Autoduality of the Hitchin system and the Geometric Langlands Programme

Michael Groechenig
Wadham College
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To my parents, Andrea and Ernst.

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Name: Michael Groechenig

College: Wadham

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Abstract

This thesis is concerned with the study of the geometry and derived categories associated to the moduli problems of local systems and Higgs bundles in positive characteristic. As a cornerstone of our investigation, we establish a local system analogue of the *BNR correspondence* for Higgs bundles. This result (Proposition 4.3.1) relates flat connections to certain modules of an Azumaya algebra on the family of spectral curves. We prove properness over the semistable locus of the *Hitchin map for local systems* introduced by Laszlo–Pauly (Theorem 4.4.1). Moreover, we show that with respect to this Hitchin map, the moduli stack of local systems is étale locally equivalent to the moduli stack of Higgs bundles (Theorem 4.6.3) (with or without stability conditions). Subsequently, we study two-dimensional examples of moduli spaces of parabolic Higgs bundles and local systems (Theorem 5.2.1), given by *equivariant Hilbert schemes* of cotangent bundles of elliptic curves. Furthermore, the Hilbert schemes of points of these surfaces are equivalent to moduli spaces of parabolic Higgs bundles, respectively local systems (Theorem 5.3.1). The proof for local systems in positive characteristic relies on the properness results for the Hitchin fibration established earlier. The *Autoduality Conjecture* of Donagi–Pantev follows from Bridgeland–King–Reid's McKay equivalence in these examples. The last chapter of this thesis is concerned with the construction of derived equivalences, resembling a *Geometric Langlands Correspondence* in positive characteristic, generalizing work of Bezrukavnikov–Braverman. Away from finitely many primes, we show that over the locus of integral spectral curves, the derived category of coherent sheaves on the stack of local systems is equivalent to a derived category of coherent D-modules on the stack of vector bundles. We conclude by establishing the *Hecke eigenproperty* of Arinkin's autoduality and thereby of the Geometric Langlands equivalence in positive characteristic.

Contents

Chapter 1

Introduction

Before delving into abstract theory, it seems justified to give a panoramic overview of the rich landscape given by the Geometric Langlands Programme and related fields. For more detailed introductions we refer the reader to the survey of T. Gómez [Gom10], E. Frenkel [Fre07], and D. Gaitsgory [Gai03]. Needless to say that these sources have shaped the author's viewpoint of the field, and therefore also influenced the presentation given in Section 1.2 below. It seems almost impossible to give a streamlined introduction to this area without freely referencing the language of moduli spaces and stacks. In order to break this vicious circle of abstraction, we have added an *introduction to the introduction*: a biased (anti-)historical account of the development of the theory of moduli spaces. We emphasize that the use of the term *history* in Section 1.1 is by no means to be understood as an academic statement. It should be seen as a form of mathematical storytelling, which we hope to serve the entertainment of the reader.

1.1 Moduli spaces and stacks: a biased history

It is not always possible to achieve a classification of geometric structures by means of discrete invariants. In many instances, solutions to a given problem are controlled by several parameters (or *moduli*). These parameters themselves can often be packaged into a space, giving rise to a rich geometry.

The first use of the word *modulus* in this context, is attributed to B. Riemann ([Rie57, chap. 12][1]). Interpreting *loc. cit.* in modern language, Riemann determined the dimension of the moduli space of complex structures on a compact Riemann surface of genus g to be $3g-3$. However, this formula can only be taken for granted if $g \geq 2$. This discrepancy between low-genus cases and general Riemann surfaces will be repaired by the introduction of *stacks*, where the appearing non-positive dimensions take into account the size of the automorphism groups of objects.

More elementary examples of moduli problems are given by projective spaces $\mathbb{P}(V)$, which parametrize one-dimensional quotients $V \twoheadrightarrow W$ of a given finite-dimensional vector space V. In order to make this statement precise, one introduces the notion of *families* of one-dimensional quotients.

Definition 1.1.1. *Let S be a variety, and V a finite-dimensional vector space. An S-family of one-dimensional quotients of V is given by a line bundle W on S, together with a surjection of sheaves $\mathcal{O}_S \otimes_k V \twoheadrightarrow W$.*

There is a natural notion of equivalence of S-families, which allows us to speak of the set of isomorphism classes. The functor

$$F : (\mathrm{Var})^{op} \to \mathrm{Set},$$

is defined by sending a variety S to the set of isomorphism classes of S-families of

[1] see `http://www.maths.tcd.ie/pub/HistMath/People/Riemann/AbelFn/AbelFn.pdf` for a transcribed version

one-dimensional quotients of V.

Definition 1.1.2. *A variety $\mathcal{M}$ is a (fine)* moduli space *for the* moduli problem *given by a functor $F : (\mathrm{Var})^{op} \to \mathrm{Set}$, if we have a natural equivalence $F(S) \cong \mathrm{Hom}_{\mathrm{Var}}(S, \mathcal{M})$ for every variety S.*

In the context of one-dimensional quotients of a vectors space V, a moduli space exists, and is usually denoted by $\mathbb{P}(V)$. We refer the reader to Theorem 7.1(b) in [Har77] for a proof of this representability statement in the context of scheme theory. This result has been subsumed by Grothendieck's construction of Hilbert and Quot schemes in [Gro95], which still constitute important building blocks of more general moduli spaces.

Mumford's sophisticated analysis of quotients in algebraic geometry, as given by *Geometric Invariant Theory* (GIT) ([MFK94]), allowed for the first time the *industrial production* of moduli spaces. The rough outline of Mumford's recipe is easily given: for a given moduli problem one first constructs a variety V together with an S-family $x \in F(S)$. One shows that the variety can be chosen to be big enough, such that every possible solution to the moduli problem arises from at least one point of V. Subsequently, one endows V with the action of a reductive group G, such that two points lie in the same orbit if and only if they give rise to equivalent solutions of the moduli problem. GIT allows now to construct *a quotient* $V//G$, depending on the *linearization* of the action. While GIT has had a great impact and accelerating effect on the field of moduli spaces, there are two problems that motivate the search for a different approach. First of all, Mumford's GIT is only well-equipped to handle quotients of reductive group actions (see [DK07] for a generalization). The second problem is that each of Mumford's GIT quotients $V//G$ is really just a quotient of an open G-invariant subscheme $U \subset V$ of *(semi)stable* points. This subscheme depends on the choice of a linearization of the action, and will in general only capture the generic solution to the original moduli problem. In the context of vector bundles on curves, Mumford's *stability* admits a geometrically appealing reformulation in bundle-theoretic terms. But often it is backbreaking work, to understand stability in the sense of GIT in geometric terms. Furthermore, it might not always be appropriate to discard unstable solutions to a moduli problem.

The theory of (algebraic) stacks has been successful in overcoming these difficulties. Instead of relying on elaborate quotient constructions within the category of varieties, one adjoins *universal quotients* to the category of varieties. However, this comes at the price of packaging slightly more information into a moduli problem. Instead of only remembering the set of isomorphism classes, it is necessary to think in terms of groupoids (i.e. categories where every morphism is an isomorphism).

The power of these abstract ideas of the above paragraph can be demonstrated for the moduli problem of line bundles. Let $F : (\mathrm{Var})^{op} \to \mathrm{Set}$ be the functor sending a variety V to the set of isomorphism classes of line bundles on it. We will see that varieties are too *rigid* to represent this functor. Assuming that $\mathcal{M}$ is a moduli space for this problem, and L a line bundle on V, we obtain a corresponding map $f : V \to \mathcal{M}$. Since line bundles are Zariski locally trivial, we can choose an open covering $\bigcup_{i \in I} U_i = V$, which trivializes L. In particular we see that all the maps $f|_{U_i}$ are constant with the same value. This implies that $f : V \to \mathcal{M}$ is constant itself, and hence L must be the trivial line bundle.

This problem can be rectified by assigning to a variety the groupoid of line bundles. This will not define a functor in the strict sense, since for a chain of maps $f : U \to V$ and $g : V \to W$ we have that f^*g^*L and $(g \circ f)^*L$ are not equal but canonically equivalent. Unlike for sets, categories are too strict to capture the mathematical properties of groupoids. Instead it is appropriate to utilize 2-categories, where we have objects, 1-morphisms, and 2-morphisms at our disposal. For instance one can build a 2-category with objects being groupoids, 1-morphisms being functors, and 2-morphisms being natural transformations. The above moduli problem of line bundles is then expressed as the 2-functor

$$F : (\mathrm{Var})^{op} \to \mathrm{Gpd},$$

sending a variety to the groupoid of line bundles. This is the definition of the *stack* of line bundles. This turns out to be exactly enough information to resolve the problem above. Although every line bundle is Zariski locally trivial, a map to a stack also remembers the glueing data along the overlaps. Hence we have just seen the example of a moduli problem which admits a moduli stack but cannot be represented within the realm of varieties.

Similarly, the action of an algebraic group G on a variety V gives rise to a *quotient stack*. Instead of passing from a set X with a G-action to the set of orbits, the stacky quotient constructs a groupoid. Objects are given by elements $x \in X$, with every group element $g \in G$ giving rise to a morphism $g : x \to gx$. After suitably generalizing these group actions, one obtains a class of stacks constructed as quotients of schemes, which is referred to as *algebraic stacks*. This viewpoint allows us to bypass the time consuming GIT construction and thereby to directly focus on geometry. Analogously to the theory of varieties, we can speak of quasi-coherent sheaves, cohomology, and derived categories for algebraic stacks.

Now that we have the language of stacks at our disposal, we introduce the moduli problem that lies at the heart of this thesis.

Definition 1.1.3. *Let X be a smooth projective curve. A* Higgs bundle *on X is a pair consisting of a vector bundle E on X together with a* Higgs field $\theta : E \to E \otimes \Omega^1_X$. *The moduli stack of Higgs bundles will be denoted by $\mathcal{M}_{\mathrm{Dol}}$.*

Higgs bundles were introduced by N. Hitchin in [Hit87a] and C. Simpson in [Sim87]. While Hitchin was led to these ideas through the study of dimensional-reductions of the Yang-Mills equations, Simpson was interested in variations of Hodge structures.

In 2010, B.-C. Ngô used the moduli problem of Higgs bundles in [Ngô10] to prove the Fundamental Lemma and thereby making a groundbreaking contribution to the classical Langlands programme.

The study of Higgs bundles can be understood as linear algebra *relative* to a curve, where we view the Higgs field as a twisted endomorphism. This perspective has been present since the beginning, and certainly played a motivating role for Hitchin when he introduced the morphism that bears his name. The *Hitchin map* is defined by sending a Higgs bundle (E, θ) to the characteristic polynomial of the Higgs field. For fixed rank, the admissible polynomials form a vector space, with the corresponding affine space being denoted by $\mathcal{A}$.

Definition 1.1.4. *The map sending a Higgs bundle (E, θ) to its characteristic polynomial, induces a map of stacks*

$$\chi_{\mathrm{Dol}} : \mathcal{M}_{\mathrm{Dol}} \to \mathcal{A},$$

and will be referred to as the Hitchin map*.*

The zero set of the characteristic polynomial a of the Higgs field θ cuts out a closed subscheme Y_a of the cotangent bundle T^*X. The scheme Y_a is of dimension one, and is called a *spectral curve*. Let $\pi_a : Y_a \to X$ denote the canonical projection to X; we obtain the following description of Higgs bundles on X due to Beauville–Narasimhan–Ramanan for integral Y_a ([BNR89]) and C. Simpson ([Sim94b, Lemma 6.8]) in the general case (see Section 2.3 for a proof).

Theorem 1.1.5 (BNR correspondence)**.** *There exists a canonical equivalence between the stack of rank n Higgs bundles with characteristic polynomial a and coherent sheaves L on Y_a satisfying that $\pi_{a,*}L$ is locally free of rank n.*

N. Nitsure performed in his article [Nit91] the GIT construction of the moduli space of semistable Higgs bundles, and showed that the Hitchin map yields a proper morphism from the GIT moduli space to $\mathcal{A}$. Properness is verified by checking Grothendieck's valuative criterion ([Har77, Thm. 4.7]), following an argument of S. Langton for moduli of torsion free sheaves [Lan75]. We will reproduce this proof in Section 2.4.

The Hitchin map has many other interesting properties, its total space carries a natural symplectic form, and χ_{Dol} can be viewed as an algebraic completely integrable system. This perspective has been explored in Hitchin's article [Hit87b]. An important consequence of this observation is that the generic fibre of the Hitchin morphism is an abelian variety. Additionally, this variety can be shown to be self-dual, which turns out to be closely related to the Geometric Langlands Conjecture (see Section 1.2). Self-duality of the generic fibre is actually a consequence of the BNR correspondence 1.1.5. If the spectral curve Y_a is smooth, then Higgs bundles with characteristic polynomial a correspond precisely to line bundles L on Y_a. In particular we have established a connection between the generic Hitchin fibre and the moduli space of line bundles on spectral curves.

1.2 Overview: Geometric Langlands

Langlands correspondences are duality phenomena that can be observed in an array of mathematical areas, ranging from arithmetics over algebraic geometry to geometric representation theory and mathematical physics. Unfortunately, this mathematical treasury happens to be covered by so many layers of difficult theory, that accessing it has turned into a seemingly infeasible task.

One of the very first instances of Langlands duality can nonetheless be explained after a second-year topology course. If Σ_g denotes an orientable compact surface of genus g, and x an arbitrary point of Σ_g, then its fundamental group Π_g is known to be given by

$$\Pi_g := \pi_1(\Sigma_g, x) \cong < A_1, \dots, A_g, B_1, \dots, B_g | [A_1, B_1] \cdots [A_g, B_g] > .$$

The Jacobian J_g of Σ_g can be defined to be the *moduli space* of one-dimensional *unitary representations* of Π_g

$$\rho : \Pi_g \to S^1.$$

Since S^1 is a commutative group, we see that the choice of a representation ρ corresponds to the choice of $2g$ elements of S^1. This allows us to identify J_g with the torus $(S^1)^{2g}$, which implies the following identity of fundamental groups:

$$\pi_1(\Sigma_g)^{ab} \cong \pi_1(J_g). \tag{1.1}$$

So far, equation (1.1) should either be taken with a grain of salt, or stated as a comparison for ranks of free abelian groups

$$\operatorname{rk} \pi_1(\Sigma_g) = \operatorname{rk} \pi_1(J_g).$$

This is due to the lack of a canonical morphism $\pi_1(\Sigma_g) \to \pi_1(J_g)$, which is independent of our choice of generators for Π_g. This problem can be overcome by endowing Σ_g with the extra structure of a *Riemann surface*, i.e. by viewing Σ_g as the underlying topological space of a complex holomorphic manifold. From now on, we denote by X a compact Riemann surface of genus g, as before with a base point x. According to a special case of a theorem of M.S. Narasimhan and C. S. Seshadri ([NS64]), one-dimensional unitary representations of Π_g are equivalent to holomorphic line bundles on X, which are topologically trivial. This endows the Jacobian J_g with a complex structure; the corresponding complex manifold will be denoted by J_X. The base point

$x \in X$ allows us to define the so-called *Abel-Jacobi map*

$$A : X \to J_X,$$

which sends $y \in X$ to the line bundle $\mathcal{O}_X(y - x)$.

Theorem 1.2.1 (Geometric Class Field Theory, version 1)**.** *Let X be a compact Riemann surface with a marked point x. The Abel-Jacobi map $X \to J_X$ induces a map $\pi_1(X, x) \to \pi_1(J_X, \mathcal{O}_X)$, which yields an equivalence $\pi_1(X)^{ab} \cong \pi_1(J_X)$. In particular, a one-dimensional local system on X gives rise to a one-dimensional local system on J_X.*

The use of number-theoretic terminology as above certainly merits some explanation. There is a well-known dictionary, connecting the geometry of curves and the arithmetic of number fields. In order to emphasize the connection with Theorem 1.2.1, we remind the reader that compact Riemann surfaces are synonymous to projective curves over $\mathbb{C}$. According to this philosophy, number fields should be compared with function fields of a projective curve. The fundamental group of the underlying topological space is analogous to the unramified part of the absolute Galois group of a number field. Below we quote the main result of *unramified* Class Field Theory in the geometric formulation given by G. Laumon in [Lau90, Thm. 2.2.1]. At first we need to explain a peculiar phenomenom of positive characteristic geometry.

Let therefore $\mathbb{F}_p$ be the finite field with p elements, and with algebraic closure denoted by k. If G is a commutative algebraic group defined over $\mathbb{F}_p$, and G' its base change to k, we can define an interesting endomorphism. First of all, we consider the base change of the map $G \to G$, which is induced by raising functions to there p-th power. The base change of this morphism to G' will be called the Frobenius $F : G' \to G'$. Using the algebraic group structure on G, we can take the difference between F and the identity map id_G, which yields a morphism $L : G' \to G'$. This map is usually referred to as the *Lang isogeny* of G', and it is known to be a finite étale map of commutative algebraic groups. Moreover, the kernel of L is the finite group $A := G(\mathbb{F}_p)$, given by the $\mathbb{F}_p$-points of G. Under the additional assumption that G' is connected we can think of $G' \to G'$ as a finite étale covering of itself, with group of deck transformations given by the finite group A of rational points. This gives rise to a surjection

$$\pi_1^{et}(G', 0) \twoheadrightarrow A,$$

which will provide the link between Class Field Theory in arithmetic and its cousin (Theorem 1.2.1). If X is a smooth projective curve defined over $\mathbb{F}_p$, its Jacobian J_X is a commutative algebraic group, and therefore a Lang isogeny L for $J'_X := J_X \times_{\operatorname{Spec}\mathbb{F}_p} \operatorname{Spec} k$ can be defined. If we denote by F the function field of X, then we see that the kernel of L can naturally be identified with $J_X(\mathbb{F}_p)$.

We have $Gal(H/K) \cong \pi_1^{et}(X, \bar{\eta})$, for a suitably chosen geometric point $\bar{\eta}$, lying over the generic point of X. Since the discussion on the Lang isogeny of J_X allows us to view Cl_K as a quotient of $\pi_1^{et}(J_X)$, we see that Class Field Theory allows us to assign to every one-dimensional ℓ-adic local system, a one-dimensional ℓ-adic local system on J_X. This last interpretation provides the link to the geometric statement obtained in 1.2.1.

Theorem 1.2.2. *Let X be a smooth projective curve defined over $\mathbb{F}_p$, with a rational point $x \in X(\mathbb{F}_p)$. The construction of the Lang isogeny, and pullback along the Abel-Jacobi map $X \to J_X$ induce a bijection between*

(1) the set of characters $\chi\colon J_X(\mathbb{F}_p) \to \bar{\mathbb{Q}}_\ell^\times$,

(2) ℓ-adic character sheaves on J_X with a trivialization at $\mathcal{O} \in J_X$,

(3) ℓ-adic local systems on X with a trivialization at x.

We recall that a character sheaf in this context refers to a one-dimensional ℓ-adic local system L on a group G, together with a compatible equivalence between m^*L and $p_1^*L \otimes p_2^*L$, where $m : G \times G \to G$ denotes the multiplication map ([Gai03, sect. 4.3]). The discussion at the beginning of this section suggested that a statement similar to Theorem 1.2.2 would be true in different settings. The framework of ℓ-adic sheaves elegantly connects geometry and arithmetic; but the theory of topological (Betti) local systems or D-modules may be amenable to different techniques.

The formulation of the *Geometric Langlands Correspondence* as given in [Lau87] by G. Laumon, has been derived by similar (but more sophisticated) techniques from the Langlands Programme in number theory. Laumon attributes Drinfeld's proof of the Langlands Conjecture for GL_2 [Dri83] as a source of inspiration for this *geometrization* of the Langlands Programme. The latter is often described as a *non-abelian* generalization of Class Field Theory. This terminology refers to the fact that Class Field Theory is only of use to study group morphisms $Gal(H/F) \to G$, which factor through the abelianization of the Galois group. The Langlands Programme allows an arbitrary reductive group G as a target, and relates them to so-called *automorphic* representations of a related group, called G^L. If G is GL_n, then its dual will be GL_n. The framework of Langlands dual reductive groups plays also an important role in the Geometric Langlands Programme. In order to keep things simple, we restrict our attention nonetheless to the self-dual case of GL_n. In the hope that our discussion of Class Field Theory has provided ample justification for the claimed link with arithmetic, we will not delve further into these side directions. For this reason we also assume from now on that we are working in the context of D-modules. Local systems are henceforth pairs, consisting of a vector bundle and a flat connection.

The statement of the Geometric Langlands Conjecture strives for a higher rank analogue of Theorem 1.2.2. In the rank one case we have associated to every one-dimensional local system on X a character sheaf on the Jacobian J_X. The notion of character sheaves depends heavily on the existence of a group structure on the moduli space of line bundles, induced by the tensor product. In a higher rank situation, such a gadget does not exist, since the tensor product of two vector bundles produces a vector bundle of different rank. A pessimist might view the the search for a higher rank analogue of Geometric Class Field Theory already as grounded, but being optimists we try a different viewpoint on tensor products of line bundles, which yields a more sensible theory for general vector bundles. Such a change of perspective can be given by viewing the second factor in a tensor product $L_1 \otimes L_2$ as being induced by a divisor D. The product can then be described as $L_1(D)$, and will be understood as a *modification* of L_1. This terminology is meant to capture the intuitive idea that $L_1(D)$ is identified with L_1 away from D, with the difference of the two invertible sheaves being controlled by D. The local nature of this concept allows us to restrict attention to divisors D supported at a single closed point $x \in X$. The following definition introduces *modifications* for arbitrary vector bundles.

Definition 1.2.3 (Hecke modification)**.** *Let E and F be vector bundles of rank n on a smooth projective curve X, and $x \in X$ a closed point. An injection of coherent sheaves $E \hookrightarrow F$ is called a* Hecke modification *at x, if F/E is equivalent to the skyscraper sheaf $\mathcal{O}_x^{\oplus i}$ for some positive integer i. We refer to i as the degree of the modification.*

The local nature of Hecke modifications and the global notion of *stability* induce irreparably tension between the two concepts. The Hecke modification of a stable vector bundle is not necessarily stable, implying that the moduli space of stable vector bundles is *not* the right analogue of the Jacobian J_X in Geometric Class Field Theory. In particular, we see that a formulation of the Geometric Langlands Conjecture will require to work with all vector bundles at once, which is most conveniently packaged in the language of stacks. The corresponding stack of all vector bundles on X will be denoted by Bun.

Definition 1.2.4 (Hecke correspondence)**.** *Let $\mathcal{H}^i$ denote the stack of Hecke modi-*

fications $(E \hookrightarrow F, x)$ *of degree* i *at* x. *We have a map* $q : \mathcal{H}^i \to \mathrm{Bun}$, *sending the data above to* F. *Moreover there is a map* $p : \mathcal{H}^i \to \mathrm{Bun} \times X$, *which sends the triple* (E, F, x) *to* (E, x). *We obtain the correspondence diagram*

$$\begin{array}{ccc} & \mathcal{H}^i & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathrm{Bun} & & \mathrm{Bun} \times X. \end{array}$$

With the Hecke correspondence at our disposal it is easy to define the Hecke operator, which allows us to conveniently state the Geometric Langlands Conjecture in its various forms.

Definition 1.2.5 (Hecke operators)**.** *The composition of derived functors*

$$p_! \circ q^* : D(\mathrm{Bun}, D_{\mathrm{Bun}}) \to D(\mathrm{Bun}, D_{\mathrm{Bun}})$$

will be denoted by $\mathbb{H}^i$ *and referred to as the Hecke operator of degree* i.[2]

As we have already alluded to above, Hecke operators are used to generalize the concept of *character sheaves* in Geometric Class Field Theory. The generalization bears the name *Hecke eigensheaves.* An object M in $D(\mathrm{Bun}, D_{\mathrm{Bun}})$ is called a Hecke eigensheaf, if there exists a local system L on X, such that $\mathbb{H}^i(M) \cong M \boxtimes \bigwedge^i L$ for all $i = 1, \dots n$. In this case, we will also call L the eigenvalue of the eigensheaf M. All the character sheaves constructed in Theorem 1.2.2 turn out to be Hecke eigensheaves. Furthermore, the eigenvalue of the eigensheaf associated to a local system L on X is equivalent to L itself. It is this observation, which the Geometric Langlands Conjectures strives to generalize.

Conjecture 1.2.6 (Geometric Langlands, version 1)**.** *Let* X *be a complex smooth*

[2] Here the use of the term *degree* seems appropriate, because degree i Hecke operators shift the degree of a vector bundle by i.

projective curve, and L an irreducible rank n local system[3]*. There exists a (cuspidal) Hecke eigensheaf in $D(\mathrm{Bun}, D_{\mathrm{Bun}})$ with eigenvalue L.*

In the related context of ℓ-adic sheaves on a smooth projective curve of positive characteristic, this has been proven by Frenkel–Gaitsgory–Vilonen in [FGV02] and [Gai04]. The tamely ramified case has been discussed by J. Heinloth in [Hei04].

The work of Beilinson–Drinfeld [BD] led to a conjecture, comprising the one stated above. They conjecture the existence of suitable derived categories of D-modules $D^?(\mathrm{Bun}, D_{\mathrm{Bun}})$ and quasi-coherent sheaves on the stack of local systems $D^?(\mathcal{M}_{\mathrm{dR}}, \mathcal{O}_{\mathcal{M}_{\mathrm{dR}}})$, and natural derived equivalence between both sides.

Conjecture 1.2.7 (Geometric Langlands, version 2)**.** *There exists a derived equivalence $D^?(\mathrm{Bun}, D_{\mathrm{Bun}}) \cong D^?(\mathcal{M}_{\mathrm{dR}}, \mathcal{O}_{\mathcal{M}_{\mathrm{dR}}})$, which intertwines the Hecke operators $\mathbb{H}^i$ from Definition 1.2.5 with the operator $\mathbb{W}^i : D^?(\mathcal{M}_{\mathrm{dR}}, \mathcal{O}_{\mathcal{M}_{\mathrm{dR}}}) \to D^?(\mathcal{M}_{\mathrm{dR}} \times X, \mathcal{O}_{\mathcal{M}_{\mathrm{dR}}} \boxtimes D_X)$, sending a quasi-coherent sheaf M to $M \boxtimes \mathcal{Q}$. Here $\mathcal{Q}$ refers to the universal local system on $\mathcal{M}_{\mathrm{dR}} \times X$.*

Recently, D. Gaitsgory and D. Arinkin have succeeded in giving a precise definition of the categories $D^?$ in [AG12]. Moreover, a proof of the corresponding conjecture for GL_2 has been sketched by Gaitsgory in [Gai13].

The ring of D-modules of a smooth variety Y carries a natural filtration, whose associated graded is canonically equivalent to the coordinate ring of the cotangent bundle T^*Y. Therefore, D_Y is considered to be a quantization of T^*Y. This is one of the many justifications for calling the Geometric Langlands Conjecture a *quantum statement*. Further backup comes from the mirror symmetry considerations of Kapustin–Witten [KW07] and the geometric representation theory viewpoint of Beilinson–Drinfeld in [BD]. Inspired by this analogy, one should view $D(\mathrm{Bun}, D_{\mathrm{Bun}})$ as a quantization of the derived category $D(T^* \mathrm{Bun}, \mathcal{O}_{T^* \mathrm{Bun}})$. It is a well-known observation, due to Laumon

[3] We remind the reader that this terminology refers henceforth to a pair consisting of a vector bundle and a flat connection.

[Lau88] that $T^*\operatorname{Bun}$ is canonically equivalent to the stack of Higgs bundles (Proposition 2.2.3). This has led Donagi–Pantev to study the classical limit of the Geometric Langlands Conjecture (see [DP12]) stated below.

Conjecture 1.2.8 (Autoduality Conjecture)**.** *There exists a derived equivalence*

$$D^?(\mathcal{M}_{\mathrm{Dol}}) \cong \mathcal{D}^?(\mathcal{M}_{\mathrm{Dol}}),$$

satisfying an analogue of the Hecke eigenproperty in Conjecture 1.2.7.

At this point we prefer to leave a precise formulation of the classical limit of the Hecke eigenproperty open. However, we refer the reader to [DP12] and Section 6.3.

Conjecture 1.2.8 is backed up by the geometry of the Hitchin fibration. As we already have indicated above, there exists an open dense subset of the Hitchin base $\mathcal{A}_{\mathrm{sm}} \subset \mathcal{A}$, where the fibres are given by self-dual abelian varieties. A classical result of Mukai [Muk87] can then be applied to show that the universal line bundle on $A \times A$ gives rise to an autoequivalence $D^b_{coh}(A) \cong D^b_{coh}(A)$, where A denotes a self-dual abelian variety. The Autoduality Conjecture ask for the existence of an extension of this generically defined autoequivalence.

The open subset $\mathcal{A}_{\mathrm{sm}}$ is contained in the open subset $\mathcal{A}_{\mathrm{int}}$ of the Hitchin base $\mathcal{A}$, where the corresponding spectral curves are integral. Over this larger locus, the fibres of the Hitchin map are equivalent to compactified Jacobians $\bar{\mathrm{J}}$ (cf. [AK80]) of integral curves. Arinkin has extended Mukai's autoequivalence to compactified Jacobians of integral curves defined over a field of zero characteristic [Aria]. One of the contributions of this thesis is that we observe that this autoequivalence $D^b_{coh}(\bar{\mathrm{J}}) \cong D^b_{coh}(\bar{\mathrm{J}})$ also holds in large enough characteristic (Theorem 6.2.7).

The connections of the Autoduality Conjecture and the Geometric Langlands Programme are far-ranging. Not only can it be seen as a classical limit of the Geometric Langlands Programme, but also arises on an equal footing in the mirror symmetry considerations of Kapustin–Witten [KW07]. Another connection appears in positive characteristic, as has been demonstrated by the work of Bezrukavnikov–Braverman [BB07]. Work of Bezrukavnikov–Mirković–Rumynin [BMR] revealed that the ring of differential operators D_Y on a smooth variety Y has a big centre, which can be identified with the structure sheaf of the cotangent bundle $T^*Y^{(1)}$.[4] The sheaf of rings D_Y turns out to be an Azumaya algebra $\mathcal{D}_Y$ relative to its centre. Let us apply this result heuristically to the stack of bundles Bun on a smooth projective curve X. The cotangent stack $T^*\operatorname{Bun}$ is canonically equivalent to the stack of Higgs bundles, as has already been noticed. In particular, we see that there should be an intimate relation between the categories $D(\operatorname{Bun}, D_{\operatorname{Bun}})$ and $D(T^*\operatorname{Bun}, \mathcal{O}_{T^*\operatorname{Bun}})$. Bezrukavnikov–Braverman succeeded in making this observation precise over the locus $\mathcal{A}_{\mathrm{sm}}$.

A close relation between Higgs bundles and local systems in positive characteristic has already been observed in pre-Langlands terms. Laszlo–Pauly constructed in [LP01] a Hitchin morphism for the moduli stack of local systems on a curve of positive characteristic. Analogously to the case of Higgs bundles, they established that the zero fibre of this map is universally closed, when restricted to the case of semi-stable Higgs bundles. One of the contributions of the present thesis is to extend this properness result to the whole Hitchin base, and to show that the two Hitchin fibrations for Higgs bundles and local systems are fibrewise equivalent.

The ideas of Bezrukavnikov–Braverman ([BB07]) have been further developed by T. Nevins in [Nev09], R. Travkin in [Tra11], and H.-T. Chen and X. Zhu in [CZ12]. While Nevins investigated extensions to the so-called mirabolic case, Travkin studied the Quantum Geometric Langlands Conjecture, and Chen–Zhu an extension to arbitrary reductive groups. While we work in the more classical context of Bezrukavnikov–

[4]The superscript $^{(1)}$ refers to the Frobenius twist of a variety, as defined in Definition 3.2.8. For now, it can be safely ignored by the reader.

Braverman, the main contribution of the work of the present thesis lies therein that we do not restrict our attention to the locus of smooth spectral curves.

From the work of [BB07] one can derive a *meta principle* that over a field of positive characteristic, the Autoduality Conjecture and the Geometric Langlands Programme are equivalent. In the last chapter of this thesis, we will put this principle to test, by applying Arinkin's Autoduality Theorem to establish Geometric Langlands over the locus $\mathcal{A}_{\mathrm{int}}$ of integral spectral curves, away from finitely many primes. While many of the ideas of *loc. cit.* carry forward in a straightforward manner, the verification of the Hecke eigenproperty (Section 6.3) opens doors to new challenges.

1.3 The contribution of this thesis

After having sketched the rich mathematical landscape given by the Geometric Langlands Programme and related conjectures, it is time to explain how this thesis fits into the picture. The author's starting point was the paper [BB07] by Bezrukavnikov–Braverman, to which we have devoted the end of Section 1.2. The objective of the present thesis can be described in a single sentence: we investigate the implications of autoduality for the Hitchin system on the geometry of moduli spaces and the Geometric Langlands Programme in positive characteristic. A more detailed discussion of our research will be given in the following subsections.

1.3.1 Moduli of local systems

In chapter 4.1 we investigate the moduli stack of local systems in positive characteristic. We emphasize that local system refers in this context to a pair consisting of a vector bundle and a flat connection. Using the Azumaya picture for D-modules in positive characteristic ([BMR]), we obtain an alternative description of local systems in Section 4.3, which is in the spirit of the BNR correspondence for Higgs bundles (see

Section 2.3). In Section 4.4 we put this result to test, by showing that Laszlo–Pauly's Hitchin map is universally closed for semi-stable local systems, using Langton's strategy to verify the valuative criterion for properness ([Lan75]).

We then embark on proving the main result of this section, which relates the Hitchin fibration for local systems to the one for Higgs bundles.

Theorem 1.3.1. *Let X be a smooth projective curve, we denote by $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}(X) \to \mathcal{A}^{(1)}$ and $\chi_{\mathrm{Dol}} : \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \to \mathcal{A}^{(1)}$ the respective Hitchin systems. There exists an étale covering $\{U_i \to \mathcal{A}^{(1)}\}$ of the base, such that we have equivalences*

$$\mathcal{M}_{\mathrm{dR}}(X) \times_{\mathcal{A}^{(1)}} U_i \cong \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \times_{\mathcal{A}^{(1)}} U_i.$$

This result will be proved as Theorem 4.6.3, and subsequently be refined in Section 4.7 to include stability conditions. It is worthwhile pointing out that the theorem above implies that Hitchin fibres for local systems are (non-canonically) equivalent to Hitchin fibres for Higgs bundles. By means of faithfully flat descent theory, we obtain a second proof for the Hitchin map for local systems being universally closed. In fact we can show more, and even establish the existence of a coarse moduli space for semistable local systems, which is proper relative to the Hitchin base (Corollary 4.7.5).

1.3.2 Hilbert schemes and Higgs bundles

In chapter 5 we investigate one of the consequences of properness for Hitchin systems. We generalize a result of Gorsky–Nekrasov–Rubtsov ([GNR01]), which relates the *Hilbert scheme* of n points on the cotangent bundle of an elliptic curve E to a certain moduli space $\mathcal{M}$ of rank n parabolic bundles on X. Our strategy of proof is to utilize autoduality of the rank one Hitchin system to construct a natural map from $(T^*E)^{[n]}$ to $\mathcal{M}$. In a small but essential step, properness of the Hitchin map will be applied to show that this map is surjective, and therefore, the inverse of the derived equivalence induces an inverse map of moduli spaces. We observe that this method can be applied in several contexts, including local systems in zero and positive characteristic. Moreover, we construct more equivalences of moduli spaces in the presence of finite group actions (e.g. induced by complex multiplication). In these cases, one replaces the Hilbert scheme by the equivariant Hilbert scheme, which yields an array of surfaces, each of which gives rise to a two-dimensional Hitchin system. On top of that, the Hilbert scheme of n points of each of these surfaces turns out to be naturally equivalent to a Hitchin system itself.

P. Boalch conjectured in [Boa11, Rmk. 11.3] that the Hilbert scheme of m points of a two-dimensional moduli space of meromorphic Higgs bundles is itself diffeomorphic to a moduli space of meromorphic Higgs bundles. Our results should thus be understood as a verification of this conjecture, for a large sample of two-dimensional moduli spaces of tamely ramified Higgs bundles.

Below we give a more detailed description of our strategy to prove the result of [GNR01]. We refer the reader to Theorem 5.3.1 for a more precise statement of our result.

Elliptic curves are self-dual abelian varieties. This means, an elliptic curve E is canonically isomorphic to the moduli space $E^\vee$ of degree zero line bundles on itself. In particular we have a universal line bundle $\mathcal{P}$ on $E \times E$, called the *Poincaré* bundle. It was shown by Mukai in [Muk87] that $\mathcal{P}$ induces an equivalence of derived categories

$$\Phi_{\mathcal{P}} : D^b_{coh}(E) \cong D^b_{coh}(E)$$

of Fourier-Mukai type. A general introduction to derived equivalences will be given in Section 5.1. While it is hard to picture the transform $\Phi_{\mathcal{P}}(\mathcal{F})$ of a general sheaf, we remark that the skyscraper sheaf $\mathcal{O}_x$ at $x \in E$ is sent to the corresponding line bundle

$\mathcal{P}_x := \mathcal{P}|_{\{x\}\times E}$ on E.

By means of this self-duality and the BNR correspondence (Theorem 1.1.5), the cotangent bundle T^*E can be identified with the moduli space of degree zero rank one Higgs bundles on E. Since E is parallelizable, we obtain $T^*E \cong E \times \mathcal{A}$; here $\mathcal{A}$ is the affine space associated to the one-dimensional vector space $H^0(E, \Omega^1_E)$. Base change gives now rise to the autoduality (see Conjecture 1.2.8) of the rank one Hitchin system

$$\Phi_{\mathcal{P}} : D^b_{coh}(T^*E) \cong D^b_{coh}(T^*E).$$

A point in T^*E, corresponding to the Higgs bundle (E, θ) is sent to the line bundle on the spectral curve, associated via the BNR correspondence. In particular, we see that autoduality for the rank one Hitchin system of an elliptic curve, contains the identification of moduli spaces given by the BNR correspondence. We may see this as motivation to extract more information about moduli spaces of Higgs bundles from it. The structure sheaf of a length n subscheme T of T^*E is a torsion sheaf $\mathcal{O}_T$. Its Fourier-Mukai transform $\Phi_{\mathcal{P}}(\mathcal{O}_T)$ turns out to be a coherent sheaf on T^*E, giving rise to a rank n Higgs bundle on E. So far we have ignored the subscheme structure of $\mathcal{O}_T$, which is given by a surjection $\mathcal{O}_{T^*E} \twoheadrightarrow \mathcal{O}_T$. Applying the functor $\Phi_{\mathcal{P}}$ to this map, one can use a Serre duality argument to endow the Higgs bundle $\Phi(\mathcal{O}_T)$ with a parabolic structure. While the Higgs bundle $\Phi(\mathcal{O}_T)$ turns out to be semistable in general, the parabolic structure serves the purpose of stabilizing the Higgs bundle.

Before explaining why this procedure gives rise to an equivalence of moduli spaces, we will illustrate the geometry of this map by analyzing it birationally. A generic length n subscheme T of T^*E is given by n distinct points $x_1, \dots, x_n$. Each of these points corresponds to a rank one degree zero Higgs bundle $(L_1, \theta_1), \dots, (L_n, \theta_n)$ on E. The Higgs bundle $\Phi_{\mathcal{P}}(T)$ is given by the direct sum $(E, \theta) := \bigoplus_{i=1}^n (L_i, \theta_i)$. The parabolic structure is constructed by choosing a generic line within the zero fibre E_0.

In this context, generic is meant to avoid the subspaces given by (direct sums of the) line bundles L_i. The isomorphism class in parabolic Higgs bundles turns out to be independent of this choice. The stabilizing effect of the parabolic structure alluded to above, is captured by the fact that (E, θ) has no proper Higgs subbundle, which contains the chosen line.

We have constructed a natural map from the Hilbert scheme $f : (T^*E)^{[n]}$ to the moduli space $\mathcal{M}$ of rank n parabolic Higgs bundles on E. It seems tempting to conclude the proof simply by claiming that the inverse of the equivalence $\Phi_{\mathcal{P}}$ yields an inverse to the map f. Unfortunately, it is not immediately obvious that $\Phi_{\mathcal{P}}^{-1}((E, \theta))$ of an arbitrary Higgs bundle gives rise to a complex supported in a single degree. On the other hand, we know that this will be the case for the parabolic Higgs bundles in the image of the map f. A short argument using properness of the Hitchin map can be applied to establish surjectivity of the map f, and therefore proves that f is indeed an isomorphism of moduli spaces.

As we have already remarked above, similar descriptions also exist for moduli spaces of local systems in zero or positive characteristic. The place of Mukai's equivalence is then taken by the derived Geometric Langlands Conjecture for GL_1 (Conjecture 1.2.7), which has been proven by G. Laumon [Lau] and M. Rothstein [Rot96]. In the first case, there is no Hitchin map, and therefore no properness available. It is an old theorem of Weil (see Atiyah's exposition in [Ati57]), which comes to the rescue. The existence of a flat connection in zero characteristic, forces the degree of a vector bundle (and all of its indecomposable summands), which turns out to be what is needed for the inverse derived equivalence to give rise to an inverse map between the two moduli spaces. In positive characteristic, Weil's result ceases to hold, a loss which is compensated by the return of a proper Hitchin map.

1.3.3 Geometric Langlands in positive characteristic

In the last chapter we extend Bezrukavnikov–Braverman's Geometric Langlands Correspondence from the locus of smooth spectral curves to integral spectral curves (in large enough characteristic). This extension depends on Arinkin's autoduality result for compactified Jacobians [Aria]. We refer the reader to Theorem 6.2.1 for a proof of the theorem below.

Theorem 1.3.2. *Let X be a smooth projective curve defined over an algebraically closed field k of high enough characteristic. There exists a derived equivalence*

$$D^b_{coh}({\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}}^{(1)}, \mathcal{D}_{\mathrm{Bun}}) \cong D^b_{coh}(\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}}, \mathcal{O}_{\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}}}).$$

In order to verify the Hecke eigenproperty, we establish an analogous result for Arinkin's equivalence. Let $\bar{\mathrm{J}}$ be the compactified Jacobian of a family of integral curves C/S over a base S of finite type over k. Then there exists a Hecke operator

$$\mathbb{H} : D^b_{coh}(\bar{\mathrm{J}}) \to D^b_{coh}(\bar{\mathrm{J}} \times_S C),$$

which is defined in analogy with the Hecke operator in the Geometric Langlands Programme (Section 6.3). There is also a multiplication operator

$$\mathbb{W} : D^b_{coh}(\bar{\mathrm{J}}) \to D^b_{coh}(\bar{\mathrm{J}} \times_S C),$$

induced by the universal family on $\bar{\mathrm{J}} \times_S C$. The theorem below has also been known to Arinkin, and is proved in subsection 6.3.2 by using the techniques developed in *loc. cit.*.

Theorem 1.3.3. *The derived equivalence $D^b_{coh}(\bar{\mathrm{J}}) \cong D^b_{coh}(\bar{\mathrm{J}})$ intertwines the operators $\mathbb{H}$ and $\mathbb{W}$.*

As a consequence of the theorem above we obtain that the Geometric Langlands Correspondence of Theorem 1.3.2 intertwines the analogue of the degree one Hecke operator with the expected multiplication operator.

After having described the main contributions of this chapter, we try to give a heuristic justification for Theorem 1.3.2. As we have seen in chapter 4.1, the moduli stack of local systems is a twisted version of the moduli stack of Higgs bundles. Similarly, the Azumaya algebra $\mathcal{D}_{\mathrm{Bun}}$ on $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}(X^{(1)})$ gives rise to a twisted version of the derived category of $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}$. Since Arinkin's autoduality result induces a derived equivalence

$$D^b_{coh}(\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}) \cong D^b_{coh}(\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}),$$

we only have to show that both sides are twisted in a compatible way to obtain the statement of Theorem 1.3.2 above.

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Chapter 2

Preliminaries on Higgs bundles

Let X be a smooth projective curve defined over an algebraically closed field k. This thesis is concerned with the study of various moduli problems of vector bundles on X endowed with extra structures. Instead of relying on the intricate machinery of Mumford's *Geometric Invariant theory* [MF82] to construct moduli spaces, we directly study the geometry of the moduli problem in terms of the corresponding stack.

2.1 Vector bundles

Definition 2.1.1. *For a positive integer n and an integer d, we let $\mathrm{Bun}_{n,d}(X)$ be the stack sending a scheme S to the groupoid of rank n vector bundles of relative degree d on $X \times S$. We denote by $\mathrm{Bun}_n(X)$ the stack of rank n vector bundles of arbitrary degree, and by $\mathrm{Bun}(X)$ the stack of vector bundles of arbitrary rank and degree on X.*

The structure of $\mathrm{Bun}_1(X)$ is well-understood and is governed by the Jacobian J_X. By definition, J_X represents the moduli problem of degree zero line bundles with a trivialization at a given closed point $x \in X$. It is well-known to be an abelian variety[1].

[1] Since J_X is a group object, it suffices to show projectivity and smoothness. Theorem 9.5.4 in [FGI$^+$05] implies projectivity, smoothness is implied by Lemma 2.1.3 below.

Accounting for degree and automorphism group of a general line bundle, we obtain the following result.

