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Critical percentiles for equalizing growth

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ABSTRACT

This paper provides precise conditions under which incremental growth reduces inequality. Critical points are derived, above which incremental income increases inequality, and below which it decreases inequality. According to the Gini coefficient, the lower bound for this critical point is the median individual. Surprisingly, critical points associated with 'absolute' and 'centrist' measures of inequality are sometimes higher than those implied by 'relative' measures. The results are illustrated using data from UNU-WIDER's World Income Inequality Database. According to the Gini, critical points are typically found to lie between the 62nd and 85th percentiles, in the least, and most, unequal countries, respectively.

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1 Introduction

There has long been considerable interest, both within economics and among policy-makers, in the impact of economic growth on inequality. Today, vastly superior data are available to bring to bear on this important question than when Kuznets (1955) famously uncovered tentative evidence for an inverted U-shaped relationship, in which inequality increased in the early stages of industrialisation and subsequently declined. Equipped with such data, a common recent empirical finding has been that, on average, growth tends to be distribution neutral (see, for example, Ravallion and Chen (1997) and Dollar and Kraay (2002)). Yet, as Ravallion (2001) and many others have argued, this 'average' finding can be deceptive for development policy since, at an individual country level, experiences are diverse, and inequality rises about as often as it falls (e.g. Ravallion (2003)). Thus, there are substantial differences between countries in the extent to which people who are less well off share in the spoils of growth, and there is a need for deeper micro-empirical work on growth and distributional change.

This paper contributes to this discourse by developing a theoretical framework intended to help illuminate when it is that growth is likely to increase inequality, and when it is likely to reduce it. In essence, the approach taken is to consider how incremental increases in income, at specific points in the income distribution, impact inequality, as captured by a wide range of inequality indices. This enables us to provide precise conditions under which increments of growth increase, or decrease, inequality. For any income distribution, we are able to derive a critical point in the distribution, above which incremental income increases inequality, and below which it decreases inequality. These critical points, and their corresponding percentiles of the income distribution, can be interpreted as social reference levels for inequality, somewhat analogous to poverty lines, above which increases to income increase inequality, and below which they decrease inequality.

The overwhelming majority of previous studies on growth and inequality have employed the Gini coefficient as the inequality indicator. As a number of recent studies have emphasised though, and as has long been recognised in the literature on social choice (e.g. Atkinson (1970), Kolm (1976)), the Gini embodies just one of many ways of conceptualising income inequality. Thus, it is important to avoid unnecessarily restricting the discourse on inequality to a single measure (e.g. Ravallion (2003), Subramanian and Jayaraj (2014), Atkinson and Brandolini (2010) and Bosmans et al. (2014)).[1] We there-

[1] See also Amiel and Cowell (1992, 1999), who have demonstrated in experimental work that people have a diverse range of views regarding how distributions should be ranked with respect to inequality.

fore derive our results on critical points for a number of different inequality measures - two 'relative' measures (the Gini and the Mean Log Deviation (MLD)), two 'absolute' measures (the Absolute Gini and the Variance) and a 'centrist' measure (the Krtscha).[2] The measures chosen deliberately encompass wildly different normative underpinnings with respect to how inequality should be measured, though our approach can easily be extended further to include virtually any inequality measure from the literature.[3]

The results on critical points provide practical tools for policymakers. Given the Gini coefficient, for example, it can be inferred immediately from our results, *ceteris paribus*, what the impact of growth in any given percentile of the income distribution will be on inequality, according to that measure. This may have important implications for a policymaker with a concern for both growth and inequality, since certain growth-promoting policies are likely to predominantly benefit particular sectors of the economy, or specific geographic regions. In turn, such benefits are likely to accrue mainly to those within certain ranges of the income distribution, which may or may not lie below our critical percentiles.

The results also highlight some interesting properties of the inequality measures employed. It turns out that the lower bound for the critical point associated with the Gini is the median income. Thus, *ceteris paribus*, increases in the median individual's income can never increase inequality, regardless of how poor individuals even slightly below the median may be. Opinions may differ on the extent to which this is a drawback for an inequality measure but, at the very least, it clearly illustrates the point that growth and falling inequality need not *necessarily* result in reduced poverty.[4]

An unexpected finding is that critical points associated with 'absolute' and 'centrist' measures of inequality are not necessarily lower than those of 'relative' measures, and are sometimes substantially higher. This is surprising since, following a seminal work

[2]'Relative' inequality measures are those which are invariant under equiproportional increases in all incomes; 'absolute' inequality measures are those which register no change when the same absolute amount of income is added to all incomes; 'centrist' inequality measures are those which register an increase in inequality if all incomes increase equiproportionally, and a decrease if the same absolute amount of income is added to all incomes.

[3]There are also some specific advantages of the particular measures chosen from each of these three broad classes. The Gini is chosen primarily since it is by far the most widely used inequality measure; the MLD has the advantage of being decomposable into sub-groups, with population-share weights; the Absolute Gini is chosen since it is one of the better known 'absolute' measures and on account of its intuitive relationship with the Gini; the Variance has the advantage of being the only known 'absolute' measure which is both decomposable (Chakravarty, 2001) and unit-consistent (Zheng, 2007), advantages which are shared by the Krtscha (Zheng, 2007).

[4]For example, in OECD countries the official poverty line is 60% of the median income. Our results imply that any growth which occurs between such a poverty line and the median income would reduce inequality. It would not, however, have any impact on poverty as captured by any commonly used poverty measure.

by Kolm (1976), 'relative' measures of inequality have long been interpreted as being "leftist" and 'absolute' measures as "rightest." These interpretations seem rather inconsistent with their behaviour in relation to incremental increases in income at specific points in the distribution; one might reasonably expect a "leftist" measure to require an increment of income to be added further down the distribution, in order for it to be inequality-decreasing, than would a "rightest" measure. Another surprising result, and one which seems rather a drawback for the Absolute Gini measure, is that, in the context of critical points, it is completely insensitive to the distribution. Its associated critical point turns out to always be equal to the median income, regardless of how unequal the distribution is deemed to be. Moreover, while the Absolute Gini is the only measure we study which is completely uncorrelated with its associated critical values, the Gini is the only one which is perfectly correlated with its critical values. For a sufficiently large population, the critical value associated with the Gini coefficient is uniquely determined by the size of the Gini.

An empirical illustration of the results is provided using data from the latest version of UNU-WIDER's World Income Inequality Database (WIID). In practice, according to the Gini, critical points are found to lie between roughly the 62nd and 85th percentile of the distribution, in the most and least equal countries, respectively. Analogous critical points for the other measures employed are provided.

The rest of the paper is organised as follows. Some notation and the basic framework are provided in Section 2. The paper's central results are derived in Section 3. The empirical application is provided in Section 4. Some concluding remarks are offered in Section 5. All proofs are deferred to the Appendices.

2 Notation and Basic Framework

Consider a society of $n \geq 2$ individuals. A profile $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ represents the distribution of incomes within the society. An inequality measure is a function that assigns to each income profile a non-negative number. Thus $I : \bigcup_{n \in \mathbb{N}} \mathbb{R}^n_+ \to \mathbb{R}_+$.

