A(x)^3 + 27B(x)^2 \in k[x]$. Since $F \neq 0$ by assumption, $D(F) \neq \emptyset$ and thus $V(F) \subsetneq \operatorname{Spec} k[x]$. To show $V(F) \neq \emptyset$, it suffices to show that $\forall c \in k^* = k[x]^*, F(x) \not\equiv c$. Assume for a contradiction that $\exists c \in k^*, F(x) = 4A(x)^3 + 27B(x)^2 = c$. Denote $p = \operatorname{Char} k$. Note that by assumption $A(x)$ and $B(x)$ are non-constant polynomials, then there exists $\alpha, \beta \geq 0$ and non-constant polynomials $a(x), b(x) \in k[x]$ s.t.

$$A(x) = a(x^{p^\alpha}), B(x) = b(x^{p^\beta})$$

with $a'(x) \neq 0$ and $b'(x) \neq 0$.

Assume $\alpha \leq \beta$. Let $t = x^{p^\alpha}$, denote $B_0(t) := b(t^{p^{\beta-\alpha}}) = B(x)$. We have $4a(t)^3 + 27b_0(t)^2 = c$. So $\gcd(a(t)^3, b_0(t)^2) = 1 = \gcd(a(t), b_0(t))$. Take the derivative, we have $12a(t)^2a'(t) + 54b_0(t)b_0'(t) = 0$ (the fact Char$k \neq 2, 3$ is used here). Thus $b_0(t)|a'(t)$ and $a(t)|b_0'(t)$. Since $a'(t) \neq 0$, we have $54b_0(t)b_0'(t) = -12a(t)^2a'(t) \neq 0$ so $b_0'(t) \neq 0$.

So $\deg b_0 \leq \deg a' < \deg a \leq \deg b_0' < \deg b_0$, contradiction. The case $\alpha > \beta$ is similar. $\square$

Every proper closed subset of $\mathbb{A}_k^1$ must be Noetherian of dimension 0, so must be a finite set of closed points of $\mathbb{A}_k^1$, whose residue field is a finite extension of k. Over locally finite type k-scheme, closed points exactly correspond to points with residue field that is a finite extension of k. Hence Z is also closed in $\mathbb{P}_k^1$. So the fibers of $\mathcal{E}$ over $D_+(s)$ are semistable reduction if and only if $V(4A(\frac{t}{s})^3 + 27B(\frac{t}{s})^2) \subset D(A(\frac{t}{s})B(\frac{t}{s}))$, or equivalently $V(4A(\frac{t}{s})^3 + 27B(\frac{t}{s})^2) \cap V(A(\frac{t}{s})B(\frac{t}{s})) = V((4A(\frac{t}{s})^3 + 27B(\frac{t}{s})^2, A(\frac{t}{s})B(\frac{t}{s}))) = \emptyset$, or equivalently $4A(\frac{t}{s})^3 + 27B(\frac{t}{s})^2$ and $A(\frac{t}{s})B(\frac{t}{s})$ are coprime in $k[\frac{t}{s}]$, or equivalently $A(\frac{t}{s})$ and $B(\frac{t}{s})$ are coprime in $k[\frac{t}{s}]$.

So we can suppose that $A(T)$ and $B(T)$ are coprime over k.

Similarly the fiber $\mathcal{E}_{[0:1]}$ is a semistable reduction if and only if either $4a_n^3 + 27b_m^2 \neq 0$ or $\{4a_n^3 + 27b_m^2 = 0$ and $a_nb_m \neq 0\}$. Since $A(T), B(T)$ are non-constant polynomials, $a_n \neq 0 \neq b_m$. So we automatically have that the fiber $\mathcal{E}_{[0:1]}$ is a semistable reduction.

Let $\mathcal{E}_s^{sm}$ (respectively $\mathcal{E}_t^{sm}$) be the open subscheme of $\mathcal{E}_s$ (respectively $\mathcal{E}_t$) obtained by deleting singular points on the singular fibers.

Claim 4.2. $\mathcal{E}_s^{sm}$ is smooth over $D_+(s)$.

Proof. By [5, tag 01V4], it suffices to show that $\mathcal{E}_s^{sm}$ is flat over $D_+(s)$. It follows from [9, Cor 14.25] with $(X, Y, S) = (\mathcal{E}_s^{sm}, D_+(s), \operatorname{Spec} k)$. $\square$

Similarly $\mathcal{E}_t^{sm}$ is smooth over $D_+(t)$ so their glued scheme $\mathcal{E}^{sm}$ is smooth over $\mathbb{P}_k^1$. By the same proof in [29, IV, 5.3] and [29, IV,6.3], we can see that $\mathcal{E}_s^{sm}$ (respectively $\mathcal{E}_t^{sm}$) is a commutative group scheme over $D_+(s)$ (respectively $D_+(t)$) and their group scheme structure agree on overlapped open subset, so their glued scheme $\mathcal{E}^{sm}$ is a commutative group scheme over $\mathbb{P}_k^1$.

Claim 4.3. The scheme $\mathcal{E}^{sm}$ is a semi-abelian scheme over $\mathbb{P}_k^1$.

Proof. Recall that we say $\pi : \mathcal{A} \to C$ is a **semi-abelian scheme** if $\mathcal{A}$ is a smooth separated commutative group scheme over C via π with geometrically connected fibers, such that each fiber $\mathcal{A}_v$, where $v \in C$, is an extension of an abelian variety $\mathcal{B}_v$ by a torus T_v over the residue field $\kappa(v)$, i.e. a short exact sequence (abbr. SES) $0 \to T_v \to \mathcal{A}_v \to \mathcal{B}_v \to 0$. We have shown that $\mathcal{E}^{sm}$ is a smooth commutative group scheme over $\mathbb{P}_k^1$. And each fiber $\mathcal{E}_v^{sm}$ where $v \in \mathbb{P}_k^1$, is either an elliptic curve over $\kappa(v)$ (so an abelian variety of dimension 1) or a torus over $\kappa(v)$. It remains to show that $\mathcal{E}^{sm}$ is separated over $\mathbb{P}_k^1$.

Since the property of being separated is local on the target (see [9, 9.13]), it remains to show separateness over $D_+(s)$ and $D_+(t)$. Over $D_+(s)$ (respectively $D_+(t)$), the open immersion $\mathcal{E}_s^{sm} \hookrightarrow \mathcal{E}_s$ (respectively $\mathcal{E}_t^{sm} \hookrightarrow \mathcal{E}_t$) and the projective morphism $\mathcal{E}_s \to D_+(s)$ (respectively $\mathcal{E}_t \to D_+(t)$) are both separated. The result follows from the fact that composition of separated morphisms is separated. $\square$

Denote $k_0 = k$, $C = \mathbb{P}_k^1$ and $\mathcal{A} = \mathcal{E}^{sm}$. So we have a desired example of $\mathcal{A}$ over C.

4.2. Calculation of $\mathrm{Hom}_C(F_C^*\omega_{\ker F_{\mathcal{A}/C}/C}, \Omega^1_{C/k_0}(E))$

Here E is the closed subset of C s.t. $E \cap D_+(s)$ is defined by $4A(\frac{t}{s})^3 + 27B(\frac{t}{s})^2 = 0$ and $E \cap D_+(t)$ is defined by $4\alpha(\frac{s}{t})^3 + 27\beta(\frac{s}{t})^2 = 0$.

Note that the unit section of $\mathcal{A}$, $e : C \to \mathcal{A}$ comes from the map $C \to \mathbb{P}^2_k \times_k C$ that is the pullback of the map $\operatorname{Spec} k \to \mathbb{P}^2_k$ defined by the k-rational point $[0 : 1 : 0]$.

We have $\mathrm{Hom}_C(F_C^*\omega_{\ker F_{\mathcal{A}/C}/C}, \Omega^1_{C/k_0}(E)) = \Gamma(C, F_C^*\omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k_0}(E))$, so will calculate $F_C^*\omega^\vee_{\ker F_{\mathcal{A}/C}/C}$ and $\Omega^1_{C/k_0}(E)$ first.

Since we are over $\mathbb{P}^1_k = C$, the following lemma will be helpful.

Lemma 4.4. *Let $\mathcal{F}$ be a locally free $\mathcal{O}_C$-module of rank 1 s.t. $\exists f_s \in \mathcal{F}(D_+(s)), \exists f_t \in \mathcal{F}(D_+(t))$,*

$$\mathcal{F}(D_+(s)) = f_s\mathcal{O}_C(D_+(s)) = f_s \cdot k[\frac{t}{s}]$$

$$\mathcal{F}(D_+(t)) = f_t\mathcal{O}_C(D_+(t)) = f_t \cdot k[\frac{s}{t}]$$

$$\mathcal{F}(D_+(st)) = f_s \cdot k[\frac{t}{s}, \frac{s}{t}] = f_t \cdot k[\frac{t}{s}, \frac{s}{t}]$$

with $f_s = (\frac{s}{t})^n \cdot f_t$ over $D_+(st)$ for some $n \in \mathbb{Z}$. Then we have $\mathcal{F} \cong \mathcal{O}_C(n)$.

Proof. See [10, 5.1.4,1.19]. $\square$

4.2.1. Calculation of $F_C^*\omega^\vee_{\ker F_{\mathcal{A}/C}/C}$

Claim 4.5. We have $\omega_{\ker F_{\mathcal{A}/C}/C}$ is a locally free $\mathcal{O}_C$-sheaf of rank 1, and we have an isomorphism $\omega_{\ker F_{\mathcal{A}/C}/C} \cong \mathcal{O}_C(d) = \mathcal{O}_{\mathbb{P}^1_k}(d)$. So $\omega^\vee_{\ker F_{\mathcal{A}/C}/C} \cong \mathcal{O}_C(-d)$.

Proof. We intend to use [5, tag 01UX] to calculate $\omega_{\ker F_{\mathcal{A}/C}/C}$. Next we just need to find a way to reduce the problem to the affine case. We first reduce it to problems over $\mathcal{A}$, then $\mathcal{E}$, which is projective over C so still not affine over C. Denote $\mathcal{E}_y = \{y \neq 0\} \hookrightarrow \mathcal{E}$ the affine part of $\mathcal{E}$ where $y \neq 0$, clearly it is affine over C, and the unit section (corresponding to $[0 : 1 : 0]$) factors through $\mathcal{E}_y$, so it is the perfect candidate for us to calculate $\omega_{\ker F_{\mathcal{A}/C}/C}$.

Step 1: Reduce to $\mathcal{A}$.

Denote $N := \ker F_{\mathcal{A}/C}$, $l : N \to \mathcal{A}$ the canonical map. Clearly the unit section of $\mathcal{A}/C$ factors through N/C since N is a subgroup of $\mathcal{A}$. Denote $e_N : C \to N$ the unit section of N/C. Consider the following commutative diagram

$$\begin{tikzcd}
C \arrow[r, "e_N"] \arrow[dr, "e"'] & N \arrow[r] \arrow[d, "l"] \arrow[dr, phantom, "\square"] & C \arrow[r, "F_C"] \arrow[d, "e^{(p)}"] \arrow[dr, phantom, "\square"] & C \arrow[d, "e"] \\
 & \mathcal{A} \arrow[r, "F_{\mathcal{A}/C}"] \arrow[d] & \mathcal{A}^{(p/C)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{A} \arrow[d] \\
 & C \arrow[r, equal] & C \arrow[r, "F_C"] & C
\end{tikzcd}$$

the square symbol above indicates that the corresponding square is Cartesian.

So we have

$$\begin{aligned}\omega_{N/C} &= e_N^*\Omega^1_{N/C}\\ &\cong e_N^* l^* \Omega^1_{F_{\mathcal{A}/C}}\\ &\cong e^*\Omega^1_{F_{\mathcal{A}/C}}\end{aligned}$$

Step 2: Reduce to $\mathcal{E}$.

Denote $j : \mathcal{A} \hookrightarrow \mathcal{E}$ the open immersion, $e_{\mathcal{E}} := j \circ e : C \to \mathcal{E}$. Consider the following commutative diagram

$$\begin{tikzcd}
C \arrow[r, "e"] \arrow[dr, "e_{\mathcal{E}}"'] & \mathcal{A} \arrow[r, "F_{\mathcal{A}/C}"] \arrow[d, hook, "j"] \arrow[dr, phantom, "\square"] & \mathcal{A}^{(p/C)} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{A} \arrow[d, hook, "j"] \\
 & \mathcal{E} \arrow[r, "F_{\mathcal{E}/C}"] \arrow[d] & \mathcal{E}^{(p/C)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E} \arrow[d] \\
 & C \arrow[r, equal] & C \arrow[r, "F_C"] & C
\end{tikzcd}$$

Among above squares, only the Cartesianness of the top left square is non-trivial. It is the left square of the following diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F_{\mathcal{A}/C}"'] \arrow[rr, bend left, "F_{\mathcal{A}}"] \arrow[d, hook, "j"] \arrow[dr, phantom, "\square"] & \mathcal{A}^{(p/C)} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{A} \arrow[d, hook, "j"] \\
\mathcal{E} \arrow[r, "F_{\mathcal{E}/C}"] \arrow[rr, bend right, "F_{\mathcal{E}}"'] & \mathcal{E}^{(p/C)} \arrow[r] & \mathcal{E}
\end{tikzcd}$$

Given that the right square is Cartesian, it reduces to showing the big square is Cartesian, which is equivalent to showing that $j : \mathcal{A} \to \mathcal{E}$ has isomorphic relative Frobenius morphism. We know j is an open immersion, so a flat monomorphism by [5, tag 01L7], and thus a weakly étale morphism by [5, tag 094X], hence j has isomorphic relative Frobenius morphism by [5, tag 0F6W].

So we have

$$\begin{aligned}\omega_{N/C} &\cong e^*\Omega^1_{F_{\mathcal{A}/C}}\\ &\cong e^* j^* \Omega^1_{F_{\mathcal{E}/C}}\\ &\cong e_{\mathcal{E}}^*\Omega^1_{F_{\mathcal{E}/C}}\end{aligned}$$

Step 3: Reduce to $\mathcal{E}_y$.

Denote $q : \mathcal{E}_y \hookrightarrow \mathcal{E}$ the open immersion, and $e_y : C \to \mathcal{E}_y$ the map s.t. $q \circ e_y = e_{\mathcal{E}} : C \to \mathcal{E}$. Consider the following commutative diagram

$$\begin{tikzcd}
C \arrow[r, "e_y"] \arrow[dr, "e_{\mathcal{E}}"'] & \mathcal{E}_y \arrow[r, "F_{\mathcal{E}_y/C}"] \arrow[d, hook, "q"] \arrow[dr, phantom, "\square"] & \mathcal{E}_y^{(p/C)} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{E}_y \arrow[d, hook, "q"] \\
 & \mathcal{E} \arrow[r, "F_{\mathcal{E}/C}"] \arrow[d] & \mathcal{E}^{(p/C)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{E} \arrow[d] \\
 & C \arrow[r, equal] & C \arrow[r, "F_C"] & C
\end{tikzcd}$$

Similarly only the Cartesianness of top left square is non-trivial. It also follows from the fact that q is an open immersion and so it has isomorphic relative Frobenius morphism.

So we have

$$\begin{aligned} \omega_{N/C} &\cong e_{\mathcal{E}}^* \Omega^1_{F_{\mathcal{E}/C}} \\ &\cong e_y^* q^* \Omega^1_{F_{\mathcal{E}/C}} \\ &\cong e_y^* \Omega^1_{F_{\mathcal{E}_y/C}} \end{aligned}$$

Step 4: Calculation.