Example 2.1.2. *Every k-point $x \in X(k)$ induces an equivalence of stacks*

$$\mathrm{Bun}_1(X) \cong \mathbb{Z} \times J_X \times B\,\mathbb{G}_m\,.$$

The map is obtained by sending a line bundle $\mathcal{L}$ to the triple

$$(\deg \mathcal{L}, \mathcal{L}(-(\deg \mathcal{L})x), \iota_x^* \mathcal{L}),$$

where $\iota_x : \operatorname{Spec} k \hookrightarrow X$ denotes the inclusion of x and $\iota_x^* \mathcal{L}$ is the one-dimensional k-vector space given by the fibre of $\mathcal{L}$ at x.

We show below that $\mathrm{Bun}(X)$ satisfies reasonable geometric properties. The reader can also find a proof of these properties in Theorem 4.6.2.1 in [LMB00].

Lemma 2.1.3. *The stack $\mathrm{Bun}(X)$ is locally of finite presentation and formally smooth.*

Proof. We refer the reader to the definition of morphisms locally of finite presentation given in A.1.1. On the level of isomorphism classes this is checked as in the proof of Proposition 9.4.17 in [FGI$^+$05]. The extra structure coming from the automorphism groups of objects can be dealt with by applying [Gro66, Corollaire 8.5.2.5].

We have to show that the structural morphism $\mathrm{Bun}(X) \to \operatorname{Spec} k$ is formally smooth in the sense of Definition A.1.2. So let V be an affine scheme and $U \to V$ be a closed immersion given by a quasi-coherent square-zero sheaf of ideals $\mathcal{I}$. For a given locally free sheaf $\mathcal{E}_U$ on $X \times U$ we have to show that there exists a lifting to a locally free sheaf $\mathcal{E}_V$ on $X \times V$, which is a problem in deformation theory. According to Theorem 8.5.3(b) in [FGI$^+$05], the obstruction to finding such an $\mathcal{E}_V$ lies in $H^2(X_U, \mathcal{I} \otimes \mathrm{End}(E_U))$, which vanishes since X is a curve. □

Most importantly the stack $\mathrm{Bun}(X)$ can be shown to be algebraic. In order to establish algebraicity of $\mathrm{Bun}(X)$, one uses an atlas produced by quot-schemes ([Gro95]). It has been shown by Grothendieck in *loc. cit.* that for a coherent sheaf $\mathcal{F}$ on a projective scheme X the functor

$$\mathrm{Quot}_X(\mathcal{F})(S) := \{\mathrm{pr}_X^* \mathcal{F} \twoheadrightarrow \mathcal{M} \,|\, \mathcal{M} \text{ is } S\text{-flat}\},$$

where $\mathrm{pr}_X : X \times S \to X$ is the canonical projection, is represented by a projective scheme.

Proposition 2.1.4. *The stack* $\mathrm{Bun}(X)$ *is algebraic.*

Proof. Let $\mathcal{L}$ be an ample line bundle on X. By [Har77, Prop. 5.3] we know that for every coherent sheaf $\mathcal{F}$ on X there exists a positive integer $n_{\mathcal{F}}$, such that $\mathcal{F} \otimes \mathcal{L}^m$ is generated by global sections and $H^1(X, \mathcal{F} \otimes \mathcal{L}^m) = 0$ for $m \geq n_{\mathcal{F}}$. In particular we may conclude that there is a surjection

$$\mathcal{O}_X^k \twoheadrightarrow \mathcal{F} \otimes \mathcal{L}^m .$$

Twisting by $\mathcal{L}^{-\otimes m}$ we obtain a surjection

$$(\mathcal{L}^{-\otimes m})^k \twoheadrightarrow \mathcal{F} .$$

The analysis above implies in particular that the obvious morphism

$$p : \coprod_{k,m \in \mathbb{N}} \mathrm{Quot}_X((\mathcal{L}^{-\otimes m})^k) \to \mathrm{Bun}(X)$$

is surjective.

The Semicontinuity Theorem [Har77, Thm.12.8] implies that the stack of vector bundles $\mathcal{E}$, satisfying $H^1(X, \mathcal{E} \otimes \mathcal{L}^{\otimes m}) = 0$ for a given $m \in \mathbb{N}$, is an open substack of

$\mathrm{Bun}(X)$. So we may restrict p to an appropriate open subscheme

$$\mathcal{U} \subset \coprod_{k,m\in\mathbb{N}} \mathrm{Quot}_X((\mathcal{L}^{-\otimes m})^k)$$

obtained by stipulating this vanishing condition. We have already seen surjectivity of $p|_{\mathcal{U}}$ in the paragraph above, it remains to check that $p|_{\mathcal{U}}$ is smooth. As in the proof of Lemma 2.1.3 we will check the criterion for formal smoothness, as given in Definition A.1.2. Note that the morphism is obviously locally of finite presentation, since both the quot-scheme and $\mathrm{Bun}(X)$ are locally of finite presentation (Lemma 2.1.3).

So let $U \to V$ be a closed immersion of affine schemes given by a square-zero quasi-coherent sheaf of ideals $\mathcal{I}$ on V. Let $\mathcal{E}_V$ be a locally free sheaf on $X \times V$, and $\mathcal{E}_U$ its pullback to $X \times U$. Given a surjection

$$(\mathcal{L}^{-\otimes m})^k \twoheadrightarrow \mathcal{E}_U,$$

we have to show that it can be lifted to a surjection

$$(\mathcal{L}^{-\otimes m})^k \twoheadrightarrow \mathcal{E}_V\,.$$

The obstruction lies in the global sections of the sheaf $R^1p_{U,*}(\mathcal{E}(m)\otimes\mathcal{I})$. We already know that $H^1(X, E(m)) = 0$ for all the fibres of E, and we would like to conclude that the higher direct image above vanishes. This vanishing statement will be derived in the next paragraph.

Let $u \in U$, it is shown in Theorem 12.11 of [Har77] that the canonical maps

$$\phi^i(u) : R^ip_{U,*}(\mathcal{E}(m)) \otimes k(u) \to H^i(X \otimes k(u), \mathcal{E}(m))$$

are bijective if and only if they are surjective; moreover, if $\phi^i(u)$ is surjective then surjectivity of $\phi^{i-1}(u)$ is equivalent to $R^i p_{U,*}(\mathcal{E}(m))$ being locally free. For $i \geq 1$, the cohomology group on the right hand side vanishes by assumption, and for $i \geq 2$ the groups on the left vanish for dimension reasons. Hence $\phi^1(u)$ is a bijection and $R^1 p_{U,*}(\mathcal{E}(m))$ is locally free of rank zero. We conclude that $\phi^0(u)$ is a bijection. Since $\phi^{-1}(u)$ is a surjection (negative cohomology groups vanish by definition), we conclude that $R^0 p_{U,*}(\mathcal{E}(m))$ is a locally free sheaf on U. Proposition 12.5 and Corollary 12.6 of [Har77] allow us to conclude that

$$R^1 p_{U,*}(\mathcal{E}(m) \otimes \mathcal{I}) \cong [R^1 p_{U,*}(\mathcal{E}(m))] \otimes \mathcal{I} \cong 0.$$

□

2.2 Higgs bundles

The theory of Higgs bundles was introduced by N. Hitchin in [Hit87a] and C. Simpson in [Sim87]. A completely algebraic treatment of the moduli problem was given by N. Nitsure in [Nit91] and by C. Simpson in [Sim94a]. A modern algebraic treatment for general structure groups appeared in B.-C. Ngô's [Ngô10].

In the following we fix a smooth proper k-scheme X and a positive integer $n \in \mathbb{N}$. For a scheme S we define the notion of an S-family of Higgs bundles below. We denote by Ω^1_{X} the cotangent sheaf on X and by $\mathrm{pr}_X : X \times S \to X$ the canonical projection. Of particular importance to us is the case where X is a curve.

Definition 2.2.1. *An S-family of rank n* Higgs bundles *on X consists of a locally free sheaf $\mathcal{E}$ of rank n on $X \times S$ and a* Higgs field *given by a morphism of $\mathcal{O}_X$-modules*

$$\theta : \mathcal{E} \to \mathcal{E} \otimes \mathrm{pr}_X^* \Omega^1_{\mathrm{X}},$$

which satisfies the integrality condition

$$0 = \theta \wedge \theta : E \to E \otimes \Omega^2_X.$$

The preceding definition, when X is a curve, describes the moduli problem central to this thesis. We remark that $\dim X = 1$ implies vanishing of the sheaf Ω^2_X, hence the integrality condition is trivially satisfied in this case.

Below we follow the standard procedure of assigning an abstract stack to a given moduli problem.

Definition 2.2.2. *The stack classifying S-families of rank n Higgs bundles (Definition 2.2.1) will be denoted by $\mathcal{M}_{\mathrm{Dol}n}(X)$. By definition it sends the scheme S to the groupoid of S-families of Higgs bundles on X. The stack of Higgs bundles of arbitrary rank will be denoted by $\mathcal{M}_{\mathrm{Dol}n}(X)$.*

For a smooth projective curve X, the stack $\mathcal{M}_{\mathrm{Dol}n}(X)$ can be related to the cotangent stack of $\mathrm{Bun}_n(X)$ (see Definition A.2.1). An insight which is contained in G. Laumon's [Lau88] and is also stated in section 2.2.3 of [BD96].

Proposition 2.2.3. *Let X be a smooth projective curve, there is a canonical equivalence between the stack of Higgs bundles $\mathcal{M}_{\mathrm{Dol}}(X)$ and the cotangent stack $T^* \mathrm{Bun}(X)$.*

Proof. Let S be an affine scheme. We analyze the groupoid $\mathcal{M}_{\mathrm{Dol}}(X)(S)$ by studying the fibration

$$p : \mathcal{M}_{\mathrm{Dol}}(X)(S) \to \mathrm{Bun}(X)(S).$$

Over a given S-family of vector bundles $\mathcal{E}$, the fibre $p^{-1}(\mathcal{E})$ is given by $Hom(\mathcal{E}, \mathcal{E} \otimes \mathrm{pr}_X^* \Omega^1_{\mathrm{X}})$, according to Definition 2.2.1. Serre duality allows us to identify $p^{-1}(\mathcal{E})$ with

$$Ext^1(\mathcal{E}, \mathcal{E})^\vee.$$

This allows us to identify the fibration of groupoids $T^* \operatorname{Bun}(X)(S) \to \operatorname{Bun}(X)(S)$ with $\mathcal{M}_{\mathrm{Dol}}(X)(S) \to \operatorname{Bun}(X)(S)$. $\square$

Using Lemma A.2.2 we obtain the Corollary stated below.

Corollary 2.2.4. *For X a smooth projective curve, the stack $\mathcal{M}_{\mathrm{Dol}}(X)$ is algebraic.*

If we perceive the Higgs field θ as a twisted endomorphism of $\mathcal{E}$ we see that the expression

$$a(\lambda) := \det(\lambda - \theta)$$

is well-defined and is a polynomial

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0,$$

where $a_i \in H^0(X, \Omega_X^{\otimes(n-i)})$.

Definition 2.2.5. *Let $\mathcal{A}$ be the affine space associated to the vector space*

$$\bigoplus_{i=0}^{n-1} H^0(X, \Omega_X^{\otimes(n-i)}),$$

it is called the Hitchin base*. The morphism of stacks*

$$\chi_{\mathrm{Dol}} : \mathcal{M}_{\mathrm{Dol}}(X) \to \mathcal{A},$$

sending a Higgs bundle $(\mathcal{E}, \theta)$ to the characteristic polynomial $a(\lambda)$ of θ is called the Hitchin morphism.

In the case of line bundles on curves, all the concepts introduced in this section become particularly simple. The reader can deduce the following statement from Example 2.1.2.

Example 2.2.6. *For a curve X the stack $\mathcal{M}_{\mathrm{Dol}1}(X)$ is (non-canonically) equivalent to*

$$\mathbb{Z} \times T^* J_X \times B\,\mathbb{G}_m,$$

where J_X denotes the Jacobian of the curve X, i.e. the moduli space of degree zero line bundles on X. The identification above depends on the choice of a k-point $x \in X(k)$. We have $T^* J_X \cong J_X \times H^0(X, \Omega^1_{\mathrm{X}})$, and the Hitchin map is given by projection to the component $H^0(X, \Omega^1_{\mathrm{X}})$.

This example illustrates that the stack $\mathcal{P}\mathrm{ic}(X) = \mathrm{Bun}_1(X)$ is not good in the sense of Definition A.2.3, due to the $B\,\mathbb{G}_m$ component. It's rigidified version is however a scheme, and therefore in particular good.

2.3 The BNR correspondence and the compactified Jacobian

A Higgs field θ on a vector bundle E on X is a section of $\underline{\mathrm{End}}(E) \otimes \Omega^1_{\mathrm{X}}$. As we have seen in section 2.2, we associate to θ its characteristic polynomial

$$a(\lambda) = \det(\lambda - \theta),$$

with coefficients

$$a_i \in H^0(X, \Omega_X^{\otimes n-i})$$

for $i = 0, \ldots, n-1$. The corresponding spectral cover Y_a is the closed subscheme of T^*X defined by the equation

$$\lambda^n + a_{n-1}\lambda^{n-1} \cdots + a_0 = 0,$$

where λ denotes the tautological section of the pullback $\pi^* \Omega^1_{\mathrm{X}}$ to the cotangent space T^*X. The affine space of spectral covers is denoted by

$$\mathcal{A} = \bigoplus_{i=1}^{n} H^0(X, \Omega_X^{\otimes i}).$$

Repeating this construction for a family of Higgs bundles parametrized by the scheme S, we obtain a morphism

$$a : S \to \mathcal{A},$$

which will also be referred to as the characteristic polynomial of the family of Higgs bundles. The corresponding spectral cover is a family of curves $Y_a \to S$.

The scheme $\mathcal{A}$ parametrizes the universal family $\phi : Y \to \mathcal{A}$ of spectral covers and in particular gives rise to a finite morphism

$$\pi : Y \to X \times \mathcal{A}\,.$$

Zariski locally on X we can trivialize the sheaf of tangent vector fields Θ_X and see that

$$\pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{n-1} p_1^* \Theta_X^{\otimes i}$$

is locally free. In particular we obtain that π and ϕ are flat morphisms. Moreover using [Gro63, Prop. 7.8.4] we see that ϕ cohomologically flat in degree zero. We record this observation for later use.

Lemma 2.3.1. *For X a smooth projective curve, the morphism $\phi : Y \to \mathcal{A}$ is proper, flat and cohomologically flat in degree zero. Moreover its geometric fibres are locally planar curves, i.e. can Zariski locally be embedded into $\mathbb{A}^2$.*

Given a quasi-coherent sheaf L on a spectral cover Y_a then its push-forward $\pi_* L$ is naturally endowed with a Higgs field. The Higgs sheaf $\pi_* L$ is a Higgs bundle if and only if $\pi_* L$ is locally free. This construction induces a natural bijection between Higgs bundles and certain coherent sheaves on spectral covers. It is usually referred to as the BNR correspondence (see [BNR89] for the case of integral spectral curves and [Sim94b, Lemma 6.8] for the general case). The statement given below is slightly weaker than the one given in *loc. cit.*, which relates Higgs bundles to so-called pure sheaves. The version stated here has the advantage of being easier to prove and is sufficient for our purposes.

Theorem 2.3.2 (BNR correspondence)**.** *Let S be an arbitrary scheme. There is a natural bijection between S-families of rank n Higgs bundles (E, θ) with characteristic polynomial $a = \chi(\theta)$ and coherent sheaves on the spectral cover Y_a/S, such that $\pi_{a,*}L$ is a locally free sheaf of rank n.*

Proof. Let $\theta : E \to E \otimes p_1^* \Omega^1_{\mathrm{X}}$ be a Higgs field on a locally free sheaf of rank n on $X \times S$, with characteristic polynomial a. Equivalently we can think of θ as a map $p_1^* \Theta_X \otimes E \to E$, where Θ_X denotes the sheaf of tangent vector fields on the variety X. This is the case since both Ω^1_{X} and Θ_X are locally free sheaves, which are dual to each other. Giving such a map, satisfying the integrality condition $\theta \wedge \theta = 0$, on the other hand is equivalent to endowing E with the structure of a $\mathrm{Sym}^\bullet p_1^* \Theta_X$-module. The Cayley-Hamilton Theorem implies that E is moreover a $\mathrm{Sym}^\bullet p_1^* \Theta_X/(a)$-module, which agrees with $\pi_* \mathcal{O}_{Y_a}$, where $\pi : Y_a \to X$ denotes the natural projection. Since π is an affine morphism, we have that π_* induces an equivalence between the category of coherent sheaves L on Y_a, satisfying that $\pi_* L$ is locally free of rank n, and the category of rank n Higgs bundles. $\square$

From now on we will assume that X is a curve. The following remark illustrates how this result can be put to use in order to gain a better understanding of the fibres of the Hitchin map.

Remark 2.3.3. *Let $a \in \mathcal{A}$, since $\pi : Y_a \to X$ is a finite morphism and $\pi_* L$ is locally free, we can conclude that L is a Cohen-Macaulay sheaf (Lemma A.4.2). If Y_a is a reduced spectral curve, then we know in particular that L is torsion free (e.g. Lemma 2.5 in [BD08]). Moreover if Y_a is smooth, then L has to be a line bundle and we conclude that $\chi^{-1}(a) = \mathcal{P}\mathrm{ic}(Y_a)$.*

The remark above might serve as motivation to study open subsets of the Hitchin base $\mathcal{A}$, defined in terms of algebraic properties of spectral curves.

Definition 2.3.4. *Let X be a curve, the open subsets defined in terms of*

$$\mathcal{A}_{\mathrm{sm}} := \{a \in \mathcal{A} \,|\, Y_a \textit{ is smooth}\}$$

and

$$\mathcal{A}_{\mathrm{int}} := \{a \in \mathcal{A} \,|\, Y_a \textit{ is integral}\}$$

will be referred to as the locus of smooth spectral curves*, respectively the* locus of integral spectral curves*. We denote by $\mathcal{M}^{\mathrm{sm}}_{\mathrm{Dol}}(X)$ the base change $\mathcal{M}_{\mathrm{Dol}}(X) \times_{\mathcal{A}} \mathcal{A}_{\mathrm{sm}}$ and similarly for $\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}(X)$.*

Remark 2.3.3 can now be read as stating the existence of a canonical equivalence of stacks

$$\mathcal{M}^{\mathrm{sm}}_{\mathrm{Dol}}(X) \cong \mathcal{P}\mathrm{ic}(Y_{sm}/\, \mathcal{A}_{\mathrm{sm}}).$$

Moreover, the same remark reveals that such a simple picture does not generalize to the full Hitchin base $\mathcal{A}$, since more general Cohen-Macaulay sheaves can give rise to Higgs bundles. Nonetheless, this description motivates a perspective on Higgs bundles, where they are understood to be generalized line bundles on the spectral curves Y_a. For general spectral curves it is difficult to make this notion precise; while for integral curves, one has the theory of *compactified Jacobians*. For a detailed introduction to this subject we refer the reader to [AK80].

Let us recall that for an integral curve C, the Jacobian variety J denotes the moduli space of degree zero line bundles on C, and the compactified Jacobian the moduli space of torsion-free rank one sheaves $\mathcal{F}$ on C of degree zero. The rank condition simply means that the sheaf is a line bundle when restricted to the smooth locus of the curve, the degree condition refers to

$$\chi(\mathcal{F}) = \chi(\mathcal{O}_C). \tag{2.1}$$

In *loc. cit.* the reader can find a more detailed definition applicable in the case of families of curves. Analogously one can define a compactified Picard stack $\overline{\mathcal{P}\mathrm{ic}}$.

Definition 2.3.5. *Let Y and S be schemes of finite type over k, and $\phi : Y \to S$ a flat proper morphism with geometrically integral fibres of dimension 1, i.e. an S-family of integral curves. Let us denote by $U \subset Y$ the maximal open subscheme, such that $\phi|U$ is smooth.*

(a) The compactified Picard stack $\overline{\mathcal{P}\mathrm{ic}}(Y/S)$ is the S-stack which sends an S-scheme $T \to S$ to the groupoid of geometrically fibrewise torsionfree sheaves on

$$Y \times_S T,$$

which restrict to invertible sheaves on $U \times_S T$.

(b) We have a natural action of the stack of line bundles on S, denoted $B\,\mathbb{G}_m \times S$, on $\overline{\mathcal{P}\mathrm{ic}}(Y/S)$, which sends $(\mathcal{F}, L) \mapsto \mathcal{F} \otimes \phi^* L$. The stack-theoretic quotient of this action is a space and will be denoted by $\overline{\mathcal{P}\mathrm{ic}}(Y/S)^{rig}$.

(c) The open subspace of $\overline{\mathcal{P}\mathrm{ic}}(Y/S)^{rig}$ of sheaves satisfying (2.1) is called the compactified Jacobian and will be denoted by $\bar{\mathrm{J}}(Y/S)$.

As in Example 2.1.2, a section s of $Y^{sm} \to S$ induces an identification

$$\overline{\mathcal{P}\mathrm{ic}}(Y/S) \cong \mathbb{Z} \times \bar{\mathrm{J}} \times B\,\mathbb{G}_m\,.$$

The BNR correspondence for Higgs bundles, more precisely remark 2.3.3, implies that for a reduced spectral curve Y_a we get a description of the Hitchin fibre as

$$\chi_{\mathrm{Dol}}{}^{-1}(a) \cong \overline{\mathcal{P}\mathrm{ic}}(Y_a).$$

Moreover, we have the following:

Remark 2.3.6. *The BNR correspondence (Theorem 2.3.2) gives rise to a canonical identification of stacks*

$$\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}(X^{(1)}) \cong \overline{\mathcal{P}\mathrm{ic}}(Y^{(1)}/\,\mathcal{A}^{(1)}).$$

2.4 Stability and properness

In this section we investigate the notion of stability for vector bundles and Higgs bundles. Exposition and terminology are inspired by the recent notion of Bridgeland's stability conditions ([Bri]) for triangulated categories.

Since we will study stability conditions on various geometric objects (vector bundles, Higgs bundles, parabolic Higgs bundles, etc.) it seems sensible to introduce them first in a very general framework. We will consider an abstract exact category $\mathcal{C}$ (see [Bue10] for an exposition of the theory of exact categories). The precise definition of an exact category is of no concern to us. For our purposes it is more than sufficient to know that an exact category is an additive category endowed with a notion of *short exact sequences.* All the abstract manipulations and arguments related to short exact sequences will be familiar from the exact category of vector bundles on a variety. Moreover we demand the following properties, some of which are already a consequence of exactness:

Assumption 2.4.1. *(a) If $0 \to 0 \to B \to C \to 0$ is an admissible short exact sequence, then $B \cong C$.*

(b) *Given two admissible subobjects $B_1 \hookrightarrow D \hookleftarrow B_2$, then there exist a largest admissible subobject $A \hookrightarrow D$, contained in both B_1 and B_2 as subobjects. Moreover we have an admissible short exact sequence*

$$0 \to A \to B_1 \oplus B_2 \to C \to 0,$$

such that the induced map $C \to D$ realizes C as the minimal subobject of D containing both B_1 and B_2.

(c) *A map $A \to B$ is called an* admissible monomorphism*, if it appears in a short exact sequence $A \to B \to C$. If the composition $A \to B \to C$ and the map $B \to C$ are admissible monomorphisms, then $A \to B$ is an admissible monomorphism as well.*

Definition 2.4.2. *Let $\mathcal{C}$ be a small exact category, satisfying assumption 2.4.1. A* stability condition *on $\mathcal{C}$ is a functor*

$$Z : \mathcal{C} \to \mathbb{H} \cup \{0\},$$

where the upper half-plane $\mathbb{H} \subset \mathbb{C}$ is viewed as a category with trivial Hom-spaces. Moreover, Z is required to satisfy the following conditions:

(a) *We have $Z(A) = 0$ if and only if $A \cong 0$.*

(b) *If $0 \to A \to B \to C \to 0$ is a non-trivial admissible short exact sequence in $\mathcal{C}$, then $Z(B) = Z(A) + Z(C)$.*

(c) *The functor Z takes values in the lattice $\mathbb{Z}[i]$.*

Condition (b) above could be replaced by defining Z instead to be a function from the Grothendieck K-group $K_0(\mathcal{C}) \to \mathbb{C}$, such that every non-zero object in $\mathcal{C}$ is sent to an element in $\mathbb{H}$. Sometimes, Z is called *central charge*.

Lemma 2.4.3. *Let $A \subset B$ be an admissible subobject of B, such that* $\operatorname{Im} Z(A) = \operatorname{Im} Z(B)$. *Then we have $A = B$.*

Proof. Let C be the quotient, i.e. we have an admissible short exact sequence $0 \to A \to B \to C \to 0$. By (b) of Definition 2.4.2 we have $\operatorname{Im} Z(C) = 0$, which implies $Z(C) = 0$ and hence $C \cong 0$, according to (a). We conclude $A \cong B$. □

In the following definition this will be linked with the more traditional viewpoint on stability for vector bundles.

Definition 2.4.4. *Let Z be a stability condition on an exact category $\mathcal{C}$. Given a non-zero object $A \in \mathcal{C}$, we define its* slope *to be*

$$\mu(A) := \tan \arg Z(A).$$

The object A is called semi-stable, *if for every non-trivial admissible subobject $A' \subset A$, we have*

$$\mu(A') \leq \mu(A).$$

In case of strict inequality, we call A stable.

For the sake of concreteness we give an example of a stability condition of relevance to the present text.

Example 2.4.5. *Let $\mathcal{C}$ be the exact category of vector bundles on a curve X. We define $Z : \mathcal{C} \to \mathbb{H}$ to be the functor sending a vector bundle E to $-\deg E + i \operatorname{rk} E$. In particular we see that $\mu(E) = \frac{\deg E}{\operatorname{rk} E}$.*

We emphasize the difference between admissible subobjects and subobjects in the example above. A subobject of a vector bundle on a curve X, is a coherent subsheaf. An admissible subobject of a vector bundle is a subbundle, i.e. a subsheaf, such that the corresponding quotient is a vector bundle as well.

Lemma 2.4.6. *Let B be an object in an exact category $\mathcal{C}$ endowed with a stability condition. We assume that the set of slopes of admissible subobjects is finite. Then, there exists a unique* maximal destabilising subobject*, i.e. an admissible subobject A, which is the maximal semistable subobject of B.*

Proof. If B is semistable, we may set $A := B$, so let us assume that B is not semistable. In particular there exists a subobject A_1, maximizing both $\operatorname{Im} Z(-)$ and slope $\mu(-)$. If A_2 is a second such subobject, then according to assumption 2.4.1, we may consider the minimal subobject $A_1 \cup A_2$ containing both A_1 and A_2. We have an exact sequence

$$0 \to A_1 \cap A_2 \to A_1 \oplus A_2 \to A_1 \cup A_2 \to 0,$$

and we know that $Z(A_1) = Z(A_2)$ and $\mu(A_1 \cap A_2) < \mu(A_1)$, as well as $\mu(A_1 \cup A_2) < \mu(A_1)$. But since we also have additivity for Z in the above short exact sequence, we arrive at the contradiction

$$\arg(Z(A_1 \cap A_2) + Z(A_1 \cup A_2)) < \arg Z(A_1 \oplus A_2) = \arg Z(A_1).$$

Therefore we must have $A_1 = A_2$, which implies uniqueness of the maximal destabilizing object. □

In the study of moduli spaces and stacks one is given more than just an exact category $\mathcal{C}$. Usually one works with a presheaf of exact categories, i.e. a 2-functor

$$\mathcal{C}_- : \mathrm{Sch}^{op} \to \mathrm{Ex},$$

sending a scheme S to an exact category $\mathcal{C}_S$. For instance, $\mathcal{C}_S$ could be the category of (families of) vector bundles or Higgs bundles on $X \times S$, where X is a curve. In all the cases of relevance to us, $\mathcal{C}_-$ will actually be a *sheaf of exact categories* with respect to the fpqc topology on Sch. This simply means that we can glue objects and morphisms between them along fpqc coverings, and that the notion of admissibility is local in the fpqc topology.

Definition 2.4.7. *Let $\mathcal{C}_{-}$ be a sheaf of exact categories. A stability condition on $\mathcal{C}_{-}$ is understood to be a stability condition Z on $\mathcal{C}_{\operatorname{Spec} K}$ for every field K. Moreover, for every morphism* $\operatorname{Spec} L \to \operatorname{Spec} K$ *we have a commutative diagram of functors*

$$\begin{array}{ccc} \mathcal{C}_{\operatorname{Spec} K} & \longrightarrow & \mathcal{C}_{\operatorname{Spec} L} \\ & {\scriptstyle Z}\searrow & \downarrow{\scriptstyle Z} \\ & & \mathbb{H}. \end{array}$$

For a scheme S, an object A of $\mathcal{C}_S$ is said to be (semi)stable, if its pullback to $C_{\operatorname{Spec} K}$ for every morphism $\operatorname{Spec} K \to S$*, where K is a field, is (semi)stable.*

Let S be a scheme, for an object $V \in \mathcal{C}_S$ we define the S-sheaf $\operatorname{Gr}_{\mathcal{C}}(V)$ of sets to be the functor sending $T \to S$ to the set of admissible subobjects of the pullback of V to T.

Although the abstract situation of Definition 2.4.7 allows for the definition of stacks of (semi)stable objects, some geometric implications of stability need more firm grounds to build on. Therefore, we introduce the following definition, of a sheaf of exact categories of *decorated vector bundles* on a curve X. In the following we denote by Vec_X the sheaf of exact categories of vector bundles on a curve X.

Definition 2.4.8. *Let $\mathcal{C}_{-}$ be a sheaf of exact categories, together with a functor*

$$F\colon\ \mathcal{C}_{-} \to \operatorname{Vec}_X.$$

We say that this data defines a sheaf of exact categories of decorated vector bundles *on X, if for every object $V \in \mathcal{C}_S$ the map*

$$\mathrm{Gr}_{\mathcal{C}}(V) \to \mathrm{Gr}_{\mathrm{Vect}_X}(F(V))$$

is a closed immersion.

Every such sheaf of exact categories inherits a stability condition from the exact category Vect_X.

Lemma 2.4.9. *Let $\mathcal{C}_{-}$ be a sheaf of exact categories, satisfying the condition of Definition 2.4.8. Given a morphism of spectra of fields $\pi : \operatorname{Spec} L \to \operatorname{Spec} K$ and an object $A \in \mathcal{C}_{\operatorname{Spec} K}$ then A is semistable if and only if its pullback to L is semistable.*

Proof. We follow the proof of Theorem 1.3.7 in [HL97]. Let M denote the maximal destabilizing object of $\pi^* A$ (Lemma 2.4.6). If the field extension L/K is purely transcendental of separable, then the unicity of M endows it with a canonical descent data. According to the theory of faithfully flat descent, we see that M arises as a pullback of a subobject of A. Since A is semistable, and pullback preserves slopes, we must have $M = \pi^* A$, i.e. $\pi^* M$ is semistable itself.

If L/K is neither of the above, we may assume that it is purely inseparable and apply the theory of Jacobson descent (see [Jac44]). We have to show that for every $\delta \in \mathrm{Der}(L/K)$, the maximal destabilizing sheaf M of $\pi^* A$ is preserved by δ. As in [HL97] we observe that the composition

$$M \to \pi^* A \xrightarrow{\delta} \pi^* A \to \pi^* A/M$$

is $\mathcal{O}$-linear. Lemma 1.3.3 of *loc. cit.* implies that this compostion has to be equal to the zero map, which implies the assertion. $\square$

At this point of the subsection we specialize to the more concrete framework of $\mathcal{C}_{-}$ being the sheaf of exact categories of Higgs bundles on a smooth projective curve

X, as introduced in Definition 2.2.1. In light of Example 2.4.5, it is not difficult to guess that we will endow this sheaf of exact categories with the stability condition induced from the functor sending a Higgs bundle (E, θ) to $-\deg E + i \operatorname{rk} E$. We will show that the stack of semistable objects $\mathcal{M}_{\text{Dol}}^{\text{ss}}$ is an open substack of $\mathcal{M}_{\text{Dol}}$, which implies algebraicity (using Corollary 2.2.4). Moreover, we verify that the Hitchin map $\chi : \mathcal{M}_{\text{Dol}}^{\text{ss}} \to \mathcal{A}$ is universally closed.

Proposition 2.4.10. *Let S be a scheme of finite presentation, and $(\mathcal{E}, \theta)$ an S-family of rank n and degree d Higgs bundles on X. Then there exists an open subscheme $S' \subset S$ satisfying the following property: for every spectrum of a field* $\operatorname{Spec} K$ *mapping to S, the pullback of $(\mathcal{E}, \theta)$ to $X \times \operatorname{Spec} K$ is (semi)stable if and only if* $\operatorname{Spec} K \to S$ *factors through the inclusion $S' \hookrightarrow S$. In particular we obtain that*

$$\mathcal{M}_{\text{Dol}}^{\text{s}} \hookrightarrow \mathcal{M}_{\text{Dol}}^{\text{ss}} \hookrightarrow \mathcal{M}_{\text{Dol}}$$

are open, hence algebraic substacks.

Proof. The open subscheme S' can easily be described in terms of certain relative quot-schemes $\operatorname{Quot}_X(\mathcal{E}, n', d')$, parametrizing quotients of $\mathcal{E}$ of rank n' and degree d'. The Higgs field θ cuts-out a closed subscheme of invariants quotients, which we will denote by $\operatorname{Quot}_X(\mathcal{E}, n', d')^\theta$. Properness of quot-schemes implies therefore that the image of $\operatorname{Quot}_X(\mathcal{E}, n', d')^\theta$ in S is closed. The semistable locus can therefore be described as the complement of a union of closed subsets, and it suffices to show that the union is taken over a finite set. This is Lemma 1.7.9 in [HL97]. □

An section E of a sheaf of exact categories $\mathcal{C}_-$ is called *simple*, if $Hom_S(E_S, E_S) \cong \mathcal{O}_S$. Stable object with respect to an arbitrary central charge are simple ([Rei08a, Lemma 4.2(d)]).

Lemma 2.4.11. *Let X be a smooth projective curve. The substacks of $\mathcal{M}_{\mathrm{Dol}}(X)$ of simple objects is smooth. In particular, the stack $\mathcal{M}^{\mathrm{s}}_{\mathrm{Dol}}(X)$ is smooth.*

Proof. This follows from a study of the deformation theory of Higgs bundles, as given in Biswas–Ramanan's [BR94]. □

The following Theorem states that the Hitchin morphism becomes universally closed when restricted to the substack $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}$. It was first proven by Nitsure [Yok91], following an argument of Langton [Lan75]. We will follow the slick proof given in [HL97, chapter 2.B], using the additional simplifications obtained by representing Higgs bundles in terms of the BNR correspondence (2.3).

Theorem 2.4.12. *The Hitchin morphism $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}} \to \mathcal{A}$ is universally closed.*

The proof uses the valuative criterion for universally closed maps (Theorem 7.3 in [LMB00]), and therefore reduces to verifying Proposition 2.4.14 below. In order to apply this criterion, we have to establish first that $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}$ is an algebraic stack of finite type over the field k.

Lemma 2.4.13. *Let X be a smooth projective curve over an algebraically close field k. The algebraic stacks $\mathrm{Bun^{ss}}_{n,d}(X)$ and $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}n,d}(X)$ are of finite type over k.*

Proof. We reproduce the proof of Theorem 5.6.1 of [LP97]. Let $\mathrm{Bun}^{ss+\alpha}_{n,d}(X)$ denote the open substack of $\mathrm{Bun}_{n,d}(X)$, given by vector bundles E, which satisfy the variation of the stability condition

$$\mu(F) \leq \mu(E) + \alpha,$$

for every subbundle $F \subset E$. For a fixed point $x \in X$, we will show that there exists an integer m, such that $E(mx)$ is globally generated, and has vanishing first cohomology. This allows us to represent E as a point of a Quot-scheme, which are known to be of finite type by Grothendieck's construction [Gro95].

Every rank one factor F of E satisfies

$$\deg F \geq \mu - (n-1)\alpha,$$

by assumption on E. If ω_X denotes the canonical line bundle of X, we have $\underline{\mathrm{Hom}}(E(mx), \omega_X) = 0$, for $m \geq 2g - 1 - \mu + (n-1)\alpha$. Serre duality implies $H^1(X, E(mx)) = 0$ under this assumption, and we obtain the assertion by choosing m appropriately.

The case of Higgs bundles follows from the statement above. One can show that there exists an integer α, such that the underlying vector bundle E of a semistable Higgs bundle $(E, \theta) \in \mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}n,d}(X)$ lies in $\mathrm{Bun}^{ss+\alpha}_{n,d}(X)$. This is spelled out in the proof of Lemma 4.2 in [LP01], where the authors discuss the more general case of t-connections. □

Proposition 2.4.14. *Let R be a discrete valuation ring with fraction field K, let a_K be a K-point of the Hitchin base $\mathcal{A}$, that extends to an R-point a_R. Given a* $\mathrm{Spec}\, K$*-family of semistable Higgs bundles $(\mathcal{E}_K, \theta_K)$ with characteristic polynomial a_K, there exists a* $\mathrm{Spec}\, R$*-family of semistable Higgs bundles $(\mathcal{E}_R, \theta_R)$ with characteristic polynomial a_R, which extends the family $(\mathcal{E}_K, \theta_K)$.*

As a first step we extend $(\mathcal{E}_K, \theta_K)$ to a $\mathrm{Spec}\, R$-family of Higgs bundles (without semistability being assumed).

Lemma 2.4.15. *Let R be a discrete valuation ring with fraction field K and residue field k', let a_K be a K-point of the Hitchin base $\mathcal{A}$, that extends to an R-point a_R. Given a* $\mathrm{Spec}\, K$*-family of Higgs bundles $(\mathcal{E}_K, \theta_K)$ with characteristic polynomial a_K, there exists a* $\mathrm{Spec}\, R$*-family of Higgs bundles $(\mathcal{E}_R, \theta_R)$ with characteristic polynomial a_R, which extends the family $(\mathcal{E}_K, \theta_K)$.*

This result is easily identified to be the analogue of Proposition 2.4.14 for the stack $\mathcal{M}_{\mathrm{Dol}}(X)$ in place of $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(X)$. We emphasize that $\mathcal{M}_{\mathrm{Dol}}(X)$ is in general only locally of finite type, but not globally. Hence, the result of Lemma 4.4.3 does not have the same impact to the theory as Proposition 2.4.14.

Proof of Lemma 2.4.15. The characteristic polynomial $a_R : \operatorname{Spec} R \to \mathcal{A}$ gives rise to a spectral curve $\pi : Y_{a_R} \to X \times \operatorname{Spec} R$. Inside of it we have the open dense subset given by the spectral curve $\iota : Y_{a_K} \hookrightarrow Y_{a_R}$. According to Theorem 2.3.2, $(\mathcal{E}_K, \theta_K)$ gives rise to a coherent sheaf $\mathcal{F}_K$ on Y_{a_K}. It suffices to extend $\mathcal{F}_K$ to a coherent sheaf $\mathcal{F}_R$ on Y_{a_R}, such that $\pi_* \mathcal{F}_{a_R}$ is locally free. The quasi-coherent sheaf $\iota_* \mathcal{F}_{a_K}$ can be written as a filtered union of coherent subsheaves $\mathcal{F}_i$, where $i \in \mathbb{N}$. There must be an $n \in \mathbb{N}$, such that $\pi_* \mathcal{F}_i |_{X \times \operatorname{Spec} K}$ equals $\mathcal{E}_K$ for $i \geq n$. If $\pi_* \mathcal{F}_n$ is locally free, we are done. If not, we use the fact that Y_r is a Cohen-Macaulay surface and and therefore the bidual $\mathcal{F}_n^{\vee\vee}$ is maximal Cohen-Macaulay (Lemma A.4.4), and hence $\pi_*(\mathcal{F}_n^{\vee\vee})$ is locally free (Lemma A.4.3). □

In order to obtain the semistable extension of the family $(\mathcal{E}_K, \theta_K)$, required to prove Proposition 2.4.14, we will modify the extension obtained in Lemma 4.4.3. We can show that for every extension $(\mathcal{E}_R, \theta_R)$, there exists a subsheaf $(\mathcal{E}'_R, \theta'_R)$, which violates the stability condition less. Applying this procedure iteratively will eventually lead to a semistable extension.

Proof of Proposition 2.4.14. We will freely use the BNR correspondence of Theorem 2.3.2 to change between Higgs bundles $(\mathcal{E}, \theta)$ with given characteristic polynomial a and corresponding coherent sheaf $\mathcal{F}$ on the spectral curve Y_a. As further abuse of language, we will not distinguish between $\mathcal{F}$ and $(\mathcal{E}, \theta)$.

Let $\mathcal{F}_R$ be the extension of the Higgs bundle $\mathcal{F}_K$, constructed in Lemma 4.4.3. Assume the assertion was false, then we could construct a descending chain of coherent subsheaves

$$\cdots \supset \mathcal{F}_R^n \supset \mathcal{F}_R^{n+1} \supset \ldots,$$

such that the restriction $\mathcal{F}^n_K$ to $X \times \operatorname{Spec} K$ equals $\mathcal{F}_K$ and the restriction $\mathcal{F}^n_{k'}$ to $X \times \operatorname{Spec} k'$ is unstable. By induction, we assume that $\mathcal{F}^n_R$ has been already defined. In order to define $\mathcal{F}^{n+1}_R$ we consider the maximal destabilizing subobject B^n of $\mathcal{F}^n_{k'}$ (Lemma 2.4.6). Moreover, we define G^n to be the quotient $\mathcal{F}^n_{k'}/B^n$. Let us denote by π a generator of the maximal ideal of R (note that R is a principal ideal domain, since it is a discrete valuation ring). By definition, we have

$$\pi\,\mathcal{F}^n \subset \mathcal{F}^{n+1} \subset \mathcal{F}^n,$$

and $\mathcal{F}^n/\pi\,\mathcal{F}^n = \mathcal{F}^n_{k'}$. The sheaf $\mathcal{F}^{n+1}$ is now defined through the following pullback diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi\,\mathcal{F}^n & \longrightarrow & \mathcal{F}^{n+1} & \longrightarrow & B_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi\,\mathcal{F}^n & \longrightarrow & \mathcal{F}^n & \longrightarrow & \mathcal{F}_n & \longrightarrow & 0
\end{array}$$

Note that $\mathcal{F}^{n+1}$ is R-flat, since R is a discrete valuation ring and $\mathcal{F}^{n+1}$ is a subsheaf of $\mathcal{F}^n$.

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi\,\mathcal{F}^{n+1} & \longrightarrow & \pi\,\mathcal{F}^n & \longrightarrow & G^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi\,\mathcal{F}^{n+1} & \longrightarrow & \mathcal{F}^{n+1} & \longrightarrow & \mathcal{F}^{n+1}_{k'} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & B^n & \longrightarrow & B^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{2.2}$$

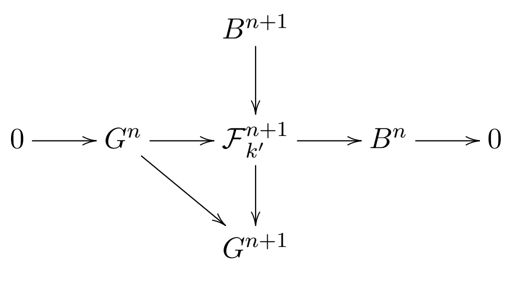
The first column is obviously an exact sequence, also the second column is exact by definition of $\mathcal{F}^{n+1}$. By an easy diagram-chase argument (tic-tac-toe lemma), we obtain that the third column is exact as well.

If we define $C^n := G^n \cap B^{n+1}$ we obtain from the exact sequence given by the third column of (2.2) that B^{n+1}/C_n is isomorphic to a subsheaf of B^n. It can be shown to be non-zero, since its slope is nontrivial. To see this one observes that $\mu(C^n) \leq \mu(F^{n+1}_{k'})$ (as C^n is a subsheaf of G^n, which is a subsheaf of $\mathcal{F}_{n+1}$), and that we have $\mu(\mathcal{F}^{n+1}_{k'}) < \mu(B^{n+1})$. Therefore we obtain the inequality

$$\mu(B^{n+1}) \leq \mu(B^{n+1}/C^n) \leq \mu(B^n),$$

where equality is equivalent to $C^n = 0$.

Since the sequence $(\mu(B^n))$ can only attain finitely many values and is monotonously decreasing, we may assume without loss of generality that it is constant. In particular we may assume without loss of generality that $C^n = 0$, which implies that we have inclusions of subbundles $B^{n+1} \subset B^n$ and $G^n \subset G^{n+1}$. Since the rank of these bundles is bounded from above, we can assume without loss of generality that these inclusions are actually equivalences. Let us denote the corresponding limit objects simply by B and G. As one sees in the diagram



an equivalence $G^n \cong G^{n+1}$ induces a splitting of the short exact sequence. We obtain that $\mathcal{F}^n_{k'} \cong B \oplus G$.

If we define $Q^n := \mathcal{F}_R \,/\, \mathcal{F}^n$, we have short exact sequences

$$0 \to G \to Q^{n+1} \to Q^n \to 0,$$

which is a $R/(\pi^n)$-flat sheaf, according to the local criterion for flatness (Lemma 2.1.3 in [HL97]). According to the formal GAGA principle (Theorem 8.4.2 in [FGI$^+$05]), we obtain a destabilizing quotient Q of $\mathcal{F}_R|_{X \times \operatorname{Spec} \widehat{R}}$, hence also for $\mathcal{F}_R|_{X \times \operatorname{Spec} \widehat{K}}$. But since $\mathcal{F}_K$ is supposed to be semistable, this contradicts the descent behaviour of semistability stated in Lemma 2.4.9. □

Every semistable Higgs bundle admits a (non-canonical) filtration, with each factor being stable (Jordan-Hölder filtration, see [Nit91]).