For a permutation ρ of the indices in $\{1, \ldots, n\}$, such that $\rho_i := \rho(i)$, we write $\rho(\mathbf{x})$ for the profile $\mathbf{z}$ with $z_i = x_{\rho_i}$ for $i = 1, \ldots, n$. The inequality measures $I(\cdot)$we use satisfy a property of *symmetry* (or *anonymity*): $I(\mathbf{x}) = I(\rho(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^n_+$. From here on, for simplicity, when we write $\mathbf{x} \in \mathbb{R}^n_+$ it is implicitly assumed that $x_1 \leq \cdots \leq x_n$. Since $I(\cdot)$ satisfies symmetry, this is not a restriction. Further, we write $w_i\mathbf{x}$ for the profile $\mathbf{x}$ with x_i replaced by w; whenever we use this notation we implicitly assume that the ordering of incomes remains from lowest to highest, i.e., $x_{i-1} \leq w \leq x_{i+1}$ if

$i \in \{2, \ldots, n-1\}$, $x_{i-1} \leq w$ if $i = n$, and $w \leq x_{i+1}$ if $i = 1$. For example, for $\varepsilon \geq 0$ when we write $(x_i + \varepsilon)_i \mathbf{x}$, we implicitly require constraints on the admissible values of ε, so that $x_i + \varepsilon \leq x_{i+1}$ whenever $i \in \{1, \ldots, n-1\}$; in particular, $\varepsilon = 0$ if $x_i = x_{i+1}$. Whenever we write $(x_i + \varepsilon)_i \mathbf{x}$ with $\varepsilon > 0$ it is implicitly assumed that $x_i < x_{i+1}$ if $i \neq n$.

Finally, we denote the mean of profile $\mathbf{x} \in \mathbb{R}^n_+$ by $\mu = \frac{1}{n} \cdot \sum_{i=1}^n x_i$.

Our five inequality measures can be defined as follows.

The Gini coefficient is given by:

$$I_G(\mathbf{x}) = 1 - \frac{1}{n} \left[\frac{\sum_{k=1}^n 2\left(n - k + \frac{1}{2}\right) x_k}{\sum_{i=1}^n x_i} \right]. \tag{1}$$

The MLD is given by:

$$I_T(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n \ln\left(\frac{\mu}{x_i}\right). \tag{2}$$

The Absolute Gini is given by:

$$I_{AG}(\mathbf{x}) = \mu \cdot I_G(\mathbf{x}). \tag{3}$$

The Variance is given by:[5]

$$I_V(\mathbf{x}) = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2. \tag{4}$$

The Krtscha is given by:

$$I_K(\mathbf{x}) = \frac{1}{n\mu} \sum_{i=1}^n (x_i - \mu)^2. \tag{5}$$

3 Growth and Inequality

Suppose that some individual $l \in \{1, \ldots, n\}$'s income increases by an amount $\varepsilon > 0$. This gives rise to a new income distribution $(x_l + \varepsilon)_l \mathbf{x}$. What happens to growth and inequality? For growth the answer is straightforward; aggregate income has increased by ε. The effect on inequality depends on the distribution of income, and on where exactly individual l lies in that distribution. Typically, if l lies at the bottom of the distribution

[5] Occasionally, where it is not explicitly being interpreted as an inequality indicator, the variance will be denoted simply by $\sigma^2_{\mathbf{x}}$.

$(l = 1)$ we would expect inequality to decrease; if l lies at the top of the distribution $(l = n)$ we would expect inequality to increase. In general though, for $\mathbf{x} \in \mathbb{R}_+^n$ and $l \in \{1, \ldots, n\}$, it is far from clear what the impact should be. Before turning to the paper's first main result, some further notation and definitions will be useful.

For any inequality measure $I(\cdot)$, let $\triangle I(\mathbf{x}; l, \varepsilon) = I((x_l + \varepsilon)_l \mathbf{x}) - I(\mathbf{x})$. This can be interpreted as the change in inequality associated with an incremental increase in individual l's income. We now define the notion of a critical rank. A critical rank $c \in \mathbb{R}_+$ exists if, *ceteris paribus*, an incremental increase in the income of any individual ranked above c increases inequality, while an incremental increase in the income of any individual ranked below c decreases inequality.

Definition 1

Given income profile $\mathbf{x} \in \mathbb{R}_+^n$, and inequality measure $I(\cdot)$, $c \in \mathbb{R}_+$is a *critical rank* if (i) and (ii) hold:

(i) $\triangle I(\mathbf{x}; l, \varepsilon) > 0$ for all $l > c$ and all $\varepsilon > 0$.

(ii) $\triangle I(\mathbf{x}; l, \varepsilon) < 0$ for all $l < c$ and all $\varepsilon > 0$.

Analogously, we define the notion of a critical income. A critical income $m \in \mathbb{R}_+$ exists if, *ceteris paribus*, an incremental increase, $\varepsilon > 0$, in the income of any individual with an income above m increases inequality, while an incremental increase in the income of any individual with an income less than m decreases inequality.

Definition 2

Given income profile $\mathbf{x} \in \mathbb{R}_+^n$, $\varepsilon > 0$, and inequality measure $I(\cdot)$, $m \in \mathbb{R}_+$is a *critical income* if (i) and (ii) hold:

(i) $\triangle I(\mathbf{x}; l, \varepsilon) > 0$ for all $x_l > m$.

(ii) $\triangle I(\mathbf{x}; l, \varepsilon) < 0$ for all $x_l < m$.[6]

We can now write our first main result, which derives a critical rank for the Gini coefficient.

[6] In contrast to the critical rank definition, the critical income is conditional on the size of the income increment ε. Thus, it may be possible for the addition of a small increment to an individual's income to decrease inequality, while giving them a larger increment would increase inequality, even if the larger increase respects the rank-ordering requirement.

Theorem 1

Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, *any* $l \in \{1, \ldots, n\}$, *and any* $\varepsilon > 0$, *the following hold:*

(i) $\triangle I_G(\mathbf{x}; l, \varepsilon) = \frac{2\varepsilon\left[l\sum_{i=1}^n x_i - \sum_{k=1}^n kx_k\right]}{n\left(\varepsilon + \sum_{i=1}^n x_i\right)\sum_{i=1}^n x_i}$

(ii)$\triangle I_G(\mathbf{x}; l, \varepsilon) > 0$ *if and only if* $l > \frac{x_1 + 2x_2 + \cdots + nx_n}{x_1 + \cdots + x_n} = \frac{\sum_{k=1}^n kx_k}{\sum_{i=1}^n x_i}$; $c_G = \frac{\sum_{k=1}^n kx_k}{\sum_{i=1}^n x_i}$ *is a critical rank.*

Whenever an income increment is added to any individual ranked above c_G then, *ceteris paribus*, inequality, as measured by the Gini, increases. Conversely, inequality decreases when an income increment is added to any individual ranked below this critical value. If an individual is ranked exactly equal to the critical value, increases in their income will have no impact on inequality, though the critical value is seldom likely to be a natural number.

It seems intuitive that higher levels of inequality should be associated with higher critical ranks but can we say anything more precise about the relationship between $I_G(\mathbf{x})$ and $c_G(\mathbf{x})$? As the next result shows, the critical rank is uniquely determined by the Gini coefficient and the number of individuals.

Theorem 2

Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, *the following identity holds:* $c_G(\mathbf{x}) = \frac{nI_G(\mathbf{x}) + (n+1)}{2}$.

Corollary 1

Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, $c_G(\mathbf{x}) \in \left[\frac{n+1}{2}, \frac{2n+1}{2}\right]$.

(i) The lower bound is attained iff $I_G(\mathbf{x}) = 0$.

(ii) The upper bound is attained iff $I_G(\mathbf{x}) = 1$.

It is striking that the lower bound for $c_G(\cdot)$ is as high as the median income; according to the Gini, inequality can never be increased by growing the median income. It is not hard to imagine situations where this may seem rather perverse. Consider, for example, the following stylized example, where $\mathbf{x} = (0, 0, 10, 11, 12)$. In this imaginary five-person economy, the poorest two individuals have no income, while the remaining three individuals have much higher, and rather similar, incomes. According to the Gini, increasing the third person's income reduces inequality. Many people might have some difficulty accepting such a normative judgement.