By considering the sequence

$$\begin{tikzcd} \mathcal{E}_y \arrow[r, "F_{\mathcal{E}_y/C}"] \arrow[rr, bend right, "F_{\mathcal{E}_y}"'] & \mathcal{E}_y^{(p/C)} \arrow[r] & \mathcal{E}_y \end{tikzcd} ,$$

using [5, tag 01UX], we have an exact sequence

$$F^*_{\mathcal{E}_y/C} \Omega^1_{\mathcal{E}_y^{(p/C)}/\mathcal{E}_y} \to \Omega^1_{F_{\mathcal{E}_y}} \to \Omega^1_{F_{\mathcal{E}_y/C}} \to 0.$$

Apply the right exact functor e_y^*, we have an exact sequence

$$e_y^* F^*_{\mathcal{E}_y/C} \Omega^1_{\mathcal{E}_y^{(p/C)}/\mathcal{E}_y} \to e_y^* \Omega^1_{F_{\mathcal{E}_y}} \to e_y^* \Omega^1_{F_{\mathcal{E}_y/C}} \to 0.$$

Since we have $e_y^* F^*_{\mathcal{E}_y/C} \Omega^1_{\mathcal{E}_y^{(p/C)}/\mathcal{E}_y} \cong \mathrm{id}_C^* \Omega^1_{F_C} = \Omega^1_{F_C}$, the sequence reduces to

$$\Omega^1_{F_C} \overset{\alpha}{\to} e_y^* \Omega^1_{F_{\mathcal{E}_y}} \to e_y^* \Omega^1_{F_{\mathcal{E}_y/C}} \to 0.$$

To be continued. ☐

To calculate sections of $\Omega^1_{F_C}$ and $\Omega^1_{F_{\mathcal{E}_y}}$, the following lemma will be helpful.

Lemma 4.6. *Let k be a perfect field of characteristic p, and $A = \frac{k[x_1,\ldots,x_n]}{(f)}$ where $f \in k[x_1, \ldots, x_n]$ is a non-constant polynomial. Then*

$$\Omega^1_{F_A} \cong \frac{\bigoplus_{i=1}^n A dx_i}{A df}$$

where $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \in \bigoplus_{i=1}^n A dx_i$.

Proof. Since k is perfect, F_k is an isomorphism. Let $\alpha : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ be the ring map induced by F_k, then it is an isomorphism. Let $\beta : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ be the ring map induced by $x_i \mapsto x_i^p$. Then $F_{k[x_1,\ldots,x_n]} = \beta \circ \alpha$.

We have the induced isomorphism $\tilde{\alpha} : \frac{k[x_i]}{(f)} \to \frac{k[x_i]}{(\alpha(f))}$ and the induced ring map $\tilde{\beta} : \frac{k[x_i]}{(\alpha(f))} \to \frac{k[x_i]}{(f)}, x_i \mapsto x_i^p$. And we have $F_A = \tilde{\beta} \circ \tilde{\alpha}$. So $\Omega^1_{F_A} = \Omega^1_{\tilde{\beta}}$.

Compose $\tilde{\beta}$ with the isomorphism $\frac{k[x_i]}{(f)} \overset{\sim}{\to} \frac{\frac{k[x_i]}{(\alpha(f))}[X_i]}{(X_i^p - x_i, f)}, x_i \mapsto X_i$, we have a map $\beta' : \frac{k[x_i]}{(\alpha(f))} \to \frac{\frac{k[x_i]}{(\alpha(f))}[X_i]}{(X_i^p - x_i, f)} =: C, x_i \mapsto x_i$. Then by [8, 21.2.E], we have

$$\Omega^1_{\tilde{\beta}} = \Omega^1_{\beta'} = \frac{\bigoplus_{i=1}^n C dX_i}{C df(X_i)}$$

since $d(X_i^p - x_i) = dX_i^p = pX_i^{p-1}dX_i = 0$. Using the isomorphism $\frac{k[x_i]}{(f)} \xrightarrow{\sim} \frac{\frac{k[x_i]}{(\alpha(f))}[X_i]}{(X_i^p - x_i, f)}$, we have $\Omega^1_{F_A} \cong \frac{\bigoplus_{i=1}^n Adx_i}{Adf}$. ☐

Claim 4.7. We have

$$\Omega^1_{F_C}(D_+(s)) = k[\frac{t}{s}]d\frac{t}{s}$$
$$\Omega^1_{F_C}(D_+(t)) = k[\frac{s}{t}]d\frac{s}{t}$$
$$\Omega^1_{F_C}(D_+(st)) = k[\frac{t}{s}, \frac{s}{t}]d\frac{t}{s} = k[\frac{t}{s}, \frac{s}{t}]d\frac{s}{t}$$

with the obvious restriction map. Note that over $D_+(st)$, we have $d\frac{t}{s} = -(\frac{t}{s})^2 d\frac{s}{t}$, so $\Omega^1_{F_C} \cong \mathcal{O}_C(-2)$.

Proof. We know $\Omega^1_{F_C}(D_+(s)) = \Omega^1_{F_{k[\frac{t}{s}]}}$ and $\Omega^1_{F_C}(D_+(t)) = \Omega^1_{F_{k[\frac{s}{t}]}}$. To show the first two equalities it suffices to show that $\Omega^1_{F_{k[x]}} = k[x]dx$, which follows from above lemma. For the last equality, since $d\frac{t}{s} \in \Omega^1_{F_C}(D_+(s))$ forms the $\mathcal{O}_C(D_+(s))$-basis of $\Omega^1_{F_C}(D_+(s))$, it descends to a $\mathcal{O}_C(D_+(st))$-basis of $\Omega^1_{F_C}(D_+(st))$. ☐

To evaluate the sections of $e_y^*\Omega^1_{F_{\mathcal{E}_y}}$, we need to name a few maps and schemes.

Let $\mathcal{E}_{ys}$ (respectively $\mathcal{E}_{yt}$) be the open subscheme of $\mathcal{E}_y$ over $D_+(s)$ (respectively $\mathcal{E}_{yt}$), so we have $\mathcal{E}_{ys} = \mathcal{E}_y \cap \mathcal{E}_s$ (respectively $\mathcal{E}_{yt} = \mathcal{E}_y \cap \mathcal{E}_t$). In particular, the map $\mathcal{E}_{ys} \to D_+(s)$ corresponds to the ring map

$$k[\frac{t}{s}] \to \frac{k[\frac{t}{s}][x,z]}{(z - x^3 - A(\frac{t}{s})xz^2 - B(\frac{t}{s})z^3)}.$$

And the map $\mathcal{E}_{yt} \to D_+(t)$ corresponds to the ring map

$$k[\frac{s}{t}] \to \frac{k[\frac{s}{t}][X,Z]}{(Z - X^3 - \alpha(\frac{s}{t})XZ^2 - \beta(\frac{s}{t})Z^3)}.$$

Over $D_+(ts) = \operatorname{Spec} k[\frac{t}{s}, \frac{s}{t}]$, they agree via the map

$$\begin{array}{ccc} \frac{k[\frac{t}{s}, \frac{s}{t}][x,z]}{(z - x^3 - A(\frac{t}{s})xz^2 - B(\frac{t}{s})z^3)} & \xrightarrow{\sim} & \frac{k[\frac{t}{s}, \frac{s}{t}][X,Z]}{(Z - X^3 - \alpha(\frac{s}{t})XZ^2 - \beta(\frac{s}{t})Z^3)} \\ x & \mapsto & (\frac{s}{t})^d X \\ z & \mapsto & (\frac{s}{t})^{3d} Z \end{array}$$

The pullback of $e_y : C \to \mathcal{E}_y$ w.r.t. $D_+(s)$ is $e_{ys} : D_+(s) \to \mathcal{E}_{ys}$ with the corresponding ring map

$$\frac{k[\frac{t}{s}][x,z]}{(z - x^3 - A(\frac{t}{s})xz^2 - B(\frac{t}{s})z^3)} \to k[\frac{t}{s}]$$

defined by sending x and z to 0.

Similarly the pullback of $e_y : C \to \mathcal{E}_y$ w.r.t. $D_+(t)$ is $e_{yt} : D_+(t) \to \mathcal{E}_{yt}$ with the corresponding ring map

$$\frac{k[\frac{s}{t}][X,Z]}{(Z - X^3 - \alpha(\frac{s}{t})XZ^2 - \beta(\frac{s}{t})Z^3)} \to k[\frac{s}{t}]$$

defined by sending X and Z to 0.

Claim 4.8. We have

$$e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(s)) = k[\frac{t}{s}]d\frac{t}{s} \oplus k[\frac{t}{s}]dx$$
$$e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(t)) = k[\frac{s}{t}]d\frac{s}{t} \oplus k[\frac{s}{t}]dX$$
$$e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(st)) = k[\frac{t}{s}, \frac{s}{t}]d\frac{t}{s} \oplus k[\frac{t}{s}, \frac{s}{t}]dx = k[\frac{t}{s}, \frac{s}{t}]d\frac{s}{t} \oplus k[\frac{t}{s}, \frac{s}{t}]dX$$

with the obvious restriction maps and $dx = (\frac{s}{t})^d dX + d \cdot (\frac{s}{t})^{d-1} X d\frac{s}{t}, d\frac{t}{s} = -(\frac{t}{s})^2 d\frac{s}{t}$ over $D_+(st)$.

Proof. Consider the following commutative diagram over $D_+(s)$:

$$\begin{tikzcd}
D_+(s) \arrow[r, "e_{ys}"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{E}_{ys} \arrow[r, "F_{\mathcal{E}_{ys}}"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{E}_{ys} \arrow[d, hook] \\
C \arrow[r, "e_y"] & \mathcal{E}_y \arrow[r, "F_{\mathcal{E}_y}"] & \mathcal{E}_y
\end{tikzcd}$$

So we have $e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(s)) = e_{ys}^*\Omega^1_{F_{\mathcal{E}_{ys}}}(D_+(s))$. Let $M = \frac{k[\frac{t}{s}][x,z]}{(z - x^3 - A(\frac{t}{s})xz^2 - B(\frac{t}{s})z^3)}$, then

$$e_{ys}^*\Omega^1_{F_{\mathcal{E}_{ys}}}(D_+(s)) = \Omega^1_{F_M} \otimes_M k[\frac{t}{s}].$$

Using lemma 4.6, we have

$$\Omega^1_{F_M} = \frac{Md\frac{t}{s} \oplus Mdx \oplus Mdz}{Mdf}$$

where

$$\begin{aligned}
df &= d(z - x^3 - A(\frac{t}{s})xz^2 - B(\frac{t}{s})z^3) \\
&= (-A'(\frac{t}{s})xz^2A'(\frac{t}{s}) - B'(\frac{t}{s})z^3)d\frac{t}{s} \\
&+ (-3x^2 - A(\frac{t}{s})z^2)dx \\
&+ (1 - 2A(\frac{t}{s})xz - 3B(\frac{t}{s})z^2)dz.
\end{aligned}$$

So we have an exact sequence

$$M \xrightarrow[\alpha]{1 \mapsto df} Md\tfrac{t}{s} \oplus Mdx \oplus Mdz \longrightarrow \Omega^1_{F_M} \longrightarrow 0$$

It stays exact after we tensor with $k[\frac{t}{s}]$ via the map $M \to k[\frac{t}{s}], x, z \mapsto 0$, we have

$$k[\tfrac{t}{s}] \xrightarrow[\overline{\alpha}]{1 \mapsto \overline{df}} k[\tfrac{t}{s}]d\tfrac{t}{s} \oplus k[\tfrac{t}{s}]dx \oplus k[\tfrac{t}{s}]dz \longrightarrow \Omega^1_{F_M} \otimes_M k[\tfrac{t}{s}] \longrightarrow 0$$

Since $\overline{df} = dz$, we have $\Omega^1_{F_M} \otimes_M k[\frac{t}{s}] \cong k[\frac{t}{s}]d\frac{t}{s} \oplus k[\frac{t}{s}]dx$. Hence $e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(s)) = k[\frac{t}{s}]d\frac{t}{s} \oplus k[\frac{t}{s}]dx$. Similarly we have $e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(t)) = k[\frac{s}{t}]d\frac{t}{s} \oplus k[\frac{s}{t}]dX$. By restriction to $D_+(st)$, we have

$$e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(st)) = k[\frac{t}{s}, \frac{s}{t}]d\frac{s}{t} \oplus k[\frac{t}{s}, \frac{s}{t}]dx = k[\frac{t}{s}, \frac{s}{t}]d\frac{s}{t} \oplus k[\frac{t}{s}, \frac{s}{t}]dX,$$

with $dx = d(\frac{s}{t})^d X = (\frac{s}{t})^d dX + Xd(\frac{s}{t})^d = (\frac{s}{t})^d dX + d \cdot (\frac{s}{t})^{d-1} X d\frac{s}{t}$. □

Claim 4.9. The map $\Omega^1_{F_C} \to e_y^*\Omega^1_{F_{\mathcal{E}_y}}$ restricting to $D_+(s), D_+(t)$ are

$$\begin{array}{rcl} \Omega^1_{F_C}(D_+(s)) & \to & e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(s)) \\ k[\frac{t}{s}]d\frac{t}{s} & \to & k[\frac{t}{s}]d\frac{t}{s} \oplus k[\frac{t}{s}]dx \\ d\frac{t}{s} & \mapsto & d\frac{t}{s} \end{array}$$

$$\begin{array}{rcl} \Omega^1_{F_C}(D_+(t)) & \to & e_y^*\Omega^1_{F_{\mathcal{E}_y}}(D_+(t)) \\ k[\frac{s}{t}]d\frac{s}{t} & \to & k[\frac{s}{t}]d\frac{s}{t} \oplus k[\frac{s}{t}]dX \\ d\frac{s}{t} & \mapsto & d\frac{s}{t}. \end{array}$$

Proof. Over $D_+(s)$, the map $\Omega^1_{F_C} \to e_y^*\Omega^1_{F_{\mathcal{E}_y}}$ is $\Omega^1_{F_{D_+(s)}} \cong e_{ys}^*F^*_{\mathcal{E}_{ys}/D_+(y)}\Omega^1_{\mathcal{E}_{ys}^{(p)}/\mathcal{E}_{ys}} \to e_{ys}^*\Omega^1_{F_{\mathcal{E}_{ys}}}$ where $F^*_{\mathcal{E}_{ys}/D_+(y)}\Omega^1_{\mathcal{E}_{ys}^{(p)}/\mathcal{E}_{ys}} \to \Omega^1_{F_{\mathcal{E}_{ys}}}$ is the map induced by the maps $\mathcal{E}_{ys} \overset{F_{\mathcal{E}_{ys}/D_+(y)}}{\to} \mathcal{E}_{ys}^{(p)} \to \mathcal{E}_{ys}$. Again let $M = \frac{k[\frac{t}{s}][x,z]}{(z-x^3-A(\frac{t}{s})xz^2-B(\frac{t}{s})z^3)}$. Then we have

$$\begin{array}{rcl} F^*_{\mathcal{E}_{ys}/D_+(y)}\Omega^1_{\mathcal{E}_{ys}^{(p)}/\mathcal{E}_{ys}} & \to & \Omega^1_{F_{\mathcal{E}_{ys}}} \\ M \otimes_{M,p} \Omega^1_{M\otimes_{k[\frac{t}{s}],p}k[\frac{t}{s}]/M} & \to & \Omega^1_{F_M} \\ a \otimes d(b \otimes c) & \mapsto & a \cdot d(cb^p). \end{array}$$

Applying e_{ys}^*, we have

$$\begin{array}{rcl} e_{ys}^*F^*_{\mathcal{E}_{ys}/D_+(y)}\Omega^1_{\mathcal{E}_{ys}^{(p)}/\mathcal{E}_{ys}} & \to & e_{ys}^*\Omega^1_{F_{\mathcal{E}_{ys}}} \\ k[\frac{t}{s}] \otimes_M M \otimes_{M,p} \Omega^1_{M\otimes_{k[\frac{t}{s}],p}k[\frac{t}{s}]/M} & \to & k[\frac{t}{s}] \otimes_M \Omega^1_{F_M} \\ h \otimes 1 \otimes d(b \otimes c) & \mapsto & h \otimes d(cb^p). \end{array}$$