Definition 2.4.16. *Two semistable Higgs bundles are called S-equivalent, if the associated graded Higgs bundles are equivalent. For a noetherian base scheme T, one calls two T-families of semistable Higgs bundles S-equivalent, if they are S-equivalent over all geometric points of T.*

The GIT construction of Nitsure ([Nit91]) implies the existence of a quasi-projective *coarse* moduli space $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{coarse}}(X)$ of semistable Higgs bundles up to S-equivalence. As a corollary of Theorem 2.4.12 one obtains that $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{coarse}}(X) \to \mathcal{A}$ is proper, and therefore projective. In characteristic zero, G. Faltings has given in [Fal93] a GIT-free proof of projectivity of the Hitchin map.[2]

[2]The author thanks J. Heinloth for pointing out this reference.

2.5 Parabolic structures

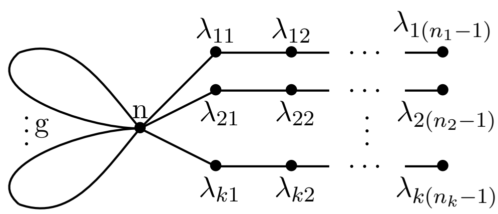
As before X denotes a smooth projective curve of genus g over an algebraically closed field k. We assume that

$$D = (n_1 - 1)p_1 + \cdots + (n_k - 1)p_k$$

is an effective divisor[3] on X, i.e. $n_i - 1 > 0$ for all i. The tuple $\widehat{X} = (X, D)$ will be referred to as a *weighted curve*. We fix a positive integer $n \in \mathbb{N}$ and for $i = 1, \ldots, k$ partitions $\lambda_{i\bullet}$

$$n = \lambda_{i0} \geq \cdots \geq \lambda_{in_i} = 0.$$

We will also write $l(\lambda_i) = n_i$ to denote the length of a partition. Following [HLRV08, Sect. 2.2] this numerical data will be encoded in the following diagram, henceforth referred to as a *comet-shaped* graph (with dimension vector).



Let S be a scheme. Below we define the notion of an S-family of parabolic vector bundles. We denote by $\iota_i : S \to X \times S$ the base change of the closed immersion

$$p_i : \operatorname{Spec} k \to X,$$

corresponding to the marked points $p_i \in X(k)$.

Definition 2.5.1. *An S-family of quasi-parabolic vector bundles on $\widehat{X}$ consists of a*

[3] This cumbersome notation will be convenient when passing from weighted curves to orbifolds (section 2.6).

locally free sheaf $\mathcal{E}$ of rank n on $X \times S$ together with flags

$$0 = F_{in_i} \subset F_{i(n_i-1)} \subset \cdots \subset F_{i0} \subset \iota_i^* \mathcal{E}$$

of locally free subsheaves of $\iota_i^* \mathcal{E}$ of rank

$$\operatorname{rk} F_{ij} = \lambda_{ij},$$

such that the successive quotients are locally free. The moduli stack of quasi-parabolic vector bundles will be denoted by $\mathrm{Bun}(\widehat{X}) = \mathrm{Bun}(\widehat{X}, n, \lambda_\bullet)$.

Algebraicity of $\mathrm{Bun}(\widehat{X})$ is a simple consequence of algebraicity of $\mathrm{Bun}(X)$, since the stack of parabolic structures on a given family of vector bundles is given by an iterated quot-scheme construction (flag scheme).

Lemma 2.5.2. *The forgetful map* $\mathrm{Bun}(\widehat{X}) \to \mathrm{Bun}(X)$ *is schematic. In particular,* $\mathrm{Bun}(\widehat{X})$ *is algebraic.*

Quasi-parabolic structures allow for an alternative, but equally useful, description in terms of filtered locally free sheaves. Given a vector bundle E on X, a point $x \in X(k)$, and a subspace $L \subset E_x$ of the fibre $E_x = E/E(-x)$, we can define a locally free sheaf E_L by the formula

$$E_L := \ker(E \to E_x/L),$$

where $E \to E_x/L$ is the obvious map factoring through $E \to E_x$. This process can be reversed, since

$$L = \ker(E_x \to \operatorname{coker}(E_L \to E)).$$

The process described above enables us to understand quasi-parabolic vector bundles as flags of locally free sheaves. For $i = 1, \ldots, k$ and $j = -n+1, \ldots, 0$ we define E_{ij}

to be the locally free subsheaf of E given by $F_{j+n} \subset E_{p_i}$. For arbitrary $j \in \mathbb{Z}$ we can write $j = j' + m$, where $-n < j' \leq 0$ and $m \in \mathbb{Z}$, and define $E_{ij} := E_{ij'} \otimes \mathcal{O}_X(mp_i)$. We conclude that there is an alternative description of parabolic bundles in terms of nested sequences of locally free sheaves ([Sim90, Sect. 3]).

Lemma 2.5.3. *The stack* $\mathrm{Bun}(\widehat{X}, n, \lambda_\bullet)$ *is equivalent to the stack of families of* $\mathbb{Z}^k$*-indexed sequences of locally free sheaves* $(V_i)_{i\in\mathbb{Z}^k}$ *on* X *satisfying*

$$V_i \subset V_{i+e_j},$$

where (e_j) *denotes the canonical basis of* $\mathbb{Z}^k$*, and*

$$V_{i+n_je_j} = V_i \otimes \mathcal{O}_X(p_j).$$

The interpretation of parabolic vector bundles as sequences of locally free sheaves $\mathcal{V} = (V_i)_{i\in\mathbb{Z}^k}$ suggests a definition for the *dual* quasi-parabolic vector bundle $\mathcal{V}^\vee$ given by the sequence

$$V_i^\vee := (V_{-i})^\vee.$$

This description of the dual quasi-parabolic bundle is easily seen to be compatible with the following definition.

Definition 2.5.4. *Let* $(\widehat{E}, F_{\bullet\bullet})$ *be a quasi-parabolic bundle on* $\widehat{X}$*. The dual quasi-parabolic bundle* $\widehat{E}^\vee$ *has underlying vector bundle* $E^\vee$ *and flag data given by*

$$F_{ij}^\vee = \ker(E_{x_i}^\vee \to F_{i(n-j)}^\vee).$$

Lemma 2.5.5. *If* $\widehat{E} = (\widehat{E}, F_{\bullet\bullet})$ *is a parabolic bundle on* $\widehat{X}$ *corresponding to the sequence* $\mathcal{V} = (V_i)_{i\in\mathbb{Z}^k}$ *then the dual* $\widehat{E}^\vee$ *corresponds to the dual sequence* $\mathcal{V}^\vee = ((V_{-i})^\vee)_{i\in\mathbb{Z}^k}$.

Proof. For every $i = 1, \dots, k$ we denote by e_i the canonical basis element in $\mathbb{Z}^k$. We need to compute

$$\ker(E_{x_i}^\vee \to \operatorname{coker}(V_{-je_i}^\vee \to E^\vee))$$

for $j = 0, \dots, n_i - 1$. But $\operatorname{coker}(V_{-je_i}^\vee \to E^\vee)) = \operatorname{coker}(E \to V_{je_i})^\vee = F_{i(n-j)}^\vee$. □

A natural family of stability conditions for quasi-parabolic vector bundles is parametrized by finite increasing sequences of positive real numbers $(\alpha_{ij}) \in [0, 1)$, where $i = 1, \dots, k$ and $j = 0, \dots n_i - 1$.

Definition 2.5.6. *A pair consisting of a quasi-parabolic vector bundle and an increasing sequences of positive real numbers $(\alpha_{ij}) \in [0, 1)$, where $i = 1, \dots, k$ and $j = 0, \dots n_i - 1$, is called a parabolic vector bundle.*

Following [BY96] we denote by $m_{ij} = \lambda_{ij} - \lambda_{ij+1}$ and define the parabolic degree of a parabolic vector bundle $\widehat{E}$ to be

$$\deg \widehat{E} = \deg E + \sum_{i=1}^{k} \sum_{j=0}^{n_i - 1} \alpha_{ij} m_{ij}.$$

The definition below is analogous to Example 2.4.5.

Definition 2.5.7. *The sheaf of exact categories of parabolic vector bundles carries a stability condition induced by the central charge $Z(\widehat{E}) = -\deg \widehat{E} + i \operatorname{rk} \widehat{E}$.*

Now that parabolic vector bundles have been introduced, the next step is to decorate them with a Higgs field.

Definition 2.5.8. *An S-family of parabolic Higgs bundles on $\widehat{X}$ consists of an S-family of parabolic vector bundles $(\mathcal{E}, \mathcal{F}_{\bullet\bullet})$ and a parabolic Higgs field given by a morphism of $\mathcal{O}_X$-modules*

$$\theta : \mathcal{E} \to \mathcal{E} \otimes p_X^* \,\Omega^1_{\mathrm{X}}(p_1 + \cdots + p_k).$$

The latter is required to satisfy the condition that

$$\operatorname{res}_{p_i} \theta(\mathcal{F}_{ij}) \subset \mathcal{F}_{ij+1}.$$

A parabolic Higgs bundle is stable (respectively semistable) if the condition of Definition 2.5.7 is satisfied for all proper subbundles F preserved by the Higgs field θ. The moduli stack of parabolic Higgs bundles will be denoted by $\mathcal{M}_{\mathrm{Dol}}(\widehat{X}) = \mathcal{M}_{\mathrm{Dol}}(\widehat{X}, n, \lambda_\bullet)$. The moduli space of stable parabolic Higgs bundles $\mathcal{M}^{\mathrm{s}}_{\mathrm{Dol}}$ is constructed as the rigidification of the open substack of stable Higgs bundles of $\mathcal{M}_{\mathrm{Dol}}(X)$.

The theory of parabolic Higgs bundles being analogous to the one of Higgs bundles, one can establish by the same means a relation between the stack of parabolic Higgs bundles and the cotangent stack of the stack of parabolic bundles.

Proposition 2.5.9. *There exists a canonical equivalence of the moduli stack of parabolic Higgs bundles $\mathcal{M}_{\mathrm{Dol}}(\widehat{X})$ and the cotangent stack $T^* \operatorname{Bun}(\widehat{X})$.*

Proof. As in the proof of Proposition 2.2.3 this is shown using Serre duality. A parabolic version of Serre duality is described in section 2.2 of [BY96]. □

Analogously to Lemma 2.4.11, one obtains the following result. The proof uses results on the deformation theory of parabolic Higgs bundles, which are discussed by Yokogawa in [Yok95].

Lemma 2.5.10. *Let $\widehat{X}$ be a smooth projective weighted curve. The substacks of $\mathcal{M}_{\mathrm{Dol}}(\widehat{X})$ of simple objects is smooth. In particular, the stack $\mathcal{M}^{\mathrm{s}}_{\mathrm{Dol}}(\widehat{X})$ is smooth.*

Proof. This follows from a study of the deformation theory of Higgs bundles, as given in Biswas–Ramanan's [BR94]. □

As before there exists a Hitchin base

$$\mathcal{A} := \bigoplus_{i=0}^{n-1} H^0(X, \Omega_X^{i+1}(ip_1 + \cdots + ip_k))$$

and a Hitchin morphism

$$\chi_{\mathrm{Dol}} : \mathcal{M}_{\mathrm{Dol}}(\widehat{X}) \to \mathcal{A},$$

which sends a parabolic Higgs bundle to the characteristic polynomial of its Higgs field θ. As in the proof of Proposition 2.4.14 one shows that the parabolic Hitchin map is proper. This has been done by Yokogawa in Corollary 5.13 and 1.6 of [Yok93].

Theorem 2.5.11 (Yokogawa)**.** *Let $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widehat{X})$ be a moduli space of semistable parabolic Higgs bundles of a fixed type (i.e. degree, rank, and ranks of flags). Then the Hitchin map $\chi : \mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widehat{X}) \to \mathcal{A}$ is universally closed.*

An important consequence of the properness results for Hitchin maps, is that they allow us to study the geometry of moduli spaces of Higgs bundles, using a natural $\mathbb{G}_m$-action. Let us assume for the remainder of this section that we are in a coprime situation of rank/type and degree, such that $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widehat{X}) = \mathcal{M}^{\mathrm{s}}_{\mathrm{Dol}}(\widehat{X})$. According to Lemma 2.5.10 and Theorem 2.5.11, we have then a smooth space $\mathcal{M}^{\mathrm{s}}_{\mathrm{Dol}}$, which is relatively proper over the Hitchin base $\mathcal{A}$.

The $\mathbb{G}_m$-action is present without these additional assumptions, and is simply given by scaling the Higgs field

$$t \cdot (E, \theta) := (E, t \cdot \theta).$$

This action covers a positive weight action on the Hitchin base $\mathcal{A}$. Properness of the Hitchin map χ_{Dol} implies now that for every semistable Higgs bundle (E, θ) the limit

$$\lim_{t \to 0} (E, t \cdot \theta)$$

exists, and lies in ${\chi_{\mathrm{Dol}}}^{-1}(0)$.

However, working under the assumption that $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widehat{X}) = \mathcal{M}^{\mathrm{s}}_{\mathrm{Dol}}(\widehat{X})$, smoothness of the space in question, allows to apply the results of Białynicki-Birula ([BB73]) to construct an interesting stratification of the moduli space. The following corollary of this viewpoint is well-known, and can be extracted from Proposition 2.1 in [GHS11].

Proposition 2.5.12. *Let $\mathcal{M}$ denote a connected component of $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widehat{X})$, such that stable and semistable locus agree. Then, the dimension of every fibre of the Hitchin system is $\leq \dim {\chi_{\mathrm{Dol}}}^{-1}(0) = \frac{1}{2}\dim \mathcal{M}$. Consequently, $\mathcal{M}$ dominates $\mathcal{A}$ via the Hitchin map.*

Proof. Proposition 2.1 in *loc. cit.* implies that $\dim {\chi_{\mathrm{Dol}}}^{-1}(0) = \frac{1}{2}\dim \mathcal{M}$. We can restrict the map $\chi_{\mathrm{Dol}} : \mathcal{M} \to \mathcal{A}$ to the closure of an arbitrary $\mathbb{G}_m$-orbit in $\mathcal{A}$. Away from the origin, the $\mathbb{G}_m$ action guarantees that the dimension of the fibres is constant. Since the fibrewise dimension in a morphism of varieties can only go up, we obtain the asserted dimension estimate for the general Hitchin fibre. If there is a connected component of $\mathcal{M}$, which does not dominate the Hitchin base, the dimension of this component would have to be strictly smaller than $\dim \mathcal{M}$. This would contradict the dimension calculations of Yokogawa [Yok95]. □

2.6 Parabolic bundles and orbifolds

Given a weighted curve $\widehat{X}$ we can associate to it an orbifold $\widetilde{X}$, assuming that the characteristic of k is large enough. We emphasize that the word orbifold refers to a smooth Deligne-Mumford stack ([LMB00, Def. 4.1]) in our context.

The orbifold $\widetilde{X}$ is defined by the following glueing data: let $\mathbb{D}_x$ denote the formal disc $\operatorname{Spec} \widehat{\mathcal{O}}_x$ around a point $x \in X(k)$. Given an effective divisor $D \subset X$ represented by the effective linear combination $(n_1 - 1)p_1 + \ldots (n_k - 1)p_k$, we denote by $\mathbb{D}_i := \mathbb{D}_{p_i}$

and let

$$U := X - D.$$

The fibre product $U \times_X \mathbb{D}_i$ is given by the punctured formal disc

$$\mathbb{D}_i^\bullet := \mathrm{Frac}\, \widehat{\mathcal{O}}_x.$$

The disc $\mathbb{D}$ is endowed with a natural action by the group scheme μ_m of m-th roots of unity. Let us denote by

$$[m] : \mathrm{Spec}\, k((t)) \to \mathrm{Spec}\, k((t))$$

the faithfully flat morphism given by $t \mapsto t^m$. Note that this is an étale μ_m-equivariant morphism if and only if the characteristic p of k does not divide m. Picking a formal coordinate t_i around every point p_i we obtain morphisms

$$[n_i] : \quad \mathbb{D}_i^\bullet \longrightarrow \mathbb{D}_i^\bullet$$

for every $i = 1, \dots, n$.

We define an algebraic stack $\widetilde{X}$ by glueing the quotients of the discs $[\mathbb{D}_i/\mu_{n_i}]$ back to U, but using the map $[n_i]$.

According to Theorem 6.1 in [Art74] this defines an algebraic stack independently of the characteristic p of k. Nonetheless this is a smooth Deligne-Mumford stack if either $p = 0$ or $\gcd(p, n_i) = 1$ for all i.

Assumption 2.6.1. *The field k is algebraically closed and its characteristic p is either zero or satisfies $\gcd(p, n_i) = 1$ for all i.*

It is a result of Furuta–Steer ([FS92, sect. 5]) that vector bundles on the orbifold $\widetilde{X}$ correspond to parabolic vector bundles on the weighted curve $\widehat{X}$. Nasatyr–Steer ([NS95, Sect. 5A]) discuss the analogous result for Higgs bundles. In the remaining part of this section we explain how this is proved in the realm of algebraic geometry instead of the analytic theory of Riemann surfaces used in [FS92] and [NS95].

Orbicurves as considered here can also be seen as certain root stacks associated to weighted curves (and this is what we will be doing implicitly). The correspondence described above is therefore reminiscent of a correspondence between parabolic vector bundles and vector bundles on root stacks, as established by N. Borne in [Bor].

The correspondence between vectorbundles on the orbifold $\widetilde{X}$ and parabolic bundles on $\widehat{X}$ is based on the following two observations: The natural morphism

$$\tau : \widetilde{X} \to X$$

realizes X as the coarse moduli stack for the Deligne-Mumford stack $\widetilde{X}$. The second observation is that the functor τ_* from quasi-coherent sheaves on $\widetilde{X}$ to quasi-coherent sheaves on X is not faithful. Nonetheless it sends a vector bundle $\widetilde{E}$ on $\widetilde{X}$ to a vector bundle $E := \tau_* \widetilde{E}$, since every torsion-free sheaf on a smooth curve is locally free.

Example 2.6.2. *Let μ_r be the cyclic group of r-th roots of unity. It acts on $\mathbb{D} :=$ $\operatorname{Spec} k[[t]]$ via $\xi \cdot t = \xi t$. If $\widetilde{X}$ is the quotient stack*

$$[\mathbb{D}/\mu_r]$$

we can identify the coarse moduli space X with $\operatorname{Spec} k[[t^r]]$. The functor τ_* sends a μ_r-equivariant $k[[t]]$-module M to the $k[[t^r]]$-module M^{μ_r}.

To reconcile the loss of information under the map $\widetilde{E} \mapsto E$ we define a $\mathbb{Z}$-indexed sequence of line bundles $(L_i)_{i \in \mathbb{Z}}$ for every orbifold point of the orbifold $\widetilde{X}$, satisfying

$$L_i \subset L_{i+1}$$

for all $i \in \mathbb{Z}$, and send $\widetilde{E}$ to the parabolic vector bundle associated to the filtered locally free sheaf $(\tau_*(\widetilde{E} \otimes L_i))_{i \in \mathbb{Z}}$.

Definition 2.6.3. *Let $\widehat{X}$ be a weighted curve and $\widetilde{X}$ the associated orbicurve. For every marked point p_i of $\widehat{X}$ we pick an n-th root L_{i1} of $\tau^* \mathcal{O}_X(p_j)$. The line bundle L_{ij} is defined to be*

$$L_{ij} := L_{i1}^j.$$

The existence of L_{i1} can be seen locally on X using the notation of Example 2.6.2. Let x be the origin of the disc $\mathbb{D}$. Since $\tau^* \mathcal{O}_X(x)$ is given by the $k[[t]]$-module $t^{-n}k[[t]]$, we see that $t^{-1}k[[t]]$ is an n-th root of $\tau^* \mathcal{O}_X(x)$.

We can show the following remark by a local argument:

Remark 2.6.4. *Let n_i denote the order of the stabilizer group of the point x_i, respectively the weight of p_i. Then we have $\tau^* \mathcal{O}_X(p_i) = L_{in_i}$.*

Using this remark and Lemma 2.5.3, it is a consequence of the projection formula

$$\tau_* \tau^* \mathcal{O}_X(p_i) \cong \mathcal{O}_X(p_i) \otimes \tau_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X(p_i)$$

that the sequence of locally free sheaves

$$E_{ij} := \tau_*(\widetilde{E} \otimes L_{ij})$$

gives rise to a parabolic vector bundle $\widehat{E} := (E, F_{\bullet\bullet})$ on $\widehat{X}$. We denote the map sending an orbibundle $\widetilde{E}$ to the parabolic bundle $\widehat{E}$ by A.

Proposition 2.6.5 (Furuta–Steer)**.** *The association*

$$A : \widetilde{E} \mapsto \widehat{E}$$

described above gives rise to an equivalence of groupoids of vector bundles on the orbicurve $\widetilde{X}$ and parabolic vector bundles on the weighted curve $\widehat{X}$.

The following Lemma is well-known.

Lemma 2.6.6. *Under the equivalence of Proposition 2.6.5 the degree of an orbibundle* $\widetilde{E}$ *is equal to the parabolic degree of the parabolic bundle* $\widehat{E}$ *with respect to the so-called* canonical weights $\alpha_{ij} := \frac{j}{n_i}$.

Proof. By Assumption 2.6.1 we have $R\tau_* = \tau_*$, in particular we see that we obtain an equality of Euler characteristics

$$\chi(\widetilde{E}) = \chi(\tau_*\widetilde{E}) =: \chi(E).$$

The right hand side can be computed with the help of the Riemann-Roch theorem for curves. We obtain

$$\chi(\tau_*\widetilde{E}) = \deg E + n(1-g).$$

To compute the left hand side we have to apply a Riemann-Roch formula valid for Deligne-Mumford stacks. Such a formula has been proved in a very general context in [Toe99, Cor. 4.14]. We also refer the reader to the elementary proof given in [AGV08, sect. 7.2]. If we denote by m_{ij} the dimension of the χ^j-part of the μ_{n_i}-representation $\widetilde{E}_{x_i}$ we obtain the expression

$$\chi(\widetilde{E}) = \deg \widetilde{E} + n(1-g) - \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} \alpha_{ij} m_i j.$$

This proves that $\deg \widetilde{E} = \deg \widehat{E}$. □

In the following Remark we make the above correspondence more explicit using the notation of Example 2.6.2.

Remark 2.6.7. *Let* E *be a* μ_r*-equivariant vector bundle on* $\mathbb{D}$*. The projection formula* *implies that we have*

$$E^\Gamma \otimes \mathcal{O} \,/\, \mathcal{O}(-x) \cong (E \otimes \mathcal{O} \,/ L^{-r})^\Gamma.$$

Using this we may identify the corresponding parabolic vector bundle $\widehat{E}$ on $\widehat{\mathbb{D}}$ with the one given by the vector bundle E^Γ together with the flags

$$F_i := (E \otimes L^{-i}/L^{-r})^\Gamma \subset (E \otimes \mathcal{O} \,/ L^{-r})^\Gamma \cong E^\Gamma \otimes \mathcal{O} \,/\, \mathcal{O}(-x).$$

The next Lemma and its proof should clarify how the equivariant structure of an vector bundle on an orbicurve corresponds to the flag data of a parabolic vector bundle.

Lemma 2.6.8. *Let $\Gamma = \mu_r$ be the finite cyclic group of order r acting on $\mathbb{D} = \operatorname{Spec} k[[t]]$ through $\xi \cdot t = \xi t$, where ξ is an r-th root of unity. Then the isomorphism classes of rank n parabolic vector bundles on the weighted curve $\widehat{[\mathbb{D}/\Gamma]}$ correspond to isomorphism classes of representations of Γ on an n-dimensional vector space. The regular representation of Γ corresponds to a rank r vector bundle with parabolic structure given by a complete flag.*

Proof. Let us denote by χ the character associated to the zero fibre of the line orbibundle L. By assumption we have $\chi(\xi) = \xi^{-1}$. If E is a bundle on $[\mathbb{D}/\Gamma]$ and $(E_i)_{i\in\mathbb{Z}}$ denotes the corresponding parabolic bundle on $\widehat{[\mathbb{D}/\Gamma]}$. A section s of E_i non-vanishing at $0 \in \mathbb{D}$ corresponds to a Γ-invariant section of $E \otimes L^i$. This gives rise to an eigenline in E_0 on which Γ acts by χ^{-i}.

Vice versa given an eigenline $k \cdot v \subset (E)_0$ on which Γ acts by χ^k then this gives rise to an eigenline in $(E \otimes L^{-k})_0$, on which Γ acts trivially. This in turn gives rise to a section of E_k. We see that the parabolic structure encodes the Γ-action on the zero fibre E_0.

To verify the last assertion we only have to observe that the regular representation of Γ is the direct sum

$$\bigoplus_{k=0}^{r} V_{\chi^k},$$

where V is a one-dimensional vector space with Γ acting on it through the character specified in the subscript. □

As a next step we investigate what happens to extra structures like a Higgs field or a connection under the transition $\widetilde{E} \mapsto \widehat{E}$. At first we have to specify what these structures mean in the present context.

Remark 2.6.9. *The Definition 2.2.1 of Higgs bundles is of étale local nature with respect to the curve X. Therefore we can make sense of Higgs bundles on an orbicurve $\widetilde{X}$.*

Proposition 2.6.10 (Nasatyr–Steer)**.** *Under the correspondence of Proposition 2.6.5 a Higgs field $\widetilde{\theta}$ on an orbibundle $\widetilde{E}$ is transformed to a parabolic Higgs field $\widehat{\theta}$ on $\widehat{E}$. This defines a natural equivalence of groupoids between S-families of Higgs bundles on the orbicurve $\widetilde{X}$ and S-families of parabolic Higgs bundles on the weighted curve $\widehat{X}$.*

The proof of this Proposition will be given below.

Lemma 2.6.11. *Let $\widetilde{X}$ be an orbicurve and $\tau : \widetilde{X} \to X$ the canonical morphism to a coarse moduli space with marked points $x_1, \ldots, x_k$. Then we have that the canonical morphism*

$$\tau^*\Omega^1_X \to \Omega^1_{\widetilde{X}}$$

induces an isomorphism

$$\tau^*\Omega^1_X(x_1 + \cdots + x_k) \cong \Omega^1_{\widetilde{X}} \otimes L_1 \otimes \cdots \otimes L_k.$$

In particular we have

$$\tau^*\Omega^1_X \cong \Omega^1_{\widetilde{X}} \otimes L_{1(1-n_1)} \otimes \cdots \otimes L_{k(1-n_k)}.$$

Proof. Away from the marked points $x_1, \ldots, x_k$ the map τ is an isomorphism, and the assertion follows. To prove the Lemma around a marked point x we may assume without loss of generality that we are in the situation of Example 2.6.2. Let $\mathbb{D}$ denote $\operatorname{Spec} k[[t]]$ with $\Gamma = \mu_r$-action given by $\xi \cdot t = \xi t$, where ξ denotes an r-th root of unity. Then τ is the obvious morphism

$$[\mathbb{D}/\Gamma] \to \mathbb{D}$$

induced by $k[[t]] \to k[[t]]$, which sends $t \to t^r$. In particular we see that the logarithmic 1-form $\frac{dt}{t}$ is mapped to $rt^{r-1}\frac{dt}{t^r} = r\frac{dt}{t}$, which proves the first assertion. The second claim follows directly from the first one, since $\tau^*\mathcal{O}(x_i) = L_i^{n_i}$. □

Lemma 2.6.12. *Let $\Gamma = \mu_r$ be the finite cyclic group of order r acting on $\mathbb{D} = \operatorname{Spec} k[[t]]$ through $\xi \cdot t = \xi t$, where ξ is an r-th root of unity. Then a Γ-equivariant Higgs bundle on $\mathbb{D}$ induces a parabolic Higgs bundle on the coarse moduli space $\mathbb{D}$ via push-forward along $\tau : \widetilde{D} = [\mathbb{D}/\Gamma] \to \mathbb{D}$, and similarly for Γ-equivariant local systems.*

Proof. A Higgs field $\widetilde{\theta}$ on $\widetilde{E}$ induces maps

$$\theta_i : \widetilde{E} \otimes L^i \to \widetilde{E} \otimes L^i \otimes \Omega^1_{\widetilde{\mathbb{D}}}.$$

According to Lemma 2.6.11 this gives rise to

$$\tau_*(\widetilde{E} \otimes L^i) \to \tau_*(\widetilde{E} \otimes L^i \otimes \Omega^1_{\widetilde{\mathbb{D}}}) \cong \tau_*(\widetilde{E} \otimes L^{i-1}) \otimes \Omega^1_{\mathbb{D}}(x),$$

where we have used the projection formula and Lemma 2.6.11 for the second equivalence. In particular we see that $\operatorname{res}\theta$ maps E_x to F_1. Replacing $\widetilde{E}$ by $\widetilde{E} \otimes L^i$ we conclude that $\operatorname{res}\theta$ is nilpotent with respect to the flag $F_\bullet$. □

Proof of Proposition 4.2.3. In Lemma 2.6.12 we have already seen that a Higgs field on an orbibundle $\widetilde{E}$ induces a parabolic Higgs field on the corresponding parabolic bundle $\widehat{E}$.

Using the notation of the proof of Proposition 2.6.5 we check that the Higgs field on the orbibundle is induced by the parabolic Higgs field $E_i \to E_{i-1} \otimes \Omega^1_X(x)$. Pulling this map back via τ and using Lemma 2.6.11 we see that this gives rise to a map

$$\tau^* E_i \otimes L^{-i} \to E_{i-1} L^{-i} \otimes L^1 \otimes \Omega^1_{\widetilde{X}}.$$

The direct sum of these maps respects the relation $t \cdot m = \iota(t \otimes m)$ from the proof of Proposition 2.6.5. In particular, it descends to a Higgs field $\widetilde{\theta}$ on $\widetilde{E}$. □

Chapter 3

D-modules in positive characteristic

In this chapter we survey the results of [BMR], which give a structure theorem for the ring of differential operators in positive characteristic.

3.1 Azumaya algebras, Morita equivalences, and gerbes

3.1.1 Azumaya algebras

We begin our exposition by studying a class of sheaves of algebras behaving like matrix algebras, which are widely known as *Azumaya algebras*. Our final goal is to get a better understanding of these algebras in order to study D-modules in positive characteristic.

Definition 3.1.1. *A coherent sheaf of algebras $\mathcal{D}$ on a scheme X is called an* Azumaya algebra*, if there exists a fppf covering $\{f_i : U_i \to X\}_{i \in I}$, such that $f_i^* \mathcal{D}$ is isomorphic to the endomorphism algebra $\underline{\mathrm{End}}(E_i)$ for a vector bundle E_i on U_i.*

From faithfully flat descent theory we see that Azumaya algebras are actually locally free. An alternative formulation of the definition could be that Azumaya algebras are fppf locally equivalent to $M_n(\mathcal{O}_{U_i})$.

Definition 3.1.2. *Let $\mathcal{D}$ be an Azumaya algebra on a scheme X, and E a locally free sheaf on X. A pair (ϕ, E) is called a* splitting *of $\mathcal{D}$, if ϕ is an isomorphism $\mathcal{D} \to \underline{\mathrm{End}}(E)$.*

Given such a splitting (ϕ, E) of $\mathcal{D}$, we can easily produce others by twisting them with a line bundle L. This makes sense, since there is a natural isomorphism $\underline{\mathrm{End}}(E) \cong \underline{\mathrm{End}}(E \otimes L)$. We see that the set of splittings can be naturally endowed with the structure of a category: An arrow between two splittings $(\phi, E) \to (\psi, F)$ is a pair (γ, L), where L is a line bundle and γ an isomorphism $E \to F \otimes L$, such that the diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\phi} & \underline{\mathrm{End}}(E) \\ \downarrow{\scriptstyle id_{\mathcal{D}}} & & \downarrow{\scriptstyle (\gamma^{-1})^* \otimes \gamma} \\ \mathcal{D} & \xrightarrow{\psi} & \underline{\mathrm{End}}(F) \end{array}$$

commutes. This category is actually a groupoid, the inverse of the morphism (γ, L) being given by (γ^{-1}, L^{-1}).

Lemma 3.1.3. *Let E, F be two vector bundles over a scheme X, which are both splittings of an Azumaya algebra $\mathcal{D}$. Then there exists a morphism (γ, L) between the two splittings.*

Proof. This is an easy consequence of the fact that for a commutative ring R the center of the matrix ring $M_n(R)$ is given by $R \cdot \mathrm{id}$. We know that $\underline{\mathrm{Hom}}_{\mathcal{D}}(F, -)$ is a categorical equivalence between the categories of $\mathcal{D}$-modules and the category of $\mathcal{O}_X$-modules (Proposition 3.1.9). Since E is another splitting, and Zariski locally both are equivalent, we need to have that $L := \underline{\mathrm{Hom}}_{\mathcal{D}}(F, E)$ is a Zariski locally free, thus it is a line bundle. On the other hand, $L \otimes_{\mathcal{O}_X} F \cong E$, therefore we get a morphism (γ, L) between both splittings. □

We can use this lemma to establish the following nice correspondence between Azumaya algebras and projective bundles.

Proposition 3.1.4. *There is a natural correspondence between Azumaya algebras $\mathcal{D}$ of rank n^2 on X and PGL_n-bundles $\mathcal{E}$ over X.*

Proof. From lemma 3.1.3 we know that there is no canonical splitting of $\mathcal{D}$. Actually there does not need to be a global splitting at all. Fppf local splittings always exist, and are well-defined up to twisting by a line bundle. Therefore we see that we can associate to $\mathcal{D}$ canonically a PGL_n-torsor $\mathcal{E}$. Vice versa given such a PGL_n-bundle $\mathcal{E}$, we may choose an étale covering $\{U_i\}$, such that on each open étale U_i we can represent the pull-back of $\mathcal{E}_i$ by a vector bundle E_i. Moreover, E_i is unique up to a line bundle. Therefore all $\underline{\mathrm{End}}(E_i)$ are well-defined and patch together to an Azumaya algebra on X. □

Besides demonstrating the correspondence between PGL-bundles and Azumaya algebras, proposition 3.1.4 indicates also how a general Azumaya algebra is build from elementary building blocks (matrix algebras). We cover our scheme X by coordinate charts $(U_i)_{i\in I}$ (in an fppf sense). Over each U_i the algebra $\mathcal{D}$ is isomorphic to the matrix algebra $M_n(\mathcal{O}_{U_i})$. In the "intersection" $U_{ij} := U_i \times_X U_j$ of two charts we get two a priori different isomorphisms to matrix algebras. By lemma 3.1.3 this yields a 1-cocycle of line bundles $(L_{ij})_{i,j\in I^2}$. Such an object is the cocycle description of a $\mathbb{G}_m$-gerbe.

Definition 3.1.5. *A $\mathbb{G}_m$-gerbe $\mathcal{X}$ over a scheme X is an X-stack $\mathcal{X}$ together with a right action $B\,\mathbb{G}_m \times \mathcal{X} \to \mathcal{X}$, such that there exists a fppf covering $\{f_i : U_i \to X\}_{i\in I}$ and equivalences $\phi_i : U_i \times_X \mathcal{X} \to B\,\mathbb{G}_m \times_k U_i$ of $B\,\mathbb{G}_m$-torsors. A gerbe $\mathcal{X}$ is called neutral, if there exists a section $X \to \mathcal{X}$ of its structure map.*

Taking into account that $B\,\mathbb{G}_m$ is a group stack, one could say that a $\mathbb{G}_m$-gerbe is something like a principal $B\,\mathbb{G}_m$-bundle. The following corollary is an easy consequence of Lemma 3.1.3.

Corollary 3.1.6. *The stack of splittings of an Azumaya algebra $\mathcal{D}$ has a natural structure of a $\mathbb{G}_m$-gerbe. It will be denoted by $\mathcal{Y}_{\mathcal{D}}$.*

In the following we will neglect $\mathbb{G}_m$ from the notation and simply refer to $\mathbb{G}_m$-gerbes as *gerbes*. Gerbes are a higher notion of line bundles, in the sense of isomorphism classes of line bundles correspond to elements of $H^1(X, \mathcal{O}_X^\times)$ and isomorphism classes of gerbes to elements of $H^2_{et}(X, \mathcal{O}_X^\times)$.

Proposition 3.1.7. *An Azumaya algebra $\mathcal{D}$ on X splits if and only if the gerbe of splittings is neutral, i.e. if and only if the corresponding class in $H^2_{et}(X, \mathcal{O}_X^\times)$ vanishes.*

Proof. This follows directly from the definition. A gerbe is neutral if its structure map admits a section. Such a section corresponds to a global splitting in our case. □

In the light of proposition 3.1.4 this discussion is summarised in the following short exact sequence of group schemes.

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

The corresponding long exact sequence includes

$$H^1_{et}(X, \mathbb{G}_m) \to H^1_{et}(X, \mathrm{GL}_n) \to H^1_{et}(X, \mathrm{PGL}_n) \to H^2_{et}(X, \mathbb{G}_m)$$

This short exact sequence tells us that a PGL_n-bundle (i.e. an Azumaya algebra) splits if and only if its characteristic gerbe is trivial. Moreover in that case $H^1_{et}(X, \mathbb{G}_m) = Pic(X)$ acts transitively on the set of isomorphism classes of all splittings.

3.1.2 Morita theory

The vector space $\mathbb{K}^n$ with the canonical $M_n(\mathbb{K})$-action serves as a machine to produce others, by taking the tensor product $\mathbb{K}^n \otimes_{\mathbb{K}} V$, where V is a vector space. Analogously we can also go backward, by applying the functor $\mathrm{Hom}_{\mathcal{M}_n(\mathbb{K})}(\mathbb{K}^n, -)$ to a $M_n(\mathbb{K})$-module W. It is an easy exercise to show that those two functors are actually inverse to each other. We have established an equivalence between the category of $\mathbb{K}$-modules and $M_n(\mathbb{K})$-modules.

Two rings are called Morita equivalent if their categories of modules are equivalent. For commutative rings it is known that Morita equivalence occurs only for isomorphic rings. But this is not the case for non-commutative rings. As explained above, an important example is given by the following proposition.

Proposition 3.1.8. *The algebras $\mathbb{K}$ and $M_n(\mathbb{K})$ are Morita equivalent.*

Proof. We consider the bimodule $S := \mathbb{K}^n$ with the obvious actions. It allows us to define two mutually inverse functors $F : \mathbb{K} - Mod \to M_n(\mathbb{K}) - Mod$ and $G : M_n(\mathbb{K}) - Mod \to \mathbb{K} - Mod$, given by $F : V \mapsto V \otimes S$ and $G : W \mapsto Hom_{M_n(\mathbb{K})}(S, W)$. □

Taking into account that over a geometric point $\mathbb{K}$ equals the structure sheaf and $M_n(\mathbb{K})$ a split Azumaya algebra, we obtain the following generalization of this classical statement.

Proposition 3.1.9. *Let X be a scheme and $\mathcal{D}$ an Azumaya algebra, which has a splitting S. Then we have an equivalence of categories*

$$\Psi_S : \mathrm{QCoh}(X, \mathcal{D}) \to \mathrm{QCoh}(X),$$

given by $Hom_{\mathcal{D}}(S, -)$.

Proof. This time we define $F : \mathcal{O} - \mathrm{Mod} \to \mathcal{D} - \mathrm{Mod}$ and $G : \mathcal{D} - \mathrm{Mod} \to \mathcal{O} - \mathrm{Mod}$ by $F : V \mapsto V \otimes S$ and $G : W \mapsto Hom_{M_n(\mathbb{K})}(S, W)$, where S is a splitting of $\mathcal{D}$. Using the result for affine schemes, i.e. for commutative k-algebras proved in [Gin, ch 2], we conclude that those two functors are inverse to each other. $\square$

To an abstract $\mathbb{G}_m$-gerbe $\mathcal{Y} \to \mathcal{X}$ on an algebraic stack X (Definition 3.1.5) one can associate a category of quasi-coherent sheaves twisted by $\mathcal{Y}$ (see Definition 3.1.11 below). We will see in Proposition 3.1.12 that for every Azumaya algebra $\mathcal{D}$ there is a canonical equivalence between $\mathcal{D}$-modules and quasi-coherent sheaves twisted by the corresponding gerbe of splittings. We refer the reader to Section A.3 for a definition of derived categories of stacks.

Lemma 3.1.10. *Let $\mathcal{Y} \to \mathcal{X}$ be a $\mathbb{G}_m$-gerbe on an algebraic stack. The categories* $\mathrm{QCoh}(\mathcal{Y})$ *and* $D_{qcoh}(\mathcal{Y})$ *decompose naturally into a some of* weight categories

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{QCoh}(\mathcal{Y})_n$$

and

$$\bigoplus_{n\in\mathbb{Z}} D_{qcoh}(\mathcal{Y})_n,$$

with respect to the action of $B\,\mathbb{G}_m$.

The proof is left to the reader, but we remark that it makes use of the observation that quasi-coherent sheaves on $B\,\mathbb{G}_m$ correspond to $\mathbb{G}_m$-representations, i.e. $\mathbb{Z}$-graded vector spaces.

Definition 3.1.11. *Let $\mathcal{Y} \to \mathcal{X}$ be a $\mathbb{G}_m$-gerbe on an algebraic space X. The full subcategory of* $\mathrm{QCoh}(\mathcal{Y})$ *given by the weight one part of the $B\,\mathbb{G}_m$-action is referred to as the category of quasi-coherent sheaves on $\mathcal{X}$* twisted by $\mathcal{Y}$ *and will be denoted by* $\mathrm{QCoh}(\mathcal{X}, \mathcal{Y})$*. Similarly one defines the derived category of quasi-coherent sheaves* $D_{qcoh}(\mathcal{X}, Y)$*.*

This allows us to formulate the precise relation between modules of an Azumaya algebra and quasi-coherent sheaves twisted by a gerbe.

Proposition 3.1.12. *Let $\mathcal{X}$ be an algebraic space and $\mathcal{D}$ an Azumaya algebra on $\mathcal{X}$ with gerbes of splittings $Y_{\mathcal{D}}$. There is a canonical equivalence of categories*

$$\mathrm{QCoh}(X, \mathcal{Y}_{\mathcal{D}}) \cong \mathrm{QCoh}(\mathcal{X}, \mathcal{D}),$$

and similarly

$$D_{qcoh}(\mathcal{X}, \mathcal{Y}_{\mathcal{D}}) \cong D_{qcoh}(\mathcal{X}, \mathcal{D}).$$

3.2 D-modules

Recall that a connection ∇ on a quasi-coherent sheaf E is a morphism of sheaves of vector spaces $\nabla : E \to E \otimes \Omega^1_X$, which satisfies the *Leibniz identity*

$$\nabla(fs) = (df)s + f\nabla s.$$

A connection is said to be *flat*, if the *curvature* morphism $\nabla \circ \nabla : E \to E \otimes \Omega^2_X$ vanishes.

Alternatively we can describe ∇ as a morphism of sheaves of Lie algebroids

$$\theta_X \to \mathcal{D}\mathrm{er}(E),$$

where $\mathcal{D}\mathrm{er}(E)$ denotes the Lie algebroid of derivations of E, i.e. the subsheaf of $\underline{\mathrm{End}}_{\mathbb{K}}(E)$ given by sections satisfying the Leibniz rule. By the universal property of the universal enveloping algebra of a Lie algebroid this is the same as a D_X-module, where D_X denotes the sheaf of universal enveloping algebras.

Definition 3.2.1. *Let X be a smooth variety, the* universal enveloping sheaf of algebras *of the Lie algebroid θ_X will be denoted by D_X, and referred to as the* ring of differential operators. *By definition we have*

$$D_X := T^\bullet\theta_X/(\partial\otimes f - f\otimes\nabla = \nabla f,\ \partial\otimes\partial' - \partial'\otimes\partial = [\partial,\partial']),$$

where $T^\bullet$ denotes the tensor sheaf of k-linear algebras, f is a section of $\mathcal{O}_X$, and ∂ a section of θ_X.

A much more down-to-earth description of this sheaf of algebras is given by the following picture. An order 0 differential operator is a function on X, an order 1 differential operator is a section of $\mathcal{D}\mathrm{er}(\mathcal{O})$. Nonetheless, D_X is not a graded algebra, but only filtered.

Definition 3.2.2. *The filtration on $T^\bullet\theta_X$, induced by the natural grading, descends to $\mathcal{D}_X$, since the ideal in Definition 3.2.1, is filtered.*

This is the general definition of the *order filtration* on the ring of differential operators. As in the theory of Lie algebras, there exists a Poincaré-Birkhoff-Witt Theorem (see section 1.1 in [HTT08]).

Proposition 3.2.3 (PBW)**.** *The associated graded of D_X with respect to the order filtration is equivalent to $\pi_*\mathcal{O}_{T^*X}$, where $\pi : T^*X \to X$ denotes the canonical projection.*

So far the restriction on the characteristic of the base field being positive has not been used. The essential difference arising in this context is that the natural morphism of algebras $D_X \to \underline{\mathrm{End}}(\mathcal{O})$ stands no chance of being injective. This is mostly due to the fact that a differential operator is defined to be a *formal* linear combination of iterated derivations. For every derivation ∂ we can easily see that applying it iteratively p times still gives a derivation

$$\partial^p(fg) = \sum_{k=0}^{p}\binom{p}{k}(\partial^{p-k}f)(\partial^k g) = (\partial^p f)g + f(\partial^p g).$$

Definition 3.2.4. *The p-th power of a derivation ∂ will be denoted by $\partial^{[p]}$.*

The formal expression $\partial^p - \partial^{[p]}$ does not vanish as a section of D_X but it does as a section of $\underline{\mathrm{End}}_{\mathbb{K}}(\mathcal{O})$.