The percentile of the income distribution in which $c_G(\cdot)$ occurs is simply $100 \cdot c_G(\cdot)/n$. Theorem 2 thus provides a tool that could potentially help indicate the likely impact on inequality of policies intended to promote growth. For example, growth of certain sectors might be expected to increase incomes mainly in percentiles below $100 \cdot c_G(\cdot)/n$, while growth in other sectors might be expected to increase incomes primarily in percentiles above it. Provided the Gini coefficient and population size are known, the critical rank and its corresponding percentile of the distribution can be estimated immediately. In fact, as can be inferred from the following corollary to Theorem 2, in most empirical applications, only the Gini coefficient need be known to estimate the critical percentile; any detailed knowledge of the population size is unlikely to be needed.

Corollary 2
Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, *the following holds:*

(i) $\frac{100 \cdot c_G(\mathbf{x})}{n} = \frac{100 \cdot I_G(\mathbf{x})}{2} + \frac{100 \cdot (n+1)}{2n}$.

(ii) In the limit, we have $\lim_{n \to \infty} \frac{100 \cdot c_G(\mathbf{x})}{n} = \frac{100 \cdot (I_G(\mathbf{x})+1)}{2} \in [50, 100]$.

We now provide analogous results for our second 'relative' inequality measure, the MLD.

Theorem 3
Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, *any* $l \in \{1, \ldots, n\}$, *and any* $\varepsilon > 0$, *the following hold:*

(i) $\triangle I_T(\mathbf{x}; l, \varepsilon) = \ln\left(\mu + \frac{\varepsilon}{n}\right) - \ln(\mu) + \frac{1}{n}\left(\ln(x_l) - \ln(x_l + \varepsilon)\right)$

(ii) $\triangle I_T(\mathbf{x}; l, \varepsilon) > 0$ *if and only if* $x_l > \frac{\varepsilon}{\left(1+\frac{\varepsilon}{n\mu}\right)^n - 1}$; $m_T = \frac{\varepsilon}{\left(1+\frac{\varepsilon}{n\mu}\right)^n - 1}$ *is a critical income.*

(iii)$\lim_{\varepsilon\to 0} m_T = \mu$.

While Theorem 1 (and also Theorem 2) provide the critical rank according to the Gini, Theorem 3 provides the critical income according to the MLD. In contrast to the result for the Gini, this result for the MLD does not permit us to determine where, in general, the critical income corresponds to rank-positions in the income distribution. Instead, this must be inferred empirically, based on where the critical income (for small increments $\varepsilon > 0$, essentially the mean) would be ranked in a particular distribution. As we will see in Section 4, our empirical evidence suggests that the percentiles in which the MLD's critical incomes lie are consistently lower than the percentiles in which the Gini's critical ranks lie.

We now extend the approach above to our 'absolute' inequality measures, starting with the Absolute Gini.

Theorem 4
Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, *any* $l \in \{1, \ldots, n\}$, *and any* $\varepsilon > 0$, *the following hold:*

(i) $\triangle I_{AG}(\mathbf{x}; l, \varepsilon) = \frac{\varepsilon(2l-n-1)}{n^2}$.

(ii)$\triangle I_{AG}(\mathbf{x}; l, \varepsilon) > 0$ *if and only if* $l > \frac{n+1}{2}$; $c_{AG} = \frac{n+1}{2}$ *is a critical rank.*

It is perhaps to be expected that the critical rank according to a "leftist" measure like the Absolute Gini should be lower than the corresponding critical rank for a "rightest" measure like the Gini. In that sense it is not entirely surprising that c_{AG} is the lower bound for c_G. Nevertheless, it might seem unexpected both that the critical rank never changes, regardless of distribution, and that it is as high as the median. The former property, essentially a disregard for distribution, seems like an unfortunate property for any inequality measure, and the latter property somewhat incongrous with a supposedly "leftist" normative underpinning.

The analogous results for our second 'absolute' inequality measure, the Variance, are very different, but also surprising.

Theorem 5

Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, *any* $l \in \{1, \ldots, n\}$, *and any* $\varepsilon > 0$, *the following hold:*

(i) $\triangle I_V(\mathbf{x}; l, \varepsilon) = \frac{1}{n}\left(2\varepsilon(x_l - \mu) + \left(\frac{n-1}{n}\right)\varepsilon^2\right)$.

(ii)$\triangle I_V(\mathbf{x}; l, \varepsilon) > 0$ *if and only if* $x_l > \mu + \frac{1}{2}\left(\frac{n-1}{n}\right)\varepsilon$; $m_V = \mu + \frac{1}{2}\left(\frac{n-1}{n}\right)\varepsilon$ *is a critical income.*

(iii)$\lim_{\varepsilon \to 0} m_V = \mu$.

The second part of Theorem 5 indicates that, at least for small increments ε, the critical income is likely, in most practical cirumstances, where n is fairly large, to be approximately equal to the mean income μ (and, in fact, this is also both its lower bound and its limit as $\varepsilon \to 0$). This is quite a striking result. The Variance is an 'absolute' measure of inequality, and, as noted, 'absolute' measures are widely regarded as "leftist," yet the mean income is typically quite far up the income distribution, well above even the median income. In this context, the interpretation of the Variance as a "leftist" inequality measure seems rather misplaced. It has the property that incremental increases to incomes below the mean can never increase inequality. For those who would regard the distribution within, say, the bottom half of the distribution as an important aspect of inequality, this might seem unsatisfactory. Indeed, together with Theorem 3 part (iii), we know that the critical income implied by the Variance is essentially identical to that implied by the MLD, a 'relative,' supposedly "rightest," measure. Furthermore, we have seen above that the Gini, another 'relative' measure, can have a critical rank as low as the median. It is possible, in fact, for the Gini to assign a lower critical percentile to an income distribution than would the Variance.[7]

Where then might the critical rank for a supposedly 'centrist' measure like the Krtscha lie?

[7] As a stylized example of this, consider a five-person economy with an income profile $\mathbf{x} = (20, 21, 22, 23, 30)$. Here, $c_G = 3.19$, indicating that incremental increases to the second highest income of 23 would increase inequality. However, $\lim_{\varepsilon \to 0} m_V = \mu = 23.2$, higher than the second highest income of 23, and indicating that sufficiently small incremental increases to that person's income would reduce inequality.

Theorem 6

Given any profile $\mathbf{x} \in \mathbb{R}^n_+$, *any* $l \in \{1, \ldots, n\}$, *and any* $\varepsilon > 0$, *the following hold:*

(i) $\triangle I_K(\mathbf{x}; l, \varepsilon) = \frac{\varepsilon}{n\mu+\varepsilon}\left(2(x_l - \mu) - \frac{\sigma^2_{\mathbf{x}}}{\mu} + \frac{\varepsilon(n-1)}{n}\right)$.

(ii)$\triangle I_K(\mathbf{x}; l, \varepsilon) > 0$ *if and only if* $x_l > \mu + \frac{\sigma^2_{\mathbf{x}}}{2\mu} - \frac{\varepsilon(n-1)}{2n}$. $m_K = \mu + \frac{\sigma^2_{\mathbf{x}}}{2\mu} - \frac{\varepsilon(n-1)}{2n}$*is a critical income.*

(iii)$\lim_{\varepsilon \to 0} m_K = \mu + \frac{\sigma^2_{\mathbf{x}}}{2\mu}$.

One way of thinking of the critical income in part (ii) is as being approximately equal to half the Krtscha measure plus the mean, and part (iii) confirms that this is its limit. Bearing in mind that mean income is typically well above median income, and that the Krtscha measure is usually well above zero (since the variance is usually quite large compared to the mean), it is immediately apparent that the critical income for the Krtscha appears to be typically very high indeed.