And the isomorphism $\Omega^1_{F_{D_+(s)}} \cong e_{ys}^*F^*_{\mathcal{E}_{ys}/D_+(y)}\Omega^1_{\mathcal{E}_{ys}^{(p)}/\mathcal{E}_{ys}}$ is just

$$\begin{array}{rcl} \Omega^1_{F_{D_+(s)}} & \to & e_{ys}^*F^*_{\mathcal{E}_{ys}/D_+(y)}\Omega^1_{\mathcal{E}_{ys}^{(p)}/\mathcal{E}_{ys}} \\ \Omega^1_{k[\frac{t}{s}]} & \to & k[\frac{t}{s}] \otimes_M M \otimes_{M,p} \Omega^1_{M\otimes_{k[\frac{t}{s}],p}k[\frac{t}{s}]/M} \\ adb & \mapsto & 1 \otimes 1 \otimes ((1 \otimes a)d(1 \otimes b)). \end{array}$$

So we have

$$\begin{tikzcd} \Omega^1_{F_{D_+(s)}} \arrow[r] & e_{ys}^*F^*_{\mathcal{E}_{ys}/D_+(y)}\Omega^1_{\mathcal{E}_{ys}^{(p)}/\mathcal{E}_{ys}} \arrow[r] & e_{ys}^*\Omega^1_{F_{\mathcal{E}_{ys}}} \\ \Omega^1_{k[\frac{t}{s}]} \arrow[r] & k[\frac{t}{s}] \otimes_M M \otimes_{M,p} \Omega^1_{M\otimes_{k[\frac{t}{s}],p}k[\frac{t}{s}]/M} \arrow[r] & k[\frac{t}{s}] \otimes_M \Omega^1_{F_M} \\ d\frac{t}{s} \arrow[r, mapsto] & 1 \otimes 1 \otimes d(1 \otimes \frac{t}{s}) \arrow[r, mapsto] & 1 \otimes d\frac{t}{s} \end{tikzcd}$$

The case over $D_+(t)$ is similar. ☐

continued proof of claim 4.5. By above claim, we can see that

$$\begin{aligned} e_y^*\Omega^1_{F_{\mathcal{E}_y/C}}(D_+(s)) &\cong k[\frac{t}{s}]dx \\ e_y^*\Omega^1_{F_{\mathcal{E}_y/C}}(D_+(t)) &\cong k[\frac{s}{t}]dX \\ e_y^*\Omega^1_{F_{\mathcal{E}_y/C}}(D_+(st)) &\cong k[\frac{t}{s},\frac{s}{t}]dx = k[\frac{t}{s},\frac{s}{t}]dX \end{aligned}$$

with $dx = (\frac{s}{t})^d dX$. Moreover, we know

$$\begin{aligned} \mathcal{O}_C(d)(D_+(s)) &\cong s^d k[\frac{t}{s}] \\ \mathcal{O}_C(d)(D_+(t)) &\cong t^d k[\frac{s}{t}] \\ \mathcal{O}_C(d)(D_+(st)) &\cong s^d k[\frac{t}{s},\frac{s}{t}] = t^d k[\frac{t}{s},\frac{s}{t}] \end{aligned}$$

with $s^d = (\frac{s}{t})^d t^d$, see [10, 5.1.4,1.19]. Hence we have an isomorphism $\omega_{N/C} \cong \mathcal{O}_C(d)$. □

To calculate $F_C^*\omega_{N/C}^\vee$, the following lemma will be helpful.

Lemma 4.10. *Let S be a scheme of characteristic p, and let $F : S \to S$ be the absolute Frobenius map. Then for every invertible sheaf $\mathcal{L}$ on S we have $F^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$.*

Proof. See [30]. □

Claim 4.11. We have $F_C^*\omega_{N/C}^\vee \cong \mathcal{O}_C(-dp)$.

Proof. We have

$$F_C^*\omega_{N/C}^\vee \cong F_C^*\mathcal{O}_C(-d) \cong \mathcal{O}_C(-d)^{\otimes p} \cong \mathcal{O}_C(-dp).$$

□

4.2.2. Calculation of $\Omega^1_{C/k_0}(E)$

We have $k_0 = k$, to simplify the notation we will stick to k. By definition we have $\Omega^1_{C/k}(E) = \Omega^1_{C/k} \otimes_{\mathcal{O}_C} \mathcal{O}(E)$. Since C is a curve smooth over k, $\Omega^1_{C/k}$ is locally free of rank 1.

Claim 4.12. We have $\Omega^1_{C/k} \cong \mathcal{O}_C(-2)$.

Proof. It follows from the fact $\Omega^1_{F_C} \cong \mathcal{O}_C(-2)$ and we can identify $\Omega^1_{C/k}$ and $\Omega^1_{F_C}$ if we identify C and C' ($C' = C$ but it is viewed as C-scheme via $F_C : C' \to C$). □

Next we calculate $\mathcal{O}(E)$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{O}_C$ defined by $4A(\frac{t}{s})^3+27B(\frac{t}{s})^2 = 0$ over $D_+(s)$ and $4\alpha(\frac{s}{t})^3+27\beta(\frac{s}{t})^2$ over $D_+(t)$. Then the closed subset E with the reduced induced subscheme structure is determined by $\mathrm{rad}(\mathcal{I})$. Denote $F(x) = 4A(x)^3+27B(x)^3 \in k[x]$ and $G(x) = 4\alpha(x)^3+27\beta(x)^3 \in k[x]$. We know $k[x]$ is a PID and UFD. Let

$$F = cf_1^{n_1}\cdots f_l^{n_l}$$

be the decomposition of F into irreducible monic polynomials in $k[x]$, with $c \in k^*$. Let $F_0 := f_1\cdots f_l$, so that $(F_0) = \mathrm{rad}(F)$.

For each i, denote $g_i(x) := x^{\deg f_i} f_i(x^{-1})$. There are two cases: If $f_i(x) = x$ then $g_i(x) = 1$; otherwise f_i has a non-zero constant term and we can see that g_i is also irreducible of the same degree (if $g_i = uv$ for some non-constant polynomials u, v, then $f_i(x) = (x^{\deg u} u(x^{-1})) \cdot (x^{\deg v} v(x^{-1}))$, by irreducibility of f_i, say $x^{\deg u} u(x^{-1})$ is a constant, so $u(x) = c_0 x^{\deg u}$ for some $c_0 \in k^*$, thus g_i has no constant term which implies f_i has no leading term, contradiction).

Clearly $G = c g_1^{n_1} \cdots g_l^{n_l}$, and this is a decomposition of G into irreducible polynomials in $k[x]$, with possibly one of g_i is 1. Let $G_0 := g_1 \cdots g_l$, so $(G_0) = \mathrm{rad}(G)$.

Claim 4.13. We have $\mathcal{O}(E) \cong \mathcal{O}_C(\deg F_0)$. In particular, if $4a_n^3 + 27b_m^2 \neq 0$ and $\gcd(F, F') = 1$, then $\deg F_0 = 12d$ and $\mathcal{O}(E) \cong \mathcal{O}_C(12d)$.

Proof. Since the closed subset E with the reduced induced subscheme structure is determined by $\mathcal{I}_E := \mathrm{rad}(\mathcal{I})$, with

$$\mathrm{rad}(\mathcal{I})(D_+(s)) = F_0(\frac{t}{s}) \cdot \mathcal{O}_C(D_+(s))$$

$$\mathrm{rad}(\mathcal{I})(D_+(t)) = G_0(\frac{s}{t}) \cdot \mathcal{O}_C(D_+(t))$$

and over $D_+(st)$, we have $F_0(\frac{t}{s}) = (\frac{t}{s})^{\deg F_0} \cdot G_0(\frac{s}{t})$. So $\mathcal{I}_E \cong \mathcal{O}_C(-\deg F_0)$. By definition we have $\mathcal{O}(E) \cong \mathcal{I}_E^\vee \cong \mathcal{O}_C(-\deg F_0)^\vee = \mathcal{O}_C(\deg F_0)$.

The leading term of $F(x) = 4A(x)^3 + 27B(x)^3$ is $(4a_n^3 + 27b_m^2)x^{12d}$, so if $4a_n^3 + 27b_m^2 \neq 0$ we have $\deg F = 12d$. If $\gcd(F, F') = 1$, then F has no multiple factors so $F = cF_0$ and thus $\deg F_0 = \deg F = 12d$, hence $\mathcal{O}(E) \cong \mathcal{O}_C(12d)$. ☐

Claim 4.14. We have $\Omega^1_{C/k}(E) \cong \mathcal{O}_C(\deg F_0 - 2)$.

4.2.3. The final calculation

Claim 4.15. We have $F_C^* \omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k}(E) \cong \mathcal{O}_C(\deg F_0 - 2 - dp)$. In particular, $\deg F_0 \leq 12d$ so $\deg F_0 - 2 - dp \leq d(12 - p) - 2$.

Proof. It follows from that $F_C^* \omega^\vee_{\ker F_{\mathcal{A}/C}/C} \cong \mathcal{O}_C(-dp)$ and $\Omega^1_{C/k}(E) \cong \mathcal{O}_C(\deg F_0 - 2)$. ☐

Remark 4.16. If we set $p \geq 13$, then $\deg F_0 - 2 - dp < 0$ and so $\Gamma(C, F_C^* \omega^\vee_{\ker F_{\mathcal{A}/C}/C} \otimes_{\mathcal{O}_C} \Omega^1_{C/k}(E)) = \mathrm{Hom}_C(F_C^* \omega_{\ker F_{\mathcal{A}/C}/C}, \Omega^1_{C/k_0}(E)) = 0$. Hence $A^{(p)}(K)/F_{A/K}(A(K))$ is trivial and thus $A^{(p)}(K) \cong A(K)$ by theorem 3.1 where A is the generic abelian variety of $\mathcal{A}/C$ and K is the function field of C.

The author has taken efforts to find a way to compute the Selmer group for this class of examples. Since we are working with isogeny of the form $F_{A/K} : A \to A^{(p)}$ instead of the traditional form $[m] : A \to A$, the results about calculating Selmer groups for $[m]$ do not generalize to $F_{A/K}$, at least not easily. If a method is found, it is probably worth to be included in a new paper.

A. Properties preserved by fppf-torsors

The results in this appendix support the proof of lemma 2.5.

Definition A.1. (cf. [5, tag 022B])Let T be a scheme. An *fpqc* covering of T is a family of morphisms $\{f_i : T_i \to T\}_{i \in I}$ of schemes such that each f_i is flat and such that for every affine open $U \subset T$ there exists $n \geq 0$, a map $a : \{1, \dots, n\} \to I$ and affine opens $V_j \subset T_{a(j)}, j = 1, \dots, n$ with $\bigcup_{j=1}^{n} f_{a(j)}(V_j) = U$.

The definition of *fppf* covering is mentioned in definition 1.7.

Lemma A.2. *Any fppf covering is an fpqc covering.*

Proof. See [5, tag 022C]. ☐

Definition A.3. Let $\mathcal{P}$ be a property of morphisms of schemes over a base. Let $\tau \in \{\text{fpqc}, \text{fppf}\}$. We say $\mathcal{P}$ is τ local on the base if for any τ-covering $\{Y_i \to Y\}_{i \in I}$ and any morphism of schemes $f : X \to Y$ over S we have

$$f \text{ has } \mathcal{P} \Leftrightarrow \text{ each } Y_i \times_Y X \to Y_i \text{ has } \mathcal{P}.$$

Using lemma A.2, we know that any property of morphisms of schemes over a base that is *fpqc* local on the base must be *fppf* local on the base. There is a list of properties of morphisms local in the fpqc topology on the base in [5, tag 02YJ], in particular it contains finite [5, tag 02LA], flat [5, tag 02L2] and surjective [5, tag 02KV].

Definition A.4. (cf. [5, tag 022Z])Let S be a scheme and (G, m, i, e) be a group scheme over S. Then an action of G on the scheme X/S is a morphism $\mathrm{ac}_X : G \times_S X \to X$ over S satisfying the following two diagrams:

$$\begin{tikzcd}
G \times_S G \times_S X \arrow[r, "m \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \mathrm{ac}_X"] & G \times_S X \arrow[d, "\mathrm{ac}_X"] \\
G \times_S X \arrow[r, "\mathrm{ac}_X"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[rr, "e \times \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"'] & & G \times_S X \arrow[dl, "\mathrm{ac}_X"] \\
& X &
\end{tikzcd}$$

Definition A.5. Let S be a scheme and (G, m, i, e) be a group scheme over S. Then an *fppf* G-torsor is a scheme X/S with action morphism $\mathrm{ac}_X : G \times_S X \to X, (g, x) \mapsto g \cdot x$ and an *fppf* covering $\{S_i \to S\}_{i \in I}$ s.t.

1. The induced morphism $(\mathrm{ac}_X, \mathrm{pr}_X) : G \times_S X \to X \times_S X, (g, x) \mapsto (g \cdot x, x)$ is an isomorphism over S;

2. $\forall i \in I$, the morphism $X_{S_i} \to S_i$ has a section.

Lemma A.6. *Let S be a scheme and G be a group scheme over S. Let X be an fppf G-torsor over S. Let $\mathcal{P}$ be a property of morphisms of schemes over a base that is fppf local on the base, then we have*

$$G \to S \textit{ has } \mathcal{P} \Rightarrow X \to S \textit{ has } \mathcal{P}.$$

In particular, $\mathcal{P}$ can be finite, flat or surjective.

Proof. By definition of *fppf* G-torsors, there exists an action morphism $\mathrm{ac}_X : G\times_S X \to X, (g,x) \mapsto g\cdot x$ and an *fppf* covering $\{S_i \to S\}_{i\in I}$ s.t. $(\mathrm{ac}_X, \mathrm{pr}_X) : G \times_S X \to X \times_S X, (g,x) \mapsto (g\cdot x, x)$ is an isomorphism over S and the morphism $X_{S_i} \to S_i$ has a section for all $i \in I$.

Since $\mathcal{P}$ is *fppf* local on the base, we have that that $G_{S_i} \to S_i$ has $\mathcal{P}$ for all $i \in I$ and it suffices to show that $P_{S_i} \to S_i$ has $\mathcal{P}$ for all $i \in I$. It reduces to show that for each $i \in I$, there exists an isomorphism of G_{S_i} and P_{S_i} over S_i. Fix $i \in I$, we may assume $S = S_i$ and that $X \to S$ has a section, now we want to show that G is isomorphic to X over S.

Let $x_0 : S \to X$ be the section of $X \to S$. We construct two morphisms over S as following:

$$\begin{tikzcd}
\alpha :\ G \arrow[r, "g\mapsto(g{,}x_0)"] & G\times_X X \arrow[r, "(g{,}x)\mapsto(g\cdot x{,}x)"] & X\times_S X \arrow[r, "(x{,}y)\mapsto x"] & X \\
\beta :\ X \arrow[r, "x\mapsto(x{,}x_0)"] & X\times_X X \arrow[r, "(x{,}y)\mapsto(g{,}y)", "\text{with } g\cdot y=x"'] & G\times_S X \arrow[r, "(g{,}x)\mapsto g"] & G
\end{tikzcd}$$

The α-path is $g \to (g, x_0) \to (g\cdot x_0, x_0) \to g\cdot x_0$. And the β-path is $x \to (x, x_0) \to (g, x_0) \to g$ with $g \cdot x_0 = x$. Thus $\alpha \circ \beta(x) = \alpha(g) = g\cdot x_0 = x$ and $\beta\circ\alpha(g) = \beta(g\cdot x_0) = g$. So $\alpha\circ\beta = \mathrm{id}_X$ and $\beta\circ\alpha = \mathrm{id}_G$. Hence $G \cong X$ over S and the result follows. ☐

B. Rigid Analytic Space

This appendix covers the lemmas and results about rigid analytic space used in lemma 3.7. Throughout this appendix, K denotes a field complete with respect to a non-trivial non-Archimedean absolute value. Basic knowledge about rigid analytic space is assumed. For a general reference about rigid analytic space, see [31] and [32].

B.1. Analytification functor and torsors over finite rigid K-group

We first introduce the analytification functor and show that it induces an equivalence between finite K-schemes and finite rigid K-spaces.

Lemma B.1. *There exists an analytification functor*

$$X \rightsquigarrow X^{\mathrm{an}}$$

from the category of locally of finite type K-schemes to the category of rigid K-spaces.