Taking into account that this violation of injectivity naturally occurred in order $\leq p$, we may still hope that injectivity holds for differential operators of strictly lower order. One can prove that this is true in general by the use of local coordinates and a simple computation for affine space.

Lemma 3.2.5. *The natural morphism of sheaves of vector spaces $(D_X)_{\leq p-1} \to \underline{\mathrm{End}}_{\mathbb{K}}(\mathcal{O}_X)$ is injective.*

Apart from being a curious fact about differential operators in positive characteristic, the map of sheaves $\iota : \theta_X \to D_X$ sending $\partial \mapsto \partial^p - \partial^{[p]}$ will be used to describe the centre of the ring of differential operators.

Definition 3.2.6. *We denote by $\iota : \theta_X \to D_X$ the map of sheaves of sets sending a tangent vector field ∂ to $\partial^p - \partial^{[p]}$.*

A priori it is not clear that it is a well-behaved map, respecting any structure of the sheaves in question.

Proposition 3.2.7. *The map ι as defined above is p-linear, i.e. it satisfies $\iota(\partial_1 + f\partial_2) = \iota(\partial_1) + f^p\iota(\partial_2)$.*

Proof. The strategy of the proof is quickly explained: show that both sides act in the same way on functions and that the difference of both sides is of order $\leq p-1$. According to lemma 3.2.5 we get equality. The first step is obvious, because the image of ι acts trivially on functions, by Definition 3.2.4. For the second step we calculate

$$(\partial_1 + f\partial_2)^p = \partial_1^p + f^p\partial_2^p + ?,$$

where ? denotes a differential operator of order $\leq p-1$. □

The fact that ι is not linear, but p-linear might look a bit cumbersome on first sight, but is easily repaired by regarding ι instead as a map $Fr_*T_{X^{(1)}} \to D_X$, where $X^{(1)}$ denotes the Frobenius twisted version of X and $Fr : X \to X^{(1)}$ the Frobenius morphism. We define these new notions below.

Definition 3.2.8. *Let U be a scheme defined over an algebraically closed field k of positive characteristic. The* Frobenius twist *$U^{(1)}$ of U is given by the following cartesian diagram*

$$\begin{array}{ccc} U^{(1)} & \longrightarrow & U \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{F} & \operatorname{Spec} k, \end{array}$$

where $F : k \to k$ denotes the Frobenius map $\lambda \mapsto \lambda^p$. There exists a canonical k-linear morphism $Fr_U : U \to U^{(1)}$, which is referred to as Frobenius morphism. *It is a homeomorphism on the underlying topological spaces, and is given by $f \mapsto f^p$ on the level of functions.*

If U is the zero set of a polynomial

$$f(x_1, \dots x_n) = \sum a_{i_1 \dots i_n} x^{i_1} \cdots x^{i_n},$$

then $U^{(1)}$ is the zero set of the polynomial

$$\sum a^p_{i_1 \dots i_n} x^{i_1} \cdots x^{i_n},$$

and Fr_U is given by $x \mapsto x^p$.

A standard procedure allows us to extend ι to a map of sheaf of algebras

$$\pi_* \mathcal{O}_{T^*X^{(1)}} \to Fr_* D_X,$$

where $\pi : T^*X^{(1)} \to X^{(1)}$ is again the canonical projection. The image of this map comprises the centre of the ring $Fr_* D_X$. We present the proof of [BMR].

Proposition 3.2.9. $\mathrm{Im}(\iota : (Fr \circ \pi)_* \mathcal{O}_{T^* X^{(1)}} \to D_X) = Z(Fr_* \mathcal{D}_X)$.

Proof. We use the same strategy as in the proof of Proposition 3.2.7 to show that the image lies inside the centre, for this it suffices to show that for every vector field ∂, we have that $\iota(\partial)$ is central with respect to functions f and other vector fields ∂'. We compute now

$$[\iota(\partial), f] = [\partial^p, f] - [\partial^{[p]}, f] = [\partial^p, f] - \partial^{[p]} f = [\partial^p, f],$$

which equals

$$\sum_{k=0}^{p-1} \partial^{p-k-1} [\partial, f] \partial^k = \sum_{k=0}^{p-1} \partial^{p-k-1} (\partial f) \partial^k.$$

It is a differential operator of degree $\leq p-1$ acting trivially on functions, thus is zero itself. Similarly we calculate

$$[\iota(\partial), \partial'] = [\partial^p, \partial'] - [\partial^{[p]}, \partial'] = [\partial^p, \partial'] - \partial'',$$

which is

$$-\partial'' + \sum_{k=0}^{p-1} \partial^{p-k-1} [\partial, \partial'] \partial^k = -\partial'' + \sum_{k=0}^{p-1} \partial^{p-k-1} \partial'' \partial^k.$$

And therefore is again a differential operator of degree $\leq p-1$ acting trivially on functions.

We will have to evoke methods from symplectic geometry to prove that the image is equal to the centre.

We recall that the sheaf of differential operators D_X comes with a natural filtration $0 \subset \mathcal{O}_X \subset (D_X)_0 \subset (D_X)_1 \subset \dots$ defined by the notion of order (Definition 3.2.2). The associated graded ring is naturally isomorphic to the push-forward of the structure sheaf of the cotangent bundle $\pi_* \mathcal{O}_{T^* X}$. Sections of it are sometimes referred to as symbols of differential operators. The commutator of D_X induces a bracket $\{\cdot,\cdot\}$ on the associated graded, which satisfies the axiom of a Poisson bracket, i.e. for every section f we have that $\{f,\cdot\}$ is a derivation. A calculation using for instance étale local coordinates, shows that this Poisson structure on $\mathcal{O}_{T^*X}$ equals the one obtained from the standard symplectic structure ω on T^*X by means of

$$\{f,g\} := \omega(df,dg).$$

The next step is to think about how the map $\iota : \partial \mapsto \partial^p - \partial^{[p]}$ acts on symbols. To do this we choose a local coordinate system x_i (in an étale sense) and take $\partial_i := \frac{d}{dx_i}$. Here, local coordinates refer to the fact that up to étale coverings, we can assume that X is an open subset of affine space. All other vector fields are étale locally given by an $\mathcal{O}_X$-linear combination of the ∂_i. And p-linearity of ι, i.e. proposition 3.2.7, allows us to compute the map ι simply by evaluating $\iota(\partial_i)$. Coordinate vector fields have the advantage that the derivation $\partial^{[p]}$ is zero, since taking iteratively the derivative of a function p times yields the zero function. We see that $\iota(\partial) = \partial^p$ and as a consequence ι acts on symbols as the p-th power map.

Taking a central element z of order m, its symbol X must lie in the Poisson centre of the associated graded, meaning that $\{X,\cdot\}$ is the zero map. According to what we stated above this tells us that $\omega(dX,\cdot)$ is zero, but ω is non-degenerate, thus $dX = 0$. Over a perfect field of characteristic p the only functions with vanishing derivative are p-th powers. Therefore, we have shown that the associated graded of the map

$$\iota : \pi_*^{(1)}\,\mathcal{O}_{T^*X^{(1)}} \to Z(Fr_* D_X)$$

is an isomorphism. This implies that ι itself is an isomorphism. □

We remind the reader that for D a quasi-coherent sheaf of algebras on a scheme

Y and $\pi : Y \to X$ an affine morphism of schemes. Then there is a natural equivalence of abelian categories, between D-modules on Y and $\pi_* D$-modules on X. This follows directly from Grothendieck's classical identification of quasi-coherent sheaves on $\operatorname{Spec} A$ with A-modules ([Har77, Cor II.5.5]).

Using this theorem, we interpret the sheaf of differential operators as living over a different base scheme, which enables us to successfully apply the machinery of Azumaya algebras. We refer the reader to the more detailed exposition in [BMR], together with applications to the representation theory of semisimple Lie algebras in positive characteristic.

We denote the canonical projection $T^* X^{(1)} \to X^{(1)}$ by $\pi^{(1)}$. The preceding proposition tells us that $\iota : \pi^{(1)}_* \mathcal{O}_{T^* X^{(1)}} \to Z(Fr_* D_X)$ is a well-defined isomorphism. As a consequence we see that $Fr_* D_X$ is also a sheaf of $\pi^{(1)}_* \mathcal{O}_{T^* X^{(1)}}$-algebras. Since $\pi^{(1)}$ is an affine morphism we obtain a well-defined sheaf $\mathcal{D}_X$ of $\mathcal{O}_{T^* X^{(1)}}$-algebras on $T^* X^{(1)}$, such that $\pi^{(1)}_* \mathcal{D}_X = Fr_* D_X$.

Proposition 3.2.10. *The functors Fr_* and $\pi^{(1)}_*$ induce an equivalence of categories*

$$\operatorname{QCoh}(X, D_X) \cong \operatorname{QCoh}(T^* X^{(1)}, \mathcal{D}_X),$$

between D_X-modules on X, and $\mathcal{D}_X$-modules on $T^* X^{(1)}$.

Proof. This is due to the fact that Fr and $\pi^{(1)}$ are affine maps. Hence they induces equivalences $Fr_* : \operatorname{QCoh}(X, D_X) \cong \operatorname{QCoh}(X^{(1)}, Fr_* D_X)$, and $\pi^{(1)}_* : \operatorname{QCoh}(T^* X^{(1)}, \mathcal{D}_X) \to \operatorname{QCoh}(X^{(1)}, \pi^{(1)}_* \mathcal{D}_X)$. Using the identification $\pi^{(1)}_* \mathcal{D}_X \cong \operatorname{Fr}_* D_X$, the assertion follows. □

We move on by to study the rank of the sheaf $\mathcal{D}_X$. We chose once again étale local coordinates (x_i) on X, which naturally induce the same kind of coordinates $x_i^{(1)}$ on $X^{(1)}$. Let us first describe the sheaf $Fr_* D_X$. Over the coordinate charts its sections are generated (as an $Fr_* \mathcal{O}_X$-algebra) by vector fields ∂_i, where $\partial := \frac{d}{dx_i}$. The map ι sends ∂ to ∂^p, because $\partial^{[p]}$ is the zero derivation. The sheaf of algebras $\pi^{(1)}_* \mathcal{O}_{T^*X^{(1)}}$ is thus generated by ∂_i^p. Therefore we obtain that the $\pi^{(1)}_* \mathcal{O}_{T^*X^{(1)}}$-algebra $Fr_* D_X$ is generated as a module (!) by the $p^{2\dim X}$ sections $x_i^k \partial_j^l$, where $i = 1, \dots, n$, and $j, k = 0, \dots p-1$.

We conclude with the observation that $\mathcal{D}_X$ is a locally free sheaf of rank p^2. This being a square number is our first analogy with matrix rings.

The following cartesian diagram of schemes is of high relevance to the theory under consideration:

$$\begin{array}{ccc} T^{*,1}X & \xrightarrow{\eta} & T^*X \\ \downarrow & & \downarrow{\scriptstyle \pi^{(1)}} \\ X & \xrightarrow{Fr} & X^{(1)} \end{array}$$

The scheme $T^{*,1}X$ has not shown up so far. Indeed the statement of this diagram being cartesian should be read as a definition. Since all morphisms in the diagram above are affine, the push-forward of the structure sheaf $(\pi^{(1)} \circ \eta)_* \mathcal{O}_{T^{*,1}X}$ is equal to the tensor product $Fr_* \mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \pi^{(1)}_* \mathcal{O}_{T^*X^{(1)}}$. The universal property of the tensor product tells us now that there is a natural morphism $(\pi^{(1)} \circ \eta)_* \mathcal{O}_{T^{*,1}X} \to \pi^{(1)}_* \mathcal{D}_X$. Because this morphism commutes with the action of the centre, we obtain a natural morphism

$$\eta_* \mathcal{O}_{T^{*,1}X} \to \mathcal{D}_X \,.$$

This yields an $\mathcal{O}_{T^{*,1}X}$-module E_X on $T^{*,1}X$, such that $\eta_* E_X = \mathcal{D}_X$. Let us emphasize that E_X does not have a natural algebra structure because the map $\eta_* \mathcal{O}_{T^{*,1}X} \to \mathcal{D}_X$ is not central. The following proposition lies at the heart of the structure of the sheaf of algebras $\mathcal{D}_X$.

Proposition 3.2.11. *There exists a natural isomorphism $\eta^* \mathcal{D}_X \cong \underline{\mathrm{End}}(E_X)$. In particular $\mathcal{D}_X$ is an Azumaya algebra on $T^*X^{(1)}$.*

Proof. We begin by describing the sheaf E_X in terms of an étale local coordinate x on X. This is easily accomplished by comparing it with $\pi_*^{(1)}\mathcal{O}_{T^*X^{(1)}}$, which was generated by ∂_i^p as an $\mathcal{O}_{X^{(1)}}$-algebra, where $\partial := \frac{d}{dx_i}$. The only difference this time is that we consider the tensor product $Fr_*\mathcal{O}_X \otimes_{\mathcal{O}_{X^{(1)}}} \pi_*^{(1)}\mathcal{O}_{T^*X^{(1)}}$, and thus we get a generating family ∂_i^p as $Fr_*\mathcal{O}_X$-algebra. For this reason, the sheaf E_X is locally free with basis $1, \partial_i, \ldots, \partial_i^{p-1}$ over the coordinate chart. So the assertion makes sense.

Let us now come to the definition of the action of $\eta^*\mathcal{D}_X$ on E_X. We consider the action of $\mathcal{D}_X$ on itself by right multiplication, which yields a map $\mathcal{D}_X \to \underline{\mathrm{End}}(\eta_* E_X) = \eta_*\underline{\mathrm{End}}(E_X)$. Equivalently we have a map $\eta^*\mathcal{D}_X \to \underline{\mathrm{End}}(E_X)$.

The remaining local verification needed for this proof can be found in [BMR, section 2.2]. □

3.3 Push-forward and !-pullback

We refer to [BB07] for a detailed discussion of the functors $q^!$ and p_* in the context of D-modules in positive characteristic.

Every morphism of smooth schemes $\pi : V \to U$ induces a morphism $d\pi : V \times_U T^*U \to T^*V$. On the Frobenius twist $(V \times_U T^*U)^{(1)}$ we thus get two Azumaya algebras: $d\pi^{(1),*}\mathcal{D}_V$ and $p_2^*\mathcal{D}_U$. In Proposition 3.7 of [BB07] it is shown that those two Azumaya algebras are canonically Morita equivalent, which allows us to identify the category of $d\pi^{(1),*}\mathcal{D}_V$-modules with the category of $p_2^*\mathcal{D}_U$-modules.

We obtain a natural functor

$$\pi_* : \mathrm{QCoh}(V, D_V) \to \mathrm{QCoh}(U, D_U).$$

If $M \in \mathrm{QCoh}(V, D_V)$, we pull it back along $d\pi^{(1)}$ and obtain a $p_2^*\mathcal{D}_U$-module by the natural identification cited above. Pushing it forward we get a $\mathcal{D}_U$-module.

Similarly we have a functor

$$\pi^! : \mathrm{QCoh}(U, D_U) \to \mathrm{QCoh}(V, D_V).$$

This time we start with a D_U-module N, pull it back along p_2 and obtain a $d\pi^{(1),*}\,\mathcal{D}_V$-module. By pushing it forward along $d\pi^{(1)}$ we obtain a $\mathcal{D}_V$-module.

The two definitions above are easily captured in a diagram:

$$\begin{array}{ccc}
\mathrm{QCoh}(V^{(1)} \times_{U^{(1)}} T^*U^{(1)}, p_2^*\,\mathcal{D}_U) & = & \mathrm{QCoh}(V^{(1)} \times_{U^{(1)}} T^*U^{(1)}, d\pi^{(1),*}\,\mathcal{D}_V) \\
{\scriptstyle p_2^*}\uparrow\downarrow{\scriptstyle p_{2,*}} & & {\scriptstyle d\pi^{(1),*}}\uparrow\downarrow{\scriptstyle d\pi^{(1)}_*} \\
\mathrm{QCoh}(T^*U^{(1)}, \mathcal{D}_U) & \underset{\pi_*}{\overset{\pi^!}{\rightleftarrows}} & \mathrm{QCoh}(T^*V^{(1)}, \mathcal{D}_V)
\end{array}$$

3.4 D-modules on stacks

Since the category of D-modules is local in the smooth topology, it is straight-forward to define the category of D-modules on a smooth stack $\mathcal{Y}$ (see the beginning of A.3 for the case of quasi-coherent sheaves) via descent along smooth maps. We refer the reader to [BD, sect. 1.1.5] where this definition is spelled out. If $\mathcal{Y}$ is defined over a field k of positive characteristic, one would like to study D-modules on $\mathcal{Y}$ in terms of modules for an Azumaya algebra $\mathcal{D}$ on $T^*\,\mathcal{Y}$ (see A.2 for a definition). While the construction for smooth schemes presented in this section does not immediately generalize to stacks, the following result is true ([BB07, sect. 3.13]).

Proposition 3.4.1 (Bezrukavnikov–Braverman). *Let $\mathcal{Y}$ be a good smooth stack (Definition A.2.3), such that $T^*\,\mathcal{Y}$ has an nonempty open dense Deligne-Mumford substack $\mathcal{U}$. There exists a sheaf of algebras $\mathcal{D}_{\mathcal{Y}}$ on $T^*\,\mathcal{Y}^{(1)}$, which restricts to an Azumaya algebra on $\mathcal{U}^{(1)}$, and with category of $\mathcal{D}_{\mathcal{Y}}\,|_{\mathcal{U}^{(1)}}$-modules being equivalent to a localization of the category of D-modules on $\mathcal{Y}$.*

Proposition 3.4.1 is also applicable, if $f : \mathcal{X} \to \mathcal{Y}$ realizes $\mathcal{X}$ as a smooth gerbe over a smooth stack $\mathcal{Y}$ satisfying the condition of the proposition. In this case, $T^* \mathcal{X}$ is naturally a smooth gerbe over $T^* \mathcal{Y}$, and we can pullback the sheaf of algebras $\mathcal{D}_{\mathcal{Y}}$ to $T^* \mathcal{X}^{(1)}$.

Chapter 4

Local systems in positive characteristic

In this chapter we develop the theory of local systems, in analogy with the theory of Higgs bundles. The first two subsections are literature reviews, from section 4.3 on, we present our contributions.

4.1 Local systems

Let us introduce the moduli problem of *local systems.*

Definition 4.1.1. *Let X be a smooth Deligne-Mumford stack, S a scheme. An S-family of* local systems *on X is a pair $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a vector bundle on $X \times S$, and $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X \times S/S}$ is an $\mathcal{O}_S$-linear map of sheaves satisfying the Leibniz rule $\nabla(fs) = (d_{X \times S/S} f)s + f \nabla s$, where f is a section of $\mathcal{O}_{X \times S}$ and s a section of $\mathcal{E}$. Moreover, ∇ has to satisfy the integrality condition*

$$0 = \nabla^2 : E \to E \otimes \Omega^2_X,$$

i.e. be a flat connection.

From now on, we let X be an orbicurve with projective coarse moduli space, unless stated otherwise. We emphasize that it is not necessary to impose a flatness condition on the connection ∇ in this case. For dimension reasons, the sheaf $\Omega^2_{X\times S/S}$ vanishes, hence every connection on a curve is automatically flat.

Definition 4.1.2. *The stack of local systems on X will be denoted by $\mathcal{M}_{\mathrm{dR}}(X)$. By definition, it is given by the functor, sending a scheme S to the groupoid of S-families of local systems on X.*

It has been shown by Laszlo and Pauly in [LP01, Cor. 3.1] that the stack $\mathcal{M}_{\mathrm{dR}}(X)$ is algebraic, if X is an algebraic curve. Nonetheless, algebraicity will also follow from the main result of this chapter, which relates stacks of local systems to stacks of Higgs bundles (Theorem 4.6.3).

Lemma 4.1.3. *The stack $\mathcal{M}_{\mathrm{dR}}(X)$ is algebraic.*

As for Higgs bundles, local systems inherit a stability condition from the one of bundles (Example 2.4.5).

In the article [Ati57], it was shown by M. Atiyah, that the characteristic classes of a vector bundle defined over a compact Kähler manifold, can be related to the obstruction of the existence of analytic connections. The same arguments also cover the case of a vector bundles defined on projective varieties of zero characteristic. As a corollary (Theorem 10 in *loc. cit.*) he reproves a theorem of Weil, which states that a vector bundle defined over a curve of zero characteristic carries an algebraic connection, if and only if the degrees of all indecomposable summands are zero. We conclude from this discussion, that a local system in zero characteristic, is always semi-stable, since we have seen that a vector bundle carrying an algebraic connection has to be of degree zero.

The same arguments do not apply in the context of positive characteristic geometry. Nonetheless, it can be shown that over a perfect infinite field of characteristic p, a connection exists on a vector bundle if and only if for every indecomposable summand congruence relation

$$\deg E \equiv 0 \bmod p$$

holds ([BS06]).

Definition 4.1.4. *Using the central charge on the category of local systems induced from the one on vector bundles (Example 2.4.5), we denote the stack of semi-stable local systems by $\mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}(X)$, the stack of stable local systems by $\mathcal{M}^{\mathrm{s}}_{\mathrm{dR}}(X)$. They are obtained by rigidifying the corresponding substacks of semi-stable, respectively stable objects.*

As indicated by the discussion above, local systems in positive characteristic will not be automatically semi-stable, like it is the case in zero characteristic.

The following proposition can be verified by the same means as Proposition 2.4.10. We remark that it can also be derived from the main results of this chapter (Theorem 4.6.3 and Lemma 4.7.4).

Proposition 4.1.5. *The stacks $\mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}(X)$ and $\mathcal{M}^{\mathrm{s}}_{\mathrm{dR}}(X)$ are open substacks of the rigidification $[\mathcal{M}_{\mathrm{dR}}(X)/B\,\mathbb{G}_m]$. In particular, they are algebraic.*

We have already mentioned a central difference between the theory of local systems in zero and positive characteristic, when discussing the admissible degrees a local system can have (see the paragraph following Lemma 4.1.3). Another significant difference lies in the existence of a Hitchin map for local systems, emphasizing the relation with the theory of Higgs bundles. It has been studied by Laszlo and Pauly in [LP01] and is based on the notion of p-curvature for flat connections.

Definition 4.1.6. *Let E be a quasi-coherent sheaf on a smooth scheme X, endowed with a flat connection ∇. The p-*curvature *of ∇ is given by*

$$\Psi_\nabla(\partial) : s \mapsto (\nabla_\partial)^p s - \nabla_{\partial^{[p]}} s,$$

where ∂ is a tangent vector field of X, $\partial^{[p]}$ is the tangent vector field given by the p-th power of the derivation ∂ (Definition 3.2.4), and s denotes a section of E.

The definition of the map ι (3.2.6) has been inspired by the notion of p-curvature. Since a D_X-module is precisely a quasi-coherent sheaf on X with a flat connection, we see that $\Psi_\nabla(\partial)(s)$ is given by applying the element $\iota(\partial)$ to the section s. Using this interpretation of p-curvature, we obtain the following reformulation of Proposition 3.2.7, which is contained in the paper [Kat, Prop. 5.2] by N. Katz.

Corollary 4.1.7. *For a quasi-coherent sheaf E on a smooth k-scheme X, with flat connection ∇, the p-curvature is p-*linear*, i.e. subject to the relation*

$$\Psi_\nabla(f\partial_1 + \partial_2) = f^p \Psi_\nabla(\partial_1) + \Psi_\nabla(\partial_2),$$

where f is a section of $\mathcal{O}_X$, and ∂_1 and ∂_2 are sections of Θ_X. This allows us to view Ψ_∇ as a morphism

$$E \to E \otimes Fr^* \Omega^1_X,$$

where $Fr : X \to X^{(1)}$ denotes the Frobenius morphism (Definition 3.2.8).

It is worthwile comparing the computationally hard proof given in *loc. cit.* with the one of Proposition 3.2.7 to appreciate the use of the theory of D-modules.

While a priori, p-curvature measures the deviation from ∇ being a map of *restricted p-Lie algebras*; therre is a second interpretation as capturing the obstruction for E descending along the Frobenius morphism $Fr : X \to X^{(1)}$. In order to make this observation precise, we have to endow Frobenius pullbacks Fr^*E' with a connection.

Lemma 4.1.8. *Every pullback Fr^*E' of a quasi-coherent sheaf E' on $X^{(1)}$ can be canonically endowed with a connection ∇^{can}, which is well-defined by the property that*

$$\nabla(Fr^*s) \equiv 0,$$

for every section pulled back along the Frobenius. The connection ∇^{can} will be called the canonical connection *on Fr^*E. This construction gives rise to a functor*

$$Fr^*_\nabla \operatorname{QCoh}(X^{(1)}) \to \operatorname{QCoh}(X, D_X).$$

Proof. Since every section of Fr^*E' can be written as $\mathcal{O}_X$-linear combinations of sections of the sheaf $F^{-1}E'$, the Leibniz rule implies uniqueness of ∇^{can}. In order to show existence it suffices to treat the case of free sheaves (possibly of infinite rank). Zariski locally, every quasi-coherent sheaf E' on $X^{(1)}$ can be written as cokernel of a map $\mathcal{O}_{X^{(1)}}^{\oplus I} \to \mathcal{O}_{X^{(1)}}^{\oplus J}$, where I and J are arbitrary sets. Since Fr^* is right exact, the cokernel will inherit a connection by functoriality. Uniqueness of ∇^{can} implies that the induced connection is independent of this local presentation, and that the local construction patch together to a global object ∇^{can} on Fr^*E'. Henceforth we may assume that $E' = \mathcal{O}_{X^{(1)}}$, i.e. $Fr^*E' = \mathcal{O}_X$, where the canonical connection is given by the universal derivation $d : \mathcal{O}_X \to \Omega^1_X$. □

Now we can state the descent-theoretic interpretation of p-curvature. It is a result of Cartier, which Katz states as Theorem 5.1 in *loc. cit.*. The proof given here relies on the interpretation of D_X as Azumaya algebra over its centre (Proposition 3.2.11).

Theorem 4.1.9 (Cartier descent)**.** *Let X be a smooth scheme. The functor Fr^*_∇ induces an equivalence between the category of quasi-coherent sheaves on $X^{(1)}$ and D-modules on X with* vanishing p-curvature*.*

Proof. Let E be a D_X-module. We remind the reader of Proposition 3.2.10, which established a correspondence between D_X-modules E and $\mathcal{D}_X$-modules F. Moreover, $\mathcal{D}_X$ was shown to be an Azumaya algebra in Proposition 3.2.11.

Interpreting Ψ_∇ as action of the centre $Z(Fr_*(D_X)) = \pi_*^{(1)} \mathcal{O}_{T^*X^{(1)}}$ (Proposition 3.2.9) on Fr_*E, we see that $\Psi_\nabla \equiv 0$ is equivalent to the $\mathcal{D}_X$-module F being supported on the zero fibre $X^{(1)} \hookrightarrow T^*X^{(1)}$.

The $\mathcal{D}_X$-module S associated to the trivial local system $(\mathcal{O}_X, d)$ induces a splitting of $\mathcal{D}_X$ when restricted to $X^{(1)} \hookrightarrow T^*X^{(1)}$. In particular, we see that F can be uniquely written as $F \cong S \otimes E'$. The projection formula reveals an equivalence

$$Fr_*E \cong S \otimes E' \cong Fr_*(\mathcal{O}_X) \otimes E' \cong Fr_*(Fr^*E'),$$

respecting the Fr_*D_X-module structure. We can therefore conclude that $E \cong Fr^*E'$.

The construction of the canonical connection in Lemma 4.1.8 allows us to conclude that this is an equivalence of categories as claimed. □

The next paragraph summarizes the definition of the Hitchin map of Laszlo–Pauly [LP01]. We refer the reader to *loc. cit.* for a more detailed account of this approach. In Definition 4.3.2 we will redefine the Hitchin map for local systems, using the BNR correspondence (Proposition 4.3.1).

As we have seen in Corollary 4.1.7, the object Ψ_∇ is a map $E \to E \otimes \mathrm{Fr}^* \Omega^1_{X^{(1)}}$, in resemblance with the definition of Higgs bundles (2.2.1). It has been shown by Laszlo and Pauly in [LP01, Prop. 3.2] that the characteristic polynomial of the p-curvature Ψ_∇ of a connection, is itself the pullback of an element of $\mathcal{A}^{(1)}$, the Hitchin base of the curve $X^{(1)}$. In order for this statement to make sense, we recall that points of the affine space $\mathcal{A}^{(1)}$ correspond to elements of the vector space

$$\bigoplus_{i=0}^{n-1} H^0(X^{(1)}, \Omega^1_{X^{(1)}}).$$

Definition 4.1.10. *The Hitchin map for local systems, as defined by Laszlo–Pauly will be denoted by* $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}(X) \to \mathcal{A}^{(1)}$.

It has been shown in Proposition 5.1 of *loc. cit.* that the zero fibre of χ_{dR} restricted to the semistable locus, i.e. the stack of semistable *nilpotent connections*, is universally closed.

Theorem 4.1.11 (Laszlo–Pauly)**.** *The stack of semistable nilpotent connections* ${\chi_{\mathrm{dR}}}^{-1}(0)^{ss}$ *is universally closed.*

We will generalize this result to the full Hitchin map χ_{dR}, using the BNR correspondence developed in the next section.

To illustrate this theory we may consider the example[1] of line bundles on an elliptic curve X. The moduli space of flat degree zero line bundles on X is a group scheme. It arises as an extension of the Jacobian J_X by the one-dimensional vector space $H^0(X, \Omega^1_{\mathrm{X}})$. The Hitchin morphism

$$\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}} \to \mathcal{A}^{(1)}$$

maps down to the one-dimensional vector space $\mathcal{A}^{(1)} = H^0(X^{(1)}, \Omega^1_{\mathrm{X}^{(1)}})$. There is an action of $J_{X^{(1)}}$ on $\mathcal{M}_{\mathrm{dR},1,0}(X)$, respecting the morphism χ_{dR}. The pullback Fr^*L of a line bundle L on $X^{(1)}$ is endowed with a canonical connection ∇^{can} of zero p-curvature, as we have seen in Lemma 4.1.8. In particular, we can tensor an arbitrary rank one local system (E, ∇) with (Fr^*L, ∇^{can}), leaving the p-curvature of (E, ∇) invariant. The Azumaya-algebra viewpoint of chapter 3.2 provides us with an alternative understanding of this action. According to Proposition 3.2.10, we can think of $\mathcal{M}_{\mathrm{dR},1,0}(X)$ as the stack of splittings of $\mathcal{D}_X$ relative to the family of curves $X^{(1)} \times \mathcal{A}^{(1)} \to \mathcal{A}^{(1)}$. As we have seen in Lemma 3.1.3, the stack of splittings is naturally acted on by $\mathcal{P}\mathrm{ic}(X^{(1)} \times \mathcal{A}^{(1)} \,/\, \mathcal{A}^{(1)})$.

[1] The author thanks Christian Pauly for explaining this example to him.

Example 4.1.12. *For an elliptic curve X, the $\mathcal{A}^{(1)}$-stack $\mathcal{M}_{\mathrm{dR},1,0}(X)$ is étale locally equivalent to the $\mathcal{A}^{(1)}$-stack $\mathcal{M}_{\mathrm{Dol},1,0}(X^{(1)})$.*

Before stating this example we have already illustrated how to give a proof using the Azumaya picture of differential operators. Below we give a more elementary discussion.

Proof. The proof proceeds by constructing a section of the Hitchin map χ_{dR}, after an étale base change. As *ansatz* to construct such a section, we study flat connection on the trivial line bundle $(\mathcal{O}, d+\omega)$ with given p-curvature. This corresponds to finding a right-inverse to the map $H^0(X,\Omega^1) \to \mathcal{M}_{\mathrm{dR}} \to H^0(X^{(1)},\Omega^1_{\mathrm{X}})$. According to formula 2.1.16 in [Ill79] this map is the sum of a p-linear and a linear map of vector spaces. Without loss of generality we may assume it is the map $\lambda \mapsto \lambda^p - \lambda$, known as the *Artin-Schreier* morphism, i.e. it is étale. By construction, we see that the codomain of the map $\mathcal{A} \to \mathcal{A}^{(1)}$ parametrizes a family of connections $(\mathcal{O}_X, d+\omega)$ on the trivial vector rank one bundle with prescribed p-curvature. This allows us to conclude that after base-change of χ_{dR} along this étale map, there exists a section s of χ_{dR}.

Using this section we can construct a morphism

$$\mathcal{M}_{\mathrm{dR},1,0}(X) \times_{\mathcal{A}^{(1)}} \mathcal{A} \to J_{X^{(1)}}.$$

A pair consisting of a local system (E,∇) with p-curvature $a' \in \mathcal{A}^{(1)}$, and an Artin-Schreier lift a of a', is sent to

$$(\mathcal{O}, d+\omega_a)^\vee \otimes (E,\nabla),$$

which is itself a rank one local system with p-curvature zero. By Cartier descent (Theorem 4.1.9), it corresponds to a unique line bundle on $X^{(1)}$. This construction gives rise to a map

$$\mathcal{M}_{\mathrm{dR},1,0}(X) \times_{\mathcal{A}^{(1)}} \mathcal{A} \to \mathcal{M}_{\mathrm{Dol},1,0}(X^{(1)}) \times_{\mathcal{A}^{(1)}} \mathcal{A}\,.$$

An obvious inverse to this map can now be constructed, by using the map

$$\mathcal{A} \times \mathcal{P}\mathrm{ic}(X^{(1)}),$$

which sends (a, L) to the local system $(\mathcal{O}, d + \omega_a) \otimes (Fr^* L, \nabla^{can})$. □

4.2 Parabolic local systems

There is a theory of parabolic local systems (see Simpson's [Sim90]), which is paralleling the one of parabolic Higgs bundles, developed in Section 2.5.

Definition 4.2.1. *For a weighted curve $\widehat{X}$ let $\omega_{ij} \in k$ be a tuple of scalars, where i corresponds to marked points in $D \subset X$ and $0 \leq j \leq n_i - 1$. An S-family of* parabolic local system *on $\widehat{X}$ with eigenvalues (ω_{ij}) consists of an S-family of parabolic vector bundles $(\mathcal{E}, \mathcal{F}_{\bullet\bullet})$ and a parabolic flat connection given by a morphism of k-linear sheaves*

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes p_X^* \Omega^1_{\mathrm{X}}(p_1 + \dots p_k),$$

satisfying the Leibniz identity and the condition that

$$\mathrm{res}_{p_i} \nabla(\mathcal{F}_{ij}) \subset \mathcal{F}_{ij},$$

with eigenvalues of the residue at x_i is given by the scalars ω_{ij} on the j-th graded of $\iota_i^* \mathcal{E}$.

Unless stated otherwise, we will always take $\omega_{ij} = \frac{j}{n_j}$. Definition 2.5.7 induces a stability condition on the category of parabolic local systems.

Definition 4.2.2. *The moduli stack of parabolic local systems will be denoted by* $\mathcal{M}_{\mathrm{dR}}(\widehat{X}) = \mathcal{M}_{\mathrm{dR}}(\widehat{X}, n, \lambda_\bullet)$. *The moduli space of semi-stable parabolic local systems* $\mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}$ *is constructed as open substack of semi-stable local systems of the rigidification* $[\mathcal{M}_{\mathrm{dR}}(X)/B\,\mathbb{G}_m]$.

In analogy with Proposition 4.2.3, which establishes a correspondence between Higgs bundle on the orbicurve $\widetilde{X}$ and parabolic Higgs bundles on the weighted curve $\widehat{X}$, we have a correspondence between local system on orbicurves and parabolic local systems ([BH12]).

Proposition 4.2.3 (Biswas–Heu)**.** *Under the correspondence of Proposition 2.6.5 a connection* $\widetilde{\nabla}$ *on an orbibundle* $\widetilde{E}$ *gets transformed to a parabolic connection* $\widehat{\nabla}$ *with eigenvalues* $\omega_{ij} = \frac{j}{n_j}$ *on* $\widehat{E}$. *This defines a natural equivalence of groupoids between* S*-families of local systems on the orbicurve* $\widetilde{X}$ *and* S*-families of parabolic local systems on the weighted curve* $\widehat{X}$.

4.3 The BNR correspondence

We can now establish an analogue of Theorem 2.3.2 for local systems. In the case of smooth spectral curves this description was contained in the proof of Lemma 4.8 in [BB07]. We denote by $\pi : Y^{(1)} \to X^{(1)} \times \mathcal{A}^{(1)}$ the universal spectral cover of $X^{(1)}$ parametrized by $\mathcal{A}^{(1)}$, where the superscript $^{(1)}$ denotes the Frobenius twist of a scheme, as defined in Definition 3.2.8.

Proposition 4.3.1 (BNR for local systems)**.** *Giving an* S*-family of local systems of rank* n *on a smooth proper Deligne-Mumford stack is equivalent to giving a morphism* $a : S \to \mathcal{A}^{(1)}$, *a coherent sheaf* $\mathcal{E}$ *on* $Y^{(1)} \times_{\mathcal{A}^{(1)}} S$, *carrying a structure of a* $\mathcal{D}_X$*-module, and satisfying that* $(\pi_S)_*\,\mathcal{E}$ *is locally free of rank* pn.

Before delving into the proof below, we will sketch the main ideas and difficulties.

We use the equivalence of categories of Proposition 3.2.10

$$\mathrm{QCoh}(X, D_X) \cong \mathrm{QCoh}(T^*X^{(1)}, \mathcal{D}_X),$$

where $\mathcal{D}_X$ is an Azumaya algebra on $T^*X^{(1)}$. Recall that under this equivalence, a local system (E, ∇) is sent to the $Fr_*\mathcal{D}_X$-module Fr_*E, which gives rise to a $\mathcal{D}_X$-module $\mathcal{E}$. The p-curvature endows the rank pn vector bundle Fr_*E on $X^{(1)}$ with a Higgs field. The BNR correspondence for Higgs bundles (Theorem 2.3.2) allows us now to relate the Higgs bundle Fr_*E to a coherent sheaf $\mathcal{F}$ supported on a spectral curve $Y_b^{(1)}$. The degree of the polynomial b equals the rank pn of Fr_*E.

As in the proof of *loc. cit.* the main difficulty arises when showing that the characteristic polynomial b of the Higgs bundle Fr_*E together with the p-curvature as Higgs field, can be written as $b = a^p$. In *loc. cit.* this is shown by a computation, while our proof below makes use of the fact that D_X gives rise to an Azumaya algebra $\mathcal{D}_X$ on $T^*X^{(1)}$.

Proof of Proposition 4.3.1. The coherent sheaf Fr_*E is locally free of rank pn, the p-curvature of ∇ endows Fr_*E with a Higgs field. An S-family (E, ∇) of local systems gives therefore rise to a morphism

$$b : S \to \mathcal{A}_{pn}^{(1)},$$

i.e. a characteristic polynomial of degree pn. According to the BNR correspondence for Higgs bundles (Theorem 2.3.2) the sheaf $\mathcal{E}$ on $S \times T^*X^{(1)}$ is supported on the spectral cover $Y_b^{(1)}$ corresponding to the characteristic polynomial b.

Since $\operatorname{rk} Fr_*E = pn$ we have $\deg b = pn$, which is a p-th multiple of the degree. We need to show that $b = a^p$ for a degree n characteristic polynomial a, and that $\mathcal{E}$ is supported on the corresponding spectral cover. This relation refers to the product of characteristic polynomials, where the coefficients are viewed as elements of the graded ring

$$\bigoplus_{i=0}^{\infty} H^0(X^{(1)}, (\Omega^1_{X^{(1)}})^{\otimes i}).$$

Let x be a geometric point of $X \times S$ and $U = \operatorname{Spec} \mathcal{O}$ the spectrum of the corresponding henselian ring. We consider the base change $Y^{(1)} \times_X U = V \to U$. Because this arrow is a finite morphism and $\mathcal{O}$ is henselian, we conclude that V is the spectrum of a product of local henselian algebras ([Mil80, Thm 4.2(b)]). In particular we know that $\mathcal{D}|_V$ is split. There exists an isomorphism $\mathcal{D}|_V \cong \underline{\mathrm{End}}(M)$, where M denotes a rank p vector bundle on V. Since $\Gamma(\mathcal{O}_V)$ is a product of local rings, M is free. Thus we may identify $\mathcal{D}|_V$ with the matrix algebra $M_p(\mathcal{O}_V)$.

This implies the existence of a coherent sheaf $\mathcal{F}$, such that we have non-canonically

$$\mathcal{E} = \bigoplus_{i=1}^{p} \mathcal{F},$$

which in turn implies a decomposition of the Higgs bundle Fr_*E as a direct sums of p copies of the same Higgs bundle $(\widetilde{E}, \widetilde{\theta})$ (we denote by $\widetilde{\theta}$ the Higgs field induced by the BNR correspondence). In particular we obtain that $\operatorname{rk} \widetilde{E} = n$ and therefore that $(\widetilde{E}, \widetilde{\theta})$ is supported on the spectral cover corresponding to a degree n polynomial a, satisfying $b = a^p$. We conclude that $\mathcal{F}$ and $\mathcal{E}$ are supported on the spectral cover $Y_a^{(1)}$. □

One consequence of the above discussion is the existence of a canonical morphism from the stack of flat connections $\mathcal{M}_{\mathrm{dR}}$ to the affine space $\mathcal{A}^{(1)}$, providing an alternative to Definition 4.1.10. Recall that we have seen in Proposition 4.3.1 that the data of a local system can be encoded by a $\mathcal{D}_X$-module $\mathcal{E}$, supported on a spectral cover. The morphism defined below simply forgets the sheaf $\mathcal{E}$, respectively sends it to its scheme-theoretic support.

Definition 4.3.2. *The morphism* $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}(X) \to \mathcal{A}^{(1)}$

$$(a : S \to \mathcal{A}^{(1)}, \mathcal{E}) \mapsto (a : S \to \mathcal{A}^{(1)})$$

is called the twisted Hitchin morphism*.*

4.4 Properness of the Hitchin map

In section 2.4 we have seen properness of the Hitchin map for local systems, as one of the main geometric consequences of the notions of stability. With the analogous definition of semistable local systems at hand (4.1.4), and a BNR correspondence for local systems (4.3.1), we will be able to establish the same result for the Hitchin morphism χ_{dR} for local systems of Definition 4.3.2.

Theorem 4.4.1. *The Hitchin morphism* $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}^{\mathrm{ss}} \to \mathcal{A}^{(1)}$ *is universally closed.*

As in the proof of Theorem 2.4.12 we use the valuative criterion for universally closed maps (Theorem 7.3 in [LMB00]). This reduces the proof of the theorem to the following Proposition.

Proposition 4.4.2. *Let* R *be a discrete valuation ring with fraction field* K*, let* a_K *be a* K*-point of the Hitchin base* $\mathcal{A}^{(1)}$*, that extends to an* R*-point* a_R*. Given a* $\operatorname{Spec} K$*-family of semistable local systems* $(\mathcal{E}_K, \nabla_K)$ *with characteristic polynomial* a_K*, there exists a* $\operatorname{Spec} R$*-family of semistable local systems* $(\mathcal{E}_R, \nabla_R)$ *with characteristic polynomial* a_R*, which extends the family defined over* $\operatorname{Spec} K$*.*

The starting point is as in the proof of Proposition 2.4.14 to extend $(\mathcal{E}_K, \nabla_K)$ to a $\operatorname{Spec} R$-family of local systems (without semistability being assumed). This is where the importance of the BNR correspondence of Proposition 4.3.1 lies.

Lemma 4.4.3. *Let* R *be a discrete valuation ring with fraction field* K *and residue field* k'*, let* a_K *be a* K*-point of the Hitchin base* $\mathcal{A}$*, that extends to an* R*-point* a_R*.*

Given a $\operatorname{Spec} K$*-family of local systems* $(\mathcal{E}_K, \nabla_K)$ *with characteristic polynomial* a_K*, there exists a* $\operatorname{Spec} R$*-family of local systems* $(\mathcal{E}_R, \nabla_R)$ *with characteristic polynomial* a_R*, which extends the family defined over* $\operatorname{Spec} K$*.*

Proof. The characteristic polynomial $a_R : \operatorname{Spec} R \to \mathcal{A}$ gives rise to a spectral curve $\pi^{(1)} : Y_{a_R} \to X \times \operatorname{Spec} R$. Inside of it we have the open dense subset given by the spectral curve $\iota : Y_{a_K} \hookrightarrow Y_{a_R}$. According to Proposition 4.3.1, $(\mathcal{E}_K, \nabla_K)$ gives rise to a coherent $\mathcal{D}_X$-module $\mathcal{F}_K$ on Y_{a_K}. It suffices to extend $\mathcal{F}_{a_K}$ to a coherent $\mathcal{D}_X$-module $\mathcal{F}_R$ on Y_{a_R}, such that $\pi^{(1)}_* \mathcal{F}_{a_R}$ is locally free. The quasi-coherent $\mathcal{D}_X$-module $\iota_* \mathcal{F}_{a_K}$ can be written as a filtered union of coherent $\mathcal{D}_X$-submodules $\mathcal{F}_i$, where $i \in \mathbb{N}$. Since $\mathcal{F}_i|_{Y \times \operatorname{Spec} K} \subset \mathcal{F}_{a_K}$, and $\mathcal{F}_{a_K}$ is coherent, there must be an $n \in \mathbb{N}$, such that $\pi^{(1)}_* \mathcal{F}_i|_{X \times \operatorname{Spec} K}$ equals $\mathcal{E}_K$ for $i \geq n$. If $\pi^{(1)}_* \mathcal{F}_n$ is locally free, we are done. If not, we use the fact that X_R is a surface and and therefore the bidual $\mathcal{E}'_R := (\pi^{(1)}_* \mathcal{F}_n)^{\vee\vee}$ is maximal Cohen-Macaulay (Lemma A.4.4) and hence, is locally free (Lemma A.4.3). $\square$

Almost verbatim to the proof of Proposition 2.4.14 one can then apply Langton's strategy to conclude the existence of a semistable extension. It is verified in Lemma 4.2 in [LP01] that $\mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}(X)$ is of finite type, hence we can apply the valuative criterion.