4 Empirical Application

In this section we provide an empirical illustration of our approach, conducted using the latest version of the UNU-WIDER World Income Inequality Database (WIID).[8] The WIID contains repeated cross-country information on Gini coefficients and income (or consumption) quantiles for 156 countries, spanning the period 1950-2012. It is arguably the most comprehensive and complete database of worldwide distributional data currently available.[9] We focus here on income decile data for 2010 from three quite different groups of countries, namely Nordic, Anglo-Saxon and BRICS. Our five inequality measures, and the corresponding critical percentiles, are estimated for a selection of countries from each of these three broad groups, and displayed in Table 1.

The inequality estimates based on the decile share data were performed by creating, for each country, a synthetic income distribution, using a smoothing algorithm within quantiles developed by Shorrocks and Wan (2008). This approach has been shown to provide better estimates than the simple approach of assuming that all individuals within the same quantile have the same income (which biases inequality estimates downwards).

[8] The data are from the WIID3.0b, which was released in September 2014. The dataset is available on the following link: http://www.wider.unu.edu/research/WIID3- 0B/en_GB/database/

[9] See Jenkins (2014) for an assessment of the WIID.

Table 1: Inequality & critical percentiles in 2010

	Inequality					Critical percentiles				
	I_G	I_T	I_{AG}	I_V	I_K	p_G	p_T	p_{AG}	p_V	p_K
Nordic										
Norway	0.235	0.097	10,982	452.355	9,671	61.7	58.4	50	58.4	68.3
Sweden	0.241	0.101	8,211	241.151	7,067	62.0	56.4	50	56.4	66.6
Denmark	0.268	0.160	8,664	276.287	8,533	63.4	54.9	50	54.9	67.0
Anglo-Saxon										
UK	0.328	0.183	10,751	496.620	15,147	66.4	62.5	50	62.5	75.7
Ireland	0.332	0.186	12,195	649.859	17,666	66.6	62.9	50	62.9	77.6
USA	0.409	0.315	17,974	1445.318	32,886	70.5	63.2	50	63.2	78.7
BRICS										
Russia	0.397	0.260	5,652	153.472	10,786	69.9	68.0	50	68.0	80.1
India	0.417	0.287	1,283	9.607	3126	70.9	70.2	50	70.2	86.2
Brazil	0.536	0.525	5,405	207.683	20,577	76.8	72.5	50	72.5	89.8
S. Africa	0.696	0.990	6,628	505.560	53,125	84.8	78.0	50	78.0	94.7

NOTES: (i) Source: WIID / Author's calculations; (ii) I_V is expressed in millions; (iii) Critical percentiles p_T, p_V, and p_K, are based on the limits of the corresponding critical incomes, i.e., respectively, $\lim_{\varepsilon\to 0} m_T$, $\lim_{\varepsilon\to 0} m_V$, and $\lim_{\varepsilon\to 0} m_K$.

As expected, the results in Table 1 indicate both significant differences in inequality levels between countries, and between the rankings the different inequality measures give. The 'relative' inequality measures broadly agree that the Nordic countries are generally the most equal and the BRICS the most unequal. The 'absolute' measures agree instead that India is the most equal country, and the USA the most unequal. The Krtscha, our 'centrist' measure, agrees with the 'relative' measures that South Africa is the most unequal country but, like the 'absolute' measures, finds that India is the most equal country. All measures agree that the USA is more unequal than any of the countries in the sample outside of the BRICS.

Of particular interest to this study, the results also give some indication of the approximate range of critical percentiles that are likely to be observed in the real world for each measure. As predicted by Corollary 2 to Theorem 2, the Gini's critical percentiles all lie between the median and top of the distribution, and in practice this turns out to be between the 62nd and 85th percentile. Consistent with Theorem 5, the critical values for the Variance are confirmed to be surprisingly high for a supposedly "leftist" measure. In this sample, they range from the 55th to the 78th percentile. While lower than the corresponding critical percentiles for the Gini, it seems remarkable (though, as discussed above, is implied by Theorems 3 and 5) that, for each country, these critical percentiles conform exactly with those resulting from another 'relative,' supposedly

Table 2: Correlations between inequality measures and critical percentiles

	I_G	I_T	I_{AG}	I_V	I_K	p_G	p_T	p_{AG}	p_V	p_K
I_G	1	0.968***	-0.301	0.046	0.789***	1	0.944***	0	0.944***	0.950***
I_T		1	-0.255	0.056	0.855***	0.968***	0.860***	0	0.860***	0.855***
I_{AG}			1	0.910***	0.274	-0.301	-0.456	0	-0.456	-0.396
I_V				1	0.527	0.046	-0.115	0	-0.115	-0.039
I_K					1	0.788***	0.603*	0	0.603*	0.620*
p_G						1	0.944***	0	0.944***	0.950***
p_T							1	0	1	0.986***
p_{AG}								1	0	0
p_V									1	0.986***
p_K										1

NOTES: (i) Source: WIID / Author's calculations; (ii) *, ** and *** indicate statistical significance at the 10%, 5% and 1% levels, respectively; (iii) "1" and "0" indicate exact relationship, i.e., respectively, perfect and zero correlation

"rightest," measure, the MLD. Consistent with Theorem 6, the Krtscha's critical percentiles are remarkably high for a supposedly 'centrist' measure - always above the Gini's. As expected from Theorem 2, there is evidence of a monotonic (in fact, linear) relation between the Gini and its critical percentile. Consistent with our theoretical results, there is no such monotonic relation for any of the other measures. To further explore the relationships between the measures and their implied critical percentiles, correlation coefficients between each of the measures and their critical percentiles are reported in Table 2.

As expected, the Gini is the only measure which has a correlation of exactly unity with its implied critical percentile. There is, however, a strong and highly statistically significant positive correlation (0.860) between the MLD and its critical percentile, and a moderately strong positive and borderline statistically significant correlation (0.620) between the Krtscha and its critical percentile. As implied by Theorem 4, there is no correlation between the Absolute Gini and its critical percentile, and we find no empirical evidence of any statistically significant correlation between the Variance and its critical percentile (though the point estimate is negative).

Strikingly, it is apparent from both Table 1 and, especially, from Table 2, that while the various measures rank countries quite differently with respect to inequality, ordering the countries according to the measures' critical percentiles provides remarkably similar rankings. The point estimates of the pairwise correlation coefficients for the critical percentiles corresponding to each of the Gini, MLD, Variance and Krtscha with one another, are all higher than 0.94 and statistically significant at the 1% level. (There is no correlation, of course, between the Absolute Gini's critical percentiles and any of the other measures' critical percentiles, since the Absolute Gini's critical percentile is constant).

It is also apparent from Table 1 and Table 2 that ordering countries according to the size of their critical percentiles, as implied by any from the Gini, MLD, Variance or Krtscha, provides very similar rankings to the Gini coefficient itself. As we have already seen, the rankings implied by the Gini's critical percentiles are identical to those of the Gini itself. The pairwise correlation coefficients between the Gini and the critical percentiles implied by the MLD, Variance and Krtscha are, respectively, 0.944, 0.944 and 0.950, all significant at the 1% level. Thus, apart from the Absolute Gini, the measures are roughly in agreement that the lower the 'relative' inequality is according to the Gini, the further down the income distribution increments to growth must be in order for inequality to decrease. Where the measures do not agree, is in quite how far down the income distribution this point must be and, unexpectedly, it is not necessarily the case that the 'absolute' and 'centrist' measures require this point in the distribution to be lower than do 'relative' measures.