Basic results about the analytification functor can be found in [27, Chap. 5]. As this functor preserves finite products (including the final object), it preserves group objects (resp. commutative group objects). So if G is a locally of finite type K-group scheme (resp. commutative K-group scheme), then G^{an} is a rigid K-group (resp. commutative rigid K-group).

Lemma B.2. *Let X be a locally of finite type K-scheme. Then $X(K) = X^{\mathrm{an}}(K)$.*

Proof. We first show that for any rigid space Y, there is a 1-1 correspondence between $Y(K)$ and the set of points of Y with residue field K. Given a map $f : \mathrm{Sp}\, K \to Y$. Denote $y = f(\mathrm{Sp}\, K) \in |Y|$. Then $f^\sharp$ induces a local homomorphism of local rings $f^\sharp_y : \mathcal{O}_{Y,y} \to K$ over K. So $f^\sharp_y$ is surjective. Hence K is the residue field of $\mathcal{O}_{Y,y}$. Thus y is a point of Y with residue field K.

Reversely given a point y of Y with residue field K, there exists a canonical quotient map $\mathcal{O}_{Y,y} \to K$. Pick an open affinoid $U = \operatorname{Sp} A$ around y, the map $A \to \mathcal{O}_{Y,y} \to K$ defines a morphism of rigid K-spaces $\operatorname{Sp} K \to \operatorname{Sp} A \hookrightarrow Y$. The two constructions are clearly inverse to each other.

We know that there exists a canonical continuous map of Grothendieck topological spaces $i : X^{\mathrm{an}} \to X$ which is a bijection onto the set of closed points of X, see [27, 5.1.2]. Plus the fact that $X(K)$ is the set of K-rational points of X, it suffices to show i preserves residue fields.

From [27, 5.1.2], we know the natural local homomorphism of local rings $\mathcal{O}_{X,i(x)} \to \mathcal{O}_{X^{\mathrm{an}},x}$ induces an isomorphism on completions, and so on residue fields by [5, tag 05GG]. The result follows. □

Lemma B.3. *The analytification functor restricting to the category of proper locally of finite type K-schemes is fully faithful. Further it induces an equivalence between the category of finite K-schemes and the category of finite rigid K-spaces.*

Proof. The first statement is one of the classical GAGA results, see [33]. For the second statement, if $X \to \operatorname{Spec} K$ is finite (so quasi-compact), then $X^{\mathrm{an}} \to \operatorname{Sp} K$ is finite by [27, 5.2.1 (2)]. So it suffices to show essentially surjectiveness of the restricted analytification functor.

By [34, 2.2.8], every finite rigid K-space X is isomorphic to $\operatorname{Spec}^{\mathrm{an}} \mathcal{A}$ for some coherent $\mathcal{O}_{\operatorname{Sp} K}$-algebra $\mathcal{A}$. Then $\mathcal{A}$ is associated to a finite dimensional K-algebra A by [32, 9.4.3, 3]. Denote $\mathcal{A}_0$ the coherent $\mathcal{O}_{\operatorname{Spec} K}$-algebra associated to A, then we have $\mathcal{A} = \mathcal{A}_0^{\mathrm{an}}$. By [34, 2.2.5 (3)], $\operatorname{Spec}^{\mathrm{an}}(\mathcal{A}_0^{\mathrm{an}}) \cong (\operatorname{Spec} \mathcal{A}_0)^{\mathrm{an}} = (\operatorname{Spec} A)^{\mathrm{an}}$. So $X \cong (\operatorname{Spec} A)^{\mathrm{an}}$ where $\operatorname{Spec} A$ is a finite K-scheme. □

Later we will interact the analytification functor with Frobenius base change, so we want to know if the analytification functor is sensitive to finite extension of the ground field. The answer to this is no.

Lemma B.4. *The analytification functor is insensitive to finite extension of the ground field, i.e. if L is a finite field extension of K and X is a locally of finite type L-scheme, denote $X^{\mathrm{an}/L}$ the analytification of X as a L-scheme and $X^{\mathrm{an}/K}$ the analytification of X as a K-scheme, then $X^{\mathrm{an}/L} \cong X^{\mathrm{an}/K}$ as both rigid K-spaces and rigid L-spaces.*

Proof. Denote K-maps (resp. L-maps) the maps of locally G-ringed K-spaces (resp. L-spaces). Assume X is affine first. Clearly the canonical L-map $X^{\mathrm{an}/L} \to X$ is also K-map, hence uniquely factors through the canonical K-map $X^{\mathrm{an}/K} \to X$. So we have a K-map $X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}$.

To find a L-map (which will automatically be a K-map) from $X^{\mathrm{an}/K}$ to $X^{\mathrm{an}/L}$, it suffices to show that the canonical K-map $X^{\mathrm{an}/K} \to X$ is actually a L-map, i.e. $X^{\mathrm{an}/K}$ is a rigid L-space.

We know X is affine, say $X \cong \operatorname{Spec} \frac{K[x_1,...,x_n]}{I}$. Pick $c \in K$ s.t. $|c| > 1$. Denote $T_n^{(i)} = K\langle c^{-i}x_1, ..., c^{-i}x_n\rangle$. We know $X^{\mathrm{an}/K} = \bigcup_{i\geq 0} \operatorname{Sp} \frac{T_n^{(i)}}{(I)}$, and $X^{\mathrm{an}/K} \to X$ is determined by the compatible maps $\frac{K[x_1,...,x_n]}{I} \to \frac{T_n^{(i)}}{(I)}$. Let $\frac{T_n^{(i)}}{(I)}$ equips with a L-algebra structure via $L \to \frac{K[x_1,...,x_n]}{I} \to \frac{T_n^{(i)}}{(I)}$ (note that any ring map from a field to a ring is injective). Then $\{\frac{K[x_1,...,x_n]}{I} \to \frac{T_n^{(i)}}{(I)}\}_{i\geq 0}$ are compatible maps of L-algebras. So $X^{\mathrm{an}/K}$ is a rigid L-space and $X^{\mathrm{an}/K} \to X$ is actually a L-map and it induces a L-map $X^{\mathrm{an}/K} \to X^{\mathrm{an}/L}$ which is also a K-map.

By the universal property of $X^{\mathrm{an}/K}$, we have an equality of K-maps

$$(X^{\mathrm{an}/K} \to X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}) = (X^{\mathrm{an}/K} \xrightarrow{\mathrm{id}} X^{\mathrm{an}/K})$$

Next we show that the K-map $X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}$ is also a L-map so we can prove that the L-map $X^{\mathrm{an}/L} \to X^{\mathrm{an}/K} \to X^{\mathrm{an}/L}$ is the identity map hence an identity K-map, it follows that $X^{\mathrm{an}/L} \cong X^{\mathrm{an}/K}$ both as rigid K-spaces and rigid L-spaces.

Just like $X^{\mathrm{an}/K} = \bigcup_{i\geq 0} \operatorname{Sp} \frac{T_n^{(i)}}{(I)}$, we have $X^{\mathrm{an}/L} = \bigcup_{j\geq 0} \operatorname{Sp} A_j$ for some L-algebras A_j. It suffices to show that $\operatorname{Sp} A_j \to X^{\mathrm{an}/K}$ is a L-map for all j. Fix $j \geq 0$, then there exists $\alpha(j) \geq 0$ s.t. $\operatorname{Sp} A_j \to X^{\mathrm{an}/K} \to X$ is induced by

$$\frac{K[x_1, ..., x_n]}{I} \to \frac{T_n^{(\alpha(j))}}{(I)} \to A_j$$

(see [31, 5.4.3]) where the resulting composition is a map of L-algebras, hence all intermediate maps are of L-algebras, and $X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}$ is a L-map and we have finished the affine case.

For a general scheme X locally of finite type over L, again we have a canonical K-map $\psi : X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}$. Now we show $X^{\mathrm{an}/K}$ is a rigid L-space. Pick an open affine cover $\{X_i\}_{i\in I}$ of X. For each pair (i, j), $X_{ij} = X_i \cap X_j$ can be covered by simultaneously distinguished open affines $\{D_{ijk}\}_{k\in S_{ij}}$. We know that $X^{\mathrm{an}/K}$ is obtained by glueing $\{X_i^{\mathrm{an}/K}\}_{i\in I}$ and $X^{\mathrm{an}/L}$ is obtained by glueing $\{X_i^{\mathrm{an}/L}\}_{i\in I}$. To show $X^{\mathrm{an}/K}$ is a rigid L-space, it suffices to show the glueing data of $X^{\mathrm{an}/K}$ over K (see [31, 5.3.5]),

$$\{U_i := X_i^{\mathrm{an}/K}\}_{i\in I},\ \{U_{ij}\}_{i,j\in I},\ \{\phi_{ij} : U_{ij} \overset{\sim}{\to} U_{ji}\}_{i,j\in I}$$

is also over L (i.e. the spaces and the maps are over L): We know $U_i = X_i^{\mathrm{an}/K}$ is a rigid L-space from the affine case, hence all U_{ij}'s are rigid L-spaces, so we just need to show ϕ_{ij} is a L-map. Denote $X_i \cong \operatorname{Spec} R_i$ and $D_{ijk} \cong \operatorname{Spec}(R_i)_{g_k} \cong \operatorname{Spec}(R_j)_{h_k}$ over K so that ϕ_{ij} is glued from

$$\{\phi_{ijk} : (\operatorname{Spec}(R_i)_{g_k})^{\mathrm{an}/K} \to (\operatorname{Spec}(R_j)_{h_k})^{\mathrm{an}/K}\}_{k\in S_{ij}}$$

[32, 9.3.3, Prop. 1]. We know $(\operatorname{Spec}(R_i)_{g_k})^{\mathrm{an}/K}$ and $(\operatorname{Spec}(R_j)_{h_k})^{\mathrm{an}/K}$ are rigid L-spaces, and $(R_i)_{g_k} \cong (R_j)_{h_k}$ as L-algebras, so ϕ_{ijk} are L-maps as well, and so is ϕ_{ij}. It results that the glueing data of $X^{\mathrm{an}/K}$ is valid over L hence $X^{\mathrm{an}/K}$ is a rigid L-space, and there exists a canonical L-map (hence a K-map) $X^{\mathrm{an}/K} \to X^{\mathrm{an}/L}$. So we must have

$$(X^{\mathrm{an}/K} \to X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}) = (X^{\mathrm{an}/K} \overset{\mathrm{id}}{\to} X^{\mathrm{an}/K})$$

Now we use the pasting lemma for L-maps again [32, 9.3.3, Prop. 1]. The K-map $\psi : X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}$ descends to a glueing data of K-maps:

$$\{\psi_i : X_i^{\mathrm{an}/L} \to X_i^{\mathrm{an}/K}\}_{i\in I},\ \{\psi_{ij} : X_{ij}^{\mathrm{an}/L} \to X_{ij}^{\mathrm{an}/K}\}_{i,j\in I}$$

In particular, ψ_i is already a L-map by the proof for affine case. Thus ψ_{ij} is a L-map as a restriction map. So we have a glueing data of L-maps, then we obtain a L-map $\psi_L : X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}$ s.t. $\psi_L = \psi$ as K-maps. So both of the maps $X^{\mathrm{an}/L} \to X^{\mathrm{an}/K}$ and $X^{\mathrm{an}/K} \to X^{\mathrm{an}/L}$ are L-maps, and we must have

$$(X^{\mathrm{an}/L} \to X^{\mathrm{an}/K} \to X^{\mathrm{an}/L}) = (X^{\mathrm{an}/L} \overset{\mathrm{id}}{\to} X^{\mathrm{an}/L})$$

The result follows. ☐

Let G be a group object in a suitable category, then in general a principal homogeneous space over G (or a G-torsor) is an object P with an action map $\mathrm{ac}_P : G \times P \to P, (g, t) \mapsto g \cdot t$ s.t. $(\mathrm{ac}_P, \mathrm{pr}_P) : G \times_K P \to P \times_K P, (g, t) \mapsto (g \cdot t, t)$ is an isomorphism. A map of G-torsors is a map compatible with G-actions. And the set of isomorphism classes of principal homogeneous spaces over G will be denoted $\mathrm{PHS}(G)$. Later we will see that $\mathrm{PHS}(G) \cong \mathrm{PHS}(G^{\mathrm{an}})$ for finite K-group scheme G.

Definition B.5. Let (G, m, i, e) be a rigid K-group. Then a *principal homogeneous space* over G (or a G-torsor) is a rigid K-space P with an action morphism $\mathrm{ac}_P : G \times_K P \to P, (g, t) \mapsto g \cdot t$ s.t. $(\mathrm{ac}_P, \mathrm{pr}_P) : G \times_K P \to P \times_K P, (g, t) \mapsto (g \cdot t, t)$ is an isomorphism. Explicitly, the action map should satisfy the following commutative diagrams

$$\begin{tikzcd}
G \times_K G \times_K P \arrow[r, "m \times \mathrm{id}_P"] \arrow[d, "\mathrm{id}_G \times \mathrm{ac}_P"] & G \times_K P \arrow[d, "\mathrm{ac}_P"] \\
G \times_K P \arrow[r, "\mathrm{ac}_P"] & P
\end{tikzcd}$$

$$\begin{tikzcd}
P \arrow[rr, "e \times \mathrm{id}_P"] \arrow[dr, "\mathrm{id}_P"'] & & G \times_K P \arrow[dl, "\mathrm{ac}_P"] \\
& P &
\end{tikzcd}$$

A morphism of G-torsors is a morphism of rigid K-spaces compatible with G-actions. Denote the set of isomorphism classes of principal homogeneous spaces over G as $\mathrm{PHS}(G)$.

Lemma B.6. *If G is a finite rigid K-group, then there exists a finite K-group scheme H s.t. $G \cong H^{\mathrm{an}}$, and it's unique up to unique isomorphism. And we have* $\mathrm{PHS}(H^{\mathrm{an}}) \cong \mathrm{PHS}(H)$*. If moreover G is commutative, then H is commutative and we have* $\mathrm{PHS}(H^{\mathrm{an}}) \cong \mathrm{H}^1_{\mathrm{fppf}}(K, H)$.

Proof. Using the equivalence of categories between finite K-schemes and finite rigid K-spaces, there exists a finite K-scheme H s.t. $G \cong H^{\mathrm{an}}$ and it's unique up to unique isomorphism. The inverse map, multiplication map and identity section are all algebraizable. So they are the analytification of the maps $i : H \to H, m : H \times_K H \to H, e : \operatorname{Spec} K \to H$ (fiber product of finite rigid K-space is finite, see [32, 9.4.4]). The axioms of H being a (resp. commutative) group object can be checked by passing to the category of finite rigid K-spaces via analytification, which is already true. So H is a finite (resp. commutative) K-group scheme.

It's not hard to see that the analytification functor preserves the axioms of being torsors as well. So there is a canonical map sending an H-torsor Q to an H^{an}-torsor Q^{an}, which induces a well-defined map $\mathrm{PHS}(H) \to \mathrm{PHS}(H^{\mathrm{an}})$. We first show that every H^{an}-torsor arise this way, i.e. $\mathrm{PHS}(H) \to \mathrm{PHS}(H^{\mathrm{an}})$ is surjective.

Let P be an H^{an}-torsor with action map $\mathrm{ac}_P : H^{\mathrm{an}} \times_K P \to P$ and isomorphism $(\mathrm{ac}_P, \mathrm{pr}_P) : H^{\mathrm{an}} \times_K P \to P \times_K P,\ (g, t) \mapsto (g \cdot t, t)$. As $H^{\mathrm{an}} \times_K P \cong P \times_K P$ is finite over P by base change, we want to use a descent argument to show P is finite over K, then it follows that $P \cong Q^{\mathrm{an}}$ for some finite K-scheme Q, and it is easy to see it equips with an H-torsor structure.

For the descent argument, we plan to use the result in [34, 4.2.7]. So it suffices to show the structure map $f : P \to \operatorname{Sp} K$ is a faithfully flat map that admits local fpqc quasi-sections in the sense of [34, 4.2.1]. Clearly, f is faithfully flat. Pick any point $x \in P$, its residue field K' is a finite extension of K and it induces a canonical K-map $\operatorname{Sp} K' \to P$. The composition $\operatorname{Sp} K' \to P \to \operatorname{Sp} K$ is clearly faithfully flat and quasi-compact, so f admits a fpqc quasi-section hence admits local fpqc quasi-sections.