Assuming the existence of a coarse moduli space $\mathcal{M}^{\mathrm{coarse}}_{\mathrm{dR}}$ for semistable local systems up to S-equivalence, analogous to Definition 2.4.16, one could now conclude properness of the Hitchin map $\mathcal{M}^{\mathrm{coarse}}_{\mathrm{dR}} \to \mathcal{A}^{(1)}$. Unfortunately, we do not know a reference for a GIT construction of the moduli space of semistable local systems in positive characteristic. In 4.7 we will take a different path, and derive the existence of $\mathcal{M}^{\mathrm{coarse}}_{\mathrm{dR}}$ from the one of $\mathcal{M}^{\mathrm{coarse}}_{\mathrm{Dol}}$ (Corollary 4.7.5).

At the end of section 2.5 on parabolic Higgs bundles, we have explained how properness of the Hitchin map and the theory of $\mathbb{G}_m$-actions by Białynicki-Birula [BB73] yields dimension estimates for Hitchin fibres. In the present context we do not have a $\mathbb{G}_m$-action. However, it is possible to consider the larger moduli space of semistable t-connections, as done for instance by Laszlo–Pauly in [LP01]. On this larger space, one has again a natural $\mathbb{G}_m$-action. C. Simpson succeeded in [Sim10] in showing that the limit

$$\lim_{t \to 0} (E, t \cdot \nabla)$$

exists for every semistable local system (E, ∇). This allows us to compare the set of connected components of the moduli space of semistable local systems with the set of connected components of the moduli space of semistable Higgs bundles.

4.5 Relative splittings of Azumaya algebras

Our main result in this subsection is a generalization of a classical theorem, which is established, using a result in Galois theory due to Tsen. Below we give a proof with a more geometric flavour, which explains how to pass from Tsen's result in Galois cohomology to the geometric setup of curves. We refer the reader to Example 2.22 (d) in [Mil80]) for an alternative argument.

Theorem 4.5.1 (Tsen). *Let X be a smooth proper curve defined over an algebraically closed field k of arbitrary characteristic. Then we have $H^2_{et}(X, \mathbb{G}_m) = 0$, hence every Azumaya algebra defined over X splits.*

We have seen in Proposition 3.1.7 that an Azumaya algebra is split if and only if the corresponding class in $H^2_{et}(X, \mathbb{G}_m)$ vanishes. Therefore the first assertion above implies the second one. In this particular instance, also the converse is true. Theorem 2.16 of [Mil80] implies every class of $H^2_{et}(X, \mathbb{G}_m)$ on a quasi-compact scheme can be represented by an Azumaya algebra, when restricted to the complement of a closed subscheme of codimension > 1. In particular we see that on a curve, every class in $H^2_{et}(X, \mathbb{G}_m)$ arises from an Azumaya algebra.

First proof of Theorem 4.5.1. According to Proposition 3.1.4, an Azumaya algebra of rank r^2 can be viewed as a PGL_r-bundle. It is a theorem of Steinberg that a G-bundle on a smooth curve, for G a semisimple group scheme, trivializes when restricted to an arbitrary affine open subset (see the more general result in [DS95] by Drinfeld–Simpson). This allows us to associate to the Azumaya algebra a Zariski cohomology class in $H^2_{Zar}(X, \mathrm{PGL}_n)$ and therefore a Zariski gerbe, given by a Zariski cohomology class in $H^2_{Zar}(X, \mathbb{G}_m)$. It is a vanishing theorem of Grothendieck (see Theorem III.2.7 in [Har77]) that the cohomology of a Zariski sheaf on a noetherian space vanishes in degrees above the dimension of the space. In particular, $H^2_{Zar}(X, \mathbb{G}_m) = 0$, since X is of dimension one. □

Tsen's Theorem 4.5.1 will be generalized in this section in two different directions. First of all we remove the smoothness assumption, and secondly, we further relativize this statement. Moreover, we allow our curves to acquire orbifold points, where Assumption 2.6.1 is supposed to hold (or alternatively, we assume the Deligne-Mumford stacks to be tame). The validity of Tsen's theorem in an orbifold context is guaranteed by a result of F. Poma ([Pom13, Cor. 4.15]).

Definition 4.5.2. *Given a morphism of stacks $\pi : Y \to X$ and an Azumaya algebra $\mathcal{D}$ over Y we say that $\mathcal{D}$* splits relatively over *X, if there exists an étale covering $(U_i)_{i \in I}$ of X, such that $\mathcal{D}$ is split over every fibre product $U_i \times_X Y$.*

Our main result in this subsection is a generalization of Theorem 4.5.1. We will remove the smoothness assumption and further prove the corresponding statement relative to a more general base.

Definition 4.5.3. *A* relative orbicurve *is a morphism of finite type tame Deligne-Mumford stacks $\pi : Y \to X$ over k, which is proper, flat, cohomologically flat in degree zero, and has geometric fibres of dimension one.*

We can now state the main result of this section.

Theorem 4.5.4. *Let $\pi : Y \to X$ be a relative orbicurve, and $\mathcal{D}$ an Azumaya algebra on Y. Then every Azumaya algebra over Y splits relatively over X.*

We begin by treating the abolute case (i.e. with base being Spec k), which is stated in [FK88], in a remark after Lemma I.5.2.

Theorem 4.5.5. *Let X be a proper noetherian algebraic space over an algebraically closed field k of dimension ≤ 1, then $H^2_{et}(X, \mathbb{G}_m) = 0$. In particular, every Azumaya algebra defined over X splits.*

Proof. At first we reduce to the case where X is reduced. Let $\mathcal{I}$ be a quasi-coherent sheaf of ideals on X satisfying $\mathcal{I}^2 = 0$. We consider the closed immersion $j : Y \to X$ given by this sheaf of ideals in $\mathcal{O}_X$ and study the truncated exponential sequence

$$0 \longrightarrow \mathcal{I} \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow j_* \mathcal{O}_Y^\times \longrightarrow 1.$$

The exponential function $\exp : \mathcal{I} \to \mathcal{O}_X^\times$ is defined by the expression $f \mapsto 1 + f$ and satisfies $\exp(f + g) = \exp(f)\exp(g)$, $\exp(0) = 1$. In particular the map takes values in the sheaf of abelian groups of units. The corresponding long exact sequence implies that $H^2_{et}(X, \mathbb{G}_m) = H^2_{et}(Y, \mathbb{G}_m)$, since $H^i(X, \mathcal{I}) = 0$ for $i > 1$. There exists a quasi-coherent sheaf of ideals $\mathcal{J}$ on X, such that $\mathcal{O}_X / \mathcal{J} = \mathcal{O}_{X^{red}}$. Since $\mathcal{J}$ consists of nilpotent elements and X is noetherian, there exists a positive integer k, such that $\mathcal{J}^k = 0$. Let us define X_i to be the scheme with structure sheaf $\mathcal{O}_X / \mathcal{J}^i$. By definition we have $X_k = X$ and $X_0 = X^{red}$. Moreover $\mathcal{I}_i := \mathcal{J}^{i-1}/\mathcal{J}^i$ defines a quasi-coherent sheaf of ideals on X_i satisfying $\mathcal{I}_i^2 = 0$. From the discussion above we may conclude that

$$H^2_{et}(X_k, \mathbb{G}_m) = \cdots = H^2_{et}(X_0, \mathbb{G}_m).$$

Assuming that X is reduced, we denote by $j : Y \to X$ the normalization map. The morphism j is finite, therefore the functor j_* is exact (cf. [Mil80, Cor II.3.6]).

We can now apply the étale cohomology functor to the short exact sequence

$$1 \to \mathcal{O}_X^\times \to j_* \mathcal{O}_Y^\times \to j_* \mathcal{O}_Y^\times / \mathcal{O}_X^\times \to 1$$

and obtain a long exact sequence

$$\cdots \to H^i_{et}(X, \mathcal{O}_X^\times) \to H^i_{et}(Y, \mathcal{O}_Y^\times) \to H^i(j_* \mathcal{O}_Y^\times / \mathcal{O}_X^\times) \to H^{i+1}_{et}(X, \mathcal{O}_X^\times) \to \dots .$$

The quotient sheaf $j_* \mathcal{O}_Y^\times / \mathcal{O}_X^\times$ is supported at finitely many closed points, therefore all of its higher cohomology groups have to vanish. As a consequence we obtain that $H^2_{et}(X, \mathcal{O}_X^\times) = 0$.

As has been argued above, we may assume that X is smooth, and therefore we are in the situation of Theorem 4.5.1. □

Recall that a splitting of an Azumaya algebra $\mathcal{D}$ on X is a pair (ϕ, E), where E is a locally free sheaf on X and

$$\phi : \underline{\mathrm{End}}(E) \to \mathcal{D}$$

is an isomorphism.

Definition 4.5.6. *Given an Azumaya algebra $\mathcal{D}$ over Y and a morphism $Y \to X$, we define a 2-functor $\mathcal{S}$ from the category Sch/X to the 2-category of groupoids sending a scheme $U \to X$ to the groupoid of splittings of $\mathcal{D}$ over $U \times_X Y$. Here a morphism between two splittings (ϕ, E), (ψ, F) is defined to be a pair (γ, L) where L is a line bundle on $U \times_X Y$ and $\gamma : E \to F \otimes L$ is an isomorphism. This is a stack referred to as the stack of relative splittings of $\mathcal{D}$ along $\pi : Y \to X$.*

To deduce the relative from the absolute case, we need to study the deformation theory of splittings.

Lemma 4.5.7 (Commutation with inverse limits)**.** *Let $\pi : Y \to X$ be a relative orbicurve (Definition 4.5.3) and $\mathcal{D}$ an Azumaya algebra over Y. Given a complete noetherian local k-algebra $\bar{A}$ together with a morphism $\operatorname{Spec} \bar{A} \to X$, then $\mathcal{D}$ splits over the base change $\operatorname{Spec} \bar{A} \times_X Y$. Furthermore, we obtain for the 2-functor $\mathcal{S}$ that*

$$\mathcal{S}(\bar{A}) \to \varprojlim \mathcal{S}(\bar{A}/\mathfrak{m}^n)$$

is an isomorphism.

Proof. In the course of the proof we will assume that $X = \operatorname{Spec} \bar{A}$. We denote the base change $Y \times_X \operatorname{Spec} A/\mathfrak{m}_A^n$ by Y_n. The corresponding formal scheme is denoted by $\widehat{Y}$.

Grothendieck's Existence Theorem states that the abelian category of coherent sheaves on Y is equivalent to the abelian category of coherent sheaves on the formal scheme $\widehat{Y}$, which is a projective 2-limit category of the categories $\mathrm{Coh}(Y_n)$ (cf. [Ill05, Thm 8.4.2] for the case of schemes, and [AV02, App. A] for tame Deligne-Mumford stacks). According to Theorem 4.5.5 there exists a splitting S_n of $\mathcal{D}|_{Y_n}$ for every n. In order to obtain a splitting of $\mathcal{D}$ we need to choose compatible splittings S_n. This is always possible, as the splitting $S_{n+1}|_{Y_n}$ differs from S_n by a twist by a line bundle. Since $\pi : Y \to X$ is a relative orbicurve, we can lift the difference line bundle from Y_n to Y_{n+1}. Therefore, a sequence of compatible splittings S_n of $\mathcal{D}_{Y_n}$ exists.

To prove the asserted equivalence, we use the existence of a splitting and reduce to the analogous statement for line bundles. This is an easy consequence of Grothendieck's Existence Theorem quoted above, since line bundles are characterized as invertible coherent sheaves. □

We refer the reader to [Art74] for the definition of the deformation theory of a stack. Note that in our case the deformation problem is unobstructed, as we are dealing with a relative orbicurve. According to a Theorem of M. Artin (cf. [Art74, Thm 5.3]), the deformation theory of a stack allows us to decide wether it is algebraic or not.

Theorem 4.5.8. *Let $\pi : Y \to X$ be a relative orbicurve and let $\mathcal{D}$ be an Azumaya algebra over Y. The 2-functor $\mathcal{S}$ is representable by an algebraic stack.*

Proof. The stack $\mathcal{S}$ is by definition equivalent to the *Hom-stack* $\mathrm{Map}_{\mathcal{A}^{(1)}}(Y^{(1)}, \mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}}})$. Algebraicity of such Hom-stacks has been shown under quite general assumptions by M. Aoki in [Aok06b], respectively [Aok06a]. The family of spectral curves $Y^{(1)}/\mathcal{A}^{(1)}$ is known to be proper, and Lemma 4.5.7 guarantees that the additional assumption of [Aok06a] is satisifed. □

Spelling out an alternative proof of algebraicity, one uses the fact that $\mathcal{S}$ is a $\mathcal{P}$ic quasi-torsor. By this we mean that if a splitting exists, then it gives rise to an identification of the groupoid of splittings with the groupoid of line bundles. Moreover Lemma 4.5.7 shows that $\mathcal{D}$ splits on formal fibres, rendering the restriction of $\mathcal{S}$ to formal fibres a $\mathcal{P}$ic-torsor. It has been shown in [Art69] that $\mathcal{P}$ic is an algebraic stack, by a close study of the deformation theory of $\mathcal{P}$ic. But since $\mathcal{D}$ splits on formal fibres, $\mathcal{P}$ic and $\mathcal{S}$ cannot be distinguished on the level of deformation theory, which allows us to conclude that $\mathcal{S}$ is an algebraic stack too.

Lemma 4.5.9. *Let $\pi : Y \to X$ be a relative orbicurve and let $\mathcal{D}$ be an Azumaya algebra over Y. Then the algebraic stack $\mathcal{S}$ is smooth over X.*

Proof. This is a simple verification of the criterion for formal smoothness. We have already seen that there exists no obstruction to lifting splittings in the curve case in the proof of Lemma 4.5.7. Therefore the structural morphism $\mathcal{S} \to X$ is smooth. □

Proof of Theorem 4.5.4. According to Lemma 4.5.9, the morphism of stacks $\mathcal{S} \to X$ is smooth. In particular, it has a section étale locally on X. □

Lemma 4.5.10. *The stack $\mathcal{S}$ is étale locally isomorphic to $\mathcal{P}\mathrm{ic}(Y/X)$. Moreover, it is locally of finite presentation, smooth and universally open over the base.*

Proof. We have seen in Theorem 4.5.4 that every Azumaya algebra splits relatively over X. Therefore there exists an étale cover (U_i) of X, such that $\mathcal{D}$ splits over $U_i \times_X Y$. Since two splittings of an Azumaya algebra are related by line bundles this gives rise to an isomorphism $U_i \times_X \mathcal{S} \cong \mathcal{P}\mathrm{ic}(U_i \times_X Y/U_i)$.

See Proposition 9.4.17 in [Kle05] for a proof that $\mathcal{P}\mathrm{ic}(Y/X)$ is locally of finite presentation. Moreover $\mathcal{P}\mathrm{ic}(Y/X)$ is smooth as the deformation theory of line bundles on curves is unobstructed. A flat morphism which is locally of finite presentation is universally open, according to Proposition 2.4.6 in EGA IV.2 (cf. [Gro65]). □

4.6 Local equivalence of moduli stacks

Let X be a smooth projective orbicurve, satisfying Assumption 2.6.1. In this subsection we use the BNR correspondence 4.3.1 and Theorem 4.5.4 to show that the moduli stack of Higgs bundles is étale locally equivalent to the moduli stack of local systems over the Hitchin base $\mathcal{A}^{(1)}$. This is based on the following observation.

Lemma 4.6.1 (Splitting principle)**.** *Let S be a k-scheme locally of finite type and $a : S \to \mathcal{A}^{(1)}$ be an S-family of spectral curves, such that the Azumaya algebra $\mathcal{D}_X$ has a splitting P, when pulled back to $Y_a^{(1)} := Y^{(1)} \times_{\mathcal{A}^{(1)}} S$. Then the equivalence Ψ_P from Proposition 3.1.9 induces an equivalence of the groupoid of rank n local systems on X with characteristic polynomial a and rank n Higgs bundles on $X^{(1)}$ with characteristic polynomial a.*

Proof. We denote by $\pi : Y_a^{(1)} \to X^{(1)} \times S$ the projection of the corresponding spectral curve to $X \times S$, which is a finite morphism. We will use the BNR correspondence for local systems (Prop. 4.3.1) and the BNR correspondence for Higgs bundles (Thm. 2.3.2). In particular, we would like to relate $\mathcal{D}$-modules $\mathcal{E}$ on $Y_a^{(1)}$, satisfying $\pi_* \mathcal{E}$

being a locally free sheaf of rank pn, with coherent sheaves $\mathcal{F}$, such that $\pi_* \mathcal{F}$ is a locally free sheaf of rank n. As in Lemma 3.1.9 we send such a sheaf $\mathcal{F}$ to

$$\mathcal{E} := P \otimes \mathcal{F} .$$

Since Proposition 3.1.9 already provides us with an equivalence of $\mathcal{D}_X$-modules with quasi-coherent sheaves on Y_a, we only have to take care of the push-forward condition to conclude the proof. For every $x \in X \times S$ we replace $X \times S$ by the spectrum of the henselization of the local ring $\mathcal{O}_{X\times S,x}$. In particular, we have that the base change of the spectral cover $Y_a^{(1)}$ is the spectrum of a product of local rings (Thm. 4.2 in [Mil80]). This implies that P is a free sheaf of rank p. In particular we see that on the level of underlying sheaves we have

$$\mathcal{E} \cong \bigoplus_{i=0}^{p-1} \mathcal{F} .$$

We may therefore conclude that $\pi_* \mathcal{E}$ is locally free of rank pn, if and only if $\pi_* \mathcal{F}$ is locally free of rank n. □

As an immediate corollary of this splitting principle we obtain the existence of an isomorphism of Hitchin fibres for local systems and Higgs bundles.

Corollary 4.6.2. *Let $\chi_{\mathrm{Dol}} : \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \to \mathcal{A}^{(1)}$ be the Hitchin fibration mapping a Higgs bundle to the characteristic polynomial of its Higgs field, and $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}} \to \mathcal{A}^{(1)}$ the Hitchin fibration of Definition 4.3.2. Then we have that for every $a \in \mathcal{A}^{(1)}$ there exists a non-canonical isomorphism of stacks $\chi^{-1}_{\mathrm{Dol}}(a) \cong \chi^{-1}_{\mathrm{dR}}(a)$.*

Proof. This follows from applying the Splitting Principle 4.6.1 to the morphism $a : \operatorname{Spec} k \to \mathcal{A}^{(1)}$. The existence of a splitting of an Azumaya algebra on a non-necessarily smooth but projective orbicurve, is guaranteed by Theorem 4.5.5. □

Let X be a complete smooth orbicurve over an algebraically closed field k of characteristic $p > 0$. We denote by $\chi_{\text{Dol}} : \mathcal{M}_{\text{Dol}}(X^{(1)}) \to \mathcal{A}^{(1)}$ the Hitchin fibration mapping a Higgs bundle to the characteristic polynomial of its Higgs field, and $\chi_{\text{dR}} : \mathcal{M}_{\text{dR}} \to \mathcal{A}^{(1)}$ the deformed Hitchin fibration of Definition 4.3.2. These two morphisms induce the structure of an $\mathcal{A}^{(1)}$-stack on their domains.

Theorem 4.6.3. *Let us denote by $\mathcal{S}$ the stack of splittings of the Azumaya algebra $\mathcal{D}_X$ relative to the family of spectral curves $Y^{(1)}/\mathcal{A}^{(1)}$ (see the sections 4.3 and 6.1). There exists a canonical isomorphism*

$$\mathcal{S} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{dR}} \cong \mathcal{S} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{Dol}} .$$

Moreover the $\mathcal{A}^{(1)}$-stacks $\mathcal{M}_{\text{dR}}(X)$ and $\mathcal{M}_{\text{Dol}}(X^{(1)})$ are étale locally equivalent, i.e. there exists an étale cover $\{U_i \to \mathcal{A}^{(1)}\}_{i\in I}$ and isomorphisms of U_i-stacks

$$U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{Dol}}(X^{(1)}) \cong U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{dR}}(X).$$

Proof. The first part of this theorem follows from the Splitting Principle (Lemma 4.6.1). We remind the reader that it states that every S-family of spectral curves $a : S \to \mathcal{A}^{(1)}$, which is endowed with a splitting P of $\mathcal{D}_X$ pulled back to $S \times_{\mathcal{A}^{(1)}} Y^{(1)}$, gives rise to an identification of S-families of Higgs bundles on $X^{(1)}$ with spectral curve a, and S-families of local systems on X with spectral curve X.

Lemma 2.3.1 and Theorem 4.5.4 imply that we can choose an étale cover $(U_i)_{i\in I}$ and splittings S_i of $\mathcal{D}_X$ on $U_i \times_{\mathcal{A}^{(1)}} Y^{(1)}$. We obtain isomorphisms

$$U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{Dol}}(X^{(1)}) \cong U_i \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\text{dR}}(X).$$

□

4.7 Stability

Unless stated otherwise, we work with a smooth projective curve X in this section. We will remark on the analogous results for orbicurves at the end of this section.

We investigate the interaction of the local equivalence of moduli stacks in Theorem 4.6.3 with stability. Unmindful choice of a splitting in the proof of this theorem will lead to the degree of the underlying bundles to be scaled and shifted. If the spectral curves has several components these shifts might differ between the components, which could certainly mean that stability is not preserved. Nonetheless, it is possible to single out a connected component $\mathcal{S}^0 \subset \mathcal{S}$ of good splittings, where the degree of the underlying bundles and their Higgs subbundles will only be scaled. First we analyze the case of smooth spectral curves.

Lemma 4.7.1. *Let $a \in \mathcal{A}^{(1)}$ be the characteristic polynomial of a smooth spectral curve $Y_a^{(1)} \to X^{(1)}$. Given a splitting S of $\mathcal{D}_X$ on $Y_a^{(1)}$ the induced isomorphism of Hitchin fibres $\chi^{-1}_{\mathrm{Dol}}(a) \to \chi^{-1}_{\mathrm{dR}}(a)$ sends a degree d Higgs bundle to a degree*

$$pd - (1-p)(1-h)n + \deg S$$

local system, where h denotes the genus of X.

Proof. Given a local system (E, ∇) the first step is to push it forward along the Frobenius morphism $\mathrm{Fr} : X \to X^{(1)}$. This being a finite morphism we obtain $\chi(E) = \chi(\mathrm{Fr}_* E)$, and using Riemann-Roch for both sides we deduce the equality

$$\deg E + n(1-h) = \deg \mathrm{Fr}_* E + pn(1-h). \tag{4.1}$$

In particular we have $\deg \mathrm{Fr}_* E = \deg E + n(1-p)(1-h)$. If $\pi : Y_a^{(1)} \to X^{(1)}$ denotes the finite morphism from the spectral curve to the base curve, we can write $\mathrm{Fr}_* E = \pi_* \mathcal{E} = \pi_*(L \otimes S)$, where L is a line bundle on $Y_a^{(1)}$. Recall that $\mathcal{E}$ denotes the

$\mathcal{D}_X$-module corresponding to E, which we can write as $L \otimes S$, since S is a splitting of $\mathcal{D}_X$ on $Y_a^{(1)}$. Using again the identity $\chi(\pi_*(L \otimes S)) = \chi(L \otimes S)$ and Riemann-Roch, we obtain

$$\deg \mathrm{Fr}_* E + pn(1-h) = p \deg L + \deg S + p(1-g). \tag{4.2}$$

Here we use g to denote the genus of the spectral curve. On the other hand, we compute for the Higgs bundle corresponding to (E, ∇) and S, i.e. for $\pi_* L$ the Euler characteristic

$$\deg \pi_* L + n(1-h) = \chi(\pi_* L) = \chi(L) = \deg L + (1-g). \tag{4.3}$$

From equations (4.2) and (4.3) we conclude that $p \deg \pi_* L + \deg S = \deg \pi_*(L \otimes S) = \deg Fr_* E$. Equation (4.1) implies now that $\deg E = p \deg \pi_* L - (1-p)(1-h)n + \deg S$. $\square$

If we knew it was possible to choose $\deg S = (1-p)(1-h)n$, then we would obtain local isomorphism of moduli stacks which only multiply the degree of underlying bundles by p. But a priori it is not obvious that such a splitting exists. The reason for this is that $\mathcal{S}(\mathcal{D}/C_a)$ is a $\mathcal{P}\mathrm{ic}(C_a)$-torsor. If we tensor S with a line bundle L the degree is $\deg L \otimes S = \deg S + p \deg L$. Therefore S can only attain degrees in a certain congruence class of p.

Definition 4.7.2. *Let $Y_{0,m}^{(1)}$ denote the spectral curve corresponding to the equation $\lambda^m = 0$. Note that we have a closed immersion*

$$Y_{0,1}^{(1)} \hookrightarrow Y_{0,m}^{(1)},$$

given by a nilpotent ideal. From the proof of Theorem 4.5.5 we obtain that every splitting of $\mathcal{D}_X$ on $Y_{0,1}^{(1)}$ extends to a splitting of $\mathcal{D}_X$ on $Y_{0,m}^{(1)}$. We denote by $\mathcal{S}^0(Y^{(1)}/\mathcal{A}^{(1)})$ the connected component of $\mathcal{S}(Y^{(1)}/\mathcal{A}^{(1)})$, which contains any such lifting of the split-

ting on $Y^{(1)}_{0,n}$ corresponding to the trivial local system $(\mathcal{O}_X, d)$.

There exists an alternative characterisation of this open substack.

Lemma 4.7.3. *For a smooth spectral curve $Y^{(1)}_a$ the set of splittings of degree $(1-p)(1-h)n$ is nonempty. The connected component of $\mathcal{S}$ containing the open substack of $\mathcal{S}(Y^{(1)}_{sm}/\mathcal{A}^{(1)}_{\mathrm{sm}})$ given by degree $(1-p)(1-h)n$ splittings on smooth spectral curves, agrees with $\mathcal{S}^0|_{\mathcal{A}_{\mathrm{sm}}}$.*

Proof. Let S be a splitting of $\mathcal{D}_X$ on the nilpotent spectral curve $Y^{(1)}_{0,n}$, as constructed in Definition 4.7.2. According to the smoothness of $\mathcal{S}(Y^{(1)}/\mathcal{A}^{(1)}) \to \mathcal{A}^{(1)}$ (Lemma 4.5.9), we can lift S to a splitting defined over the formal neighbourhood $\operatorname{Spec}\mathcal{O}$ of $0 \in \mathcal{A}^{(1)}$. Since the generic fibre of $Y^{(1)}|_{\operatorname{Spec}\mathcal{O}}$ is a smooth spectral curve, and the degree of a flat family of locally free sheaves is locally constant in a flat family, we obtain from Lemma 4.7.1 that $\mathcal{S}^0(Y^{(1)}/\mathcal{A}^{(1)})$ contains the asserted connected component. Since $\mathcal{S}^0(Y^{(1)}/\mathcal{A}^{(1)})$ is a connected component itself by Definition 4.7.2, we obtain the claim. □

Lemma 4.7.4. *Let $U \to \mathcal{A}^{(1)}$ be an étale morphism and assume that there is a splitting $S \in \mathcal{S}^0(\mathcal{D}/U)$. Then the induced morphism*

$$U \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}(X^{(1)}) \to U \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{dR}}(X)$$

changes the degree of the Higgs bundles, and Higgs subbundles, by multiplication with p. In particular, it preserves the notion of (semi)stability.

Proof. To see that the degree of the underlying bundles is changed by multiplication with p, we may apply Lemma 4.7.3 and Lemma 4.7.1 to conclude this property over the locus of smooth spectral curves $\mathcal{A}^{(1)}_{\mathrm{sm}}$. If $a \in \mathcal{A}^{(1)}$ is arbitrary and S is a splitting of $\mathcal{S}(Y^{(1)}_a)$, then we can lift S to a splitting defined over the formal neighbourhood of a according to Lemma 4.5.7. Since the generic fibre of this formal neighbourhood corresponds to a smooth spectral curve, we obtain from Lemma 4.7.1 and the fact that the degree of a family of locally free sheaves stay locally constant in a flat family, the assertion made above.

The more general case concerning the degrees of Higgs subbundles requires a different analysis. For $n = k + l$ we consider the morphism $\phi : \mathcal{B} := \mathcal{A}_k \times \mathcal{A}_l \to \mathcal{A}_n$ given by polynomial multiplication. This is motivated by the fact that a Higgs subbundle gives rise to a factorization of the characteristic polynomial of the Higgs field. Forming the base change of U we obtain $U' \to \mathcal{B}^{(1)}$ and a splitting S' of $\mathcal{D}$ on the base change of the rank k spectral curve family. We know that the splitting S' induces a transition between local system of rank k and Higgs bundles of rank k, which multiplies the degree with p and shifts it. We may now consider the formal neighbourhood $V \to \mathcal{B}^{(1)}$ of $0 \in \mathcal{B}^{(1)}$, since the component of good splitting $\mathcal{S}^0$ is defined through lifting the rank one splitting corresponding to the tautological local system $(\mathcal{O}_X, d)$ (Definition 4.7.2). In particular we see that S' is also a good splitting for rank k local systems, which implies that stability is preserved. □

A splitting from the substack $\mathcal{S}^0$ multiplies the degree of a Higgs bundle and its subbundles by p, and therefore preserves the notion of semistability.

Corollary 4.7.5. *There exists an algebraic space $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)$, which is a coarse moduli space for the moduli problem of S-equivalence classes of semistable local system.*

Proof. We will first construct an algebraic space $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)$ relative to the base $\mathcal{A}^{(1)}$, using descent theory. Then we check that it verifies the two axioms of a coarse moduli space listed below. Let us fix $n \in \mathbb{N}$ and $d \in \mathbb{Z}$. From now on all the bundles will be of rank n, every Higgs bundle of degree d and every local system of degree pd. According to Theorem 5.10 in [Nit91] a coarse moduli space $\mathcal{M}_{\mathrm{Dol}}^{\mathrm{coarse}}$ of S-equivalences of Higgs bundles exists.

Let $\{U_i \to \mathcal{A}^{(1)}\}_{i \in I}$ be an étale covering together with splittings S_i of $\mathcal{D}_X|_{U_i \times_{\mathcal{A}^{(1)}} Y^{(1)}}$, which lie in $\mathcal{S}^0(U_i)$. According to Lemma 4.7.4 and Theorem 4.6.3, S_i induces an equivalence

$$\phi_i : \chi_{\mathrm{dR}}^{-1}(U_i) \to \chi_{\mathrm{Dol}}^{-1}(U_i),$$

preserving the notion of semistability and scaling the degree by p. For pairs $(i,j) \in I^2$ we thus obtain equivalences

$$\psi_{ij} := \phi_i \circ \phi_j^{-1} : \mathcal{M}_{\mathrm{Dol}}^{\mathrm{coarse}}(X^{(1)}) \times_{\mathcal{A}^{(1)}} U_i \times_{\mathcal{A}^{(1)}} U_j \to \mathcal{M}_{\mathrm{Dol}}^{\mathrm{coarse}}(X^{(1)}) \times_{\mathcal{A}^{(1)}} U_i \times_{\mathcal{A}^{(1)}} U_j.$$

By construction the $\{\psi_{ij}\}$ satisfies the cocycle condition, which allows us to assemble them to a new algebraic space $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)$, together with natural maps

$$\psi_i : \mathcal{M}_{\mathrm{Dol}}^{\mathrm{coarse}}(X^{(1)}) \times_{\mathcal{A}^{(1)}} U_i \to \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X).$$

Let us check next that $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}$ satisfies the two axioms for being an coarse moduli space for S-equivalences of local systems on X.

(1) Let $F : \mathrm{Sch}^{op} \to \mathrm{Set}$ denote the functor sending U to the set of S-equivalence classes of U-families of semistable local systems on X. Then there is a natural transformation $F \to \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)(-)$, which is universal in the sense that for every algebraic space N together with a natural transformation $F \to N$ we have a unique map $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X) \to N$, such that the diagram

$$\begin{array}{ccc} F & & \\ \downarrow & \searrow & \\ \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)(-) & \longrightarrow & N(-) \end{array}$$

commutes.

(2) For every algebraically closed field k', we have an identification of $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)(k')$ with the set of S-equivalence classes of local systems, induced by the above

natural transformation.

Let

$$G : \mathrm{Sch}^{op} \to \mathrm{Set}$$

denote the set-valued functor which sends a scheme U to the set of isomorphism classes of U-families of semistable Higgs bundles on $X^{(1)}$. We have a universal natural transformation

$$G \to \mathcal{M}_{\mathrm{Dol}}^{\mathrm{coarse}}(X^{(1)})(-).$$

The Hitchin morphism induces a map $U \to \mathcal{A}^{(1)}$. We denote by V_i the base change $U \times_{\mathcal{A}^{(1)}} U_i$. The splitting S_i induces an equivalence of sets

$$F(V_i) \cong G(V_i).$$

In order to construct the required natural transformation

$$F \to \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)(-)$$

we have to give a construction, which assigns to every U-family of semistable local systems on X a map

$$U \to \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X).$$

Using the above equivalence of sets and the glueing construction of $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)$, we obtain maps

$$V_i \to \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X),$$

which agree on the fibre products $V_i \times_U V_j$ and therefore give rise to a morphism

$$U \to \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)$$

by descent theory.

Using a similar argument it is easy to establish that this natural transformation verifies the first condition of being a coarse moduli space. The second condition follows from the fact that every geometric point

$$\operatorname{Spec} k' \to \mathcal{A}^{(1)}$$

factors through one of the étale morphisms $U_i \to \mathcal{A}^{(1)}$. In particular S_i induces a splitting of $\mathcal{D}_X$ on the corresponding spectral curve, which gives rise to an identification

$$F(k') \cong G(k'),$$

and therefore verifying the second assertion. □

Corollary 4.7.6. *The deformed Hitchin map* $\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}} \to \mathcal{A}^{(1)}$ *is proper.*

Proof. It has been shown by Nitsure (cf. [Nit91, Thm. 6.1]) that the Hitchin map

$$\chi_{\mathrm{Dol}} : \mathcal{M}_{\mathrm{Dol}}^{\mathrm{ss}}(X^{(1)}) \to \mathcal{A}^{(1)}$$

is proper (see Theorem 2.4.12). From the descent theoretic construction of $\mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X)$, given in the proof of Corollary 4.7.5, we immediately obtain properness for

$$\chi_{\mathrm{dR}} : \mathcal{M}_{\mathrm{dR}}^{\mathrm{coarse}}(X) \to \mathcal{A}^{(1)}\,.$$

□

Corollary 4.7.6 is a generalization of a result of Laszlo and Pauly. In their paper [LP01] they prove that the zero fibre of the deformed Hitchin fibration from the stack of semi-stable t-connections to the Hitchin base is universally closed.

Let now $\widehat{X}$ be an orbicurve, satisfying Assumption 2.6.1. We fix a choice of stability condition, as given by parabolic weights on the corresponding weighted curve (see Definition 2.5.7). The natural map $\widehat{X} \to X$ to a coarse moduli space, will be denoted by τ.

Definition 4.7.2 also applies to orbicurves, and the Riemann-Roch calculations of Lemma 4.7.4 imply that splittings in $\mathcal{S}^0$ preserve the degree of the vector bundle $\tau_* E$ on the coarse moduli space X. The numbers m_{ij} that enter the definition of the parabolic degree 2.5.7, are permuted (which corresponds to changing the type $\lambda_\bullet$ of the parabolic bundle). However, this allows us to state the following local equivalence result.

Theorem 4.7.7. *Let $\widehat{X}$ be a smooth projective weighted curve, and let (α_{ij}) be a given choice of parabolic weights on the associated weighted curve. The moduli stacks of semistable parabolic local systems on $\widehat{X}$ for fixed rank n and degree d, is étale locally equivalent to the moduli stack of semistable parabolic Higgs bundles on $\widehat{X}^{(1)}$ of rank n and degree pd.*

Chapter 5

Higgs bundles and crepant resolutions

5.1 Derived Equivalences

5.1.1 Fourier-Mukai transform

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be algebraic stacks, which we assume to be quasi-compact and having affine diagonal. We denote by $D_{qcoh}(\mathcal{X})$ the unbounded derived category of quasi-coherent sheaves on $\mathcal{X}$, as defined in A.3. Let $\mathcal{X} \to \mathcal{Z}$ and $\mathcal{Y} \to \mathcal{Z}$ be morphisms of stacks and $K \in D_{qcoh}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ a complex on the fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$. If we denote by $p_{\mathcal{X}} : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X}$ the canonical projection, and similarly for $p_{\mathcal{Y}}$, we obtain an exact functor

$$\Phi_K : D_{qcoh}(\mathcal{X}) \to D_{qcoh}(\mathcal{Y}),$$

which sends the complex of sheaves $F \in D_{qcoh}(\mathcal{X})$ to

$$\Phi_K(F) := Rp_{\mathcal{Y},*}(Lp_{\mathcal{X}}^{*}F \otimes^{L} K).$$

Functors between derived categories of this type are referred to as (generalized) Fourier-Mukai transforms and were introduced by Mukai ([Muk87]). The following statement is proved as in [Muk87], but using a slightly more general base change formula (e.g. [BZFN10, Prop. 3.10]). The proof can also be extracted from the proof of Lemma 5.1.4.

Lemma 5.1.1. *Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be $\mathcal{W}$-stacks. We assume that all of these stacks are algebraic, quasi-compact and have affine diagonal; moreover we require that $\mathcal{X} \to \mathcal{W}$, $\mathcal{Y} \to \mathcal{W}$ and $\mathcal{Z} \to \mathcal{W}$ are representable flat morphisms. For $L \in D_{qcoh}(\mathcal{X} \times_{\mathcal{W}} \mathcal{Y})$ and $K \in D_{qcoh}(\mathcal{Y} \times_{\mathcal{W}} \mathcal{Z})$ we define*

$$L * K := Rp_{\mathcal{X}\,\mathcal{Z},*}(Lp^*_{\mathcal{X}\,\mathcal{Y}} L \otimes^L Lp^*_{\mathcal{Y}\,\mathcal{Z}} K).$$

There exists a natural equivalence between the functors $\Phi_K \circ \Phi_L$ and Φ_{L*K}.

As we are mainly dealing with generalized Fourier-Mukai functors, i.e. integral kernels living on a fibre product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, we have to investigate how the kernel changes if we replace the base $\mathcal{Z}$ along a morphism $\mathcal{Z} \to \mathcal{W}$. The behaviour of integral kernels under this change of base stack is expressed in the well-known lemma below, which is proved by application of the projection formula (e.g. [BZFN10, Prop. 3.10]).

Lemma 5.1.2. *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ be algebraic stacks which are quasi-compact and have affine diagonal. We assume that $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{Z}$-stacks, and that there is a schematic morphism $\mathcal{Z} \to \mathcal{W}$. Let*

$$f := (p_{\mathcal{X}}, p_{\mathcal{Y}}) : \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \to \mathcal{X} \times_{\mathcal{W}} \mathcal{Y}$$

be the canonical morphism, and $K \in D_{qcoh}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$. Then the Fourier-Mukai transform Φ_K is naturally equivalent to Φ_{Rf_*K}.

We will also have to understand the behaviour of Fourier-Mukai transform with respect to base change.

Lemma 5.1.3. *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\mathcal{W}$ be perfect algebraic stacks which are quasi-compact and have affine diagonal. We assume that $\mathcal{X}$ and $\mathcal{Y}$ are flat $\mathcal{Z}$-stacks, and that there is a schematic morphism $\pi : \mathcal{W} \to \mathcal{Z}$. Every Fourier-Mukai equivalence*

$$\Phi_K : D_{qcoh}(\mathcal{X}) \cong D_{qcoh}(\mathcal{Y})$$

relative to $\mathcal{Z}$ induces a Fourier-Mukai equivalence relative to $\mathcal{W}$

$$\Phi_{\pi^* K} : D_{qcoh}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{W}) \cong D_{qcoh}(\mathcal{Y} \times_{\mathcal{Z}} \mathcal{W})$$

by pulling back the kernel K.

Proof. Let $L \in D_{qcoh}(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})$ so that $K * L \cong \Delta_* \mathcal{O}_{\mathcal{X}}$ and $L * K \cong \Delta_* \mathcal{O}_{\mathcal{Y}}$. The base change formula implies that the same relations hold for $\mathcal{X} \times_{\mathcal{Z}} \mathcal{W}$ and $\mathcal{Y} \times_{\mathcal{Z}} \mathcal{W}$. □

The next lemma tells us that if X_i and Y_i are Fourier-Mukai partners for $i = 1, 2$, then then products $X_1 \times X_2$ and $Y_1 \times Y_2$ are Fourier-Mukai partners. Using the formalism of stable ∞-categories, this is simply a consequence of Theorem 1.2 in [BZFN10].

Lemma 5.1.4. *For $i = 1, 2$ let $\mathcal{X}_i$, $\mathcal{Y}_i$, $\mathcal{Z}_i$ and $\mathcal{W}$ be perfect algebraic stacks which are quasi-compact and have affine diagonal. We assume that $\mathcal{X}_i$, $\mathcal{Y}_i$, $\mathcal{Z}_i$ are $\mathcal{W}$-stacks and that the structural morphisms are flat. Let $D_{qcoh}(\mathcal{X}_i) \cong D_{qcoh}(\mathcal{Y}_i)$ be derived equivalences of Fourier-Mukai type, induced by integral kernels $K_i \in D_{qcoh}(\mathcal{X}_i \times_{\mathcal{Z}_i} \mathcal{Y}_i)$. Then $K_1 \boxtimes^L K_2$ induces a derived equivalence*

$$D_{qcoh}(\mathcal{X}_1 \times_{\mathcal{W}} \mathcal{X}_2) \cong D_{qcoh}(\mathcal{Y}_1 \times_{\mathcal{W}} \mathcal{Y}_2),$$

relative to $\mathcal{Z}_1 \times_{\mathcal{W}} \mathcal{Z}_2$.

Proof. According to Lemma 5.1.3 we know that the equivalences $D_{qcoh}(\mathcal{X}_i) \cong D_{qcoh}(\mathcal{Y}_i)$ induce equivalences

$$D_{qcoh}(\mathcal{X}_1 \times_{\mathcal{W}} \mathcal{Y}_1) \cong D_{qcoh}(\mathcal{X}_2 \times_{\mathcal{W}} \mathcal{Y}_1)$$

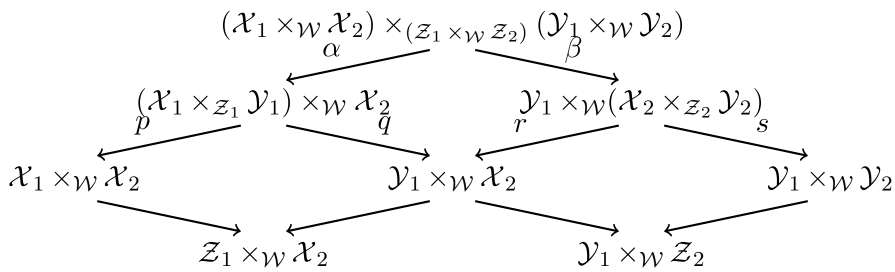
and

$$D_{qcoh}(\mathcal{X}_2 \times_{\mathcal{W}} \mathcal{Y}_1) \cong D_{qcoh}(X_2 \times_{\mathcal{W}} Y_2).$$

By juxtaposition we obtain a derived equivalence

$$D_{qcoh}(\mathcal{X}_1 \times_{\mathcal{W}} \mathcal{Y}_1) \cong D_{qcoh}(\mathcal{X}_2 \times_{\mathcal{W}} \mathcal{Y}_2).$$

In order to obtain a better understanding of the integral kernel of this composition we take a look at the following commutative diagram with cartesian squares:



Let $M \in D_{qcoh}(\mathcal{X}_1 \times \mathcal{Y}_1)$, we denote by

$$c : (\mathcal{X}_1 \times_{\mathcal{Z}_1} \mathcal{Y}_1) \times \mathcal{X}_2 \to \mathcal{X}_1 \times_{\mathcal{Z}_1} \mathcal{Y}_1$$

and

$$d : \mathcal{Y}_1 \times (\mathcal{X}_2 \times_{\mathcal{Z}_2} \mathcal{Y}_2) \to \mathcal{X}_2 \times_{\mathcal{Z}_2} \mathcal{Y}_2$$

the canonical projection; the base change formula reveals now that

$$Rs_*((Lr^* Rq_*(Lp^* \otimes^L Lc^* K_1)) \otimes^L Ld^* K_2) \cong Rs_*(R\beta_*(L\alpha^* Lp^* M \otimes^L K_1) \otimes^L Ld^* K_2).$$

Using the projection formula we obtain

$$Rs_*R\beta_*(L\alpha^*Lp^*M \otimes L\alpha^*Lc^*K_1 \otimes^L L\beta^*Ld^*K_2).$$

In particular we see that the integral kernel is given by $K_1 \boxtimes^L K_2$. □

The lemma below is reminiscent from Lemma A.3.3, but is stated in a more classical context.