Taking this approach then, apart from the Absolute Gini, the measures broadly agree that critical percentiles are generally highest in the BRICS (the countries with the highest 'relative' inequality) and lowest in the Nordic countries (the countries with the lowest 'relative' inequality). 'Relative' inequality in South Africa is found to be so high that, all else equal, even increasing incomes in the 77th, 77th, 84th and 94th percentiles would reduce inequality according to, respectively, the MLD, Variance, Gini and Krtscha. By way of contrast, in Sweden, increasing incomes in the 56th, 56th, 62nd and 67th percentiles would increase these respective inequality measures.

5 Concluding Remarks

This paper has proposed a new framework for thinking about the impact of growth on inequality. Growth and changes in inequality occur simultaneously, as two sides of the same phenomenon. The results indicate the specific parts of the income distribution in which growth must occur if it is to be accompanied by a fall in inequality.

Of course growth is never, in reality, confined to a single individual or percentile, at least not over any meaningful time-frame. Nevertheless, the critical percentiles may be suggestive of the likely impact on inequality of certain growth-promoting policies. For example, in many developing countries, programmes to improve the quality of education, or infrastructure, in lagging rural areas might be expected to promote growth predominantly in parts of the income distribution below any of the critical percentiles derived in this paper. Conversely, in many contexts, deregulating employment law, or providing tax incentives for investing in securities, might be expected to promote growth primarily above critical percentiles.

There has been extensive discussion in the development policy literature on how to make growth more 'pro-poor.'[10] While a degree of commonality might be expected between policies which help promote 'pro-poor' growth and those which drive 'equalizing' growth, an important implication of the paper's results is that 'equalizing' growth need not necessarily be poverty-reducing. It is quite possible to imagine policies that could help grow the incomes of those far above official poverty lines, yet below our critical percentiles. This sounds a cautionary note for policymakers with a concern for poverty. Growth and redistributive policies are rightly widely recognised as the only means of reducing poverty. Our results imply though that these are not always sufficient; it is possible to have both a growing economy and a falling Gini, while making no impact on poverty.

Finally, as many have argued, it is important to bring measures other than the Gini into the discourse on inequality, and this applies also to the discourse on its relationship with economic growth. This paper has demonstrated though that inequality measures do not always behave in the way that they might be expected to when economies grow. Of the measures employed here, only the Gini's critical percentiles are consistent with the measure itself, in the sense that higher inequality necessarily means that growth can take place in higher percentiles before it becomes disequalizing. This property seems a highly desirable one for an inequality measure, and one that might be formalized and employed in subsequent axiomatic work on inequality measurement. As it turns out, the precise way in which the other measures studied here stray from this consistency principle results in a surprising degree of consensus among them as to where in the income distribution growth must occur if inequality is to decrease. The fact that these judgements are broadly consistent with the Gini, a 'relative,' "rightest" measure, for which critical points are never below the median, indicates something of a failure, in this context at least, of the "leftist" measures to live up to their billing.

[10] See Ravallion (2004) for an introduction to this literature. For example, in the context of India, Ravallion and Datt (1996, 2002) have stressed the importance of growing the rural economy, and of the interaction between initial conditions and the sectoral composition of growth.

Appendices

A Proof of Theorem 1

Take an arbitrary $\mathbf{x} \in \mathbb{R}^n_+$, $l \in \{1, \ldots, n\}$, and $\varepsilon > 0$. It follows from (1) that

$$I_G\left((x_l+\varepsilon)_l\mathbf{x}\right) = 1 - \frac{1}{n}\left[\frac{\sum_{k=1,k\neq l}^{n} 2\left(n-k+\frac{1}{2}\right)x_k + 2\left(n-l+\frac{1}{2}\right)(x_l+\varepsilon)}{\varepsilon + \sum_{i=1}^{n} x_i}\right]$$

By definition, $\triangle I_G(\mathbf{x}; l, \varepsilon) = I_G\left((x_l+\varepsilon)_l\mathbf{x}\right) - I_G(\mathbf{x})$.

It follows that:

$$\triangle\, I_G(\mathbf{x}; l, \varepsilon) = -\frac{1}{n}\left[\frac{\sum_{i=1}^{n} x_i\left(\sum_{k=1,k\neq l}^{n} 2\left(n-k+\frac{1}{2}\right)x_k + 2\left(n-l+\frac{1}{2}\right)(x_l+\varepsilon)\right)}{\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i}\right] + \frac{1}{n}\left[\frac{\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k}{\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i}\right]$$

$$\begin{aligned}
\triangle I_G(\mathbf{x}; l, \varepsilon) &= \frac{\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k - \sum_{i=1}^{n} x_i\left(\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k + 2\left(n-l+\frac{1}{2}\right)\varepsilon\right)}{n\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i} \\
&= \frac{\varepsilon\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k - 2\left(n-l+\frac{1}{2}\right)\varepsilon\sum_{i=1}^{n} x_i}{n\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i} \\
&= \frac{\varepsilon\left[\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k - 2\left(n-l+\frac{1}{2}\right)\sum_{i=1}^{n} x_i\right]}{n\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i} \\
&= \frac{\varepsilon\left[-2\sum_{k=1}^{n} kx_k + 2l\sum_{i=1}^{n} x_i\right]}{n\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i} \\
&= \frac{2\varepsilon\left[l\sum_{i=1}^{n} x_i - \sum_{k=1}^{n} kx_k\right]}{n\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i} \qquad (6)
\end{aligned}$$

This concludes part (i).

The expression in (6) is (strictly) positive if and only if:

$$l\sum_{i=1}^{n} x_i > \sum_{k=1}^{n} kx_k$$

yielding the necessary and sufficient condition:

$$l > \frac{x_1 + 2x_2 + \cdots + nx_n}{x_1 + \cdots + x_n} = \frac{\sum_{k=1}^{n} kx_k}{\sum_{i=1}^{n} x_i}$$

This concludes part (ii) and the proof of Theorem 1.

B Proof of Theorem 2

Take an arbitrary $\mathbf{x} \in \mathbb{R}_{+}^{n}$. It follows from (1) that:

$$\begin{aligned}
nI_G(\mathbf{x}) &= n\left[1 - \frac{1}{n}\left(\frac{\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k}{\sum_{i=1}^{n} x_i}\right)\right] \\
&= n - \frac{\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k}{\sum_{i=1}^{n} x_i} \\
&= n - \frac{\sum_{k=1}^{n} 2nx_k - 2kx_k + x_k}{\sum_{i=1}^{n} x_i} \\
&= n - \left[\frac{\sum_{k=1}^{n}(2n+1)x_k - \sum_{k=1}^{n} 2kx_k}{\sum_{i=1}^{n} x_i}\right] \\
&= n - \left[\frac{(2n+1)\sum_{i=1}^{n} x_i - 2\sum_{k=1}^{n} kx_k}{\sum_{i=1}^{n} x_i}\right] \\
&= n - \left[(2n+1) - \frac{2\sum_{k=1}^{n} kx_k}{\sum_{i=1}^{n} x_i}\right] \\
&= n - 2n - 1 + \frac{2\sum_{k=1}^{n} kx_k}{\sum_{i=1}^{n} x_i} \\
&= -n - 1 + 2c_G
\end{aligned}$$

Rearranging, we have

$$c_G = \frac{nI_G(\mathbf{x}) + (n+1)}{2}$$

which concludes the proof of Theorem 2.