Now we show that $\mathrm{PHS}(H) \to \mathrm{PHS}(H^{\mathrm{an}})$ is injective. If $Q_1^{\mathrm{an}} \cong Q_2^{\mathrm{an}}$ as G-torsors for two H-torsors Q_1 and Q_2, we have $Q_1 \cong Q_2$ as H-torsors using the equivalence in lemma B.3. Hence we have proved that $\mathrm{PHS}(H) \cong \mathrm{PHS}(H^{\mathrm{an}})$. Moreover every H-torsor must be finite over K.

Every H-torsor in the categorical sense (in this case it must be finite over K) is equivalent to an H-torsor in the sense of fppf-topology so there is no ambiguity, see [12, III, 4.1]. If H is commutative then $\mathrm{PHS}(H) \cong \mathrm{H}^1_{\mathrm{fppf}}(K, H)$, see [12, III, (2.10)&(4.7)]. The result follows. □

If $f : G_1 \to G_2$ is a homomorphism of finite commutative rigid K-groups which is the analytification of $g : H_1 \to H_2$ between finite commutative K-group schemes, we can define $\mathrm{PHS}(G_1) \to \mathrm{PHS}(G_2)$ using $\mathrm{H}^1_{\mathrm{fppf}}(K, H_1) \to \mathrm{H}^1_{\mathrm{fppf}}(K, H_2)$. This map is clearly functorial in G.

Given a homomorphism of rigid K-groups $f : G_1 \to G_2$, define $\ker f := G_1 \times_{G_2,e_2} \operatorname{Sp} K$ where $e_2 : \operatorname{Sp} K \to G_2$ is the identity section of G_2. It's not hard to see that it represents the kernel of f: $(\ker f)(T) = \ker(G_1(T) \to G_2(T))$. We can construct an inverse map, a multiplication map and an identity section for it from those on G_1, so that $\ker f$ is a well-defined rigid K-group and the canonical morphism $\ker f \to G_1$ is a categorical monomorphism.

Just like the scheme case, every K-rational point of G_2 lying in $\operatorname{Im} f$ induces a $\ker f$-torsor.

Lemma B.7. *Let $f : G \to H$ be a homomorphism of rigid K-groups, let $t \in H(K)$ be a K-rational point s.t. $G \times_{H,t} \operatorname{Sp} K$ is non-empty, then $G \times_{H,t} \operatorname{Sp} K$ is a principal homogeneous space over* $\ker f$. *(The proof is purely categorical so the lemma extends to any category with fiber products and final objects)*

Proof. Denote $G_t := G \times_{H,t} \operatorname{Sp} K$ together with the projection maps $\mathrm{pr}_1 : G_t \to G$ and $\mathrm{pr}_2 : G_t \to \operatorname{Sp} K$. Denote $g : G_e := \ker f \to G$ the canonical projection map $G_e = G \times_{H,e_H} \operatorname{Sp} K \to G$ and $h : G_e \to \operatorname{Sp} K$ the structure map.

The following commutative diagram

$$\begin{tikzcd}
G_e \times_K G_t \arrow[r, "g \times \mathrm{pr}_1"] \arrow[d, "h \times \mathrm{pr}_2"] & G \times_K G \arrow[r, "m_G"] \arrow[d, "f \times f"] & G \arrow[d, "f"] \\
\operatorname{Sp} K \times_K \operatorname{Sp} K \arrow[r, "e_H \times t"] \arrow[d, equal] & H \times_K H \arrow[r, "m_H"] & H \\
\operatorname{Sp} K \arrow[urr, "t"'] & &
\end{tikzcd}$$

induces the candidate for the action morphism $\rho : G_e \times_K G_t \to G_t$. Next we check that $\rho \circ (m_{G_e} \times \mathrm{id}_{G_t}) = \rho \circ (\mathrm{id}_{G_e} \times \rho)$. As $\mathrm{pr}_1 : G_t \to G$ is a monomorphism (by checking the injection of $G_t(T) \to G(T)$), it suffices to check $\mathrm{pr}_1 \circ \rho \circ (m_{G_e} \times \mathrm{id}_{G_t}) = \mathrm{pr}_1 \circ \rho \circ (\mathrm{id}_{G_e} \times \rho)$, which can be shown in the following commutative diagram:

$$\begin{tikzcd}
G_e \times_K G_e \times_K G_t \arrow[r, "m_{G_e} \times \mathrm{id}_{G_t}"] \arrow[d, "g \times g \times \mathrm{pr}_1"] & G_e \times_K G_t \arrow[r, "\rho"] \arrow[d, "g \times \mathrm{pr}_1"] & G_t \arrow[d, "\mathrm{pr}_1"] \\
G \times_K G \times_K G \arrow[r, "m_G \times \mathrm{id}_G"] \arrow[d, equal] & G \times_K G \arrow[r, "m_G"] & G \arrow[d, equal] \\
G \times_K G \times_K G \arrow[r, "\mathrm{id}_G \times m_G"] & G \times_K G \arrow[r, "m_G"] & G \\
G_e \times_K G_e \times_K G_t \arrow[r, "\mathrm{id}_{G_e} \times \rho"] \arrow[u, "g \times g \times \mathrm{pr}_1"] & G_e \times_K G_t \arrow[r, "\rho"] \arrow[u, "g \times \mathrm{pr}_1"'] & G_t \arrow[u, "\mathrm{pr}_1"]
\end{tikzcd}$$

Next, we check $\rho \circ (e_{G_e} \circ \mathrm{pr}_2, \mathrm{id}_{G_t}) = \mathrm{id}_{G_t}$, again it suffices to check $\mathrm{pr}_1 \circ \rho \circ (e_{G_e} \circ \mathrm{pr}_2, \mathrm{id}_{G_t}) = \mathrm{pr}_1 \circ \mathrm{id}_{G_t}$, which can be shown in the following commutative diagram:

$$\begin{tikzcd}
G_t \arrow[r] \arrow[d, "\mathrm{pr}_1"] & \mathrm{Sp}\, K \times_K G_t \arrow[r, "e_{G_e} \times \mathrm{id}_{G_t}"] \arrow[d, "\mathrm{id}_K \times \mathrm{pr}_1"] & G_e \times_K G_t \arrow[r, "\rho"] \arrow[d, "g \times \mathrm{pr}_1"] & G_t \arrow[d, "\mathrm{pr}_1"] \\
G \arrow[r] \arrow[d, equal] & \mathrm{Sp}\, K \times_K G \arrow[r, "e_G \times \mathrm{id}_G"] & G \times_K G \arrow[r, "m_G"] & G \arrow[d, equal] \\
G \arrow[rrr, "\mathrm{id}_G"] & & & G \\
G_t \arrow[u, "\mathrm{pr}_1"] \arrow[rrr, "\mathrm{id}_{G_t}"] & & & G_t \arrow[u, "\mathrm{pr}_1"']
\end{tikzcd}$$

Hence ρ is indeed an action morphism.

Now we check that $(\rho, \mathrm{pr}_{G_t}) : G_e \times_K G_t \to G_t \times_K G_t, (\alpha, \beta) \mapsto (\alpha \cdot \beta, \beta)$ is an isomorphism. It suffices to construct an inverse representing $(\alpha, \beta) \mapsto (\alpha \cdot \beta^{-1}, \beta)$, or just $(\alpha, \beta) \mapsto \alpha \cdot \beta^{-1}$ embedding in G. Consider the following commutative diagram:

$$\begin{tikzcd}
G_t \times_K G_t \arrow[r, "\mathrm{pr}_1 \times \mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2 \times \mathrm{pr}_2"] & G \times_K G \arrow[r, "\mathrm{id}_G \times i_G"] \arrow[d, "f \times f"] & G \times_K G \arrow[r, "m_G"] \arrow[d, "f \times f"] & G \arrow[d, "f"] \\
\mathrm{Sp}\, K \times_K \mathrm{Sp}\, K \arrow[r, "t \times t"] \arrow[d, equal] & H \times_K H \arrow[r, "\mathrm{id}_H \times i_H"] & H \times_K H \arrow[r, "m_H"] & H \\
\mathrm{Sp}\, K \arrow[rrru, "e_H"] & & &
\end{tikzcd}$$

We can see the diagram induces the desired map $G_t \times_K G_t \to G_e, (\alpha, \beta) \mapsto \alpha \cdot \beta^{-1}$. Hence $(\rho, \mathrm{pr}_{G_t}) : G_e \times_K G_t \to G_t \times_K G_t$ is an isomorphism and G_t is a G_e-torsor. ☐

Recall that a short exact sequence of abelian sheaves on a site $\mathcal{C}$

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

has an associated long exact sequence

$$0 \to \mathrm{H}^0(\mathcal{C}, \mathcal{F}_1) \to \mathrm{H}^0(\mathcal{C}, \mathcal{F}_2) \to \mathrm{H}^0(\mathcal{C}, \mathcal{F}_3) \to \mathrm{H}^1(\mathcal{C}, \mathcal{F}_1) \to \cdots$$

and $\mathrm{H}^1(\mathcal{C}, \mathcal{F}_1)$ is canonically bijective with the set of isomorphism classes of $\mathcal{F}_1$-torsors, see [5, tag 03AJ]. The following lemma is a similar result in rigid K-spaces.

Proposition B.8. *Let $f : G \to H$ be a homomorphism of rigid K-groups which is surjective on the level of topological space (actually $H(K) \subset \mathrm{Im}\, f$ will suffice), then there exists a natural map of sets $\delta : H(K) \to \mathrm{PHS}(\ker f)$. If moreover $\ker f$ is finite and everything is commutative, then δ is a group homomorphism.*

Proof. For the first statement, with the previous lemma, it suffices to show that for all $t \in H(K)$, $E_t := G \times_{H,t} \mathrm{Sp}\, K$ is non-empty. As f is surjective, there exists a point $x \in G$ s.t. $f(x) = t$. The residue field K' at x is a finite extension of K and induces a map $\mathrm{Sp}\, K' \to G$. So we have a commutative square

$$\begin{tikzcd}
\mathrm{Sp}\, K' \arrow[r] \arrow[d, "x"] & \mathrm{Sp}\, K \arrow[d, "t"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

which induces a map $\operatorname{Sp} K' \to G \times_{H,t} \operatorname{Sp} K = E_t$. Hence E_t contains at least one point so it is non-empty.

Now assume $\ker f$ is finite and everything is commutative so $\ker f \cong B^{\text{an}}$ for some finite commutative K-group scheme B and $\operatorname{PHS}(\ker f) \cong \mathrm{H}^1_{\text{fppf}}(K, B)$ has a group structure. To show δ is a group homomorphism, we need to show $\delta(ab) = \delta(a) \cdot \delta(b)$ where $a, b \in H(K)$.

Let $a, b \in H(K)$ then $a \cdot b = m_H \circ (a, b)$. We want to show $E_a \cdot E_b = E_{a \cdot b}$ in $\mathrm{H}^1_{\text{fppf}}(K, B)$. We know E_a is finite by descent, so it is of the form P_a^{an} for some finite K-scheme P_a. There exists an isomorphism $B \times_K P_a \cong P_a \times_K P_a$. Take $\mathrm{id}_{P_a} \in P_a(P_a)$, there exists a unique $s_a \in B(P_a \times_K P_a)$ s.t. $s_a \cdot (\mathrm{id}_{P_a})|_1 = (\mathrm{id}_{P_a})|_2$ (since $P_a(P_a \times_K P_a)$ is a $B(P_a \times_K P_a)$-torsor). So s_a^{an} is a map in $\ker f(E_a \times_K E_a)$ s.t.

$$s_a^{\text{an}} \cdot (E_a \times_K E_a \overset{\mathrm{pr}_1}{\to} E_a) = (E_a \times_K E_a \overset{\mathrm{pr}_2}{\to} E_a).$$

Explicitly, we can embed E_a into G to calculate s_a^{an} (by abusing notation, we will write s_a for s_a^{an}), $s_a : E_a \times_K E_a \to \ker f$ is induced by the following commutative diagram:

$$\begin{tikzcd}
E_a \times_K E_a \arrow[r] \arrow[d] & G \times_K G \arrow[r, "(u,v)\mapsto vu^{-1}"] \arrow[d] & G \arrow[d] \\
\operatorname{Sp} K \times_K \operatorname{Sp} K \arrow[r, "a\times a"] \arrow[d, equal] & H \times_K H \arrow[r, "(u,v)\mapsto vu^{-1}"] & H \\
\operatorname{Sp} K \arrow[urr, "e_H"'] & &
\end{tikzcd}$$

(or we have $s_a = E_a \times_K E_a \overset{(u,v)\to(v,u)}{\to} E_a \times_K E_a \overset{\sim}{\to} \ker f \times_K E_a \to \ker f$, their equality can be proved by composing with $\ker f \to G$ and showing that $E_a \times_K E_a \to G$ equals the map defined by $(u, v) \mapsto (uv^{-1})$.)

Actually, we can consider s_a as an element of $G(E_a \times_K E_a)$, which is just

$$s_a : E_a \times_K E_a \to G \times_K G \overset{(u,v)\mapsto vu^{-1}}{\longrightarrow} G.$$

It defines a 1-cocycle for B relative to the covering $\{P_a \to \operatorname{Spec} K\}$, see [12, pp. 122-123].

Similarly, after viewing $\ker f(E_a \times_K E_a)$ as a subset of $G(E_a \times_K E_a)$ we have

$$s_b : E_b \times_K E_b \to G \times_K G \overset{(u,v)\mapsto vu^{-1}}{\longrightarrow} G$$

defining a 1-cocycle for B relative to the covering $\{P_b \to \operatorname{Spec} K\}$, and

$$s_{a\cdot b} : E_{a\cdot b} \times_K E_{a\cdot b} \to G \times_K G \overset{(u,v)\mapsto vu^{-1}}{\longrightarrow} G$$

defining a 1-cocycle for B relative to the covering $\{P_{a\cdot b} \to \operatorname{Spec} K\}$.

Now we just need to find a way to compare them relative to the same covering. Consider the following commutative diagram:

$$\begin{tikzcd}
E_a \times_K E_b \arrow[r] \arrow[d] & G \times_K G \arrow[r, "m_G"] \arrow[d] & G \arrow[d] \\
\operatorname{Sp} K \times_K \operatorname{Sp} K \arrow[r, "a\times b"] \arrow[d, equal] & H \times_K H \arrow[r, "m_H"] & H \\
\operatorname{Sp} K \arrow[urr, "a\cdot b"'] & &
\end{tikzcd}$$

We can see there exists a unique morphism $\omega : E_a \times_K E_b \to E_{a\cdot b}$. So we can pull each covering back to the covering $\{P_a \times_K P_b \to \operatorname{Spec} K\}$.