Lemma 5.1.5. *Let X, Y and Z be quasi-projective smooth k-varieties, proper and flat over Z, endowed with the action of an abstract finite group Γ, such that the characteristic of k does not divide Γ; we assume that there is a functor of Fourier-Mukai type*

$$\Phi_K : D^b_{coh}(X) \to D^b_{coh}(Y),$$

given by an integral kernel $K \in D_{qcoh}(X \times_Z Y)$ of finite Tor-dimension. Moreover we assume that K is endowed with a Γ-equivariant structure; in the sense that $K \cong f^*L$ for

$$L \in D^b_{coh}([X/\Gamma] \times_{[Z/\Gamma]} [Y/\Gamma]),$$

and the obvious map

$$f : X \times_Z Y \to [X/\Gamma] \times_{[Z/\Gamma]} [Y/\Gamma].$$

Then Φ_K is an equivalences of categories if and only if Φ_L is.

We emphasize that every functor of Fourier-Mukai type $D_{qcoh}(\mathcal{X}) \to D_{qcoh}(\mathcal{Y})$ lifts to an ∞-functor $QC(\mathcal{X}) \to QC(\mathcal{Y})$. And for a large class of stacks every functor between ∞-categories $QC(\mathcal{X}) \to QC(\mathcal{Y})$ is obtained from a Fourier-Mukai transform ([BZFN10, Thm. 1.2(2)]). For our purposes it is therefore merely a matter of taste whether one utilizes the theory of stable ∞-categories or derived categories and functors of Fourier-Mukai type.

5.1.2 The McKay correspondence

In the paper [BKR] an important special case of the Fourier-Mukai transform has been considered to establish a form of the derived McKay correspondence. We denote by X a smooth quasi-projective variety with an abstract finite group Γ acting on it. Moreover we assume that the characteristic of k is zero or $p > |\Gamma|$. Let us denote by $Y \subset X^{[|\Gamma|]}$ the subscheme of the Hilbert scheme, representing the functor given by Γ-equivariant subschemes

$$Z \to X,$$

such that there exists a surjection of Γ-equivariant sheaves $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z$ and Γ acts on $H^0(Z, \mathcal{O}_Z)$ as the regular representation. Moreover we remove redundant irreducible components, so that we are left with the irreducible component containing the free Γ-orbits.

The universal Γ-cluster living on the fibre product $Y \times [X/\Gamma]$ will be denote by $\mathcal{Z}$.

Theorem 5.1.6 ([BKR, Thm. 1.1])**.** *We assume that the Γ-Hilbert scheme Y of X is smooth and satisfies the estimate*

$$\dim Y \times_{X/\Gamma} Y \leq \dim Y + 1,$$

where X/Γ denotes the GIT quotient. Then the structure sheaf of the universal family $\mathcal{O}_{\mathcal{Z}}$ of Γ-clusters *on*

$$Y \times [X/\Gamma]$$

induces an equivalence of k-linear derived categories of Fourier-Mukai type

$$D^b_{coh}(Y) \cong D^b_{coh}([X/\Gamma]).$$

In [BKR] a slightly more general Theorem is proved for $k = \mathbb{C}$. The reason for this restriction is the use of the so-called *New Intersection Theorem* due to Roberts ([Rob89]) and Peskine–Szpiro, which guarantees smoothness of Y. While this Theorem holds in positive characteristic, [BKR] uses an addendum proved in [BI] in the right generality. Nonetheless, in cases of interest to us, Y will be already known to be smooth for different reasons.

In the next lemma we observe how this Fourier-Mukai transform interacts with a given morphism $X \to S$. This will be important for our analysis of autoduality of the Hitchin fibration in Theorem 5.2.9 and 5.3.4.

Lemma 5.1.7. *Let $\pi : X \to S$ be a flat morphism of smooth quasi-projective varieties, endowed with the action of a finite group Γ, such that π is Γ-equivariant. If X satisfies the conditions of Theorem 5.1.6, Y denotes the Γ-equivariant Hilbert scheme as before and S/Γ the GIT quotient, then the natural equivalence of derived categories*

$$D^b_{coh}(Y) \cong D^b_{coh}([X/\Gamma])$$

is of Fourier-Mukai type relative to S/Γ.

Proof. We only need to check that $\mathcal{O}_{\mathcal{Z}}$ is supported on the fibre product

$$Y \times_{S/\Gamma} [X/\Gamma],$$

which one expects to be a consequence of the Γ-equivariance of π. To show this we may cover S/Γ by Zariski open affine subsets U_i and cover S by the fibre products

$$S_i := U_i \times_{S/\Gamma} S,$$

which are still affine, as $S \to S/\Gamma$ is finite. Using quasi-projectivity of X we can cover $X \times_S S_i \subset X$ by Zariski open affine subsets V_i, which are Γ-invariant (as before by taking preimages of affine open subsets of the GIT quotient). Henceforth we may assume without loss of generality that $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} D$ are affine varieties endowed with the action of an abstract group Γ.

Let now C be another algebra endowed with the trivial Γ-action and B a C-flat quotient of $A \otimes C$ sitting in a short exact sequence

$$0 \longrightarrow I \longrightarrow A \otimes C \longrightarrow B \longrightarrow 0$$

such that I is a Γ-invariant ideal of $A \otimes C$. In particular this is a short exact sequence of Γ-modules. Moreover we assume that $C \to A \otimes C \to B$ induces an isomorphism

$$C \cong B^{\Gamma}.$$

Every C-point of the Γ-Hilbert scheme of X gives rise to this data.

Since A is a D-algebra we obtain a natural morphism

$$D^{\Gamma} \to B^{\Gamma} = C,$$

endowing C with the structure of a D^{Γ}-algebra. By construction, the $A \otimes C$-module B the action of D^{Γ} via A agrees with the action via C. Thus B is actually a $A \otimes_{D^{\Gamma}} C$-module, which is what we wanted to show. □

An important example is given by Hilbert schemes of surfaces. To see how those relate to equivariant Hilbert schemes we quote the following result of M. Haiman ([Hai01, Thm. 6]), which is a corollary of his proof of the $n!$-conjecture.

Remark 5.1.8. *If H_n denotes the Hilbert scheme of length n points on $\mathbb{A}^2$, Haiman's result shows that a certain natural morphism $X_n \to H_n$ is finite and flat. The scheme X_n is referred to as the* isospectral Hilbert scheme, *and we may also state the aforementioned result of Haiman as sayiong that X_n is Cohen-Macaulay. Again the original source states the relevant theorem only in the case of characteristic zero, but the proof* *is easily adapted to $p > n$. The only characteristic-sensitive part of Haiman's proof is the use of Maschke's theorem for the symmetric group S_n, which is true as long as $p > n$ ([Lan02, Thm. XVIII.1.2]). The combinatorial backbone of Haiman's work, the Polygraph Theorem, has already been proved over $\mathbb{Z}$ in the original publication.*

Theorem 5.1.9 (Haiman)**.** *Let X be a surface defined over a field of characteristic $p > n$ or zero. Let us denote by $X^{[n]}$ the Hilbert scheme of length n subschemes and by Y_n the S_n-Hilbert scheme of X^n with respect to the natural group action of the symmetric group S_n given by permuting factors. Then there is a natural isomorphism*

$$Y_n \cong X^{[n]}.$$

Combining this result with Theorem 5.1.6 we obtain a well-known derived equivalence.

Corollary 5.1.10. *If X denotes a surface defined over a field of characteristic $p > n$ or zero, and $X^{[n]}$ denotes the Hilbert scheme of length n subschemes, then we have a natural derived equivalence*

$$D^b_{coh}(X^{[n]}) \cong D^b_{coh}([X^n/S_n]).$$

Note that the required dimension estimate follows from the classical result of Briançon ([Bri77]) and Iarrabino (Corollary 1 in [Iar72]) that for the punctual Hilbert scheme $(\mathbb{A}^2)^{[m]}_0$ we have

$$\dim(\mathbb{A}^2)^{[m]}_0 = m - 1.$$

5.1.3 Duality for elliptic curves with symmetries

Let A be an abelian variety, the dual abelian variety $A^\vee$ is equivalent to the stack $Map_{grp}(A, B\,\mathbb{G}_m)$ representing morphisms of group stacks $A \to B\,\mathbb{G}_m$ ([Ser88, p.

184]). Equivalently, we can say that $A^\vee$ classifies extensions of A by $\mathbb{G}_m$. This construction is analogous to the dual of a vector space $V^\vee := Hom(V,k)$, with $B\,\mathbb{G}_m$ taking the place of the one-dimensional vector space k. For the same reason as there is a canonical morphism $V \to V^{\vee\vee}$ for vector spaces there is a a canonical morphism

$$\psi_A : A \to A^{\vee\vee},$$

which is an isomorphism. This in turn gives rise to a morphism

$$A \times A^\vee \to B\,\mathbb{G}_m\,.$$

As the stack $B\,\mathbb{G}_m$ classifies line bundles, we see that there is a canonical line bundle $\mathcal{P}$ on $A \times A^\vee$, called *Poincaré bundle*. There is a general duality theory for group stacks, an exposition of which is given in Section 6.1 and [Ari08].

It has been shown by Mukai in [Muk87] that the Poincaré line bundle $\mathcal{P}$ induces a natural equivalence of categories

$$D^b_{coh}(A) \cong D^b_{coh}(A^\vee). \tag{5.1}$$

If $\phi : A \to B$ is a morphism we obtain a dual morphism $\phi^\vee : B^\vee \to A^\vee$ which sends an S-point $f : B \times S \to B\,\mathbb{G}_m \times S$ of $B^\vee$ to the composition

$$A \times S \xrightarrow{\phi \times \mathrm{id}_S} B \times S \xrightarrow{f} B\,\mathbb{G}_m \times S.$$

By definition, the diagram

$$\begin{array}{ccc} A & \longrightarrow & A^{\vee\vee} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi^{\vee\vee}} \\ B & \longrightarrow & B^{\vee\vee} \end{array}$$

is commutative. In particular, we conclude that if Γ is a finite group acting on A then

Mukai's equivalence (5.1) is Γ-equivariant; in the strong sense that the integral kernel $\mathcal{P}$ is endowed with a Γ-equivariant structure.

If E denotes an elliptic curve we may identify

$$T^*E = E \times \mathcal{A} = \mathcal{M}_{\mathrm{Dol}}(E^\vee, 1),$$

where $\mathcal{A}$ denotes the Hitchin base. Note that there is a canonical identification of elliptic curves $E \cong E^\vee$, given by the Abel-Jacobi map. This induces an identification of Hitchin bases

$$\mathcal{A}(E) = \mathcal{A}(E^\vee).$$

Using this autoequivalence, the remarks above, Lemma 5.1.2 and Lemma 5.1.5 we arrive at the following well-known observation.

Proposition 5.1.11. *There is a canonical equivalence of derived categories of Fourier-Mukai type relative to* $\mathcal{A}$

$$D^b_{coh}(T^*E) \cong D^b_{coh}(T^*E^\vee).$$

If E is equipped with a Γ-action this equivalence respects the Γ-action, in particular we have an equivalence of derived categories of Fourier-Mukai type relative to $\mathcal{A}/\Gamma$

$$D^b_{coh}([T^*E/\Gamma]) \cong D^b_{coh}([T^*E^\vee/\Gamma]).$$

Using the categorification of geometric class field theory obtained in [Lau] and [Rot96] one can prove analogous results for categories of D-modules on an elliptic curve. In the following we denote by $D_{qcoh}(X, D_X)$ the derived category of quasi-coherent D_X-modules on a smooth variety X.

Theorem 5.1.12 (Laumon & Rothstein)**.** *If A is an abelian variety defined over an algebraically closed field of characteristic zero we denote by $A^\sharp$ the moduli space of* *rank one local systems on A. Then there exists a canonical equivalence of derived categories*

$$\Phi_{CFT} : D_{qcoh}(A^\sharp) \cong D_{qcoh}(A^\vee, D_{A^\vee}).$$

The analogue of the above Theorem in positive characteristic is proved in [CZ12, Cor. 3.8], using the techniques surveyed in chapter 3.2.

Theorem 5.1.13 (Chen–Zhu)**.** *If A is an abelian variety defined over an algebraically closed field of positive characteristic we denote by $A^\sharp$ the moduli space of rank one local systems on A. Then there is a canonical equivalence of derived categories*

$$D_{qcoh}(A^\sharp) \cong D_{qcoh}(A^\vee, D_{A^\vee}).$$

In this thesis we will only be interested in the case where $A = E$ is an elliptic curve. This special case is also covered by [BB07, Thm. 4.10(2)].

5.2 Two-dimensional moduli spaces

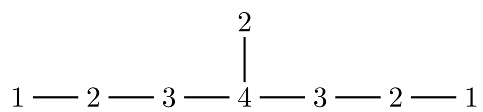
5.2.1 Higgs bundles and crepant resolutions

Let Q be a Dynkin diagram, such that the corresponding affine Dynkin diagram $\tilde{Q}$ is comet-shaped (see section 2.5). The only Dynkin diagrams satisfying this assumption are A_0, D_4, E_6, E_7 and E_8. In Definition 2.5.8 we explained how comet-shaped graphs together with a dimension vector encode moduli problems for parabolic (Higgs) bundles. The graphs $\tilde{Q}$ listed above together with the basic imaginary root $\lambda_\bullet$, are exactly the ones corresponding to the moduli spaces of parabolic Higgs bundles of dimension 2 which we will study in this section.

The A_0-case is the simplest one, corresponding to Higgs bundles of rank one on an elliptic curve E. In this case, the moduli space is the cotangent bundle T^*E. Nonetheless there are many other examples of two-dimensional moduli spaces of Higgs bundles that are somehow reminiscent of this one. To each such moduli space of parabolic Higgs bundles we can associate a finite group Γ and a graph. For D_4, E_6, E_7 and E_8 these groups are $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. Let now E be an appropriate elliptic curve with a Γ-action. In the D_4-case, E is an arbitrary elliptic curve with $\mathbb{Z}/2\mathbb{Z}$ acting on it via $x \mapsto -x$. In all the other cases, the Γ-action stems from complex multiplication on the curve E. In the $\widetilde{E}_7$-case we pick an elliptic curve E with an automorphism of order 4. This is an elliptic curve with a special form of complex multiplication, which corresponds to the lattice of Gaussian integers $\mathbb{Z}[i] \subset \mathbb{C}$. According to [Sil94, p. 483] this elliptic curve corresponds to the equation

$$y^2 = x^3 + x,$$

and the $\mathbb{Z}/4\mathbb{Z}$-action is generated by $(x, y) \mapsto (-x, iy)$. Taking the scheme-theoretic quotient of E as in the example above, we obtain $\mathbb{P}^1$ with three marked points. And it turns out that the Γ-Hilbert scheme of T^*E is a moduli space for rank 4 orbifold degree 0 parabolic Higgs bundles on this marked rational curve.



For the graphs $\widetilde{E}_6$ and $\widetilde{E}_8$ we proceed analogously with an elliptic curve E with a non-trivial $\mathbb{Z}/3\mathbb{Z}$- and $\mathbb{Z}/6\mathbb{Z}$-action respectively. Over the field of complex numbers such a curve is given by the lattice of Eisenstein integers $\mathbb{Z}[\omega] \subset \mathbb{C}$, where ω is a primitive third root of unity. According to [Sil94, p. 483] an explicit equation for this curve is given by

$$y^2 + y = x^3.$$

The $\mathbb{Z}/6\mathbb{Z}$-action is generated by $\mathbb{Z}/3\mathbb{Z}$-action given by $\xi \cdot (x, y) = (\xi x, y)$ for every third root of unity ξ, and the $\mathbb{Z}/2\mathbb{Z}$-action induced by the inverse map of E. The corresponding parabolic bundles are of rank 3, respectively rank 6 and orbifold degree 0. The respective parabolic structure is encoded in the diagrams:

This allows us to formulate the following folklore theorem which will be proven below.

Theorem 5.2.1. *We consider the weighted curve associated to the orbifold* $[E/\Gamma]$*, for the marked points away from* $0 \in E$ *we consider canonical weights. At the marked point corresponding to* $0 \in E$ *we work with the weights* $\alpha_i := \frac{i}{r}$ *for* $i < r-1$ *and* $1 > \alpha_{r-1} > \frac{r-1}{r}$*, where* $r := |\Gamma|$*. The moduli space of stable parabolic Higgs bundles* $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(Q, \lambda)$ *of orbifold degree zero with respect to these weights is naturally isomorphic to the* Γ*-Hilbert scheme of the surface* T^*E*.*

A formula for the dimension of $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widehat{X}, n, \lambda_\bullet)$ is given in [BY96, p. 3], assuming the moduli space is non-empty:

$$2(g-1)n^2 + 2 + \sum_{p \in D}(n^2 - \sum_{i=1}^{n_p}(\lambda_{pi+1} - \lambda_{pi})^2). \tag{5.2}$$

We have the estimate

$$\sum_{i=1}^{n_p}(\lambda_{pi+1} - \lambda_{pi})^2 \leq n^2$$

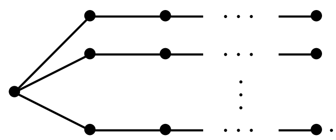
which follows from the inequality

$$\sum_{i=1}^{n} x_i^2 \leq (\sum_{i=1}^{n} x_i)^2, \tag{5.3}$$

where $x_i \geq 0$. In particular we see that there are two possible cases, where the expression (5.2) specializes to 2. If $g = 1$ and $D = 0$, since the inequality (5.3) is strict if there are two non-zero summands; and $g = 0$ and

$$-2n^2 + \sum_{p \in D} (n^2 - \sum_{i=1}^{n_p - 1} (\lambda_{pi+1} - \lambda_{pi})^2) = 0.$$

This expression on the other hand is $-2q$, where q denotes the quadratic form associated to the star-shaped graph



If Q is of affine Dynkin type, we see in particular that all such dimension vectors are multiples of the basic imaginary root α. Next we give a technical definition, which will be essential for relating the two kinds of moduli spaces with each other.

Definition 5.2.2. *Let X be an orbicurve and (E, θ) a Higgs bundle on it. A composition series for (E, θ) is an increasing filtration by Higgs subbundles*

$$(E^\bullet, \theta) \subset (E, \theta),$$

such that the successive quotients E^{i+1}/E^i, called factors, are locally free and have no non-trivial Higgs bundle as a quotient. The Higgs bundle (E, θ) is said to be admissible *if there exists a composition series, such that all factors are of rank one and degree zero. An S-family of Higgs bundles is called admissible if it is admissible over every geometric point of S. We denote the stack of rank n admissible Higgs bundles on an orbicurve X by $\mathcal{M}^{\mathrm{ad}}_{\mathrm{Dol}}(X, n)$.*

It is a well-known fact (Lemma 4.2(1) in [Rei08b]) that an extension of semistable objects of the same slope is again semistable. We record the following implication for admissible Higgs bundles for later use.

Remark 5.2.3. *Admissible Higgs bundles are semistable of slope zero.*

A priori it is not clear wether this stack is algebraic. We will see later that admissible Higgs bundles on the cotangent bundle of an elliptic curve correspond to torsion sheaves supported on the cotangent bundle of the dual elliptic curve. This also proves algebraicity in this particular case.

Definition 5.2.4. *For an orbisurface S we denote by $\mathcal{T}(S, n)$ the stack of length n torsion sheaves on S, i.e. the 2-functor $\mathrm{Aff}^{op} \to Grpd$ which sends an affine scheme T to the groupoid of quasi-coherent sheaves $\mathcal{F}$ on $S \times T$, such that $\pi : \mathrm{supp}\, \mathcal{F} \to T$ is finite and $\pi_* \mathcal{F}$ is locally free of rank n on S.*

In the next lemma we formulate how admissible Higgs bundles are related to torsion sheaves.

Lemma 5.2.5. *The equivalence $D^b_{coh}(T^*E) \cong D^b_{coh}(T^*E^\vee)$ (Proposition 5.1.11) gives rise to an equivalence of stacks $\mathcal{T}(T^*E, n) \cong \mathcal{M}^{\mathrm{ad}}_{\mathrm{Dol}}(E^\vee, n)$.*

Proof. Given a T-point $\mathcal{F}$ of $\mathcal{T}(T^*E, n)$ we have to verify that the Fourier-Mukai transform $\Phi(\mathcal{F})$ on $T^*E^\vee$ is a T-family of quasi-coherent sheaves on $T^*E^\vee$, corresponding to a T-family Higgs bundle on $E^\vee$ via the BNR correspondence (Theorem 2.3.2). This formulation is justified, as we know from Lemma 5.1.3 that for every k-scheme T there is an induced Fourier-Mukai transform

$$\Phi : D^b_{coh}(T^*E \times T) \cong D^b_{coh}(T^*E^\vee \times T).$$

If $\pi : T^*E^\vee \to E^\vee$ denotes the canonical projection, we need to verify that $\pi_*\Phi(\mathcal{F})$ is a locally free sheaf of rank n. This push-forward can be calculated as the Fourier-Mukai transform of $\mathcal{F}$ along the functor

$$\Psi : D^b_{coh}(T^*E \times T) \to D^b_{coh}(E^\vee \times T)$$

induced by the Poincaré bundle $\mathcal{P}$ on $E \times E^{\vee}$. Let $p_1 : T^*E \times E^{\vee} \to T^*E$ and $p_2 : T^*E \times E^{\vee} \to E^{\vee}$ denote the canonical projections onto the factors, respectively their base changes with respect to T. Then we have

$$\pi_* \Phi(\mathcal{F}) = \Psi(F) = p_{2,*}(p_1^* \mathcal{F} \otimes \mathcal{P}).$$

But since $\operatorname{supp} \mathcal{F} \to T$ is finite and $\mathcal{P}$ is a line bundle, we see that this is a locally free sheaf of rank n on $E^{\vee}$. A similar Fourier-Mukai set-up was used in [GNR01] to define the vector bundle underlying the Higgs bundle constructed from a torsion sheaf.

We also need to check that the Fourier-Mukai transform $\Phi(\mathcal{F})$ is a family of admissible Higgs bundles on $[E/\Gamma]$. For this we may replace S by a geometric point and therefore assume that $\mathcal{F}$ has a composition series $\mathcal{F}^\bullet$, such that the successive quotients $\mathcal{F}^{i+1} / \mathcal{F}^i$ are skyscraper sheaves of length one. This composition series can be encoded in a sequence of distinguished triangles

$$\mathcal{F}^i \longrightarrow \mathcal{F}^{i+1} \longrightarrow \mathcal{F}^{i+1} / \mathcal{F}^i \overset{\bullet}{\longrightarrow}.$$

Applying the equivalence Φ to $\mathcal{F}$ we see that $\Phi(\mathcal{F})$ may be filtered by distinguished triangles

$$\Phi(\mathcal{F}^i) \to \Phi(\mathcal{F}^{i+1}) \to \Phi(\mathcal{F}^{i+1} / \mathcal{F}^i) \overset{\bullet}{\to}.$$

By assumption $\Phi(\mathcal{F}^{i+1} / \mathcal{F}^i)$ is a quasi-coherent sheaf $T^*E^{\vee}$, corresponding to a rank one degree zero Higgs bundle on $E^{\vee}$ via the BNR correspondence (Theorem 2.3.2). By induction on n we obtain that $\Phi(\mathcal{F})$ corresponds to an admissible Higgs bundle.

Similarly, we see that an admissible Higgs bundle of rank n on $E^{\vee}$ is sent to a length n torsion free sheaf on T^*E. □

We have found a way of relating torsion sheaves on the surface T^*E to Higgs bundles on the dual elliptic curve $E^{\vee}$. As a next step we investigate the transform of a point of the Γ-Hilbert scheme Y of T^*E. Such a point gives rise to a Γ-equivariant torsion sheaf $\mathcal{F}$ on T^*E together with a Γ-equivariant surjection $s : \mathcal{O}_{T^*E} \twoheadrightarrow \mathcal{F}$. As a first approximation we expect to obtain a Γ-equivariant Higgs bundle of rank $|\Gamma|$ on $E^\vee$, due to the functoriality of the construction described above. In the proof below we investigate the structure corresponding to the surjection s.

Proof of Theorem 5.2.1. The Γ-Hilbert scheme Y of T^*E can be defined in terms of $\mathcal{T} = \mathcal{T}(T^*[E/\Gamma])$. An S-point of Y consists of an S-point $\mathcal{F}$ of $\mathcal{T}$ together with a surjection

$$s : \mathcal{O}_{[T^*E/\Gamma]\times S} \twoheadrightarrow \mathcal{F}\,.$$

Moreover we demand that the Γ-representation

$$Hom(\mathcal{O}_{[T^*E/\Gamma]\times S}, \mathcal{F})$$

is the regular S-linear representation of Γ. We can now try to understand how s transforms under the equivalence of categories Φ.

Let us denote by T^*_0E the closed subscheme of T^*E given by the fibre over zero of $T^*E \to E$. The equivalence $D^b_{coh}(T^*E) \cong D^b_{coh}(T^*E^\vee)$ sends $\mathcal{O}_{T^*E}$ to $\mathcal{O}_{T^*_0E^\vee}[-1]$. In particular we see that $v := \Phi(s)$ is a morphism

$$v : \mathcal{O}_{T^*_0E^\vee}[-1] \to \Phi(\mathcal{F}).$$

Serre duality tells us that this is equivalent to a morphism

$$v' : (\Phi(\mathcal{F})_0)^\vee \to k,$$

i.e. an element $v \in \Phi(\mathcal{F})_0$. Under this equivalence a morphism $\mathcal{O}_{\mathcal{M}} \to \mathcal{F}$ corresponds to a linear map $\tau_*(V^\vee \otimes \mathcal{O}\,/L_0^{-1}) \to k$, where (V, θ) is the Γ-equivariant Higgs bundle associated to $\mathcal{F} \in \mathcal{T}$. But

$$\tau_*(V^\vee \otimes \mathcal{O}/L_0^{-1}) \cong E_0^\vee / F_1^\vee.$$

In particular we obtain a non-trivial linear map

$$k \to F_{n_0-1}$$

by dualizing, i.e. a nonzero vector $v \in F_{n_0-1}$.

The vector space $Hom(\mathcal{O}_{[T^*E/\Gamma]\times S}, \mathcal{F})$ corresponds to $\Phi(\mathcal{F})_0$, as the argument given above tell us. In particular we see that Y is equivalent to the moduli stack of the data

$$(\mathcal{E}, \theta, v),$$

where $(\mathcal{E}, \theta)$ is an admissible Higgs bundle on $[T^*E/\Gamma]$, $\mathcal{E}_0$ is the regular Γ-representation and $v \in \mathcal{E}_0^\Gamma$ is a non-zero vector spanning the invariant part of $\mathcal{E}_0$. The latter is naturally equivalent to the moduli space of admissible Higgs bundles on $[T^*E/\Gamma]$, such that $\mathcal{E}_0$ carries the regular representation. Now we may apply Lemma 2.6.8 to see that this corresponds exactly to the required type of parabolic bundles.

Stability of the parabolic Higgs bundles follows from the fact that all Γ-invariant subbundles are of orbifold slope ≤ 0 (see Remark 5.2.3) and that $\Phi(\mathcal{F})$ is the only degree zero subbundle containing v. Since the weights are the canonical weights except from α_{n-1}, stability follows.

Note that Y is naturally a $\mathcal{A}/\Gamma$-space with respect to the structural morphism

$$Y \to T^*E/\Gamma \to \mathcal{A}/\Gamma.$$

Here the first morphism is the Hilbert-Chow morphism, which sends a subscheme to it's weighted support. The fact that $\Phi : D^b_{coh}([T^*E/\Gamma]) \cong D^b_{coh}([T^*E^\vee/\Gamma]$ is defined relative to $\mathcal{A}/\Gamma$ implies that the morphism $Y \to \mathcal{M}([\widehat{E^\vee/\Gamma}], Q, \lambda)$ is a morphism of $\mathcal{A}/\Gamma$-spaces. We observe as well that this map is proper.

Therefore we have a morphism

$$Y \to \mathcal{M}([\widehat{E^\vee/\Gamma}], Q, \lambda)$$

of proper $\mathcal{A}/\Gamma$-spaces. Since both spaces are of equal dimension and connected we conclude that it is surjective. In particular we obtain that every Higgs bundle in the moduli space $\mathcal{M}([\widehat{E^\vee/\Gamma}], Q, \lambda)$ is admissible. This allows us to conclude that

$$Y \cong \mathcal{M}([\widehat{E^\vee/\Gamma}], Q, \lambda),$$

with the inverse map provided by the inverse of the Fourier-Mukai transform Φ^{-1}. □

5.2.2 Local systems and crepant resolutions

For an abelian variety A, we define $A^\sharp$ to be the moduli space of rank one degree zero local systems on A.

As before we start by relating torsion sheaves on the surface $\mathcal{M}_{\mathrm{dR}}(E, 1) = E^\sharp$ with local systems on $E^\vee$. Although the next Proposition is completely analogous to Lemma 5.2.5, it is more powerful, since every local system defined over an algebraically closed field of characteristic zero is admissible due to the fact that every vector bundle on a curve supporting an algebraic connection has degree zero.

Proposition 5.2.6. *Let k be an algebraically closed field of characteristic zero. The equivalence Φ_{CFT} of Theorem 5.1.12 (respectively 5.1.13) induces an equivalence of stacks*

$$\mathcal{T}(E^\sharp, n) \cong \mathcal{M}_{\mathrm{dR}}(E^\vee, n),$$

relating length n torsion sheaves on the surface $E^\sharp$ to rank n local systems on $E^\vee$.

Proof. Let us denote by S an affine scheme, X an arbitrary smooth scheme and by $D_{qcoh}(X \times S, p_X^* D_X)$ the derived category of $p_X^* D_X$-modules, where $p_X : X \times S \to X$ is the canonical projection. Objects of this category should be thought of as S-families of complexes of D_X-modules. It is clear that we also have an equivalence of derived categories

$$D_{qcoh}(A^\sharp \times S) \cong D_{qcoh}(A^\vee \times S, p_{A^\vee}^* D_{A^\vee}),$$

as it follows for instance from Proposition 4.1 in [BZFN10] and the fact that the above equivalence of Laumon and Rothstein can be lifted to the canonical enhancements as stable ∞-categories.

Using the forgetful functor

$$\Psi : D_{qcoh}(E^\vee, D_{E^\vee}) \to D_{qcoh}(E^\vee)$$

we can describe the underlying quasi-coherent sheaf $(\Psi \circ \Phi_{CFT})(\mathcal{F})$ as the Fourier-Mukai transform

$$D_{qcoh}(E^\sharp) \to D_{qcoh}(E^\vee)$$

with integral kernel given by the universal flat connection $\mathcal{L}$ on $E^\sharp \times E$. As in the proof of Lemma 5.2.5 we obtain therefore that $\Phi_{CFT}(\mathcal{F})$ is a complex of a family of D-modules concentrated in a single degree.

Vice versa starting with a family of local systems (V, ∇) on $E^\vee$ we see from the existence of a composition series for (V, ∇) as in the proof of lemma 5.2.5 that $\Phi_{CFT}^{-1}(V)$ is a torsion sheaf on $E^\sharp$. □

For $k = \mathbb{C}$ Proposition 5.2.6 seems natural from a complex analytic viewpoint. Since $\pi_1(E) \cong \mathbb{Z}^2$, the Riemann-Hilbert correspondence implies that $\mathcal{M}_{\mathrm{dR}}(E, n)$ is complex analytically isomorphic to the quotient stack

$$[\{(A,B) \in \mathrm{GL}_n \times \mathrm{GL}_n \,|AB = BA\}/\,\mathrm{GL}_n].$$

This algebraic quotient stack in turn is equivalent to $\mathcal{T}(\mathbb{C}^\times \times \mathbb{C}^\times, n)$, as we record below.

Remark 5.2.7. *Let k be an algebraically closed field. There exists a canonical equivalence of stacks*

$$\mathcal{T}(\mathbb{G}_m^r \times \mathbb{A}^s, n) \cong [\{(A_1, \dots, A_r, A_{r+1}, \dots, A_{r+s}) \in \mathrm{GL}_n^r \times \mathfrak{gl}_n^s | [A_i, A_j] = 0 \;\forall\; (i,j)\}/\,\mathrm{GL}_n],$$

where GL_n acts by conjugation on this variety of matrices.

Proof. The data of a length n torsion sheaf on $\mathbb{G}_m^r \times \mathbb{A}^s$ is equivalent to a rank n k-vector space V, endowed with the structure of a $k[X_1^{\pm 1}, \dots, X_r^{\pm 1}, X_{r+1}, \dots, X_{r+s}]$-module. This in turn is tautologically the same thing as a k-vector space V together with $r+s$ pairwise commuting endomorphisms $(A_i)_{i=1,\dots r+s}$, such that $\det A_i \neq 0$ for $i \leq r$. As the same statements hold in families, we conclude the proof of the assertion. □

On the other hand, the surface $\mathbb{C}^\times \times \mathbb{C}^\times$ is complex analytically equivalent to $E^\sharp$, which induces an isomorphism of complex analytic stacks

$$\mathcal{T}(E^\sharp, n) \cong \mathcal{T}(\mathbb{C}^\times \times \mathbb{C}^\times, n) \cong \mathcal{M}_{\mathrm{dR}}(E, n).$$

Proposition 5.2.6 allows us to prove a version of Theorem 5.2.1 for moduli spaces of parabolic local systems, by the exact same methods.

Theorem 5.2.8. *Let k be an algebraically closed field of characteristic zero or $p > |\Gamma|$. The moduli space of stable parabolic local systems $\mathcal{M}_{\mathrm{dR}}^{\mathrm{ss}}(Q, \lambda)$ of orbifold degree* *zero, associated to the orbifold $[E/\Gamma]$ with the same weights as in Theorem 5.2.1 and eigenvalues given by the canonical weights, is naturally isomorphic to the Γ-Hilbert scheme $(E^\sharp)^{[\Gamma]}$ of the surface $E^\sharp$.*

Proof. As in the proof of Theorem 5.2.1 we use Proposition 5.2.6 and Serre duality to construct a morphism

$$(E^\sharp)^{[\Gamma]} \to \mathcal{M}_{\mathrm{dR}}^{\mathrm{ss}}(Q,\lambda).$$

In order to show that this is an isomorphism it suffices to establish that every local system in $\mathcal{M}_{\mathrm{dR}}^{\mathrm{ss}}(Q,\lambda)$ is admissible. In characteristic zero this is automatically satisfied, since every vector bundle admitting an algebraic flat connection is of degree zero. In positive characteristic, having a proper Hitchin map at one's disposal, as explained in Section 4.4 and Corollary 4.7.6, one proceeds as in the Higgs bundle case (Theorem 5.2.1). □

In the $\widetilde{D}_4$-case similar two-dimensional moduli spaces of flat connections have been studied by Okamoto [Oka87] in the context of the sixth Painlevé equation.

5.2.3 Derived equivalences

In Proposition 5.1.11 we have shown that there is a derived equivalence

$$D^b_{coh}([T^*E^\vee/\Gamma]) \cong D^b_{coh}([T^*E/\Gamma]).$$

Using Theorem 5.2.1 and the derived equivalence of Theorem 5.1.6 we arrive at a string of equivalences

$$D^b_{coh}(\mathcal{M}) \cong D^b_{coh}([T^*E^\vee/\Gamma]) \cong D^b_{coh}([T^*E/\Gamma]) \cong D^b_{coh}(\mathcal{M}^\vee),$$

where $\mathcal{M}$ and $\mathcal{M}^\vee$ denote the respective moduli spaces of parabolic Higgs bundles.

Theorem 5.2.9. *Let $\mathcal{M} := \mathcal{M}_{\mathrm{Dol}}(\widehat{[E/\Gamma]}, Q)$ denote the moduli space over the Hitchin base $\mathcal{A}$ studied in section 5.2.1. We have a natural equivalence of derived categories of Fourier-Mukai type*

$$\Phi : D^b_{coh}(\mathcal{M}) \cong D^b_{coh}(\mathcal{M}^\vee),$$

relative to $\mathcal{A}$, extending the Fourier-Mukai transform for dual abelian varieties over the locus $\mathcal{A}^{\mathrm{sm}}$. The corresponding Fourier-Mukai kernel is given by a Cohen-Macaulay sheaf $\bar{\mathcal{P}}$ on the fibre product $\mathcal{M} \times_{\mathcal{A}} \mathcal{M}^\vee$.

Proof. This is an equivalence of Fourier-Mukai type relative to $\mathcal{A}$ by construction. Therefore we only need to verify the second assertion, namely that the integral kernel $\bar{\mathcal{P}}$ restricts to the Fourier-Mukai transform associated to the Poincaré bundle $\mathcal{P}$ on

$$\mathcal{M}^{\mathrm{sm}} \times_{\mathcal{A}^{\mathrm{sm}}} \mathcal{M}^{\vee,\mathrm{sm}}.$$

Over the smooth locus $\mathcal{A}^{\mathrm{sm}}$ the two morphisms

$$\begin{array}{ccc} \mathcal{M} & & [T^*E^\vee/\Gamma] \\ & \searrow \quad \swarrow & \\ & T^*E^\vee/\Gamma & \end{array}$$

are actually isomorphisms and the restriction of the equivalence of Theorem 5.1.6 to the smooth locus (which is possible because of Lemma 5.1.7) is the equivalence induced from this isomorphism. Étale locally on $\mathcal{A}^{\mathrm{sm}}$ we may identify the relative abelian variety given by the Hitchin fibration with $E^\vee$. We see that the equivalence in question is just Fourier-Mukai duality for the abelian variety E.

To verify the last assertion we need to show that the equivalence $D^b_{coh}(\mathcal{M}^\vee) \cong D^b_{coh}(\mathcal{M})$ sends the $\mathcal{M}^\vee$-family of quasi-coherent sheaves on $\mathcal{M}^\vee$ given by the structure sheaf of the diagonal $\Delta_* \mathcal{O}_{\mathcal{M}^\vee}$ to a Cohen-Macaulay sheaf. This equivalence can be divided into several steps

$$D^b_{coh}(\mathcal{M}) \cong D^b_{coh}([T^*E^\vee/\Gamma]) \cong D^b_{coh}([T^*E/\Gamma]) \cong D^b_{coh}(\mathcal{M}^\vee).$$

According to Theorem 5.2.1, the composition of the first two equivalences send $\Delta_* \mathcal{O}_{\mathcal{M}^\vee}$ to the universal family $\bar{\mathcal{Q}}$ of Higgs orbibundles on $\mathcal{M}^\vee \times [T^*E/\Gamma]$. If

$$\pi : [T^*E/\Gamma] \to [E/\Gamma]$$

denotes the canonical projection, we have that

$$(\mathrm{id}_{\mathcal{M}} \times \pi)_* \bar{\mathcal{Q}}$$

the $\mathcal{M}$-family of vector bundles underlying the universal family of Higgs bundles $\bar{\mathcal{Q}}$. In particular, since $\pi : \operatorname{supp} \bar{\mathcal{Q}} \to [E/\Gamma]$ is finite, we see that $\bar{\mathcal{Q}}$ is Cohen-Macaulay. Therefore we need to show that the equivalence $\Psi : D^b_{coh}([T^*E/\Gamma]) \cong D^b_{coh}(\mathcal{M}^\vee)$ sends $\bar{\mathcal{Q}}$ to a Cohen-Macaulay sheaf $\bar{\mathcal{P}}$ on $\mathcal{M} \times_{\mathcal{A}} \mathcal{M}^\vee$.

$$\begin{array}{ccc} & \mathcal{Z} & \\ {}^{p}\swarrow & & \searrow^{q} \\ [T^*E/\Gamma] & & \mathcal{M}^\vee \end{array}$$

The universal Γ-cluster is endowed with a line bundle (up to shift) $\mathcal{K}$ and Ψ can be written as

$$Rq_*(Lp^* - \otimes^L \mathcal{K}),$$

this is explained on p. 16 of [BKR]. Because q is a finite morphism and $\mathcal{K}$ is a line bundle,

$$\Psi(\bar{\mathcal{Q}}) = R(\mathrm{id} \times q)_*(L(\mathrm{id} \times p)^* \bar{\mathcal{Q}} \otimes^L \mathcal{K})$$

is Cohen-Macaulay if and only if $Lp^* \bar{\mathcal{Q}}$ is Cohen-Macaulay. Lemma 2.3 of [Aria] implies Cohen-Macaulyness of this pullback, if $\mathcal{M} \times_{\mathcal{A}} [T^*E/\Gamma]$ is Gorenstein, $(\mathrm{id} \times p)$ is Tor-finite and $\mathcal{M} \times_{\mathcal{A}} \mathcal{Z}$ is Cohen-Macaulay. Tor-finiteness of p follows from smoothness of $\mathcal{M}$ and is preserved by base change along a flat morphism. The two fibre products $\mathcal{M} \times_{\mathcal{A}} [T^*E/\Gamma]$ and $\mathcal{M} \times_{\mathcal{A}} \mathcal{M}$ are locally complete intersections (see tags 01UH, 01UI in [The]), and

$$\mathcal{M} \times_{\mathcal{A}} \mathcal{Z} \to \mathcal{M} \times_{\mathcal{A}} \mathcal{M}$$

is a finite and flat morphism, which implies Cohen-Macaulayness of $\mathcal{M} \times_{\mathcal{A}} \mathcal{Z}$. □

We obtain a similar result for moduli spaces of flat connections, which should be seen as an instance of the Geometric Langlands correspondence.

Theorem 5.2.10. *Let $\mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}(\widehat{[E/\Gamma]}, Q, \lambda)$ denote the moduli space of local systems studied in section 5.2.1. We have a natural equivalence of derived categories*

$$\Phi_{GL} : D_{qcoh}(\mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}(\widehat{[E/\Gamma]}, Q, \lambda)) \cong D_{qcoh}([E^\vee/\Gamma], D_{[E^\vee/\Gamma]}).$$

5.3 Hilbert schemes as moduli spaces

If Q is a graph with a marked vertex v, we denote by Q' the quiver obtained by adjoining an extra edge, linking v with a new vertex v'. If λ is a dimension vector for Q, we denote by λ' the dimension vector satisfying

$$\lambda'|Q = \lambda$$

and $\lambda'(v') = 1$.

If Q is a Dynkin diagram, then the associated affine Dynkin diagram $\widetilde{Q}$ has a marked vertex v, called the affine vertex. In this section we discuss the geometric analogue for moduli spaces of Higgs bundles and local systems of the transition

$$\widetilde{Q} \rightsquigarrow \widetilde{Q}'.$$

5.3.1 Identification of moduli spaces

Theorem 5.3.1. *Let k be an algebraically closed field of characteristic zero or $p > \max(|\Gamma|, n)$. We denote by $\mathcal{M}$ the moduli spaces of parabolic Higgs bundles $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(Q, \lambda)$ from Theorem 5.2.1. Then the Hilbert scheme $\mathcal{M}^{[n]}$ is again a moduli space of Higgs bundles. More precisely, we have*

$$\mathcal{M}^{[n]} \cong \mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widetilde{Q}', (n\lambda)'),$$

where the weights at the marked point corresponding to $0 \in E$ are $\alpha_i := \frac{i}{n}$ for $i < n$ and $1 > \alpha_n > \frac{n-1}{n}$, and all the other weights are given by the canonical weights; and the orbifold degree is zero. The Hitchin map $\mathcal{M}^{[n]} \to \mathcal{A}_n$ factors through the Hilbert-Chow map

$$\mathcal{M}^{[n]} \to \mathcal{M}^{(n)} \to \mathcal{A}_1^{(n)} = \mathcal{A}_n,$$

where $\mathcal{M}^{(n)} \to \mathcal{A}_1^{(n)}$ is the map induced by $\mathcal{M}^n \to \mathcal{A}_1^n$.

In the case of $Q = \widetilde{A}_0$ this is a theorem of Gorsky–Nekrasov–Rubtsov ([GNR01, Sect. 5.1]).

Proof. Theorem 5.1.6 and Theorem 5.2.9 imply that we have an equivalence

$$D^b_{coh}(\mathcal{M}) \cong D^b_{coh}([T^*E/\Gamma]),$$

defined relative to $\mathcal{A}_1$. In particular we can show as in Lemma 5.2.5 that the moduli stack of length n torsion sheaves on $\mathcal{M}$ is equivalent to the moduli stack of Γequivariant rank n admissible Higgs bundles on E:

$$\mathcal{T}(\mathcal{M}) \cong \mathcal{M}_{\mathrm{Dol}}([E/\Gamma], n).$$

As in the proof of Theorem 5.2.1 we see that under this equivalence a morphism $\mathcal{O}_{\mathcal{M}} \to \mathcal{F}$ corresponds to a linear map $\tau_*(V^\vee \otimes \mathcal{O}/L_0^{-1}) \to k$, where (V, θ) is the Γ-equivariant Higgs bundle associated to $\mathcal{F} \in \mathcal{T}$. But

$$\tau_*(V^\vee \otimes \mathcal{O}/L_0^{-1}) \cong E_0^\vee / F_1^\vee.$$

In particular we obtain a non-trivial linear map

$$k \to F_{n_0 - 1}$$

by dualizing, i.e. a nonzero vector $v \in F_{n_0-1}$.

We claim that the condition that $\mathcal{O}_{\mathcal{M}} \to \mathcal{F}$ is surjective, equivalent is to the fact that (V, θ, v) does not contain any degree zero Higgs subbundles containing v.

Let us assume that $\mathcal{O}_{\mathcal{M}}$ is surjective. If (V, θ, v) contains a non-trivial degree zero Higgs subbundle, which contains v, then there is a smallest such Higgs subbundle (V', θ, v) of rank $k < n$. In particular its transform $\mathcal{G}$ gives rise to a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{M}} & \longrightarrow & \mathcal{F} \\ & \searrow \quad \nearrow & \\ & \mathcal{G}\,. & \end{array}$$

Because the horizontal arrow is surjective and W is a length k torsion sheaf, this is a contradiction.