C Proof of Theorem 3

Take an arbitrary $\mathbf{x} \in \mathbb{R}^n_+$, $l \in \{1, \ldots, n\}$, and $\varepsilon > 0$. It follows from (2) that

$$\begin{aligned} \triangle\, I_T(\mathbf{x}; l, \varepsilon) = \frac{1}{n}\left[\sum_{i=1, i\neq l}^{n} \left(\ln\left(\mu + \frac{\varepsilon}{n}\right) - \ln(x_i)\right) - \sum_{i=1}^{n} (\ln(\mu) - \ln(x_i))\right] \\ + \frac{1}{n}\left[\ln\left(\mu + \frac{\varepsilon}{n}\right) - \ln(x_l + \varepsilon)\right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{n}\sum_{i=1}^{n} \ln\left(\mu + \frac{\varepsilon}{n}\right) - \frac{1}{n}\sum_{i=1}^{n} (\ln(\mu) - \ln(x_i)) - \frac{1}{n}\sum_{i=1, i\neq l}^{n} \ln(x_i) - \frac{1}{n}\ln(x_l + \varepsilon) \\ &= \frac{1}{n}\left[\sum_{i=1}^{n} \left(\ln\left(\mu + \frac{\varepsilon}{n}\right) - \ln(\mu) + \ln(x_i)\right)\right] - \frac{1}{n}\sum_{i=1}^{n} \ln(x_i) + \frac{1}{n}\ln(x_l) - \frac{1}{n}\ln(x_l + \varepsilon) \\ &= \frac{1}{n}\left[\sum_{i=1}^{n} \left(\ln\left(\mu + \frac{\varepsilon}{n}\right) - \ln(\mu)\right)\right] + \frac{1}{n}(\ln(x_l) - \ln(x_l + \varepsilon)) \\ &= \ln\left(\mu + \frac{\varepsilon}{n}\right) - \ln(\mu) + \frac{1}{n}(\ln(x_l) - \ln(x_l + \varepsilon)) \end{aligned}$$

and part (i) is proved.

We then have:

$$
\begin{aligned}
\triangle I_T(\mathbf{x};l,\varepsilon)>0 &\Longleftrightarrow \ln\left(\mu+\frac{\varepsilon}{n}\right)-\ln(\mu)>\frac{1}{n}\left[\ln(x_l+\varepsilon)-\ln(x_l)\right]\\
&\Longleftrightarrow \ln(x_l+\varepsilon)-\ln(x_l)<n\left[\ln\left(\mu+\frac{\varepsilon}{n}\right)-\ln(\mu)\right]\\
&\Longleftrightarrow \ln\left(\frac{x_l+\varepsilon}{x_l}\right)<n\ln\left(\frac{\mu+\frac{\varepsilon}{n}}{\mu}\right)\\
&\Longleftrightarrow \ln\left(\frac{x_l+\varepsilon}{x_l}\right)<n\ln\left(1+\frac{\varepsilon}{n\mu}\right)\\
&\Longleftrightarrow \ln\left(\frac{x_l+\varepsilon}{x_l}\right)<\ln\left(1+\frac{\varepsilon}{n\mu}\right)^n\\
&\Longleftrightarrow \frac{x_l+\varepsilon}{x_l}<\left(1+\frac{\varepsilon}{n\mu}\right)^n\\
&\Longleftrightarrow 1+\frac{\varepsilon}{x_l}<\left(1+\frac{\varepsilon}{n\mu}\right)^n\\
&\Longleftrightarrow \frac{\varepsilon}{x_l}<\left(1+\frac{\varepsilon}{n\mu}\right)^n-1\\
&\Longleftrightarrow \frac{x_l}{\varepsilon}>\frac{1}{\left(1+\frac{\varepsilon}{n\mu}\right)^n-1}\\
&\Longleftrightarrow x_l>\frac{\varepsilon}{\left(1+\frac{\varepsilon}{n\mu}\right)^n-1}
\end{aligned}
$$

This concludes part (ii).

Part (iii) follows immediately since function $m_T(\varepsilon)=\frac{\varepsilon}{\left(1+\frac{\varepsilon}{n\mu}\right)^n-1}$ is of the form $f(\varepsilon)=\frac{\varepsilon}{\left(1+\frac{\varepsilon}{y}\right)^z-1}$, where $y=n\mu$ and $z=n$, and it is a standard result that $\lim_{\varepsilon\to 0}f(\varepsilon)=\frac{y}{z}=\mu$.

This concludes the proof of Theorem 3.

D Proof of Theorem 4

Take an arbitrary $\mathbf{x}\in\mathbb{R}^n_+$, $l\in\{1,\dots,n\}$, and $\varepsilon>0$. It follows from (3) that

$$\triangle I_{AG}(\mathbf{x};l,\varepsilon)=\mu\left[I_G(x_l+\varepsilon)_l\,\mathbf{x}-I_G(\mathbf{x})\right]+\frac{\varepsilon}{n}I_G(x_l+\varepsilon)_l\,\mathbf{x}$$

$$= \frac{\mu\varepsilon\left[\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k - 2\left(n-l+\frac{1}{2}\right)\sum_{i=1}^{n} x_i\right]}{n\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i} + \frac{\varepsilon}{n}$$

$$- \frac{\varepsilon}{n^2}\left[\frac{\sum_{k=1,k\neq l}^{n} 2\left(n-k+\frac{1}{2}\right)x_k + 2\left(n-l+\frac{1}{2}\right)(x_l+\varepsilon)}{\varepsilon+\sum_{i=1}^{n} x_i}\right]$$

$$\triangle\, I_{AG}\left(\mathbf{x};l,\varepsilon\right) = \frac{\mu\varepsilon\left[\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k - 2\left(n-l+\frac{1}{2}\right)\sum_{i=1}^{n} x_i\right]}{n\left(\varepsilon+\sum_{i=1}^{n} x_i\right)\sum_{i=1}^{n} x_i} + \frac{\varepsilon}{n}$$

$$- \frac{\varepsilon}{n^2}\left[\frac{\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k + 2\left(n-l+\frac{1}{2}\right)\varepsilon}{\varepsilon+\sum_{i=1}^{n} x_i}\right]$$

$$= \frac{\mu\varepsilon\left[\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k - 2\left(n-l+\frac{1}{2}\right)n\mu\right]}{n\left(\varepsilon+n\mu\right)n\mu} + \frac{\varepsilon}{n}$$

$$- \frac{\varepsilon}{n^2}\left[\frac{\sum_{k=1}^{n} 2\left(n-k+\frac{1}{2}\right)x_k + 2\left(n-l+\frac{1}{2}\right)\varepsilon}{\varepsilon+n\mu}\right]$$

$$= \frac{\mu\varepsilon\left[2n\sum_{k=1}^{n} x_k - 2\sum_{k=1}^{n} kx_k + n\mu - 2n^2\mu + 2ln\mu - n\mu\right]}{n^2\mu\left(\varepsilon+n\mu\right)} + \frac{\varepsilon}{n}$$

$$- \frac{\varepsilon}{n^2}\left[\frac{2n\sum_{k=1}^{n} x_k - 2\sum_{k=1}^{n} kx_k + n\mu + 2n\varepsilon - 2l\varepsilon + \varepsilon}{\varepsilon+n\mu}\right]$$

$$= \frac{\mu\varepsilon\left[2n^2\mu - 2\sum_{k=1}^{n} kx_k - 2n^2\mu + 2ln\mu\right]}{n^2\mu\left(\varepsilon+n\mu\right)} + \frac{\varepsilon}{n}$$

$$- \frac{\varepsilon}{n^2}\left[\frac{2n^2\mu - 2\sum_{k=1}^{n} kx_k + n\mu + 2n\varepsilon - 2l\varepsilon + \varepsilon}{\varepsilon+n\mu}\right]$$