Define t_a, t_b and $t_{a\cdot b}$ as following

$$\begin{aligned} t_a &= s_a \circ (E_a \times_K E_b \times_K E_a \times_K E_b \to E_a \times_K E_a), \\ t_b &= s_b \circ (E_a \times_K E_b \times_K E_a \times_K E_b \to E_b \times_K E_b), \\ t_{a\cdot b} &= s_{a\cdot b} \circ (E_a \times_K E_b \times_K E_a \times_K E_b \overset{\omega\times\omega}{\to} E_{a\cdot b} \times_K E_{a\cdot b}). \end{aligned}$$

They are the corresponding 1-cocyles for B relative to the covering $\{P_a \times_K P_b \to \operatorname{Spec} K\}$. It suffices to show $t_{a\cdot b} = t_a \cdot t_b$, which can be obtained from the following commutative diagram:

$$\begin{tikzcd}
E_a \times E_b \times E_a \times E_b \arrow[r, red, "\omega\times\omega"] \arrow[d] & E_{a\cdot b} \times E_{a\cdot b} \arrow[r, red] \arrow[d] & G \times G \arrow[r, red, "(s,t)\mapsto ts^{-1}"] & G \arrow[d, equal] \\
G \times G \times G \times G \arrow[r, "m_G \times m_G"] \arrow[d, equal] & G \times G \arrow[rr, "(s,t)\mapsto ts^{-1}"] & & G \arrow[d, equal] \\
G \times G \times G \times G \arrow[rrr, "(s,t,u,v)\to(st,uv)\to(uvt^{-1}s^{-1})"] \arrow[d, equal] & & & G \arrow[d, equal] \\
G \times G \times G \times G \arrow[rrr, "(s,t,u,v)\to(s,u,t,v)\to(us^{-1},vt^{-1})\to(us^{-1}vt^{-1})"] & & & G \\
E_a \times E_b \times E_a \times E_b \arrow[r, blue] \arrow[u] & E_a \times E_a \times E_b \times E_b \arrow[r, blue] & G \times G \times G \times G \arrow[r, blue, "(s,t,u,v)\mapsto(ts^{-1},vu^{-1})"] & G \times G \arrow[u, blue, "m_G"']
\end{tikzcd}$$

where the red arrows represent $t_{a\cdot b}$ and the blue arrows represent $t_a \cdot t_b$. Note that we have $uvt^{-1}s^{-1} = us^{-1}vt^{-1}$ because G is commutative. Hence δ is a group homomorphism. □

Corollary B.9. *δ is functorial with respect to surjective homomorphism of commutative rigid K-groups with finite kernel. Explicitly, assume there exists a commutative diagram of commutative rigid K-groups whose columns are surjective*

$$\begin{tikzcd}
G' \arrow[r, "i"] \arrow[d, "f"] & H' \arrow[d, "g"] \\
G'' \arrow[r, "j"] & H''
\end{tikzcd}$$

and both $\ker f_2 \cong G^{\mathrm{an}}$ and $\ker g_2 \cong H^{\mathrm{an}}$ are finite. Then the following diagram of groups commutes:

$$\begin{tikzcd}
G''(K) \arrow[r, "j'"] \arrow[d, "\delta_G"] & H''(K) \arrow[d, "\delta_H"] \\
\mathrm{PHS}(G^{\mathrm{an}}) \arrow[r, "\psi"] & \mathrm{PHS}(H^{\mathrm{an}})
\end{tikzcd}$$

Proof. Take $a \in G''(K)$, we define $E_a = G' \times_{G'',a} \operatorname{Sp} K$ and denote $E_a \cong P_a^{\mathrm{an}}$. By arguments from the previous proposition, we know the morphism $s_a \in G^{\mathrm{an}}(E_a \times_K E_a) \subset G'(E_a \times_K E_a)$ (we abuse the notation s_a here)

$$s_a : E_a \times_K E_a \to G' \times_K G' \overset{(u,v)\mapsto vu^{-1}}{\longrightarrow} G'$$

defines a 1-cocycle for G relative to the covering $\{P_a \to \operatorname{Spec} K\}$.

Then $i \circ s_a \in H'(E_a \times_K E_a)$ defines the 1 cocycle $\psi(\delta_G(a))$ for H relative to the covering $\{P_a \to \operatorname{Spec} K\}$ because it clearly factors through H^{an}. In the other direction, define $E_{j'(a)} = H' \times_{H'', j'(a)} \operatorname{Sp} K$ and denote $E_{j'(a)} \cong P^{\text{an}}_{j'(a)}$. Similarly we have $s_{j'(a)} \in H'(E_{j'(a)} \times_K E_{j'(a)})$

$$s_{j'(a)} : E_{j'(a)} \times_K E_{j'(a)} \to H' \times_K H' \overset{(u,v)\mapsto vu^{-1}}{\longrightarrow} H'$$

defining a 1-cocyle for H relative to the covering $\{P_{j'(a)} \to \operatorname{Spec} K\}$.

Clearly there exist a canonical morphism $\omega : E_a \to E_{j'(a)}$. So we just need to identify $i \circ s_a$ with $s_{j'(a)} \circ (\omega \times \omega)$. So it suffices to show the commutativity of the following diagram.

$$\begin{tikzcd}
E_a \times_K E_a \arrow[r] \arrow[d, "\omega\times\omega"] & G' \times_K G' \arrow[r, "(u,v)\mapsto vu^{-1}"] \arrow[d, "i\times i"] & G' \arrow[d, "i"] \\
E_{j'(a)} \times_K E_{j'(a)} \arrow[r] & H' \times_K H' \arrow[r, "(u,v)\mapsto vu^{-1}"] & H'
\end{tikzcd}$$

The commutativity of both squares are automatic. The result follows. ☐

B.2. Analytification and Frobenii in the category of rigid K-spaces

Throughout this subsection we assume $\operatorname{Char} K = p$ and that the absolute Frobenius map F_K on K is a **finite extension**.

Denote K' ($K^{\frac{1}{p}}$ in some reference) by the Frobenius extension of K via $F_K : K \to K'$, and denote by K^p the sub-field of K which is the image of the absolute Frobenius map. Since K' is a finite field extension of K, it is an affinoid K-algebra. So $F_{\operatorname{Sp} K} : \operatorname{Sp} K' \to \operatorname{Sp} K$ is a well-defined morphism of rigid K-spaces.

Similarly let A be an affinoid K-algebra, together with an epimorphism $K\langle x_1, ..., x_m\rangle \to A$. Denote $A' = A$ as a K-algebra via $K \to K' \to A'$ and we want to show that A' is an affinoid K-algebra, i.e. there exists a K-epimorphism $K\langle y_1, ..., y_n\rangle \to A'$. Clearly $K'\langle x_1, ..., x_m\rangle \to A'$ is an epimorphism of K'-algebras hence an epimorphism of K-algebras. We know $K'\langle x_1, ..., x_m\rangle \cong K'\hat{\otimes}_K K\langle x_1, ..., x_m\rangle$ (see [31, Appx. B, 5]) is also affinoid with an epimorphism $K\langle y_1, ..., y_n\rangle \to K'\hat{\otimes}_K K\langle x_1, ..., x_m\rangle$. Hence there exists an epimorphism of K-algebra

$$K\langle y_1, ..., y_n\rangle \to K'\hat{\otimes}_K K\langle x_1, ..., x_m\rangle \to A'$$

making A' affinoid over K.

And we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "F_A"] & A' \\
K \arrow[r] \arrow[u] & K' \arrow[u]
\end{tikzcd}$$

of affinoid K-algebras. In particular we have a forgetful functor from the category of affinoid K'-algebras to the category of affinoid K-algebras. And it induces a forgetful functor from the category of rigid K'-spaces to the category of rigid K-space.

Explicitly, let $(X, \mathcal{O}_X)$ be a rigid K-space, then we have the rigid K'-space X' s.t. $(X', \mathcal{O}_{X'}) = (X, \mathcal{O}_X)$ as a locally G-ringed space, but unlike $\mathcal{O}_X$ which is a sheaf of K-algebras, we have that $\mathcal{O}_{X'}$ is a sheaf of K'-algebras. We can also view $\mathcal{O}_{X'}$ as a sheaf of K-algebras via F_K: for any admissible open subset

V, its K-algebra structure is $K \to K' \to \mathcal{O}_{X'}(V)$. If $\{X_i \cong \operatorname{Sp} A_i\}_{i\in I}$ is an admissible covering of X by affinoid spaces, then $\{X_i \cong \operatorname{Sp} A_i'\}_{i\in I}$ is an admissible covering of X' by affinoid spaces, where $K \to K' \to A_i'$ is an affinoid K-algebra.

We can define the *absolute Frobenius morphism* on X as

$$(F_X, F_X^*) : (X', \mathcal{O}_{X'}) \to (X, \mathcal{O}_X)$$

where $F_X : X' \to X$ is the identity map on the topological space and $F_{X,V}^* : \mathcal{O}_X(V) \to \mathcal{O}_{X'}(V)$, $t \mapsto t^p$ is the absolute Frobenius morphism on the algebra $\mathcal{O}_X(V)$. Any absolute Frobenius homomorphism on a local ring is a local homomorphism of local rings, so it is a well defined map of locally G-ringed space. Clearly this construction is compatible with the algebraic Frobenius map when X is affinoid.

Lemma B.10. *Let $f : X \to Y$ be a morphism of rigid K-spaces, then the following diagram commutes*

$$\begin{tikzcd}
X' \arrow[r, "F_X"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
Y' \arrow[r, "F_Y"] & Y
\end{tikzcd}$$

Proof. It suffices to check the commutativity as locally G-ringed K-spaces. Clearly it commutes on the level of topological spaces. For morphisms of sheaves, let V be an admissible open subset of Y, then the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_Y(V) \arrow[r, "F_{Y,V}^*"] \arrow[d, "f_V^*"] & \mathcal{O}_{Y'}(V) \arrow[d, "(f')_V^*"] \\
\mathcal{O}_X(f^{-1}(V)) \arrow[r, "F_{X,f^{-1}(V)}^*"] & \mathcal{O}_{X'}(f^{-1}(V))
\end{tikzcd}$$

This is compatible with restriction maps. So the morphisms of sheaves commute. The result follows. □

Definition B.11. Let X be a rigid K-space with structure morphism $f : X \to \operatorname{Sp} K$. Denote $X^{(p)} := X \times_{K, F_{\operatorname{Sp} K}} \operatorname{Sp} K$ where the second factor $\operatorname{Sp} K$ is endowed with a structure of rigid K-space via $F_{\operatorname{Sp} K} : \operatorname{Sp} K \to \operatorname{Sp} K$. So besides the obvious rigid K-space structure of $X^{(p)}$ as a fiber product, we can endow $X^{(p)}$ with the structure of a rigid K-space via the second projection $X \times_{K, F_{\operatorname{Sp} K}} \operatorname{Sp} K \to \operatorname{Sp} K$. We always assume its structure morphism is the second one unless otherwise stated. We define the relative Frobenius morphism $F_{X/K} : X \to X^{(p)}$ as the following uniquely induced map

$$\begin{tikzcd}
X \arrow[rrd, bend left, "F_X"] \arrow[rd, dotted, "F_{X/K}"'] \arrow[rdd, bend right, "f"'] & & \\
& X^{(p)} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \operatorname{Sp} K \arrow[r, "F_{\operatorname{Sp} K}"] & \operatorname{Sp} K
\end{tikzcd}$$

It is a morphism of rigid K-spaces.

Lemma B.12. *Let $f : X \to Y$ be a morphism of rigid K-spaces. Then the following diagram commutes*

$$\begin{tikzcd}
X \arrow[r, "F_{X/K}"] \arrow[d, "f"] & X^{(p)} \arrow[d, "f^{(p)}"] \\
Y \arrow[r, "F_{Y/K}"] & Y^{(p)}
\end{tikzcd}$$

Proof. It follows from lemma B.10 and definitions. □

Lemma B.13. *Let X be a rigid K-group, then $X^{(p)}$ is a rigid K-group and $F_{X/K}$ is a homomorphism of rigid K-groups.*

Proof. As X is a rigid K-group, $X^{(p)}$ is a rigid K-group as the base change of X with $F_{\operatorname{Sp} K} : \operatorname{Sp} K \to \operatorname{Sp} K$. Denote the multiplication map of X as $m : X \times_K X \to X$. Then the multiplication map of $X^{(p)}$ is $m^{(p)} : X^{(p)} \times_K X^{(p)} \to X^{(p)}$. So to show the lemma, it suffices to show the following diagram commutes:

$$\begin{tikzcd}
X \times_K X \arrow[r, "m"] \arrow[d, "F_{X/K} \times F_{X/K}"'] & X \arrow[d, "F_{X/K}"] \\
X^{(p)} \times_K X^{(p)} \arrow[r, "m^{(p)}"] & X^{(p)}
\end{tikzcd}$$

Consider above lemma and the fact $(X \times_K X)^{(p)} = (X \times_K X) \times_{K, F_{\operatorname{Sp} K}} \operatorname{Sp} K \cong X^{(p)} \times_K X^{(p)}$, it suffices to show that $F_{X/K} \times F_{X/K} = F_{(X \times_K X)/K} : X \times_K X \to (X \times_K X)^{(p)}$. We can use the universal property of fiber products, denote the projection maps $\operatorname{pr}_1 : (X \times_K X)^{(p)} \to X \times_K X$ and $\operatorname{pr}_2 : (X \times_K X)^{(p)} \to \operatorname{Sp} K$, it suffices to show $\operatorname{pr}_i \circ (F_{X/K} \times F_{X/K}) = \operatorname{pr}_i \circ F_{(X \times_K X)/K}$ for $i = 1, 2$.

The case $i = 2$ is trivial as both $F_{X/K} \times F_{X/K}$ and $F_{(X \times_K X)/K}$ are morphisms of rigid K-spaces so that we get the structure morphism of the source after precomposing with the structure morphism of the target.

In the case $i = 1$, we have

$$\begin{aligned}
& X \times_K X \overset{F_{X/K} \times F_{X/K}}{\to} X^{(p)} \times_K X^{(p)} \overset{\operatorname{pr}_1}{\to} X \times_K X \\
= {} & X \times_K X \overset{F_{X/K} \times F_{X/K}}{\to} X^{(p)} \times_K X^{(p)} \overset{\operatorname{pr} \times \operatorname{pr}}{\to} X \times_K X \\
= {} & X \times_K X \overset{F_X \times F_X}{\to} X \times_K X
\end{aligned}$$

and $\operatorname{pr}_i \circ F_{(X \times_K X)/K} = F_{X \times_K X}$. So it reduces to showing $F_X \times F_X = F_{X \times_K X}$. When X is affinoid, the equality trivially follows from the fact $F_{A \hat{\otimes}_R B} : A \hat{\otimes}_R B \to A \hat{\otimes}_R B$ is induced by $A \overset{F_A}{\to} A \hookrightarrow A \hat{\otimes}_R B$ and $B \overset{F_B}{\to} B \hookrightarrow A \hat{\otimes}_R B$.

For general rigid K-space X, choose an admissible covering $(X_i)_{i \in I}$ of X by affinoid spaces, then both $F_X \times F_X|_{X_i \times_K X_j}, F_{X \times_K X}|_{X_i \times_K X_j} : X_i \times_K X_j \to X \times_K X$ factor through the open subspace $X_i \times_K X_j$. They agree by the affinoid result. So $F_X \times F_X$ and $F_{X \times_K X}$ have the same glueing data, and so they must be the same map, see [31, 5.3.6]. The result follows. □

Moreover, Frobenius base change and the relative Frobenius morphism commute with the analytification functor.

Lemma B.14. *Let X be a locally of finite type K-scheme, then $(X^{(p)})^{\mathrm{an}} \cong (X^{\mathrm{an}})^{(p)}$, $F_X^{\mathrm{an}} = F_{X^{\mathrm{an}}}$, $F_{X/K}^{\mathrm{an}} = F_{X^{\mathrm{an}}/K}$.*

Proof. As analytification is insensitive to finite extension of ground field (see lemma B.4) and by assumption F_K is finite, the analytification of $X^{(p)}$ as a K'-scheme via the projection $X^{(p)} \to \operatorname{Sp} K'$ is the same with the analytification as K-scheme via $X^{(p)} \to \operatorname{Sp} K' \overset{F_{\mathrm{Sp}}}{\to} \operatorname{Sp} K$. So $(X^{(p)})^{\mathrm{an}} = (X \times_{K, F_K} K')^{\mathrm{an}}$ is a rigid K-space and a rigid K'-space. Clearly $(X \times_{K, F_K} K')^{\mathrm{an}} \cong X^{\mathrm{an}} \times_{K, F_{\operatorname{Sp} K}} \operatorname{Sp} K' = (X^{\mathrm{an}})^{(p)}$ as rigid K-spaces, now we just need to show this isomorphism is actually over K', which is trivial because the canonical map $(X \times_{K,F_K} K')^{\mathrm{an}} \to X^{\mathrm{an}} \times_{K,F_{\mathrm{Sp}\,K}} \mathrm{Sp}\,K'$ is determined by $(X \times_{K,F_K} K')^{\mathrm{an}} \to X^{\mathrm{an}}$ and $(X \times_{K,F_K} K')^{\mathrm{an}} \to \mathrm{Sp}\,K'$, in particular $(X \times_{K,F_K} K')^{\mathrm{an}} \to X^{\mathrm{an}} \times_{K,F_{\mathrm{Sp}\,K}} \mathrm{Sp}\,K'$ is a map over $\mathrm{Sp}\,K'$. So $(X^{(p)})^{\mathrm{an}} \cong (X^{\mathrm{an}})^{(p)}$ both as rigid K-spaces and rigid K'-spaces.