Similarly one shows that if (V, θ, v) does not contain a non-trivial degree zero Higgs subbundle containing v, then the corresponding morphism $\mathcal{O}_{\mathcal{M}} \to \mathcal{F}$ is surjective.

Namely, if it is not surjective, its image gives rise to a non-trivial Higgs subbundle of (V, θ, v) containing v. Stability is checked as in the proof of Theorem 5.2.1.

We obtain a morphism of $\mathcal{A}_n$-spaces

$$\mathcal{M}^{[n]} \to \mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widetilde{Q}', (n\lambda)'),$$

as the type of the corresponding parabolic bundle can be checked for a single point, by connectivity of the moduli spaces, for instance over the locus of smooth spectral curves. Properness of the Hitchin morphism and the fact that both spaces have equal dimension and are connected, imply that this morphism is surjective. In particular we may conclude that every parabolic Higgs bundle in $\mathcal{M}^{\mathrm{ss}}_{\mathrm{Dol}}(\widetilde{Q}', (n\lambda)')$ is admissible. This implies that the above morphism is an isomorphism, with the inverse given by the inverse Fourier-Mukai transform. □

There is an analogous statement for moduli spaces of local systems that is proved by the same means as the Theorems 5.2.8 and 5.3.1. The role of the derived equivalence $D^b_{coh}(T^*E) \cong D^b_{con}(T^*E^\vee)$ is taken by Laumon-Rothstein's geometric class field theory (Theorem 5.1.12). In characteristic zero, the proof relies on the fact that local systems are admissible. In positive characteristic this is not the case, but one proceeds as in the Higgs bundle case using properness of the Hitchin map (Section 4.4 and Corollary 4.7.6).

Theorem 5.3.2. *Let k be an algebraically closed field of characteristic zero or $p > \max(|\Gamma|, n)$. We denote by $\mathcal{M}$ one of the moduli spaces of parabolic local systems $\mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}(Q, \lambda)$ from Theorem 5.2.8 defined over k. Then the Hilbert scheme $\mathcal{M}^{[n]}$ is again a moduli space of local systems. More precisely, we have*

$$\mathcal{M}^{[n]} \cong \mathcal{M}^{\mathrm{ss}}_{\mathrm{dR}}(\widetilde{Q}', (n\lambda)'),$$

where the weights α_i are as in Theorem 5.3.1, the orbifold degree is fixed to be zero and the eigenvalues of the residues are given by $\omega_i := \alpha_i$ for $i < n$ and $\omega_n := \alpha_{n-1}$.

The reader might be wondering wether the moduli spaces of Theorem 5.3.1 and 5.3.2 are related via nonabelian Hodge theory ([Sim90]). The answer is now, according to the table on p. 720 of *loc. cit.*, the weight α and eigenvalue $b + ic$ of the residues of a parabolic Higgs bundle give rise to weight $\alpha - 2b$ and eigenvalue $\alpha + i2c$ of the corresponding parabolic local system. In our case the eigenvalues are zero in the Higgs case (nilpotent residue). In particular we see that these Higgs bundles correspond to local systems with the same weights, but eigenvalues given by the weights. Comparing weights and eigenvalues in the local system case, as given by Theorem 5.2, one sees that the two moduli spaces do not correspond to each other with respect to nonabelian Hodge theory.

5.3.2 Derived equivalences

We begin this subsection with the following observation, which has been proven in [Plo07, Prop. 8] for varieties defined over $\mathbb{C}$.

Lemma 5.3.3. *Let X and Y be two quasi-projective smooth surfaces defined over an algebraically closed field of characteristic $p > n$. If we have an equivalence of derived categories $D^b_{coh}(X) \cong D^b_{coh}(Y)$ of Fourier-Mukai type then this induces an equivalence $D^b_{coh}(X^{[n]}) \cong D^b_{coh}(Y^{[n]})$ of Fourier-Mukai type.*

Proof. Lemma 5.1.4 allows us to take the n-th power of the equivalence $D^b_{coh}(X) \cong D^b_{coh}(Y)$, yielding $D^b_{coh}(X^n) \cong D^b_{coh}(Y^n)$. On both spaces we have a natural action of the symmetric group S_n by permuting the factors. The integral kernel is a sheaf naturally endowed with an S_n-equivariant structure; therefore we may apply Lemma 5.1.5 and conclude that

$$D^b_{coh}([X^n/S_n]) \cong D^b_{coh}([Y^n/S_n]).$$

Together with Corollary 5.1.10 we obtain $D^b_{coh}(X^{[n]}) \cong D^b_{coh}(Y^{[n]})$. □

Theorem 5.3.4. *We denote by $\mathcal{M}^{[n]}$ the moduli space of parabolic Higgs bundles associated to $[E/\Gamma]$ of Theorem 5.3.1. By $\mathcal{M}^{\vee[n]}$ we denote the same moduli space for $[\widehat{E^\vee/\Gamma}]$. Both moduli spaces $\mathcal{M}^{[n]}$ and $\mathcal{M}^{\vee[n]}$ are $\mathcal{A}$-spaces, where $\mathcal{A}$ is the Hitchin base. Under the assumptions of Lemma 5.3.3 there is a canonical equivalence of derived categories*

$$D^b_{coh}(\mathcal{M}^{[n]}) \cong D^b_{coh}(\mathcal{M}^{\vee[n]}),$$

relative to $\mathcal{A}$. The integral kernel of this derived equivalence is a Cohen-Macaulay sheaf $\bar{\mathcal{P}}$ on $\mathcal{M}^{[n]} \times_{\mathcal{A}} \mathcal{M}^{\vee[n]}$, which restricts to the Poincaré bundle $\mathcal{P}$ over the locus of smooth spectral curves $\mathcal{A}^{\mathrm{sm}}$.

Proof. This is a consequence of Theorem 5.3.1 and Lemma 5.3.3. Note that the construction in the proof of this lemma respects the morphism to the Hitchin base (due to Lemma 5.1.7). The last two assertions are verified as in the proof of Theorem 5.2.9, with the single exception that this time the sheaf $\bar{\mathcal{Q}}$ cannot be thought of as a universal family of Higgs bundles. Therefore Cohen-Macaulayness has to be established by different means. The sheaf $\bar{\mathcal{Q}}$ is the transform of the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ of the universal S_n-cluster on $\mathcal{M}^{[n]} \times_{\mathcal{A}} [\mathcal{M}^n/S_n]$ along the equivalence $D^b_{qcoh}([\mathcal{M}^n/S_n]) \cong D^b_{qcoh}([\mathcal{M}^{\vee n}/S_n])$; we denote the integral kernel of the latter equivalence by $\mathcal{R}$, it is Cohen-Macaulay according to Theorem 5.2.9 and Lemma 5.1.4. Let $\iota : \mathcal{Z} \to [M^n/S_n]$ be the canonical morphism; Lemma 2.3 of [Aria] implies that $L\iota^*\mathcal{R}$ is Cohen-Macaulay, since $\mathcal{Z}$ is finite over $\mathcal{M}^{[n]}$ and therefore Cohen-Macaulay itself. The natural morphism $\pi : \mathcal{Z} \to \mathcal{M}^{[n]}$ is finite, and so is every base change thereof. In particular we obtain that the transform of $\mathcal{O}_{\mathcal{Z}}$ is Cohen-Macaulay, as we wanted. □

Similarly one obtains an analogue for local systems.

Theorem 5.3.5. *We denote by $\mathcal{M}^{[n]}$ the moduli space of parabolic local systems associated to the weighted curve $\widehat{[E/\Gamma]}$ studied in Theorem 5.3.2. Under the assumptions of Lemma 5.3.3 there is a canonical equivalence of derived categories*

$$D_{qcoh}(\mathcal{M}^{[n]}) \cong D_{qcoh}([[E/\Gamma]^n/S_n], D_{[[E/\Gamma]^n/S_n]}).$$

Chapter 6

Geometric Langlands in positive characteristic

6.1 Splitting of $\mathcal{D}_{\text{Bun}}$ on smooth Hitchin fibres

This section is entirely expository. We begin by reviewing the theory of abelian group stacks and refer the reader to [Ari08] for a more detailed treatment. In the following we fix a base scheme S and consider an abelian group S-stack Y. The dual $Y^\vee$ is defined to be the stack classifying morphisms of group stacks $Y \to B\,\mathbb{G}_m$.

Definition 6.1.1. *Let Y be an abelian group S-stack. We define $Y^\vee$ to be the group stack, which sends every S-scheme T to the groupoid of morphisms of group stacks $Y \times_S T \to B\,\mathbb{G}_m$.*

If A is an abelian S-scheme, $A^\vee$ is the dual abelian S-scheme, which can be extracted from [Ser88, p. 184]. Moreover we have $B\,\mathbb{G}_m^\vee = \mathbb{Z}$ and $\mathbb{Z}^\vee = B\,\mathbb{G}_m$. The dual group scheme functor can also be thought of as the first derived functor of the Cartier dual.

Definition 6.1.2. *The* Cartier dual Y^* *of an abelian group stack Y is stack of group* *stack morphisms* $Y \to \mathbb{G}_m$.

If Γ is a finite group scheme, its dual is given by the classifying stack $B\Gamma^*$ of the Cartier dual Γ^*.

A group stack Y is said to be *nice*, if the natural morphism $Y \to Y^{\vee\vee}$ is an equivalence. All the examples considered above are nice.

Dualizing is in general not an exact operation, as the following counter-example shows. An isogeny of abelian varieties $A \to B$ gives rise to an exact sequence of nice group stacks

$$0 \to \Gamma \to A \to B \to 0,$$

the dual sequence is $B^\vee \to A^\vee \to B\Gamma^* \to 0$, but the first arrow is the dual isogeny and certainly has a non-trivial kernel in general.

The theory of abelian group stacks is used by Bezrukavnikov–Braverman to show the following Theorem ([BB07, Thm. 4.10(1)]):

Theorem 6.1.3. *The Azumaya algebra of differential operators* $\mathcal{D}_{\mathrm{Bun}}$ *(see section 3.13 in [BB07]) carries a natural group structure over the locus of smooth spectral curves* $\mathcal{A}^{(1)}_{\mathrm{sm}}$*. In particular we have an extension of group* $\mathcal{A}^{(1)}_{\mathrm{sm}}$*-stacks*

$$0 \to B\,\mathbb{G}_m \to \mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}}} \to \mathcal{P}\mathrm{ic}(Y^{(1)}/\,\mathcal{A}^{(1)}_{\mathrm{sm}}) \to 0.$$

Here we use $\mathcal{Y}_{\mathcal{D}_{\mathrm{Bun}}}$ *to denote the gerbe associated to* $\mathcal{D}_{\mathrm{Bun}}$.

We refer the reader to [OV07, App. 5.5] for a precise definition of group structures on Azumaya algebras. For our purpose it is sufficient to know that we have an extension of group stacks as stated in the Theorem above.

Using this theorem, Bezrukavnikov–Braverman conclude in [BB07, Thm. 4.10(2)] that étale locally over the Hitchin base $\mathcal{A}^{(1)}_{\mathrm{sm}}$ the Azumaya algebra $\mathcal{D}_{\mathrm{Bun}}$ splits. By virtue of the Abel-Jacobi map

$$Y^{sm,(1)} \to \mathcal{P}\mathrm{ic}(Y^{sm,(1)} / \mathcal{A}^{(1)}_{\mathrm{sm}}),$$

sending $y \in Y_Y$ to the line bundle $\mathcal{O}_{Y_a}(-y)$, they compare splittings of $\mathcal{D}_{\mathrm{Bun}}$ respecting the group structure on the Hitchin fibres, with splittings of $\mathcal{D}_X$ on the spectral curve.

Corollary 6.1.4 (Bezrukavnikov–Braverman)**.** *The pullback of* $\mathcal{D}_{\mathrm{Bun}}$ *along the Abel-Jacobi map is canonically Morita equivalent to* $\mathcal{D}_X$*. This implies the existence of a natural isomorphism*

$$\mathcal{S}_{\mathrm{grp}}(\mathcal{D}_{\mathrm{Bun}} / \mathcal{A}^{(1)}_{\mathrm{sm}}) \cong \mathcal{S}(\mathcal{D}_X / \mathcal{A}^{(1)}_{\mathrm{sm}}),$$

where $\mathcal{S}_{\mathrm{grp}}$ *refers to the stack of relative splittings respecting the group structure. In particular, the Azumaya algebra* $\mathcal{D}_{\mathrm{Bun}}$ *splits étale locally over the base* $\mathcal{A}^{(1)}_{\mathrm{sm}}$*.*

We refer the reader to [OV07, App. 5.5] for the notion of a splitting respecting the group structure. All that matters to us, is that we can relate splittings of $\mathcal{D}_X$ on smooth spectral curves, to splittings of $\mathcal{D}_{\mathrm{Bun}}$ on the corresponding Hitchin fibres.

6.2 The Langlands correspondence over $\mathcal{A}_{\mathrm{int}}$

This section is devoted to the proof of Theorem 6.2.1, which is an extension of the main result of [BB07]. We use the notation $\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}} = \mathcal{M}^{\mathrm{int}}_{\mathrm{dR}}(X)$ and $\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}} = \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}(X^{(1)})$ for the moduli stacks of local systems on X, respectively Higgs bundles on $X^{(1)}$, of rank n with integral spectral curves. The corresponding open dense subset of $\mathcal{A}^{(1)}$ will be denoted by $\mathcal{A}^{(1)}_{\mathrm{int}}$. We will freely use results and notation about Fourier-Mukai transforms from subsection 5.1.1. As we have seen in Proposition 3.1.12, it is possible to replace derived categories of $\mathcal{D}$-modules, where $\mathcal{D}$ is an Azumaya algebra, by derived categories twisted by a gerbe. In particular we see that the level of generality of subsection 5.1.1 is high enough to include Fourier-Mukai transform for derived categories of modules of an Azumaya algebra.

Theorem 6.2.1. *There exists a canonical $p_2^*\mathcal{D}_{\mathrm{Bun}}$-module $\bar{\mathcal{L}}$ on*

$$\mathcal{M}_{\mathrm{dR}}^{\mathrm{int}} \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}$$

inducing an equivalence of derived categories

$$D^b_{\mathrm{coh}}(\mathcal{M}_{\mathrm{dR}}^{\mathrm{int}}, \mathcal{O}) \cong D^b_{\mathrm{coh}}(\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}, \mathcal{D}_{\mathrm{Bun}}).$$

6.2.1 Relation with the Arinkin-Poincaré sheaf

This section contains an introduction to autoduality for compactified Jacobians. We refer the reader to Section 2.3, for an overview on the theory of compactified Jacobians. In the following we denote by S a scheme of finite type over k, which parametrizes a flat family of locally planar, integral curves $\pi : C \to S$. We denote by $\overline{\mathcal{P}\mathrm{ic}}(C/S)$ the compactified Picard stack, and by $\bar{\mathrm{J}}(C/S)$ the compactified Jacobian, as defined in Definition 2.3.5. The universal family of torsion free sheaves on $\overline{\mathcal{P}\mathrm{ic}}(C/S) \times_S C$ will be referred to as $\bar{\mathcal{Q}}$. Similarly, we denote the universal line bundle on $\mathcal{P}\mathrm{ic}(C/S) \times_S C$ by $\mathcal{Q}$. Let $(C/S)^{[d]}$ denote the relative Hilbert scheme of d points. Below we define a natural morphism $(C/S)^{[d]} \to \overline{\mathcal{P}\mathrm{ic}}^d(C/S)$ to the degree d component of the compactified Picard stack. The following proposition can be found in [AK80, sect. 5].

Proposition 6.2.2 (Abel map)**.** *The Abel map*

$$A^d : (C/S)^{[d]} \to \overline{\mathcal{P}\mathrm{ic}}^d(C/S)$$

sends a family of subschemes D of C/S to the dual of its ideal sheaf $\mathcal{I}_D^\vee$. For $d \geq 2g-1$, where g denotes the arithmetic genus of the family C/S, the map A is smooth and surjective.

On a smooth curve X, elements of $X^{[d]}$ are usually referred to as degree d effective divisors. A general divisor on X can be represented as a formal difference $D - D'$ of two effective divisors. The constructions given by the inverse of a line bundle, and tensor products, allow to extend the Abel map to a map from divisors to $\mathcal{P}\text{ic}(X)$. In order to extend this definition to integral, locally planar curves C, we have to restrict the divisors D' to lie in the smooth locus C^{sm} of C/S.

Corollary 6.2.3. *Let* $\text{Div}(C/S)$ *denote the* S*-space of* divisors *given by the sheafification of the functor, which sends an* S*-scheme* T *to the set of formal linear combinations* $D - D'$*, where*

$$D, D' \in \bigcup_{d \geq 0} (C/S)^{[d]}(T),$$

and $D' \subset C^{sm}$*. The Abel map (Proposition 6.2.2) induces a surjective map*

$$\text{Div}(C/S) \to \mathcal{P}\text{ic}(C/S),$$

given by $D - D' \mapsto A(D) \otimes A(D')^{\vee}$*. If* $\text{Div}^{sm}(C/S)$ *denotes the subspace of* smooth divisors*, corresponding to formal differences* $D - D'$*, where* $D \subset C^{sm}$*, then we obtain a surjective map*

$$\text{Div}^{sm}(C/S) \to \mathcal{P}\text{ic}(C/S).$$

In the next lemma we use the universal subscheme correspondence

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{j} & (C^{sm}/S)^{[d]} \times_S C^{sm} \\ {\scriptstyle g}\downarrow & & \\ (C^{sm}/S)^{[d]}, & & \end{array}$$

where g is a finite flat morphism of degree d, and j is a closed immersion. Moreover, we will make use of graded linear algebra.

Definition 6.2.4. *A* graded line *is a pair* (L, n)*, where* L *is a line and* n *an integer.*

The tensor product of graded line bundles is defined by means of the tensor product of lines and addition of integers. We impose the following commutativity constraint

$$(L, n) \otimes (L', n') \cong (L', n') \otimes (L, n),$$

by twisting the standard commutativity constraint for the tensor product of vector spaces by the sign $(-1)^{nn'}$. *The corresponding commutative group stack of will be denoted by* $B\,\mathbb{G}_m^{\mathbb{Z}}$. *The graded determinant*

$$\mathbb{Z}\det : \mathrm{Vect} \to B\,\mathbb{G}_m^{\mathbb{Z}},$$

is the natural functor sending a vector space V *of rank* n *to the pair* $(\det V, n)$.

The motivation to introduce graded determinants is that the canonical morphism

$$\det(V \oplus W) \cong \det V \otimes \det W,$$

for finite dimensional vector spaces V and W, is sensitive to the ordering of the two factors. Graded determinants repair this defect, thereby simplifying the definition of the Poincaré line bundle $\mathcal{P}$ on $\overline{\mathcal{P}\mathrm{ic}} \times_S \mathcal{P}\mathrm{ic}$. Although the graded version of $\mathcal{P}$ appears a priori only as a consequence of our reluctance to worry about sign issues, we will show later that the grading has a natural geometric interpretation (see Lemma 6.2.6(c)).

Lemma 6.2.5 (Poincaré bundle). *The graded line bundle* $\mathcal{L}$ *on* $\overline{\mathcal{P}\mathrm{ic}}(C/S) \times_S (C^{sm}/S)^{[d]}$ *given by*

$$\mathbb{Z}\det[(id \times_S g)_*(id \times_S j)^*\, \mathcal{Q}] \otimes (\mathbb{Z}\det[(id \times_S g)_*(id \times_S j)^*\, \mathcal{O}])^{-1},$$

descends to a graded line bundle on $\overline{\mathcal{P}\mathrm{ic}}(C/S) \times_S \mathcal{P}\mathrm{ic}(C/S)$.

Before proving this statement, we try to get some understanding of the nature of

$\mathcal{P}$ for a single integral curve C. For $F \in \overline{\mathcal{P}\mathrm{ic}}(C)$ and $D = \sum_{x \in C^{sm}} a_x x \in Div^{sm}(C)$, the one-dimensional space $\mathcal{P}\,|_{(F,\mathcal{O}(D))}$ can be described after deciphering the formula above as

$$\bigotimes_{x \in C^{sm}} (F|_x)^{\otimes a_x}.$$

We observe that for $F = \mathcal{O}_X$, this space is always canonically trivial. Hence we can ignore the second factor in the above definition of $\mathcal{P}$, in our fibrewise analysis. However, in a family of integral curves it has a nontrivial contribution.

The lemma above could be either proved by explicitly verifying the descent condition, or by showing the equivalence with the definition given by Arinkin in [Arib]. We will take the second approach here.

Proof of Lemma 6.2.5. The graded determinant map $\mathbb{Z}\det : \mathrm{Vect} \to B\,\mathbb{G}_m^{\mathbb{Z}}$ extends naturally to a map $\mathbb{Z}\det : \mathrm{Perf} \to B\,\mathbb{G}_m^{\mathbb{Z}}$, sending a perfect complex $A^\bullet$ to

$$\bigotimes_{i \in \mathbb{Z}} (-1)^i \mathbb{Z}\det(A^i).$$

Every distinguished triangle of perfect complexes

$$A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$$

gives rise to an equivalence

$$\mathbb{Z}\det(A^\bullet) \otimes \mathbb{Z}\det(C^\bullet) \cong \mathbb{Z}\det(B^\bullet).$$

We claim that the graded line bundle $\mathcal{L}$ is naturally equivalent to the pullback of the graded line bundle $\mathcal{P}$ given by

$$\frac{\mathbb{Z}\det(Rp_{12,*}(\bar{\mathcal{Q}} \boxtimes \mathcal{Q})) \otimes \mathbb{Z}\det(Rp_{12,*}\,\mathcal{O})}{\mathbb{Z}\det(Rp_{12,*}\,\mathcal{Q}) \otimes \mathbb{Z}\det(Rp_{12,*}\,\bar{\mathcal{Q}})}.$$

Before continuing with the argument, we note that for an individual curve C, and a pair $(F, L) \in \overline{\mathcal{P}\mathrm{ic}}(C) \times \mathcal{P}\mathrm{ic}(C)$, this agrees with the expression

$$\frac{\mathbb{Z}\det(H^\bullet(C, F \otimes L)) \otimes \mathbb{Z}\det(H^\bullet(C, \mathcal{O}_C))}{\mathbb{Z}\det(H^\bullet(C, L)) \otimes \mathbb{Z}\det(H^\bullet(C, F))}.$$

The proof commences by exhibiting a descent condition for the line bundle obtained by restricting to effective divisors of high enough degree $\overline{\mathcal{P}\mathrm{ic}} \times_S (C/S)^{[d]}$. The universal divisor $\mathcal{Z}$ gives rise to a short exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(\mathcal{Z}) \to \mathcal{O}_{\mathcal{Z}} \to 0$$

of coherent sheaves on $\overline{\mathcal{P}\mathrm{ic}} \times_S C$. We tensor this sequence with the universal torsion free bundle $\bar{\mathcal{Q}}$, and apply the push-forward functor $Rp_{12,*}$ along the projection map to the first and second component. Since these functors are exact, we obtain a distinguished triangle giving rise to the identity

$$\mathbb{Z}\det(Rp_{12,*}\, \bar{\mathcal{Q}}(\mathcal{Z})) \cong \mathbb{Z}\det(Rp_{12,*}\, \bar{\mathcal{Q}}) \otimes \mathbb{Z}\det(Rp_{12,*}(\bar{\mathcal{Q}} \otimes \mathcal{O}_{\mathcal{Z}}))$$

for the determinants lines. The leftmost factor is easily identified with $\mathbb{Z}\det[(id \times_S g)_*(id \times_S j)^* \mathcal{Q}]$, which allows us to equate the bundle $\mathcal{L}$ with the pullback of the line bundle on $\overline{\mathcal{P}\mathrm{ic}} \times \mathcal{P}\mathrm{ic}$ given by

$$\frac{\mathbb{Z}\det(Rp_{12,*}(\bar{\mathcal{Q}} \boxtimes \mathcal{Q})) \otimes \mathbb{Z}\det(Rp_{12,*}\, \mathcal{O})}{\mathbb{Z}\det(Rp_{12,*}\, \mathcal{Q}) \otimes \mathbb{Z}\det(Rp_{12,*}\, \bar{\mathcal{Q}})}.$$

The argument above settles the descent question for the high degree components $d >> 0$. In order to conclude the proof for the general case, we pick (étale locally) a section $x : S \to C^{sm}$ and consider the divisors $D - kx$, where D is effective. According to Corollary 6.2.3, every line bundle on C/S will eventually be representable by a divisor of this form. The commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{\mathcal{Q}}(-kx) & \longrightarrow & \bar{\mathcal{Q}}(\mathcal{Z}-kx) & \longrightarrow & \bar{\mathcal{Q}}(-kx)\otimes\mathcal{O}_{\mathcal{Z}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{\mathcal{Q}} & \longrightarrow & \bar{\mathcal{Q}}(\mathcal{Z}) & \longrightarrow & \bar{\mathcal{Q}}\otimes\mathcal{O}_{\mathcal{Z}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{\mathcal{Q}}\otimes\mathcal{O}_{kx} & \longrightarrow & \bar{\mathcal{Q}}(\mathcal{Z})\otimes\mathcal{O}_{kx} & \longrightarrow & \bar{\mathcal{Q}}\otimes\mathcal{O}_{kx\cap\mathcal{Z}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

allows us to conclude the general case by a straightforward variation of the above calculations. □

In Lemma 6.2.5 we have seen two different descriptions of the (graded) line bundle $\mathcal{P}$ on $\overline{\mathcal{P}\mathrm{ic}}(C/S)\times_S\mathcal{P}\mathrm{ic}(C/S)$. Each viewpoint has its advantages, and we conclude by stating the following properties of $\mathcal{P}$ (see [Arib] for (a) and (b)), which can be proved by invoking either of the definitions above.

Lemma 6.2.6. *The Poincaré line bundle $\mathcal{P}$ on $\overline{\mathcal{P}\mathrm{ic}}(C/S)\times_S\mathcal{P}\mathrm{ic}(C/S)$ has the following properties:*

(a) Its restriction to $\mathcal{P}\mathrm{ic}(C/S)\times_S\mathcal{P}\mathrm{ic}(C/S)$ is symmetric under change of coordinates. In particular, $\mathcal{P}$ extends to a line bundle on $\overline{\mathcal{P}\mathrm{ic}}(C/S)\times_S\mathcal{P}\mathrm{ic}(C/S)\cup\mathcal{P}\mathrm{ic}(C/S)\times_S\overline{\mathcal{P}\mathrm{ic}}(C/S)$, which will be referred to by the same notation.

(b) For every $F\in\overline{\mathcal{P}\mathrm{ic}}(C/S)$, the total space of the corresponding line bundle $\mathcal{P}\mid_{\{F\}\times\mathcal{P}\mathrm{ic}}$ is endowed with a natural group structure, and thus gives rise to a central extension of $\mathcal{P}\mathrm{ic}$.

(c) Since $\overline{\mathcal{P}\mathrm{ic}}(C/S)\times_S\mathcal{P}\mathrm{ic}(C/S)\to\overline{\mathcal{P}\mathrm{ic}}^{rig}(C/S)\times_S\mathcal{P}\mathrm{ic}^{rig}(C/S)$ is a $\mathbb{G}_m$-gerbe, the

category of quasi-coherent sheaves obtains a natural $\mathbb{Z}$-grading (see Definition 3.1.11). The grading on $\mathcal{P}$ of Lemma 6.2.5 is compatible with this one.

According to the first and last point of the lemma above, we can restrict $\mathcal{P}$ to a line bundle on $\bar{\mathrm{J}} \times_S \mathrm{J} \cup \mathrm{J} \times_S \bar{\mathrm{J}}$. In [Aria] Arinkin has constructed a maximal Cohen-Macaulay sheaf $\bar{\mathcal{P}}$ on $\bar{\mathrm{J}} \times_S \bar{\mathrm{J}}$, extending the Poincaré line bundle $\mathcal{P}$ defined above.

Theorem 6.2.7 (Arinkin)**.** *Let k be an algebraically closed field of characteristic zero or $p > 2g - 1$. Let S be a k-scheme locally of finite type and $\pi : C \to S$ a flat family of integral curves with planar singularities and arithmetic genus g. Then there exists a coherent sheaf $\bar{\mathcal{P}}$ on $\bar{\mathrm{J}} \times_s \bar{\mathrm{J}}$, which is Cohen-Macaulay over every closed point $s \in S$ and extends the Poincaré line bundle $\mathcal{P}$ on $\bar{\mathrm{J}} \times_S \mathrm{J} \cup \mathrm{J} \times_S \bar{\mathrm{J}}$. Relative Fourier-Mukai duality with respect to the kernel $\bar{\mathcal{P}}$ gives rise to an equivalence of bounded derived categories*

$$\Phi_{\bar{\mathcal{P}}} : D^b_{\mathrm{coh}}(\bar{\mathrm{J}}) \to D^b_{\mathrm{coh}}(\bar{\mathrm{J}}).$$

The proof of this theorem can be found in [Aria] for the case of characteristic zero. The reason for restricting the characteristic is that Haiman's celebrated n! Theorem (cf. Remark 5.1.8 and [Hai01]) plays a role in the construction of $\bar{\mathcal{P}}$. In order to construct the Arinkin-Poincaré sheaf one needs Cohen-Macaulayness of the isospectral Hilbert scheme X_n for $n = 2g - 1$. This requires $p > 2g - 1$. But since representation theory of the symmetric group is also used in the defining formula [Aria, (4.1)] of the Arinkin-Poincaré sheaf, Arinkin's methods depend on the restriction $p > 2g - 1$ a second time.

Over the smooth open subscheme $J \subset \bar{\mathrm{J}}$, the Arinkin-Poincaré sheaf $\bar{\mathcal{P}} / \bar{\mathrm{J}} \times \bar{\mathrm{J}}$ restricts to the Poincaré line bundle $\mathcal{P} \,/\, \mathrm{J} \times \bar{\mathrm{J}}$. Moreover if the generic member of the family $C \to S$ is smooth, the codimension of the complement of $J \subset \bar{\mathrm{J}}$ is ≥ 2. A Cohen-Macaulay sheaf can be recovered as the push-forward of the restriction of an open subset, which is the complement of a closed subset of codimension ≥ 2. This allows us to reconstruct the original sheaf by push-forward from the smooth locus

$$\bar{\mathcal{P}} = i_* \mathcal{P}\,.$$

The following Theorem is a direct consequence of [Aria, Thm. C].

Theorem 6.2.8. *With the same assumptions as in Theorem 6.2.7 there exists a natural automorphism of derived categories*

$$\Phi_{\bar{\mathcal{P}}} : D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}}) \to D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}})$$

given by an integral kernel $\bar{\mathcal{P}}$, which is a Cohen-Macaulay coherent sheaf on the stack $\overline{\mathcal{P}\mathrm{ic}} \times_S \overline{\mathcal{P}\mathrm{ic}}$. Moreover $\bar{\mathcal{P}}$ is the push-forward of the Poincaré line bundle $\mathcal{P}$ on

$$\overline{\mathcal{P}\mathrm{ic}} \times_S \mathcal{P}\mathrm{ic} \cup \mathcal{P}\mathrm{ic} \times_S \overline{\mathcal{P}\mathrm{ic}}\,.$$

Proof. According to Lemma 6.2.6, a section of the degree one component $\pi : \mathcal{P}\mathrm{ic}^1(C/S) \to S$ induces not only a splitting of the extension

$$0 \to J \to \mathcal{P}\mathrm{ic}^{rig} \to \mathbb{Z} \to, 0$$

but also of the group gerbe $\overline{\mathcal{P}\mathrm{ic}} \to \overline{\mathcal{P}\mathrm{ic}}^{rig}$. Since π is a smooth map of stacks, such a section can be chosen étale locally. This yields an étale local identification of $\overline{\mathcal{P}\mathrm{ic}}$ with $\mathbb{Z} \times \bar{\mathrm{J}} \times B\,\mathbb{G}_m$, under which $\mathcal{P}$ just restricts componentwise to the Poincaré line bundle on $J \times_S J_b \cup \bar{\mathrm{J}} \times_S J$. In particular, we see that $i_* \mathcal{P}$ is a maximal Cohen-Macaulay sheaf on $\overline{\mathcal{P}\mathrm{ic}} \times_S \overline{\mathcal{P}\mathrm{ic}}$ and induces an equivalence of derived categories. □

6.2.2 Twisting Arinkin's equivalence

We would like to construct an integral kernel, i.e. a sheaf $\bar{\mathcal{L}}$ on the fibre product $\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}} \times_{\mathcal{A}^{(1)}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}$, which is endowed with the structure of a $p_2^*\mathcal{D}_{\mathrm{Bun}}$-module. As soon as this is done, we are able to set-up a Fourier-Mukai functor between derived categories, analogous to [Muk87].

Let us denote by

$$\phi : \mathcal{S}^0_{\mathrm{int}} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{dR}} \to \mathcal{S}^0_{\mathrm{int}} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}$$

the restriction of the isomorphism of Theorem 4.6.3 to the integral locus $\mathcal{A}^{(1)}_{\mathrm{int}}$. According to Corollary 6.1.4 there exists a $p_2^*\mathcal{D}_{\mathrm{Bun}}$-splitting $\mathcal{L}$ on $\mathcal{S} \times_{\mathcal{A}^{(1)}_{\mathrm{sm}}} \mathcal{M}^{\mathrm{sm}}_{\mathrm{Dol}}$. Since the total space of $\mathcal{S} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}$ is smooth we can extend $\mathcal{L}$ to a splitting on $\mathcal{S} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \overline{\mathcal{P}\mathrm{ic}}$ (Corollary 2.6 in [Mil80]). On every connected component this extension is unique up to tensoring by a line bundle $\chi^* L$ pulled back from $\mathcal{A}^{(1)}_{\mathrm{int}}$. But $\mathcal{P}\mathrm{ic}(\mathcal{A}^{(1)}_{\mathrm{int}}) = 1$, since $\mathcal{A}^{(1)}_{\mathrm{int}} \subset \mathcal{A}^{(1)}$ is an open subscheme of affine space. We choose one and denote it again by $\mathcal{L}$.

Lemma 6.2.9. *We have* $\underline{\mathrm{Hom}}_{p_3^*\mathcal{D}_{\mathrm{Bun}}}(p_{13}^*\mathcal{L}, p_{23}^*\mathcal{L}) = \phi^* p_{23}^*\mathcal{P}$ *on*

$$\mathcal{S}^0_{\mathrm{int}} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{dR}} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}} \cong \mathcal{S}^0_{\mathrm{int}} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}.$$

Proof. All constituents in the above identity are either splittings or line bundles, which arise as pullbacks of some extension of the same object defined over $\mathcal{A}^{(1)}_{\mathrm{sm}}$. In particular they are well-defined up to twisting by the same line bundle defined on $\mathcal{A}^{(1)}$. We may therefore conclude that it suffices to check the identity over $\mathcal{A}^{(1)}_{\mathrm{sm}}$. Over the locus of smooth spectral curves it holds by means of the Abel-Jacobi map defining the families of line bundles, respectively splittings, given by $\mathcal{P}$ and $\mathcal{L}$ (Corollary 6.1.4). □

We denote by

$$j : \mathcal{S} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}} \to \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}} \times_{\mathcal{A}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}$$

the inclusion of this open substack. We observe that its complement has codimension ≥ 2, since χ_{dR} is flat and j is surjective over the locus of smooth spectral curves $\mathcal{A}_{\mathrm{sm}}^{(1)}$. Theorem 6.2.8 motivates the next definition.

Definition 6.2.10. *We define the $p_2^* \mathcal{D}_{\mathrm{Bun}}$-module $\bar{\mathcal{L}}$ to be $j_* \mathcal{L}$.*

Using this identification we obtain the identity

$$(\phi \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathrm{id}_{\mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}}})^* \, \underline{\mathrm{Hom}}_{p_3^* \mathcal{D}_{\mathrm{Bun}}}(p_{13}^* \mathcal{L}, p_{23}^* \bar{\mathcal{L}}) = p_{13}^* \bar{\mathcal{P}}$$

on

$$\mathcal{S}_{\mathrm{int}}^0 \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{dR}}^{\mathrm{int}} \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}} \cong \mathcal{S}_{\mathrm{int}}^0 \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}} \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{Dol}}^{\mathrm{int}} .$$

Étale locally on $\mathcal{A}_{\mathrm{int}}^{(1)}$ we can choose a section of $\mathcal{S}_{\mathrm{int}}^0 \to \mathcal{A}_{\mathrm{int}}^{(1)}$. Using descent theory, we conclude the following lemmas.

Lemma 6.2.11. *The sheaf $\bar{\mathcal{L}}$ is Cohen-Macaulay.*

Proof. As we have seen above it is possible to choose a splitting of $\mathcal{D}$ étale locally on $\mathcal{A}_{\mathrm{int}}^{(1)}$, which sends $\bar{\mathcal{L}}$ to $\bar{\mathcal{P}}$. Since the latter sheaf is Cohen-Macaulay and it differs from $\bar{\mathcal{L}}$ only by tensoring with a locally free sheaf, we conclude that étale locally $\bar{\mathcal{L}}$ is Cohen-Macaulay too. According to Corollary 2.1.8 of [BH93], the coherent sheaf $\bar{\mathcal{L}}$ is Cohen-Macaulay if and only if its restriction to the formal neighbourhood of each closed point is Cohen-Macaulay. But since k is assumed to be algebraically closed, there are no non-trivial étale coverings of formal neighbourhoods of closed points ([Mil80, Prop. 4.4]). We may therefore conclude that $\bar{\mathcal{L}}$ is Cohen-Macaulay when restricted to the formal fibres of $\mathcal{M}_{\mathrm{dR}}^{\mathrm{int}} \times_{\mathcal{A}_{\mathrm{int}}^{(1)}} \mathcal{M}_{\mathrm{dR}}^{\mathrm{int}} \to \mathcal{A}_{\mathrm{int}}^{(1)}$, which allows us to conclude that it is Cohen-Macaulay itself. ☐

Lemma 6.2.12. *The Fourier-Mukai functor* $\Phi_{\bar{\mathcal{L}}} : D^b_{coh}(\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}}, \mathcal{O}) \to D^b_{coh}(\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}, \mathcal{D}_{\mathrm{Bun}})$ *is an isomorphism.*

Proof. According to Theorem C in [Aria] the inverse integral kernel to $\bar{\mathcal{P}}$ is a complex supported in a single cohomological degree. This allows us to evoke faithfully flat descent in order to construct the inverse kernel for $\bar{\mathcal{L}}$, which would require more care for a general object of the derived category.

We denote by $\phi : S \to \mathcal{A}^{(1)}_{\mathrm{int}}$ an étale covering, such that $\mathcal{D}$ splits when pulled back to $\phi^* \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}} := S \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}$. The existence of such a ϕ is guaranteed by Theorem 4.6.3. In order to simplify notation we denote the base change of an arbitrary morphism $Z \to \mathcal{A}^{(1)}$ along $S \to \mathcal{A}^{(1)}$ by $\phi^* Z \to S$. The same convention applies to objects defined over Z. The Fourier-Mukai transform $\Phi_{\phi^* \bar{\mathcal{L}}}$ is an equivalence of categories of Fourier-Mukai type, according to the fact that $\phi^* \bar{\mathcal{L}}$ can be naturally related to $\bar{\mathcal{P}}$. Moreover we see that there is an integral kernel $\bar{\mathcal{K}}'$ defined over

$$\phi^* \mathcal{M}^{\mathrm{int}}_{\mathrm{dR}} \times_S \phi^* \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}},$$

which gives rise to the quasi-inverse equivalence $\Phi_{\phi^* \bar{\mathcal{L}}}$. Since $\Phi_{\phi^* \bar{\mathcal{L}}}$ is a Fourier-Mukai transform corresponding to a kernel pulled back along ϕ, it is naturally equipped with descent data along the morphism ϕ. The quasi-inverse $\Phi^{-1}_{\phi^* \bar{\mathcal{L}}}$ is therefore endowed with the same descent data, which its integral kernel $\bar{\mathcal{K}}'$ inherits. As we have already pointed out above, Theorem C in [Aria] implies that $\bar{\mathcal{K}}'$ is a coherent sheaf up to shift, which allows us to descend it to a complex $\bar{\mathcal{K}}$ on

$$\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}} \times_{\mathcal{A}^{(1)}_{\mathrm{int}}} \mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}$$

by faithfully flat descent. We conclude that $\Phi_{\bar{\mathcal{L}}}$ and $\Phi_{\bar{\mathcal{K}}}$ are inverse to each other, since the cohomological computations involved in checking this can be verified étale locally on the base according to the flat base change theorem. □

In order to explain the restriction we have to put on the characteristic of the base field, we need to calculate the arithmetic genus of spectral curves. From the characteristic zero theory one expects the arithmetic genus of spectral curves to be $n^2(h-1)+1$, where n denotes the rank of the Higgs bundles respectively local systems, since the genus of a smooth spectral curve equals the dimension of its Picard, i.e. the corresponding Hitchin fibre. Due to the Lagrangian property of a Hitchin fibre, this dimension is half the dimension of the total space, i.e. the same as the dimension of the moduli space of vector bundles $n^2(h-1)+1$. For general fields we arrive at the same number by a simple Riemann-Roch computation.

Lemma 6.2.13. *The arithmetic genus g of a spectral curve Y_a of a curve X of genus h and a Higgs bundle of rank n is given by $g = n^2(h-1)+1$*

Proof. Because the arithmetic genus is constant in flat families it suffices to calculate the genus of smooth spectral curves. Let $\pi : Y_a \to X$ denote the finite morphism of a smooth spectral curve to X. Since π is finite, we know that $\chi(\pi_* \mathcal{O}_{Y_a}) = \chi(\mathcal{O}_{Y_a})$. The right hand side is given by $1-g$ according to the Riemann-Roch formula. The left hand side is constant in flat families and thus may be computed for a particular spectral curve. If Θ_X denotes the sheaf of tangent vector fields on X, then

$$\pi_* \mathcal{O}_{Y_a} = \bigoplus_{i=0}^{n-1} \Theta_X^{\otimes i}.$$

Combining this with the Riemann-Roch formula we compute

$$\chi(\pi_* \mathcal{O}_{Y_0}) = \sum_{i=0}^{n-1} (1-h+2i(1-h)) = n^2(1-h).$$

In particular we obtain that the arithmetic genus of a spectral curve is given by

$$n^2(h-1)+1.$$

□

This concludes the proof of Theorem 6.2.1. We finish this section by an important remark, which is proved by exactly the same methods as Theorem 6.2.1.

The proposition below is a relativization of Lemma 6.2.12. The case $S = X$ is of particular importance, since it allows us to formulate the Hecke eigenproperty.

Proposition 6.2.14. *Let S be a smooth k-scheme locally of finite type. Then we have an equivalence of derived categories*

$$D^b_{coh}(\mathcal{M}^{\text{int}}_{\text{Dol}} \times T^*S^{(1)}, \mathcal{D}_{\text{Bun}\,\times S}) \cong D^b_{\text{coh}}(\mathcal{M}^{\text{int}}_{\text{dR}} \times T^*S^{(1)}, \mathcal{O}_{\mathcal{M}_{\text{dR}}} \boxtimes \mathcal{D}_S),$$

which is induced by the pullback of $\bar{\mathcal{L}}$.

Proof. Let us denote by $\phi : \mathcal{A}^{(1)}_{\text{int}} \times S \to \mathcal{A}^{(1)}_{\text{int}}$ the canonical projection to the first component. Since $S \to \operatorname{Spec} k$ is faithfully flat, the same holds for the base change ϕ. As in the proof of Lemma 6.2.12 we denote by $\phi^* Z$ the base change of the $\mathcal{A}^{(1)}_{\text{int}}$-scheme Z along the map ϕ. The $\mathcal{O}_{\phi^*\mathcal{M}^{\text{int}}_{\text{dR}}} \boxtimes \phi^* \mathcal{D}_{\text{Bun}}$-module $\phi^* \bar{\mathcal{L}}$ induces a functor

$$D^b_{coh}(\mathcal{M}^{\text{int}}_{\text{Dol}} \times T^*S^{(1)}, \mathcal{D}_{\text{Bun}\,\times S}) \to D^b_{\text{coh}}(\mathcal{M}^{\text{int}}_{\text{dR}} \times T^*S^{(1)}, \mathcal{O}_{\mathcal{M}_{\text{dR}}} \boxtimes \mathcal{D}_S).$$

Using the descent argument of the proof of Lemma 6.2.12 we conclude that this Fourier-Mukai transform is an equivalence. □

6.3 The Hecke eigenproperty

The equivalence of Theorem 6.2.1 can be shown to intertwine two canonical functors with each other. This is expected from the geometric Langlands conjecture over $\mathbb{C}$.

Let $\mathcal{E}$ denote the universal vector bundle over $\mathcal{M}_{\mathrm{dR}} \times X$. It gives rise to a multiplication functor.

$$\begin{aligned} \mathbb{W}: \quad D^b_{\mathrm{coh}}(\mathcal{M}_{\mathrm{dR}}, \mathcal{O}) &\to D^b_{\mathrm{coh}}(\mathcal{M}_{\mathrm{dR}} \times X, \mathcal{O} \boxtimes \mathcal{D}_X) \\ M &\mapsto Lp_1^* M \otimes \mathcal{E} \end{aligned}$$

We define the stack $\mathcal{H}$ to be the classifying stack of the data (E, F, ι, x), such that $E, F \in \mathrm{Bun}$, $\iota : E \to F$ is an injection, $x \in X$ and $\operatorname{coker} \iota$ is a skyscraper sheaf of length one.