$$
\begin{aligned}
&= \frac{\varepsilon}{n} + \frac{-2\varepsilon \sum_{k=1}^{n} kx_k + 2\varepsilon ln\mu - 2\varepsilon\mu n^2 + 2\varepsilon \sum_{k=1}^{n} kx_k - \varepsilon n\mu - 2n\varepsilon^2 + 2l\varepsilon^2 - \varepsilon^2}{n^2(\varepsilon + n\mu)} \\
&= \frac{\varepsilon}{n} + \frac{2\varepsilon ln\mu - 2\varepsilon\mu n^2 - \varepsilon n\mu - 2n\varepsilon^2 + 2l\varepsilon^2 - \varepsilon^2}{n^2(\varepsilon + n\mu)} \\
&= \frac{\varepsilon n(\varepsilon + n\mu) + 2\varepsilon ln\mu - 2\varepsilon\mu n^2 - \varepsilon n\mu - 2n\varepsilon^2 + 2l\varepsilon^2 - \varepsilon^2}{n^2(\varepsilon + n\mu)} \\
&= \frac{\varepsilon^2 n + \varepsilon n^2\mu + 2\varepsilon ln\mu - 2\varepsilon\mu n^2 - \varepsilon n\mu - 2n\varepsilon^2 + 2l\varepsilon^2 - \varepsilon^2}{n^2(\varepsilon + n\mu)} \\
&= \frac{-\varepsilon^2 n + \varepsilon n^2\mu + 2\varepsilon ln\mu - 2\varepsilon\mu n^2 - \varepsilon n\mu + 2l\varepsilon^2 - \varepsilon^2}{n^2(\varepsilon + n\mu)} \\
&= \frac{-\varepsilon^2 n - \varepsilon\mu n^2 + \varepsilon n\mu(2l-1) + \varepsilon^2(2l-1)}{n^2(\varepsilon + n\mu)} \\
&= \frac{(2l-1)\left(\varepsilon n\mu + \varepsilon^2\right) - \left(\varepsilon^2 n + \varepsilon\mu n^2\right)}{n^2(\varepsilon + n\mu)} \\
&= \frac{(2l-1)\varepsilon(n\mu + \varepsilon) - \varepsilon n(\varepsilon + \mu n)}{n^2(\varepsilon + n\mu)} \\
&= \frac{\varepsilon(2l-1-n)}{n^2}
\end{aligned}
$$

and part (i) is proved.

We then have that

$$\triangle I_{AG}(\mathbf{x}; l, \varepsilon) > 0 \Longleftrightarrow l > \frac{n+1}{2}$$

This concludes part (ii) and the proof of Theorem 4.

E Proof of Theorem 5

Take an arbitrary $\mathbf{x} \in \mathbb{R}_+^n$, $l \in \{1, \dots, n\}$, and $\varepsilon > 0$. It follows from (4) that

$$I_V((x_l + \varepsilon)_l \mathbf{x}) = \frac{1}{n}\left[\sum_{i=1, i\neq l}^{n} \left(x_i - \left(\mu + \frac{\varepsilon}{n}\right)\right)^2 + \left(x_l + \varepsilon - \left(\mu + \frac{\varepsilon}{n}\right)\right)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i=1, i\neq l}^{n}\left(x_i^2 - 2x_i\left(\mu + \frac{\varepsilon}{n}\right) + \mu^2 + \frac{2\mu\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right) + \right.$$

$$\left. x_l^2 + 2x_l\varepsilon + \varepsilon^2 - 2\left(x_l + \varepsilon\right)\left(\mu + \frac{\varepsilon}{n}\right) + \mu^2 + \frac{2\mu\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right]$$

$$= \frac{1}{n}\left[\sum_{i=1, i\neq l}^{n}\left(x_i^2 - 2x_i\mu - 2x_i\frac{\varepsilon}{n} + \mu^2 + \frac{2\mu\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right) + \right.$$

$$\left. x_l^2 + 2x_l\varepsilon + \varepsilon^2 - 2x_l\mu - 2x_l\frac{\varepsilon}{n} - 2\varepsilon\mu - 2\frac{\varepsilon^2}{n} + \mu^2 + \frac{2\mu\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right]$$

$$= \frac{1}{n}\left[\sum_{i=1}^{n}\left(x_i^2 - 2x_i\mu - 2x_i\frac{\varepsilon}{n} + \mu^2 + \frac{2\mu\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right) + 2x_l\varepsilon + \varepsilon^2 - 2\varepsilon\mu - 2\frac{\varepsilon^2}{n}\right]$$

We then have that

$$\triangle I_V\left(\mathbf{x}; l, \varepsilon\right) = \frac{1}{n}\left[\sum_{i=1}^{n}\left(x_i^2 - 2x_i\mu - 2x_i\frac{\varepsilon}{n} + \mu^2 + \frac{2\mu\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right) + 2x_l\varepsilon + \varepsilon^2 - 2\varepsilon\mu - 2\frac{\varepsilon^2}{n}\right]$$
$$- \frac{1}{n}\left[\sum_{i=1}^{n}\left(x_i^2 - 2x_i\mu + \mu^2\right)\right]$$

$$
\begin{aligned}
&= \frac{1}{n}\left[\sum_{i=1}^{n}\left(-2x_i\frac{\varepsilon}{n}+\frac{2\mu\varepsilon}{n}+\frac{\varepsilon^2}{n^2}\right)+2x_l\varepsilon+\varepsilon^2-2\varepsilon\mu-2\frac{\varepsilon^2}{n}\right] \\
&= \frac{1}{n}\left[-\frac{2\varepsilon}{n}\sum_{i=1}^{n}x_i+2\mu\varepsilon+\frac{\varepsilon^2}{n}+2x_l\varepsilon+\varepsilon^2-2\varepsilon\mu-2\frac{\varepsilon^2}{n}\right] \\
&= \frac{1}{n}\left[-2\varepsilon\mu+2\mu\varepsilon+\frac{\varepsilon^2}{n}+2x_l\varepsilon+\varepsilon^2-2\varepsilon\mu-2\frac{\varepsilon^2}{n}\right] \\
&= \frac{1}{n}\left(2x_l\varepsilon-2\varepsilon\mu+\varepsilon^2-\frac{\varepsilon^2}{n}\right) \\
&= \frac{1}{n}\left(2\varepsilon\left(x_l-\mu\right)+\left(1-\frac{1}{n}\right)\varepsilon^2\right) \\
&= \frac{1}{n}\left(2\varepsilon\left(x_l-\mu\right)+\left(\frac{n-1}{n}\right)\varepsilon^2\right)
\end{aligned}
$$

and part (i) is proved.

We then have that

$$
\begin{aligned}
\triangle I_V\left(\mathbf{x};l,\varepsilon\right)>0 &\Longleftrightarrow 2\varepsilon\left(x_l-\mu\right)>\left(\frac{n-1}{n}\right)\varepsilon^2 \\
&\Longleftrightarrow 2x_l-2\mu>\left(\frac{n-1}{n}\right)\varepsilon \\
&\Longleftrightarrow x_l-\mu>\frac{1}{2}\left(\frac{n-1}{n}\right)\varepsilon \\
&\Longleftrightarrow x_l>\mu+\frac{1}{2}\left(\frac{n-1}{n}\right)\varepsilon
\end{aligned}
$$

This concludes part (ii) and it is clear that part (iii) follows immediately. This completes the proof of Theorem 5.