Now consider $F_X^{\mathrm{an}},\ F_{X^{\mathrm{an}}} : X^{\mathrm{an}} \to X^{\mathrm{an}}$, they are comparable (i.e. their sources and targets agree respectively) because analytification is insensitive to finite extensions of ground fields. Using [32, 9.3.3, Prop. 1], we can safely assume $X \cong \mathrm{Spec}\,A$ is affine, so $X^{\mathrm{an}} = \bigcup_{i \geq 0} \mathrm{Sp}\,A_i$ for some K-algebras. Using [32, 9.3.3, Prop. 1] again, it suffices to show $F_X^{\mathrm{an}}|_{\mathrm{Sp}\,A_i} = F_{X^{\mathrm{an}}}|_{\mathrm{Sp}\,A_i}$ for all i.

Clearly, $F_{X^{\mathrm{an}}}$ restricting to $\mathrm{Sp}\,A_i$ is determined by $A_i \overset{F_{A_i}}{\to} A_i$. And $F_X^{\mathrm{an}}|_{\mathrm{Sp}\,A_i}$ is determined by $\mathrm{Sp}\,A_i \hookrightarrow X^{\mathrm{an}} \to X \overset{F_X}{\to} X$, which is induced by $(A \overset{F_A}{\to} A \to A_i) = (A \to A_i \overset{F_{A_i}}{\to} A_i)$. Therefore $F_X^{\mathrm{an}}|_{\mathrm{Sp}\,A_i}$ is also determined by $A_i \overset{F_{A_i}}{\to} A_i$. So $F_X^{\mathrm{an}} = F_{X^{\mathrm{an}}}$.

Next we consider $F_{X/K}^{\mathrm{an}},\ F_{X^{\mathrm{an}}/K} : X^{\mathrm{an}} \to (X^{(p)})^{\mathrm{an}} \cong (X^{\mathrm{an}})^{(p)}$. Denote the projection maps $\mathrm{pr}_1 : (X^{\mathrm{an}})^{(p)} \to X^{\mathrm{an}}$ and $\mathrm{pr}_2 : (X^{\mathrm{an}})^{(p)} \to \mathrm{Sp}\,K$. Then it suffices to show that $\mathrm{pr}_i \circ F_{X/K}^{\mathrm{an}} = \mathrm{pr}_i \circ F_{X^{\mathrm{an}}/K}$ for $i = 1, 2$. The case $i = 2$ is trivial as both $F_{X/K}^{\mathrm{an}}$ and $F_{X^{\mathrm{an}}/K}$ are morphisms of rigid K-spaces so that we get the structure morphism of the source after precomposing with the structure morphism of the target. For the case $i = 1$, we have

$$\begin{aligned} \mathrm{pr}_1 \circ F_{X/K}^{\mathrm{an}} &= F_X^{\mathrm{an}} \\ &= F_{X^{\mathrm{an}}} \\ &= \mathrm{pr}_1 \circ F_{X^{\mathrm{an}}/K} \end{aligned}$$

The result follows. □

B.3. Frobenii in the category of formal schemes and completion

We used formal schemes in our proof, e.g. the exact sequence of formal groups (3.3). Let R be a valuation ring of a suitable field K, then the path $X \to \hat{X} \to \hat{X}^{\mathrm{rig}}$ from a suitable R-scheme to a formal R-scheme (by completion) to a rigid K-space (by a rigidification functor to be introduced in next subsection), provides an alternative method to construct rigid K-spaces. And we will study their interaction with the Frobenius operators at each step to make sure an absolute (resp. relative) Frobenius map stays absolute (resp. relative) Frobenius all the way to the end.

For preliminary knowledge on formal schemes, we refer to [35, Chap. I, 10] and [25, 8]. A formal scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is called of characteristic p if there exists a morphism of formal schemes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathrm{Spf}\,\mathbb{F}_p$ where $\mathbb{F}_p$ is seen as an admissible topological ring with discrete topology.

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a formal scheme of characteristic p, we define the absolute Frobenius morphism of $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ as

$$(F_{\mathcal{X}}, F_{\mathcal{X}}^*) : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

where $F_{\mathcal{X}} : \mathcal{X} \to \mathcal{X}$ is the identity map on the topological space and $F_{\mathcal{X},V}^* : \mathcal{O}_{\mathcal{X}}(V) \to \mathcal{O}_{\mathcal{X}}(V),\ t \mapsto t^p$ is the absolute Frobenius morphism on the algebra $\mathcal{O}_{\mathcal{X}}(V)$. As the multiplication map on a topological ring is continuous, so is the map $t \mapsto t^p$. Any absolute Frobenius homomorphism on a local ring is a local homomorphism of local rings, so it is a well defined map of locally topologically ringed spaces. If $\mathcal{X} = \mathrm{Spf}\,A$ is an affine formal scheme, then $F_{\mathcal{X}}$ is induced by the absolute Frobenius morphism on A, $F_A : A \to A,\ a \mapsto a^p$.

Lemma B.15. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of formal schemes of characteristic p, then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F_{\mathcal{X}}"] \arrow[d, "f"] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{Y} \arrow[r, "F_{\mathcal{X}}"] & \mathcal{Y}
\end{tikzcd}$$

Proof. It suffices to check the commutativity as locally topologically ringed spaces. Clearly it commutes on the level of topological spaces. For morphisms of sheaves, let V be an open subset of $\mathcal{Y}$, then the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{Y}}(V) \arrow[r, "F^*_{\mathcal{Y},V}"] \arrow[d, "f^*_V"] & \mathcal{O}_{\mathcal{Y}}(V) \arrow[d, "f^*_V"] \\
\mathcal{O}_{\mathcal{X}}(f^{-1}(V)) \arrow[r, "F^*_{\mathcal{X},f^{-1}(V)}"] & \mathcal{O}_{\mathcal{X}}(f^{-1}(V))
\end{tikzcd}$$

This is compatible with restriction maps. So the morphisms of sheaves commutes. The result follows. ☐

Definition B.16. Let $\mathcal{S}$ be a formal scheme of characteristic p. Let $\mathcal{X}$ be a formal $\mathcal{S}$-scheme with structure morphism $f : \mathcal{X} \to \mathcal{S}$. Denote $\mathcal{X}^{(p)} := \mathcal{X} \times_{\mathcal{S},F_{\mathcal{S}}} \mathcal{S}$ where the second factor $\mathcal{S}$ is endowed with the structure of formal S-scheme via $F_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$. We define the relative Frobenius morphism $F_{\mathcal{X}/\mathcal{S}} : \mathcal{X} \to \mathcal{X}^{(p)}$ as the following uniquely induced map

$$\begin{tikzcd}
\mathcal{X} \arrow[dr, dotted, "F_{\mathcal{X}/\mathcal{S}}"] \arrow[drr, bend left, "F_{\mathcal{X}}"] \arrow[ddr, bend right, "f"'] & & \\
& \mathcal{X}^{(p)} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "f"] \\
& \mathcal{S} \arrow[r, "F_{\mathcal{S}}"] & \mathcal{S}
\end{tikzcd}$$

It is a morphism of formal $\mathcal{S}$-schemes.

Lemma B.17. *Let $\mathcal{S}$ be a formal scheme of characteristic p. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of formal $\mathcal{S}$-schemes. Then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F_{\mathcal{X}/\mathcal{S}}"] \arrow[d, "f"] & \mathcal{X}^{(p)} \arrow[d, "f^{(p)}"] \\
\mathcal{Y} \arrow[r, "F_{\mathcal{Y}/\mathcal{S}}"] & \mathcal{Y}^{(p)}
\end{tikzcd}$$

Proof. It follows from lemma B.15 and definitions. ☐

Lemma B.18. *Let $\mathcal{S}$ be a formal scheme of characteristic p. Let $\mathcal{X}$ be a formal $\mathcal{S}$-group (group object in the category of formal $\mathcal{S}$-scheme), then $\mathcal{X}^{(p)}$ is a formal $\mathcal{S}$-group and $F_{\mathcal{X}/\mathcal{S}}$ is homomorphism of formal $\mathcal{S}$-group.*

Proof. The proof is parallel to the one in lemma B.13. ☐

Let S be a locally Noetherian scheme and $S' \subset S$ be a closed subset. Then we can form a locally Noetherian formal scheme $S_{/S'}$ in the sense of [25, 8.1.6] or [35, Chap. I, 10.9], called the formal completion of S along S'. It is sometimes denoted $\hat{S}$ when no confusion arises. Let $f : X \to Y$ be a morphism of locally Noetherian schemes, X' (resp. Y') a closed subset of X (resp. Y) such that $f(X') \subset Y'$. Then we can extend f to a morphism of locally Noetherian formal schemes:

$$\hat{f} : X_{/X'} \to Y_{/Y'}.$$

There is a generalization of the original completion defined in [35, Chap. I, 10]: X is a locally finitely presented scheme over the valuation ring R of a complete field K with a non-trivial non-Archimedean absolute value (so X may not be locally Noetherian) and we form the formal completion $\mathcal{X}$ of X w.r.t. an ideal of definition I of R, see [27, 5.3]. This version is more compatible with rigid analytic geometry.

Next we show that the completion process commutes with the Frobenius operators in every possible sense, the following lemmas also work in the above generalized version of formal completion with parallel proofs. In our case everything is locally Noetherian and the two versions of completion agree so we don't need to worry about which version we are using.

Lemma B.19. *Let S be a locally Noetherian scheme of characteristic p with a closed subset S', then $\hat{F_S} = F_{S_{/S'}}$.*

Proof. It suffices to assume S is affine. Say $S = \operatorname{Spec} A$. Choose an ideal I of A s.t. the closed subscheme defined by I has S' as an underlying space. We know I must be finitely generated as A is Noetherian. Then we have $S_{/S'} = \operatorname{Spf} \hat{A}$ where $\hat{A}$ is the completion of A with respect to the I-adic topology. Clearly the Frobenius morphism on A, $F_A : A \to A$ induces the Frobenius morphism on $\hat{A}$, $F_{\hat{A}} : \hat{A} \to \hat{A}$. So we are done. □

Lemma B.20. *Let $f : X \to Y$ be a morphism of locally Noetherian schemes of characteristic p, X' (resp. Y') a closed subset of X (resp. Y) such that $f(X') \subset Y'$. Denote $\mathrm{pr}_1 : X^{(p)} \to X$ and $\mathrm{pr}_2 : X^{(p)} \to Y$ where $X^{(p)}$ is relative to $f : X \to Y$, denote $Z = \mathrm{pr}_1^{-1}(X')$ which is a closed subset of $X^{(p)}$. Suppose $X^{(p)}$ is a locally Noetherian scheme, then we have*

1. $(X^{(p)})_{/Z} = X_{/X'} \times_{Y_{/Y'}, F_{\hat{Y}}} Y_{/Y'}$;

2. $F_{X/Y}(X') \subset Z$;

3. $(F_{X/Y})\hat{} = F_{\hat{X}/\hat{Y}} : X_{/X'} \to (X^{(p)})_{/Z}$.

Proof. For the first statement, denote $Z' = \mathrm{pr}_1^{-1}(X') \cap \mathrm{pr}_2^{-1}(Y')$, we know $(X^{(p)})_{/Z'} = X_{/X'} \times_{Y_{/Y'}, F_{\hat{Y}}} Y_{/Y'}$ from [35, Chap. I, 10.9.7]. So it suffices to show $\mathrm{pr}_1^{-1}(X') = \mathrm{pr}_1^{-1}(X') \cap \mathrm{pr}_2^{-1}(Y')$. And it reduces to show $\mathrm{pr}_1^{-1}(X') \subset \mathrm{pr}_2^{-1}(Y')$. Let $x \in \mathrm{pr}_1^{-1}(X')$ so that $\mathrm{pr}_1(x) \in X'$ hence $f \circ \mathrm{pr}_1(x) = F_Y \circ \mathrm{pr}_2(x) \in Y'$. Since F_Y is an identity map on the underlying space, we have $\mathrm{pr}_2(x) \in Y'$ so $x \in \mathrm{pr}_2^{-1}(Y')$. The result follows. The second statement $F_{X/Y}(X') \subset \mathrm{pr}_1^{-1}(X')$ follows from the fact $\mathrm{pr}_1 \circ F_{X/Y}(X') = F_X(X') \subset X'$.

For the third statement, we can simply use the universal property of fiber products. From [35, Chap. I, 10.9.7], we know the projection morphisms of $X_{/X'} \times_{Y_{/Y'}, F_{\hat{Y}}} Y_{/Y'}$ are exactly $(\mathrm{pr}_1)^\wedge$ and $(\mathrm{pr}_2)^\wedge$.

$$\begin{aligned}(\mathrm{pr}_1)^\wedge \circ (F_{X/Y})^\wedge &= (F_{X/Y} \circ \mathrm{pr}_1)^\wedge \\ &= \hat{F_X} \\ &= F_{\hat{X}} \\ &= (\mathrm{pr}_1)^\wedge \circ F_{\hat{X}/\hat{Y}}\end{aligned}$$

$$\begin{aligned}(\mathrm{pr}_2)^\wedge \circ (F_{X/Y})^\wedge &= (F_{X/Y} \circ \mathrm{pr}_2)^\wedge \\ &= \hat{f} \\ &= (\mathrm{pr}_2)^\wedge \circ F_{\hat{X}/\hat{Y}}\end{aligned}$$

Hence we must have $(F_{X/Y})^\wedge = F_{\hat{X}/\hat{Y}}$. □

The additional assumption that $X^{(p)}$ is a locally Noetherian scheme is necessary because it does not automatically follow from locally Noetherianness of X and Y. For a counter example consider $K' \otimes_{K,F_K} K'$ where $K = \mathbb{F}_p(x_1, ..., x_n, ...)$. By [36, Thm. 11], $K' \otimes_{K,F_K} K'$ is Noetherian if and only if $F_K : K \to K'$ is a ring map of finite type. But F_K is not a ring map of finite type, which can be proved easily by contradiction, because there is no surjection from $K[y_1, ..., y_m] \to K'$.

We end this subsection with a lemma about completing a (resp. commutative) group scheme.

Lemma B.21. *Let S be a locally Noetherian scheme of characteristic p, and (G, m, e, i) be a locally of finite type S-group (resp. commutative S-group) scheme with structure morphism $f : G \to S$. Let S' be a closed subset of S and denote $G' = f^{-1}(S')$. Denote $G'' = (f \times f)^{-1}(S') \subset G \times_S G$ and $G''' = (f \times f \times f)^{-1}(S')$.*

1. *Then G, $G \times_S G$, $G \times_S G \times_S G$ are locally Noetherian schemes.*

2. *Denote $\hat{S} = S_{/S'}, \hat{G} = G_{/G'}$, $(G \times_S G)^\wedge = (G \times_S G)_{/G''}$, $(G \times_S G \times_S G)^\wedge = (G \times_S G \times_S G)_{/G'''}$. We have $(G \times_S G)^\wedge \cong \hat{G} \times_{\hat{S}} \hat{G}$ and $(G \times_S G \times_S G)^\wedge \cong \hat{G} \times_{\hat{S}} \hat{G} \times_{\hat{S}} \hat{G}$.*

3. *Then $(\hat{G}, \hat{m}, \hat{e}, \hat{i})$ is a locally Noetherian formal group (resp. commutative group) scheme over $\hat{S}$ with structure morphism $\hat{f} : \hat{G} \to \hat{S}$.*

Proof. The property of being locally of finite type is stable under base change and composition, so G, $G \times_S G$ and $G \times_S G \times_S G$ are locally of finite type over S. Any scheme that is locally of finite type over a locally Noetherian scheme is locally Noetherian, the first statement follows trivially. For the second statement, denote the i-th projection map $\mathrm{pr}_i : G \times_S G \to G, i = 1, 2$. We know $f \times f = f \circ \mathrm{pr}_i : (G \times_S G \to S \times_S S \cong S)$, so

$$\begin{aligned}G'' &= (f \times f)^{-1}(S') \\ &= (f \circ \mathrm{pr}_i)^{-1}(S') \\ &= \mathrm{pr}_i^{-1}(G')\end{aligned}$$

In particular $G'' = \mathrm{pr}_1^{-1}(G') \cap \mathrm{pr}_2^{-1}(G')$ and we can apply [35, Chap. I, 10.9.7] to get $(G \times_S G)^\wedge \cong \hat{G} \times_{\hat{S}} \hat{G}$. A similar argument can be used to show $(G \times_S G \times_S G)^\wedge \cong \hat{G} \times_{\hat{S}} \hat{G} \times_{\hat{S}} \hat{G}$.