Note that $\mathcal{H}$ is equipped with two natural morphisms

$$\begin{array}{ccc} & \mathcal{H} & \\ {}^{q}\swarrow & & \searrow^{p} \\ \mathrm{Bun} & & \mathrm{Bun} \times X \end{array}$$

sending $(E, F, \iota, x) \mapsto F$ respectively $(E, F, \iota, x) \mapsto (E, x)$.

The following remark will be of use later to motivate the definition of Hecke operators in positive characteristic.

Remark 6.3.1. *The stack of Hecke operators $\mathcal{H}$ is actually a moduli stack for a certain type of (quasi-)parabolic bundles (see Section 2.5). According to Proposition 2.5.9, the corresponding moduli stack of (quasi-)parabolic Higgs bundles is equivalent to the cotangent stack $T^* \mathcal{H}$.*

We define the Hecke operator $\mathbb{H}$ to be the functor:

$$\begin{aligned} \mathbb{H}: \quad D^b_{\mathrm{coh}}(\mathrm{Bun}, D) &\to D^b_{\mathrm{coh}}(\mathrm{Bun} \times X, D) \\ M &\mapsto LRp_* LRq^! M. \end{aligned}$$

Whereas the definition of $\mathbb{W}$ makes immediately sense for the smaller stack $\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}}$, it is not obvious that $\mathbb{H}$ descends to a functor

$$D^b_{\mathrm{coh}}(\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}, \mathcal{D}) \to D^b_{\mathrm{coh}}(\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}} \times X, \mathcal{D}_{\mathrm{Bun} \times X}).$$

In order to see that this is the case we need to remind the reader of the definition of the functors p_* and $q^!$, respectively their derived versions LRp_* and $LRq^!$, as defined in Section 3.3.

6.3.1 A reformulation of the Hecke operator

In order to define the functor $q^! : \mathcal{D}_{\mathrm{Bun}} - \mathrm{Mod} \to \mathcal{D}_{\mathcal{H}} - \mathrm{Mod}$ we have to consider the morphism

$$dq^{(1)} : q^{(1),*} T^* \mathrm{Bun}^{(1)} \to T^* \mathcal{H}^{(1)},$$

and use that $q^{(1),*} \mathcal{D}_{\mathrm{Bun}}$ and $dq^{(1),*} \mathcal{D}_{\mathcal{H}}$ are canonically Morita equivalent.

Analogously we need to consider

$$dp^{(1)} : p^{(1),*} T^* (\mathrm{Bun} \times X)^{(1)} \to T^* \mathcal{H}^{(1)},$$

and the Morita equivalence of $dp^{(1),*} \mathcal{D}_{\mathcal{H}}$ with $p^{(1),*} \mathcal{D}_{\mathrm{Bun} \times X}$ to define

$$p_* : \mathcal{D}_{\mathcal{H}} - \mathrm{Mod} \to \mathcal{D}_{\mathrm{Bun} \times X} - \mathrm{Mod}\,.$$

The most natural way to deal with those two morphisms simultaneously is to look at their fibre product

$$\begin{array}{ccc} Z & \longrightarrow & q^*T^*\,\mathrm{Bun} \\ \downarrow & & \downarrow \\ p^*T^*(\mathrm{Bun}\times X) & \longrightarrow & T^*\,\mathcal{H}\,. \end{array}$$

The stack Z is the domain of the morphisms[1]

$$\alpha_1 = q \circ \mathrm{pr}_1 : Z \to \mathcal{M}_{\mathrm{Dol}}(X)$$

and

$$\alpha_2 = p \circ \mathrm{pr}_2 : Z \to \mathcal{M}_{\mathrm{Dol}}(X) \times T^*X.$$

On $Z^{(1)}$ we then have three natural Azumaya algebras, $\alpha_1^*\,\mathcal{D}_{\mathrm{Bun}}$, $\alpha_2^*\,\mathcal{D}_{\mathrm{Bun}\,\times X}$ and $\pi^*\,\mathcal{D}_{\mathcal{H}}$, where $\pi : Z^{(1)} \to \mathcal{H}^{(1)}$ denotes the structural morphism of the fibre product $Z^{(1)}$. By construction, all these algebras are pairwise Morita equivalent.

The base change formula implies that

$$\mathbb{H} : M \mapsto R\alpha_{2,*}L\alpha_1^*M,$$

where the application of Morita equivalences (which involves tensoring by a splitting) has been suppressed to simplify notation. We will later turn this into a definition of $\mathbb{H}$. Let us record the following observation of [BB07, 4.16].

Lemma 6.3.2. *The stack Z is given by the lax 2-functor sending an $\mathcal{A}$-scheme $S \to \mathcal{A}$ to the groupoid classifying $\{x : S \to X \times \mathcal{A}, L_1 \subset L_2\}$, such that $\pi_*L_1, \pi_*L_2 \in \mathcal{M}_{\mathrm{Dol}}$ and $x^*(L_2/L_1)$ is locally free of rank 1. Here we make use of the BNR correspondence (Theorem 2.3.2) to describe Higgs bundles in terms of sheaves on the spectral curves.*

Proof. We prove this lemma on the level of k-points and leave the only notationally

[1]Note that we use the notation p and q to denote morphisms which are really base changes thereof.

different case of S-families to the reader. According to remark 6.3.1 the stack $T^* \mathcal{H}$ classifies the data

$$(E, F, \theta, x, \xi),$$

where $(x, \xi) \in T^* X^{(1)}$, $(E, F, x) \in \mathcal{H}$ and

$$\theta : F \to F \otimes \Omega^1_{\mathrm{X}^{(1)}}(x),$$

such that $\operatorname{res}\theta$ is a nilpotent endomorphism of the fibre $F \otimes k_x$ factoring through a linear map

$$F/E \to E \otimes k_x.$$

The morphism $T^*(\mathrm{Bun} \times X) \times_{\mathrm{Bun} \times X} \mathcal{H} \to T^* \mathcal{H}$ is given by

$$[(F, \theta, x, \xi), (E, F, x)] \mapsto (E, F, \theta, x, \xi).$$

Note that $\operatorname{res}\theta = 0$ in this particular case, since θ is a non-singular Higgs field on F. The morphism $T^* \mathrm{Bun} \times_{\mathrm{Bun}} \mathcal{H} \to T^* \mathcal{H}$ can be described as

$$[(E, \theta), (E, F)] \mapsto (E, F, \theta', x, \xi),$$

where we use that $E|_{X-\{x\}} \cong F|_{X-\{x\}}$ and therefore the Higgs field θ on E induces a Higgs field θ', possibly having a simple pole at x on F. By construction this is a (quasi-)parabolic Higgs bundle. The 1-form ξ at x is the eigenvalue of the Higgs field θ' on the length one quotient F/E. Note that this is a sensible definition since θ' preserves E by construction, and that $\operatorname{res}(\theta) : E/F \to E/F$ is the zero map according to the axioms of parabolic Higgs bundles.

Computing the base change Z now, with this information at hand, we obtain that

Z classifies

$$(E, F, \theta, x, \xi),$$

where (F, θ) is a Higgs bundle, $E \subset F$ is preserved by θ and F/E is a length one sheaf acted on by θ with eigenvalue ξ. □

Finally we obtain a definition of $\mathbb{H}$ which can be used in our context. We observe that the two morphisms to the Hitchin base $Z \to \mathcal{A}$, given by $\chi \circ \alpha_1$ and $\chi \circ \mathrm{pr}_1 \circ \alpha_2$ agree. This is a consequence of Lemma 6.3.2, as a point of Z consists of two Higgs bundles identified away from a point x. In particular they have the same characteristic polynomial. This allows us to view $Z^{(1)}$ as $\mathcal{A}^{(1)}$-stack, and in particular to form the base change over the integral locus.

Remark 6.3.3. *From now on all $\mathcal{A}^{(1)}$-stacks are understood to be restricted to $\mathcal{A}^{(1)}_{\mathrm{int}}$. In order to simplify notation, we will omit the sub- and superscripts specifying that we work over the integral locus.*

Using $Z^{(1)}$ as a correspondence, we obtain a functor

$$\mathbb{H} : D^b_{coh}(\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}, \mathcal{D}_{Bun}) \to D^b_{coh}(\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}} \times T^*X^{(1)}, \mathcal{D}_{Bun \times X}).$$

Interpreting the category on the left-hand-side as a derived category of D-modules on Bun supported on the integral locus, and the right-hand-side as an analogous category of D-modules on $\mathrm{Bun} \times X$, we can be satisfied with $\mathbb{H}$ as a positive characteristic analogue of Hecke functors. The remainder of this paper is devoted to the proof of the following theorem, which is a formal consequence of Theorem 6.3.6 below.

Theorem 6.3.4. *The equivalence of Theorem 6.2.1 intertwines $\mathbb{H}$ with $\mathbb{W}$, i.e. it* *gives rise to the following 2-commutative diagram of derived categories*

$$\begin{array}{ccc} D^b_{\mathrm{coh}}(\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}}, \mathcal{O}) & \longrightarrow & D^b_{\mathrm{coh}}(\mathcal{M}^{\mathrm{int}}_{\mathrm{dR}} \times T^*X^{(1)}, \mathcal{O}_{\mathcal{M}_{\mathrm{dR}}} \boxtimes \mathcal{D}_X) \\ \downarrow & & \downarrow \\ D^b_{\mathrm{coh}}(\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}}, \mathcal{D}_{\mathrm{Bun}}) & \longrightarrow & D^b_{\mathrm{coh}}(\mathcal{M}^{\mathrm{int}}_{\mathrm{Dol}} \times T^*X^{(1)}, \mathcal{D}_{\mathrm{Bun}\times X}). \end{array}$$

6.3.2 The Hecke eigenproperty of the Arinkin-Poincaré sheaf

In order to establish that the equivalence of Theorem 6.2.1 intertwines the Hecke operator $\mathbb{H}$ with a multiplication operator, we will show that the Arinkin-Poincaré sheaf $\bar{\mathcal{P}}$ satisfies a similar property. Given a k-scheme S of finite type and a flat family of integral curves with planar singularities $C \to S$ we can construct the compactified relative Picard stack $\overline{\mathcal{P}\mathrm{ic}} \to S$, which classifies flat families of rank one torsion free sheaves L on C/S (Section 2.3). Let us denote the universal family on $\overline{\mathcal{P}\mathrm{ic}} \times_S C$ by $\bar{\mathcal{Q}}$. The fibre product $\overline{\mathcal{P}\mathrm{ic}} \times_S \overline{\mathcal{P}\mathrm{ic}}$ is endowed with a Cohen-Macaulay sheaf $\bar{\mathcal{P}}_{C/S}$, which induces an equivalence $\Phi_{\bar{\mathcal{P}}_{C/S}} : D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}}) \to D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}})$ ([Aria, Thm. C], Theorem 6.2.8).

Definition 6.3.5. *The stack $\mathcal{H}$ classifies quadruples (L_1, L_2, ι, x), such that $L_i \in \overline{\mathcal{P}\mathrm{ic}}$ and $\iota : L_1 \subset L_2$ and* $\operatorname{coker} \iota$ *is a length one coherent sheaf supported at x. We have a natural morphism $\mathcal{H} \to \overline{\mathcal{P}\mathrm{ic}} \times \overline{\mathcal{P}\mathrm{ic}} \times C$ and composing it with the projections p_2, respectively p_{13} we obtain morphisms $q : \mathcal{H} \to \overline{\mathcal{P}\mathrm{ic}}$ and $p : \mathcal{H} \to \overline{\mathcal{P}\mathrm{ic}} \times_S C$. This allows us to define the* Hecke functor

$$\mathbb{H}_{C/S} = Rp_* \circ Lq^* : D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}}) \to D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}} \times_S C).$$

With this definition at hand, we can state the following theorem, which has been known to Arinkin.

Theorem 6.3.6. *The Fourier-Mukai transform $\Phi_{\bar{\mathcal{P}}}$ intertwines the Hecke functor*

$\mathbb{H}_{C/S}$ *with the multiplication functor* $-\boxtimes^L \bar{\mathcal{Q}}$*. In other words, we have a 2-commutative diagram of categories:*

$$\begin{array}{ccc} D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}}) & \xrightarrow{-\boxtimes^L \bar{\mathcal{Q}}} & D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}} \times_S C) \\ {\scriptstyle \Phi_{\bar{\mathcal{P}}}} \downarrow & & \downarrow {\scriptstyle \Phi_{\bar{\mathcal{P}}_{C\times_S C/C}}} \\ D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}}) & \xrightarrow[\mathbb{H}_{C/S}]{} & D^b_{\mathrm{coh}}(\overline{\mathcal{P}\mathrm{ic}} \times_S C) \end{array}$$

The subsequent proof uses the strategy developed in [Aria]. By the virtue of the base change theorem, we replace S by the moduli stack $\mathcal{M}_g$ of integral curves with planar singularities of arithmetic genus g and study the universal family $\mathcal{C} \to \mathcal{M}_g$. In this case $\overline{\mathcal{P}\mathrm{ic}}$ is a smooth stack (cf. [FGvS99, Thm B.2]).

Translating Theorem 6.3.6 to integral kernels, we see that it suffices to show the following proposition.

Proposition 6.3.7. *We have* $R(p \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})_* L(q \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})^* \bar{\mathcal{P}} = \bar{\mathcal{P}} \boxtimes^L \bar{\mathcal{Q}}$.

We denote by $\mathcal{H}^{\mathrm{sm}}$ the open substack of $\mathcal{H}$ given by the preimage $p^{-1}(\mathcal{P}\mathrm{ic} \times C)$. By definition $\mathcal{H}^{\mathrm{sm}}$ classifies quadruples (L_1, L_2, ι, x), where L_2 is a line bundle. Note that L_1 is then uniquely defined through L_2 and x. It is given by $\mathcal{I}_x \otimes L_2$, i.e. the twist of the ideal sheaf of x by the line bundle L_2. This implies the following Lemma.

Lemma 6.3.8. *The restriction* $p|_{\mathcal{H}^{\mathrm{sm}}}$ *is an isomorphism onto its image.*

Let us now restrict Proposition 6.3.7 to the open substack $\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} C$. The Hecke functor on the left-hand-side may be replaced by $L(q|_{\mathcal{H}^{\mathrm{sm}}} \times \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})^*$, since p is an isomorphism. Both sides of the identity are underived, as $\mathcal{P}$ is an invertible sheaf.

Lemma 6.3.9. *Let* $q : \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C} \to \overline{\mathcal{P}\mathrm{ic}}$ *be the morphism sending a pair* (L, x) *consisting of a line bundle* L *on a curve* C*, and a point* $x \in C$*, to the twist* $L(x) =$

$\underline{\mathrm{Hom}}(\mathcal{I}_x, L)$, *where* $\mathcal{I}_x$ *denotes the ideal sheaf of* $x \in X$*. Then we have*

$$L(q \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})^* \mathcal{P} = \mathcal{P} \boxtimes \mathcal{Q},$$

which implies Proposition 6.3.7 when restricted to $\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}}$*, respectively* $\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \mathcal{C}$.

Proof. According to Lemma 6.3.12,

$$L(q \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})^* \mathcal{P} \cong (q \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})^* \mathcal{P}$$

is a maximal Cohen-Macaulay sheaf. Since

$$\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C}^{\mathrm{sm}} \subset \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \mathcal{C}$$

has a complement of codimension 2, it is thus sufficient to check the identity restricted to

$$\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C}^{\mathrm{sm}}.$$

Over this locus we are able to describe $\mathcal{P}\,|_{\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic}}$ as a family of $\mathbb{G}_m$-extensions of $\mathcal{P}\mathrm{ic}$. In particular, if

$$(m, id) : \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \to \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic}$$

is given by identity in the second component and multiplication in the first, we have an isomorphism

$$(m, id)^* \mathcal{P} \cong p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}.$$

If $A : \mathcal{C}^{\mathrm{sm}} \to \mathcal{P}\mathrm{ic}$ denotes the Abel-Jacobi map $x \mapsto \mathcal{O}_C(x)$ then $q = m \circ (id, A, id)$

and

$$(id, A)^*(\mathcal{P}\,|_{\mathcal{P}\mathrm{ic}\times_{\mathcal{M}_g}\mathcal{P}\mathrm{ic}}) \cong \mathcal{Q}\,|_{\mathcal{P}\mathrm{ic}\times_{\mathcal{M}_g}\mathcal{C}}.$$

Therefore, we obtain

$$(q\times id)^*\,\mathcal{P} \cong (id, A, id)^*(m, \mathrm{id})^*\,\mathcal{P} \cong (id, A, id)^* p_{13}^*\,\mathcal{P} \otimes p_{23}^*\,\mathcal{P} \cong p_{13}^*\mathcal{P} p_{23}^*\,\mathcal{Q}\,.$$

□

Lemma 6.3.10. *For every $C \in \mathcal{M}_g$ there exists a versal deformation $\mathcal{C}'$ along a complete local ring $U = \operatorname{Spec}\hat{R} \to \mathcal{M}_g$, such that $U \times_{\mathcal{M}_g} \mathcal{C}$ has a Zariski covering $\{V_i\}_{i\in I}$ and for every i we have an open immersion $V_i \to U \times \mathbb{A}^2$.*

Proof. This is a combination of Lemma A.2 and Proposition A.3 in [FGvS99]. □

Lemma 6.3.11. *The morphism $\mathcal{H} \to \mathcal{M}_g$ is syntomic[2] of relative dimension $g+1$. Moreover it is fibrewise irreducible.*

Proof. As we have seen in Proposition 6.2.2, for $d \geq 2g-1$ the morphism $(\mathcal{C}\,/\,\mathcal{M}_g)^{[d]} \to \overline{\mathcal{P}\mathrm{ic}}^d(\mathcal{C})$ is smooth and faithfully flat. Under the dictionary between effective divisors and torsion sheaves, provided by the Abel map, $\mathcal{H}$ corresponds to the nested Hilbert scheme $(\mathcal{C}\,/\,\mathcal{M}_g)^{n+1,n}$. It is easy to conclude that the relative dimension of $\mathcal{H}$ is $g+1$. We only need to show that for an integral curve C of arithmetic genus g and planar singularities, $\dim C^{[n+1,n]} = n+1$. This can be done as in the proof of Theorem 5 in [AIK77], complementing the estimate (5.1) by the same inequality for nested Hilbert schemes (Proposition 2.3 in [EL99]).

We will show that $(\mathcal{C}\,/\,\mathcal{M}_g)^{n+1,n}$ is a locally complete intersection. As we have seen in Lemma 6.3.10 we can pick a versal deformation $\mathcal{C}' \to U$ of $C \in \mathcal{M}_g$ together with a Zariski covering $\bigcup_{i\in I} V_i = \mathcal{C}'$ and open immersions $V_i \to U \times \mathbb{A}^2$. Therefore,

[2] i.e. locally of finite presentation, flat and fibrewise of locally complete intersection

we obtain a cartesian square

$$\begin{array}{ccc} (V_i/U)^{d+1,d} & \longrightarrow & (U \times \mathbb{A}^2/U)^{[d+1,d]} \\ \downarrow & & \downarrow \\ (V_i/U)^{[d+1]} & \longrightarrow & (U \times \mathbb{A}^2/U)^{[d+1]}. \end{array}$$

It is known (cf. [AIK77, Cor. 7]) that $(V_i/U)^{[d+1]} \subset (U \times \mathbb{A}^2/U)^{[d+1]}$ is a locally complete intersection. Since its base change $(\mathcal{C}'/U)^{[d+1]}$ has the same codimension in $(U \times \mathbb{A}^2/U)^{[d+1,d]}$, and the latter stack is smooth[3], we have shown that the total space $\mathcal{H}$ is a locally complete intersection, in particular it is Cohen-Macaulay. Because $\mathcal{M}_g$ is smooth ([Arib, Prop. 4]) we see that $\mathcal{H} \to \mathcal{M}_g$ is flat, since the dimension of the fibres is constant. Applying the above argument fibrewise, we see that the fibres are locally complete intersections. Irreducibility is verified as in the proof of Theorem 5 in [AIK77]. □

Lemma 6.3.12. *The complex* $L(q \times \mathrm{id})^* \bar{\mathcal{P}}$ *is a Cohen-Macaulay sheaf* $(q \times \mathrm{id})^* \bar{\mathcal{P}}$.

Proof. Lemma 2.3 in [Aria] states that derived pullback along Lf^* along a morphism $f : Y \to X$ of schemes preserves maximal Cohen-Macaulay sheaves, if X is Gorenstein of pure dimension, Y is Cohen-Macaulay, and f is Tor-finite.

We already know that $\mathcal{H} \to \mathcal{M}_g$, and hence its base change $\mathcal{H} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}}$ are syntomic. In particular we can conclude that both spaces are locally complete intersections (i.e. also Cohen-Macaulay), since $\mathcal{M}_g$ is smooth. Similarly one concludes that $\overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}}$ is a locally complete intersection (i.e. Gorenstein). According to Theorem B.2 in [FGvS99] the total space of $\overline{\mathcal{P}\mathrm{ic}}$ is smooth. For this reason every morphism mapping into $\overline{\mathcal{P}\mathrm{ic}}$ is Tor-finite. Since Tor-finite morphisms are preserved by flat base change, we may conclude that $\mathcal{H} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}} \to \overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}}$ is Tor-finite. □

[3] This is proved in [Che98], see section 0.2 for a statement of his result. Although the author assumes $k = \mathbb{C}$ this statement and its proof are true for general algebraically closed fields.

The final ingredient in the proof of the Heck eigenproperty is the following characterization of Cohen-Macaulay sheaves as objects in the derived category, given in Proposition A.4.5. We intend to apply this result to the integral kernel Θ of the Fourier-Mukai transform $\Phi_{\bar{\mathcal{P}}^\vee} \circ \mathbb{H}_{C/\mathcal{M}_g} \circ \Phi_{\bar{\mathcal{P}} \boxtimes \mathcal{O}_{\mathcal{C}}}$. If we can show that it is a Cohen-Macaulay sheaf of the right codimension, then it suffices to determine it in a complement of a closed subset (of the support) of codimension 2.

Lemma 6.3.13. *We have* $\operatorname{codim} \operatorname{supp} \Theta \geq g$ *and every maximal-dimensional component intersects* $\Delta_{\overline{\mathcal{P}\mathrm{ic}}/\mathcal{M}_g} \times_{\mathcal{M}_g} \mathcal{C}$.

Proof. Let $(F_1, F_2, x) \in \operatorname{supp} \Theta$. By base change this is the case if and only if there exists $i \in \mathbb{Z}$, such that $\mathbb{H}^i(\mathcal{H}, \mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee) \neq 0$. Similarly to the proof of Proposition 7.2 of [Aria] we claim that $\mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee$ is T-equivariant, where T denotes the $\mathbb{G}_m$-extension of $\mathcal{P}\mathrm{ic}$ associated to $\mathcal{P}_1 \otimes \mathcal{P}_2^\vee$.

We denote by T_i the $\mathbb{G}_m$-extension of $\mathcal{P}\mathrm{ic}$ associated to $\bar{\mathcal{P}}_{F_i}$. The sheaf $\bar{\mathcal{P}}_{F_i}$ has a natural T_1-equivariant structure ([Aria, Lemma 6.5]). The morphisms p and q in the correspondence diagram defining $\mathbb{H}$ are T_1-equivariant. Consequently $\mathbb{H}(\bar{\mathcal{P}}_{F_1})$ is an element of the T_1-equivariant derived category (i.e. an object in the derived category of the stack $[(\overline{\mathcal{P}\mathrm{ic}} \times C_s)/T_1]$). Consequently, the tensor product $\mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee$ lies in the T-equivariant derived category.

The hypercohomology group $\mathbb{H}^i(\mathcal{H}, \mathbb{H}(\bar{\mathcal{P}}_{F_1}) \otimes \bar{\mathcal{P}}_{F_2}^\vee) \neq 0$ carries an induced T-action, such that the $\mathbb{G}_m$-part acts tautologically. If this group was non-zero, there would be a one-dimensional T-invariant subspace, as T is abelian. This would provide a splitting of the extension $0 \to \mathbb{G}_m \to T \to \mathcal{P}\mathrm{ic} \to 0$. We conclude that $F_1|_{C^{\mathrm{sm}}} = F_2|_{C^{\mathrm{sm}}}$ by pulling back along the Abel-Jacobi map. If $\tilde{g}$ denotes the genus of the normalization of C, then the dimension of the subspace of pairs of line bundles of rank 1 satisfying $F_1|_{C^{\mathrm{sm}}} = F_2|_{C^{\mathrm{sm}}}$ is $2g - \tilde{g}$. But by Proposition 6 in [Arib] the strata $\mathcal{M}^{\tilde{g}}$ of curves of geometric genus $\tilde{g}$ has codimension $\geq g - \tilde{g}$. This proves the first part of the claim.

To prove the second assertion it suffices to note that Lemma 6.3.9 implies

$$\operatorname{supp}\Theta \cap \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C}^{\mathrm{sm}} = \Delta_{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \mathcal{C}^{sm}.$$

This is sufficient, since an irreducible component of $\operatorname{supp}\Theta$, which does not intersect this smooth locus, must have even higher codimension. Hence we see that every top-dimensional irreducible component intersects $\Delta_{\overline{\mathcal{P}\mathrm{ic}}/\mathcal{M}_g} \times_{\mathcal{M}_g} \mathcal{C}$. □

Lemma 6.3.14. *We have* $\Theta \in D^{\leq g}(\mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{P}\mathrm{ic} \times_{\mathcal{M}_g} \mathcal{C})$.

Proof. We denote by $\mathbb{H}\,\bar{\mathcal{P}}$ the complex

$$R(p \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})_* L(q \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})^* \,\bar{\mathcal{P}} \in D^b_{coh}(\overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \mathcal{C}).$$

We already know from Lemma 6.3.12 that $L(q \times_{\mathcal{M}_g} \mathrm{id}_{\overline{\mathcal{P}\mathrm{ic}}})^* \,\bar{\mathcal{P}}$ is a Cohen-Macaulay sheaf. As seen in Lemma 6.3.11 the morphism $\mathcal{H} \to \mathcal{M}_g$ is fibrewise irreducible and of dimension $g+1$. In particular we conclude that the dimension of fibres of $\mathcal{H} \to \overline{\mathcal{P}\mathrm{ic}}$ is bounded by g, hence $\operatorname{supp} H^i(\mathbb{H}\,\bar{\mathcal{P}})$ is of relative dimension $\leq g-i$ over the parametrizing component $\overline{\mathcal{P}\mathrm{ic}}$.

The integral kernel Θ is given by convolution

$$\bar{\mathcal{P}}^\vee * \mathbb{H}\,\bar{\mathcal{P}} = Rp_{13,*}(Lp_{12}^*\,\bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^*\,\mathbb{H}\,\bar{\mathcal{P}}).$$

The dimension estimate above implies that $H^i(\Theta) = 0$ if $i > g$. □

Proof of Theorem 6.3.6. We apply Proposition A.4.5 to the integral kernel $\Theta[g]$. We have already checked two of the three necessary conditions in Lemma 6.3.13 and 6.3.14. Moreover we know that the theorem is true when restricted to the complement of a codimension two subvariety (cf. Lemma 6.3.9). Therefore it suffices to check the last condition of Proposition A.4.5: we need to show that $H^i(\mathbb{D}\Theta[g]) = 0$ if $i > g$. From

Grothendieck-Serre duality it follows that

$$\mathbb{D}\Theta = \mathbb{D}Rp_{13,*}(Lp_{12}^*\,\bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^*\,\mathbb{H}\,\bar{\mathcal{P}}) = (Rp_{13,*}\mathbb{D}(Lp_{12}^*\,\bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^*\,\mathbb{H}\,\bar{\mathcal{P}}))[g],$$

which in turn can be simplified as

$$Rp_{13,*}\mathbb{D}(Lp_{12}^*\,\bar{\mathcal{P}}^\vee \otimes^L Lp_{23}^*\,\mathbb{H}\,\bar{\mathcal{P}}) = Rp_{13,*}[\omega_{\mathcal{P}\mathrm{ic}^3}p_{23}^*\omega_{\mathcal{P}\mathrm{ic}^2}^{-1}Lp_{12}^*\,\bar{\mathcal{P}} \otimes^L (Lp_{23}^*\mathbb{D}\,\mathbb{H}\,\bar{\mathcal{P}})].$$

Using Grothendieck-Serre duality again we see that

$$\mathbb{D}\,\mathbb{H}\,\bar{\mathcal{P}} = (p\times \mathrm{id})_*\mathbb{D}(q\times\mathrm{id})^*\,\bar{\mathcal{P}}\,.$$

According to Lemma 6.3.12 the sheaf $(q\times\mathrm{id})^*\,\bar{\mathcal{P}}$ is Cohen-Macaulay, therefore $\mathbb{D}(q\times \mathrm{id})^*\,\bar{\mathcal{P}}$ is a sheaf itself. Applying the same reasoning as in Lemma 6.3.14 we see that $\mathbb{D}\Theta \in D^{\leq 0}(\overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \overline{\mathcal{P}\mathrm{ic}} \times_{\mathcal{M}_g} \mathcal{C})$. We conclude that $\Theta[g]$ is a Cohen-Macaulay sheaf. □

Appendix A

Appendix

A.1 Properties of algebraic stacks

The following definition is reminiscent of Proposition 4.15(i) in [LMB00].

Definition A.1.1. *Let $\mathcal{X}$ be a stack. We say that it is* locally of finite presentation*, if for every filtered inverse system of affine schemes (T_i), the obvious morphism of groupoids*

$$\varinjlim \mathcal{X}(T_i) \to \mathcal{X}(\varprojlim T_i)$$

is an equivalence.

By definition, stacks which are locally of finite presentation, are defined by their restriction to the subcategory of affine schemes of finite presentation. This turns out to be useful in the study of moduli problems, where stacks often can be shown to be locally of finite presentation, using the concrete moduli problem at hand. In turn it suffices to restrict attention to families parametrized by affine schemes of finite presentation, since the general case can be described in terms of the colimit above.

The definition of formally smooth morphism is contained in Proposition 4.15(ii) in [LMB00].

Definition A.1.2. *A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is called* formally smooth *if for every affine scheme V and every closed immersion $U \to V$ given by a quasi-coherent square-zero sheaf of ideals, there exists a dotted morphism as in the diagram below:*

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{X} \\ \downarrow & \nearrow & \downarrow \\ V & \longrightarrow & \mathcal{Y}\,. \end{array}$$

It is called formally étale*, if the lifting is moreover unique.*

It is illuminating to take a closer look at this definition in case of $\mathcal{X}$ and $\mathcal{Y}$ being smooth $\mathbb{C}$-schemes, therefore giving rise to smooth manifolds X and Y. A smooth map f between smooth schemes gives rise to a *submersion* of manifolds. The Inverse Function Theorem tells us that every point y of Y has a neighbourhood W in the standard topology, over which a section $s : W \to X$ exists, sending y to a chosen point $x \in f^{-1}(y)$. Since W is defined in terms of inequalities, it is unreasonable to expect a similar statement to hold for Zariski neighbourhoods. Nonetheless, Definition A.1.2 should be read as stating that a section exists in infinitesimally small formal neighbourhoods of every point $y \in Y(\mathbb{C})$. Let us denote by $\mathbb{D}_y := \operatorname{Spec} \widehat{\mathcal{O}}_{Y,y}$ the formal neighbourhood of the complex point y. It is a formal scheme, given by the system of Artinian schemes $U_i := \operatorname{Spec} \mathcal{O}_{Y,y} / \mathfrak{m}_y^{i+1}$. Since $U_0 = y$, we set $s_0 = x \in f^{-1}(y)(\mathbb{C})$. Applying the lifting lemma for $U = U_{i+1}$ and $V = U_i$ we can then recursively construct a compatible system of sections $s_i : U_i \to X$, sending y to x. In particular we obtain such a section $s : \mathbb{D}_y \to X$.

A.2 Cotangent stacks

A quasi-coherent sheaf of commutative algebras A on an algebraic stack $\mathcal{X}$ is given by a compatible choice of quasi-coherent sheaves of algebras A_U on every affine scheme

U mapping into $\mathcal{X}$. Since U is affine, each A_U is given by an actual commutative algebra. Taking the spectrum of A_U we obtain a map of affine schemes $\operatorname{Spec} A_U \to U$. The stack $\operatorname{Spec}_{\mathcal{X}} A$ is by definition given by the functor sending U to the set of sections $Hom_U(U, \operatorname{Spec} A_U)$. Every atlas for $\mathcal{X}$ induces an atlas for $\operatorname{Spec}_{\mathcal{X}} A$ by means of the above construction, implying algebraicity of $\operatorname{Spec}_{\mathcal{X}} A$.

In chapter 17 of [LMB00] the cotangent complex of an algebraic stack is defined. Taking the dual of that complex and then zeroth cohomology, one obtains the quasi-coherent sheaf θ_X of tangent vector fields.

Definition A.2.1. *Let $\mathcal{X}$ be an algebraic stack. The relative spectrum of* $\operatorname{Sym} \theta_{\mathcal{X}}$ *will be denoted by* $\mathrm{T}^*\mathcal{X}$ *and is referred to as the cotangent stack of* $\mathcal{X}$.

The first paragraph's discussion implies the lemma below.

Lemma A.2.2. *The cotangent stack of an algebraic stack $\mathcal{X}$ is algebraic.*

Proof. By definition, the morphism $\mathrm{T}^*\mathcal{X} \to \mathcal{X}$ is affine. Hence it is in particular representable, which implies algebraicity of $\mathrm{T}^*\mathcal{X}$. □

It is important to point out that one cannot expect $\mathrm{T}^*\mathcal{X}$ to be smooth, even if $\mathcal{X}$ is a smooth algebraic stack. In general, the dimension of $\mathrm{T}^*\mathcal{X}$ is bounded from below by twice the dimension of $\mathcal{X}$.

Definition A.2.3 (Beilinson-Drinfeld). *A smooth stack is called* good*, if*

$$\dim \mathrm{T}^*\mathcal{X} = 2 \dim \mathcal{X} .$$

It is called very good*, if it has an open dense Deligne-Mumford substack.*

The concept of good stacks has been introduced by Beilinson and Drinfeld in their book [BD, p. 6]. Also the following results are due to them and can be found in *loc. cit.*

Lemma A.2.4. *If* $\mathcal{X}$ *is a good stack, then* $\mathrm{T}^*\mathcal{X}$ *is a* locally complete intersection stack *of dimension* $2\dim\mathcal{X}$*.*

A quotient stack BG, where G is an algebraic group space, is good if and only if G is discrete. Nonetheless, moduli stacks of bundles often turn out to be good.

Proposition A.2.5 (Beilinson-Drinfeld)**.** *Let* G *be a semisimple algebraic group and* X *a curve defined over an algebraically closed field of zero characteristic. Then* $\mathrm{Bun}_G(X)$ *is a very good stack.*

This is Proposition 2.1.2 in [BD].

A.3 Derived categories

We begin by reviewing the theory of quasi-coherent sheaves and their derived categories on stacks. A good summary of this theory, together with theoretical justification for some of the definitions given below, is contained in [AB09, Sect. 2].

The data of a quasi-coherent sheaf F on a prestack $\mathcal{X}$ is equivalent to a collection of quasi-coherent sheaves $F_{U\to\mathcal{X}}$ for every affine scheme U with a morphism $U\to\mathcal{X}$, in a way compatible with pullback. This compatibility condition stipulates the existence of isomorphisms

$$\phi_{V\to U}:\psi^*F_{U\to\mathcal{X}}\to F_{V\to\mathcal{X}}$$

for every morphism $\psi:V\to U$ of $\mathcal{X}$-schemes, which are required to obey a compatibility law of their own. In the language of category theory we have exhibited the category of quasi-coherent sheaves on $\mathcal{X}$ as the 2-limit of the categories $\mathrm{QCoh}(U)$ of quasi-coherent sheaves on U

$$\mathrm{QCoh}(\mathcal{X}):=\lim_{U\in\mathrm{Aff}/\mathcal{X}}\mathrm{QCoh}(U).$$

If $\mathcal{X}$ is an algebraic stack it is possible to replace the above 2-limit by a less intimidating one. Let $Y \to \mathcal{X}$ be an atlas, i.e. a smooth surjective morphism, where Y is a scheme. Faithfully flat descent theory implies that $\mathrm{QCoh}(\mathcal{X})$ is equivalent to the 2-limit

$$\mathrm{QCoh}(\mathcal{X}) \cong \lim[\mathrm{QCoh}(Y) \rightrightarrows \mathrm{QCoh}(Y \times_{\mathcal{X}} Y) \Rrightarrow \mathrm{QCoh}(Y \times_{\mathcal{X}} Y \times_{\mathcal{X}} Y)\ ].$$

This 2-limit amounts to the simple fact that the data of a quasi-coherent sheaf on $\mathcal{X}$ is equivalent to a quasi-coherent sheaf F_Y on the atlas Y endowed with descent data. In the special case that $\mathcal{X}$ is a global quotient stack $[Y/G]$, where G is a smooth algebraic group scheme, this descent data amounts to a G-equivariant structure on F_Y ([FGI+05, Def. I.3.46]).

Below we give a definition of the bounded derived category of coherent sheaves $D^b_{coh}(\mathcal{X})$ on a stack $\mathcal{X}$. In the cases of interest to us this definition is equivalent to the one given in [LMB00], but in the case of the unbounded derived category $D_{qcoh}(\mathcal{X})$ of quasi-coherent sheaves, we prefer to use slightly more machinery.

Definition A.3.1. *Let $\mathcal{X}$ be a quasi-compact algebraic stack with affine diagonal and atlas $Y \to X$. We define the bounded derived category $D^b_{coh}(\mathcal{X})$ of coherent sheaves on $\mathcal{X}$ to be the full subcategory of the derived category of* $\mathrm{QCoh}(\mathcal{X})$ *of complexes $F^\bullet$ whose cohomology sheaves are coherent when pulled back to Y and vanish for almost all degrees.*

It is a well-known fact that the non-functoriality of cones leads to technical complications in the theory of derived categories. For instance, it is not possible to obtain $D_{qcoh}(\mathcal{X})$ as a 2-limit of the derived categories $D_{qcoh}(U)$ for affine schemes $U \to \mathcal{X}$ as we did it for the abelian category above. And neither is the category of G-equivariant objects in $D_{qcoh}(Y)$ equivalent to the derived category of the quotient stack $[Y/G]$. This defect of $D_{qcoh}(\mathcal{X})$ can be fixed by replacing the derived category by an *enhancement*, i.e. a closely related object, from which $D_{qcoh}(\mathcal{X})$ can be fully recovered, but which possesses a functorial construction of cones. One way to do this is by using the theory of stable ∞-categories [Lur]. Every affine scheme U has an associated stable ∞-category $QC(U)$, whose homotopy category is the derived category of quasi-coherent sheaves on U. For a prestack $\mathcal{X}$ one defines $QC(\mathcal{X})$ as the homotopy limit of ∞-categories

$$QC(\mathcal{X}) := \lim_{U \in \mathrm{Aff}/\mathcal{X}} QC(U),$$

in analogy with the definition of the category of quasi-coherent sheaves $\mathrm{QCoh}(\mathcal{X})$ given at the beginning of this section.

Definition A.3.2. *Let $\mathcal{X}$ be an algebraic stack, the derived category of quasi-coherent sheaves $D_{qcoh}(\mathcal{X})$ is defined to be the homotopy category of the stable ∞-category $QC(\mathcal{X})$.*

The inherent functoriality in the language of stable ∞-categories allows straightforward constructions, which would be more intricate in the world of triangulated categories.

Lemma A.3.3. *Let X and Y be two schemes, endowed with an action of an abstract finite group Γ; we assume that there is an equivalence of ∞-categories*

$$QC(X) \cong QC(Y),$$

which is Γ-equivariant. This induces an equivalence

$$QC([X/\Gamma]) \cong QC([Y/\Gamma]).$$

Proof. Since $X \to [X/\Gamma]$ is an atlas for the stack $[X/\Gamma]$ it is possible to write $QC([X/\Gamma])$ as the homotopy limit

$$\lim_{J \in \Delta^{op}} QC(X^{[J]}),$$

where Δ denotes the category of finite non-empty ordered sets and

$$X^{[J]} := X \times \Gamma^J.$$

Let $B\Gamma$ be the nerve of the groupoid associated to the group Γ. The Γ-action on X induces an action on $QC(X)$, which is encoded by an ∞-functor from $B\Gamma$ to the ∞-category of ∞-categories

$$\mathsf{act} : B\Gamma \to \infty - \mathrm{Cat}.$$

The above homotopy limit can be rewritten as

$$\lim_{B\Gamma} \mathsf{act},$$

which is a purely ∞-categorical construction, and therefore depends only on the ∞-category $QC(X)$ and the Γ-action up to equivalence. In general we refer to such a limit as the ∞-category of Γ-equivariant objects in a ∞-category. As the equivalence $QC(X) \cong QC(Y)$ respects the Γ-action, we obtain that the ∞-categories of Γ-equivariant objects in $QC(X)$ and $QC(Y)$ must be equivalent. In particular we have

$$QC([X/\Gamma]) \cong QC([Y/\Gamma]).$$

□

Even more generally, for an ∞-groupoid G, which is pointed and connected, and an ∞-functor $\mathsf{act} : G \to \infty - \mathrm{Cat}$, we should think of the homotopy limit $\mathcal{C}^\Gamma := \lim_G \mathsf{act}$ as an ∞-category of G-equivariant objects in an ∞-category $\mathcal{C}$. If $\mathcal{C}$ is stable (in particular its homotopy category is triangulated) then so is $\mathcal{C}^\Gamma$ according to Theorem 5.4 in [Lur]. In [Sos11] an alternative *linearization* procedure is described for triangulated categories having a strongly pre-triangulated dg-model. Using this definition of linearization, P. Sosna also obtains an analogue of Lemma A.3.3 in *loc. cit.*

A.4 Cohen-Macaulay sheaves

Definition A.4.1. *Let X be a scheme of pure dimension of finite type over a field k, with a dualizing functor*

$$\mathbb{D}_X : D^b_{coh}(X)^{op} \to D^b_{coh}(X).$$

A coherent sheaf $\mathcal{F}$ on X is called Cohen-Macaulay *of codimension d, if and only if $\mathbb{D}\,\mathcal{F}[d]$ is a coherent sheaf. Cohen-Macaulay sheaves of codimension zero will also be called* maximal*. The category of maximal Cohen-Macaulay sheaves on X will be denoted by* $\mathrm{CM}(X)$*.*

We emphasize that we use the convention where $\mathbb{D}\,\mathcal{O}_X \cong \omega_X$ is the canonical bundle for X being Gorenstein. It is also possible to introduce Cohen-Macaulay sheaves avoiding the formalism of duality. We refer the reader to [BH93] and [BD08] for an approach within the world of commutative algebra.

Lemma A.4.2. *Let X and Y be two schemes of finite type over a field k (with dualizing functor $\mathbb{D}_X$, respectively $\mathbb{D}_Y$), which are moreover of pure dimension. If $\pi : Y \to X$ is a finite morphism, then a sheaf $\mathcal{F}$ on Y is Cohen-Macaulay of codimension d if and only if $\pi_*\mathcal{F}$ is Cohen-Macaulay of codimension d.*

Proof. The dualizing functor commutes with push-forward along finite morphisms

$$\pi_* \circ \mathbb{D}_Y \cong \mathbb{D}_X \circ \pi_*.$$

Moreover π_* allows us to detect in which degrees a complex is supported, for π being finite. Therefore we conclude that $\mathbb{D}_Y \mathcal{F}[d]$ is a coherent sheaf if and only if $\mathbb{D}\,\pi_* \mathcal{F}$ is. □

A proof of the lemma below is given in [BD08, Cor. 2.11].

Lemma A.4.3. *If X is a regular scheme, then every maximal Cohen-Macaulay module is locally free.*

We refer the reader to [BD08, Prop. 3.2] for a proof of the lemma below. The construction described there, allows us to associate to every coherent sheaf $\mathcal{F}$ on a surface X a Cohen-Macaulay sheaf, which is generically isomorphic to $\mathcal{F}$.

Lemma A.4.4. *Let X be a surface, and $\mathcal{F}$ a coherent sheaf on X. The bidual $\mathcal{F}^{\vee\vee}$ is maximal Cohen-Macaulay and the functor $\mathcal{F} \mapsto \mathcal{F}^{\vee\vee}$ is the left adjoint to the inclusion functor $\mathrm{CM}(X) \to \mathrm{Coh}(X)$.*

It is important to remark that by virtue of the fact that the Cohen-Macaulay property is local in the smooth topology, we can lift the definitions and results of this appendix to algebraic stacks of finite type over a field k.

The following result is Lemma 7.7 in [Aria]. It gives a characterization of Cohen-Macaulay sheaves on X amongst the objects of the derived category $D^b_{coh}(X)$.

Proposition A.4.5. *Let X be an algebraic stack of finite typer over a field k, which is moreover of pure dimension. Then $\mathcal{F}^\bullet \in D^b_{coh}(X)$ is a Cohen-Macaulay sheaf of codimension d if and only if the following conditions are satisfied*

(a) $\operatorname{codim} \operatorname{supp} \mathcal{F}^\bullet \geq d$,

(b) $H^i(\mathcal{F}^\bullet) = 0$ *for* $i > 0$,

(c) $H^i(\mathbb{D}_X \mathcal{F}^\bullet) = 0$ *for* $i > d$.

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