F Proof of Theorem 6

Take an arbitrary $\mathbf{x} \in \mathbb{R}^n_+$, $l \in \{1, \dots, n\}$, and $\varepsilon > 0$. It follows from (5) that

$$I_K\left((x_l+\varepsilon)_l \mathbf{x}\right) = \frac{1}{n\left(\mu+\frac{\varepsilon}{n}\right)}\left[\sum_{i=1, i\neq l}^{n}\left(x_i - \left(\mu+\frac{\varepsilon}{n}\right)\right)^2 + \left(x_l+\varepsilon-\left(\mu+\frac{\varepsilon}{n}\right)\right)^2\right]$$

$$\triangle I_K\left(\mathbf{x}; l, \varepsilon\right) = \frac{1}{n\mu+\varepsilon}\left[\sum_{i=1, i\neq l}^{n}\left(x_i - \left(\mu+\frac{\varepsilon}{n}\right)\right)^2 + \left(x_l+\varepsilon-\left(\mu+\frac{\varepsilon}{n}\right)\right)^2\right] - \frac{1}{n\mu}\sum_{i=1}^{n}\left(x_i-\mu\right)^2$$

$$= \frac{1}{n\mu+\varepsilon}\left[\sum_{i=1, i\neq l}^{n}\left(x_i^2 - 2x_i\left(\mu+\frac{\varepsilon}{n}\right) + \left(\mu+\frac{\varepsilon}{n}\right)^2\right) + \left(x_l+\varepsilon\right)^2 - 2\left(x_l+\varepsilon\right)\left(\mu+\frac{\varepsilon}{n}\right) + \left(\mu+\frac{\varepsilon}{n}\right)^2\right] - \frac{1}{n\mu}\sum_{i=1}^{n}\left(x_i^2 - 2x_i\mu + \mu^2\right)$$

$$= \frac{1}{n\mu+\varepsilon}\left[\sum_{i=1, i\neq l}^{n}\left(x_i^2 - 2x_i\mu - 2x_i\frac{\varepsilon}{n} + \mu^2 + 2\mu\frac{\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right) + x_l^2 + 2x_l\varepsilon + \varepsilon^2 - 2x_l\mu - 2x_l\frac{\varepsilon}{n} - 2\varepsilon\mu - 2\frac{\varepsilon^2}{n} + \mu^2 + 2\mu\frac{\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right] - \frac{1}{n\mu}\sum_{i=1}^{n}\left(x_i^2 - 2x_i\mu + \mu^2\right)$$

$$= \frac{1}{n\mu+\varepsilon}\left[\sum_{i=1}^{n}\left(\mu^2 + 2\mu\frac{\varepsilon}{n} + \frac{\varepsilon^2}{n^2}\right) + \sum_{i=1, i\neq l}^{n}\left(x_i^2 - 2x_i\mu - 2x_i\frac{\varepsilon}{n}\right) + x_l^2 - 2x_l\mu - 2x_l\frac{\varepsilon}{n} + 2x_l\varepsilon + \varepsilon^2 - 2\varepsilon\mu - 2\frac{\varepsilon^2}{n}\right] - \frac{1}{n\mu}\sum_{i=1}^{n}\left(x_i^2 - 2x_i\mu + \mu^2\right)$$

$$= \frac{1}{n\mu+\varepsilon}\left[n\left(\mu^2+2\mu\frac{\varepsilon}{n}+\frac{\varepsilon^2}{n^2}\right)+\sum_{i=1}^{n}\left(x_i^2-2x_i\mu-2x_i\frac{\varepsilon}{n}\right)+2x_l\varepsilon+\varepsilon^2-2\varepsilon\mu-2\frac{\varepsilon^2}{n}\right]$$
$$-\frac{1}{n\mu}\sum_{i=1}^{n}\left(x_i^2-2x_i\mu+\mu^2\right)$$

$$= \frac{1}{n\mu+\varepsilon}\left[n\mu^2+2\mu\varepsilon+\frac{\varepsilon^2}{n}+\sum_{i=1}^{n}\left(x_i^2-2x_i\mu-2x_i\frac{\varepsilon}{n}\right)+2x_l\varepsilon+\varepsilon^2-2\mu\varepsilon-2\frac{\varepsilon^2}{n}\right]$$
$$-\frac{1}{n\mu}\sum_{i=1}^{n}\left(x_i^2-2x_i\mu+\mu^2\right)$$

$$= \frac{1}{n\mu+\varepsilon}\left[n\mu^2+\varepsilon^2\left(1-\frac{1}{n}\right)+2x_l\varepsilon+\sum_{i=1}^{n}x_i^2-2\mu\sum_{i=1}^{n}x_i-2\varepsilon\sum_{i=1}^{n}\frac{x_i}{n}\right]$$
$$-\frac{1}{n\mu}\left[\sum_{i=1}^{n}x_i^2-2\mu\sum_{i=1}^{n}x_i+n\mu^2\right]$$

$$= \frac{1}{n\mu+\varepsilon}\left[n\mu^2+\varepsilon^2\left(1-\frac{1}{n}\right)+2x_l\varepsilon+\sum_{i=1}^{n}x_i^2-2n\mu^2-2\varepsilon\sum_{i=1}^{n}\frac{x_i}{n}\right]$$
$$-\frac{1}{n\mu}\left[\sum_{i=1}^{n}x_i^2-2n\mu^2+n\mu^2\right]$$

$$
\begin{aligned}
&= \frac{1}{n\mu+\varepsilon}\left[-n\mu^2+\varepsilon^2\left(1-\frac{1}{n}\right)+2x_l\varepsilon+\sum_{i=1}^{n}x_i^2-2\varepsilon\sum_{i=1}^{n}\frac{x_i}{n}\right]-\frac{1}{n\mu}\left[\sum_{i=1}^{n}x_i^2-n\mu^2\right]\\
&= \left(\sum_{i=1}^{n}x_i^2-n\mu^2\right)\left(\frac{1}{n\mu+\varepsilon}-\frac{1}{n\mu}\right)+\frac{1}{n\mu+\varepsilon}\left[\varepsilon^2\left(1-\frac{1}{n}\right)+2x_l\varepsilon-2\varepsilon\sum_{i=1}^{n}\frac{x_i}{n}\right]\\
&= \left(\sum_{i=1}^{n}x_i^2-n\mu^2\right)\left(\frac{n\mu-(n\mu+\varepsilon)}{n\mu\,(n\mu+\varepsilon)}\right)+\frac{1}{n\mu+\varepsilon}\left[\varepsilon^2\left(1-\frac{1}{n}\right)+2x_l\varepsilon-2\frac{\varepsilon}{n}\cdot n\mu\right]\\
&= \left(\sum_{i=1}^{n}x_i^2-n\mu^2\right)\left(\frac{-\varepsilon}{n\mu\,(n\mu+\varepsilon)}\right)+\frac{1}{n\mu+\varepsilon}\left[\varepsilon^2\left(1-\frac{1}{n}\right)+2x_l\varepsilon-2\varepsilon\mu\right]\\
&= n\sigma_{\mathbf{x}}^2\left(\frac{-\varepsilon}{n\mu\,(n\mu+\varepsilon)}\right)+\frac{\varepsilon}{n\mu+\varepsilon}\left[\frac{\varepsilon\,(n-1)}{n}+2x_l-2\mu\right]\\
&= \frac{\varepsilon}{n\mu+\varepsilon}\left(2\,(x_l-\mu)-\frac{\sigma_{\mathbf{x}}^2}{\mu}+\frac{\varepsilon\,(n-1)}{n}\right)
\end{aligned}
$$

and part (i) is proved.

We then have that

$$
\begin{aligned}
\triangle I_K(\mathbf{x};l,\varepsilon)>0 &\iff -\frac{\sigma_{\mathbf{x}}^2}{\mu}+\frac{\varepsilon\,(n-1)}{n}+2x_l-2\mu>0\\
&\iff 2x_l>2\mu+\frac{\sigma_{\mathbf{x}}^2}{\mu}-\frac{\varepsilon\,(n-1)}{n}\\
&\iff x_l>\mu+\frac{\sigma_{\mathbf{x}}^2}{2\mu}-\frac{\varepsilon\,(n-1)}{2n}
\end{aligned}
$$

This concludes part (ii) and it is clear that part (iii) follows immediately. This completes the proof of Theorem 6.

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