For the final statement, we first show that those completions of morphisms are well defined. Let X, Y be schemes over S with structure morphisms g, h respectively, and let $\alpha : X \to Y$ be a morphism of S-schemes, we claim that $\alpha(g^{-1}(S')) \subset h^{-1}(S')$. This claim trivially follows from the fact that $g^{-1}(S') = (h \circ \alpha)^{-1}(S') = \alpha^{-1}(h^{-1}(S'))$. Hence if both X and Y are locally Noetherian schemes, α induces a morphism of locally Noetherian formal schemes $\hat{\alpha} : X_{/g^{-1}(S')} \to Y_{/h^{-1}(S')}$ over $\hat{S}$. Now we would like to show $(\hat{G}, \hat{m}, \hat{e}, \hat{\imath})$ satisfies the following three rules (resp. four rules in the commutative case):

1. $\hat{m} \circ (\mathrm{id}_{\hat{G}} \times \hat{m}) = \hat{m} \circ (\hat{m} \times \mathrm{id}_{\hat{G}}) : \hat{G} \times_{\hat{S}} \hat{G} \times_{\hat{S}} \hat{G} \to \hat{G}$;
2. $\hat{m} \circ (\hat{e} \circ \hat{f}, \mathrm{id}_{\hat{G}}) = \hat{m} \circ (\mathrm{id}_{\hat{G}}, \hat{e} \circ \hat{f}) = \mathrm{id}_{\hat{G}} : \hat{G} \to \hat{G}$;
3. $\hat{m} \circ (\mathrm{id}_{\hat{G}}, \hat{\imath}) = \hat{m} \circ (\hat{\imath}, \mathrm{id}_{\hat{G}}) = \hat{e} \circ \hat{f} : \hat{G} \to \hat{G}$;
4. In the commutative case, $\hat{m} = \hat{m} \circ \mathrm{swap}_{\hat{G}/\hat{S}} : \hat{G} \times_{\hat{S}} \hat{G} \to \hat{G}$ where $\mathrm{swap}_{\hat{G}/\hat{S}} : \hat{G} \times_{\hat{S}} \hat{G} \to \hat{G} \times_{\hat{S}} \hat{G}$ swaps the order of the factors.

Clearly it suffices to show the completion operation commutes with the identity morphism, product of morphisms (in both sense of (α, β) and $\alpha \times \beta$), and the swap morphism under suitable conditions. Obviously $\mathrm{id}_{\hat{G}} = (\mathrm{id}_G)^\wedge$. Now let X, Y, Z be locally Noetherian schemes over S with structure morphisms g, h, l respectively, and suppose $X \times_S Y$ is also a locally Noetherian scheme. Let $\alpha : Z \to X$ and $\beta : Z \to Y$ be a morphisms of S-schemes. They induce a morphism $(\alpha, \beta) : Z \to X \times_S Y$ and then the equality

$$(\alpha, \beta)^\wedge = (\hat{\alpha}, \hat{\beta}) : Z_{/l^{-1}(S')} \to (X \times_S Y)_{/(g \times h)^{-1}(S')} = X_{/g^{-1}(S')} \times_{\hat{S}} Y_{/h^{-1}(S')}$$

follows from the universal property of fiber products. Now we consider the completion of $\alpha \times \beta : Z \times_S Z \to X \times_S Y$, it is essentially the same as above, since $\alpha \times \beta = (\alpha \circ \mathrm{pr}_1, \beta \circ \mathrm{pr}_2)$. So we have $(\alpha \times \beta)^\wedge = \hat{\alpha} \times \hat{\beta}$. Similarly $\mathrm{swap}_{G/S} = (\mathrm{pr}_2, \mathrm{pr}_1)$, so they commute with the completion operation. The result follows. ☐

B.4. Rigidification functor and its relation with Frobenius operators

Let $(R, \mathfrak{m})$ be a valuation ring of height 1 with field of fraction K. We view R as a topological ring by taking the system of its non-zero ideals as a basis of neighborhoods of 0. It follows that R is automatically separated (i.e. Hausdorff. Let $r \in R \backslash \{0\}$, if $r \in R$ is an unit then $(r + \mathfrak{m}) \cap \mathfrak{m} = \emptyset$; if r is not a unit, then $r \notin (r^2)$ so $(r + (r^2)) \cap (r^2) = \emptyset$). We know $\mathfrak{m}$ is the only non-trivial prime ideal so the topology of R coincides with the t-adic topology for any non-zero element $t \in \mathfrak{m}$ by [31, 7.1.7]. Assume further that R is complete with this topology, i.e. every Cauchy sequence converges, then R is a separated and complete adic ring and K is a complete field with a non-trivial non-Archimedean absolute value, see [31, 7.1, p. 154]. So it makes sense to talk about formal R-schemes and rigid K-spaces.

In this subsection we introduce the rigidification functor and investigate its relation with Frobenius operators.

Lemma B.22. *Let R be a complete valuation ring of height 1 with field of fraction K. Then the functor $A \mapsto A \otimes_R K$ on R-algebras A of topologically finite type gives rise to a rigidification functor $(\cdot)^{\mathrm{rig}} : \mathcal{X} \rightsquigarrow \mathcal{X}^{\mathrm{rig}}$ from the category of formal R-schemes that are locally of topologically finite type, to the category of rigid K-spaces.*

Proof. See [31, 7.4, Prop. 3]. ☐

Since the rigidification functor preserves fiber products, in particular finite products (i.e. finite fiber products over the terminal object), it preserves group objects (resp. commutative group objects). We fix R and further assume that R is of characteristic p and F_R is a **finite ring map**, so that F_K is also finite as a localization of F_R. Next we show that the rigidification functor commutes with the Frobenius operators in every possible sense.

Lemma B.23. *Let $\mathcal{X}$ be a locally of topologically finite type formal R-schemes, then $F_{\mathcal{X}^{\mathrm{rig}}} = (F_{\mathcal{X}})^{\mathrm{rig}}$.*

Proof. To use the notion $(F_{\mathcal{X}})^{\mathrm{rig}}$, first we need to show that $F_{\mathcal{X}}$ is a map in the category of formal R-schemes that are locally of topologically finite type. It suffices to check any topological ring A that is a topologically finite type R-algebra with structure map $R \xrightarrow{f} A$ is still of topologically finite type over R when the structure map switches to $R \xrightarrow{f} A \xrightarrow{F_A} A$.

Denote $R' = R$ as the R-algebra via $F_R : R \to R$, and $A' = A$ as the R-algebra with structure map $R \xrightarrow{f} A \xrightarrow{F_A} A$. Fix a non-zero element $t \in \mathfrak{m}$, then R is t-adic. As a topological R-algebra, the topology on A' is t^pA'-adic; as a topological R'-algebra, the topology on A' is tA'-adic. Since $(tA')^p = t^pA'$, the two adic topologies coincide on A'. Also F_A is clearly continuous as the multiplication map is continuous on any topological ring.

As A is of topologically finite type over R, A is isomorphic to $\frac{R\langle x_1,...,x_n\rangle}{I}$ endowed with the t-adic topology for some ideal I of $R\langle x_1, ..., x_n\rangle$. By assumption R' is finite over R, let $r_1, ..., r_m$ be the generators of R' as an R-module. Then we can construct a surjection of topological R-algebras:

$$\begin{array}{rcl} R\langle y_1, ..., y_m, x_1, ..., x_n\rangle & \to & \dfrac{R'\langle x_1, ..., x_n\rangle}{I} \\ r & \mapsto & r^p \\ y_i & \mapsto & r_i \\ x_j & \mapsto & x_j \end{array}$$

So A' is also of topologically finite type over R. Hence $F_{\mathcal{X}}$ is indeed a map in the category of formal R-schemes that are locally of topologically finite type.

To show $F_{\mathcal{X}^{\mathrm{rig}}} = (F_{\mathcal{X}})^{\mathrm{rig}}$, it suffices to show they agree on a covering of affinoid spaces. So it reduces to the case $\mathcal{X}$ is affine formal. Say $\mathcal{X} = \operatorname{Spf} A$ where A is a topologically finite type R-algebra then $F_{\mathcal{X}}$ is induced by F_A. Then the result follows from the fact that $F_A \otimes_R K = F_{A\otimes_R K}$ (localization of a Frobenius map is again a Frobenius map). ☐

Lemma B.24. *Let $\mathcal{X}$ be a locally of topologically finite type formal R-scheme, then $(\mathcal{X}^{(p)})^{\mathrm{rig}} = (\mathcal{X}^{\mathrm{rig}})^{(p)}$ and $F_{\mathcal{X}^{\mathrm{rig}}/K} = (F_{\mathcal{X}/R})^{\mathrm{rig}}$.*

Proof. Clearly the property of being locally of topologically finite type is stable under base change, so $(\mathcal{X}^{(p)})^{\mathrm{rig}}$ makes sense and $F_{\mathcal{X}/R}$ is a map in the category of formal R-schemes that are locally of topologically finite type. To show $(\mathcal{X}^{(p)})^{\mathrm{rig}} = (\mathcal{X}^{\mathrm{rig}})^{(p)}$, it is safe to assume $\mathcal{X} = \operatorname{Spf} A$ is affine formal where A is a topologically finite type R-algebra. It suffices to show $A \otimes_{R,F_R} R \otimes_R K = A \otimes_R K \otimes_{K,F_K} K$ which can be seen by identifying $R \xrightarrow{F_R} R \to K$ with $R \to K \xrightarrow{F_K} K$.

To show $F_{\mathcal{X}^{\mathrm{rig}}/K} = (F_{\mathcal{X}/R})^{\mathrm{rig}}$, it suffices to show they agree on a covering of affinoid spaces. Again it reduces to the case $\mathcal{X} = \operatorname{Spf} A$ is affine formal where A is a topologically finite type R-algebra. Then the result follows from the fact that $F_{A/R} \otimes_R K = F_{A\otimes_R K/K}$ (more generally we have $F_{X/S} \times \mathrm{id}_{S'} = F_{X\times_S S'/S'}$ where S is a scheme of characteristic p and X, S' are S-schemes). ☐

B.5. A lemma supporting lemma 3.7

Let R and K be as in last subsection. Let X be a locally of finitely presented R-scheme with generic fiber X_K and formal completion $\mathcal{X}$ w.r.t. an ideal of definition of R, then there is a morphism of rigid K-spaces:

$$i_X : \mathcal{X}^{\mathrm{rig}} \to X_K^{\mathrm{an}}$$

which is functorial in X, respecting the formation of fiber products, see [27, 5.3.1]. When X is separated and finitely presented over R, i_X is a quasi-compact open immersion. When X is proper over R, i_X is an isomorphism.

Lemma B.25. *Let G be a separated, finitely presented commutative group scheme over R with generic fiber G_K and formal completion $\mathcal{G}$ w.r.t. an ideal of definition of R, then there exists a canonical isomorphism of finite rigid K-spaces $\alpha : \ker F_{\mathcal{G}^{\mathrm{rig}}/K} \to \ker F_{G_K^{\mathrm{an}}/K}$.*

Proof. By lemma B.21 (in the generalized setting of formal completion), $\mathcal{G}$ is a commutative group object so $\mathcal{G}^{\mathrm{rig}}$ is a commutative group object. Similarly G_K is commutative group object and so is G_K^{an}. And the relative Frobenius morphism is a group homomorphism so we can talk about their kernels.

Denote $m : G \times_R G \to G$ the multiplication morphism of G. As i_X respects the formation of fiber products and is functorial in X, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{G}^{\mathrm{rig}} \times_K \mathcal{G}^{\mathrm{rig}} \arrow[r, "i_{G\times G}", "=i_G\times i_G"'] \arrow[d, "\hat{m}^{\mathrm{rig}}"] & G_K^{\mathrm{an}} \times_K G_K^{\mathrm{an}} \arrow[d, "m_K^{\mathrm{an}}"] \\
\mathcal{G}^{\mathrm{rig}} \arrow[r, "i_G"] & G_K^{\mathrm{an}}
\end{tikzcd}$$

which shows that i_G is a group homomorphism of commutative rigid K-groups. Then we obtain α by taking the map of kernels of columns of following commutative diagram

$$\begin{tikzcd}
\mathcal{G}^{\mathrm{rig}} \arrow[r, "i_G"] \arrow[d, "F_{\mathcal{G}^{\mathrm{rig}}/K}"] & G_K^{\mathrm{an}} \arrow[d, "F_{G_K^{\mathrm{an}}/K}"] \\
(\mathcal{G}^{\mathrm{rig}})^{(p)} \arrow[r, "i_G^{(p)}"] & (G_K^{\mathrm{an}})^{(p)}
\end{tikzcd}$$

Denote $H = \ker F_{G/R}$ with generic fiber H_K and formal completion $\mathcal{H}$ w.r.t. an ideal of definition of R. We first show $\ker F_{\mathcal{G}^{\mathrm{rig}}/K} \cong \mathcal{H}^{\mathrm{rig}}$ and $\ker F_{G_K^{\mathrm{an}}/K} \cong H_K^{\mathrm{an}}$ then prove that $\alpha = i_H$ so that it's an isomorphism by properness of H.

First, we have

$$\begin{aligned}
\ker F_{\mathcal{G}^{\mathrm{rig}}/K} &\cong \ker(F_{\mathcal{G}/R})^{\mathrm{rig}} \text{ by lemma B.24} \\
&\cong (\ker F_{\mathcal{G}/R})^{\mathrm{rig}} \text{ as } (\cdot)^{\mathrm{rig}} \text{ commutes with fiber products} \\
&\cong (\ker(F_{G/R})^{\wedge})^{\mathrm{rig}} \text{ by lemma B.20} \\
&\cong ((\ker F_{G/R})^{\wedge})^{\mathrm{rig}} \text{ as generalized completion commutes with fiber products} \\
&= \mathcal{H}^{\mathrm{rig}}
\end{aligned}$$

and

$$\begin{aligned}\ker F_{G_K^{\mathrm{an}}/K} &\cong \ker(F_{G_K/K})^{\mathrm{an}} \text{ by lemma B.14}\\ &\cong (\ker F_{G_K/K})^{\mathrm{an}} \text{ as } (\cdot)^{\mathrm{an}} \text{ commutes with fiber products}\\ &\cong (\ker(F_{G/R} \otimes_R K))^{\mathrm{an}}\\ &\cong ((\ker F_{G/R}) \otimes_R K)^{\mathrm{an}}\\ &= H_K^{\mathrm{an}}\end{aligned}$$

Using the universal property of fiber products on $H_K^{\mathrm{an}} = G_K^{\mathrm{an}} \times_{(G_K^{\mathrm{an}})^{(p)}} K$, we can see that any morphism between $\mathcal{H}^{\mathrm{rig}}$ and H_K^{an} that making the following square commute must be α.

$$\begin{tikzcd}\mathcal{H}^{\mathrm{rig}} \arrow[r, dotted] \arrow[d] & H_K^{\mathrm{an}} \arrow[d] \\ \mathcal{G}^{\mathrm{rig}} \arrow[r, "i_G"] & G_K^{\mathrm{an}}\end{tikzcd}$$

In particular, $i_H = \alpha$. As G is locally of finite type over R, $F_{G/R}$ is finite by [5, tag 0CCD], so H is finite by base change, hence proper. Thus α is an isomorphism. ☐

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